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Three-dimensional time-dependent water flows with constant non-vanishing vorticity and depth dependent density

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ABSTRACT

We show that the movement of a time-dependent gravity water flow with constant non-zero vorticity and continuously depth dependent density satisfying the three-dimensional water wave equations is essentially two-dimensional: the velocity field, the pressure and the free surface do not change in the direction orthogonal to the direction of propagation. Our result is true both for the inviscid as well as for the viscous water wave problem.

1. Introduction

The mathematical study of the propagation of water waves is tremendously intricate. Difficulties arise since the evolution of waves has to be tracked along free boundaries (at the waters surface or as internal waves along an interface), and the governing equations as well as the boundary conditions are nonlinear. If in addition to that one takes into account the stratification of the fluid and the presence of underlying currents or swirling motions of the fluid due to the presence of vorticity, the analysis becomes truly challenging.

Despite all of these potentially complicated fluid motions, it is sometimes possible to observe simpler wave patterns: ocean swell, generated as wind waves at a distant source and aligned due to dispersive effects, forms a regular wave train with no variation in the direction orthogonal to the flow direction. This was mathematically first analysed by Constantin [1,2], who observed that the mere presence of non-zero constant vorticity renders the flow beneath the surface wave essentially two-dimensional: he showed that under the assumption of a steady periodic free surface profile, the velocity, surface and pressure have no variation in the direction orthogonal to the direction of propagation, and the vorticity points orthogonal to the flow direction. For similar results concerning solitary water waves we refer the reader to [3] for the irrotational case and to [4] for the rotational scenario. Non-existence results for two-dimensional solitary waves can be found in [5–7].

Moreover, the three-dimensional case where the free surface has a traveling wave character in both horizontal directions was dealt with by Wahlén [8]. More recently it was shown that the result is still true even without the steadyness assumption [9], which highlights that the main driver of this phenomenon is indeed the non-zero constant vorticity.

Vorticity is omnipresent in fluid flows. It can be viewed as a measure of the local infinitesimal rotation of fluid elements (with no implication on the global rotation, see [10]), similar to the angular momentum in solids. In vector calculus terms, this feature is captured by the vorticity vector which is defined as the curl of the velocity vector. Vorticity gives rise to currents (in its simplest form, a linear shear can be used to model tidal currents [11,12]) whose interaction with waves is a delicate and subtle problem to analyze not only mathematically [10], but also from a numerical and experimental perspective [11,13–16].

In the current study, we aim to investigate the combined effect of non-vanishing vorticity and stratification. Many water wave models do not take into account density variations in the fluid, arguing on the grounds of negligible compressibility of water. While this is often a reasonable simplification, there are certainly regions, for example near the Equator, where the water density varies considerably due to changes in temperature and salinity [17,18]. This results in a layering of the fluid and the appearance of a pycnocline [17,19,20], which separates the lighter fluid above from the heavier fluid below. The pycnocline can be modeled as an interface using a density distribution with a jump

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discontinuity at the interface, which facilitates the study of the propagation of internal waves as perturbations of the pycnocline [21–25]. In contrast to that, the present study focuses on a *continuous density distribution* [26–33] that allows the density to vary with depth and time [34]. From a historical perspective, stratified water waves have been studied by Dubreil-Jacotin [35], who constructed two-dimensional small-amplitude stratified traveling gravity waves by means of power series expansions. Following [35], major contributions to the field of stratified water flows have been made by Ter-Krikorov [36] and Yanowitch [37], Amick [38], Amick–Turner [39,40], Benjamin [41,42], Bona–Bose–Turner [43], James [44], Kirchgässner [45], Sun [46], Vanden-Broeck & Turner [47]. More recently, substantial contributions to water waves with stratifications concerning existence (and qualitative properties) of exact solutions by Abrashkin & Constantin [48], Constantin and Johnson [16], Ambrose, Strauss and Wright [49], Chen–Walsh [50], Chen–Walsh–Wheeler [51], Haziot [52], Henry and B.-V. Matioc [27, 53], Henry and A.-V. Matioc [28], Nilsson [54], Sinambela [55], and Walsh [56–58].

Our result extends previous results in homogeneous fluids [1,9,59] and in two-layered fluids (in the rigid-lid and free-surface cases) [60–63] to continuously stratified fluids, where we assume that the density depends only on the water depth and on time. The main outcome is that, in the presence of non-zero constant vorticity, the flow beneath a free surface–allowed to depend in the most general way on time and space–is effectively two dimensional: the velocity field, the surface and the pressure exhibit no variation in the direction orthogonal to the flow.

The outline of the paper is as follows: after introducing the three dimensional inviscid gravity water wave problem with non-constant continuously stratified density in Section 2, we present in Section 3 the result regarding the dimensionality reduction of the flow. The viscous case is treated in Section 4. The Appendix contains the proof of the invariance of the water wave problem under rotations in the inviscid and viscous case, which is of great importance in Sections 3 and 4.

2. Preliminaries

The following analysis concerns surface gravity water waves propagating above a three-dimensional water flow bounded below by a rigid flat bottom. In Cartesian coordinates (x, y, z) and denoting time by the variable t, the surface elevation is described by the equation $z = \eta(x, y, t)$ and the flat bed by $z = -d$, with $d > 0$. Given that typical Reynolds numbers in (geophysical) fluid dynamics are typically very large [64], with nonlinear effects dominating over viscosity, the inviscid theory is suitable for water waves that are not near breaking, cf. [65]. In the following, we state the governing equations for the motion of an inviscid fluid, which will be analyzed in Section 3, while the viscous situation will be considered in Section 4. Denoting the velocity field by $\mathbf{u} = (u, v, w)$, the pressure by P, the gravitational constant by g, and the density by ρ, the motion of an inviscid fluid is governed (see e.g. [20,66,67]) by the conservation of momentum equation,

$$\begin{aligned} u_t + uu_x + vu_y + wu_z &= -\frac{P_x}{\rho}, \\ v_t + uv_x + vv_y + wv_z &= -\frac{P_y}{\rho}, \\ w_t + uw_x + vw_y + ww_z &= -\frac{P_z}{\rho} - g, \end{aligned} \tag{2.1}$$

as well as by the equation of mass conservation,

$$\rho_t + (\rho u)_x + (\rho v)_y + (\rho w)_z = 0. \tag{2.2}$$

The system of equations that hold in the bulk of the fluid is completed by the boundary conditions. These are the kinematic boundary conditions

$$w = \eta_t + u\eta_x + v\eta_y \quad \text{on} \quad z = \eta(x, y, t) \tag{2.3}$$

and

$$w = 0 \quad \text{on} \quad z = -d, \tag{2.4}$$

and the dynamic boundary condition

$$P = P_{atm} \quad \text{on} \quad z = \eta(x, y, t), \tag{2.5}$$

where P_{atm} denotes the constant atmospheric pressure.

Note that since the density is not assumed to be constant in (2.2), the velocity field is allowed to be compressible (the case of incompressible flows is analyzed in Section 3.1.). The well-posedness for models of compressible flows has been studied, for instance, in [68–72]. Moreover, allowing for compressibility raises the question about an equation of state. As discussed in Talley et al. [73], to serve practical purposes in theoretical and numerical models, the equation of state is sometimes approximated as linear and its pressure dependence is ignored. That is, an adequate form of the equation of state, advocated also by Abrashkin & Constantin [48], is

$$\rho \approx \rho_0 - \alpha(T - T_0) + \beta(\mathfrak{s} - \mathfrak{s}_0), \tag{2.6}$$

where T is temperature, $\mathfrak{s}$ is salinity, ρ_0, T_0 and $\mathfrak{s}_0$ are mean values for the region being modeled, and α is the thermal expansion coefficient and β is the saline contraction coefficient. For instance, Abrashkin & Constantin [48] argue that the Southern Ocean presents variations in salinity and temperature with depth: the salinity in the mentioned area ranges from 34 to 35 ppt, cf. also [74], while the temperature difference between the surface and bottom is around 4 °C, cf. also [75]. Since we are interested in the behavior of the velocity field, pressure and density, we can absorb the differences in salinity and temperature in the density function. Moreover, the density may also depend on time, which can be justified, for instance, by the seasonal and diurnal variability: the temperature can vary by 6° C, cf. [76]. Thus, we will take the density to vary with respect to time and depth, that is $\rho = \rho(z, t)$, which we consider as the equation of state of the problem (2.1)–(2.5), see also Remark 2.2. This discussion justifies the following assumption.

Assumption 2.1. We will assume throughout the paper that

$$\rho = \rho(z, t),$$

that is, the density ρ depends only on the time variable t and on the depth variable z.

Remark 2.2. Depth-dependent density stratification (as considered in this work) represents an intrinsic characteristic of large-scale ocean movements, cf. [20,67,77]. Density fluctuations in the ocean, driven by variations in temperature and salinity, lead to stratification, where fluid layers of differing densities are arranged with higher-density layers below lower-density ones, cf. [16,17,24,78]. In line with the previous statement, recent findings [79] suggest that the evolution of hydroacoustic waves in weakly compressible fluids is significantly influenced by depth variations of the sound speed profile.

Remark 2.3. We note that in this paper we are interested in three-dimensional water flows from an analytical perspective (and are not concerned at all with atmospheric flows). That said, including a more general equation of state $\rho = \rho(T, S, p)$ as suggested, for instance, by Griffies et al. [80], Griffies [81] and McDougall et al. [82] is of great interest, but beyond the scope of the methods presented in this paper.

Assumption 2.4. Throughout the paper we make the assumption that

$$\sup_{(x,y)\in\mathbb{R}^2} P_z(x, y, \eta(x, y, t), t) < 0, \tag{2.7}$$

at all times t. Moreover, we will assume that the tuple (η, u, v, w, P) is a *bounded* solution of the water wave problem (2.1)–(2.5) and that u, v and w are two times differentiable with respect to x, y and z.

Remark 2.5. The condition in (2.7) is a restatement of the fact that the water pressure increases with depth near the surface. Indeed, since the pressure function is assumed to be continuous, the condition (2.7) implies that $P_z(x,y,z) < 0$ for $z \in (f(x,y), \eta(x,y)]$ where f is some real valued function of $(x,y) \in \mathbb{R}^2$. Therefore, the pressure at a depth below the free surface is bigger than the constant atmospheric pressure, which is what one expects. Hence, assuming P_z negative on the surface can be made without loss of generality, see also [8,67]. We emphasize that we need this assumption only at the free surface, not in whole bulk of the fluid.

As underlined in the introduction, we will take into account the effect of the flow's local swirling motions, which are encompassed in the vorticity vector, defined as the curl of the velocity field:

$$\omega = (w_y - v_z, u_z - w_x, v_x - u_y) =: (\omega_1, \omega_2, \omega_3). \tag{2.8}$$

Owing to the invariance of the water wave problem under rotations around the z axis, cf. Appendix, we can assume without loss of generality that one of the horizontal components of the vorticity vector vanishes. Therefore, the following assumption is justified.

Assumption 2.6. The vorticity vector ω is constant and non-vanishing throughout the flow such that $\omega_1 = 0$.

3. The inviscid case

Lemma 3.1. *Assume that the vorticity vector is constant and non-vanishing throughout the flow. Then the third component of the vorticity vector vanishes:* $\omega_3 = 0$.

Proof. We will conduct a proof by contradiction. Hence, we assume that $\omega_3 \neq 0$. First we consider the curl of the Euler Eqs. (2.1) and obtain the system

$$\frac{D\omega}{Dt} = (\omega\cdot\nabla)\mathbf{u} - (\nabla\cdot\mathbf{u})\omega + \frac{1}{\rho^2}\cdot\nabla\rho\times\nabla P, \tag{3.1}$$

see for instance [10], where $\frac{D}{Dt} = \partial_t + u\partial_x + v\partial_y + w\partial_z$ denotes the material derivative. Note that, unlike for homogeneous flows, the pressure term does not vanish from Eq. (3.1) in view of the fact that the density ρ is not constant and hence $\nabla\times\frac{1}{\rho}\nabla P = -\frac{1}{\rho^2}\nabla\rho\times\nabla P$. The constant vorticity vector assumption yields the vorticity equation in the form

$$(\omega\cdot\nabla)\mathbf{u} - (\nabla\cdot\mathbf{u})\omega + \frac{1}{\rho^2}\cdot\nabla\rho\times\nabla P = 0,$$

which can be expanded as

$$\begin{aligned}
&\omega_1 u_x + \omega_2 u_y + \omega_3 u_z - \omega_1(u_x + v_y + w_z) - \frac{\rho'(z)}{\rho^2(z)}P_y = 0,\\
&\omega_1 v_x + \omega_2 v_y + \omega_3 v_z - \omega_2(u_x + v_y + w_z) + \frac{\rho'(z)}{\rho^2(z)}P_x = 0,\\
&\qquad\omega_1 w_x + \omega_2 w_y + \omega_3 w_z - \omega_3(u_x + v_y + w_z) = 0.
\end{aligned} \tag{3.2}$$

From the equation of mass conservation (2.2) we find that

$$-(\nabla\cdot(u,v,w)) = \frac{1}{\rho}(\rho_t + \rho_z w).$$

Considering now w along the characteristic curves $s \mapsto \bar{x}(s)$, $s \mapsto \bar{y}(s)$, $s \mapsto \bar{z}(s)$ defined by the system $\frac{d\bar{x}}{ds} = \omega_1$, $\frac{d\bar{y}}{ds} = \omega_2$, $\frac{d\bar{z}}{ds} = \omega_3$ and using the above equation, we can rewrite the third equation in (3.2) as

$$\frac{d}{ds}w(\bar{x}(s),\bar{y}(s),\bar{z}(s),t) = -\frac{\omega_3}{\rho(\bar{z}(s),t)}\left(\rho_t(\bar{z}(s),t) + \rho_z(\bar{z}(s),t)w(\bar{x}(s),\bar{y}(s),\bar{z}(s),t)\right).$$

Integrating the previous equation we find that

$$\begin{aligned}
w(x,y,z,t) =& e^{-\omega_3\int_0^{\frac{z+d}{\omega_3}}\frac{\rho_z}{\rho}(\bar{z}(\xi),t)d\xi}\times\\
&\left[-\omega_3\int_0^{\frac{z+d}{\omega_3}}\frac{\rho_t}{\rho}(\bar{z}(\tau),t)e^{\omega_3\int_0^{\tau}\frac{\rho_z}{\rho}(\bar{z}(\xi),t)d\xi}d\tau\right.\\
&\left.+f\left(x - \frac{\omega_1}{\omega_3}(z+d), y - \frac{\omega_2}{\omega_3}(z+d), t\right)\right]
\end{aligned}$$

for some function $(q,p,t) \mapsto f(q,p,t)$. From the kinematic boundary condition on the bed (2.4) we see immediately that

$$f(x,y,t) = 0,$$

for all x,y,t. The latter yields that

$$w(x,y,z,t) = e^{-\omega_3\int_0^{\frac{z+d}{\omega_3}}\frac{\rho_z}{\rho}(\bar{z}(\xi),t)d\xi}\times\left[-\omega_3\int_0^{\frac{z+d}{\omega_3}}\frac{\rho_t}{\rho}(\bar{z}(\tau),t)e^{\omega_3\int_0^{\tau}\frac{\rho_z}{\rho}(\bar{z}(\xi),t)d\xi}d\tau\right],$$

from which we infer that

$$w_x(x,y,z,t) = w_y(x,y,z,t) = 0, \tag{3.3}$$

for all (x,y,z,t) for which (x,y,z) belongs to the fluid domain. Eq. (3.3) implies that the third equation in the vorticity Eq. (3.2) reduces to

$$u_x + v_y = 0 \tag{3.4}$$

at all points in the flow. Differentiating with respect to x in (3.4) we obtain (after taking into account the definition of ω_3 in (2.8)) that

$$u_{xx} + u_{yy} = 0 \tag{3.5}$$

throughout the fluid domain. Similarly, we also see that

$$v_{xx} + v_{yy} = 0 \tag{3.6}$$

within the flow. Now, from (3.5) and (3.6) we infer that the functions

$$(x,y) \mapsto u(x,y,z,t) \quad \text{and} \quad (x,y) \mapsto v(x,y,z,t)$$

are harmonic in $\mathbb{R}^2$ (as functions of x,y) for all $t \geq 0$ and all $z \in [-d, z_0]$ where z_0 is such that $-d < z_0 < \inf\{\eta(x,y,t) : (x,y) \in \mathbb{R}^2, t \geq 0\}$. Since they are also bounded, it follows by Liouville's theorem [83] that there exist functions $(z,t) \mapsto \mathfrak{u}(z,t)$ and $(z,t) \mapsto \mathfrak{v}(z,t)$ such that

$$u(x,y,z,t) = \mathfrak{u}(z,t) \quad \text{and} \quad v(x,y,z,t) = \mathfrak{v}(z,t)$$

in the domain $D_0 := \{(x,y,z) : (x,y) \in \mathbb{R}^2, -d \leq z \leq z_0\}$. Since ω_3 is assumed constant, we have $\omega_3 = \omega_3|_{D_0} = \mathfrak{v}_x - \mathfrak{u}_y = 0$. That is, we have reached a contradiction with the assumption that $\omega_3 \neq 0$. Thus, we conclude that $\omega_3 = 0$. □

Remark 3.2. Instrumental in proving Lemma 3.1, the vorticity Eq. (3.1) plays critical roles in many scientific and engineering flows, such as astrophysical flows, inertial confinement fusion, scramjet, and many other cases, cf. e.g. [84–86].

We are now ready to state the main result.

Theorem 3.3. *Assume that* (η, u, v, w, P) *represents a bounded solution of the water wave problem* (2.1)–(2.5) *with arbitrary density* $\rho = \rho(z,t)$ *and constant non-vanishing vorticity vector* ω. *Then, under the assumption* (2.7), *we have that the horizontal velocity component* v *is constant, and* u, w, P *and the free surface* η *are independent of* y.

Proof. In view of the fact that $\omega_3 = 0$ by Lemma 3.1, it follows from the vorticity Eq. (3.2) that

$$\begin{aligned}
&\omega_2 u_y - \omega_1 v_y - \omega_1 w_z - \frac{\rho'}{\rho^2}P_y = 0\\
&\omega_1 v_x - \omega_2 u_x - \omega_2 w_z + \frac{\rho'}{\rho^2}P_x = 0\\
&\qquad\omega_1 w_x + \omega_2 w_y = 0.
\end{aligned} \tag{3.7}$$

Therefore, system (3.7) is rewritten as

$$\begin{aligned}
\omega_2 u_y - \frac{\rho'}{\rho^2}P_y &= 0,\\
-\omega_2 u_x - \omega_2 w_z + \frac{\rho'}{\rho^2}P_x &= 0,\\
\omega_2 w_y &= 0,
\end{aligned}$$

which immediately implies that $w_y = 0$ and, using the definition of ω_1 in (2.8), we also have that $v_z = 0$ within the fluid domain.

Since we assume that the density function ρ depends only on z and t, we have that the equation of mass conservation (2.2) can be simplified to

$$\rho_t + \rho(u_x + v_y + w_z) + \rho_z w = 0.$$

Applying now a y-derivative to the latter equation provides us with the relation

$$u_{xy} + v_{yy} + w_{yz} = 0,$$

which by the previous considerations and in view of the vanishing of ω_3 implies that $v_{xx} + v_{yy} = 0$. This means that the function $(x, y) \mapsto v(x, y, z, t)$ is harmonic in $\mathbb{R}^2$ for all $t \geq 0$ and all $z \in [-d, z_0]$ where z_0 is such that $-d < z_0 < \inf\{\eta(x, y, t) : (x, y) \in \mathbb{R}^2, t \geq 0\}$. Since $(x, y) \mapsto v(x, y, z, t)$ is also bounded, we have by Liouville's theorem [83], that there exists a function $(z, t) \mapsto \mathfrak{v}(z, t)$ such that $v(x, y, z, t) = \mathfrak{v}(z, t)$ within the domain $\{(x, y, z) : (x, y) \in \mathbb{R}^2,\ z \leq z_0\}$. Hence,

$$v_x(x, y, z, t) = v_y(x, y, z, t) = 0$$

for all $(x, y) \in \mathbb{R}^2$, for all $t \geq 0$ and for all $z \in [-d, z_0]$. The arguments given by now show that v_x and v_y are harmonic functions in the fluid domain $\{(x, y, z, t) : t \geq 0,\ -d \leq z \leq \eta(x, y, t)\}$ which equal 0 on the open subset $\{(x, y, z) : (x, y) \in \mathbb{R}^2,\ -d < z < z_0\}$ at every time instant $t \geq 0$. In view of the real analyticity of harmonic functions, we conclude that $v_x = v_y = 0$ at all points of the fluid domain. Since we have already seen that $v_z = 0$ within the fluid domain, this yields that v is a function of time t alone. Furthermore, since $0 = \omega_3 = v_x - u_y$ we also have that u_y vanishes identically within the fluid. We can also infer from the second equation in (2.1) that

$$-v'(t) = \frac{P_y}{\rho}.$$

Integrating in y we find that there exists a function $(x, z) \mapsto f(x, z)$ such that

$$P(x, y, z, t) = -\rho v'(t) y + f(x, z)$$

for all x, y, z, t in the fluid domain. Since this is a linear function in y but the function $y \mapsto P(x, y, z, t)$ is bounded we conclude that $v'(t) = 0$ for all t. Therefore, $P_y = 0$ and v is constant in the fluid domain. Finally, differentiating the dynamic boundary condition (2.5) in y we find that

$$P_y(x, y, \eta(x, y, t), t) + P_z(x, y, \eta(x, y, t), t)\eta_y(x, y, t) = 0.$$

In view of the vanishing of P_y and the assumption on the pressure gradient (2.7) this implies that $\eta_y(x, y, t) = 0$ for all x, y in the fluid domain at all times t. ☐

3.1. Incompressible flows

In the case of incompressible flows, for which we assume that in addition to the governing equation (2.1)–(2.5) for the inviscid water wave problem the velocity field is divergence free, that is,

$$u_x + v_y + w_z = 0, \tag{3.8}$$

we obtain a stronger conclusion than the one stated in Theorem 3.3, even without the assumption (2.7) on the pressure.

Proposition 3.4. *Assuming an arbitrary density $\rho = \rho(z, t)$ such that $\rho_z < 0$ and constant non-zero vorticity, the only bounded solutions of the water wave problem (2.1)–(2.5) with a divergence free velocity field (3.8) are parallel shear flows of the form $u(z) = \omega_2 z + c_1$, $v = c_2$, $w = 0$, $c_i \in \mathbb{R}$, with a flat free surface.*

Before proceeding to the proof of Proposition 3.4 a few remarks are in order.

Remark 3.5. We would like to point out that similar shear flows were found in the context of (piecewise) constant vorticity for the case of the three-dimensional water wave equations with rigid lid boundary conditions in [60]. Also, in the presence of Coriolis effects, a parallel flow solution with two non-vanishing horizontal velocities was presented in [87]. The assumption that $\rho_z < 0$ can be justified in view of the typical density stratification present in the Ocean, see Remark 2.2.

Proof. The fact that $\omega_3 = 0$ follows more easily than in the proof of Lemma 3.1: the condition (3.8) implies that w is constant, which in view of (2.4) implies that $w \equiv 0$. Hence, we can conclude that Eq. (3.4) holds and the rest of the prove is identical to that of Lemma 3.1.

The proof that v is constant and $u_y = w_y = P_y = 0$ follows the same line of reasoning as in the proof of Theorem 3.3. To obtain the desired conclusions, we first show that the last step of that proof can be obtained without the pressure condition (2.7). We do so by extending an argument presented by Wahlén [8] to our case, which includes the time-dependence of the problem and the depth-variations of the density. More precisely, let $h(x, t) := \sup_{y\in\mathbb{R}} \eta(x, y, t)$ and note that the pressure function $P(x, z, t)$ is defined on the set $D(t) := \{(x, z, t) : -d < z < h(x, t)\}$ which is the projection of the fluid domain on the xz-plane. Assuming, for the sake of contradiction, that $\eta(x, y, t) < h(x, t)$, and using the real-analyticity of P, the lower semicontinuity of h, and the intermediate-value property of the continuous function η, we can argue, as in the proof of Lemma 3 in [8], that P is constant on $D(t)$ for all t. Now, taking into account the bottom condition (2.4), we have that $w_t(x, y, -d, t) = w_x(x, y, -d, t) = w_y(x, y, -d, t) = 0$ for all x, y, t, and, therefore the third equation in the momentum Eq. (2.1) yields the contradiction $0 = -g$ for all x, y, t. Consequently, $\eta(x, y, t) = \eta(x, t)$ for all x, y, t, that is, the surface η is independent of y as well.

Now we show that the velocity field is actually that of a shear flow and the free surface is constant. To see this, note that the equation of mass conservation can be simplified to

$$\rho_t + \rho_z w = 0, \tag{3.9}$$

where we have used that $u_x = -w_z$ in view of (3.8). Differentiating in (3.9) with respect to x and y, respectively, the assumption $\rho_z < 0$ implies that w_x and w_y vanish identically in the fluid domain. Thus, from (2.8) we obtain $u_z = \omega_2$ which implies, via $u_y = 0$, that

$$u(x, z, t) = \omega_2 z + F(x, t) \tag{3.10}$$

within the flow, where $F(x, t)$ is some differentiable function of x and t. Then, from $u_x + w_z = 0$ we infer that

$$w = -F_x(x, t)z + G(x, t)$$

for some differentiable function $(x, t) \mapsto G(x, t)$. Hence,

$$0 = w_x = -F_{xx}(x, t)z + G_x(x, t)$$

for all x, z, t with $z \in [-d, \eta(x, t)]$. The latter implies that

$$F(x, t) = c_1(t)x + c_2(t) \quad \text{and} \quad G(x, t) = c_3(t),$$

for some functions $t \mapsto c_i(t)$, $i = 1, 2, 3$. Since u is bounded, we see from (3.10) and from the previous equality that $c_1(t) = 0$ for all t, and, consequently, $u(z, t) = \omega_2 z + c_2(t)$ and $w = c_3(t)$ at all points of the fluid domain. Owing to the bottom boundary condition (2.4) we obtain that w vanishes identically within the fluid domain. From the latter assertion and using also (3.11), the first equation in (2.1) can be written as

$$P_x(x, z, t) = -c_2'(t)\rho(z, t).$$

The boundedness of P yields now that $c_2'(t) = 0$ for all t, and so $P_x = 0$ and

$$u(z, t) = \omega_2 z + c_2, \tag{3.11}$$

at all points of the flow. Notice that since $w = 0$ within the flow, the third equation in (2.1) implies that $P_z(z, t) = -g\rho(z, t)$. Hence, the dynamic boundary condition (2.5) implies that

$$P_{atm} = P(\eta(x, t), t) = -g \int_{-d}^{\eta(x)} \rho(s, t)\, ds + p(t)$$

for some function $t \mapsto p(t)$. Differentiating in the previous equality by x we obtain that $-g\rho(\eta(x,t),t)\eta_x(x,t) = 0$ for all x,t. Since the density never vanishes, it follows that $\eta_x(x,t) = 0$ for all x,t. Finally, it follows from the kinematic surface boundary condition (2.3) that $\eta_t = 0$ as well, so the surface profile η is indeed constant. □

4. The viscous case

In this section we treat the case when the viscosity of the water is not negligible and the governing equations are taken to be the Navier–Stokes equation, the equation of mass conservation (as before) and the associated boundary conditions, see [88]. Using the same notation as in previous sections, the motion of a viscous, continuously stratified fluid is governed by the conservation of momentum equation

$$\begin{aligned} u_t + uu_x + vu_y + wu_z &= -\frac{P_x}{\rho} + \nu\Delta u, \\ v_t + uv_x + vv_y + wv_z &= -\frac{P_y}{\rho} + \nu\Delta v, \\ w_t + uw_x + vw_y + ww_z &= -\frac{P_z}{\rho} - g + \nu\Delta w, \end{aligned} \tag{4.1}$$

where $\nu = \mu\rho^{-1}$ denotes the coefficient of *kinematic viscosity* and μ stands for the coefficient of *Newtonian viscosity*, and the equation of mass conservation

$$\rho_t + (\rho u)_x + (\rho v)_y + (\rho w)_z = 0. \tag{4.2}$$

In the viscous setting the fluid is exposed to normal and tangential stresses along the free surface $z = \eta(x,y,t)$ which is reflected in the boundary conditions. The normal stresses due to the ambient pressure above the surface are expressed as

$$P - 2\mu\frac{\eta_x^2 u_x + \eta_y^2 v_y - \eta_x(u_z + w_x) - \eta_y(v_z + w_y) + \eta_x\eta_y(u_y + v_x) + w_z}{1 + \eta_x^2 + \eta_y^2} = P_{atm}, \tag{4.3}$$

where as before P_{atm} denotes the constant atmospheric pressure. The tangential stresses due to the shearing action of the air at the free surface $z = \eta(x,y,t)$ can be formulated as

$$\eta_x(v_z + w_y) - \eta_y(u_z + w_x) + 2\eta_x\eta_y(u_x - v_y) - (\eta_x^2 - \eta_y^2)(u_y + v_x) = 0, \tag{4.4a}$$

and

$$\begin{aligned} &2\eta_x^2(u_x - w_z) + 2\eta_y^2(v_y - w_z) + 2\eta_x\eta_y(u_y + v_x) \\ &\quad + (\eta_x^2 + \eta_y^2 - 1)\left[\eta_x(u_z + w_x) + \eta_y(v_z + w_y)\right] = 0. \end{aligned} \tag{4.4b}$$

Finally, we the usual kinematic boundary conditions on the free surface and on the rigid bed are given respectively by

$$w = \eta_t + u\eta_x + v\eta_y \quad \text{on} \quad z = \eta(x,y,t) \tag{4.5}$$

and

$$u = v = w = 0 \quad \text{on} \quad z = -d. \tag{4.6}$$

In the following we will prove that the inclusion of viscosity in the governing equations does not alter the findings previously presented for the inviscid scenario, i.e. the fact that the flow is essentially two dimensional in the presence of constant non-zero vorticity remains true for the viscous case.

Theorem 4.1. *Assume that (η,u,v,w,P) represents a bounded solution of the viscous water wave problem (4.1)–(4.6) with arbitrary density $\rho = \rho(z,t)$ and constant non-vanishing vorticity vector ω. Then, under the assumption (2.7), we have that the third component of the vorticity vector vanishes, i.e. $\omega_3 = 0$, the horizontal velocity component v is constant, and u,w,P and the free surface η are independent of y.*

Proof. We first prove that $\omega_3 = 0$. Note that in the viscous case the vorticity equation, which is obtained by taking the curl of the Navier–Stokes equation, takes the form

$$\frac{D\omega}{Dt} = (\omega\cdot\nabla)\mathbf{u} - (\nabla\cdot\mathbf{u})\omega + \frac{1}{\rho^2}\cdot\nabla\rho\times\nabla P + \nu\Delta\omega + (-\nu_z\Delta v, \nu_z\Delta u, 0)^t,$$

where the upperscript denotes the transpose of the corresponding vector. Note that the z-dependence of the kinematic viscosity ν is inherited from ρ. Due to the assumption of constant vorticity, one of the viscosity terms vanishes and the previous equation can be rewritten as

$$\begin{aligned} &\omega_1 u_x + \omega_2 u_y + \omega_3 u_z - \omega_1(u_x + v_y + w_z) - \frac{\rho'(z)}{\rho^2(z)}P_y - \nu_z\Delta v = 0, \\ &\omega_1 v_x + \omega_2 v_y + \omega_3 v_z - \omega_2(u_x + v_y + w_z) + \frac{\rho'(z)}{\rho^2(z)}P_x + \nu_z\Delta u = 0, \\ &\qquad \omega_1 w_x + \omega_2 w_y + \omega_3 w_z - \omega_3(u_x + v_y + w_z) = 0. \end{aligned} \tag{4.7}$$

We assume for the sake of contradiction that $\omega_3 \neq 0$. Arguing as in the proof of Lemma 3.1 we obtain that $w_x = w_y = 0$, which implies that the third equation in (4.7) reduces to $u_x + v_y = 0$. From here, all further arguments go through as in Lemma 3.1 and we may conclude that $\omega_3 = 0$.

Therefore, the vorticity Eq. (4.7) is reduced to the form

$$\begin{aligned} &\omega_2 u_y - \frac{\rho'}{\rho^2}P_y + \nu_z\Delta v = 0, \\ &-\omega_2 u_x - \omega_2 w_z + \frac{\rho'}{\rho^2}P_x - \nu_z\Delta u = 0, \\ &\qquad \omega_2 w_y = 0, \end{aligned} \tag{4.8}$$

from which we readily infer that $w_y = v_z = 0$ within the flow. Utilizing the equation of mass conservation (4.2) we conclude as in the proof of Theorem 3.3 that v is a function of t alone. The latter assertion renders the first equation in (4.8) the form

$$\omega_2 u_y - \frac{\rho'}{\rho^2}P_y = 0$$

and the second of the Euler Eqs. (4.1) becomes

$$P_y = -\rho v'(t).$$

From here we are able to conclude as in the proof of Theorem 3.3 that throughout the flow $u_y = P_y = 0$. Recalling that

$$u_y = 0,\ v_x = v_y = v_z = 0,\ w_y = 0, \tag{4.9}$$

we proceed to the most difficult part of the proof, the fact that $\eta_y(x,y,t) = 0$ for all x,y,t. Let us assume that at some fixed time t_0 there exists a point (x_0,y_0) such that $\eta_y(x_0,y_0,t_0) \neq 0$. By continuity, the same is true in a ball B_r of radius r around (x_0,y_0), i.e. there exists $r > 0$ such that

$$\eta_y(x,y,t_0) \neq 0 \text{ for all } (x,y) \in B_r := \{(x,y) : (x-x_0)^2 + (y-y_0)^2 < r\}. \tag{4.10}$$

Our goal is to use the normal and tangential stress boundary conditions on the free surface to lead this assumption to a contradiction. Note that due to the conclusions on the velocity field (4.9), the tangential stress conditions (4.4) simplify to

$$\eta_y[-(u_z + w_x) + 2\eta_x u_x] = 0, \tag{4.11a}$$

and

$$2\eta_x^2(u_x - w_z) - 2\eta_y^2 w_z + (\eta_x^2 + \eta_y^2 - 1)\eta_x(u_z + w_x) = 0. \tag{4.11b}$$

In view of (4.10) we may conclude from (4.11a) that

$$u_z + w_x = 2\eta_x u_x \text{ for all } (x,y,t_0) \in B_r, \tag{4.12}$$

which together with (4.11b) yields

$$2(\eta_x^2 + \eta_y^2)(\eta_x^2 u_x - w_z) = 0.$$

In view of the assumption (4.10) this implies that

$$w_z = \eta_x^2 u_x \tag{4.13}$$

holds for all $z = \eta(x, y, t_0)$ with $(x, y) \in B_r$. Turning now to the normal stress condition (4.3), we find that in view of (4.9) it reads

$$P - 2\mu \frac{\eta_x^2 u_x - \eta_x(u_z + w_x) + w_z}{1 + \eta_x^2 + \eta_y^2} = P_{atm}.$$

In view of the Eqs. (4.12) and (4.13) the normal stress condition acquires the simple form

$$P(x, y, \eta(x, y, t_0)) = P_{atm} \text{ for all } (x, y) \in B_r.$$

Differentiating the above expression with respect to y we obtain that

$$P_y(x, y, \eta(x, y, t_0)) + P_z(x, y, \eta(x, y, t_0))\eta_y(x, y, t_0) = 0 \text{ for all } (x, y) \in B_r.$$

Using now that $P_y = 0$ throughout the flow, while $P_z < 0$ in view of the hypothesis (2.7), we conclude that $\eta_y(x, y, t_0) = 0$ for all $(x, y) \in B_r$, which is obviously a contradiction with the assumption that $\eta_y(x_0, y_0, t_0) \neq 0$. Therefore, $\eta_y(x, y, t_0) = 0$ for all $(x, y) \in \mathbb{R}^2$, and since the above analysis does not depend on the choice of t_0 the claim holds for all t, x, y. □

Remark 4.2. For incompressible viscous flows we obtain an even stronger result without the assumption (2.7) on the pressure, just as in the inviscid case presented in Section 3.1. One can show that the only solutions to the governing Eqs. (4.1)–(4.6) are parallel shear flows of the form $u = \omega_2(z + d), v = w = 0$, with a flat free surface.

CRediT authorship contribution statement

Anna Geyer: Writing – review & editing, Writing – original draft, Methodology, Investigation, Formal analysis. **Calin I. Martin:** Writing – review & editing, Writing – original draft, Methodology, Investigation, Formal analysis.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix

This section is devoted to the proof of the invariance of the water wave problem under rotations around the z-axis for the situation of a continuously stratified flow with a density function $\rho(z, t)$. We treat the inviscid as well as the viscous situation. To this end, we note that it is well-known that the Euler and Navier–Stokes equations are invariant under rotations, see [89]. To complete the assertion it remains to show the invariance (under rotations around the z-axis) of the equation of mass conservation (2.2), of the boundary conditions (2.3)–(2.5) in the inviscid case, and of the boundary conditions (4.3)–(4.6) in the viscous case, respectively.

To begin with, we set

$$Q = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

to be the rotation matrix by angle $\theta \in [0, 2\pi)$ around the z-axis. For a given solution $(u, v, w), P, \eta$ of the water wave problem (2.1)–(2.5) we define a new velocity field (U, V, W) through the formula

$$\begin{pmatrix} U(x,y,z,t) \\ V(x,y,z,t) \\ W(x,y,z,t) \end{pmatrix} = Q^T \begin{pmatrix} u(X,Y,Z,t) \\ v(X,Y,Z,t) \\ w(X,Y,Z,t) \end{pmatrix}, \tag{A.1}$$

where

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = Q \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x\cos\theta + y\sin\theta \\ -x\sin\theta + y\cos\theta \\ z \end{pmatrix}. \tag{A.2}$$

Likewise, let

$$\tilde{P}(x, y, z, t) := P(Q(x, y, z)^T, t),$$

$$\tilde{\eta}(x, y, t) := \eta(x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta, t),$$

represent a new pressure function and a new free surface, respectively. According to [89] the tuple $(U, V, W, \tilde{P}, \tilde{\eta})$ is a solution of the Euler (2.1) or Navier–Stokes (4.1) equations as soon as (u, v, w, P, η) is. Next we show that (U, V, W) satisfies the equation of mass conservation (2.2) provided (u, v, w) does. Note first that

$$U_x = (u_x\cos\theta - u_y\sin\theta)\cos\theta - (v_x\cos\theta - v_y\sin\theta)\sin\theta,$$
$$V_y = (u_x\sin\theta + u_y\cos\theta)\sin\theta + (v_x\sin\theta + v_y\cos\theta)\cos\theta,$$

which yield that $U_x + V_y = u_x + v_y$. Since $W_z = w_z$ we obtain that

$$\rho_t + \rho(U_x + V_y + W_z) + \rho_z W = \rho_t + \rho(u_x + v_y + w_z) + \rho_z w = 0,$$

which shows that (2.2) holds. A routine computation shows that $U\tilde{\eta}_x + V\tilde{\eta}_y = u\eta_x + v\eta_y$, i.e. the kinematic boundary condition on the surface (2.3) holds as well. The previous considerations show that $(U, V, W), \tilde{P}, \tilde{\eta}$ is a solution of the inviscid water wave problem (2.1)–(2.5) if and only if $(u, v, w), P, \eta$ is.

To prove that the viscous water wave problem (4.1)–(4.6) remains invariant under the rotations (A.1)–(A.2), we will show the invariance of all expressions that build up the boundary conditions at the free surface given by relations (4.3), (4.4a) and (4.4b). Availing of the expressions for U_z, V_z, W_x and W_y we obtain that

$$\tilde{\eta}_x(V_z + W_y) - \tilde{\eta}_y(U_z + W_x) = \eta_x(v_z + w_y) - \eta_y(u_z + w_x). \tag{A.3}$$

A tedious computation shows that

$$\begin{aligned} &2\tilde{\eta}_x\tilde{\eta}_y(U_x - V_y) = \\ &\quad (\eta_x^2 - \eta_y^2)(u_x - v_y)\sin(2\theta)\cos(2\theta) - 2\eta_x\eta_y(u_y + v_x)\cos(2\theta)\sin(2\theta) \\ &\quad - (\eta_x^2 - \eta_y^2)(u_y + v_x)\sin^2(2\theta) + 2\eta_x\eta_y(u_x - v_y)\cos^2(2\theta), \end{aligned}$$

and

$$\begin{aligned} &-(\tilde{\eta}_x^2 - \tilde{\eta}_y^2)(U_y + V_x) = \\ &\quad 2\eta_x\eta_y(u_y + v_x)\cos(2\theta)\sin(2\theta) - (\eta_x^2 - \eta_y^2)(u_x - v_y)\sin(2\theta)\cos(2\theta) \\ &\quad + 2\eta_x\eta_y(u_x - v_y)\sin^2(2\theta) - (\eta_x^2 - \eta_y^2)(u_y + v_x)\cos^2(2\theta). \end{aligned}$$

The previous two formulas show that

$$2\tilde{\eta}_x\tilde{\eta}_y(U_x - V_y) - (\tilde{\eta}_x^2 - \tilde{\eta}_y^2)(U_y + V_x) = 2\eta_x\eta_y(u_x - v_y) - (\eta_x^2 - \eta_y^2)(u_y + v_x). \tag{A.4}$$

We note now that (A.3) and (A.4) establish the invariance of the boundary condition (4.4a). To see that the boundary condition (4.4b) is also invariant we compute

$$\begin{aligned} &\tilde{\eta}_x^2 U_x + \tilde{\eta}_y^2 V_y = \\ &\quad (\eta_x^2 u_x + \eta_y^2 u_y)\left(1 - \frac{\sin^2(2\theta)}{2}\right) + (\eta_y^2 - \eta_x^2)(u_y + v_x)\frac{\sin(2\theta)\cos(2\theta)}{2} \\ &\quad + 2(\eta_y^2 u_x + \eta_x^2 v_y)\sin^2(\theta)\cos^2(\theta) \\ &\quad + \eta_x\eta_y(u_y + v_x)\sin^2(2\theta) + \eta_x\eta_y(v_y - u_x)\sin(2\theta)\cos(2\theta), \end{aligned} \tag{A.5}$$

and

$$\tilde{\eta}_x\tilde{\eta}_y(U_y+V_x)= \\ (\eta_x^2 u_x+\eta_y^2 v_y)\frac{\sin^2(2\theta)}{2}-(\eta_y^2 u_x+\eta_x^2 v_y)\frac{\sin^2(2\theta)}{2} \\ +\eta_x\eta_y(u_x-v_y)\cos(2\theta)\sin(2\theta) \\ +(\eta_x^2-\eta_y^2)(u_y+v_x)\frac{\sin(2\theta)\cos(2\theta)}{2}+\eta_x\eta_y(u_y+v_x)\cos^2(2\theta). \tag{A.6}$$

We find that the previous two expressions yield

$$\tilde{\eta}_x^2 U_x+\tilde{\eta}_y^2 V_y+\tilde{\eta}_x\tilde{\eta}_y(U_y+V_x)=\eta_x^2 u_x+\eta_y^2 u_y+\eta_x\eta_y(u_y+v_x).$$

Since $\tilde{\eta}_x^2+\tilde{\eta}_y^2=\eta_x^2+\eta_y^2$ and $W_z=w_z$ we conclude that

$$\tilde{\eta}_x^2(U_x-W_z)+\tilde{\eta}_y^2(V_y-W_z)+\tilde{\eta}_x\tilde{\eta}_y(U_y+V_x) \\ =\eta_x^2(u_x-w_z)+\eta_y^2(v_y-w_z)+\eta_x\eta_y(u_y+v_x). \tag{A.7}$$

Moreover, we check that

$$\tilde{\eta}_x(U_z+W_x)+\tilde{\eta}_y(V_z+W_y)=\eta_x(u_z+w_x)+\eta_y(v_z+w_y). \tag{A.8}$$

We infer that relations (A.7) together with (A.8) imply the invariance of the boundary condition (4.4b). Finally, the boundary condition for the pressure (4.3) is invariant under rotations around the z-axis in view of the expressions (A.5), (A.6) and (A.8).

We conclude this section by computing the vorticity associated with the flow (U,V,W), $\tilde{P}$, $\tilde{\eta}$. Denoting with $(\Omega_1,\Omega_2,\Omega_3)$ the vorticity vector associated with (U,V,W) we obtain

$$\begin{aligned}\Omega_2 &= U_z-W_x \\ &= u_z\cos\theta-v_z\sin\theta-W_x \\ &= (\omega_2+w_x)\cos\theta-(w_y-\omega_1)\sin\theta \\ &= \omega_2\cos\theta+\omega_1\sin\theta\end{aligned} \tag{A.9}$$

since $W_x=w_x\cos\theta-w_y\sin\theta$. Analogously we obtain

$$\Omega_1=W_y-V_z=\omega_1\cos\theta-\omega_2\sin\theta. \tag{A.10}$$

From formulas (A.9) and (A.10) we infer that, provided ω_1 and ω_2 are constants, there is a value of θ such that $\Omega_1\Omega_2=0$ and $\Omega_1^2+\Omega_2^2\neq 0$.

Data availability

No data was used for the research described in the article.

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