Pretty triangles on ugly grids

by

Lars de Hoop

to obtain the degree of Bachelor of Science
at the Delft University of Technology,
to be defended publicly on Wednesday June 26, 2024 at 9:45 AM.

Student number:	5644690	
Project duration:	April 20, 2024 – June 26, 2024	
Thesis committee:	Dr. N. Verhulst,	TU Delft, supervisor
	Dr. B van den Dries,	TU Delft

An electronic version of this thesis is available at
http://repository.tudelft.nl/.

TU Delft

Layman's summary

Equable triangles are triangles that have equal area and perimeter. This paper aims to find all equable triangles that can be placed on a grid. A grid is a set of points that is placed in a regular way, similar to, for example, a chessboard or a beehive. In this paper, we look for a general method of finding these equable triangles on an arbitrary grid. Adapting the proof by Aebi and Cairns [1], we were able to find all equable triangles on a large amount of grids, if possible. On most grids, no equable triangles could be placed. When it was possible however, the equable triangles often had integer side lengths.

Summary

Equable triangles are triangles that have equal area and perimeter. Although there are infinitely many such triangles, this paper aims to find all equable triangles, up to Euclidean motions, that can be placed on a lattice. A lattice is a set of points that is generated with integral combinations of a set of basis vectors. Examples of lattices are $\mathbb{Z}^2$ and the Eisenstein lattice. In 1980, Foss proved that there are 5 different equable triangles that can be placed on the integer lattice [6]. In 2023, Aebi and Cairns proved that there are only 2 equable triangles placeable on the Eisenstein lattice [1]. This paper tries to generalize these proofs for other 2-dimensional lattices.
We did this to all lattices generated by two unit vectors at an angle θ such that $\cos(\theta)$ is rational. To find all equable triangles on such a grid, we follow three steps:

1. Find r such that all side lengths of equable triangles are of the form rn with $n \in \mathbb{N}$.

2. Find all equable triangles with side lengths rn.

3. Check for each of these triangles if they can be placed on the grid.

When applying these steps to all grids with $\cos(\theta) = \frac{p}{q}$ for co-prime $p, q \in \mathbb{Z}$ with $0 \leq p < q \leq 100$, we found that, for most values, there were no equable triangles placeable on the grid. When there were equable that could be placed however, they often had integer side lengths. We found that can only be possible if p and q are part of a Pythagorean triple.

Contents

Chapter 1

Introduction

Grids or lattices can be found frequently in nature. Look for example at the hexagonal shape of beehives or look at the molecular structure of crystals. Apart from their occurrences in nature, they also have applications in mathematics and computer science. For example in cryptography [10]. This paper aims to examine geometric properties of lattices and in particular look for so-called equable triangles that can be placed on such lattices.

An equable triangle is a triangle that has equal area and perimeter. It can be shown that there are infinitely many such triangles. However, we look for equable triangles for which the vertices lie on a lattice. For two specific grids, the number of different equable triangles up to Euclidean motions has already been found. In 1980, Foss showed that there are 5 different triangles that can be placed on the integer lattice [6]. In 2023, Aebi and Cairns proved that there are only 2 different equable triangles possible on the so-called Eisenstein lattice [1]. This last paper inspired me to look for equable triangles on arbitrary grids.

In Chapter 2, we introduce a definition for grids (Section 2.1) and use symmetries to define different lattice types (Section 2.2). We provide the definition for equable triangles (Section 2.3) and give theorems we will use later in the paper (Section 2.4). In Chapter 3, we will look for possible candidates for equable triangles on an arbitrary grid. This is done by first placing a constraint on the side lengths of an equable triangle (Section 3.1) and then finding all equable triangles satisfying that constraint (Section 3.2). In Chapter 4, we use a brute-force approach to place these triangles on a grid. In Chapter 5, we combine these concepts to find all equable triangles on a grid by implementing them in Python. We will run this program for various grids and look for patterns.

Chapter 2

Preliminaries

The aim of this report is to find all equable triangles on arbitrary grids. Before we do this however, we first need formal definitions for these concepts. In Section 2.1, we define a grid and give some examples. In Section 2.2, we look at symmetries in grids and use them to define different lattice types. In Section 2.3, we give a definition for equable triangles and give some simple results. Lastly, in Section 2.4, we give some other well-known theorems that will be used in proofs later.

2.1 Grids

In the book "Complexity of Lattice Problems" [9], Micciancio and Goldwasser define a general lattice or grid in $\mathbb{R}^n$ as the integral combinations of linearly independent vectors. We will use the same definition in this paper.

Definition 2.1. A *lattice* or *grid* in $\mathbb{R}^n$ is a set

$$\Lambda(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m) = \Big\{ \sum_{i=1}^{m} a_i \mathbf{v}_i : a_i \in \mathbb{Z} \Big\},$$

where the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m \in \mathbb{R}^n$ are linearly independent. The set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ is called a *basis* of Λ. The values n and m are called the *dimension* and *rank* of the lattice. We say a lattice is *full rank* if $n = m$.

In this paper, we will consider full rank lattices in $\mathbb{R}^2$, i.e. lattices with two linearly independent vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.
These lattices are also sometimes defined using complex numbers instead of vectors. In that case we get a lattice $\Lambda(\omega_1, \omega_2)$ with $\omega_1, \omega_2 \in \mathbb{C}$. The complex numbers ω_1 and ω_2 are also called the fundamental pair of periods. We will use this definition in the following chapters. In this chapter however, we will mostly use the vector definition as this is common for theorems involving bases.

Example 2.2. The most basic example is the integer lattice $\mathbb{Z}^2$. This lattice can be generated by the vectors $\mathbf{e}_1 = [1, 0]^T$ and $\mathbf{e}_2 = [0, 1]^T$. It consists of all pairs of integers and is shown in Figure 2.1.

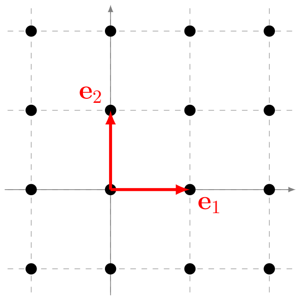
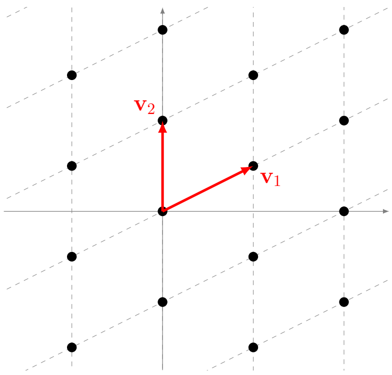


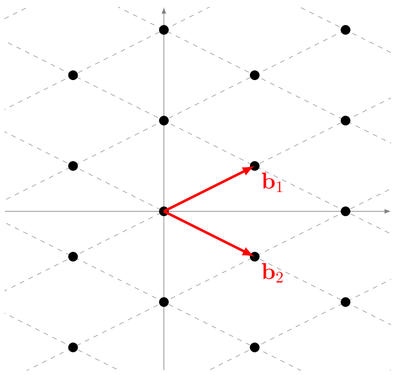
Figure 2.1: Integer Lattice from Example 2.2.

This concept can be extended to an n-dimensional integer lattice $\mathbb{Z}^n$. This can be done with vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ where $\mathbf{e}_i = [0, \ldots, 0, 1, 0, \ldots, 0]^T$ with a 1 in the i-th coordinate.

Example 2.3. We can also generate a grid with non-orthogonal vectors, for example $\mathbf{v}_1 = [2,\ 1]^T$ and $\mathbf{v}_2 = [0,\ 2]^T$. These vectors are linearly independent and therefore generate a full rank lattice. Another grid we can investigate is generated by $\mathbf{b}_1 = [2,\ 1]^T$ and $\mathbf{b}_2 = [2, -1]^T$. In Figure 2.2, we see what these two grids look like.



(a) Lattice generated with $\mathbf{v}_1$ and $\mathbf{v}_2$.



(b) Lattice generated with $\mathbf{b}_1$ and $\mathbf{b}_2$.

Figure 2.2: Lattice with two different bases from Example 2.3.

We notice that these two grids consist of the same points and are therefore the same. We conclude that lattices can have different bases. The conditions under which two bases produce the same lattice are given in Lemma 2.4.

Lemma 2.4. *Two lattice bases $B_1 = \{\mathbf{v}_1, \mathbf{v}_2\}$ and $B_2 = \{\mathbf{w}_1, \mathbf{w}_2\}$ generate the same lattice if and only if there exist values $a, b, c, d \in \mathbb{Z}$ and with $ad - bc = \pm 1$ such that*

$$\mathbf{v}_1 = a\mathbf{w}_1 + b\mathbf{w}_2$$
$$\mathbf{v}_2 = c\mathbf{w}_1 + d\mathbf{w}_2.$$

Proof. This is a special case of Theorem 1 from [8]. □

Remark 2.5. If we use this lemma, we can find that it is always possible to change the sign of a vector in a basis. That is because if we choose $b = c = 0$, $a = \pm 1$ and $d = \pm 1$, then $ad - bc = ad = \pm 1$. Therefore, the bases $B_1 = \{\mathbf{v}_1, \mathbf{v}_2\}$ and $B_2 = \{\pm\mathbf{v}_1, \pm\mathbf{v}_2\}$ produce the same lattice.

2.2 Grid symmetries

When we look again at the grids from Figure 2.1 and Figure 2.2, we can see that there is some symmetry in these grids. This is true for every lattice. In particular, every lattice has 180° rotational symmetry around the origin.

Lemma 2.6. *Every 2-dimensional lattice Λ has 180° rotational symmetry around the origin.*

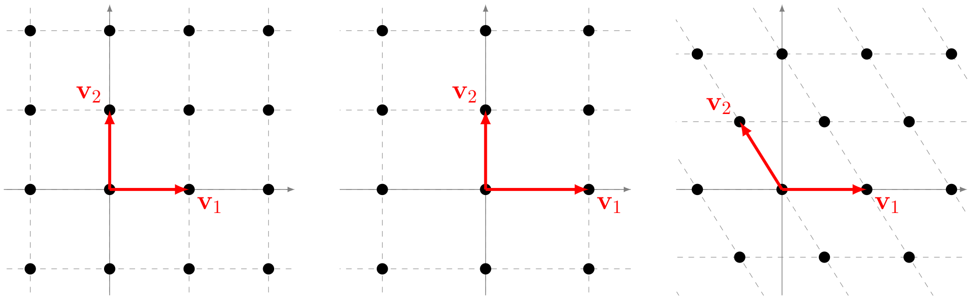
Proof. Let $\mathbf{p}$ be a point in $\Lambda(\mathbf{v}_1, \mathbf{v}_2)$ for some $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$. So there are some $a, b \in \mathbb{Z}$ such that $\mathbf{p} = a\mathbf{v}_1 + b\mathbf{v}_2$. Rotating around the origin with 180° is the same as multiplying by -1. Then rotated point $-\mathbf{p} = -a\mathbf{v}_1 - b\mathbf{v}_1 \in \Lambda(\mathbf{v}_1, \mathbf{v}_2)$. So indeed we have 180° rotational symmetry. □

Apart from this 180° rotational symmetry, some lattices also have additional symmetries. For example, the integer lattice $\mathbb{Z}^2$ also has 90° rotational symmetry, as can be seen in Figure 2.1. In the 1800s, French chemist Auguste Bravais observed this during his research into crystallography. He classified different lattices based on whether they have different symmetries. For 2-dimensional grids, there are five different types of lattices. For 3-dimensional grids, he identified 14 different lattice types. These different lattices are also called Bravais lattices [7].

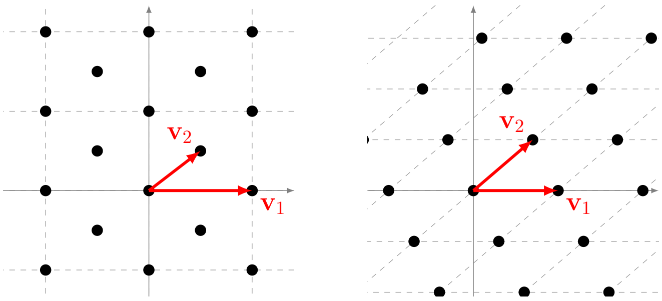
Definition 2.7. In $\mathbb{R}^2$, there are five types of Bravais lattices. Let $\Lambda(\mathbf{v}_1, \mathbf{v}_2)$ with angle ϕ between $\mathbf{v}_1$ and $\mathbf{v}_2$, then this lattice is called:

1. a *square* lattice, if $|\mathbf{v}_1| = |\mathbf{v}_2|$ and $\phi = 90°$.
2. a *rectangular* lattice, if $|\mathbf{v}_1| \neq |\mathbf{v}_2|$ and $\phi = 90°$.
3. a *hexagonal* lattice, if $|\mathbf{v}_1| = |\mathbf{v}_2|$ and $\phi = 120°$.
4. a *centered rectangular* lattice, if it is a rectangular lattice with an additional point in the center of each rectangle.
5. an *oblique* lattice, if it is no other special lattice type.

To get a more intuitive understanding of the different lattice types, examples for different Bravais lattices are given in Figure 2.3.



(a) Square lattice. (b) Rectangular lattice. (c) Hexagonal lattice.



(d) Centered rectangular lattice. (e) Oblique lattice.

Figure 2.3: Different lattice types

The definition of a centered rectangular lattice is quite vague compared to the definitions of other lattice types. This is because this lattice type is closely related to the rectangular lattice. If we have a rectangular lattice generated with $[a,\ 0]^T$ and $[0,\ b]^T$, then the corresponding centered rectangular lattice is generated with, for example, $[a,\ 0]^T$ and $\frac{1}{2}[a,\ b]^T$.

Similar to how the integer lattice $\mathbb{Z}^2$ is the unit square lattice (i.e. square lattice with basis vectors of length 1), there is also a unit hexagonal lattice. This lattice is called the Eisenstein lattice. In most literature, this lattice is defined using complex numbers instead of vectors, so we will do this as well.

Definition 2.8. The Eisenstein lattice is the hexagonal lattice generated by elements 1 and $\omega = -\frac{1}{2} + \frac{1}{2}\sqrt{3}i$.

Although this is the most used basis for the Eisenstein lattice, throughout this paper, we will use a different basis. With Lemma 2.4, we can see that a lattice generated with 1 and $\frac{1}{2} + \frac{1}{2}\sqrt{3}i$ is the same. We will use this basis because it has a positive real part.

2.3 Equable triangles

Definition 2.9. A triangle is called *equable* if the area and the perimeter are equal. We say that a triangle is *realizable* or *placeable* on Λ if it can be created with 3 points from Λ.

Example 2.10. The Pythagorean triangle with side lengths 5, 12 and 13 is an equable triangle. We can see this because it has area $\frac{1}{2} \cdot 5 \cdot 12 = 30$ and perimeter $5 + 12 + 13 = 30$.

In this paper, we are interested in the amount of different equable triangles that can be placed on a grid up to Euclidean motions or congruence. This means that a triangle cannot be transformed into another triangle using a combination of translation, rotation and reflection [13]. This is the case if and only if they have different side lengths. The total amount of equable triangles placeable on a grid has already been proven for both the integer lattice and the Eisenstein lattice.

Theorem 2.11. *On the integer lattice, there are 5 different equable triangles up to Euclidean motions. They have the following side lengths:*

	a	b	c
1	29	25	6
2	20	15	7
3	17	10	9
4	13	12	5
5	10	8	6

Table 2.1: Side lengths of all equable triangles on $\mathbb{Z}^2$.

Proof. This was proven in 1980 by Arthur Foss [6]. □

The different equable triangles on the integer lattice can be found in Figure 2.4.

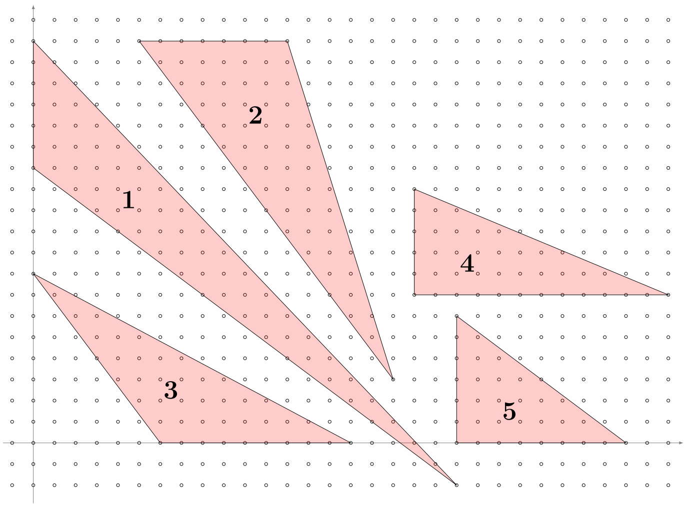


Figure 2.4: Equable triangles on the integer lattice with numbering from Table 2.1.

Theorem 2.12. *On the Eisenstein lattice, there are 2 different equable triangles up to Euclidean motions. They have the following side lengths:*

	a	b	c
1	$8\sqrt{3}$	$7\sqrt{3}$	$3\sqrt{3}$
2	$4\sqrt{3}$	$4\sqrt{3}$	$4\sqrt{3}$

Table 2.2: Side lengths of all equable triangles on the Eisenstein Lattice.

Proof. This was proven in 2023 by Christian Aebi and Grant Cairns [1]. □

The different equable triangles on the Eisenstein lattice can be found in Figure 2.5.

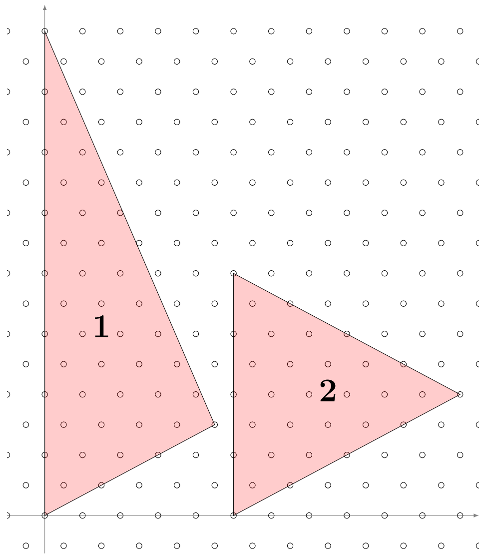


Figure 2.5: Equable triangles on the Eisenstein lattice with numbering from Table 2.2.

Apart from the equable triangles from Table 2.1 and Table 2.2, there are many more equable triangles. In fact, there are infinitely many such triangles.

Lemma 2.13. *There are infinitely many equable triangles.*

Proof. Consider a right-angled equable triangle T with side lengths a, b and hypotenuse c. Because we have a right-angled triangle, $c = \sqrt{a^2 + b^2}$ and $\text{Area}(T) =$

$\frac{1}{2}ab$. T is equable if and only if

$$\begin{aligned} a + b + \sqrt{a^2+b^2} &= \frac{1}{2}ab \\ a^2+b^2 &= (\frac{1}{2}ab - a - b)^2 \\ a^2+b^2 &= \frac{1}{4}a^2b^2 - a^2b - ab^2 + a^2 + b^2 + 2ab \\ \frac{1}{4}a^2b^2 - a^2b - ab^2 + 2ab = 0. \end{aligned}$$

Since $a, b > 0$, we can divide by ab:

$$\begin{aligned} \frac{1}{4}ab - a - b + 2 = 0 \\ (\frac{1}{4}b - 1)a = b - 2 \\ a = \frac{4b-8}{b-4}. \end{aligned}$$

So for all $b > 4$, there is some $a > 0$ such that T is equable. Therefore, there are infinitely many equable triangles. □

Although there are infinitely many equable triangles, we are only concerned with those that can be placed on a given grid. To make this easier, we make a few assumptions on the triangles and on the grids we will investigate:

1. Triangles always have one vertex on the origin.
2. The basis of a grid consists of one vector on the x-axis and another vector with non-negative x- and y-coordinates.

This first assumption is based on the translational invariance of a lattice. Suppose we have two points $\mathbf{x} = a\mathbf{v}_1 + b\mathbf{v}_2 \in \Lambda(\mathbf{v}_1, \mathbf{v}_2)$ and $\mathbf{y} = c\mathbf{v}_1 + d\mathbf{v}_2 \in \Lambda(\mathbf{v}_1, \mathbf{v}_2)$. Then $\mathbf{x} + \mathbf{y} = (a+c)\mathbf{v}_1 + (b+d)\mathbf{v}_2 \in \Lambda(\mathbf{v}_1, \mathbf{v}_2)$. Therefore, if we have a triangle that is placed on a grid, then we can always translate it such that one vertex lies in the origin.
The second assumption is based on congruence of triangles. Recall that if we rotate or reflect a triangle, it is congruent. Therefore, if an equable triangle is placeable on a grid and we apply a rotation or reflection to that grid, then there must be a congruent triangle that is placeable on this new grid. So, applying a rotation or reflection to a grid does not change the amount of equable triangles that are placeable on the grid. We can therefore rotate or reflect the basis vectors such that one vector lies on the x-axis and the other has non-negative x- and y-coordinates. This is illustrated in Example 2.14.

Example 2.14. For a given $\theta_1, \theta_2 \in [0, 2\pi)$ with $\theta_1 < \theta_2$ and $r_1, r_2 > 0$, we can look at the lattice $\Lambda(\mathbf{v}_1, \mathbf{v}_2)$ where $\mathbf{v}_1 = [r_1\cos(\theta_1),\ r_1\sin(\theta_1)]^T$ and $\mathbf{v}_2 = [r_2\cos(\theta_2),\ r_2\sin(\theta_2)]^T$. Now, we can rotate both basis vectors by $-\theta_1$ such that $\mathbf{v}_1$ lies on the x-axis. This would give us an equivalent lattice $\Lambda(\mathbf{b}_1, \mathbf{b}_2)$, where $\mathbf{b}_1 = [r_1,\ 0]^T$ and $\mathbf{b}_2 = [r_2\cos(\theta),\ r_2\sin(\theta)]^T$ with $\theta = \theta_2 - \theta_1$. Note that $\theta \neq \pi$, because otherwise $\mathbf{b}_1$ and $\mathbf{b}_2$ would be linearly dependent. Thus, we have four cases:

1. If $\theta \in (0, \frac{\pi}{2}]$, then $\cos(\theta) \geq 0$ and $\sin(\theta) > 0$ and we are done.
2. If $\theta \in (\frac{\pi}{2}, \pi)$, then $\cos(\theta) < 0$ and $\sin(\theta) > 0$. We can first flip the sign of $\mathbf{b}_2$ and then apply a reflection along the x-axis to get an equivalent lattice $\Lambda(\mathbf{w}_1, \mathbf{w}_2)$ with $\mathbf{w}_1 = [r_1,\ 0]^T$ and $\mathbf{w}_2 = [-r_2\cos(\theta),\ r_2\sin(\theta)]^T$, which satisfies our assumptions.
3. If $\theta \in (\pi, \frac{3\pi}{2})$, then $\cos(\theta) < 0$ and $\sin(\theta) < 0$. By flipping the sign of $\mathbf{b}_2$, we get an equivalent lattice $\Lambda(\mathbf{w}_1, \mathbf{w}_2)$ with $\mathbf{w}_1 = [r_1,\ 0]^T$ and $\mathbf{w}_2 = [-r_2\cos(\theta), -r_2\sin(\theta)]^T$, which satisfies our assumptions.
4. If $\theta \in [\frac{3\pi}{2}, 2\pi)$, then $\cos(\theta) \geq 0$ and $\sin(\theta) < 0$. We can apply a reflection along the x-axis to $\mathbf{b}_2$ to get an equivalent lattice $\Lambda(\mathbf{w}_1, \mathbf{w}_2)$ with $\mathbf{w}_1 = [r_1,\ 0]^T$ and $\mathbf{w}_2 = [r_2\cos(\theta), -r_2\sin(\theta)]^T$, which satisfies our assumptions.

Therefore, for every lattice, we can find an equivalent lattice generated with one vector on the x-axis and one vector under an angle $\theta \in (0, \frac{\pi}{2}]$.

2.4 Useful theorems

This section will introduce some theorems that we will use for proofs in Chapter 3. Since they are well-known, we will not prove them ourselves.

Lemma 2.15. *Let $a_1, a_2, \ldots, a_n$ be positive rational numbers and let $k_1, k_2, \ldots, k_n$ be integers greater than 1. If $\sum_{i=1}^{n} \sqrt[k_i]{a_i} \in \mathbb{Q}$, then $\sqrt[k_i]{a_i} \in \mathbb{Q}$ for all i.*

Proof. The proof of this theorem can be found in [12] exercise 2. □

We will use this to put some constraints on the side lengths of equable triangles. Next, we have two useful results about the area of a triangle.

Theorem 2.16 (Heron's Formula)**.** *Let T a triangle with side lengths a, b, c and semiperimeter $s = \frac{1}{2}(a + b + c)$. The area of the triangle is then given by:*

$$Area(T) = \sqrt{s(s-a)(s-b)(s-c)}.$$

This can also be written in the following way:

$$Area(T) = \frac{1}{4}\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}.$$

Proof. The proof of this theorem can be found in [14]. □

This formula describes the area of a triangle using its side lengths. In our case, this is especially helpful since we know that the area is equal to the perimeter, i.e. the sum of the side lengths.

Lemma 2.17 (Area of a triangle)**.** *Let T be a triangle with vertices $(0, 0)$, (x_1, y_1) and (x_2, y_2). The signed area of the triangle is then given by:*

$$\frac{1}{2}\det\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} = \frac{1}{2}\big(x_1y_2 - x_2y_1\big).$$

Proof. This is a special case of the Surveyor's formula. The proof of which can be found in [3]. □

Chapter 3

Equable triangles on grids

In this chapter, we look for possible candidates of equable triangles that can be placed on grids. We will do this by adapting the proof given by Aebi and Cairns for the Eisenstein lattice [1]. In Section 3.1, we will first put restrictions on the side lengths of equable triangles. In Section 3.2, we will find all equable triangles satisfying this constraint.

3.1 Constraint on side lengths

For the results we will see in Section 3.2, we will need some constraints on the side lengths of equable triangles. In particular we will need that the side lengths are of the form rn with $n \in \mathbb{N}$ for a given $r > 0$. We will see that for many different grids, the side lengths of equable triangles that can be realized on that grid will be of that form. We will limit ourselves to grids generated by vectors that lie on the unit circle. We first need a simple result about splitting an integer into a squared part and a square-free part. A square-free number is a number that has at most one factor for each prime in its prime factorization.

Lemma 3.1. *Let $m \in \mathbb{N}$, then there exist $u, v \in \mathbb{N}$ with v square-free such that $u^2v = m$*

Proof. Let the prime factorization of m be $m = p_1^{a_1} \dots p_l^{a_l}$. Suppose that primes are ordered such that the first k primes have even and the rest have odd exponents. We can say that if $i \leq k$, then $a_i = 2b_i$ and otherwise $a_i = 2b_i + 1$ for some $b_i \in \mathbb{N} \cup \{0\}$.

We can rewrite this as follows:

$$
\begin{aligned}
m &= p_1^{a_1} \dots p_k^{a_k} \\
m &= \prod_{i=1}^{k} \left(p_i^{a_i}\right) \prod_{i=k+1}^{l} \left(p_i^{a_i}\right) \\
m &= \prod_{i=1}^{k} \left(p_i^{2b_i}\right) \prod_{i=k+1}^{l} \left(p_i^{2b_i+1}\right) \\
m &= \prod_{i=1}^{k} \left(p_i^{2b_i}\right) \prod_{i=k+1}^{l} \left(p_i^{2b_i}\right) \prod_{i=k+1}^{l} \left(p_i\right) \\
m &= \left(\prod_{i=1}^{l} \left(p_i^{b_i}\right)\right)^2 \prod_{i=k+1}^{l} \left(p_i\right)
\end{aligned}
$$

We can define $u = \prod_{i=1}^{l} \left(p_i^{b_i}\right)$ and $v = \prod_{i=k+1}^{l} \left(p_i\right)$ to get that $m = u^2 v$ with v square-free. □

With this result, we can find a constraint on the side lengths of equable triangles.

Theorem 3.2. *Let Λ be a grid generated by 1 and $\omega = \cos(\theta) + i\sin(\theta)$ with $\theta \in (0, \frac{\pi}{2}]$ such that $\cos(\theta) = \frac{p}{q} \in \mathbb{Q}$ for co-prime $p, q \in \mathbb{Z}$ with $0 \le p < q$. We can find $u, v \in \mathbb{N}$ with v square-free such that $u^2 v = q^2 - p^2$. Then the side lengths of equable triangles realizable on Λ are of the form rn with $r = \frac{1}{q\sqrt{v}}$ and $n \in \mathbb{N}$.*

Proof. Let Λ and ω as in the claim. Let T be an equable triangle on Λ with vertices $A = a_1 + a_2\omega$, $B = b_1 + b_2\omega$ and the origin for some $a_1, a_2, b_1, b_2 \in \mathbb{Z}$. We call the side lengths of T a, b and c.
By Lemma 2.17, the signed area of T is given by:

$$
\begin{aligned}
\text{Area}(T) &= \frac{1}{2}\Big((a_1 + a_2\cos(\theta))b_2\sin(\theta) - (b_1 + b_2\cos(\theta))a_2\sin(\theta)\Big) \\
&= \frac{1}{2}\Big(a_1b_2\sin(\theta) + a_2b_2\cos(\theta)\sin(\theta) - a_2b_1\sin(\theta) - a_2b_2\cos(\theta)\sin(\theta)\Big) \\
&= \frac{\sin(\theta)}{2}\Big(a_1b_2 - a_2b_1\Big).
\end{aligned}
$$

Furthermore, if we look at the length a of one side squared, then

$$
\begin{aligned}
a^2 = A\bar{A} &= (a_1 + a_2\omega)(a_1 + a_2\bar{\omega}) \\
&= a_1^2 + a_1a_2(\omega + \bar{\omega}) + a_2^2\omega\bar{\omega} \\
&= a_1^2 + a_1a_2(\cos(\theta) + i\sin(\theta) + \cos(\theta) - i\sin(\theta))i + a_2^2 \\
&= a_1^2 + 2a_1a_2\cos(\theta) + a_2^2.
\end{aligned}
$$

Since we know that $\cos(\theta) = \frac{p}{q}$, we have that

$$qa^2 = qa_1^2 + 2a_1a_2p + qa_2^2 \in \mathbb{N}.$$

Furthermore also $q^2a^2 \in \mathbb{N}$. The same is true for side lengths b and c. By rationality of $\cos(\theta)$, we also have that $\sin^2(\theta) = 1 - \cos^2(\theta) = \frac{q^2-p^2}{q^2} \in \mathbb{Q}$. Because $\theta \in (0, \frac{\pi}{2}]$, $\sin(\theta) > 0$, so we can say that $\sin(\theta) = \frac{\sqrt{q^2-p^2}}{q}$. From Lemma 3.1, we can find $u, v \in \mathbb{N}$ such that $u^2v = q^2 - p^2$ with v square-free. We get that $\sin(\theta) = \frac{u\sqrt{v}}{q}$ Since T is equable, we can say that

$$\begin{aligned}
a + b + c &= \frac{\sin(\theta)}{2}\Big(a_1b_2 - a_2b_1\Big) \\
a + b + c &= \frac{u\sqrt{v}}{2q}\Big(a_1b_2 - a_2b_1\Big) \\
\sqrt{v}qa + \sqrt{v}qb + \sqrt{v}qc &= \frac{uv}{2}\Big(a_1b_2 - a_2b_1\Big) \\
\sqrt{vq^2a^2} + \sqrt{vq^2b^2} + \sqrt{vq^2c^2} &= \frac{uv}{2}\Big(a_1b_2 - a_2b_1\Big).
\end{aligned}$$

Because the right-hand side of the equation is rational, the left-hand side must be rational as well. From Lemma 2.15, we must then have that $\sqrt{vq^2a^2}$, $\sqrt{vq^2b^2}$ and $\sqrt{vq^2c^2}$ are all rational. Then, because vq^2a^2, vq^2b^2 and vq^2c^2 are all natural, $\sqrt{vq^2a^2}$, $\sqrt{vq^2b^2}$ and $\sqrt{vq^2c^2}$ must all be natural as well. This means that all side lengths are of the form $\frac{1}{q\sqrt{v}}n$. □

Example 3.3. The integer lattice $\mathbb{Z}^2$ can be generated with the complex numbers 1 and i, so we have that $\cos(\frac{\pi}{2}) = 0 = \frac{0}{1}$ and $q^2 - p^2 = 1$. From Theorem 3.2, we therefore get that all equable triangles on this lattice must have integer side lengths. Triangles with integer area and side lengths are also called Heronian triangles.

In Section 3.2, we will try to find equable triangles with side lengths of the form rn with $n \in \mathbb{N}$ for a given $r > 0$. To reduce the amount of candidates we will find, we want r to be as large as possible. For example, if we know that all equable triangles on a certain grid have side lengths of the form $2n$, then the side lengths are also of the form n. But this will give more possible candidate equable triangles that will definitely not lie on the grid. We have two ways to optimize this value of r.

Corollary 3.4. *Let Λ be a grid generated by 1 and $\omega = \cos(\theta) + i\sin(\theta)$ with $\theta \in (0, \frac{\pi}{2}]$ such that $\cos(\theta) = \frac{p}{q} \in \mathbb{Q}$ for co-prime $p, q \in \mathbb{Z}$ where $0 \leq p < q$ and q is even. We can find $u, v \in \mathbb{N}$ with v square-free such that $u^2v = q^2 - p^2$. Then the side lengths of equable triangles realizable on Λ are of the form rn with $r = \frac{2}{q\sqrt{v}}$ and $n \in \mathbb{N}$.*

Proof. We follow the proof of Theorem 3.2. We still have that

$$\text{Area}(T) = \frac{\sin(\theta)}{2}\Big(a_1b_2 - a_2b_1\Big)$$

and

$$a^2 = a_1^2 + 2a_1a_2\cos(\theta) + a_2^2.$$

Now we use the fact that $\cos(\theta) = \frac{p}{q}$. Since q even, we can say that $q = 2m$ for some $m \in \mathbb{N}$. So now we get that

$$ma^2 = ma_1^2 + a_1a_2p + ma_2^2 \in \mathbb{N}.$$

Since $m \in \mathbb{N}$, also $m^2a^2 \in \mathbb{N}$. Again, we have that $\sin(\theta) = \frac{u\sqrt{v}}{q} = \frac{u\sqrt{v}}{2m}$ with $u^2v = q^2 - p^2$ and v square-free. Since triangle T is equable, we must have

$$\begin{aligned}
a+b+c &= \frac{\sin(\theta)}{2}\Big(a_1b_2 - a_2b_1\Big)\\
a+b+c &= \frac{u\sqrt{v}}{4m}\Big(a_1b_2 - a_2b_1\Big)\\
\sqrt{v}ma + \sqrt{v}mb + \sqrt{v}mc &= \frac{uv}{4}\Big(a_1b_2 - a_2b_1\Big)\\
\sqrt{vm^2a^2} + \sqrt{vm^2b^2} + \sqrt{vm^2c^2} &= \frac{uv}{4}\Big(a_1b_2 - a_2b_1\Big).
\end{aligned}$$

We use the same logic as in the proof of Theorem 3.2 to derive that $\sqrt{vm^2a^2}$, $\sqrt{vm^2b^2}$ and $\sqrt{vm^2c^2}$ are natural. We therefore get that all side lengths are of the form $\frac{1}{m\sqrt{v}}n = \frac{2}{q\sqrt{v}}n$. $\square$

This already doubles the value of r compared to Theorem 3.2, but there is another, more significant optimization we can make. For this we first need a simple observation about divisibility by square-free numbers.

Lemma 3.5. *Let $n, v \in \mathbb{N}$ such that v is a divisor of n^2. If v is square-free, then v is also a divisor of n.*

Proof. Let $n = p_1^{a_1} \dots p_k^{a_k}$ and $v = q_1^1 \dots q_l^1$ be the prime factorizations of n and v. Then n^2 has prime factorization $n^2 = p_1^{2a_1} \dots p_k^{2a_k}$. Then, since v divides n^2, we must have that for every $i \in \{1, \dots l\}$, there exists some $j \in \{1, \dots k\}$ such that $q_i = p_j$. Therefore also every q_i divides n. This implies that v also divides n. $\square$

We can use this lemma to make the constraints on the side lengths even stricter.

Theorem 3.6. *Let Λ be a grid generated by 1 and $\omega = \cos(\theta) + i\sin(\theta)$ with $\theta \in (0, \frac{\pi}{2}]$ such that $\cos(\theta) = \frac{p}{q} \in \mathbb{Q}$ for co-prime $p, q \in \mathbb{Z}$ with $0 \le p < q$. We can find $u, v \in \mathbb{N}$ with v square-free such that $u^2v = q^2 - p^2$. Then the side lengths of equable triangles realizable on Λ are of the form rn with $r = \frac{\sqrt{v}}{q}$ and $n \in \mathbb{N}$.*

Proof. Consider an arbitrary side length a of the triangle. From Theorem 3.2, we have that $a = \frac{1}{q\sqrt{v}}n$ for some $n \in \mathbb{N}$. In the proof of Theorem 3.2, we also saw that $q^2a^2 = \frac{n^2}{v} \in \mathbb{N}$. This means that v is a divisor of n^2. From Lemma 3.5, we get that v must also be a divisor of n. Say $n = vs$ for some $s \in \mathbb{N}$. We then get that $a = \frac{1}{q\sqrt{v}}n = \frac{\sqrt{v}}{q}s$, so we get the required result. $\square$

This is actually a very large improvement. If we have a grid as in Theorem 3.2 with v large, then instead of $r = \frac{1}{q\sqrt{v}}$, we get $r = \frac{\sqrt{v}}{q}$ which is v times larger. This will significantly reduce the amount of equable triangle candidates.

Remark 3.7. We can combine Corollary 3.4 and Theorem 3.6 to get a better constraint on the side lengths of equable triangles. If $\cos(\theta) = \frac{p}{q}$ with $q = 2m$ even, then from Corollary 3.4, we get that the side lengths are of the form $\frac{1}{m\sqrt{v}}n$. Furthermore, we saw that $m^2a^2 \in \mathbb{N}$ for a side length a. We can therefore follow the proof of Theorem 3.6 to find that all equable triangles on this grid must have side lengths of the form rn with $r = \frac{2\sqrt{v}}{q}$.

Example 3.8. The Eisenstein lattice can be generated with the complex numbers 1 and $\frac{1}{2} + \frac{1}{2}\sqrt{3}i$. We therefore have that $p = 1$ and $q = 2$. Furthermore, we get that $q^2 - p^2 = 3$ is prime. We can use both Corollary 3.4 and Theorem 3.6 to find that the side lengths are of the form rn with $r = \sqrt{3}n$.

3.2 Finding equable triangles

In the previous section, we saw that for many grids, we can find a value r such that the side lengths of equable triangles on that grid are of the form rn with $n \in \mathbb{N}$. In this section, we will find all equable triangles subject to this constraint. We first look at an equation that must hold for these equable triangles.

Lemma 3.9. *Suppose we have an equable triangle with side lengths a, b, c of the form rn with $n \in \mathbb{N}$ for a given $r > 0$. Let $x = \frac{1}{r}(-a + b + c)$, $y = \frac{1}{r}(a - b + c)$ and $z = \frac{1}{r}(a + b - c)$. We have that $x, y, z \in \mathbb{N}$. The following equation must then be true:*

$$r^2 xyz = 16(x + y + z). \tag{3.1}$$

The side lengths of the triangle expressed in terms of x, y and z are then

$$a = \tfrac{r}{2}(y + z),\ \ b = \tfrac{r}{2}(x + z),\ \text{ and } c = \tfrac{r}{2}(x + y).$$

Proof. Let $r > 0$. Let T be an equable triangle with side lengths a, b and c of the form rn with $n \in \mathbb{N}$. From Heron's formula (Theorem 2.16), we know that

$$\begin{aligned}\text{Area}(T) &= \frac{1}{4}\sqrt{(a + b + c)(-a + b + c)(a - b + c)(a + b - c)} \\ 16 \cdot \text{Area}(T)^2 &= (a + b + c)(-a + b + c)(a - b + c)(a + b - c).\end{aligned}$$

Since T is equable, we also know that $\text{Area}(T) = a + b + c$. Therefore we have that

$$(-a + b + c)(a - b + c)(a + b - c) = 16(a + b + c). \tag{3.2}$$

Let $x = \frac{1}{r}(-a + b + c)$, $y = \frac{1}{r}(a - b + c)$ and $z = \frac{1}{r}(a + b - c)$. Then since all lengths are of the form rn, we know that $x, y, z \in \mathbb{N}$. One can show that then $a = \frac{r}{2}(y + z)$, $b = \frac{r}{2}(x + z)$, and $c = \frac{r}{2}(x + y)$.

If we substitute, x, y and z into (3.2), we get that

$$\begin{aligned}rx \cdot ry \cdot rz &= 16r(x + y + z) \\ r^2 xyz &= 16(x + y + z).\end{aligned}$$

□

We can use Equation (3.1) to find equable triangles with side lengths of the form rn. This can be done by checking whether the equation holds for certain values of x, y and z. We will show that we only need to check a finite amount of these values. We will first make some observations that will be useful in the future.

Corollary 3.10. *There are no equable triangles with side lengths of the form rn for $n \in \mathbb{N}$ if r^2 is irrational.*

Proof. If we look at Equation (3.1), we see that the right-hand side of the equation is clearly integral. Therefore, the left-hand side of the equation should also be integral. This is impossible if r^2 is irrational. $\square$

This result is quite powerful. If we observe that in some grid the equable triangles must have side lengths of the form rn with r^2 irrational, we immediately know that this grid cannot have any equable triangles on it.
The next corollary significantly reduces the amount of values for x, y and z we need to check.

Corollary 3.11. *Let x, y, z be a solution to Equation (3.1) for a certain equable triangle, then x, y and z all have the same parity.*

Proof. Since a, b and c are all of the form rn with $n \in \mathbb{N}$ for a given $r > 0$, we can say that $a = rn_1$, $b = rn_2$ and $c = rn_3$ with $n_1, n_2, n_3 \in \mathbb{N}$. We then get that

$$\begin{aligned} x &= -n_1 + n_2 + n_3 \\ y &= \quad n_1 - n_2 + n_3 \\ z &= \quad n_1 + n_2 - n_3. \end{aligned}$$

If we add two numbers or subtract two numbers, the two outcomes will have the same parity. Therefore, they all must have the same parity as they are made from adding or subtracting the same numbers. $\square$

For certain values of r, we might have additional information about the values of x, y and z. We will see later on that, in many cases, we can say that all solutions must be even.

Theorem 3.12. *Let $r > 0$. Let T be an equable triangle with side lengths of the form rn with $n \in \mathbb{N}$ and values $x, y, z \in \mathbb{N}$ as in Lemma 3.9. We can assume that $x \leq y \leq z$. Then these values are subject to*

$$\begin{aligned} x &\leq \frac{4}{r}\sqrt{3} \\ y &\leq \frac{1}{r^2 x}\left(16 + 4\sqrt{16 + r^2 x^2}\right) \\ z &= \frac{16(x+y)}{r^2 xy - 16}. \end{aligned} \tag{3.3}$$

Furthermore, the values of x, y and z must be either all odd or all even. The side lengths of the triangle expressed in terms of x, y and z are then

$$a = \tfrac{r}{2}(y+z), \ \ b = \tfrac{r}{2}(x+z), \ \ \textit{and } c = \tfrac{r}{2}(x+y).$$

Proof. From Corollary 3.10, we know that r^2 must be rational. So let $r^2 = \frac{p}{q}$ for some $p, q \in \mathbb{N}$ co-prime. Equation (3.1) then becomes

$$pxyz = 16q(x+y+z). \tag{3.4}$$

We can assume that $x \leq y \leq z$. Using Equation (3.4), we can get an inequality for y:

$$y \leq z = \frac{16q(x+y)}{pxy - 16q}.$$

Because we know that $z > 0$ and $16q(x+y) > 0$, it must be that $pxy - 16q > 0$. We can take the denominator to the other side of the inequality:

$$\begin{aligned}(pxy - 16q)y &\leq 16q(x+y)\\ pxy^2 - 16qy &\leq 16qx + 16qy\\ pxy^2 - 32qy - 16qx &\leq 0.\end{aligned}$$

This is a parabola with a minimum, so, using the quadratic formula, we can find that

$$\begin{aligned}y &\leq \frac{1}{2px}\left(32q + \sqrt{(-32q)^2 - 4px(-16qx)}\right)\\ y &\leq \frac{1}{px}\left(16q + \frac{1}{2}\sqrt{1024q^2 + 64pqx^2}\right)\\ y &\leq \frac{1}{px}\left(16q + 4\sqrt{16q^2 + pqx^2}\right). \qquad (3.5)\end{aligned}$$

Note that $16q^2 + pqx^2 > 0$. We can rewrite 3.5 to get an equation in terms of r:

$$y \leq \frac{1}{r^2 x}\left(16 + 4\sqrt{16 + r^2x^2}\right).$$

If we use Inequality (3.5) and the fact that $x \leq y$, we get that

$$\begin{aligned}x &\leq \frac{1}{px}\left(16q + 4\sqrt{16q^2 + pqx^2}\right)\\ px^2 &\leq 16q + 4\sqrt{16q^2 + pqx^2}\\ px^2 - 16q &\leq 4\sqrt{16q^2 + pqx^2}.\end{aligned}$$

Note that squaring both sides of an inequality and keeping the sign the same is valid when both sides are positive. For the right side this is trivially true, but for the left side, this may not be the case. If we assume that the left side is negative, we find the following upper bound

$$\begin{aligned}px^2 - 16q &\leq 0\\ px^2 &\leq 16q\\ x^2 &\leq \frac{16q}{p} = \frac{16}{r^2}\\ x &\leq \frac{4}{r}.\end{aligned}$$

If instead, we assume that the left side is positive, we can take a square on both sides. This gives

$$\begin{aligned} (px^2 - 16q)^2 &\leq 16(16q^2 + pqx^2) \\ p^2x^4 - 32pqx^2 + 256q^2 &\leq 256q^2 + 16pqx^2 \\ p^2x^4 - 48pqx^2 &\leq 0 \\ px^2 - 48q &\leq 0 \\ x^2 &\leq \frac{48q}{p} = \frac{48}{r^2} \\ x &\leq \frac{4}{r}\sqrt{3}. \end{aligned}$$

In both cases, we have that $x \leq \frac{4}{r}\sqrt{3}$. Therefore in total, we must have that the values $x, y, z \in \mathbb{N}$ must have the same parity and be subject to

$$\begin{aligned} x &\leq \frac{4}{r}\sqrt{3} \\ y &\leq \frac{1}{r^2x}\left(16 + 4\sqrt{16 + r^2x^2}\right) \\ z &= \frac{16(x+y)}{r^2xy - 16}. \end{aligned}$$

□

We now have a way to check a finite amount of values for x, y to see if z is integer. If we find such values with the same parity, they correspond to an equable triangle with sides $a = \frac{r}{2}(y+z)$, $b = \frac{r}{2}(x+z)$, and $c = \frac{r}{2}(x+y)$. Therefore, this also gives an upper bound for the amount of equable triangles with side lengths of form rn. This upper bound increases rapidly as r decreases, so we want r to be as large as possible.

This theorem also gives us a way to see when there are definitely no equable triangles of the form rn. We already saw that this was the case if r^2 is irrational. But also if the upper bound ($x \leq \frac{4}{r}\sqrt{3}$) for x is lower than its lower bound ($x \geq 1$), then there are no possible solutions for x. This implies that there are no solutions if $r > 4\sqrt{3}$.

Remark 3.13. Although Equation 3.5 gives a tight bound on the possible values of y, it might be easier to use a bound that is independent of the value of x. If we call the right-hand side of Equation 3.5 $f(x)$, then we can look at its derivative:

$$\begin{aligned} f'(x) &= \frac{-1}{px^2}\left(16q + 4\sqrt{16q^2 + pqx^2}\right) + \frac{1}{px}\left(\frac{8pqx}{2\sqrt{16q^2 + pqx^2}}\right) \\ &= \frac{-1}{px^2}\left(16q + 4\sqrt{16q^2 + pqx^2} - \frac{4pqx^2}{\sqrt{16q^2 + pqx^2}}\right) \\ &= \frac{-1}{px^2\sqrt{16q^2 + pqx^2}}\left(16q\sqrt{16q^2 + pqx^2} + 4\left(16q^2 + pqx^2\right) - 4pqx^2\right) \\ &= \frac{-1}{px^2\sqrt{16q^2 + pqx^2}}\left(16q\sqrt{16q^2 + pqx^2} + 64q^2\right) < 0. \end{aligned}$$

Since the derivative is negative, $f(x)$ is decreasing in terms of x. Therefore, the largest value is for $x = 1$. We get a constant upper bound for y:

$$y \leq \frac{1}{p}\Big(16q + 4\sqrt{16q^2 + pq}\Big) = \frac{1}{r^2}\Big(16 + 4\sqrt{16 + r^2}\Big).$$

Remark 3.14. From Corollary 3.11 we know that the values for x, y and z must have the same parity. We can use this fact to reduce the search space from Theorem 3.12. If $r^2 = \frac{p}{q}$ for p, q co-prime and p not a multiple of 16, then we can say that x, y and z must all be even. To see this, we look at Equation 3.4 again:

$$pxyz = 16q(x + y + z).$$

If p is not a multiple of 16, we can say that $p = 2^a m$ with $0 \leq a \leq 3$ and m odd. Then we get that

$$mxyz = 2^{4-a}q(x + y + z).$$

The right-hand side is still even. If x, y and z were all odd, then the left side would be odd, which is impossible. Therefore, x, y and z must be even.
In this case, we can make the substitution $u = \frac{1}{2}x = \frac{1}{2r}(-a + b + c)$, $v = \frac{1}{2}y = \frac{1}{2r}(a - b + c)$ and $w = \frac{1}{2}z = \frac{1}{2r}(a + b - c)$. Now instead of Equation 3.1, these values should satisfy:

$$r^2uvw = 4(u + v + w).$$

With this equation, we can follow the proof of Theorem 3.12 again to get new upper bounds. The details of this can be found in Appendix A. We can also do the same as in Remark 3.13 to get new constant upper bounds:

$$\begin{aligned} u &\leq \frac{2}{r}\sqrt{3} \\ v &\leq \frac{1}{r^2}(4 + 2\sqrt{4 + r^2}) \\ w &= \frac{4(u + v)}{r^2uv - 4}. \end{aligned} \tag{3.6}$$

Example 3.15. In Example 3.3, we have shown that, on the integer lattice $\mathbb{Z}^2$, all equable triangles have side lengths of the form $n \in \mathbb{N}$, so $r = 1$. We can use the upper bounds from Equation 3.6 to see that these triangles are $a = v + w$, $b = u + w$ and $c = u + v$ with $u, v, w \in \mathbb{N}$ and subject to

$$\begin{aligned} u &\leq 2\sqrt{3} \approx 3.46 \\ v &\leq 4 + 2\sqrt{5} \approx 8.47 \\ w &= \frac{4(u + v)}{uv - 4}. \end{aligned}$$

So then $u \leq 3$ and $v \leq 8$. If we try different values for u and v, we find only 5 feasible solutions. The results can be found in Table 3.1.

u	v	w	a	b	c
1	5	24	29	25	6
1	6	14	20	15	7
1	8	9	17	10	9
2	3	10	13	12	5
2	4	6	10	8	6

Table 3.1: All possible equable triangles on $\mathbb{Z}^2$.

These are exactly the same solutions found by Foss [6]. In Figure 2.4, we see that these can indeed be placed on the integer lattice.

Example 3.16. On the Eisenstein lattice, we can do something similar. In this case, the side lengths are of the form $\sqrt{3}n$ with $n \in \mathbb{N}$ as was shown in Example 3.8. From Remark 3.14, we get the upper bounds

$$\begin{aligned} u &\leq 2 \\ v &\leq \frac{4}{3} + \frac{2}{3}\sqrt{7} \approx 3.10 \\ w &= \frac{4(u+v)}{3uv-4}. \end{aligned}$$

So then $u \leq 2$ and $v \leq 3$. This time, we find two feasible solutions. These can be found in Table 3.2.

u	v	w	a	b	c
1	2	6	$8\sqrt{3}$	$7\sqrt{3}$	$3\sqrt{3}$
2	2	2	$4\sqrt{3}$	$4\sqrt{3}$	$4\sqrt{3}$

Table 3.2: All possible equable triangles on the Eisenstein lattice.

These are the same solutions found by Aebi and Cairns [1]. In Figure 2.5, we see that these can indeed be placed on the Eisenstein lattice.

So we see that this method for finding equable triangles works for the integer and the Eisenstein lattice. We can also see what happens for a new grid.

Example 3.17. Consider Λ generated by 1 and $\omega = \cos(\theta) + \sin(\theta)i$ such that $\cos(\theta) = \frac{p}{q} = \frac{2}{3}$. Since $q^2 - p^2 = 9 - 4 = 5$, we use Theorem 3.6 to find that all equable triangles must be of the form rn with $r = \frac{\sqrt{5}}{3}$ and $n \in \mathbb{N}$. Since the numerator of $r^2 = \frac{5}{9}$ is not a multiple of 16, we can use the bounds from Remark 3.14:

$$\begin{aligned} u &\leq \frac{6}{\sqrt{5}}\sqrt{3} \approx 4.65 \\ v &\leq \frac{9}{5}(4 + 2\sqrt{4 + \frac{5}{9}}) \approx 14.88 \\ w &= \frac{36(u+v)}{5uv-36}. \end{aligned}$$

So then $u \leq 4$ and $v \leq 14$. When we try different values for u and v, we get 7 different equable triangles. We see these in Table 3.3.

u	v	w	a	b	c
1	8	81	$89\sqrt{5}/3$	$82\sqrt{5}/3$	$3\sqrt{5}$
1	9	40	$49\sqrt{5}/3$	$41\sqrt{5}/3$	$10\sqrt{5}/3$
2	4	54	$58\sqrt{5}/3$	$56\sqrt{5}/3$	$2\sqrt{5}$
2	5	18	$23\sqrt{5}/3$	$20\sqrt{5}/3$	$7\sqrt{5}/3$
2	6	12	$6\sqrt{5}$	$14\sqrt{5}/3$	$8\sqrt{5}/3$
3	3	24	$9\sqrt{5}$	$9\sqrt{5}$	$2\sqrt{5}$
3	6	6	$4\sqrt{5}$	$3\sqrt{5}$	$3\sqrt{5}$

Table 3.3: Equable triangles with side lengths of form $\frac{\sqrt{5}}{3}n$.

Although these triangles all satisfy the constraint, they cannot all be placed on the grid. We will see why in Example 4.1.

Chapter 4

Placing triangles on a grid

In this chapter, we will try to place the candidates found in Chapter 3 on the grid. In Section 4.1, we will see that not every candidate can be placed on a grid and we will look at some theorems about placing triangles on specific grids. In Section 4.2, we will use a brute-force approach to place triangles on a grid.

4.1 Triangle placeability theorems

In Chapter 3, we presented a method to find candidate equable triangles given a specific grid. However, not all candidates can actually be realized on the grid. To see that this is indeed the case, we look at an example.

Example 4.1. We look at Λ generated by 1 and $\omega = \cos(\theta) + \sin(\theta)i$ such that $\cos(\theta) = \frac{2}{3}$. In Example 3.17, we saw that all equable triangles must be of the form $\frac{\sqrt{5}}{3}n$ with $n \in \mathbb{N}$ and that $(6\sqrt{5}, \frac{14}{3}\sqrt{5}, \frac{8}{3}\sqrt{5})$ is an equable triangle satisfying this constraint. This triangle can, however, not be placed on the grid. To see why this is the case, we look at the side with side length $\frac{8}{3}\sqrt{5}$. Let $A = a_1 + a_2\omega$ such that $|OA| = \frac{8}{3}\sqrt{5}$, then

$$\begin{aligned} a_1^2 + \frac{4}{3}a_1a_2 + a_2^2 &= (\frac{8}{3}\sqrt{5})^2 \\ a_1^2 + \frac{4}{3}a_1a_2 + a_2^2 &= \frac{320}{9} \\ 9a_1^2 + 12a_1a_2 + 9a_2^2 &= 320. \end{aligned}$$

Since the left-hand side is a multiple of 3 for all $a_1, a_2 \in \mathbb{Z}$ and the right-hand side is not, this equation is impossible. Therefore, we cannot place a side with side length $\frac{8}{3}\sqrt{5}$ on the grid and so the triangle cannot be realized on this grid.
We can use similar logic to eliminate other triangles from Table 3.3. Only two of these triangles can actually be placed. These can be found in Table 4.1.

	a	b	c
1	$4\sqrt{5}$	$3\sqrt{5}$	$3\sqrt{5}$
2	$9\sqrt{5}$	$9\sqrt{5}$	$2\sqrt{5}$

Table 4.1: Equable triangles on the grid with $\cos(\theta) = \frac{2}{3}$.

These two triangles are visualized in Figure 4.1.

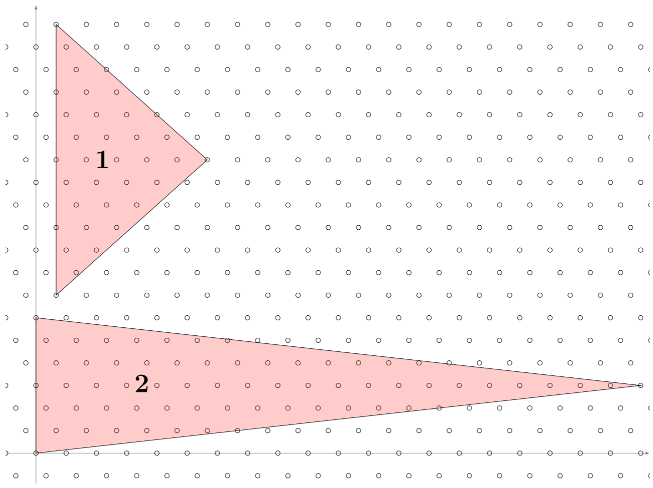


Figure 4.1: Equable triangles on a grid where $\cos(\theta) = \frac{2}{3}$ with numbering from Table 4.1.

In general, placing a triangle on a grid is a hard problem. When looking at the side lengths of a triangle it might not be immediately clear whether it can be realized on a grid or not. However, Yiu proved a condition for when this is possible on the integer lattice.

Theorem 4.2. *All Heronian triangles can be realized on the integer lattice.*

Proof. The proof can be found in [16]. □

Recall that a Heronian triangle is a triangle with integer side lengths and area. We can use this result to immediately see that all candidates found in Example 3.15 can actually be realized on the integer lattice. In 2023, Aebi and Cairns adapted the proof by Yiu to work on the Eisenstein lattice.

Theorem 4.3. *A planar triangle T with side lengths a, b, c is realizable on the Eisenstein lattice if and only if the following three conditions hold:*

(i) *the area of T is of the form $\frac{\sqrt{3}}{4}n$, where $n \in \mathbb{N}$,*

(ii) $a^2, b^2, c^2 \in \mathbb{N}$,

(iii) *one of the side lengths of T is of the form $r\sqrt{t}$, where $r, t \in \mathbb{N}$ and t has no prime divisors congruent to* 2 (mod 3).

Proof. The proof can be found in [2]. □

In Example 3.16, we found that the equable triangle candidates on the Eisenstein lattice have side lengths $(8\sqrt{3}, 7\sqrt{3}, 3\sqrt{3})$ and $(4\sqrt{3}, 4\sqrt{3}, 4\sqrt{3})$. We see that all conditions are satisfied for both candidates and so they are realizable on the Eisenstein lattice.

4.2 Brute-force approach

In general, we do not have a good way to determine whether or not a triangle can be placed on a grid. Instead, we use a brute-force approach to check possible coordinates and try to place the triangle. To do this, we need to find a way to only have to check a finite number of points.

Theorem 4.4. *Let $a > 0$ be the side length of a triangle. Let Λ be a grid generated by 1 and $\omega = \cos(\theta) + \sin(\theta)i$. Then there exists a point $A \in \Lambda$ with $|OA| = a$ if and only if there exist $a_1, a_2 \in \mathbb{Z}$ such that:*

$$\frac{-a}{\sin(\theta)} \leq a_1 \leq \frac{a}{\sin(\theta)} \tag{4.1}$$
$$a_2 = -a_1\cos(\theta) \pm \sqrt{a^2 - a_1^2\sin^2(\theta)}.$$

Then $A = a_1 + a_2\omega$.

Proof. Let $A = a_1 + a_2\omega$ for some $a_1, a_2 \in \mathbb{R}$. Note that if $|OA| = a$, then A must lie on the circle $c_a : x^2 + y^2 = a^2$. Furthermore, if we fix the value of a_1, then A must lie on line l through $(a_1, 0)$ in the direction of ω. When looking for feasible solutions, we therefore look at the intersection points between c_a and l. We see this illustrated in Figure 4.2.

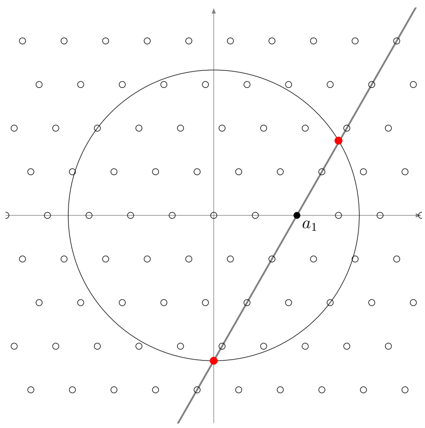


Figure 4.2: Circle c_a and line l with their intersections for given a and a_1.

Since $|OA|^2 = A\bar{A}$, the point A lies on c_a exactly when

$$a_1^2 + 2a_1a_2\cos(\theta) + a_2^2 = a^2.$$

We can rewrite this to express a_2 in terms of a_1 by completing the square:

$$\begin{aligned}(a_1\cos(\theta)+a_2)^2-a_1^2\cos^2(\theta) &= a^2-a_1^2\\ (a_1\cos(\theta)+a_2)^2 &= a^2+(\cos^2(\theta)-1)a_1^2\\ a_1\cos(\theta)+a_2 &= \pm\sqrt{a^2-a_1^2\sin^2(\theta)}\\ a_2 &= -a_1\cos(\theta)\pm\sqrt{a^2-a_1^2\sin^2(\theta)}. \qquad (4.2)\end{aligned}$$

Note that there is only a solution if $a^2 - a_1^2\sin^2(\theta) \geq 0$. Therefore, we get that $a_1^2 \leq \frac{a^2}{\sin^2(\theta)}$, so $\frac{-a}{\sin(\theta)} \leq a_1 \leq \frac{a}{\sin(\theta)}$. Note that $A \in \Lambda$ only if $a_1, a_2 \in \mathbb{Z}$.

□

With this result, we can construct a brute-force algorithm to try and place a triangle on a grid when given its side lengths a, b and c. Recall from Section 2.3, that we assumed that one vertex of the triangle lies in the origin. For the other vertices $A = a_1 + a_2\omega$ and $B = b_1 + b_2\omega$, we will find all possible pairs $(a_1, a_2) \in \mathbb{Z}^2$ and $(b_1, b_2) \in \mathbb{Z}^2$ such that $|OA| = a$ and $|OB| = b$ using Theorem 4.4. Since the amount of values that need to be checked is proportional to the side length, we will do this for the shortest two sides a and b.
Now we can try every combination of the pairs (a_1, a_2) and (b_1, b_2) to see if the other side has length c. We will check that $|AB|^2 = c^2$, so we get

$$(a_1-b_1)^2+2(a_1-b_1)(a_2-b_2)\cos(\theta)+(a_2-b_2)^2=c^2.$$

If we find any combination of the pairs (a_1, a_2) and (b_1, b_2) such that this holds, we know that the triangle can be placed on the grid.
From Lemma 2.6, we know that lattices have 180° rotational symmetry. Therefore, if the pairs (a_1, a_2) and (b_1, b_2) produce a valid triangle, then $(-a_1, -a_2)$ and $(-b_1, -b_2)$ will also produce a valid triangle. Therefore, we can reduce the computation time by forcing $a_1 \geq 0$.

Chapter 5

Implementation and observations

In this chapter, we will use the previous results to find all equable triangles on a grid. We will create a python program that will do this for us. The code can be found in Appendix B. In Section 5.1, we will explain how the code works. In Section 5.2, we run the code for a large range of values and look for patterns in the data.

5.1 Implementation

When looking for the equable triangles that are placeable on a grid, we follow three steps:

1. Find r such that all side lengths of equable triangles are of the form rn with $n \in \mathbb{N}$.
2. Find all equable triangles with side lengths rn.
3. Check for each of these triangles if they can be placed on the grid.

In our code, each of these steps is a function. In particular, these functions are called SideLength(), FindCandidates() and IsPlacable(). We only look for equable triangles on grids generated by 1 and $\omega = \cos(\theta) + \sin(\theta)i$ with $\theta \in (0, \frac{\pi}{2}]$ and $\cos(\theta)$ rational.

When calculating, for example, the value of r, we often get fractions and square roots. Python rounds these values to speed up computations. However, for our purposes, we need the exact values to be able to determine when something is integral or not. We will therefore be using a symbolic math package called SymPy [11]. This package allows us to make exact calculations.

5.1.1 SideLength()

This function calculates the largest value of r such that all equable triangles on a particular grid have side lengths of the form rn with $n \in \mathbb{N}$. The input of the

function consists of the variables p, q. These values represent the numerator and the denominator of $\cos(\theta)$, respectively.
We want to apply Theorem 3.6. To do so, we need to find the values $u, v \in \mathbb{N}$ such that $q^2 - p^2 = u^2 v$ with v square-free. This is done by first initializing $u = 1$ and $v = q^2 - p^2$. Then we check every value for $d \leq \sqrt{v}$ to see if d^2 is a divisor of v. If this is the case we update $u \mapsto ud$ and $v \mapsto v/d^2$. We then set $r = \frac{\sqrt{v}}{q}$. Lastly, we check if q is even. If this is the case, we apply Corollary 3.4 and multiply r by 2. The function then returns this value.

5.1.2 FindCandidates()

The goal of this function is to find all equable triangles that have side lengths of the form rn with $n \in \mathbb{N}$ for a given r. The input of this function is this value of r as an exact SymPy value. As output, the function will return a list of tuples containing the side lengths of each triangle.
We first check if r^2 is rational. From Corollary 3.10, we know that if this is not the case, there are no equable triangles, so we return an empty list. Otherwise we define p to be the numerator of r^2 and initialize an empty list of candidates. As was specified in Remark 3.14, we need to check different values depending on the value of p.
If p is not a multiple of 16, we use the upper bounds from Theorem A.1. We check for each value of u with $u \leq \frac{2}{r}\sqrt{3}$ and each v with $u \leq v \leq \frac{1}{r^2 u}\left(4 + 2\sqrt{4 + r^2 u^2}\right)$ whether $w = \frac{4(u+v)}{r^2 uv - 4}$ is integral. We also check if $v \leq w$ to prevent double counting of triangles. Then we calculate the values $a = r(v+w)$, $b = r(u+w)$ and $c = r(u+v)$ and add the tuple (a, b, c) to the list of candidates.
If p is a multiple of 16, we do the same thing for the upper bounds from Theorem 3.12. So we check each value of x with $x \leq \frac{4}{r}\sqrt{3}$ and each value of y with $x \leq y \leq \frac{1}{r^2 x}\left(16 + 4\sqrt{16 + r^2 x^2}\right)$ to see if $z = \frac{16(x+y)}{r^2 xy - 16}$ is integral. Here, we also need to check if the values of x, y and z are all of the same parity and that $y \leq z$. Then we calculate the values $a = \frac{r}{2}(y + z)$, $b = \frac{r}{2}(x + z)$ and $c = \frac{r}{2}(x + y)$ and add the tuple (a, b, c) to the list of candidates.

5.1.3 IsPlaceable()

The goal of this function is to determine whether a given triangle can be placed on the grid. As input, we give it a tuple containing the side lengths of a triangle and the value for $\cos(\theta)$. The output of this function will be a boolean saying whether the triangle can be placed on the grid.
To start with, we also calculate the value for $\sin(\theta)$ using the fact that $\sin(\theta) = \sqrt{1 - \cos^2(\theta)}$. We also sort the side lengths (a, b, c) such that $a \leq b \leq c$. This will reduce the amount of values we need to check.
For side lengths a and b, we calculate all possible values $a_1, a_2, b_1, b_2 \in \mathbb{Z}$ such that for $A = a_1 + a_2\omega$ and $B = b_1 + b_2\omega$, we have $|OA| = a$ and $|OB| = b$. We find these values using Theorem 4.4. We also use the 180° rotational symmetry to force $a_1 \geq 0$. Then for each combination of values (a_1, a_2) and (b_1, b_2), we will check that

$|AB|^2 = c^2$. So we check if

$$(a_1 - b_1)^2 + 2(a_1 - b_1)(a_2 - b_2)\cos(\theta) + (a_2 - b_2)^2 = c^2.$$

If there is such a combination, then we know that the triangle can be placed on the grid and we return True. Otherwise, we return False.

5.2 Observations

Now that we have implemented a way to find all equable triangles on a grid, we can use this on specific grids and analyse the results. The calculations were performed for grids generated by 1 and $\omega = \cos(\theta) + \sin(\theta)i$ with $\cos(\theta) = \frac{p}{q}$ for co-prime $p, q \in \mathbb{Z}$ with $0 \le p < q \le 100$. We can see the side lengths for different grids in Table 5.1. On other grids with these values for p and q, no equable triangles can be realized.

$\cos(\theta)$	a	b	c
0	29	25	6
	20	15	7
	17	10	9
	13	12	5
	10	8	6
1/2	$8\sqrt{3}$	$7\sqrt{3}$	$3\sqrt{3}$
	$4\sqrt{3}$	$4\sqrt{3}$	$4\sqrt{3}$
1/3	$6\sqrt{2}$	$6\sqrt{2}$	$4\sqrt{2}$
2/3	$9\sqrt{5}$	$9\sqrt{5}$	$2\sqrt{5}$
	$4\sqrt{5}$	$3\sqrt{5}$	$3\sqrt{5}$
3/5	29	25	6
	20	15	7
	17	10	9
	13	12	5
	10	8	6
4/5	29	25	6
	20	15	7
	17	10	9
	13	12	5
	10	8	6
1/9	$58\sqrt{5}/3$	$56\sqrt{5}/3$	$2\sqrt{5}$
5/13	20	15	7
	13	12	5
12/13	29	25	6
	13	12	5
15/17	17	10	9
	10	8	6

$\cos(\theta)$	a	b	c
7/25	10301/5	10201/5	102/5
	629/5	101	126/5
	1402/5	1352/5	54/5
	754/5	702/5	56/5
	226/5	152/5	78/5
	202/5	104/5	102/5
	776/5	754/5	6
	104/5	58/5	54/5
	29	25	6
	318/5	306/5	24/5
	78/5	56/5	34/5
	10	8	6
	39/5	39/5	6
24/25	2729/5	2626/5	21
	629/5	101	126/5
	754/5	702/5	56/5
	313/5	252/5	13
	229/5	156/5	77/5
	20	15	7
	78/5	56/5	34/5
	538/5	533/5	21/5
21/29	29	25	6
35/37	20	15	7
	13	12	5
49/81	$1949\sqrt{65}/9$	$1945\sqrt{65}/9$	$2\sqrt{65}/3$
77/85	17	10	9

Table 5.1: All equable triangles realizable on grids for different $\cos(\theta) = \frac{p}{q}$ with $p, q \le 100$.

5.2.1 Integer equable triangles

When we look at Table 5.1, we see that the equable triangles with integer side lengths occur frequently. In particular, we might notice that, on these grids, the values p, q are often found in Pythagorean triples. We can prove that this must always be the case.

Corollary 5.1. *If an equable triangle with integer side lengths is realizable on a grid with* $\cos(\theta) = \frac{p}{q}$*, then p and q are part of a primitive Pythagorean triple with q the largest element.*

Proof. From Theorem 3.6, we know that all equable triangles on this grid have side lengths of the form $\frac{\sqrt{v}}{q}n$ with $n \in \mathbb{N}$ and v square-free such that $u^2 v = q^2 - p^2$ for some $u \in \mathbb{N}$. Now since the side lengths of the equable triangle are integers and v square-free, we get that $v = 1$. Therefore, $q^2 - p^2 = u^2$ for some $u \in \mathbb{N}$. We find that indeed p and q are part of a Pythagorean triple with q the largest element. Since p and q are co-prime, this must be a primitive Pythagorean triple. □

From this, we can also see that, on grids with p and q part of a Pythagorean triple, the side lengths of equable triangles will be of the form $\frac{1}{q}n$. Therefore, the equable triangles with integer side lengths are valid candidates for these grids.
We can use Euclid's formula to generate such pairs (p, q) such that they are part of a primitive Pythagorean triple [15]. We have that if $m > n$ are co-prime and have different parity, then $x = m^2 - n^2$, $y = 2mn$ and $z = m^2 + n^2$ form a primitive Pythagorean triple. From this we get the two pairs (x, z) and (y, z). We can now check more of these pairs to see when the integer equable triangles can be placed on them. The results for all possible pairs (p, q) with $q \leq 1000$ is shown in Table 5.2.

Triangle	Valid grids
(10, 8, 6)	3/5
	4/5
	15/17
	7/25
	143/145
(13, 12, 5)	3/5
	4/5
	5/13
	12/13
	35/37
(17, 10, 9)	3/5
	4/5
	15/17
	77/85
	323/325

Triangle	Valid grids
(20, 15, 7)	3/5
	4/5
	5/13
	24/25
	35/37
	220/221
(29, 25, 6)	3/5
	4/5
	12/13
	7/25
	21/29
	99/101
	143/145

Table 5.2: Grids on which the equable triangles with integer side lengths could be placed for $q \leq 1000$.

5.2.2 Constraints

We can also investigate what happens to the value of r for different grids. We know that when r is smaller, the amount of values we need to check in FindCandidates() is much larger. This means that finding all equable triangles takes more time. We therefore would like some additional insight in the value of r. In Figure 5.1, the values of r are plotted for different values of $\cos(\theta)$.

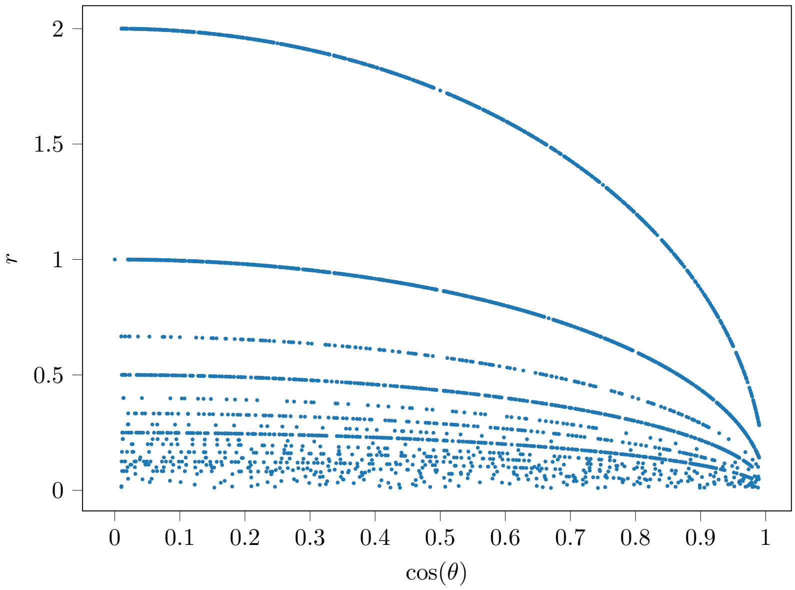


Figure 5.1: Different values of r depending on $\cos(\theta)$

We see several clear lines appearing. These come from how we calculate the value of r. Recall that from Theorem 3.6, we had that $r = \frac{\sqrt{v}}{q}$ for square-free v such that $u^2 v = q^2 - p^2$ for some $u, v \in \mathbb{N}$. We can say that $v = \frac{1}{u^2}(q^2 - p^2)$. Substituting this in the equation for r gives

$$\begin{aligned} r &= \frac{\sqrt{\frac{1}{u^2}(q^2 - p^2)}}{q} \\ &= \frac{1}{u}\sqrt{\frac{q^2 - p^2}{q^2}} \\ &= \frac{1}{u}\sqrt{1 - \cos^2(\theta)} \end{aligned}$$

Now, each line in Figure 5.1 corresponds a line $\frac{1}{u}\sqrt{1 - \cos^2(\theta)}$ for different values of u. So for example, the line starting at $r = 1$ corresponds to $u = 1$ and the line starting at $r = 0.5$ corresponds to $u = 2$. Furthermore, for every line there is also a line at twice the height. These are formed by grids with q even, since, in that case, the value of r is doubled because of Corollary 3.4.
In Figure 5.1, we also see that as $\cos(\theta) \to 1$, $r \to 0$. This is because, as $\cos(\theta)$

gets closer to 1, the two basis vectors get closer together. Therefore, the shortest distance between two points in the lattice gets smaller and r decreases.

Chapter 6

Conclusion and discussion

The goal of this paper was to find all equable triangles on arbitrary grids. We did this for all lattices generated with unit length basis vectors with angle θ such that $\cos(\theta)$ is rational. We can assume that a grid is then generated with 1 and $\omega = \cos(\theta) + \sin(\theta)i$ for some $\theta \in (0, \frac{\pi}{2}]$. To find all equable triangles on a grid, we follow three steps:

1. Find r such that all side lengths are of the form rn with $n \in \mathbb{N}$. In particular, we saw that if $\cos(\theta) = \frac{p}{q}$ for $p, q \in \mathbb{Z}$ co-prime with $0 \leq p < q$ and if $q^2 - p^2 = u^2 v$ for some $u, v \in \mathbb{N}$ with v square-free, then we have that $r = \frac{\sqrt{v}}{q}$.

2. Find all equable triangles with side lengths rn. We can do this by finding all values $x, y, z \in \mathbb{N}$ with $x \leq y \leq z$ and the same parity satisfying the constraints:

$$x \leq \frac{4}{r}\sqrt{3}$$
$$y \leq \frac{1}{r^2 x}\left(16 + 4\sqrt{16 + r^2 x^2}\right)$$
$$z = \frac{16(x+y)}{r^2 xy - 16}.$$

   These values then correspond to a triangle with side lengths a, b and c where

$$a = \tfrac{r}{2}(y+z), \ b = \tfrac{r}{2}(x+z), \ \text{and } c = \tfrac{r}{2}(x+y).$$

3. Check for each of these triangles if they can be placed on the grid. This is done by finding all points $A \in \Lambda$ such that $|OA| = a$ and all $B \in \Lambda$ such that $|OB| = b$. To do this, we find all values $a_1, a_2 \in \mathbb{Z}$ such that

$$\frac{-a}{\sin(\theta)} \leq a_1 \leq \frac{a}{\sin(\theta)}$$
$$a_2 = -a_1\cos(\theta) \pm \sqrt{a^2 - a_1^2 \sin^2(\theta)}.$$

   Then $A = a_1 + a_2\omega$. We can do the same for $B = b_1 + b_2\omega$. Then for every combination of values a_1, a_2, b_1, b_2, we can check if $|AB| = c$. If this is indeed the case, the triangle can be placed on the grid.

These three steps were implemented in Python and applied to all grids with $\cos(\theta) = \frac{p}{q}$ where $0 \leq p < q \leq 100$. For most values, there were no equable triangles that could be placed on the grid. When it was possible however, the equable triangles often had integer side lengths. It was proven that if these triangles were placeable on a grid, then p and q were part of a Pythagorean triple.

In future research, more values for p and q could be checked to gain more insight into when equable triangles can be placed on a grid. Additionally, since this paper only considered bases with unit vectors, it would also be interesting to see what happens for arbitrary vector lengths. Lastly, one could investigate whether it is possible to place equable triangles if $\cos(\theta)$ is irrational.

Appendix A

Improved upper bounds

Theorem A.1. *Let $r > 0$. Let T be an equable triangle with side lengths of the form rn with $n \in \mathbb{N}$ and values $x, y, z \in \mathbb{N}$ as in Lemma 3.9. If x, y and z are even, we can say that $x = 2u$, $y = 2v$ and $z = 2w$. We can assume that $u \leq v \leq w$. Then these values are subject to*

$$\begin{aligned} u &\leq \frac{2}{r}\sqrt{3} \\ v &\leq \frac{1}{r^2 u}\left(4 + 2\sqrt{4 + r^2u^2}\right) \\ w &= \frac{4(u+v)}{r^2uv - 4}. \end{aligned}$$

The side lengths of the triangle expressed in terms of u, v and w are then

$$a = r(v+w), \ b = r(u+w), \ \textit{and } c = r(u+v).$$

Proof. If x, y and z always even, we can substitute u, v and w into Equation 3.1 to get

$$r^2uvw = 4(u+v+w).$$

From Corollary 3.10, we know that r^2 must be rational. So let $r^2 = \frac{p}{q}$ for some $p, q \in \mathbb{N}$ co-prime. We now have that

$$puvw = 4q(u+v+w). \tag{A.1}$$

We can assume that $u \leq v \leq w$. Using this, we can get an inequality for v:

$$v \leq w = \frac{4q(u+v)}{puv - 4q}.$$

Because we know that $w > 0$ and $4q(u+v) > 0$, it must be that $puv - 4q > 0$ as well. We can take the denominator to the other side of the inequality:

$$\begin{aligned} (puv - 4q)v &\leq 4q(u+v) \\ puv^2 - 4qv &\leq 4qu + 4qv \\ puv^2 - 8qv - 4qu &\leq 0. \end{aligned}$$

This is a parabola with a minimum, so, using the quadratic formula, we can find that

$$\begin{aligned}
v &\leq \frac{1}{2pu}\left(8q + \sqrt{(-8q)^2 - 4pu(-4qu)}\right) \\
v &\leq \frac{1}{pu}\left(4q + \frac{1}{2}\sqrt{64q^2 + 16pqu^2}\right) \\
v &\leq \frac{1}{pu}\left(4q + 2\sqrt{4q^2 + pqu^2}\right). \qquad \text{(A.2)}
\end{aligned}$$

Now we use the fact that $u \leq v$. We get that

$$\begin{aligned}
u &\leq \frac{1}{pu}\left(4q + 2\sqrt{4q^2 + pqu^2}\right) \\
pu^2 &\leq 4q + 2\sqrt{4q^2 + pqu^2} \\
pu^2 - 4q &\leq 2\sqrt{4q^2 + pqu^2}.
\end{aligned}$$

Note that squaring both sides of an inequality and keeping the sign the same is valid when both sides are positive. For the right side this is trivially true, but for the left side, this may not be the case. If we assume that the left side is negative, we find the following upper bound

$$\begin{aligned}
pu^2 - 4q &\leq 0 \\
pu^2 &\leq 4q \\
u^2 &\leq \frac{4q}{p} = \frac{4}{r^2} \\
u &\leq \frac{2}{r}.
\end{aligned}$$

If instead, we assume that the left side is positive, we can take a square on both sides. This gives

$$\begin{aligned}
(pu^2 - 4q)^2 &\leq 4(4q^2 + pqu^2) \\
p^2u^4 - 8pqu^2 + 16q^2 &\leq 16q^2 + 4pqu^2 \\
p^2u^4 - 12pqu^2 &\leq 0 \\
pu^2 - 12q &\leq 0 \\
u^2 &\leq \frac{12q}{p} = \frac{12}{r^2} \\
u &\leq \frac{2}{r}\sqrt{3}.
\end{aligned}$$

In both cases, we have that $u \leq \frac{2}{r}\sqrt{3}$. Therefore in total, we must have that the values $u, v, w \in \mathbb{N}$ must be subject to

$$
\begin{aligned}
u &\leq \frac{2}{r}\sqrt{3} \\
v &\leq \frac{1}{r^2 u}\left(4 + 2\sqrt{4 + r^2 u^2}\right) \\
w &= \frac{4(u+v)}{r^2 uv - 4}.
\end{aligned}
$$

□

Appendix B

Code

In this chapter, all major functions from Section 5.1 are written as Python functions. The functions all make use of the SymPy library. On my Github [5], there is a notebook that makes use of these functions to find equable triangles for new grids. Here, there is also a pickled dictionary containing the values r, the placeable triangles and the candidate triangles for each grid investigated in Section 5.2. For those interested, I have also made a demo in Geogebra to visualize different grids [4].

B.1 SideLength

Code B.1: Finding side length constraint

```
def SideLength(p,q):
"""
Find all equable triangles of the form rn with n integral.
Args:
    - p: the numerator of cos(theta).
    - q: the denominator of cos(theta).
Returns:
    - r: a sympy value, each equable triangle on this grid must have
        side lengths of the form rn with n integral.
"""
# reduce p and q such that they are co-prime
g = math.gcd(p,q)
p,q = p // g, q // g

# apply Theorem 3.6
u,v = 1, q**2 - p**2
d = 2
while d**2 <= v:
    if v % d**2 == 0:
        u,v = u*d, v // d**2
    else:
        d+=1
r = sympy.sqrt(v) / (q)
```

```
# apply Corollary 3.4 if possible
if q % 2 == 0:
    r *= 2
return r
```

B.2 FindEquableTriangles

Code B.2: Finding equable triangles

```
def FindCandidates(r):
"""
Finds all equable triangles of the form rn with n integral.
Args:
    - r: a sympy value, each side length is of the form rn with n
        integral.

Returns:
    - candidates: a list of tuples, each tuple contains the 3 side
        lengths of the triangle as a sympy value.
"""

# r^2 must be rational
if not (r**2).is_rational:
    return []

p = (r**2).numerator
candidates = []
if p % 16 != 0 :
    # Use upper bounds from theorem A.1
    u_upper = 2 / r * sympy.sqrt(3)
    u = 1
    while u <= u_upper:
        v = u
        v_upper = (4 + 2 * sympy.sqrt(4 + r**2*u**2)) / (r**2*u)
        while v <= v_upper:
            if r**2 * u * v - 4 > 0:
                w = 4*(u+v) / (r**2 * u * v - 4)
                if v <= w:
                    if w.is_integer:
                        a,b,c = r*(v+w), r*(u+w), r*(u+v)
                        candidates.append((a,b,c))
            v+=1
        u+=1

else:
    # Use upper bounds from theorem 3.12
    x_upper = 4 / r * sympy.sqrt(3)
    x = 1
    while x <= x_upper:
```

```
        y = x
        y_upper = (16 + 4 * sympy.sqrt(16 + r**2*x**2)) / (r**2*x)
        while y <= y_upper:
            if r**2 * x * y - 16 > 0:
                z = 16*(x+y) / (r**2 * x * y - 16)
                if y <= z:
                    if z.is_integer:
                        # check if the same parity
                        if x % 2 == y % 2 == z % 2:
                            a,b,c = r/2*(y+z), r/2*(x+z), r/2*(x+y)
                            candidates.append((a,b,c))
            y+=1
        x+=1

    return candidates
```

B.3 IsPlaceable

Code B.3: Placing Triangles

```
def IsPlaceable(triangle, cos_theta):
"""
Checks if a triangle is placeable on a grid generated by 1 and
    cos(theta) + sin(theta)*i.
Args:
    - triangle: a tuple with 3 sympy values, these are the side lengths
        of the triangle we want to check.
    - cos_theta: a sympy value for cos(theta).

Returns:
    - placeable: boolean whether the triangle is placeable on the grid.
"""
# define sin_theta
sin_theta = sympy.sqrt(1 - cos_theta**2)
# sorts the side lengths to reduce search space.
triangle = sorted(triangle)

# finds all possible values for a1 and a2 to get the 1st length using
    Theorem 4.4
first_coord_options = []
a = triangle[0]
n = sympy.floor(a / sin_theta)
for a1 in range(sympy.floor(n)+1): # can ignore negative part because
    of 180 degree rotational symmetry
    a2_plus = -a1*cos_theta + sympy.sqrt(a**2 - sin_theta**2 * a1**2)
    if a2_plus.is_integer:
        first_coord_options.append((a1,a2_plus))
    a2_min = -a1*cos_theta - sympy.sqrt(a**2 - sin_theta**2 * a1**2)
    if a2_min.is_integer:
        first_coord_options.append((a1,a2_min))
```

```
# If side length a cannot be placed, triangle cannot be placed
if len(first_coord_options) == 0:
    return False

# finds all possible values for b1 and b2 to get the 2nd length Theorem
    4.4
second_coord_options = []
b = triangle[1]
n = sympy.floor(b / sin_theta)
for b1 in range(-sympy.floor(n),sympy.floor(n)+1):
    b2_plus = -b1*cos_theta + sympy.sqrt(b**2 - sin_theta**2 * b1**2)
    if b2_plus.is_integer:
        second_coord_options.append((b1,b2_plus))
    b2_min = -b1*cos_theta - sympy.sqrt(b**2 - sin_theta**2 * b1**2)
    if b2_min.is_integer:
        second_coord_options.append((b1,b2_min))

# If side length b cannot be placed, triangle cannot be placed
if len(second_coord_options) == 0:
    return False

# try different combinations of (a1,a2) and (b1,b2) to see if they make
    the 3rd side length
c = triangle[2]
for (a1,a2) in first_coord_options:
    for (b1,b2) in second_coord_options:
        c_squared = (a1-b1)**2 + 2 * (a1-b1) * (a2-b2) * cos_theta +
            (a2-b2)**2
        if c_squared == c**2:
            return True

return False
```

B.4 TryGrid

Code B.4: Trying a certain grid

```
def TryGrid(p,q):
"""
Find all equable triangles in 3 steps: 1. Find constraint on side
    lengths.
                                    2. Find equable triangles with that
                                        constraint.
                                    3. Try to place triangles on the
                                        grid.
Grid is generated with 1 and cos(theta) + sin(theta)*i, with
    cos(theta)=p/q rational.

Args:
```

```
    - p: numerator of cos(theta)
    - q: denominator of cos(theta)

Returns:
    - r: a sympy value, each side length is of the form rn with n
        integral.
    - placeable_triangles: a list of tuples containing the side lengths
        of equable triangles placeable on the grid.
    - candidates: a list of tuples containing the side lengths of
        candidate equable triangles.
"""
# find r
r = SideLength(p,q)

# find possible candidates
candidates = FindCandidates(r)

# try to place triangles on grid
placeable_triangles = []
for triangle in candidates:
    if IsPlaceable(triangle, sympy.Rational(p,q)):
        placeable_triangles.append(triangle)

return r, placeable_triangles, candidates
```

Bibliography

[1] C. Aebi and G. Cairns. Equable triangles on the eisenstein lattice, 2023. arXiv:2309.04476.

[2] C. Aebi and G. Cairns. Following in yiu's footsteps but on the eisenstein lattice, 2023. arXiv:2309.13551.

[3] B. Braden. The surveyor's area formula. *The College Mathematics Journal*, 17(4):326–337, 1986. URL: http://www.jstor.org/stable/2686282.

[4] L. de Hoop. Geogebra demo. URL: https://www.geogebra.org/m/ktr4ffpg.

[5] L. de Hoop. Github repository. URL: https://github.com/LarsdeHoop/EquableTriangles.

[6] A. H. Foss. Integer-sided triangles. *The Mathematics Teacher*, 73(5):390–392, 1980. URL: http://www.jstor.org/stable/27962041.

[7] C. Kittel. Crystal structure. In *Introduction to Solid State Physics*, pages 6–9, 2004.

[8] D. Micciancio. Lecture notes in lattice algorithms and applications, 2010. URL: https://cseweb.ucsd.edu/classes/wi10/cse206a/lec1.pdf.

[9] D. Micciancio and S. Goldwasser. *Complexity of Lattice Problems*. Springer US, jan. 2002. doi:10.1007/978-1-4615-0897-7.

[10] P. Q. Nguyen and J. Stern. The two faces of lattices in cryptology. In *Cryptography and Lattices*, pages 146–180, 2001.

[11] SymPy Development Team. Sympy. URL: https://www.sympy.org/en/index.html.

[12] V. Wang. The art of problem solving. URL: https://artofproblemsolving.com/community/c1461h1035155.

[13] Wikipedia contributors. Congruence (geometry). URL: https://en.wikipedia.org/wiki/Congruence_(geometry).

[14] Wikipedia contributors. Heron's formula. URL: https://en.wikipedia.org/wiki/Heron%27s_formula.

[15] Wikipedia contributors. Pythagorean triple. URL: https://en.wikipedia.org/wiki/Pythagorean_triple.

[16] P. Yiu. Heronian triangles are lattice triangles. *The American Mathematical Monthly*, 108(3):261, mar. 2001. doi:10.2307/2695390.