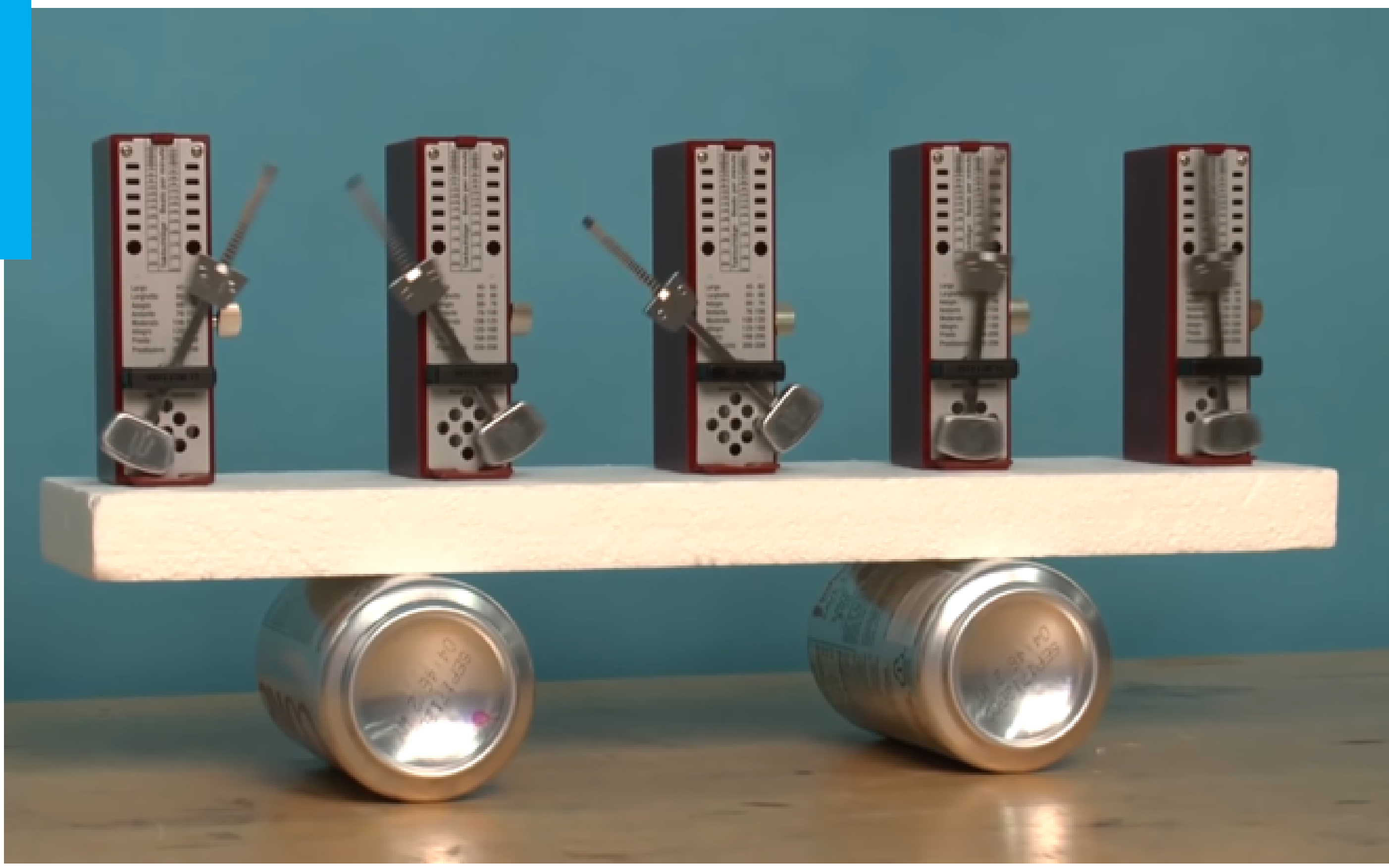
Disturbance attenuation in discrete-time Kuramoto models

J. Vlaardingerbroek

Master of Science Thesis



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Delft Center for Systems and Control

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For the degree of Master of Science in Systems and Control at Delft University of Technology

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Abstract

This thesis reports the results of research into the stability of the all-to-all coupled discrete time Kuramoto model under constant, matched input disturbances. The discrete time Kuramoto model can be used as a dynamic, decentralized multi-agent orientation coordination system: once initialized, the agents will communicate their orientations to all other agents and calculate their own step-update based on the received data. A properly controlled and undisturbed Kuramoto model can direct agents to two final, stable sets of orientations: either all agents align to the same orientation or they form a balanced set of orientations with the characteristic that the centre of gravity of all orientations on the unit circle is at the origin.

These final states will not be reached when at least one of the agents is influenced by a disturbance. Not only will this agent be affected, but because of the networked system, the disturbance in one agent will influence other agents as well. Understanding the Kuramoto model enables the design of controllers that can attenuate the influence of disturbances. All controllers are designed with the assumption of constant matched input disturbances.

The first controller is an error feedback controller. For this strategy, the original Kuramoto model had to be modified. The resulting controller can attenuate the effects of matched input disturbances in a system of agents, but individual agents with matched input disturbances will not reach a steady state.

The second controller is based on predictor-error feedback and the Kuramoto characteristic that the average orientation in a Kuramoto model is constant. The controller is augmented with an algorithm that generates a one-step ahead prediction based on the known states and inputs. Since the disturbance is assumed to be constant, its effects can be calculated and attenuated in the next time step. This controller succeeds in directing the system to the same aligned set as the undisturbed system, although via a different trajectory. The controller also succeeds in directing the system to a balanced set, but for systems with $N \geq 4$ agents that balanced set is different from the undisturbed set.

Since the second controller showed that a deviation from the undisturbed trajectory leads to a different balanced set, the third controller is designed with reference trajectories that do not use the actual states and inputs, but are generated fully autonomously. The difference between reference and actual state is processed by a proportional-integral algorithm to ensure

zero steady state error. This controller however has the possibility of destabilizing the system, when not properly tuned.

All controllers have their merits: the first controller decouples the agents, thereby preventing that a disturbed agent affects others. Under constant disturbance, the second controller guarantees stability, but will let the agents follow different trajectories than the undisturbed system, leading to different balanced sets. The third controller can direct all agents to their undisturbed trajectory, but can negatively impact the stability properties of the Kuramoto controller when improperly tuned.

Table of Contents

Preface

This is document is a part of my Master of Science graduation thesis. The idea of doing my thesis on this subject came while following courses during the master program Systems and Control. I find autonomous mobile agents very interesting and I enjoy doing research into their behavior. I read with great interest about the distributed behavioral model for flocks, herds and schools by Reynolds [1], and when dr. M. Jafarian introduced me to the Kuramoto model, I was very enthusiastic. After a lot of reading, thinking and consulting, I wanted to focus on disturbance rejection, so that mobile agents in a Kuramoto model can get closer to real world applications.

The process of starting and conducting my research during covid was difficult, and it took much more time than I had hoped. It was difficult, and I could not have done this without the loving support of my wife and sons. The result of my work is this report. I would like to thank my supervisors dr. M. Jafarian and A. Ilioudi, MSc for their assistance during the time I worked on this thesis. Please enjoy the result.

Delft, University of Technology — J. Vlaardingerbroek
June 22, 2023

Chapter 1

Introduction

1-1 Motivation

Since the late 1980s, researchers have been investigating mobile multi-agent systems [2][1]. Multi-agent systems are preferred over single agent systems when [3]:

1. the task complexity is too high for a single agent to accomplish;
2. the task is inherently distributed;
3. building several resource-bounded agents is much easier than having a single powerful agent;
4. multiple agent can solve problems faster using parallelism;
5. the introduction of multiple agents increases robustness through redundancy.

For instance, in [4] it is described how a group of heterogeneous agents can plan and allocate different tasks to different agents to construct a lunar base. In [5] an example is given of multiple homogeneous agents working together to move an object that is too large for a single agent to move. In [6] a group of agents is shown iteratively sweeping an area to provide maximum coverage. In all three examples, the success of the group of agents depends on coordination between the group members, specifically the coordination of the motions and actions of the group members.

1-2 The coordination problem

To have an agent exhibit the desired behavior its states have to be controlled. Controlling multiple mobile agents gives more challenges than just the sum of states. An agent in an

[1] In [2] the term *robots* is used. In this thesis, the more general term *agents* will be used

environment without other agents or obstacles can reach its desired location without risk of collisions. Adding one or more agents creates the risk of collisions and thus the task of collision avoidance. Decision-making structure, communication and motion planning are identified in [3], [7] and [8] as additional problems for multi-agent systems.

1-2-1 Static versus dynamic coordination

Static coordination is also known as offline coordination. In general, this type of coordination means that the mobile agents adopt a set of rules and agreements before they begin a task. In [9] this has been exemplified by giving agents traffic rules to obey. According to [7], static coordination can handle complex tasks, but is less suited for real time controlling, whereas dynamic coordination is suited for real-time controlling, but it cannot handle complex tasks as well as static coordination. The characteristics of coordination can be improved by combining static and dynamic coordination into a hybrid form.

1-3 The Kuramoto model as a multi-agent coordination system

A particular example of a dynamic, multi-agent coordination system is the discrete time representation of the Kuramoto model. The original, continuous time Kuramoto model [10] was used to model chemical oscillations, waves and turbulence for all-to-all coupled particles. Following the original research, a discrete time representation of the Kuramoto model [11] was designed. This discrete time model was then further researched and applied to the coordination of the orientation of mobile agents with constant velocities [12]. The discrete time Kuramoto model enables users to design a group of mobile agents as a leaderless swarm that can either align their orientations so that after a while all agents have the same orientation, or they can balance their orientations[2]. More recent research [13] has augmented the discrete time Kuramoto model to include collision avoidance and to shift the groups behavior from aligned to balanced as a target location is approached. Until now, the discrete time Kuramoto model has been researched with the assumption of absence of disturbances. As will be shown in chapter 2, a disturbance in any of the agents in a Kuramoto model will affect all coupled agents. Using the undisturbed Kuramoto model as reference, attenuating the disturbance and its effect is a *servo problem*: all agents must track the undisturbed reference trajectory. By investigating the servo problem, this thesis extends the research on the discrete time Kuramoto model by researching disturbance attenuation options and provides recommendations for use cases.

1-4 Outline

The subject of this thesis is disturbance attenuation in the discrete time Kuramoto model. This subject was chosen after an initial literature review of multi-agent formation control. The second chapter will give an introduction to the Kuramoto model, a review of relevant publications about the discrete time Kuramoto model and an overview of the disturbed discrete time

[2] Aligned and balanced behavior will be further explained in chapter 2.

Kuramoto model. After these preliminaries, different disturbance attenuating controllers will be designed and evaluated in chapters 3, 5 and 4. The last chapter contains conclusions and recommendations.

Chapter 2

Preliminaries: the Kuramoto model

This chapter introduces and explains the different variations of the Kuramoto model used in research and applications. The chapter starts off with the original, continuous-time Kuramoto model (2-1), followed by the zero-order hold, first order discrete-time approximation model (2-2) and the discrete-time Kuramoto model in mean field coupling form (2-4). This chapter then adds to the research in [14] by combining the relation between balanced orientations for $N = 2$ and $N = 3$ with the constant average orientation in an undisturbed Kuramoto model in order to precisely predict the final orientations. After the introduction of the undisturbed Kuramoto model, the effects of matched input disturbances are demonstrated. After these preliminaries, the chapter introduces and refines the research goals.

2-1 The original continuous-time Kuramoto model

The Kuramoto model is a mathematical model that describes the behavior of coupled phase oscillators [10]. The continuous-time model is given by[1]

$$\dot{\theta}(t) = \omega_i - \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j(t) - \theta_i(t)). \tag{2-1}$$

The symbols used in equation 2-1 are given in Table 2-1. The subscript j is for all agents that are coupled with agent i. In this thesis, unless explicitly stated otherwise, all models are assumed to be all-to-all coupled systems. For a system of $N = 4$ agents, the graph topology is then as shown in Figure 2-1, with the agents in red and the connections, including self-loops, in black.

[1]Sometimes the Kuramoto model is represented with a '+' between ω_i and $\frac{K}{N}$ instead of '-'. This has no effects on the research, since the coupling strength K can be either positive or negative, and the different characteristics of the system behavior can always be selected.

Table 2-1: Symbols from Equation 2-1

Symbol	Description
θ	Agent orientation
$\dot{\theta}$	Angular velocity
ω	Natural frequency
K	Coupling strength
N	Number of agents
t	Continuous time

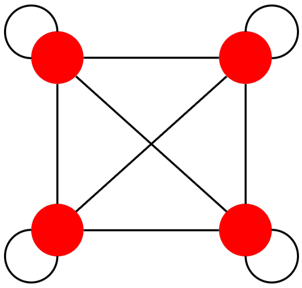


Figure 2-1: Graph for four all-to-all coupled agents

2-2 The discrete-time Kuramoto model

A discrete-time model with fixed time step can be used, if the Kuramoto model is to be used for a system of networked physical agents with intermittent communications. [11]. The zero-order hold discretized Kuramoto model is then given by:

$$\theta_i(h+1) = \theta_i(h) - \frac{K\tau}{N}\sum_{j=1}^{N}\sin(\theta_j(h) - \theta_i(h)), \tag{2-2}$$

where the new symbols are τ for the time step used for discretization and h for the time index in discrete-time. The natural frequency has been left out in equation 2-2, since all agents are assumed to be identical, and the agents can be set in a rotating frame. Since the difference in agent orientation between two consecutive time steps is determined by an algorithm that determines the characteristics of the Kuramoto model, this part of equation 2-2 ($-\frac{K\tau}{N}\sum_{j=1}^{N}\sin(\theta_j(h) - \theta_i(h))$) will be referred to as the *Kuramoto algorithm*. It is worth noting that the Kuramoto model in equation 2-2 has no random parameters and is therefore deterministic. This fact will be used later on in preditions and simulations.

The Kuramoto model in equation 2-2 has been reformulated in [12] using the phasor $\mathcal{R}$ as order parameter:

$$\mathcal{R}(\theta_i(h)) \equiv \frac{1}{N}\sum_{i=1}^{N}\begin{bmatrix}\cos(\theta_i(h))\\ \sin(\theta_i(h))\end{bmatrix}. \tag{2-3}$$

Using $\rho(h) \equiv \|\mathcal{R}(\theta_i(h))\|$ and $\psi(h) \equiv \angle\mathcal{R}(\theta_i(h))$, the discrete-time Kuramoto model in mean field coupling form becomes:

$$\theta_i(h+1) = \theta_i(h) - K\tau\rho(h)\sin(\psi(h) - \theta_i(h)). \tag{2-4}$$

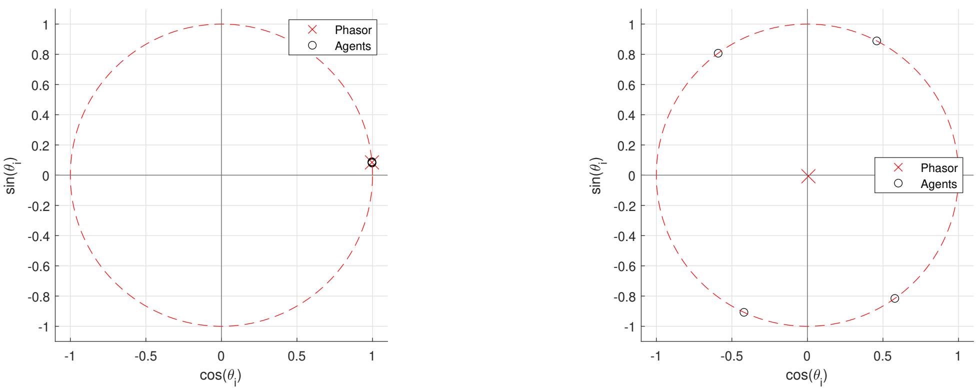
The Kuramoto algorithm is now reformulated to:

$$- K\tau\rho(h)\sin(\psi(h) - \theta_i(h)) \tag{2-5}$$

The magnitude of the phasor, $\rho(h)$, can reach values from zero to one and is an indication of the order in the system. The definitions of the aligned and balanced sets are [12]:

$$\begin{aligned} &\textit{Aligned set}: \quad \mathcal{A} = \{\theta_i(h) \mid \rho(h) = 1\} \\ &\textit{Balanced set}: \quad \mathcal{B} = \{\theta_i(h) \mid \rho(h) = 0\}. \end{aligned}$$

In the balanced set, when $\rho(h) = 0$, the phasor will not have an orientation. However, since the Kuramoto model converges asymptotically $\rho(h) = 0$ will only be reached as $h \to \infty$, so before $h = \infty$, $\psi(h)$ can still be calculated. A visual representation of the aligned and balanced set can be seen in Figure 2-2. Figure 2-2a shows that the phasor is at a distance $\rho(h) = 1$ from the origin, while in Figure 2-2b the phasor coincides with the origin.



(a) Aligned set **(b)** Balanced set

Figure 2-2: Visual representation of an aligned and a balanced set

2-3 Aligned and balanced sets

For all-to-all coupled, identical agents, the behavior of the system is determined by the value of the product $K\tau$. If $-2 < K\tau < 0$, then the agents will converge to an aligned set. If $0 < K\tau < 2$, the agents disperse until they are in a balanced set [12].

2-3-1 Characteristics of the aligned set

Asymptotic stability of the aligned set for $N = 2$ is proven in [11]. Asymptotic stability of the aligned set for any N is proven in [12]. The evolution of $N = 4$ agent orientations towards an aligned set is illustrated in Figure 2-3.

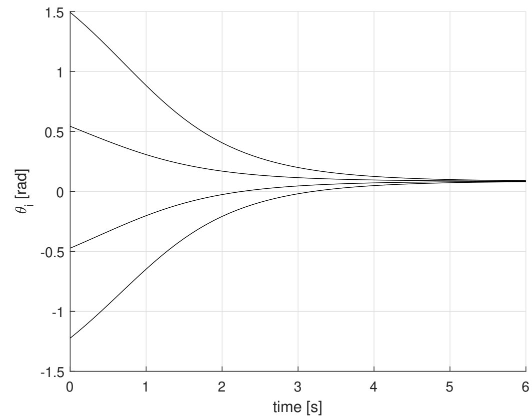


Figure 2-3: Agent orientations

The evolution of $\rho(h)$ for the same agents as used in the system in Figure 2-3 is given in Figure 2-4. To better show the asymptotic behavior of the aligned set, the vertical axis in Figure 2-4 is changed into $1 - \rho(h)$.

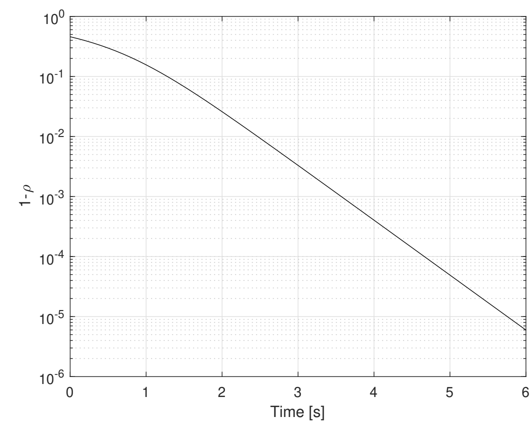


Figure 2-4: Magnitude of the phasor

An interesting observation can be made in Figure 2-5: $\theta_{avg}(h) = \frac{1}{N}\sum_{i=1}^{N} \theta_i(h))$ never changes. The reason that $\theta_{avg}(h)$ never changes is that the function that handles the interaction between agents (the sine) is an odd function, and the adjacency matrix for the symmetric graph in Figure 2-1 is undirected, resulting in a symmetric Laplacian matrix L with the property $\mathbb{1}^\top L = 0$ [15]. This will be very useful when designing controllers, since the average orientation can be checked for errors at every time step, and the final aligned orientations can be predicted.

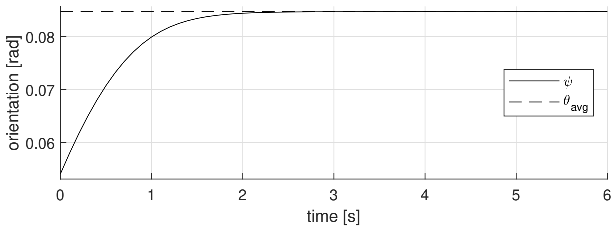


Figure 2-5: Orientations

2-3-2 Characteristics of the balanced set

Asymptotic stability of the balanced set for $N = 2$ is proven [11], while asymptotic stability of the balanced set for any N has not been proven yet, but is conjectured [12]. Further research [14] has proven for $N < 5$ that the complex exponentials of balanced orientations in the Kuramoto model sum up to zero. This is equivalent to stating that the magnitude of the phasor is zero. It also explains why it is difficult to predict balanced orientations for the Kuramoto model with $N > 3$: the complex exponential of an agents orientation will sum up to zero with that of a single other agent that has an angle of π with respect to the first agent. The remaining agents (two remaining if $N = 4$) can have any combination of orientations whose complex exponentials sum up to zero, without influencing the first two. However, under the assumption that the set of balanced states found in [14] also holds for the discrete-time Kuramoto model (2-2), it is possible to predict the balanced orientations for $N = 2$ and $N = 3$, which is useful for control. For $N \geq 4$, not enough equations can be formulated for the number of unknown variables. The final orientations for the agents can not be predicted, so they cannot be used in a feedback setting. The average orientation can be predicted, since it is constant in an undisturbed Kuramoto model.

Balanced orientations, $N = 2$

According to [14],

$$\lim_{h\to\infty} \theta_2(h) - \theta_1(h) = \pi. \tag{2-6}$$

Recalling that the average orientation in the Kuramoto model is constant

$$\lim_{h\to\infty} \frac{\theta_1(h) + \theta_2(h)}{2} = \theta_{avg}(0) = \frac{\theta_1(0) + \theta_2(0)}{2}, \tag{2-7}$$

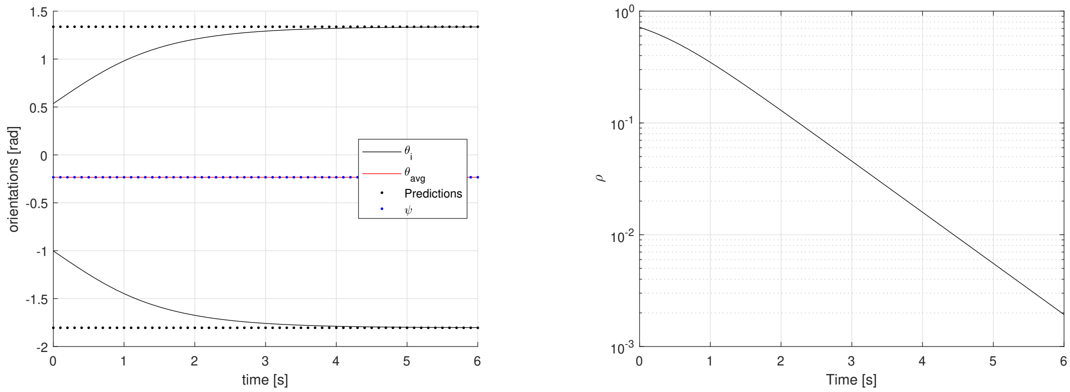
allows for combining equations 2-6 and 2-7, finding:

$$\begin{aligned} \lim_{h\to\infty} \theta_1(h) &= 2\theta_{avg}(0) - \underset{h\to\infty}{lim}\, \theta_2(h), \\ &= 2\theta_{avg}(0) - \lim_{h\to\infty} \theta_1(h) - \pi, \\ &= \theta_{avg}(0) - \frac{\pi}{2}, \end{aligned} \tag{2-8}$$

and combining equations 2-6 and 2-8 results in:

$$\lim_{h\to\infty}\theta_2(h) = \theta_{avg}(0) + \frac{\pi}{2}. \tag{2-9}$$

The results of a simulation can be seen in Figure 2-6: both agents converge to their predicted orientation and the average orientation is constant.



(a) Orientations

(b) Phasor magnitude

Figure 2-6: Simulation for $N = 2$, balanced

Balanced orientations, $N = 3$

According to [14], if one balanced orientation is determined, the other two balanced orientations can be found by adding or subtracting $\frac{2\pi}{3}$. Mathematically:

$$\begin{aligned}\lim_{h\to\infty}\theta_1(h) &= \lim_{h\to\infty}\theta_2(h) - \frac{2\pi}{3},\\ \lim_{h\to\infty}\theta_2(h) &= \lim_{h\to\infty}\theta_2(h),\\ \lim_{h\to\infty}\theta_3(h) &= \lim_{h\to\infty}\theta_2(h) + \frac{2\pi}{3}.\end{aligned} \tag{2-10}$$

Recalling that the average orientation in the Kuramoto model is constant:

$$\lim_{h\to\infty}\frac{\theta_1(h)+\theta_2(h)+\theta_3(h)}{3} = \theta_{avg}(0) = \frac{\theta_1(0)+\theta_2(0)+\theta_3(0)}{3} \tag{2-11}$$

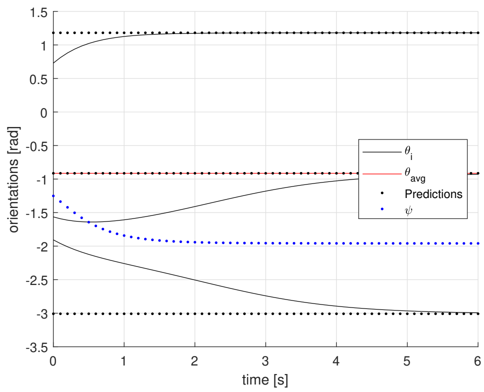
allows for combining equations 2-10 and 2-11, finding:

$$\begin{aligned}\lim_{h\to\infty}\theta_2(h) &= 3\theta_{avg}(0) - \mathop{\mathit{lim}}_{h\to\infty}\theta_1(h) - \mathop{\mathit{lim}}_{h\to\infty}\theta_3(h),\\ &= 3\theta_{avg}(0) - \lim_{h\to\infty}\theta_2(h) + \frac{2\pi}{3} - \lim_{h\to\infty}\theta_2(h) - \frac{2\pi}{3},\\ &= \theta_{avg}(0),\end{aligned} \tag{2-12}$$

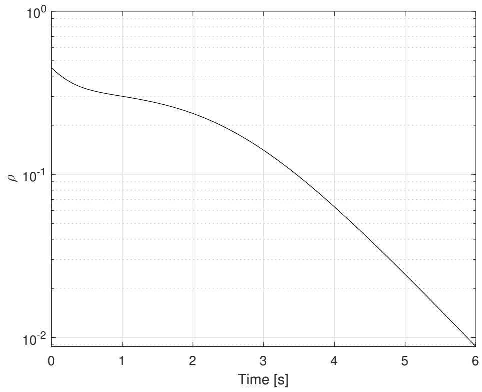
and therefore:

$$\lim_{h\to\infty} \theta_1(h) = \theta_{avg}(0) - \frac{2\pi}{3},$$
$$\lim_{h\to\infty} \theta_2(h) = \theta_{avg}(0), \tag{2-13}$$
$$\lim_{h\to\infty} \theta_3(h) = \theta_{avg}(0) + \frac{2\pi}{3}.$$

The results of a simulation are illustrated in Figure 2-7: all agents converge to their predicted orientation and the average orientation is constant[2].



(a) Orientations



(b) Phasor magnitude

Figure 2-7: Simulation for $N = 3$, balanced

2-4 Matched input disturbance in the Kuramoto model

Until now this thesis has only discussed undisturbed models. The purpose of this thesis is disturbance attenuation in discrete-time Kuramoto models. Now, a constant matched input disturbance (d_i) will be introduced, changing the model into:

$$\theta_i(h+1) = \theta_i(h) - K\tau\rho(h)\sin(\psi(h) - \theta_i(h)) + d_i \tag{2-14}$$

The system is represented in Figure 2-8. The signals and systems are given in Table 2-3.

Since the Kuramoto model only uses relative measurements between agents, disturbances may spread through the network [16]. Figures 2-9 to 2-11 show a simulation of a system that is supposed to go towards an aligned or balanced set, but one agent has been given a constant matched input disturbance.

[2]In Figure 2-6a, the Kuramoto model moving towards a balanced set shows a remarkable difference with the Kuramoto model moving towards an aligned set in Figure 2-5. In an aligned set, $\psi(h)$ always coincides with $\theta_{avg}(h)$. In a balanced set with $N > 2$, $\psi(h)$ may coincide with θ_{avg}, but it never did for any simulation for this thesis. Further exploration and proof for this observation are outside the scope of this thesis.

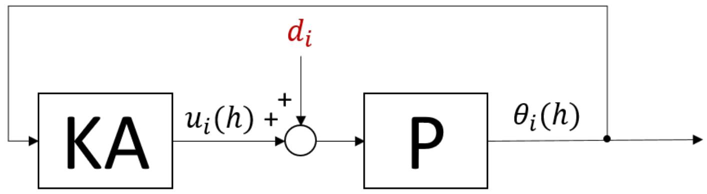


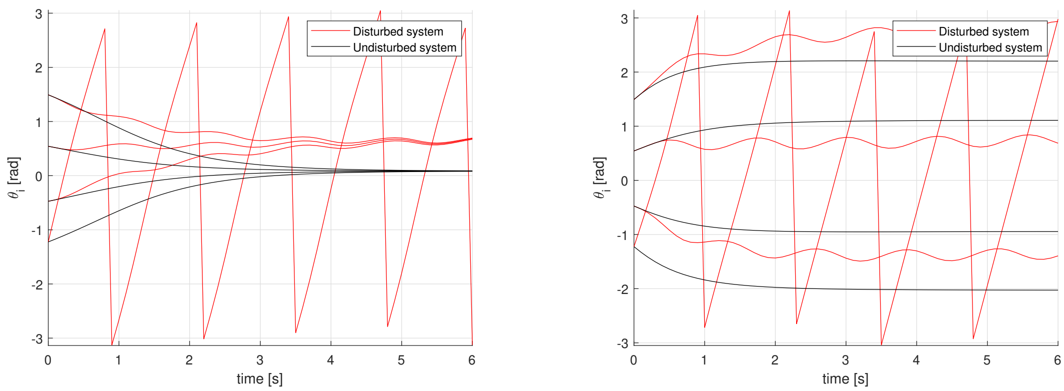
Figure 2-8: Kuramoto model with matched input disturbance

Table 2-2: Blocks from Figure 2-8

Block	Description
KA	Here the Kuramoto Algorithm (2-5) is used to calculate input $u_i(h)$
P	Plant: here the agents carry out their step-update

Table 2-3: Variables from Figure 2-8

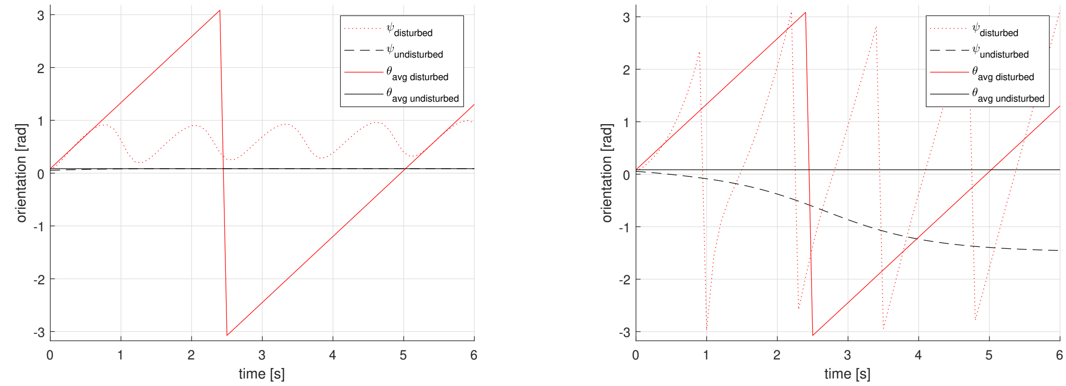
Variable or constant	Description	Equation
$u_i(h)$	calculated input	$u_i(h) = -K\tau\rho(h)\sin(\psi(h) - \theta_i(h))$
$\rho(h)$	phasor magnitude	$\rho(h) = \|\|\mathcal{R}(\theta_i(h))\|\|$
$\psi(h)$	phasor orientation	$\psi(h) \equiv \angle\mathcal{R}(\theta_i(h))$
$\mathcal{R}(\theta_i(h))$	phasor	$\mathcal{R}(\theta_i(h)) \equiv \frac{1}{N}\sum_{i=1}^{N}\begin{bmatrix}\cos(\theta_i(h))\\ \sin(\theta_i(h))\end{bmatrix}$
$\theta_i(h)$	agent state	$\theta_i(h+1) = \theta_i(h) + u_i(h) + d_i$
d_i	disturbance	unknown constant



(a) Towards aligned set **(b)** Towards balanced set

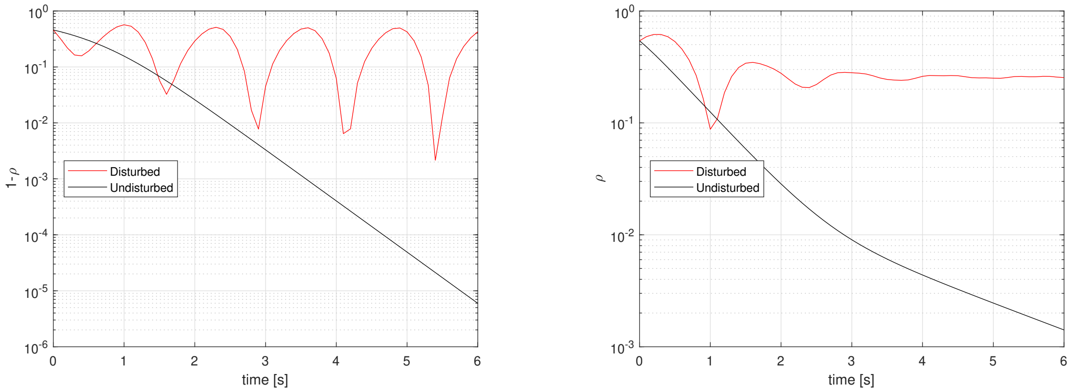
Figure 2-9: Agent orientations

In Figure 2-9 the disturbed agent displays the saw tooth movement. This is caused by angle wrapping. It is clear that the disturbance spread to other agents as well. Furthermore, Figure 2-10 shows that the average orientation is not constant in the disturbed systems. Figure 2-11 shows that the system does not converge to either an aligned or balanced state.



(a) Towards aligned set **(b)** Towards balanced set

Figure 2-10: $\psi(h)$ and $\theta_{avg}(h)$



(a) Towards aligned set **(b)** Towards balanced set

Figure 2-11: Magnitude of the phasor

2-5 Research goal

Previous publications on the discrete-time Kuramoto model [11], [12], [13] did not report about disturbances in the Kuramoto model. This thesis will contribute to the field of research into the discrete-time Kuramoto model by being the first publication to investigate the effects of disturbances and methods for disturbance attenuation. The focus of this thesis will be on matched input disturbances because of practical applications: the mobile agents as researched in [12] and [13] can be affected by real world load disturbances caused, for example, by wind or terrain. Since load disturbances typically have low frequencies [17], this thesis will focus on the lowest frequency: constant disturbances. The results of this thesis will enable the mobile agents from [12] to attenuate constant disturbances and it will be a step towards attenuating disturbances with dynamics that are much slower than the dynamics of the agents in [12] and [13]. This thesis not only designs controllers that reach asymptotic stability of the disturbed systems, but also lets the agents exhibit behavior as close to undisturbed behavior as possible. The main research goal can be defined as follows:

"The goal of this research is to design a controller that attenuates the effects of constant matched input disturbances in a discrete-time Kuramoto model"

This research goal can be further refined, because the characteristics of the term *discrete-time Kuramoto model* are not completely defined. Moving the Kuramoto model from continuous to discrete-time leaves some space for simplifications and adjustments to suit the problem at hand. For example, in the past, a one-to-all coupled controller has been implemented with successful results [12], which is not strictly according to the all-to-all characteristic of the Kuramoto model. Therefore, in this research other simplifications and adjustments will be implemented and explained to reduce model complexity and computational cost and to reach the research goal.

2-5-1 Refined research goals - aligned set

The first refined research goal is based on convergence to an aligned set. The controller should direct the agents in a way that the magnitude of the phasor in the presence of disturbances asymptotically goes to 1 if $-2 < K\tau < 0$. Mathematically:

$$\lim_{h\to\infty} ||\mathcal{R}^d(h)|| = 1 \tag{2-15}$$

Since it might be possible that the agents in a disturbed system converge (marked by the superscript d) to an aligned set with an orientation that is different from the agents in an undisturbed system (marked by the superscript u), this must be countered. The second research goal is therefore based on the characteristic of the Kuramoto model that the average orientation is constant. If $-2 < K\tau < 0$ and the magnitude of the phasor goes to 1, the orientation of all agents in the disturbed system asymptotically should go to the same orientation as the agents in the undisturbed system, that is, to the initial average orientation. Mathematically:

$$\lim_{h\to\infty} \angle\mathcal{R}^d(h) = \lim_{h\to\infty} \angle\mathcal{R}^u(h) \tag{2-16}$$

If the first two research goals are achieved by more than one controller, a third criterion can be used to compare the controllers and quantify which controller is more effective. An ideal controller would direct the evolution of agent orientations in a disturbed system to the same trajectories as in an undisturbed system. Mathematically:

$$\theta_i^d(h) - \theta_i^u(h) = 0 \tag{2-17}$$

In real world applications, with unknown disturbances, equation 2-17 cannot hold for all h. If at every time step the square of the difference between disturbed and undisturbed orientation is taken, and this difference is summed over time, then the controller with the lower deviation sum is more effective:

$$\text{Deviation sum} = \sum_{h=0}^{\infty}\sum_{i=1}^{N}(\theta_i^d(h) - \theta_i^u(h))^2 \tag{2-18}$$

2-5-2 Refined research goal - balanced set

The first research goal for the balanced set is to investigate whether the magnitude of the phasor in the presence of disturbances asymptotically goes to 0 if $0 < K\tau < 2$. Mathematically:

$$\lim_{h\to\infty} ||\mathcal{R}^d(h)|| = 0 \tag{2-19}$$

If research goal 2-19 is achieved, the second research goal for the balanced set is to investigate whether the agent orientations in the presence of disturbances asymptotically go to the same orientations as the agent orientations in the undisturbed if $0 < K\tau < 2$. Mathematically:

$$\lim_{h\to\infty} \theta_i^d(h) = \lim_{h\to\infty} \theta_i^u(h) \tag{2-20}$$

If research goals 2-19 and 2-20 are achieved by more than one controller, a third research goal for the balanced set can be used to investigate which controller is more effective. The indicator for the best controller is the controller with the lowest score on the deviation sum:

$$\text{Deviation sum} = \sum_{h=0}^{\infty}\sum_{i=1}^{N}(\theta_i^d(h) - \theta_i^u(h))^2 \tag{2-21}$$

Chapter 3

The Kuramoto model and error feedback

Feedback can be used as a means to control a system, for example to track a reference, or to attenuate disturbances [17]. The Kuramoto model is a feedback system, since it can be seen as two dynamical systems that influence each other: the agents communicate their orientation to the controller, and the control algorithm calculates the step-update for each agent. The orientation is both the state and the output of the system. The Kuramoto model can therefore be described as both a state feedback system and an output feedback system. A third feedback option is error feedback. Since the error is the difference between a reference and the actual state, a reference is required. As shown in chapter 2, the reference for all agents moving towards an aligned set can be the initial average orientation, since this is where all agents will move their orientation to. When the error is zero, the servo problem has been solved. This will be proven in section 3-2. Choosing the initial average orientation as reference for error feedback will disable the option to direct the system towards a balanced set. This will be proven in section 3-3. The reference for both the aligned and the balanced set can be the predicted final orientations of the agents. As explained in section 2-3-2, current knowledge restricts this prediction for a balanced set to a system of two or three agents, but using the predicted final orientations will enable a controller that direct a system to both an aligned and balanced set, as will be proven in section 3-4.

3-1 The error feedback model with initial average orientation as reference

For error feedback with the initial average orientation ($\theta_{avg}(0) = \frac{1}{N}\sum_{i=1}^{N}\theta_i(0)$) as reference, the undisturbed discrete-time Kuramoto model will be modified into

$$\theta_i(h+1) = \theta_i(h) - K\tau\rho(h)\sin(\theta_{avg}(0) - \theta_i(h)). \tag{3-1}$$

The modified Kuramoto algorithm in equation 3-1 is:

$$u_i(h) = -K\tau\rho(h)\sin(\theta_{avg}(0) - \theta_i(h)). \tag{3-2}$$

The system in equation 3-1 is not equal to the systems in equations 2-2 and 2-4. In equation 3-1 every agent is coupled only to the reference, $\theta_{avg}(0)$, so the new system is not all-to-all coupled. Furthermore, $K\tau\rho(h)\sin(\theta_{avg}(0) - (\theta_i(h)) \neq K\tau\rho(h)\sin(\psi(h) - \theta_i(h))$, so the step-updates are different. Although different from the original model, the controller in equation 3-1 can still direct the agents to an aligned set, as will be shown later.

The system is shown in Figure 3-1, the blocks and variables used in the system are given in Tables 3-1 and 3-2. When initializing at $h = 0$, all $\theta_i(0)$ are known, so the controller can calculate $\rho(0)$, $\theta_{avg}(0)$ and all inputs for the agents, $u_i(0)$. While the agents (in block P) update their orientations, the clock counter h is increased one step. All $\theta_i(1)$ are sent back to the controller, and calculation of the next step update can be carried out.

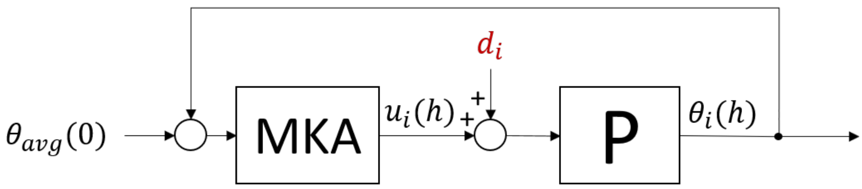


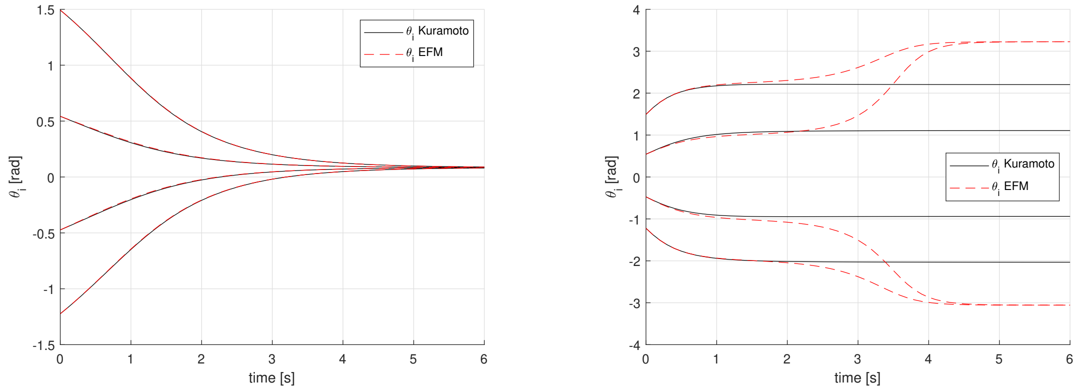
Figure 3-1: Error feedback system with the initial average orientation as reference

Table 3-1: Blocks from Figure 3-1

Block	Description
MKA	Here, a Modified Kuramoto Algorithm (3-2) is used to calculate $u_i(h)$
P	Plant: here the agents carry out their step-update.

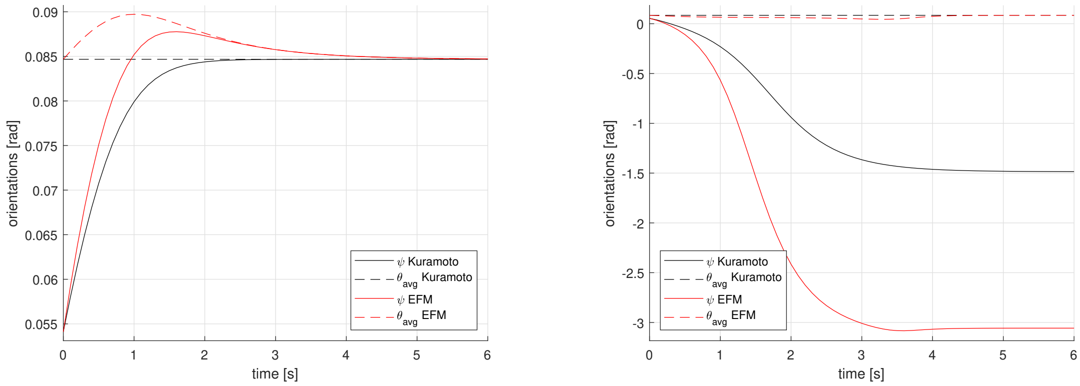
Table 3-2: Variables from Figure 3-1

Variable	Description	Equation
$\theta_i(h)$	agent orientation	$\theta_i(h+1) = \theta_i(h) + u_i(h)$
$u_i(h)$	calculated input	$u_i(h) = -K\tau\rho(h)\sin(\theta_{avg}(0) - \theta_i(h))$
$\rho(h)$	magnitude of the phasor	$\rho(h) = \|\|\mathcal{R}(\theta_i(h))\|\|$
$\mathcal{R}(\theta_i(h))$	phasor	$\mathcal{R}(\theta_i(h)) \equiv \frac{1}{N}\sum_{i=1}^{N}\begin{bmatrix}\cos(\theta_i(h))\\ \sin(\theta_i(h))\end{bmatrix}$
$\theta_{avg}(0)$	initial average orientation	$\theta_{avg}(0) = \frac{1}{N}\sum_{i=1}^{N}\theta_i(0)$



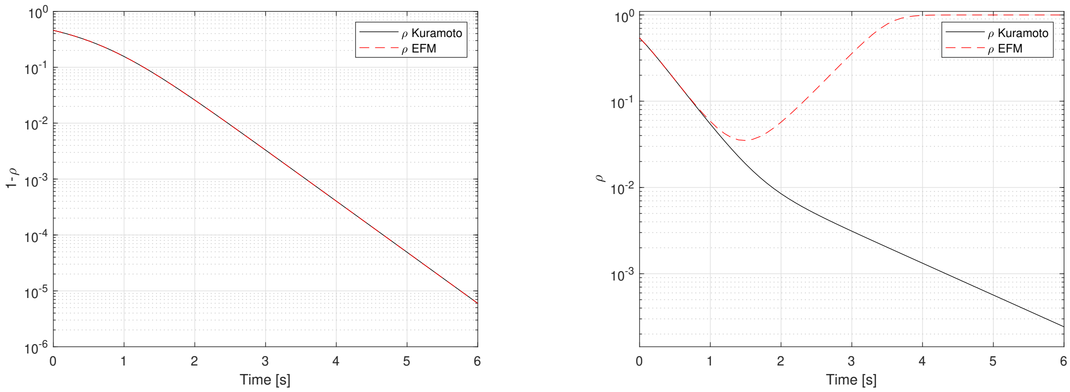
(a) K=-1 **(b)** K=3

Figure 3-2: Agent orientations



(a) K=-1 **(b)** K=3

Figure 3-3: Phasor and average orientation



(a) K=-1 **(b)** K=3

Figure 3-4: Magnitude of the phasor

3-2 Proof for convergence to an aligned set with negative coupling strength

In Figure 3-2a the agents in the error feedback model with initial average orientation as reference (3-1, indicated as EFM in the figures) appear to be following the same trajectory towards an aligned set as the agents in the Kuramoto model, but the trajectories are slightly different. The difference is best visible in Figure 3-3a. The average orientation in the Kuramoto model is constant, but in the error feedback model, the average orientation deviates from its initial value before returning to it. This shows that transforming the Kuramoto model (2-2) into an error feedback model with the initial average orientation as reference (3-1) is changing the model into another model, even when undisturbed. However, if the disturbed system in an error feedback model always converges to an aligned state at the initial average orientation, the first two research goals for the aligned set (2-15 and 2-16) are achieved, meaning that a controller based on error feedback might work.

Define the error in the i^{th} agent as the difference between its current orientation and the predictable final orientation: $\epsilon_i(h) \coloneqq \theta_i(h) - \theta_{i,fin}$. Then the evolution of that error is:

$$\begin{aligned}\epsilon_i(h+1) &= \theta_i(h+1) - \theta_{avg}(0),\\ &= \theta_i(h) - K\tau\rho(h)\sin(\theta_{avg}(0) - \theta_i(h)) - \theta_{avg}(0),\\ &= \theta_i(h) - \theta_{avg}(0) + K\tau\rho(h)\sin(\theta_i(h) - \theta_{avg}(0),\\ &= \epsilon_i(h) + K\tau\rho(h)\sin(\epsilon_i(h)) \qquad = f(\epsilon_i(h)).\end{aligned} \tag{3-3}$$

Take as Lyapunov candidate function $V(\epsilon_i(h)) \coloneqq (\epsilon_i(h))^2$. Then follow definition 3.6 from [18] to prove whether $V(\epsilon_i(h))$ is a Lyapunov equation for the system from equation 3-3. Equation 3-3 proofs that $\epsilon_i(h+1) = f(\epsilon_i(h))$. $V(\epsilon_i)$ is continuous in ϵ_i and $V(0) = 0$. $V(\epsilon_i)$ is positive definite and $\Delta V(\epsilon) = V(f(\epsilon_i)) - V(\epsilon_i)$ is negative definite:

$$\begin{aligned}\Delta V(\epsilon_i) &= V(f(\epsilon_i)) - V(\epsilon_i),\\ &= (f(\epsilon_i))^2 - (\epsilon_i)^2,\\ &= (\epsilon_i(h) + K\tau\rho(h)\sin(\epsilon_i(h)))^2 - (\epsilon_i)^2.\end{aligned} \tag{3-4}$$

Equation 3-4 is negative definite if

$$\begin{gathered}(\epsilon_i(h) + K\tau\rho(h)\sin(\epsilon_i(h)))^2 < (\epsilon_i)^2,\\ |\epsilon_i(h) + K\tau\rho(h)\sin(\epsilon_i(h))| < |\epsilon_i|,\\ |1 + K\tau\rho(h)sinc(\epsilon_i(h))| < 1\end{gathered} \tag{3-5}$$

Take as domain $\mathcal{D}$ for $\epsilon_i < -\pi, \pi >$. On $\mathcal{D}$, $0 < sinc(\epsilon_i(h)) \leq 1$. Since $-2 < K\tau < 0$ and if the balanced and aligned sets are excluded as possible initial conditions, $0 < \rho < 1$, equation 3-5 always holds. Finally, with $\phi(||\epsilon_i||) \coloneqq \frac{1}{2}(\epsilon_i)^2$,

$$0 < \phi(||\epsilon_i||) < V(\epsilon_i) \tag{3-6}$$

and since $\phi(||\epsilon_i||) \to \infty$ if $(||\epsilon_i||) \to \infty$, by theorem 3.4 in [18] the solution $\epsilon_i(h) = 0$ is asymptotically stable for all allowed initial conditions on $\mathcal{D}$. This means that all agents will converge to the initial average orientation.

3-3 Proof for convergence to an aligned set with positive coupling strength

The error feedback model with initial average orientation as reference (3-1) has also been simulated with $K = 3$ (and $\tau = 0.1$). The product $K\tau = 0.3$ would direct the discrete-time Kuramoto model (2-2) to a balanced set, but the error feedback model (3-1) directs the system to an aligned set at an angle of π with the initial average orientation, as shown in Figures 3-2b[1], 3-3b and 3-4b. Stability analysis is required to prove when the error feedback model with $0 < K\tau < 2$ converges to an aligned set.

The error feedback model (3-1) (in a possibly rotating reference frame) is in equilibrium if $\theta_i(h+1) = \theta_i(h),\ \forall\ i\ \in N$. For this to be true, $-K\tau\rho(h)\sin(\theta_{avg}(0) - \theta_i(h)) = 0$ must hold for all agents. This has two possible solutions:

1. $\rho(h) = 0$;
2. $\theta_i(h) = \theta_{avg}(0) + a\pi\ \forall\ i,$ with $a \in \{-1, 0, 1\}$.

The first solution, $\rho(h) = 0$ is a balanced state. It is, however, an unstable equilibrium: if only a single agent is moved away from its balanced position, then $\rho(h) \neq 0$, and all agents move away from $\theta_{avg}(0)$: K, τ and ρ are always positive, resulting in the $-\sin$ giving positive input for an agent with $\theta_{avg}(0) < \theta_i(h)$ and negative input to agents with $\theta_{avg}(0) > \theta_i(h)$. This is a violation of the requirements for stability from definition 3.1 in [18].

The second solution, $\theta_i(h) = \theta_{avg}(0) + a\pi\ \forall\ i,$ with $a \in \{-1, 0, 1\}$ can be divided in several sub-options:

1. Half of all agents are at $\theta_{avg}(0)$, and the other half are at $\theta_{avg}(0) \pm \pi$. This means $\rho(h) = 0$, an unstable equilibrium;
2. Both at $\theta_{avg}(0)$ and at $\theta_{avg}(0) \pm \pi$ are at least 1 agent. If any agent is perturbed, $\theta_{avg}(0)$ will change. Since then $\rho(h) \neq 0$, all agents will move towards $\theta_{avg}(0) \pm \pi$: if $\theta_i(h) > \theta_{avg}(0)(h)$, then $\sin(\theta_{avg}(0) - \theta_i(h)) < 0$, and with $0 < K\tau < 2$ this leads to $\theta_i(h+1) > \theta_i(h)$. Next, the changed average orientation has all other agents move away from the new average orientation as well. The same reasoning applies for a disturbance that leads to $\theta_i(h) < \theta_{avg}(0)$. This option thereby fails definition 3.1 for stability in [18];
3. All agents are at $\theta_{avg}(0) \pm \pi$.

[1] The orientations in Figure 3-2b have not been wrapped, since this would negatively affect the simulation. The agents in the disturbed set in Figure 3-2b at different orientations have an relative orientation of 2π.

A remarkable case is $N = 2$: both agents will move away from the average orientation at exactly the same speed, but in opposite directions. They converge to a state where their orientation exact opposites (π angle difference), but if even one of them is slightly perturbed, both start moving again towards anti-aligning with the average orientation.

The third sub-option (all agents are at $\theta_{avg}(0) \pm \pi$) describes the equilibrium that was found in the simulations in Figures 3-2 to 3-4. Using the error feedback model (3-1), let the definition of the error be the angle between the agent and anti-alignment with the initial average orientation:

$$\epsilon_i(h) = \theta_i(h) - \theta_{avg}(0) - \pi, \quad \{\epsilon, \theta_i, \theta_{avg}(0)\} \in [-\pi, \pi].^2 \tag{3-7}$$

Then the error dynamics are:

$$\begin{aligned}
\epsilon_i(h+1) &= \theta_i(h+1) - \theta_{avg}(0) - \pi, \\
&= \theta_i(h) - K\tau\rho(h)\sin(\theta_{avg}(0) - \theta_i(h)) - \theta_{avg}(0) - \pi, \\
&= \theta_i(h) - \theta_{avg}(0) - \pi - K\tau\rho(h)\sin(\theta_{avg}(0) - \theta_i(h)), \\
&= \epsilon_i(h) - K\tau\rho(h)\sin(\theta_{avg}(0) - \theta_i(h)), \\
&= \epsilon_i(h) + K\tau\rho(h)\sin(\theta_i(h) - \theta_{avg}(0)), \\
&= \epsilon_i(h) - K\tau\rho(h)\sin(\theta_i(h) - \theta_{avg}(0) - \pi), \\
&= \epsilon_i(h) - K\tau\rho(h)\sin(\epsilon_i(h)) \qquad = f(\epsilon_i(h)).
\end{aligned} \tag{3-8}$$

The candidate Lyapunov function is:

$$V(\epsilon_i(h)) = (\epsilon_i(h))^2. \tag{3-9}$$

This candidate Lyapunov function is continuous in its argument, $V(0) = 0$ and it is positive definite. Furthermore, the Lyapunov function must be negative definite:

$$\begin{aligned}
\Delta V(\epsilon_i(h) &= V(f(\epsilon_i(h))) - V(\epsilon_i(h)), \\
&= (\epsilon_i(h) - K\tau\rho(h)sin(\epsilon_i(h)))^2 - (\epsilon_i(h))^2
\end{aligned} \tag{3-10}$$

For equation 3-10 to be negative definite, the following inequality must be true:

$$\begin{aligned}
(\epsilon_i(h) - K\tau\rho(h)sin(\epsilon_i(h)))^2 &< (\epsilon_i(h))^2, \\
|\epsilon_i(h) - K\tau\rho(h)sin(\epsilon_i(h))| &< |\epsilon_i(h)|, \\
|1 - K\tau\rho(h)sinc(\epsilon_i(h))| &< 1.
\end{aligned} \tag{3-11}$$

In equation 3-11, $0 < K\tau < 2$, $0 < \rho(h) < 1$ and $0 < sinc(\epsilon_i(h)) < 1$ for all possible $\epsilon_i(h) \neq 0$, equation 3-11 is true for all sets outside the unstable equilibrium sets. Then by Lyapunov's stability theorem [18] $\theta_i(h) = \theta_{avg}(0) - \pi$ is asymptotically stable. For all initial conditions that are not unstable equilibria, the error feedback model (3-1) and $0 < K\tau < 2$ does not converge to a balanced state, but to an aligned state, with $\theta_i(h) = \theta_{avg}(0) \pm \pi, \ \forall\, i$.

[2] With angles wrapped to the stated domain, adding or subtracting π makes no difference.

3-4 Error feedback with predicted final orientations as reference

The error feedback model in equation 3-1 does not lead to a balanced state with positive coupling strength K, so the model must be improved. As shown in section 3-3, $\theta_i(h) = \theta_{avg}(0) - \pi$ is asymptotically stable for $0 < K\tau < 2$. This includes the selected reference $\theta_{avg}(0)$ for error feedback. In section 2-3-2 it is explained that the final orientations of agents in a balanced set can be predicted for $N = 2$ and $N = 3$. If these predicted final orientations are used as a reference vector $\theta_{i,fin}$, with:

$$\theta_{i,fin} = \lim_{h\to\infty} \theta_i(h), \tag{3-12}$$

then the system will be as shown in Figure 3-5.

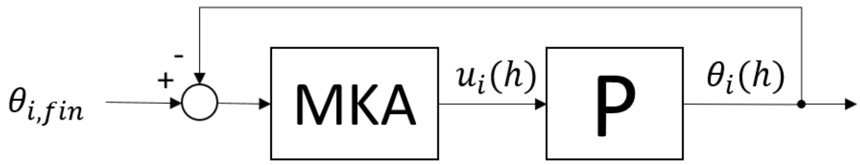


Figure 3-5: Error feedback system with the predicted final orientations as reference

The step update for the new error feedback model in Figure 3-5 is:

$$\theta_i(h+1) = \theta_i(h) - K\tau\rho(h)\sin(\theta_{i,fin} - \theta_i(h)). \tag{3-13}$$

The modified Kuramoto algorithm for the system in Figure 3-5 is:

$$u_i(h) = -K\tau\rho(h)\sin(\theta_{i,fin} - \theta_i(h)). \tag{3-14}$$

The block in Figure 3-5 are explained in Table 3-3. The variables from equation 3-13 are explained in Table 3-4.

Table 3-3: Blocks from Figure 3-5

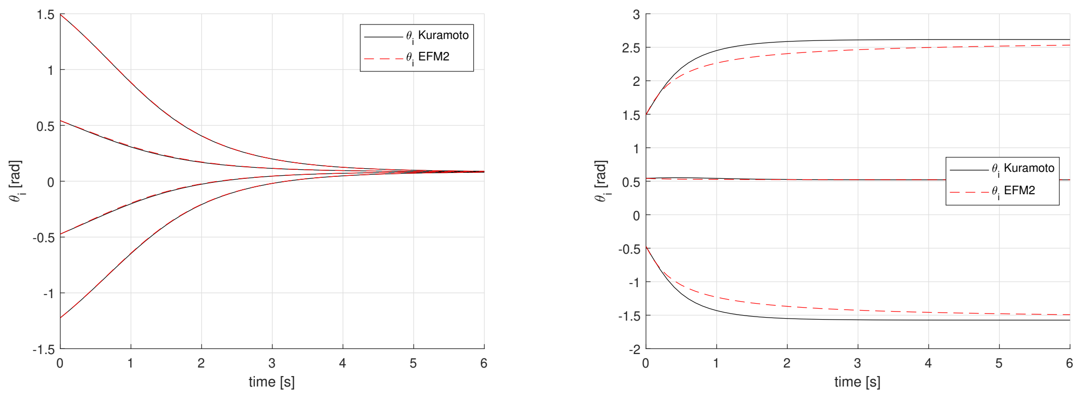
Block	Description
MKA	Here, a Modified Kuramoto Algorithm (3-14) is used to calculate $u_i(h)$
P	Plant: here the agents carry out their step-update.

The system, with predicted final orientations as reference (3-13), has been simulated for convergence towards an aligned set (Figures 3-6a to 3-8a) and towards a balanced set (Figures 3-6b to 3-8b). In these figures the evolution of the agents' orientation in the original Kuramoto model (equation 2-4) are indicated by the annotation 'Kuramoto' and the evolution of the agents moving according to the system in equation 3-13 by the annotation 'EFM2'.

The system in equation 3-13 is in equilibrium if $\theta_i(h+1) = \theta_i(h), \ \forall\ i$. For this to be true, $-K\tau\rho(h)(\sin(\theta_{i,fin} - \theta_i(h))) = 0$ must hold for all agents. This has several possible solutions:

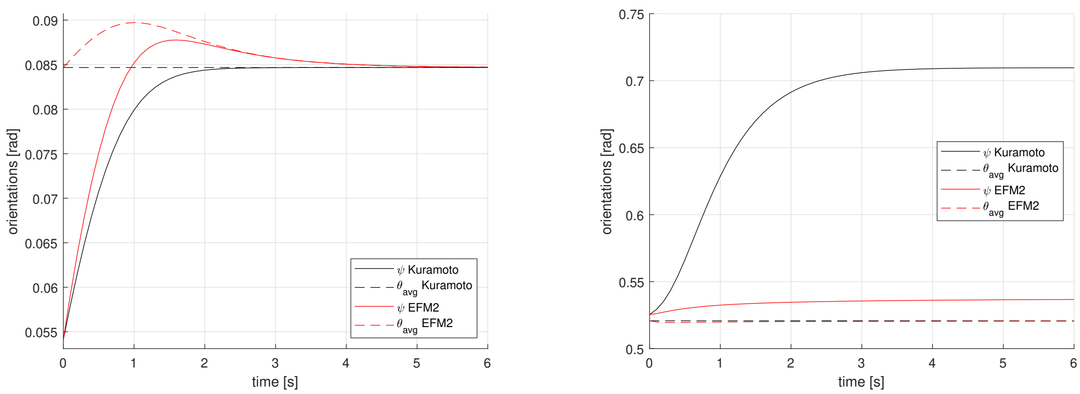
Table 3-4: Variables from Figure 3-5

Variable	Description	Equation
$\theta_i(h)$	agent orientation	$\theta_i(h+1) = \theta_i(h) + u_i(h)$
$u_i(h)$	calculated input	$u_i(h) = -K\tau\rho(h)\sin(\theta_{i,fin} - \theta_i(h))$
$\rho(h)$	magnitude of the phasor	$\rho(h) = \lvert\lvert\mathcal{R}(\theta_i(h))\rvert\rvert$
$\mathcal{R}(\theta_i(h))$	phasor	$\mathcal{R}(\theta_i(h)) \equiv \frac{1}{N}\sum_{i=1}^{N}\begin{bmatrix}\cos(\theta_i(h))\\ \sin(\theta_i(h))\end{bmatrix}$
$\theta_{i,fin}$	predicted final orientations	(see section 2-3-2)



(a) K=-1 **(b)** K=-3

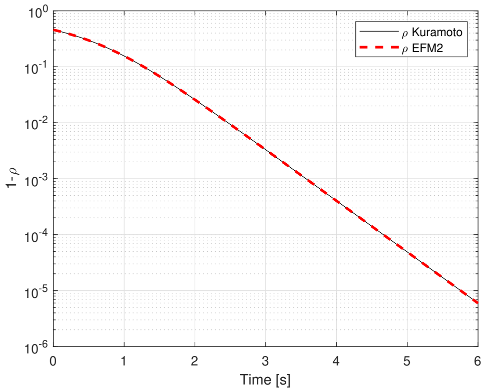
Figure 3-6: Agent orientations



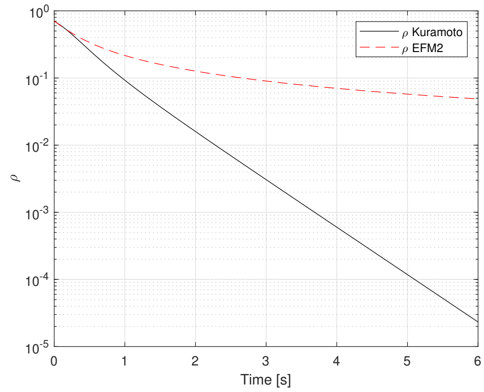
(a) K=-1 **(b)** K=-3

Figure 3-7: Phasor and average orientation

1. $\theta_i(h) = \theta_{i,fin}\ \forall\ i$ with $\rho(h) = 0$, a balanced set,
2. $\theta_i(h) = \theta_{i,fin}\ \forall\ i$ with $\rho(h) = 1$, an aligned set
3. $\theta_i(h) = \theta_{i,fin} + a\pi\ \forall\ i$, with $a \in \{-1, 0, 1\}$ and $a \neq 0$ for at least a single i, which means neither an aligned nor a balanced set.



(a) K=-1



(b) K=-3

Figure 3-8: Magnitude of the phasor

The stability of all three types of equilibria can be analyzed. If the error in the i^{th} agent is defined as the difference between its current orientation and its predicted final orientation: $\epsilon_i(h) := \theta_i(h) - \theta_{i,fin}$, then the evolution of the error is:

$$\begin{aligned} \epsilon_i(h+1) &= \theta_i(h+1) - \theta_{i,fin}, \\ &= \theta_i(h) - K\tau\rho(h)(\sin(\theta_{i,fin} - \theta_i(h))) - \theta_{i,fin}, \\ &= \theta_i(h) - \theta_{i,fin} + K\tau\rho(h)(\sin(\theta_i(h) - \theta_{i,fin})), \\ &= \epsilon_i(h) + K\tau\rho(h)(\sin(\epsilon_i(h))) \qquad = f(\epsilon_i(h)). \end{aligned} \tag{3-15}$$

The candidate Lyapunov function is:

$$V(\epsilon_i(h)) = (\epsilon_i(h))^2. \tag{3-16}$$

This candidate Lyapunov function is continuous in its argument, $V(0) = 0$ and it is positive definite. For the system to be asymptotically stable, the Lyapunov function must be negative definite:

$$\begin{aligned} \Delta V(\epsilon_i(h) &= V(f(\epsilon_i(h))) - V(\epsilon_i(h)), \\ &= (\epsilon_i(h) + K\tau\rho(h)\sin(\epsilon_i(h)))^2 - (\epsilon_i(h))^2 \end{aligned} \tag{3-17}$$

For equation 3-17 to be negative definite, the following must hold:

$$\begin{aligned} (\epsilon_i(h) + K\tau\rho(h)\sin(\epsilon_i(h)))^2 &< (\epsilon_i(h))^2, \\ |\epsilon_i(h) + K\tau\rho(h)\sin(\epsilon_i(h))| &< |\epsilon_i(h)|, \\ |1 + K\tau\rho(h)sinc(\epsilon_i(h))| &< 1, \\ -1 < 1 + K\tau\rho(h)sinc(\epsilon_i(h)) &< 1, \\ -2 < K\tau\rho(h)sinc(\epsilon_i(h)) &< 0. \end{aligned} \tag{3-18}$$

If any agent i is in any of the three equilibria, then $\epsilon_i(h) \in \{-\pi, 0, \pi\}$. Under the assumption that no agent is in one of its equilibria, then $0 < \rho(h) < 1$ and (because of angle wrapping)

$0 < sinc(\epsilon_i(h)) < 1$. Since both $\rho(h)$ and $\epsilon_i(h)$ are strictly positive and < 1, for equation 3-18 to be true, $-2 < K\tau < 0$ must be true for both a system moving to an aligned set and a system moving to a balanced set. This means that for a system moving towards an aligned set the research goals in equations 2-15 and 2-16 are fulfilled. Although the research goals in equation 2-19 and 2-20 have not been met completely (since the result of equation 3-18 indicates that $-2 < K\tau < 0$ and current knowledge is limited to predicting the final balanced orientations for $N = 2$ or 3 agents), the error feedback model can be used to direct a system of agents towards an aligned or balanced set. Because of this, it is useful to investigate the disturbance attenuation properties of the error feedback model with predicted final orientations as reference (3-13).

3-5 The error feedback model with constant disturbance

The introduction of a constant matched input disturbance to the system in Figure 3-5 will lead to the system in Figure 3-9.

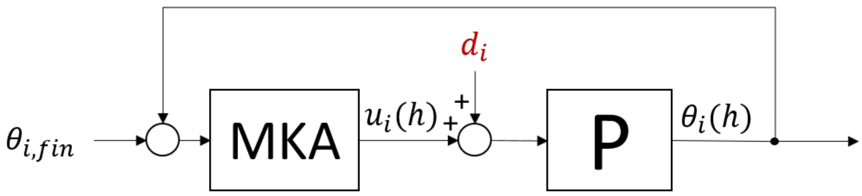


Figure 3-9: Error feedback system with the predicted final orientations as reference

The step update for the system in Figure 3-9 is:

$$\theta_i(h+1) = \theta_i(h) - K\tau\rho(h)\sin(\theta_{i,fin} - \theta_i(h)) + d_i. \tag{3-19}$$

The modified Kuramoto algorithm for the system in Figure 3-9 is:

$$u_i(h) = -K\tau\rho(h)\sin(\theta_{i,fin} - \theta_i(h)). \tag{3-20}$$

The block in Figure 3-9 are explained in Table 3-3. The variables from equation 3-13 are explained in Table 3-4.

Table 3-5: Blocks from Figure 3-9

Block	Description
MKA	Here, a Modified Kuramoto Algorithm (3-20) is used to calculate $u_i(h)$
P	Plant: here the agents carry out their step update.

In order to determine whether the disturbed error feedback model (3-19) can successfully attenuate the effects of constant matched input disturbances, and possibly converge to the same (aligned or balanced) set as the undisturbed error feedback model (3-13), first it must

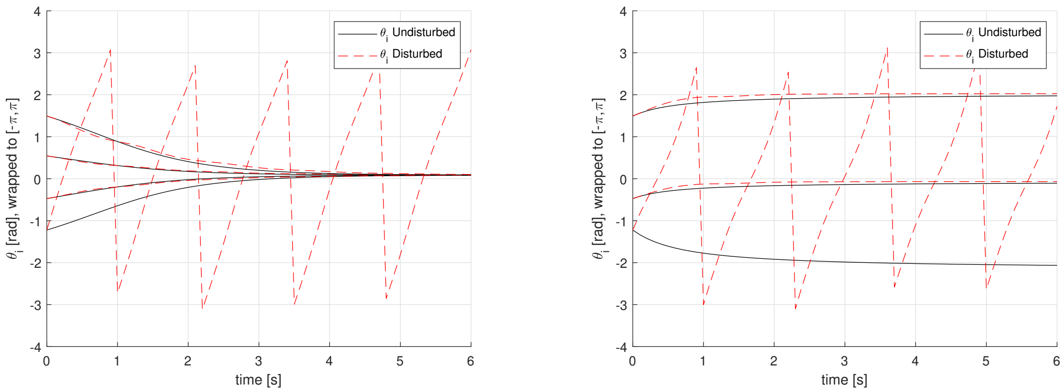
Table 3-6: Variables from Figure 3-9

Variable	Description	Equation
$\theta_i(h)$	agent orientation	$\theta_i(h+1) = \theta_i(h) + u_i(h) + d_i$
$u_i(h)$	calculated input	$u_i(h) = -K\tau\rho(h)\sin(\theta_{i,fin} - \theta_i(h))$
$\rho(h)$	phasor magnitude	$\rho(h) = \lvert\lvert\mathcal{R}(\theta_i(h))\rvert\rvert$
$\mathcal{R}(\theta_i(h))$	phasor	$\mathcal{R}(\theta_i(h)) \equiv \frac{1}{N}\sum_{i=1}^{N}\begin{bmatrix}\cos(\theta_i(h))\\ \sin(\theta_i(h))\end{bmatrix}$
$\theta_{i,fin}$	predicted final orientations	(see section 2-3-2)
d_i	constant disturbance	(unknown)

be determined whether the agents in a disturbed error feedback model reaches a steady state at all. An agent is in steady state if $\theta_i(h+1) = \theta_i(h)$, meaning that:

$$\begin{aligned} u_i(h) + d_i &= 0, \\ -K\tau\rho(h)\sin(\theta_{i,fin} - \theta_i(h)) + d_i &= 0, \\ K\tau\rho(h)\sin(\theta_{i,fin} - \theta_i(h)) &= d_i \end{aligned} \tag{3-21}$$

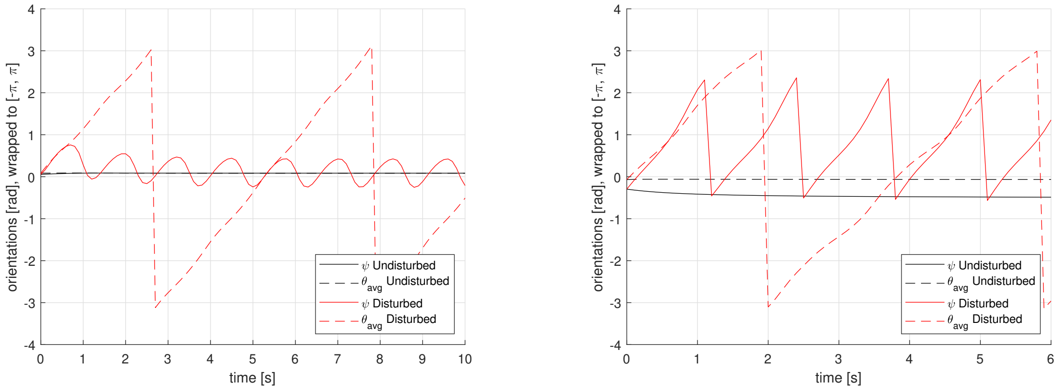
If the constant matched input disturbance is equal for all agents, then the system can be placed in a rotating reference frame. This is the same solution as was used for removing the natural frequency in section 2-2, but trivial for this thesis. The more relevant results can be found when at least a single agent has a constant matched input disturbance, that is different from the other agents. To illustrate the effects of a constant matched input disturbance, a system with the same agents and settings as in Figures 3-6 to 3-8 has been simulated, but with a disturbance in a single agent. The results can bee seen in Figures 3-10 to 3-12.



(a) K=-1 **(b)** K=-3

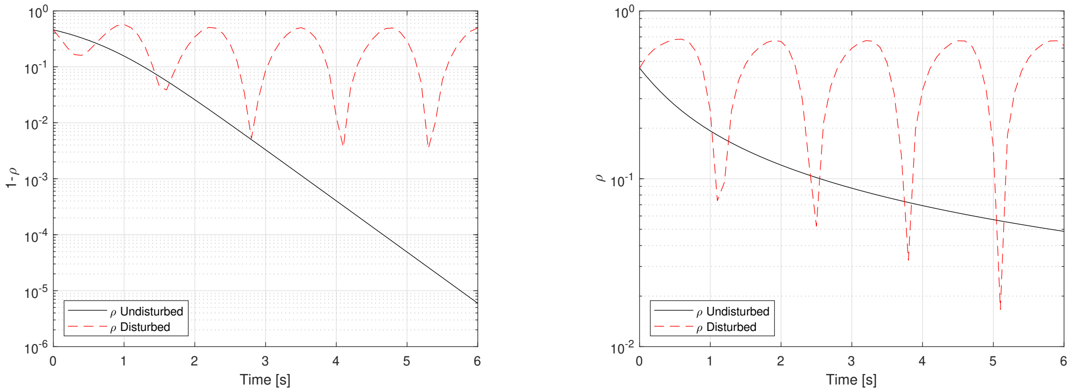
Figure 3-10: Agent orientations

When comparing the results in Figure 3-10 to those in Figure 2-9, the added value of decoupling the agents in controller (3-19) are clear: the main research goal of the thesis has been achieved, because the undisturbed agents in Figure 3-10 converge to the same orienta-



(a) K=-1 **(b)** K=-3

Figure 3-11: Phasor and average orientation



(a) K=-1 **(b)** K=-3

Figure 3-12: Magnitude of the phasor

tion as their counterparts in the undisturbed system, albeit via different trajectories.[3] The refined research goals in sections 2-5-1 and 2-5-2 have not been achieved since the effect of the disturbance on the disturbed agent itself has not been attenuated. In the system from equation 3-19, the disturbed agent will never reach a steady state. If the error in the disturbed agent is defined as the difference between the actual state and the predicted undisturbed final orientation:

$$\epsilon_i(h) = \theta_i(h) - \theta_{i,fin}, \tag{3-22}$$

then the evolution of this error is:

[3]For the undisturbed agents $d_i = 0$, meaning that 3-19 is equal to 3-13. This means that the proof for converging to the same orientation as their undisturbed counterparts in paragraph 3-4 (equations 3-15 to 3-18) can be used.

$$\begin{aligned}\epsilon_i(h+1) &= \theta_i(h+1) - \theta_{i,fin}, \\ &= \theta_i(h) - K\tau\rho(h)\sin(\theta_{i,fin} - \theta_i(h)) + d_i - \theta_{i,fin}, \\ &= \theta_i(h) - \theta_{i,fin} + K\tau\rho(h)\sin(\theta_i(h) - \theta_{i,fin}) + d_i, \\ &= \epsilon_i(h) + K\tau\rho(h)\sin(\epsilon_i(h)) + d_i\end{aligned} \tag{3-23}$$

Equation 3-23 shows that even if at any moment $\epsilon_i(h) = 0$ for the disturbed agent, the non-zero d_i will cause that at $h+1$ $\epsilon_i \neq 0$.

3-6 Conclusion and recommendation for the error feedback model

The error feedback model with predicted final orientations (3-19) can attenuate the effects of matched input disturbances in a discrete-time Kuramoto model, but this comes with adaptations of the model and limitations. First, the newly designed Kuramoto model loses the property of all-to-all coupling. Second, convergence towards an aligned or balanced set is no longer determined by the value of the product $K\tau$, but by the predicted final orientations. This currently limits the number of agents moving towards a balanced set to three, since it is still unknown how to predict the balanced final orientations for $N \geq 4$. The third limitation of 3-19 is that the the effect of the disturbance on the disturbed agent itself is not attenuated.

Because of the limitations mentioned above, it is recommended only to use the controller in situations where loss of functionality of disturbed agents is acceptable in the system. A second recommendation is to do further research into predicting the final orientations of agents in a system with $N \geq 4$ moving towards a balanced set.

Chapter 4

One step ahead prediction and deadbeat control

In control theory, many methods exist to attenuate the effects of disturbances. One of these methods is *Predictive Active Disturbance Rejecting Control* [19]. The concept of active disturbance rejecting control enables engineers to design systems that can accommodate unknown internal dynamics and disturbances. In active disturbance rejecting control, an extended state observer is used to estimate the system states and the unknown internal dynamics and disturbances. The robustness of the system as a whole is influenced by the tuning of the extended state observer gain. In [19] an improvement of active disturbance rejecting control is introduced to counter this weakness, called predictive active disturbance rejecting control. The improvement consists of a predictor for the system states, based on the input.

Predictive active disturbance rejecting control can also be used for the discrete-time Kuramoto model, but with some changes when compared to [19]. The first change is small: this thesis assumes absence of unknown internal dynamics in the system. The second is that the output of the system is the known system state, so a state observer is not required. The predictor takes as input the known current state and the known calculated input, and provides a predicted output for the next step of an undisturbed system. This is a one-step ahead predictor [20]. The difference between the predicted state and the actual state can then be used to filter the disturbance in the next step-update, without knowing or directly measuring the disturbance. If the disturbance is countered in a single step, it is known as deadbeat control [18], [20]. Deadbeat control completely counters the disturbances and leaves the Kuramoto model with all its stability, aligned and balanced properties intact. However, the nonlinear properties of the Kuramoto model may still cause the system to behave different from a completely undisturbed system.

4-1 One step ahead prediction and deadbeat control without average state as reference

The Kuramoto system with matched input disturbance can be seen in Figure 4-1. The controller is indicated by the dashed blue line. The predictor for undisturbed behavior is represented by $\hat{P}$. The undisturbed predicted states are calculated as the agents update their actual states. The filter variable for deadbeat control is represented by $\hat{d}_i(h) = \theta_i(h) - \hat{\theta}_i(h)$ and the (unmodified) Kuramoto algorithm by the block KA.

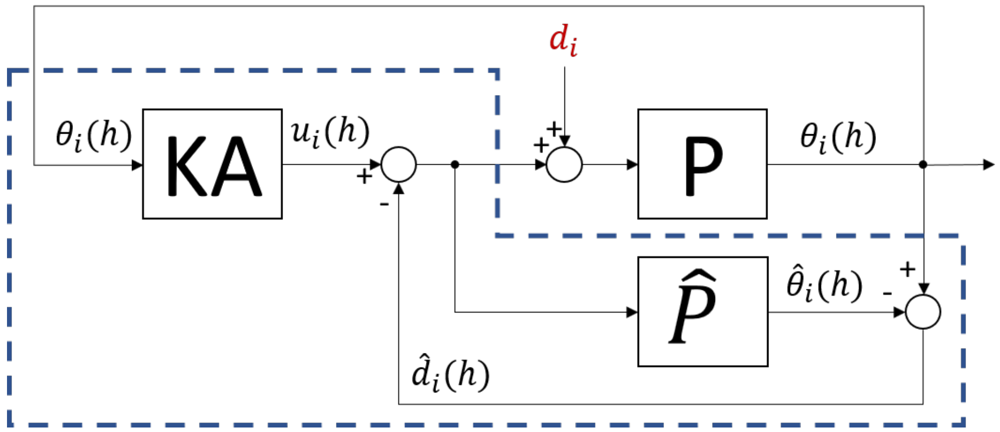


Figure 4-1: One step ahead prediction and deadbeat control

The corresponding step update is:

$$\theta_i(h+1) = \theta_i(h) - K\tau\rho(h)\sin(\psi(h) - \theta_i(h)) - \hat{d}_i(h) + d_i. \tag{4-1}$$

The blocks and variables from Figure 4-1 are given in Tables 4-1 and 4-2, the initial conditions are given in equation 4-2. When initializing at $h = 0$, the system knows all $\theta_i(0)$ and $\hat{\theta}_i(0)$. The controller can then calculate $\rho(0)$, $\psi(0)$ and all inputs $u_i(0)$. Since $\hat{d}_i(0) = 0$, this variable has no effect at $h = 0$, and since $d_i \neq 0$, the disturbed agent(s) move to a new orientation based on $\theta_i(0)$, $u_i(0)$ and d_i. While the real agents (in block P) and the modelled agents (in block $\hat{P}$) update their orientations, the clock counter h is increased one step. The predicted orientation ($\hat{\theta}_i(h)$) is subtracted from the actual orientation ($\theta_i(h)$) and this difference $\hat{d}_i(h)$ is the estimated disturbance. The controller uses all $\theta_i(1)$ to calculate $u_i(1)$ and also communicates the now nonzero $\hat{d}_i(1)$ to the actual and modelled agents. This process repeats until the system is stopped.

Table 4-1: Blocks from Figure 4-1

Block	Description
KA	Here the Kuramoto Algorithm (2-5) is used to calculate $u_i(h)$.
P	Here the agents carry out their step-update.
$\hat{P}$	Predictor: here the controller calculates the predicted, undisturbed output, based on the actual state, calculated input and estimated disturbance.

Table 4-2: Variables from Figure 4-1

Variable	Description	Equation
$\theta_i(h)$	agent orientation	$\theta_i(h+1) = \theta_i(h) + u_i(h) - \hat{d}_i(h) + d_i$
$\hat{\theta}_i(h)$	predicted output	$\hat{\theta}_i(h+1) = \theta_i(h) + u_i(h) - \hat{d}_i(h)$
$u_i(h)$	calculated input	$u_i(h) = -K\tau\rho(h)\sin(\psi(h) - \theta_i(h))$
$\rho(h)$	phasor magnitude	$\rho(h) = \lvert\lvert\mathcal{R}(\theta_i(h))\rvert\rvert$
$\psi(h)$	phasor orientation	$\psi(h) \equiv \angle\mathcal{R}(\theta_i(h))$
$\mathcal{R}(\theta_i(h))$	phasor	$\mathcal{R}(\theta_i(h)) \equiv \frac{1}{N}\sum_{i=1}^{N}\begin{bmatrix}\cos(\theta_i(h))\\ \sin(\theta_i(h))\end{bmatrix}$
$\hat{d}_i(h)$	estimated disturbace	$\hat{d}_i(h) = \theta_i(h) - \hat{\theta}_i(h)$
d_i	disturbance	unknown constant

$$\begin{aligned}\hat{\theta}_i(0) &= \theta_i(0),\\ \hat{d}_i(0) &= 0.\end{aligned} \tag{4-2}$$

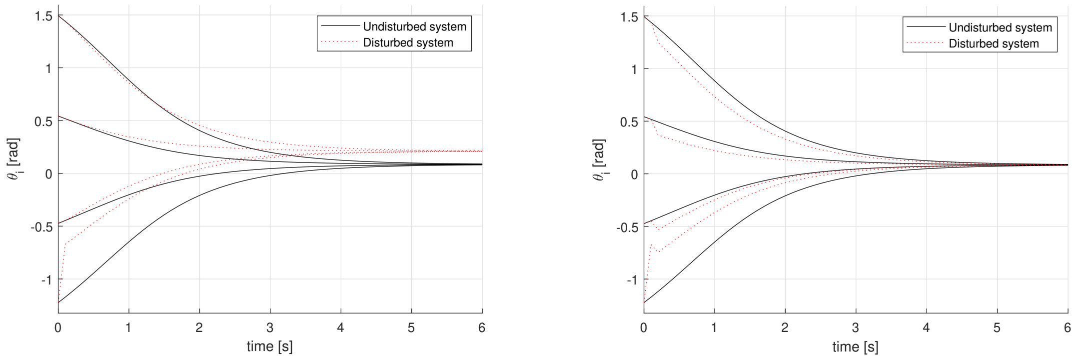
Since the disturbance is assumed to be constant, it can be countered once it has been identified. For this, a *disturbance observer* [21], [22] has been implemented. This disturbance observer (represented in Figure 4-1 by a circle) estimates the disturbance by subtracting the predicted states from the actual states. The initial disturbance estimate is calculated after the first step-update, at $h = 1$. The initial conditions in equation 4-2 show that $\hat{\theta}_i(0) = \theta_i(0)$ and $\hat{d}_i(0) = 0$. Then for $h \neq 0$:

$$\begin{aligned}\hat{d}_i(h) &= \theta_i(h) - \hat{\theta}_i(h)\\ &= (\theta_i(h-1) + u_i(h-1) + d_i) - (\theta_i(h-1) + u_i(h-1))\\ &= d_i\end{aligned} \tag{4-3}$$

Now the step-update in equation 4-1 can be reformulated for $h \neq 0$:

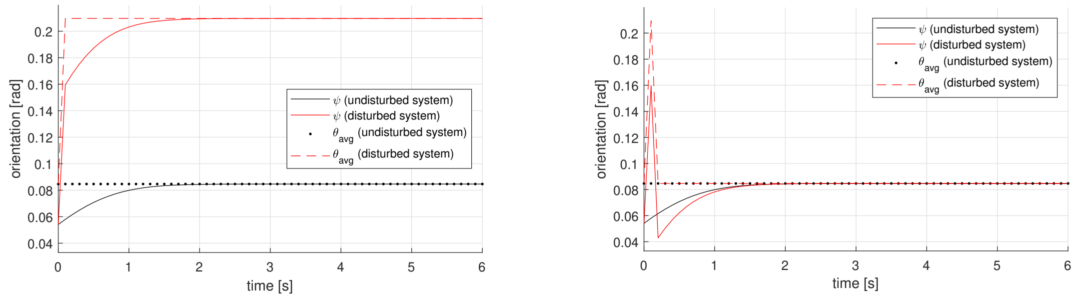
$$\begin{aligned}\theta_i(h+1) &= \theta_i(h) - K\tau\rho(h)\sin(\psi(h) - \theta_i(h)) - \hat{d}_i(h) + d_i,\\ &= \theta_i(h) - K\tau\rho(h)\sin(\psi(h) - \theta_i(h)) - d_i + d_i,\\ &= \theta_i(h) - K\tau\rho(h)\sin(\psi(h) - \theta_i(h)).\end{aligned} \tag{4-4}$$

This means that the disturbance in the disturbed agent itself is countered from $h = 2$ on, and the system behaves as if undisturbed, confirming research goal 2-15. Next to that, equation 4-4 is equal to the original discrete-time Kuramoto model as researched in [12], and therefore all proof and conjectures of stability from [12] are valid. The results of a simulation of the system from equation 4-1 with a single disturbed agent can be seen in Figures 4-2a, 4-3a and 4-4a. Figure 4-2a shows that the disturbance in a single agent also affects the trajectories of other agents, as stated as a possibility by [16]. The effect of $\theta_i(1) \neq \hat{\theta}_i(1)$ for the disturbed agent is twofold: the undisturbed agents deviate from the path in the undisturbed system after $h = 1$, and the average orientation of all agents is changed, which results in the system converging to an orientation that is different from the undisturbed system. As a consequence, the second research goal for the aligned set (equation 2-16) is not fulfilled. This is illustrated in Figures 4-2a, 4-3a and 4-4a, where although the system still converges to an aligned state, this state is different from the one in the undisturbed system.



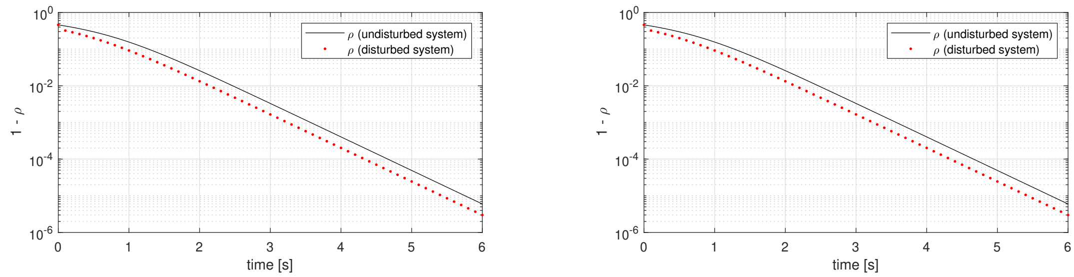
(a) Without $\theta_{avg}(0)$ reference **(b)** With $\theta_{avg}(0)$ reference

Figure 4-2: Agent orientations



(a) Without $\theta_{avg}(0)$ reference **(b)** With $\theta_{avg}(0)$ reference

Figure 4-3: Phasor and average orientation



(a) Without $\theta_{avg}(0)$ reference **(b)** With $\theta_{avg}(0)$ reference

Figure 4-4: Magnitude of the phasor

4-2 One step ahead prediction and deadbeat control with average state reference

Since section 4-1 shows that a disturbance in one agent affects all other agents via the phasor orientation in the control algorithm, the effects of a disturbance must be countered both in the disturbed agent itself and in the other agents. The known fact that the average orientation is constant in the undisturbed Kuramoto model can be used for this: the system from equation 4-1 can be augmented with the average orientation of the real agents, $\theta_{avg}(h)$ and an undisturbed reference for this state, $\theta_{avg}(0)$ and an additional filtering variable, $\tilde{\theta}_{avg}(h)$. The predictor $\hat{P}$ will not only generate undisturbed one step ahead predictions [18] as reference for each individual agent, but also the undisturbed average orientation. In the new disturbance estimator, L, the effect of the matched input disturbance will now be estimated by calculating two differences: the difference between the predicted and actual orientation for every agent, and the difference between the actual average orientation and the initial average orientation. This way, constant disturbances on the agents and the effect on the average orientation are countered by the filter variables generated by the disturbance estimator. The designed system then becomes as shown in Figure 4-5, where the controller is represented by the dashed blue line.

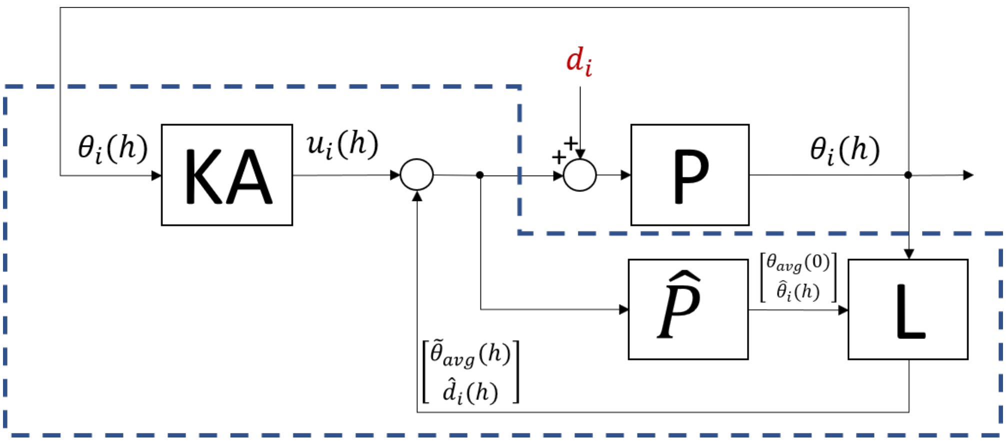


Figure 4-5: One step ahead predictor and deadbeat control with average state reference

The blocks and variables used in the system from Figure 4-5 are given in Tables 4-3 and 4-4, and the initial conditions in equation 4-2. When initializing at $h = 0$, the system knows all $\theta_i(0)$ and $\hat{\theta}_i(0)$. $\theta_{avg}(0)$ is calculated and stored, so that the internal model can use it at every time step. The controller can calculate $\rho(0)$, $\psi(0)$ and all inputs $u_i(0)$. Since $\hat{d}_i(0) = 0$ and $\tilde{\theta}_{avg}(0) = 0$, these variables have no effect at $h = 0$, and since $d_i \neq 0$, the disturbed agent(s) move to a new orientation based on $\theta_i(0)$, $u_i(0)$ and d_i. While the real agents (in block P) and the modelled agents (in block $\hat{P}$) update their orientations, the clock counter h is increased one step. After the step-updates the disturbance estimator L calculates $\theta_{avg}(1)$, $\tilde{\theta}_{avg}(1)$ and all $\hat{d}_i(1)$. $\tilde{\theta}_{avg}(1)$ and all $\hat{d}_i(1)$ are then communicated to the controller. The controller uses the control algorithm to calculate $u_i(1)$ and also communicates the now nonzero $\hat{d}_i(1)$ and $\tilde{\theta}_{avg}(1)$ to the real and modelled agents. This process repeats until the system is stopped.

The initial conditions are:

Table 4-3: Blocks from Figure 4-5

Block	Description
KA	Here the Kuramoto Algorithm (2-5) is used to calculate $u_i(h)$
P	Here the agents carry out their step-update: $\theta_i(h+1) = \theta_i(h) + u_i(h) + d_i$
$\hat{P}$	Predictor: here, the controller calculates the predicted output, based on the actual state, calculated input and filter variables
L	Disturbance estimator: here $\hat{d}_i(h)$ and $\tilde{\theta}_{avg}(h)$ are calculated

Table 4-4: Variables from Figure 4-5

Variable	Description	Equation
$\theta_i(h)$	agent orientation	$\theta_i(h+1) = \theta_i(h) + u_i(h) - \hat{d}_i(h) - \tilde{\theta}_{avg}(h) + d_i$
$\hat{\theta}_i(h)$	predicted output	$\hat{\theta}_i(h+1) = \theta_i(h) + u_i(h) - \hat{d}_i(h) - \tilde{\theta}_{avg}(h)$
$u_i(h)$	calculated input	$u_i(h) = -K\tau\rho(h)\sin(\psi(h) - \theta_i(h))$
$\rho(h)$	phasor magnitude	$\rho(h) = \|\|\mathcal{R}(\theta_i(h))\|\|$
$\psi(h)$	phasor orientation	$\psi(h) \equiv \angle\mathcal{R}(\theta_i(h))$
$\mathcal{R}(\theta_i(h))$	phasor	$\mathcal{R}(\theta_i(h)) \equiv \frac{1}{N}\sum_{i=1}^{N}\begin{bmatrix}\cos(\theta_i(h))\\ \sin(\theta_i(h))\end{bmatrix}$
$\hat{d}_i(h)$	filter variable	$\hat{d}_i(h) = \theta_i(h) - \hat{\theta}_i(h)$
$\tilde{\theta}_{avg}(h)$	filter variable	$\tilde{\theta}_{avg}(h) = \theta_{avg}(h) - \theta_{avg}(0)$
$\theta_{avg}(h)$	avg orientation	$\theta_{avg}(h) = \frac{1}{N}\sum_{i=1}^{N}\theta_i(h)$
$\theta_{avg}(0)$	initial avg orientation	$\theta_{avg}(0) = \frac{1}{N}\sum_{i=1}^{N}\theta_i(0)$
d_i	disturbance	unknown constant

$$
\begin{aligned}
\hat{\theta}_i(0) &= \theta_i(0), \\
\hat{d}_i(0) &= 0, \\
\tilde{\theta}_{avg}(0) &= 0.
\end{aligned}
\tag{4-5}
$$

The step-update for agents in the system from Figure 4-5 is:

$$
\theta_i(h+1) = \theta_i(h) - K\tau\rho(h)\sin(\psi(h) - \theta_i(h)) - \hat{d}_i(h) - \tilde{\theta}_{avg}(h) + d_i. \tag{4-6}
$$

Because the disturbance is unknown, the new filter variable $\tilde{\theta}_{avg}(h)$ is 0 when the system from equation 4-6 is initialized. The evolution of the new filter variable is:

$$\begin{aligned}\tilde{\theta}_{avg}(h+1) &= \theta_{avg}(h+1) - \theta_{avg}(0),\\ &= \frac{1}{N}\sum_i(\theta_i(h+1)) - \theta_{avg}(0),\\ &= \frac{1}{N}\sum_i\theta_i(h) + \frac{1}{N}\sum_i u_i(h) - \frac{1}{N}\sum_i\hat{d}_i(h) - \frac{1}{N}\sum_i\tilde{\theta}_{avg}(h) + \frac{1}{N}\sum_i d_i - \theta_{avg}(0).\end{aligned} \tag{4-7}$$

The input $u_i(h)$ in equation 4-6 is equal to input in the original discrete-time Kuramoto model (equation 2-4), $\frac{1}{N}\sum\limits_i u_i(h) = 0$, and can be left out from equation 4-7. Using $\theta_{avg}(h) = \frac{1}{N}\sum\limits_i \theta_i(h)$ and $\frac{1}{N}\sum\limits_i \tilde{\theta}_{avg}(h) = \tilde{\theta}_{avg}(h)$, equation 4-7 becomes:

$$\begin{aligned}\tilde{\theta}_{avg}(h+1) &= \theta_{avg}(h) - \frac{1}{N}\sum_i\hat{d}_i(h) + \frac{1}{N}\sum_i d_i - \tilde{\theta}_{avg}(h) - \theta_{avg}(0),\\ &= \theta_{avg}(h) - \theta_{avg}(0) - \tilde{\theta}_{avg}(h) - \frac{1}{N}\sum_i\hat{d}_i(h) + \frac{1}{N}\sum_i d_i,\\ &= \tilde{\theta}_{avg}(h) - \tilde{\theta}_{avg}(h) + \frac{1}{N}\sum_i d_i - \frac{1}{N}\sum_i\hat{d}_i(h),\\ &= \frac{1}{N}\sum_i d_i - \frac{1}{N}\sum_i\hat{d}_i(h),\\ &= \frac{1}{N}\sum_i(d_i - \hat{d}_i(h)).\end{aligned} \tag{4-8}$$

Equation 4-8 shows that the evolution of $\tilde{\theta}_{avg}(h)$ depends on $\hat{d}_i(h)$. The evolution of $\hat{d}_i(h)$ in equation 4-6 can be derived from Table 4-4:

$$\begin{aligned}\hat{d}_i(h+1) &= \theta_i(h+1) - \hat{\theta}_i(h+1),\\ &= d_i.\end{aligned} \tag{4-9}$$

Equation 4-9 shows that if $h \geq 1$, then $\hat{d}_i(h) = d_i$. The initial conditions state that $\tilde{\theta}_{avg}(0) = 0$. Using $h = 0$ and $\hat{d}_i(0) = 0$ in equation 4-8 gives:

$$\tilde{\theta}_{avg}(1) = \frac{1}{N}\sum_i(d_i - \hat{d}_i(0)) = \frac{1}{N}\sum_i d_i. \tag{4-10}$$

For $h \geq 1$:

$$\tilde{\theta}_{avg}(h+1) = \frac{1}{N}\sum_i(d_i - \hat{d}_i(h)) = 0. \tag{4-11}$$

Figure 4-3b illustrates the proof from equation 4-11: for $h \geq 2$ the average orientation of the disturbed system is equal to the average orientation of the undisturbed system. Using $\tilde{\theta}_{avg}(h \geq 2) = 0$ and $\hat{d}_i(h \geq 1) = d_i$ means that *from* $h = 2$ *on*, the step-update system from equation 4-6 is:

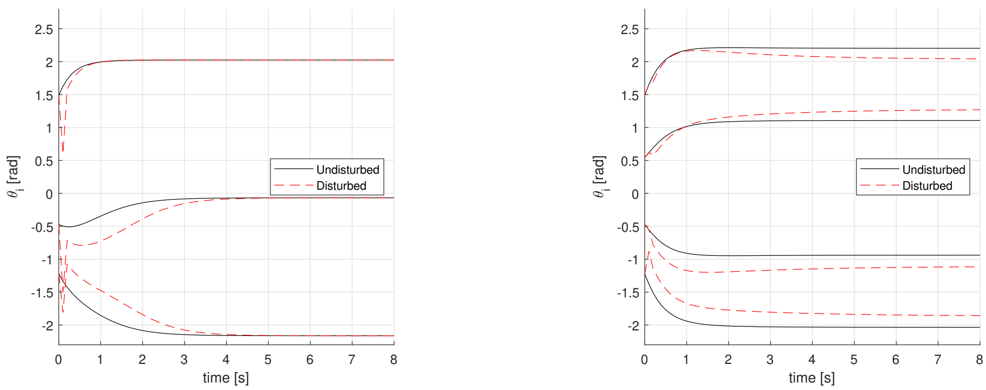
$$\begin{aligned}\theta_i(h+1) &= \theta_i(h) - K\tau\rho(h)\sin(\psi(h)-\theta_i(h)) - \hat{d}_i(h) - \tilde{\theta}_{avg}(h) + d_i, \\ &= \theta_i(h) - K\tau\rho(h)\sin(\psi(h)-\theta_i(h)).\end{aligned} \tag{4-12}$$

Equation 4-12 proofs that from $h = 2$ on behaves the same as the original discrete-time Kuramoto model as researched in [12]. This means that again all proof and conjectures of stability from [12] are valid: for $-2 < K\tau < 0$ asymptotic stability of the aligned set (the first refined research goal for the aligned set, in equation 2-15) is proven and asymptotic stability of the balanced set (the first refined research goal for the balanced set, in equation 2-19) is conjectured [12].

The second refined research goal for the aligned set (in equation 2-16) is to find out whether the agents converge to the same aligned orientations as the undisturbed system. This research goal is also achieved since equation 4-11 proves that for $h \geq 0$ the average orientation of the disturbed system is equal to that of the undisturbed system, and the agents converge to the average orientation if $-2 < K\tau < 0$.

4-3 One step ahead prediction and deadbeat control towards a balanced set

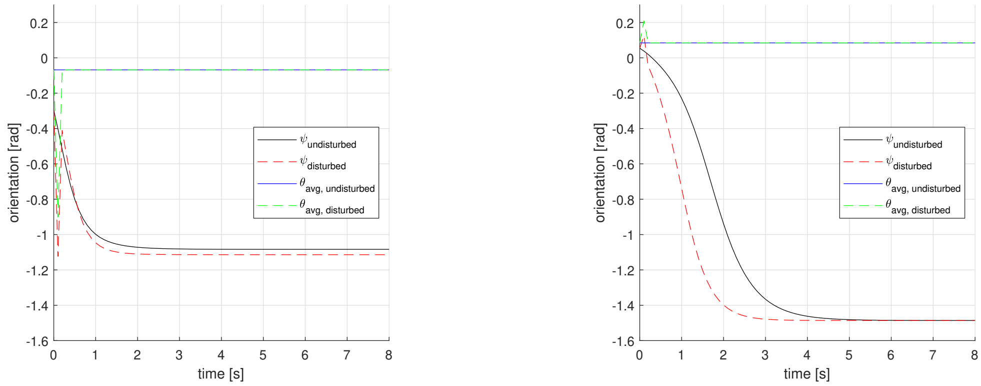
The previous sections shows the analysis of how a controller with one step ahead prediction and deadbeat control can counter a matched input disturbance and direct a system towards an aligned or balanced state, while also rejecting effects on the average orientation. Figure 4-2b also illustrates that although the agents move towards alignment with the initial average orientation, the path they follow is different. The results for movement towards a balanced set are worse: in section 2-3-2 and [14] it is explained that for $N > 3$ it is still unknown how the balanced orientations can be predicted. The results of a simulation of the system from equation 4-6 for $N = 3$ and $N = 4$, both with a single disturbed agent, can be seen in Figures 4-6 to 4-8.



(a) N=3 **(b)** N=4

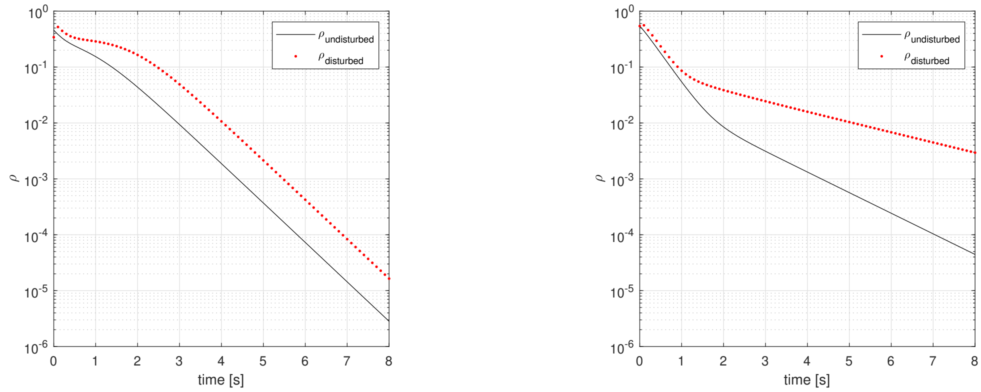
Figure 4-6: Agent orientations

Figure 4-8 and 4-7 show that the systems converge to balanced sets with the average orientation identical to that of their undisturbed counterparts. In Section 4-2, mathematical



(a) N=3 **(b)** N=4

Figure 4-7: Phasor and average orientations



(a) N=3 **(b)** N=4

Figure 4-8: Magnitude of the phasors

proof is given that this will always be true for the system from equation 4-6 with constant disturbances.

Figure 4-6a shows that for $N = 3$, the controller will direct the agents in the disturbed system to the same balanced set as their counterparts in an undisturbed system. Figure 4-6b shows that for $N = 4$, the controller will direct the agents towards a balanced set that is different from the undisturbed system. This is in line with Section 2-3-2: a balanced set for $N = 2$ or $N = 3$ agents with known and constant average orientation has only one possible solution. A balanced set for $N \geq 4$ agents with known and constant average orientation has an infinite amount of possible solutions. Even if it were possible to predict the balanced orientations, the controller in equation 4-6 will only achieve research goal 2-20 for $N = 2$ and $N = 3$ agents.

4-4 Conclusions and recommendations for one step ahead prediction and deadbeat control

In [12] asymptotic convergence of the agents in a Kuramoto model towards a balanced set has been proven, and convergence towards a balanced set has been conjectured. In [18], [20] and [23], it is explained how discrete-time predictor-feedback can reject disturbances. This thesis has added to that research the conclusion that, with a predictable consensus point and under constant disturbance, the system in equation 4-6 with $-2 < K\tau < 0$ will reach an aligned state identical to the undisturbed consensus point for any N. The agents however follow a trajectory that is different from the undisturbed trajectory. Using $0 < K\tau < 2$, the controller will direct the agents to a balanced set, but only systems with $N = 2$ or $N = 3$ agents will reach the same set as their undisturbed counterparts. It is still unknown how to predict final orientations for systems with $N \geq 4$ agents, as explained in [14] and Section 2-3-2.

It is recommended to use this system in applications where the agents need to be aligned and the trajectory is less important. Another possible application is when agents need to be in a balanced set while the individual orientations have no specific requirements.

Chapter 5

Autonomous reference trajectories and Proportional-Integral control

Since the controller in Chapter 4 does not meet the research goal of directing the agents to the same balanced orientations as their counterparts in an undisturbed system when $N > 3$, another controller must be designed. The system with one step ahead prediction and dead-beat control in equation 4-6 can direct agents in a system with matched input disturbances to the same final orientation for all aligned sets and for balanced sets if $N = 2$ or $N = 3$. The agents in those disturbed systems follow a different trajectory than their counterparts in an undisturbed system, and for balanced sets with $N \geq 4$, this results in balanced orientations that are different from their counterparts in an undisturbed system. The only feasible option to direct a system with $N \geq 4$ and matched input disturbance to the same balanced orientations as their undisturbed counterparts is to use the trajectories of the agents in the undisturbed system as reference. This means that attenuating disturbances in a Kuramoto model must be handled as a *tracking problem*.

5-1 The reference trajectory

The goal of the tracking problem is to let the output ($\theta_i(h)$ in the Kuramoto model) asymptotically track a reference trajectory. Since the Kuramoto model is deterministic (see Chapter 2), the known initial states can be used to generate a reference trajectory before the agents in the disturbed system begin their trajectories. At every time step h, the controller can use the individual orientations of all agents as references to determine the error in every agents actual orientation.

The system to generate the reference trajectories can be seen in Figure 5-1. The blocks and variables from Figure 5-1 are explained in Tables 5-1 and 5-2.

The step update for the system in Figure 5-1 is:

$$\hat{\theta}_i(h+1) = \hat{\theta}_i(h) - K\tau\hat{\rho}(h)\sin(\hat{\psi}(h) - \hat{\theta}_i(h)), \tag{5-1}$$

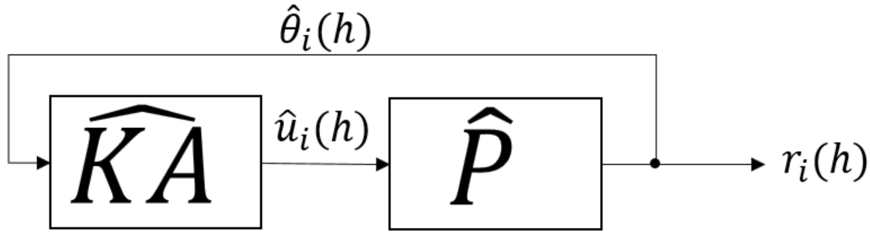


Figure 5-1: The system to generate reference trajectories

Table 5-1: Blocks from Figure 5-1

Block	**Description**
$\hat{KA}$	Here the Kuramoto algorithm (5-2) is used to calculate $\hat{u}_i(h)$
$\hat{P}$	Plant: here the agents carry out their step-update.

Table 5-2: Variables from Figure 5-1

Variable	**Description**	**Equation**
$\hat{\theta}_i(h)$	agent orientation	$\hat{\theta}_i(h+1) = \hat{\theta}_i(h) + \hat{u}_i(h)$
$\hat{u}_i(h)$	calculated input	$\hat{u}_i(h) = -K\tau\hat{\rho}(h)\sin(\hat{\psi}(h) - \hat{\theta}_i(h))$
$\hat{\rho}(h)$	phasor magnitude	$\hat{\rho}(h) = \|\|\hat{\mathcal{R}}(\hat{\theta}_i(h))\|\|$
$\hat{\psi}(h)$	phasor orientation	$\hat{\psi}(h) = \angle\hat{\mathcal{R}}(\hat{\theta}_i(h))$
$\hat{\mathcal{R}}(\hat{\theta}_i(h))$	phasor	$\hat{\mathcal{R}}(\hat{\theta}_i(h)) \equiv \frac{1}{N}\sum_{i=1}^{N}\begin{bmatrix}\cos(\hat{\theta}_i(h))\\ \sin(\hat{\theta}_i(h))\end{bmatrix}$

The unmodified Kuramoto algorithm used in equation 5-1 is equal to equation 5-2, but with hats, to indicate that these are not the actual agents:

$$\hat{u}_i(h) = -K\tau\hat{\rho}(h)\sin(\hat{\psi}(h) - \hat{\theta}_i(h)). \tag{5-2}$$

If the initial orientation of the simulated agents in the system in Figure 5-1 are equal to the initial orientation of the actual agents ($\hat{\theta}_i(0) = \theta_i(0)$), then the reference trajectories $\hat{\theta}_i(h)$ are exactly those of the undisturbed system. The stability of the reference system (5-1) is proven for $-2 < K\tau < 0$ and conjectured for $0 < K\tau < 2$ [12].

5-2 The tracking problem

A common strategy for a tracking problem is a negative feedback loop [17]. In a negative feedback loop, the actual state is subtracted from the reference. Next, this difference is used as input for the controller. With the reference trajectories $\hat{\theta}_i(h)$ from the system in Figure 5-1 known, designing a simple controller that directs the agents towards the desired trajectories is straightforward. Since the reference system is asymptotically stable for $-2 < K\tau < 0$ and $0 < K\tau < 2$, the reference agents settle to their aligned or balanced orientations, and a proportional feedback controller can asymptotically direct the actual agents to the same orientation. Since a matched input disturbance is a constant (nonzero) input, this can be countered by the integral part of a Proportional-Integral (PI) controller [18]. A well tuned PI-controller with a nonzero integral gain (K_I) will direct the actual agents in a disturbed system to their reference trajectories and as $h \to \infty$ all agents go to the same orientation as their reference counterparts. The purpose of this thesis however is to attenuate the effects of disturbances in agents that behave according to the Kuramoto model themselves. This requires a different system design.

5-3 Designing the Proportional-Integral controller

Designing the optimal PI-controller for a Kuramoto model is a difficult task. An often used technique to design a general, stable controller for closed loop systems (which the Kuramoto model is) is the Youla-parametrization [20]. A Youla-parametrization requires the z-transform of the system that is to be controlled [18], [20]. The z-transform of the discrete-time Kuramoto model would be very difficult to calculate since it is a nonlinear and multivariable system whose behavior not only depends on all states $\theta_i(h)$, but also on the the value of $k\tau$. Even if the Youla-parametrization works, the resulting class of controllers may be designed without the use of the Kuramoto algorithm or a modified version of it. The controller will therefore be designed with PI-gain and the Kuramoto algorithm in it. Stability of the system will be proven by showing that the designed system is exactly equal to the system from equation 2-2 if $d_i = 0$ and by showing that the system is stable near aligned or balanced steady states.

5-4 The system

The Kuramoto model with previously generated reference trajectories and PI-control can be seen in Figure 5-2 and the step-update in equation 5-3. The blocks and variables are further explained in Tables 5-3 and 5-4. The controller in Figure 5-2 is represented by the dashed blue line. The unmodified Kuramoto algorithm (equal to equation 5-2) generates the normal input for the agents while the PI-input algorithm counters the effects of the disturbances. The reference trajectories ($\hat{\theta}_i(h)$), the PI-input algorithm ($\tilde{u}_i(h)$) and its outcome, the Kuramoto algorithm and its outcome ($u_i(h)$) are all part of the controller. The orientations of the actual agents ($\theta_i(h)$) and the unknown disturbance (d_i) are exogenous signals.

The step-update for the system in Figure 5-2 is

$$\theta_i(h+1) = \theta_i(h) + u_i(h) - \tilde{u}_i(h) + d_i. \tag{5-3}$$

5-5 Stability of the Proportional-Integral controlled system

Stabilizeability of the system with PI-control will be shown by proving that in absence of disturbance the system from equation 5-3 is equal to the original discrete-time Kuramoto model in equation 2-2 and by proving that the system with matched input disturbance can reach a steady state at the same orientations as the undisturbed system.

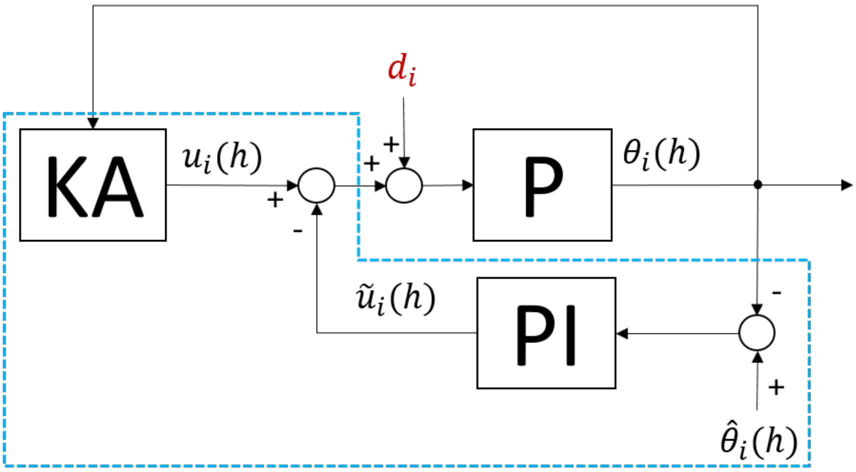


Figure 5-2: The system with autonomous reference trajectories and PI-control. The controller is indicated by the dashed blue line

Table 5-3: Blocks from Figure 5-2

Block	Description
KA	Here the Kuramoto algorithm (2-5) is used to calculate $u_i(h)$.
P	Plant: here the agents carry out their step-update.
PI	PI-controller: here the input to counter the disturbance is calculated.

Table 5-4: Variables from Figure 5-2

Variable	Description	Equation
$\theta_i(h)$	agent orientation	$\theta_i(h+1) = \theta_i(h) + u_i(h) - \tilde{u}_i(h) + d_i(h)$
$u_i(h)$	control algorithm	$u_i(h) = -K\tau\rho(h)\sin(\psi(h) - \theta_i(h))$
$\rho(h)$	phasor magnitude	$\rho(h) = \|\|\mathcal{R}(\theta_i(h))\|\|$
$\psi(h)$	phasor orientation	$\psi(h) \equiv \angle\mathcal{R}(\theta_i(h))$
$\mathcal{R}(\theta_i(h))$	phasor	$\mathcal{R}(\theta_i(h)) \equiv \frac{1}{N}\sum_{i=1}^{N}\begin{bmatrix}\cos(\theta_i(h))\\ \sin(\theta_i(h))\end{bmatrix}$
$\tilde{u}_i(h)$	input from PI-algorithm	$\tilde{u}_i(h) = K_P(\hat{\theta}_i(h) - \theta_i(h)) + K_I\sum_{j=0}^{h}(\hat{\theta}_i(j) - \theta_i(j))$
K_P	proportional gain	constant
K_I	integral gain	constant
$\hat{\theta}i(h)$	reference trajectory	see section 5-1
d_i	disturbance	unknown constant

5-5-1 Proof of equality in absence of disturbances

In absence of disturbances $d_i = 0$ and with identical initial states ($\theta_i(0) = \hat{\theta}_i(0)$), the system from equation 5-3 is equal to the system from equation 2-2:

$$\begin{aligned}\theta_i(h+1) &= \theta_i(h) + u_i(h) - \tilde{u}_i(h) + d_i, \\ &= \theta_i(h) + u_i(h) - \tilde{u}_i(h), \\ &= \theta_i(h) + u_i(h) - K_P(\hat{\theta}_i(h) - \theta_i(h)) - K_I \sum_{j=0}^{h} (\hat{\theta}_i(j) - \theta_i(j)), \\ &= \theta_i(h) - K\tau\rho(h)\sin(\psi(h) - \theta_i(h)) - K_P(\hat{\theta}_i(h) - \theta_i(h)) - K_I \sum_{j=0}^{h} (\hat{\theta}_i(j) - \theta_i(j)).\end{aligned} \tag{5-4}$$

Using the initial states $\theta_i(0) = \hat{\theta}_i(0)$, the parts in equation 5-4 with proportional (K_P) and integral (K_I) gain are set to zero and equation 5-4 becomes exactly the same as equation 2-2. This means that undisturbed, asymptotic stability of of the aligned set is proven and asymptotic stability of the balanced set is conjectured.

5-5-2 Proof of existence of steady state with matched input disturbance

The Kuramoto model in equation 2-2 can asymptotically reach steady states. This means that $\theta_i(h+1) = \theta_i(h)$, and therefore $-K\tau\rho(h)\sin(\psi(h) - \theta_i(h)) = 0$. In the aligned set this is because $\psi(h) - \theta_i(h) = 0$ and in the balanced set because $\rho(h) = 0$. In the system from equation 5-3 the difference between two consecutive time-steps $(\theta_i(h+1) - \theta_i(h))$ is given by:

$$-K\tau\rho(h)\sin(\psi(h) - \theta_i(h)) - K_P(\hat{\theta}_i(h) - \theta_i(h)) - K_I \sum_{j=0}^{h} (\hat{\theta}_i(j) - \theta_i(j)) + d_i. \tag{5-5}$$

At the aligned or balanced states, the output of the Kuramoto algorithm $(-K\tau\rho(h)\sin(\psi(h) - \theta_i(h)))$ equals zero. This leaves to proof that

$$d_i - K_P(\hat{\theta}_i(h) - \theta_i(h)) - K_I \sum_{j=0}^{h} (\hat{\theta}_i(j) - \theta_i(j)) = 0. \tag{5-6}$$

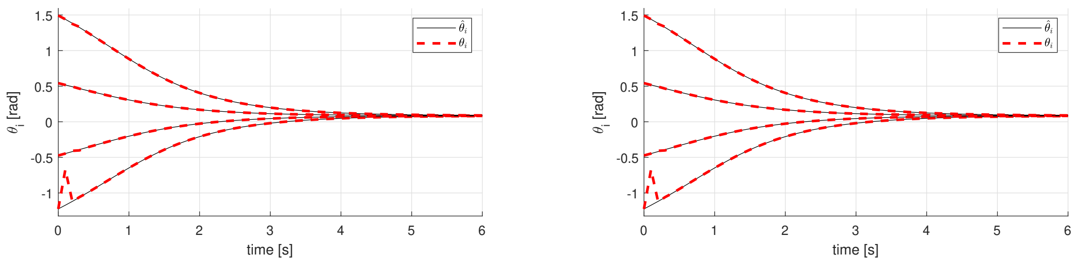
In the introduction of this section it was mentioned that the investigated steady state is at the same orientations as the undisturbed system, so $theta_i(h) = \hat{\theta}_i(h)$ and therefore the part with proportional gain drops out, leaving to proof that the following is possible:

$$d_i = K_I \sum_{j=0}^{h} (\hat{\theta}_i(j) - \theta_i(j)). \tag{5-7}$$

Since d_i can have any constant value, it is possible that it is equal to the integral of the difference between the reference and actual orientations over time. This zero input will persist in all following time-steps, proving the possibility of stability of the aligned and balanced sets for the system from equation 5-3.

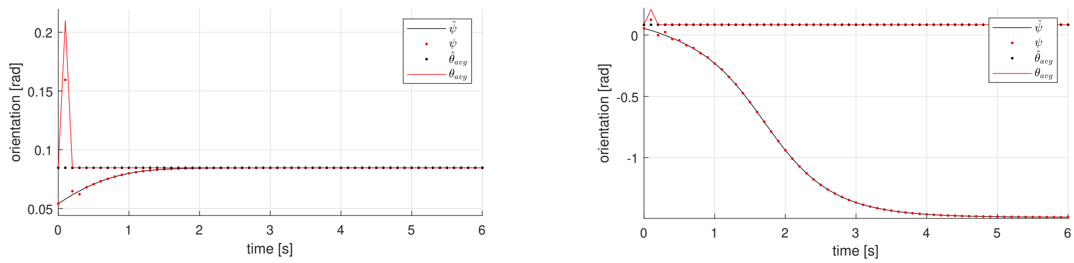
5-6 Simulations

To illustrate the stability and disturbance attenuation properties of the system from equation 5-3, the results of simulations can be seen in Figures 5-3 to 5-5.



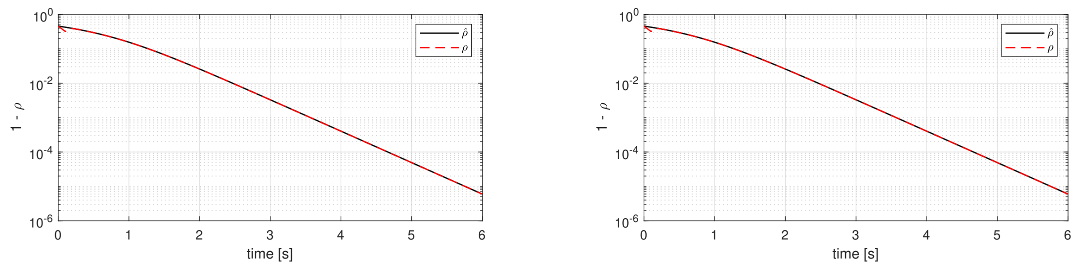
(a) Towards aligned set **(b)** Towards balanced set

Figure 5-3: Agent orientations



(a) Towards aligned set **(b)** Towards balanced set

Figure 5-4: Average and phasor orientations



(a) Towards aligned set **(b)** Towards balanced set

Figure 5-5: Magnitude of the phasor

The simulations show that the agent with the matched input disturbance initially deviates strongly from its reference trajectory, but is quickly returned to it. Because of the networked nature of the Kuramoto model, the undisturbed agents are affected through the step-update, but these agents are also quickly guided back to their reference trajectories.

5-7 Conclusions for autonomous reference trajectories and PI-control

When properly tuned, the strategy of combining *a priori* generated autonomous reference trajectories with a PI-controller will direct the discrete-time Kuramoto model under constant matched input disturbance towards an aligned or balanced state that is equal to its undisturbed counterpart. Furthermore, the agents trajectories in the disturbed system are almost identical to the trajectories in the undisturbed system. However, due to the nature of the PI-controller, it must be checked for stability. Wrong choices in parameters K_P or K_I can prevent the system from converging to consensus. Further research could investigate how the interactions between the product $K\tau$, the PI-gains K_P, K_I and the disturbance influence the stability of the system.

Chapter 6

Comparing controllers, conclusions and recommendations

6-1 Comparing controllers

The purpose of this thesis was to research the behavior of systems of identical, all-to-all coupled oscillators in the discrete-time Kuramoto model under constant, matched input disturbances. If the disturbance was found out to change the behavior of the agents, the goal of the research was to design and evaluate controllers that counter the effects of the disturbance.

6-1-1 Design choices

The academic field of control gives many possible choices for designing controllers that handle disturbances. Several options have been researched and reported in this thesis, and others have not. The options that have been researched and reported in this thesis are:

1. No additional control. This design choice is discussed in Section 2-4, where was shown that all agents are affected by a matched input disturbance in a single agent. The system can neither reach an aligned, nor a balanced state.

2. Error feedback with average initial orientation as reference. This design choice is researched and reported in Sections 3-1 - 3-3. In order to enable error feedback, the system must have a reference. This reference required a modification of the Kuramoto algorithm (2-5) that removed the agent coupling in the system and changed the effect of the product $K\tau$. Section 3-3 showed that using the initial average orientation as reference disabled the possibility to reach a balanced set, so this option was not researched further.

3. Error feedback with predicted final orientation as reference. This design choice is researched and reported in Sections 3-4 and 3-5. After modification of the Kuramoto

algorithm (2-5 into 3-14), choosing between moving towards an aligned or balanced set is no longer determined by the product $K\tau$, but by the reference orientations. Using reference orientaions also removes the all-to-all coupling characteristic of the Kuramoto model. As shown in Section 3-14, a disturbance in an agent no longer affects other agents. The undisturbed agents will reach their reference orientation, but agents with a matched input disturbance will not. As a result, the system will not reach an aligned or balanced state. In addition to this shortcoming, current understanding of the balanced set limits the possibility of predicting balanced orientations tot systems of $N = 2$ or $N = 3$ agents. For predicting balanced sets with $N \geq 4$ agents, further research is required.

4. One step ahead prediction and deadbeat control. After the previous two mentioned error feedback options, this design choice returns to using the unmodified Kuramoto algorithm (2-5) and combines it with prediction-error feedback. While current understanding of the balanced set limits the option to predict the final orientations for $N \geq 4$ agents, it is possible to predict the orientations of all agents in the next time step *as if the system were undisturbed.* Section 4-2 shows that including the initial average orientation as reference will enable directing all agents in a disturbed system to the same final orientation as their undisturbed counterparts. This is a significant improvement over the error feedback models in Chapter 3. Also, prediction-error feedback enables directing a system of any number of agents towards a balanced set. Section 4-3 explains that for $N = 2$ and $N = 3$ the balanced set will be identical to the undisturbed reference, but for $N \geq 4$ the balanced orientations will different from the undisturbed references.

5. Autonomous reference and Proportional-Integral control. This controller is designed to enable agents in a disturbed system to move to the same balanced orientations as their counterparts in an undisturbed system. Since the Kuramoto model is deterministic, reference trajectories can be generated when the initial states are known. At every time step the error for every agent can be calculated and countered by a PI-algorithm that is added to the controller. This controller gave the best results for the research goals in Section 2-5.

6-1-2 Other design options

The design choices in the previous section are not the only options. Other possible design choices for controllers that have been considered are:

1. One step ahead prediction with Proportional-Integral (PI)-control. This design combined the predictions from Chapter 4 with the PI disturbance attenuation from Chapter 5. This option was not researched further since it has the same limitations in predicting final balanced orientations as the system in Chapter 4.

2. Autonomous reference trajectories with deadbeat control. This design combined the predictions from Chapter 5 with the deadbeat control from Chapter 4. This design was not researched further since it was similar to the controller in Chapter 5, but without the integral control, resulting in steady state errors.

3. Machine learning algorithms. This would require designing a system that can learn the behavior of undisturbed systems in a Kuramoto and then learn to counter disturbances. The amount of extra research required for machine learning put this option outside the scope of this thesis.

4. Model Predictive Control (MPC) [24]. In MPC the controller will calculate the optimal input based on weighted penalties for state error and input, calculated for a set number of steps ahead. The state error has to be calculated in relation to a reference, which results in additional choices. The reference could be an autonomous system, as in Chapter 5, or a limited number of steps ahead, based on known states, as in Chapter 4. The input would have be based on the weight of the input penalty in relation to the state penalty. This becomes difficult when the scope of the research is to use the Kuramoto algorithm (2-5) or a modified version. The additional research required to design the model and augment the MPC model from [24] with disturbance attenuation resulted in leaving MPC outside the scope of this thesis.

6-2 Conclusions

The discrete-time Kuramoto model with matched input disturbances is a very interesting and challenging system. Understanding the properties of the undisturbed Kuramoto model is an essential prerequisite for designing disturbance attenuating controllers. In Chapter 2, the properties of the discrete-time Kuramoto model were introduced and explained, as were the effects of matched input disturbances in a system without additional control.

The controllers in Chapters 3 and 4 reinforce the statement in [14] that

> *balanced states are a highly underexplored class of solutions of the Kuramoto model*

The current inability to predict final balanced orientations for systems with $N \geq 4$ agents puts a limit on the usability of the error feedback models in Chapter 3 and the prediction-error feedback model in Chapter 4.

The best results are achieved when the outcome of an autonomous simulation is used as reference for a controller with a PI-algorithm to attenuate the matched input disturbances.

6-3 Recommendations

For use in real world applications, choosing between the designed controllers is a choice of desired characteristics: stability under matched input disturbances will only be guaranteed by the controller with one step ahead prediction and deadbeat control, while the controller with an autonomous reference system and PI-control will direct the system to the same balanced set as the undisturbed system with the risk of introducing instability due to bad PI-tuning.

Next to practical applications, the strategies in this thesis give several options for further research. First, matched input disturbances are only one type of disturbance, all other types can still be investigated. Second, it is possible that the strategy of an autonomous reference system with PI-control can lead to sustained oscillations or even instability, even if the product $K\tau$ is within the specified ranges for the aligned and balanced sets. Further research could be aimed at understanding the interaction between the disturbance, the PI gain parameters K_P and K_I and the product $K\tau$.

On a more fundamental level, the all-to-all coupled discrete-time Kuramoto model still provides options for research. The asymptotic stability of the balanced set has been conjectured [12] but is still to be proven. The characteristics of the balanced set have been researched [14], but prediction of balanced orientations for $N \geq 4$ is still an open research option.

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Glossary

List of Acronyms

MPC	Model Predictive Control
PI	Proportional-Integral

List of Symbols

$\dot{\theta}$	Angular velocity
ω	Natural Frequency
$\psi(h)$	Orientation of $\mathcal{R}$
$\rho(h)$	Magnitude of $\mathcal{R}$
τ	Time step used for discretization
θ	Agent orientation
$\mathcal{R}$	Phasor, $\mathcal{R}(\theta_i(h)) \equiv \frac{1}{N}\sum_{i=1}^{N}\begin{bmatrix}\cos(\theta_i(h))\\ \sin(\theta_i(h))\end{bmatrix}$
h	Time index
K	Coupling strength
N	Number of agents
t	Continuous time