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The Time-Domain Optical Theorem in Antenna Theory

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***Abstract*—A special form of the time-domain optical theorem related to a general receiving antenna system is rigorously derived. It is shown that the total energies dissipated in the antenna load and in the antenna system itself can be directly related to the electromagnetic energy of the scattered field and its time-domain far-field characteristics. A practical implication of the result in optimizing antenna scattering properties with regard to the maximum energy dissipated in the antenna loading is discussed.**

***Index Terms*—Electromagnetic scattering, receiving antenna, reciprocity principle, time-domain analysis.**

I. Introduction

THE optical theorem (to be traced back, as such, to [1]) is one of the most interesting results of the wave scattering theory. The theorem directly relates the extinction cross-section of a scatterer to the scattered field in the far-field region as observed in the direction of propagation of the incident plane wave. While the optical theorem is well explored in the frequency domain (FD) [2], [3], the time-domain (TD) counterpart was firstly published in its general form as late as 1984 by De Hoop [4]. Although the TD optical theorem, in contrast to its (standard) FD counterpart, does not require the scatterer to be linear and/or time invariant, it seems that its potentialities have not been fully appreciated so far. An exception in this respect are the interesting applications reported by Karlsson [5], [6].

This letter provides a special form of the TD optical theorem as needed for its applications in the antenna theory. Here it is shown that the total energies dissipated in the antenna load and in the the antenna system itself can be directly related to the electromagnetic energy of the scattered field and its TD far-field characteristics. This result serves a purpose in addressing practical questions such as maximizing the energy dissipated in the antenna load for a given incident pulse shape or the analysis of the antenna's scattering properties.

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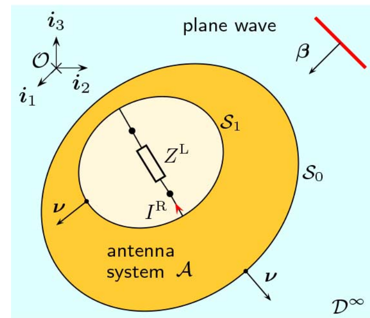

Fig. 1. Receiving one-port antenna system.

II. Problem Definition

We study the case of an antenna system in its receiving state, the configuration being described in Fig. 1. In it, position in space is given by the position vector $\boldsymbol{x} = x_1\boldsymbol{i}_1 + x_2\boldsymbol{i}_2 + x_3\boldsymbol{i}_3$ with respect to the Cartesian reference frame. Partial differentiation with respect to x_m will be denoted by ∂_m. The time coordinate is $\{t \in \mathbb{R}; t > 0\}$; the partial differentiation with respect to time is ∂_t and the time-convolution operator is denoted by $\star_t$. The time-integration operator is defined as

$$\int f_k(\boldsymbol{x}, t) \triangleq \int_{\tau=-\infty}^{t} f_k(\boldsymbol{x}, \tau)\mathrm{d}\tau. \tag{1}$$

The Dirac delta distribution is denoted by $\delta(t)$. The standard subscript notation for Cartesian tensors with the summation convention for repeated subscripts is employed [7, Sec. A.2]. The Levi-Civita tensor is $\boldsymbol{e}_{k,m,p} = 1$ for $\{k, m, p\}$ = even permutation of $\{1, 2, 3\}$, $\boldsymbol{e}_{k,m,p} = -1$ for $\{k, m, p\}$ = odd permutation of $\{1, 2, 3\}$ and $\boldsymbol{e}_{k,m,p} = 0$ in all other cases and the Kronecker tensor is $\delta_{i,j} = 1$ for $i = j$ and $\delta_{i,j} = 0$ for $i \neq j$ [7, Sec. A.7].

The antenna system is situated in the linear, homogeneous and isotropic embedding $\mathcal{D}^\infty$ whose electromagnetic properties are described by its electric permittivity $\epsilon_0 > 0$ and magnetic permeability $\mu_0 > 0$. The corresponding electromagnetic wave speed and wave admittance are $c_0 = (\epsilon_0\mu_0)^{-1/2}$ and $\eta_0 = (\epsilon_0/\mu_0)^{1/2}$, respectively. Note that this type of embedding also includes the case of free space. The antenna system occupies a bounded domain $\mathcal{A} \subset \mathbb{R}^3$ enclosed externally and internally by surfaces $\mathcal{S}_0$ and $\mathcal{S}_1$, respectively. No further restrictions as to antenna electromagnetic properties are imposed. Owing to the fact that the antenna optical theorem is entirely constructed in the time domain, nonlinear electromagnetic effects of the antenna system are not excluded in our analysis.

The antenna system is taken to be irradiated by a pulsed uniform plane wave

$$E_p^{\mathrm{i}}(\boldsymbol{x},t) = \alpha_p e^{\mathrm{i}}(t - \beta_q x_q/c_0) \tag{2}$$
$$H_j^{\mathrm{i}}(\boldsymbol{x},t) = \eta_0 \boldsymbol{e}_{j,r,s}\alpha_s\beta_r e^{\mathrm{i}}(t - \beta_q x_q/c_0) \tag{3}$$

where $\boldsymbol{E}^{\mathrm{i}}(\boldsymbol{x},t)$ and $\boldsymbol{H}^{\mathrm{i}}(\boldsymbol{x},t)$ are the *incident* electric and magnetic field strengths, respectively, $\boldsymbol{\alpha}$ is the polarization vector, $\boldsymbol{\beta}$ denotes the unit vector in the direction of propagation and the *causal* plane-wave signature $e^{\mathrm{i}}(t)$ satisfies $e^{\mathrm{i}}(t) = 0$ for $t < 0$.

For the sake of clarity, this study is confined to the case of a single-port antenna model [8], the extension to an N-port antenna system being straightforward (see [9]).

III. EM Fields in the Receiving State

The scattered ('s') field is defined as the difference between the total field ('R') in the receiving situation and the incident ('i') field

$$\left\{E_q^{\mathrm{s}}, H_p^{\mathrm{s}}\right\}(\boldsymbol{x},t) \triangleq \left\{E_q^{\mathrm{R}}, H_p^{\mathrm{R}}\right\}(\boldsymbol{x},t) - \left\{E_q^{\mathrm{i}}, H_p^{\mathrm{i}}\right\}(\boldsymbol{x},t) \tag{4}$$

for all $\boldsymbol{x} \in \mathcal{D}^\infty$ and $t > 0$. Without explicitly stating the constitutive relations, the total electromagnetic field satisfies

$$\boldsymbol{e}_{k,m,r}\partial_m H_r^{\mathrm{R}} - J_k^{\mathrm{R}} - \partial_t D_k^{\mathrm{R}} = 0 \tag{5}$$
$$\boldsymbol{e}_{j,n,q}\partial_n E_q^{\mathrm{R}} + \partial_t B_j^{\mathrm{R}} = 0 \tag{6}$$

for all $\boldsymbol{x} \in \mathcal{A}$ and $t > 0$, where $\boldsymbol{E}^{\mathrm{R}}(\boldsymbol{x},t)$ and $\boldsymbol{H}^{\mathrm{R}}(\boldsymbol{x},t)$ are the *total* electric and magnetic field strengths, respectively, $\boldsymbol{D}^{\mathrm{R}}(\boldsymbol{x},t)$ and $\boldsymbol{B}^{\mathrm{R}}(\boldsymbol{x},t)$ are the corresponding electric- and magnetic-flux densities, respectively, and $\boldsymbol{J}^{\mathrm{R}}(\boldsymbol{x},t)$ is the volume density of electric (conductive) current. The scattered field admits the causal far-field expansion

$$\begin{aligned}\{E_q^{\mathrm{s}}, H_p^{\mathrm{s}}\}(\boldsymbol{x},t) = \{&E_q^{\mathrm{s};\infty}, H_p^{\mathrm{s};\infty}\}(\boldsymbol{\xi}, t - |\boldsymbol{x}|/c_0)\\ &\times (4\pi|\boldsymbol{x}|)^{-1}[1 + O(|\boldsymbol{x}|^{-1})]\end{aligned} \tag{7}$$

as $|\boldsymbol{x}| \to \infty$ for all $t > 0$, where $\{E_q^{\mathrm{s};\infty}, H_p^{\mathrm{s};\infty}\}(\boldsymbol{\xi},t)$ are the (vectorial) far-field radiation characteristics and $\boldsymbol{\xi} = \boldsymbol{x}/|\boldsymbol{x}|$ is the unit vector in the direction of observation. The electric-field radiation characteristic can be determined from its surface-integral representation (cf. [4, Eq. (26)])

$$\begin{aligned}\textstyle\int E_k^{\mathrm{s};\infty}(\boldsymbol{\xi},t) = {}&\mu_0(\xi_k\xi_p - \delta_{k,p})\boldsymbol{e}_{p,m,j}\\ &\int_{\boldsymbol{x}'\in\mathcal{S}_0} \nu_m(\boldsymbol{x}')H_j^{\mathrm{s}}(\boldsymbol{x}', t + \xi_q x_q'/c_0)\mathrm{d}A(\boldsymbol{x}')\\ &+ c_0^{-1}\boldsymbol{e}_{k,r,s}\xi_r\boldsymbol{e}_{s,n,m}\\ &\int_{\boldsymbol{x}'\in\mathcal{S}_0} \nu_m(\boldsymbol{x}')E_n^{\mathrm{s}}(\boldsymbol{x}', t + \xi_q x_q'/c_0)\mathrm{d}A(\boldsymbol{x}')\end{aligned} \tag{8}$$

for all $\boldsymbol{\xi} \in \Omega = \{\xi_q\xi_q = 1\}$ and $t > 0$. The magnetic-field counterpart then follows from the plane-wave relation

$$H_p^{\mathrm{s};\infty}(\boldsymbol{\xi},t) = \eta_0\boldsymbol{e}_{p,q,s}\xi_q E_s^{\mathrm{s};\infty}(\boldsymbol{\xi},t) \tag{9}$$

Finally, the voltage V^{R} across the antenna load is taken to be linearly related to the electric current I^{R} flowing into it according to

$$V^{\mathrm{R}}(t) = Z^{\mathrm{L}}(t) \star_t I^{\mathrm{R}}(t) \tag{10}$$

where $Z^{\mathrm{L}}(t)$ is the impedance of the load.

IV. Time-Domain Antenna Optical Theorem

Combination of the far-field surface-integral representation of the scattered field (8) with (2)–(3) yields (see Appendix)

$$\begin{aligned}\boldsymbol{e}_{k,r,p}\int_{\boldsymbol{x}'\in\mathcal{S}_0} &\nu_k(\boldsymbol{x}')[E_r^{\mathrm{i}}(\boldsymbol{x}',t) \star_t H_p^{\mathrm{s}}(\boldsymbol{x}',-t)\\ &+E_r^{\mathrm{s}}(\boldsymbol{x}',-t) \star_t H_p^{\mathrm{i}}(\boldsymbol{x}',t)]\mathrm{d}A(\boldsymbol{x}')\\ &= \mu_0^{-1}\alpha_q e^{\mathrm{i}}(t) \star_t \textstyle\int E_q^{\mathrm{s};\infty}(\boldsymbol{\beta},-t).\end{aligned} \tag{11}$$

Taking into the account that the embedding is lossless and by using (4) it is found that

$$\begin{aligned}\boldsymbol{e}_{k,r,p}\int_{\tau\in\mathbb{R}}\mathrm{d}\tau\int_{\boldsymbol{x}'\in\mathcal{S}_0} &\nu_k(\boldsymbol{x}')[E_r^{\mathrm{R}}(\boldsymbol{x}',\tau)H_p^{\mathrm{R}}(\boldsymbol{x}',\tau)\\ &-E_r^{\mathrm{s}}(\boldsymbol{x}',\tau)H_p^{\mathrm{s}}(\boldsymbol{x}',\tau)]\mathrm{d}A(\boldsymbol{x}')\\ = \mu_0^{-1}\int_{\tau\in\mathbb{R}} &\alpha_q e^{\mathrm{i}}(\tau)\textstyle\int E_q^{\mathrm{s};\infty}(\boldsymbol{\beta},\tau)\mathrm{d}\tau.\end{aligned} \tag{12}$$

at $t = 0$. The second integral on the left-hand side of (12) is the total scattered energy, symbolically written as

$$W^{\mathrm{s}} = \boldsymbol{e}_{k,r,p}\int_{\mathbb{R}}\mathrm{d}\tau\int_{\mathcal{S}_0}\nu_k E_r^{\mathrm{s}}H_p^{\mathrm{s}}\mathrm{d}A \tag{13}$$

while making use of (5)–(6), along with application of Gauss' theorem, leads to

$$\begin{aligned}\boldsymbol{e}_{k,r,p}\int_{\mathbb{R}}\mathrm{d}\tau\int_{\mathcal{S}_0}\nu_k E_r^{\mathrm{R}}H_p^{\mathrm{R}}\mathrm{d}A + W^{\mathrm{h}}\\ = \boldsymbol{e}_{k,r,p}\int_{\mathbb{R}}\mathrm{d}\tau\int_{\mathcal{S}_1}\nu_k E_r^{\mathrm{R}}H_p^{\mathrm{R}}\mathrm{d}A\end{aligned} \tag{14}$$

where W^{h} represents the electromagnetic energy converted into heat in the antenna system $\mathcal{A}$. Upon combining Eqs. (12)–(14) together with the interfacing condition (cf. [10, Eq. (1)])

$$\boldsymbol{e}_{k,r,p}\int_{\mathcal{S}_1}\nu_k E_r^{\mathrm{R}}H_p^{\mathrm{R}}\mathrm{d}A = -V^{\mathrm{R}}I^{\mathrm{R}} \tag{15}$$

we finally end up with the sought for relation

$$\begin{aligned}W^{\mathrm{h}} + \int_{\tau\in\mathbb{R}} V^{\mathrm{R}}(\tau)I^{\mathrm{R}}(\tau)\mathrm{d}\tau + W^{\mathrm{s}}\\ = -\mu_0^{-1}\int_{\tau\in\mathbb{R}}\alpha_q e^{\mathrm{i}}(\tau)\textstyle\int E_q^{\mathrm{s};\infty}(\boldsymbol{\beta},\tau)\mathrm{d}\tau\end{aligned} \tag{16}$$

The final result thus interrelates the total energies dissipated in the antenna system and its load with the scattering properties of the antenna. The scattering far-field characteristics are explicitly expressed through the slant-stack transformation of the equivalent surface current densities on the antenna surface $\mathcal{S}_0$ (see [11, Eq. (5)]). The thus constructed antenna optical theorem may then be useful, for instance, in searching for optimal antenna scattering properties and excitation conditions to achieve the maximum energy dissipated in the antenna loading.

V. Conclusion

A special form of the time-domain optical theorem has been constructed for the time-domain plane-wave scattering by a general one-port receiving system. It was shown that the total energies dissipated in the antenna load and in the antenna system itself can be directly associated with the antenna's scattering behavior. Some practical applications have been hinted at.

Appendix A

The time correlation of the time-integrated, copolarized far-field amplitude in the forward direction with the incident plane-wave signature follows from Eq. (8) as

$$\begin{aligned}
&e^{\mathrm{i}}(t) \star_t \mu_0^{-1}\alpha_q \textstyle\int E_q^{\mathrm{s};\infty}(\boldsymbol{\beta}, -t) = \displaystyle\int_{\tau\in\mathbb{R}} \mathrm{d}\tau \\
&\Bigg[\alpha_p e^{\mathrm{i}}(t-\tau)\boldsymbol{e}_{p,j,m} \int_{\boldsymbol{x}'\in\mathcal{S}_0} \nu_m(\boldsymbol{x}') H_j^{\mathrm{s}}(\boldsymbol{x}', \beta_q x_q'/c_0 - \tau)\mathrm{d}A \\
&\eta_0 \boldsymbol{e}_{s,r,k}\alpha_k\beta_r e^{\mathrm{i}}(t-\tau) \\
&\boldsymbol{e}_{s,m,n}\int_{\boldsymbol{x}'\in\mathcal{S}_0} \nu_m(\boldsymbol{x}') E_n^{\mathrm{s}}(\boldsymbol{x}', \beta_q x_q'/c_0 - \tau)\mathrm{d}A\Bigg]
\end{aligned} \tag{17}$$

where we made use of the fact that $\alpha_q\beta_q = 0$. The successive substitutions $u = t + \beta_k x_k' - \tau$ and $u = \tau$ allow rewriting (17) as

$$\begin{aligned}
&e^{\mathrm{i}}(t) \star_t \mu_0^{-1}\alpha_q \textstyle\int E_q^{\mathrm{s};\infty}(\boldsymbol{\beta}, -t) = \displaystyle\int_{\tau\in\mathbb{R}} \mathrm{d}\tau \\
&\Bigg[\int_{\boldsymbol{x}'\in\mathcal{S}_0} \alpha_p e^{\mathrm{i}}(\tau - \beta_q x_q'/c_0)\boldsymbol{e}_{p,j,m}\nu_m(\boldsymbol{x}') H_j^{\mathrm{s}}(\boldsymbol{x}', \tau - t)\mathrm{d}A \\
&\int_{\boldsymbol{x}'\in\mathcal{S}_0} \eta_0 \boldsymbol{e}_{s,r,k}\alpha_k\beta_r e^{\mathrm{i}}(\tau - \beta_q x_q'/c_0) \\
&\boldsymbol{e}_{s,m,n}\nu_m(\boldsymbol{x}') E_n^{\mathrm{s}}(\boldsymbol{x}', \tau - t)\mathrm{d}A\Bigg]
\end{aligned} \tag{18}$$

Finally, the use of (2) and (3) in (18) leads to

$$\begin{aligned}
&e^{\mathrm{i}}(t) \star_t \mu_0^{-1}\alpha_q \textstyle\int E_q^{\mathrm{s};\infty}(\boldsymbol{\beta}, -t) = \displaystyle\int_{\tau\in\mathbb{R}} \mathrm{d}\tau \\
&\Bigg[\int_{\boldsymbol{x}'\in\mathcal{S}_0} E_p^{\mathrm{i}}(\boldsymbol{x}', \tau)\boldsymbol{e}_{p,j,m}\nu_m(\boldsymbol{x}') H_j^{\mathrm{s}}(\boldsymbol{x}', \tau - t)\mathrm{d}A \\
&\int_{\boldsymbol{x}'\in\mathcal{S}_0} H_s^{\mathrm{i}}(\boldsymbol{x}', \tau)\boldsymbol{e}_{s,m,n}\nu_m(\boldsymbol{x}') E_n^{\mathrm{s}}(\boldsymbol{x}', \tau - t)\mathrm{d}A\Bigg]
\end{aligned} \tag{19}$$

that directly corresponds to Eq. (11). The latter is used in the main text to derive the time-domain optical theorem.

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