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An alternative approach to stochastic integration in Banach spaces

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J.B.P. de Graaff

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An alternative approach to stochastic integration in Banach spaces

J.B.P. de Graaff (4446941)

Delft University of Technology

Supervisor

Prof. dr. J.M.A.M. van Neerven

Other committee members

Dr. R. Kraaij

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Abstract

In his 2019 article, Kalinichenko proposed an alternative way of doing stochastic integration in general separable Banach spaces [12]. This way circumvents the usual UMD assumption on our separable Banach space X, and instead imposes a strict condition on the integrating process $\Phi : (0,T) \times \Omega \to \mathscr{L}(H,X)$. Namely, we require the existence of an X-valued Gaussian g such that almost surely for all $x^* \in X^*$,

$$\int_0^T \|\Phi(t,\omega)^* x^*\|_H^2 \, dt \leq \mathbb{E}\langle g, x^* \rangle^2.$$

Most notably, this approach works in any separable Banach space. In this thesis we will take a closer look at the proofs used in [12], and place his article in the context of the known theory on stochastic analysis in Banach spaces (as can be read in [23], [25], [28]). We will compare Kalinichenko's approach both to the UMD and martingale type 2 situation, and discuss the advantages and disadvantages of either strategy.

Moreover, we will compare the conditions imposed in [12] on the stochastic process to the condition of γ-radonification as assumed in the UMD case [25].

Preface

This thesis was written in 2021-2022 during my last year as a master student Applied Mathematics at TU Delft. In this preface I want to thank everyone who contributed to this thesis in one way or another.

First and foremost, I want to express my thanks to my thesis advisor Jan van Neerven. I first met Jan when I attended his Applied Functional Analysis course in the first year of my master degree, and since this was the class I enjoyed the most, it was only natural that I do my thesis in a topic related to functional analysis under his supervision.

The frequency of our meetings ranged from once every three weeks to once every three days, but this turned out to be no problem at all: each and every session was very productive and it has been great working together. Due to this, I have learned a lot about the topic of stochastic integration in Banach spaces in the past year.

I would moreover like to thank Richard Kraaij for taking part in my thesis committee and Mark Veraar for a helpful discussion on tightness, and uniqueness of solutions for (real valued) stochastic differential equations.

Contents

Chapter 1

Introduction

Like so many things in life, a lot of the motivation for studying stochastic differential equations comes from money. One of the earliest works on the theory of stochastic processes can be traced back to the Ph.D. thesis of Louis Bachelier from 1900 [1], where he argued that the Paris stock market could be modeled like a random walk. Around the same time, Albert Einstein wrote his article [7] on the movement of pollen particles suspended in water, which first was observed by the biologist Robert Brown in 1827. He argued that the particles moved like a scaled random walk, due to ricocheting off of other particles in any random direction, with any random strength, at each small timestep. This random walk theory was a big deal, since at the time matter was assumed to be continuous instead of made up of small particles.

Going back to the example of mathematical finance, we assume, like Bachelier did in [1], that some asset follows the Brownian motion W. We now want to investigate different betting strategies. The following approach is called the *martingale transform*: given a partition $\mathcal{P}_n := \{t_0 = 0, t_1, \dots, t_n = T\}$ of $[0, T]$, we want to determine how much we want to invest in the asset at each time point t_k, this investing strategy is denoted by $\sigma(t)$. Our gains over the interval $[t_k, t_{k+1})$ will be

$$\sigma(t_k)(W(t_{k+1}) - W(t_k)).$$

Summing all our gains over the time interval $[0, T]$, we end up with

$$\sum_{k=0}^{n-1} \sigma(t_k)(W(t_{k+1}) - W(t_k)). \tag{1.1}$$

One such strategy is as follows. We buy the asset at time $t = 0$ for $W(0) = 0$. At each time point t_k, we want to increase our share with $t_{k-1} - t_k$, until we sell the asset either when the price hits some predetermined $a > 0$, so $W(t) = a$, or at time $t = 1$ we sell our shares anyway. In this case $\sigma(t) = t1_{(0,\tau\wedge 1)}(t)$, with τ the stopping time at $W(t) = a$. The result of this is in the following figure, where we have taken $a = \sqrt{\frac{2}{\pi}}$:

Now by taking the limit as $\text{mesh}(\mathcal{P}_n) \to 0$, we arrive exactly at the definition of the stochastic integral as first defined in the groundbreaking work of Kyoshi Itō in [15] and [16]. We then have the celebrated Itō isometry, given by

$$\mathbb{E}\Big|\int_0^T \sigma(s)\ dW(s)\Big|^2 = \mathbb{E}\int_0^T |\sigma(s)|^2\ ds.$$

The theory of Itō allows us to solve real-valued stochastic differential equations, of the form

$$dX(t) = b(t, X(t))\ dt + \sigma(t, X(t))\ dW(t),$$

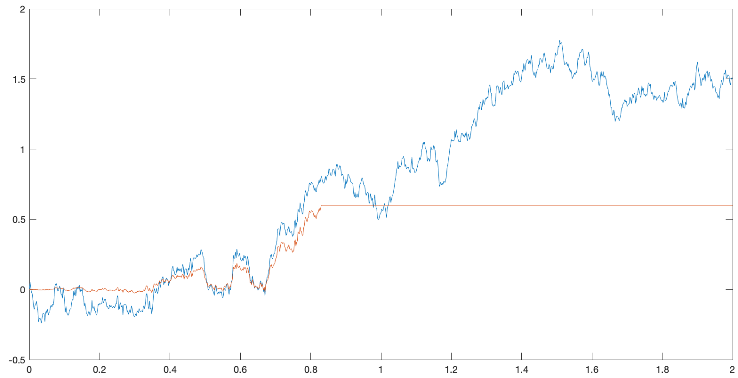


Figure 1.1: In blue our Browian motion, the red line is the result of the sum in (1.1). As can be seen, we have made some profit from our asset.

under specific conditions on b and σ. In the theory of stochastic partial differential equations (SPDEs) however, we work with equations which are, for example, of the form

$$\partial_t u(t,x) = -\Delta u(t,x) + \mathcal{W}, \quad t \geq 0, \quad x \in \mathbb{R}^d,$$

where $\mathcal{W}$ is space-time white noise. One way to deal with these problems is to transform it to a stochastic ODE, with values in some function space X. We view $U(t) = u(t,\cdot)$ as an element of X for each $t \geq 0$, and $A = -\Delta$ as an operator on X. The above equation then becomes

$$\frac{d}{dt}U(t) = AU(t) + \mathcal{W}(t).$$

In the case where X is a Hilbert space, we are in luck, as the Itō isometry can be readily extended to the Hilbert-valued case, see for example Theorem 4.23 in Da Prato-Zabcyk [6]. When X is a Banach space however, the inner product structure of the Hilbert space can obviously not be applied, and so a different approach is necessary.

One such approach is through the theory of γ-radonifying operators and UMD spaces. This approach was described in 2005 by Jan van Neerven, Mark Veraar and Lutz Weis in their article [25] and in its sequel (2008) [26]. The ideas in these articles however, date back to the work of McConnell (1989) [18], Rosinski and Suchanecki (1980) [33], Garling (1986) [8] and Montgomery-Smith (1998) [19].

A parallel theory, of stochastic integration in martingale type 2 Banach spaces, was developed by Neidhardt (1978) [30] and Brzezniak [3, 4]. We will discuss this approach in Chapter 7.

The UMD approach from [25, 26] assumes that the Banach space X has the *UMD property*. We will explain in detail what this is in Chapter 2.5, examples of such spaces include L^p-spaces and Hilbert spaces. Now assume a Brownian motion W_H taking values in a separable real Hilbert space H, and a process $\Phi : (0,T) \times \Omega \to \mathscr{L}(H,X)$ such that $\Phi \in L^p(\Omega; \gamma(L^2(0,T;H),X))$. In this case we have the following Itō isometry-esque estimate

$$\mathbb{E}\|\Phi\|^p_{L^p(\Omega;\gamma(L^2(0,T;H),X)} \eqsim_p \mathbb{E}\Big\| \int_0^T \Phi(t)\; dW_H(t)\Big\|^p_X,$$

which we will state more rigorously in the Preliminaries section. The theory from [25] and [26] allows us to solve (in some sense) a large class of stochastic differential equations in a UMD

Banach space X,

$$dU(t) = [A + F(t, U(t))]U(t)\ dt + B(t, U(t))\ dW_H(t),$$

under some additional assumptions on A, B and F. We can now solve these equations in UMD spaces (see Section 5 for what exactly we mean by "solve"), however the non-UMD case has not yet been widely studied, mostly because the class of UMD spaces is very large and thus the UMD assumption is satisfied in most applications. An overview of the theory in type p, martingale type p and UMD spaces is given in the survey [23] by the same authors.

In some cases however, it is necessary to venture out of the UMD setting. An example of this is stochastic delay equations, these are stochastic differential equations which depend on the past of the process. An example is the equation

$$\begin{cases} dU(t) & = \left(\int_{[-h,0]} U(t+s)\ d\mu(s) \right)\ dt + dW(t) \\ U(t) & = u(t), \quad t \in [-h, 0] \end{cases}.$$

Here $\mu \in M[-h, 0]$, the space of all signed measures on $[-h, 0]$. As we want the dual pairing to make sense, we need $U(t) \in C[-h, 0]$ for all $t \geq 0$, however $C[-h, 0]$ is not a UMD space. The equation is solved in [20] using semigroup theory and using a weak**-integral.

An entirely different approach to stochastic integration in non-UMD spaces was introduced by Kalinichenko in 2018 in his articles [12] and [13]. In [12], the stochastic integral is constructed in a completely new way, under a strict assumption on the process we want to integrate, but in a general, separable Banach space X.

Using chaining techniques from Talagrand [34], Kalinichenko is able to prove in his Theorem 1 in [12] that for a process $\sigma : [0, T] \times \Omega \to \mathscr{L}(H, X)$, the following bound holds for all p

$$\left(\mathbb{E} \left\| \int_0^T \sigma(t)\ dW_H(t) \right\|_X^p \right)^{1/p} \lesssim_{p,\sigma,X} \sqrt{p}.$$

Subsequently, we are able to solve, due to Theorem 2 in [12], very basic equations in X in the weak sense,

$$dx_t = b(t, x_t)\ dt + \sigma(t, x_t)\ dW_H(t),$$

where we assume, among other things, that b and σ are bounded.

In this thesis we will take a close look at the proof of Theorem 1 and 2 from [12], where we repair some inaccuracies as we find them. The technical details, which Kalinichenko puts away in his Lemma 1 and 2 of the same article, are studied in Chapter 3. The estimate for the stochastic integral, Theorem 1 in [12], is then proven in Chapter 4. We will also see that in the Hilbert space case, the result already follows from the well-known theory in [6].

Chapter 5 contains the statement and proof of Theorem 2 from the same article, the proof follows a more or less standard approach by proving relative compactness and then extracting a convergent subsequence. In this section, we will also generalize a result on the stochastic abstract Cauchy problem from [28], which is a new result.

Chapter 6 looks at [12] from a different point of view, namely we assume X to be a UMD$^-$ space, and we find that under this assumption the proof of Theorem 1 can be simplified a lot. In this section we also construct an example of a process that does not satisfy the assumptions from Kalinichenko's article, but is stochastically integrable any way. From this we can conclude that Theorem 1 is not an 'if and only if' situation. Moreover, we will make an explicit connection between the theory in [12] and the theory of γ-radonifying operators.

We continue this study in Chapter 7, where we look at the martingale type 2 case. In contrast to the UMD$^-$ case, the proof of Kalinichenko can not be simplified, and there are cases of functions that satisfy the conditions for stochastic integration in martingale type 2 spaces, but do not satisfy the conditions of Kalinichenko, and vice versa.

The first section here aims to not only stack definitions on top of each other, but tries to put the above theory in a context as well. Based on [23], [24], [25], we give an introduction to the field of stochastic integration in Banach spaces as it is mostly studied now. We treat Banach-valued Gaussian random variables, reproducing kernel Hilbert spaces, γ-radonifying operators and finally we define the stochastic integral in UMD spaces.

Chapter 2

Stochastic Integration in UMD spaces

In this section we will give some first preliminaries for understanding the proofs and concepts in [12]. We will introduce the concepts of Banach valued Gaussian random variables, γ-radonifying operators, and stochastic integration in (martingale) type p spaces. We assume knowledge on the level of a standard master course in functional analysis, and basic knowledge on real valued stochastic integration. The exposition in this section will primarily be based on the Internet Seminar notes [24], unpublished lecture notes (1997) by Jan van Neerven on the Feldman-Hajek theorem, and the survey article [23]. The subsection on UMD spaces is based on the article by Jan van Neerven, Mark Veraar and Lutz Weis [25].

2.1 Gaussian random variables in a Banach space

In this entire section, we assume $(\Omega, \mathcal{F}, \mathbb{P})$ to be a probability space and X a Banach space. We start with the theory for a Gaussian random variable in X. We introduce the concept of strong measurability, but before we do this we need the notion of a simple function taking values in X. Given a measurable space (A, Σ), we say that a function $f : (A, \Sigma) \to X$ is a Σ-simple function if it can be written as $f = \sum_{n=1}^{N} 1_{A_n} x_n$, where $A_n \in \Sigma$ and $x_n \in X$ for all $n = 1, \ldots, N$.

Definition 2.1. *Let (A, Σ) be a measurable space. A function $f : A \to X$ is called strongly Σ-measurable if there exists a sequence $(f_n)_{n \geq 1}$ of Σ-simple functions converging to f pointwise on A.*

Since this definition is not practical to work with, we usually check measurability via the Pettis measurability theorem:

Theorem 2.2 (Pettis measurability theorem)**.** *Let (A, Σ) be a measurable space and $f : A \to X$. The following are equivalent:*

1. *$f : A \to X$ is strongly Σ-measurable;*
2. *$\langle f, x^* \rangle$ is Σ-measurable for all $x^* \in X^*$.*

Definition 2.3. *A Borel measure μ on the Banach space X is called* Gaussian *if for each $x^* \in X^*$ the image measure defined by*

$$\langle \mu, x^* \rangle(A) := \mu\{x \in X : \langle x, x^* \rangle \in A\}, \quad A \subset \mathbb{R},$$

is a Gaussian measure on $\mathbb{R}$.

Now a strongly $\mathcal{F}$-measurable function $f : \Omega \to X$ is called a *Gaussian random variable* if for every $x^* \in X^*$, the real-valued random variable $\langle f, x^* \rangle$ is Gaussian.

The following theorem is an important one, which essentially states that any X-valued Gaussian random variable has finite p-th moments for all $p > 0$. We will also use it later on in this thesis.

Theorem 2.4 (Fernique). *Let f be an X-valued Gaussian random variable. Then there exists a $\beta > 0$ such that*

$$\mathbb{E}[\exp(\beta \|f\|_X^2)] < \infty.$$

As a corollary, since for every $p > 0$, there exists $\varepsilon > 0$ such that $\varepsilon x^p < \exp(\beta x^2)$ for all $x \geq 0$, we have $\mathbb{E}\|f\|_X^p < \infty$ for all $p > 0$.

Recall from finite-dimensional probability theory, that a d-dimensional Gaussian is uniquely determined by its mean vector and covariance matrix. We have something similar in the X-valued setting. In this case however, the matrix is replaced by an operator $Q : X^* \to X$ which serves a similar purpose. This operator is defined in the following way. If μ is the measure on X associated with the X-valued Gaussian random variable$f$, then for all $x^* \in X^*$,

$$Qx^* = \int_X \langle x, x^* \rangle x \; d\mu(x).$$

It follows from this definition that, if μ is centered as well, we have,

$$\mathbb{E}[\langle f, x^* \rangle^2] = \int_{\mathbb{R}} t^2 \; d\langle \mu, x^* \rangle(t) = \int_X \langle x, x^* \rangle^2 \; d\mu(x) = \langle Qx^*, x^* \rangle.$$

In fact the above equality uniquely determines the operator Q, as is seen in Proposition 2.6. Before we state the proposition, we need the following definitions:

Definition 2.5. *Let $T : X^* \to X$ be a bounded operator. We call T positive if for every $x^* \in X^*$,*

$$\langle Tx^*, x^* \rangle \geq 0.$$

We call T symmetric if for all $x^*, y^* \in X^*$ we have

$$\langle Tx^*, y^* \rangle = \langle Ty^*, x^* \rangle.$$

We now have the following proposition:

Proposition 2.6. *For an X-valued random variable f, the following assertions are equivalent*

1. *f is Gaussian;*
2. *there exists a positive, symmetric, bounded operator $Q : X^* \to X$ such that for all $x^* \in X^*$,*

$$\mathbb{E}[\exp(-i\langle f, x^* \rangle)] = \mathbb{E}[\exp(-\tfrac{1}{2}\langle Qx^*, x^* \rangle)]. \tag{2.1}$$

The operator Q is uniquely determined by Equation (2.1), moreover,

$$\mathbb{E}[\langle f, x^* \rangle^2] = \langle Qx^*, x^* \rangle, \quad x^* \in X^*.$$

We can now define the following bilinear form on the range QX^*:

$$[Qx^*, Qy^*] = \langle Qx^*, y^* \rangle.$$

This bilinear form is well-defined: if $Qy^* = 0$, then by the symmetry of Q, we have

$$[Qx^*, Qy^*] = \langle Qx^*, y^* \rangle = \langle x^*, Q^*y^* \rangle = \langle x^*, Qy^* \rangle = 0.$$

Moreover, $[\cdot, \cdot]$ is symmetric, since

$$[Qx^*, Qy^*] = \langle Qx^*, y^* \rangle = \langle x^*, Q^*y^* \rangle = \langle x^*, Qy^* \rangle = [Qy^*, Qx^*].$$

By positivity, $[Qx^*, Qx^*] = \langle Qx^*, x^* \rangle \geq 0$ for all $x^* \in X^*$. At last, assume $[Qx^*, Qx^*] = 0$. Then

$$0 = [Qx^*, Qx^*] = \langle Qx^*, x^* \rangle = \int_X \langle x, x^* \rangle^2 \, d\mu(x).$$

Thus, by the Cauchy-Schwarz inequality, we have for all $y^* \in X^*$

$$|\langle Qx^*, y^* \rangle| \leq \int_X |\langle x, x^* \rangle \langle x, y^* \rangle| \, d\mu(x) \leq \int_X \langle x, x^* \rangle^2 \, d\mu(x) \int_X \langle x, y^* \rangle^2 \, d\mu(x) = 0.$$

We can then use Hahn-Banach to conclude $Qx^* = 0$. So $[Qx^*, Qx^*] = 0$ implies $Qx^* = 0$. Thus $[\cdot, \cdot]$ defines an inner product on QX^*. By taking the closure of QX^* with respect to this inner product, we obtain a Hilbert space H_Q. We call H_Q the *reproducing kernel Hilbert space*.

This space has several properties which will be of interest later on. We have the following theorem:

Theorem 2.7. *The reproducing kernel Hilbert space H_Q has the following properties:*

1. *The inclusion $i : QX^* \to X$ extends to a compact embedding $i : H_Q \to X$;*
2. *As maps in $\mathscr{L}(X^*, H_Q)$ we have $Q = i^*$;*
3. *H_Q is separable.*

Proof. We first show that the inclusion map $i : QX^* \to X$ is continuous. For all $x^*, y^* \in X^*$ we have

$$\begin{aligned}\langle Qx^*, y^* \rangle^2 = \left(\int_X \langle x, x^* \rangle \langle x, y^* \rangle \, d\mu(x) \right)^2 &\leq \int_X \langle x, x^* \rangle^2 \, d\mu(x) \int_X \langle x, y^* \rangle^2 \, d\mu(x) \\ &\leq \|Q\| \, \|y^*\|_{X^*}^2 \|Qx^*\|_{H_Q}^2.\end{aligned}$$

Thus $i : QX^* \to X$ is continuous and can be extended to a bounded operator $i : H_Q \to X$ with $\|i\|_{\mathscr{L}(H_Q, X)} \leq \|Q\|^{1/2}$. Now we have for all $x^*, y^* \in X^*$

$$[Qx^*, i^*y^*]_{H_Q} = \langle iQx^*, y^* \rangle_X = \langle Qx^*, y^* \rangle = [Qx^*, Qy^*].$$

Since this equality holds for all $x^*, y^* \in X^*$, and QX^* is dense in H_Q, (2) follows. Moreover, Q is self-adjoint in the sense that

$$\langle Q^*x^*, y^* \rangle = \langle x^*, Qy^* \rangle = \int_X \langle x, x^* \rangle \langle x, y^* \rangle d\mu(x) = \langle Qx^*, y^* \rangle.$$

Since C^* has dense range in H, the operator i is injective. By (2), it is enough to show $Q : X^* \to H_Q$ is compact. Now let $\{x_n^*\}_{n\geq 1}$ be a bounded sequence in X^*. By the separability of X, we know that $\{x_n^*\}_{n\geq 1}$ has a weak*-convergent subsequence $\{x_{n_j}\}_{j\geq 1}$ converging to some $x^* \in X^*$. By taking $y_{n_j}^* = x_{n_j}^* - x^*$ we can, without loss of generality, assume that $x_{n_j}^*$ weak*-converges to 0. Now we have

$$\lim_{j\to 0} \|Qx_{n_j}^*\|_{H_Q}^2 = \int_X \langle x, x_{n_j}^* \rangle \; d\mu(x) = 0.$$

In the last step we have used the dominated convergence theorem, which is allowed due to Fernique's theorem. From this, (1) follows.

Now from (2) it follows that Q is weak*-to-weak continuous as an operator from X^* to H_Q. Since the unit ball B_{X^*} is separable in the weak*-topology in X^*, it follows that

$$QX^* = \bigcup_{n\geq 1} n \cdot QB_{X^*}$$

is weakly separable in H_Q, and by the Hahn-Banach theorem it is then strongly separable. Since QX^* is dense in H_Q, the separability of H_Q follows, thus proving (3). $\square$

In this way, we can write every covariance operator Q as the product of an inclusion map and its adjoint, we have $Q = ii^*$. As it turns out, the operator i is a γ-radonifying operator, which will be the key objects in stochastic integration.

2.2 Yor's counterexample

To see that stochastic integration in general separable Banach spaces requires more machinery than the real-valued case, and in particular the concept of γ-radonifying operators, we turn to the counterexample of Marc Yor [35]. This counterexample builds a function φ on a real, separable Banach space (namely ℓ^1) such that φ is bounded, but not stochastically integrable.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space with $(M_t)_{t\geq 0}$ a real-valued $\mathcal{F}_t$-martingale. For a real, separable Banach space X we let $\varphi : [0,T]\times\Omega \to X$ be an adapted process, and we say that φ is stochastically integrable with respect to M if there exists a process $\Phi : [0,T]\times\Omega \to X$ such that

$$\langle \Phi(t), x^* \rangle = \int_0^t \langle \varphi(s), x^* \rangle \; dM_s, \quad \text{a.s.}$$

For our counterexample, we take $X = \ell^1$, fix $T > 0$, and let

$$\varphi(s) := \left(1_{[\frac{T}{n+1},\frac{T}{n})}(s)\right)_{n\geq 1}.$$

Note that φ in fact maps into ℓ^1. Now let $(W_t)_{t\geq 0}$ be a real-valued Brownian motion and consider the integral

$$\Phi(t) := \int_0^t \varphi(s) \; dW_s. \tag{2.2}$$

Now computing Φ gives

$$\Phi(t) = \left(W_{\frac{T}{n}} - W_{\frac{T}{n+1}}\right)_{n\geq 1} \stackrel{d}{=} (X_n)_{n\geq 1},$$

with X_n a sequence of independent Gaussian with distribution $N\big(0, T(\frac{1}{(n+1)n})\big)$, in other words $X_n = \sqrt{\frac{1}{(n+1)n}} Y_n$ for some sequence of independent standard Gaussians. Now choose any $x > 2$.

We know

$$\frac{e^{-\frac{x^2}{2}}}{6x} \leq \mathbb{P}(Y_n \geq x) = \mathbb{P}\left(X_n \geq x\sqrt{\frac{1}{(n+1)n}}\right).$$

We can thus use Borel-Cantelli to conclude that $X_n \geq 3\sqrt{\frac{1}{(n+1)n}}$ infinitely often almost surely. Thus $(X_n)_{n\geq 1} \notin \ell^1$, so φ is not integrable in ℓ^1.

2.3 γ-Radonifying operators

As before, X is a separable Banach space, and in this case we let H be a separable Hilbert space. If $h \in H$ and $x \in X$, we write $h \otimes x$ for the operator in $\mathscr{L}(H, X)$ defined by

$$(h \otimes x)h' = [h, h']x \in X.$$

We say that an operator in $\mathscr{L}(H, X)$ is of finite rank if it is a finite linear combination of operators of the above form. We denote the set of finite rank operators in $\mathscr{L}(H, X)$ by $H \otimes X$. Now assume that an operator $T : H \to X$ has the form

$$T = \sum_{n=1}^{N} h_n \otimes x_n,$$

with the $h_1, \ldots, h_N$ orthonormal and $x_1, \ldots, x_N \in X$ arbitrary. Note that we can always write a finite rank operator T in this way, by using Gram-Schmidt orthonormalization. We now define, for $1 \leq p < \infty$ and $\gamma_1, \ldots, \gamma_N$ a Gaussian sequence, the following norm

$$\|T\|^p_{\gamma_p(H,X)} = \mathbb{E}\Big\| \sum_{n=1}^{N} \gamma_n x_n \Big\|^p_X.$$

Note that by the Kahane-Khintchine inequality, for all $1 \leq p < \infty$, the above norms are equivalent. Moreover, this norm does not depend on the choice of the orthonormal system $(h_n)_{n=1}^N$ and thus is well-defined. This can be seen in the following way. Let $(h_n)_{n=1}^N$ and $(h'_n)_{n=1}^N$ be the orthonormal systems in H, and $(x_n)_{n=1}^N$ and $(x'_n)_{n=1}^N$ be two sequences in X such that

$$T = \sum_{n=1}^{N} h_n \otimes x_n = \sum_{n=1}^{N} h'_n \otimes x'_n.$$

Now let $(\gamma_n)_{n=1}^N$ be a Gaussian sequence and set $\Gamma = \sum_{n=1}^N \gamma_n h_n$. We can then write

$$\|T\|^p_{\gamma_p(H,X)} = \mathbb{E}\|T\Gamma\|^p_X.$$

Now if $(Y_1, \ldots Y_N)$ is a Gaussian vector in $\mathbb{R}^N$ with covariance matrix $K : \mathbb{R}^N \to \mathbb{R}^N$ and $r : \mathbb{R}^N \to \mathbb{R}^N$ is a rotation, then $(rY_1, \ldots, rY_N)$ is also a Gaussian vector in $\mathbb{R}^N$ with covariance matrix rK. In other words, Gaussian distributions on $\mathbb{R}^N$ are rotationally invariant. Thus we have for another Gaussian sequence $(\gamma'_n)_{n=1}^N$ that Γ is equal in distribution to $\Gamma' = \sum_{n=1}^N \gamma'_n h'_n$. But then

$$\|T\|^p_{\gamma_p(H,X)} = \mathbb{E}\|T\Gamma\|^p_X = \mathbb{E}\|T\Gamma'\|^p_X = \mathbb{E}\Big\| \sum_{n=1}^{N} \gamma'_n x'_n \Big\|^p_X.$$

Thus the norm $\|\cdot\|_{\gamma_p(H,X)}$ is well-defined. The space of *γ-radonifying operators*, denoted by $\gamma(H, X)$, is now defined as the closure of $H \otimes X$ with respect to the norm $\|\cdot\|_{\gamma(H,X)} := \|\cdot\|_{\gamma_2(H,X)}$. Each $T \in \gamma(H, X)$ is compact, moreover, the γ-radonifying operators have the following ideal property:

Theorem 2.8. *If H' is another separable Hilbert space and X' another separable Banach space, and we have $S \in \mathscr{L}(H', H)$, $T \in \gamma(H, X)$ and $U \in \mathscr{L}(X, X')$, then $STU \in \gamma(H', X')$ and*

$$\|STU\|_{\gamma(H',X')} \leq \|S\|\|T\|_{\gamma(H,X)}\|U\|.$$

We are now ready to state the following theorem, which draws the connection between radonifying operators and covariance operators. We let $(\gamma_n)_{n\geq 1}$ be a Gaussian sequence.

Theorem 2.9. *Let $Q \in \mathscr{L}(X^*, X)$ and $R \in \mathscr{L}(H, X)$ such that $Q = RR^*$. The following are equivalent:*

1. *Q is the covariance operator of an X-valued Gaussian random variable;*
2. *$R \in \gamma(H, X)$.*

In this situation, if f is an X-valued Gaussian with covariance Q, we have

$$\mathbb{E}\|f\|_X^p = \|R\|_{\gamma_p(H,X)}^p.$$

As we have seen before, if f is an X-valued Gaussian with covariance Q and reproducing kernel space H_Q, we can write $Q = ii^*$ with $i : H_Q \to X$ the embedding. Thus we are in the above setting, and $i \in \gamma(H_Q, X)$. Moreover,

$$\mathbb{E}\|f\|_X^2 = \|i\|_{\gamma(H_Q,X)}^2.$$

Finally, we will state a domination result, which is Theorem 9.4.1 in [9].

Theorem 2.10. *Let H_1 and H_2 be separable Hilbert spaces and let $R_1 \in \mathscr{L}(H_1, X)$ and $R_2 \in \mathscr{L}(H_2, X)$ be such that for all $x^* \in X^*$,*

$$\|R_1^* x^*\|_{H_1} \leq \|R_2^* x^*\|_{H_2}.$$

Then $R_2 \in \gamma(H_2, X)$ implies $R_1 \in \gamma(H_1, X)$ and for $1 \leq p < \infty$,

$$\|R_1\|_{\gamma_p(H_1,X)} \leq \|R_2\|_{\gamma_p(H_2,X)}.$$

The flexibility in the above theorem of allowing two distinct Hilbert spaces, lets us prove statements like the following. Let f_1, f_2 be two X-valued Gaussians with covariances Q_1 and Q_2, such that for all $x^* \in X^*$,

$$\mathbb{E}\langle f_1, x^*\rangle^2 \leq \mathbb{E}\langle f_2, x^*\rangle^2. \tag{2.3}$$

Write for $j = 1, 2$, H_j for its reproducing kernel space and $i_j : H_j \to X$ for the embedding in X. We know that for $j = 1, 2$,

$$\mathbb{E}\langle f_j, x^*\rangle^2 = \langle Qx^*, x^*\rangle = \langle i_j i_j^* x^*, x^*\rangle = \|i_j^* x^*\|_{H_j}^2.$$

Thus by Equation (2.3) we have $\|i_1^* x^*\|_{H_j}^2 \leq \|i_2^* x^*\|_{H_j}^2$, and we can use Theorem 2.10 to obtain $\|i_1\|_{\gamma(H_1,X)} \leq \|i_2\|_{\gamma(H_2,X)}$. Now by the final remark in Theorem 2.9 we find that (2) implies

$$\mathbb{E}\|f_1\|_X^2 \leq \mathbb{E}\|f_2\|_X^2.$$

We are now ready to move on to stochastic integration, where we think of functions taking values in the space $\gamma(H, X)$ as the objects to integrate.

2.4 Stochastic integration in Banach spaces

We start by defining our integrator, dW, in the Banach space setting. As it turns out, it is useful to do this via a detour through a separable Hilbert space H.

Definition 2.11. *For a separable Hilbert space H, an H-isonormal process is a bounded linear mapping $W : H \to L^2(\Omega)$ with the following properties:*

1. *for all $h \in H$, the random variable Wh is Gaussian;*
2. *for all $h_1, h_2 \in H$, we have $\mathbb{E}[Wh_1 Wh_2] = (h_1, h_2)_H$.*

We can show that this operator W can be constructed for any separable Hilbert space H. To this end, let $(\eta_n)_{n\geq 1}$ be a sequence of i.i.d. standard Gaussians and let $(h_n)_{n\geq 1}$ be an ONB for H. For all $n \geq 1$, we set $Wh_n = \eta_n$ and we extend linearly, so for $h = \sum_{n\geq 1} k_n h_n$, we have $Wh = \sum_{n\geq 1} k_n \eta_n$. By Parseval $\sum_{n\geq 1} k_n^2 < \infty$, thus Wh converges in $L^2(\Omega)$ to a Gaussian random variable with variance $\sum_{n\geq 1} k_n^2$. Now for a second element $h' = \sum_{n\geq 1} k_n' h_n$ we have

$$\mathbb{E}\Big[\Big(\sum_{n=1}^{N} \eta_n k_n\Big)\Big(\sum_{m=1}^{N} \eta_m k_m'\Big)\Big] = \sum_{n,m=1}^{N} k_n k_m' \mathbb{E}[\eta_n \eta_m] = \sum_{n=1}^{N} k_n k_m'.$$

The converge again holds in $L^2(\Omega)$. Thus $W : H \to L^2(\Omega)$ is well-defined and satisfies 1 and 2.

The object against which we will integrate in our setting will be an H-cylindrical Brownian motion:

Definition 2.12. *An H-cylindrical Brownian motion W is an $L^2(\mathbb{R}_+; H)$-isonormal process.*

If W_H is an H-cylindrical Brownian motion, we will denote for any $h \in H$,

$$W_H(t)h := W(1_{[0,t]} \otimes h).$$

We can explicitly construct such an H-cylindrical Brownian motion in the following way. If $(W^{(n)})_{n\geq 1}$ is a sequence of independent Brownian motions on $\mathbb{R}_+$, and $(h_n)_{n\geq 1}$ an orthonormal basis of H. Then we can define the following operator, for each $h \in H$:

$$W_H(t)h := \sum_{n\geq 1} W^{(n)}(t)(h, h_n)_H.$$

Then W_H is an H-cylindrical Brownian motion. The convergence in $L^2(\Omega)$ for fixed t can be proven in the same way as above. This construction also ties in with our finite-dimensional intuition. After all, a d-dimensional standard Brownian motion is just a d-dimensional vector with an independent Brownian motion for each component. In the Hilbert space setting we have done the same thing, but then along all of its infinite dimensions.

We can start defining the stochastic integral for deterministic, finite rank step functions $\Phi : [0, T] \to H \otimes X$, i.e., Φ is a linear combination of functions of the form $1_{(s,t]} h \otimes x$, where $h \in H$ and $x \in X$. Then we set

$$\int_0^T 1_{(s,t]}(h \otimes x)\, dW_H := W_H(1_{(s,t]} h)x,$$

and we extend this linearly. Note that since $W_H(1_{(s,t]} h)$ is a real-valued Gaussian, the stochastic integral of Φ with respect to W_H is also an X-valued Gaussian for every finite rank step function

$\Phi : [0,T] \to H \otimes X$. Associated with each simple $\Phi : [0,T] \to H \otimes X$ is a γ-radonifying operator $R_\Phi : L^2(0,T;H) \to X$, which acts on functions $f \in L^2(0,T;H)$ in the following way:

$$R_\Phi f = \int_0^T \Phi(t) f(t)\ dt.$$

We have in this case the following Itō isometry:

Proposition 2.13 (Itō isometry). *For any separable Banach space X, and each finite rank simple function $\Phi : [0,T] \to H \otimes X$, we have for all $1 \leq p < \infty$,*

$$\mathbb{E}\Big\| \int_0^T \Phi(t)\ dW_H(t) \Big\|_X^p = \|R_\Phi\|^p_{\gamma_p(L^2(0,T;H),X)}.$$

Thus for each $1 \leq p < \infty$ we can extend the integral operator $J_T^{W_H} : \Phi \mapsto \int_0^T \Phi\ dW_H$ uniquely to an isometry $J_T^{W_H} : \gamma_p(L^2(0,T;H),X) \to L^p(\Omega;X)$. We can thus meaningfully define the stochastic integral for any operator $R \in \gamma(L^2(0,T;H),X)$. However we do not know which functions $\Phi : [0,T] \to \mathscr{L}(H,X)$ have such an operator $R_\Phi \in \gamma(L^2(0,T;H),X)$ associated to it. We define this in the following way:

Definition 2.14. *We say that a function $\Phi : [0,T] \to \mathscr{L}(H,X)$ is stochastically integrable with respect to W_H if there exists a sequence of finite rank step functions $\Phi_n : [0,T] \to H \otimes X$ such that*

1. *$\lim_{n\to\infty} \Phi_n h = \Phi h$ in measure;*
2. *there exists an X-valued random variable $\mathcal{I}_T$ such that*

$$\lim_{n\to\infty} \int_0^T \Phi_n(t)\ dW_H(t) = \mathcal{I}_T \text{ in probability.}$$

The stochastic integral of Φ is then defined as

$$\int_0^T \Phi(t)\ dW_H(t) := \lim_{n\to\infty} \int_0^T \Phi_n(t)\ dW_H(t) \text{ in probability.}$$

Before we state equivalent conditions for stochastic integrability, we first need a new definition.

Definition 2.15. *We say that a function $\Phi : [0,T] \to \mathscr{L}(H,X)$ is H-strongly measurable if for all $h \in H$, $\Phi h : [0,T] \to X$ is strongly measurable.*

We now have the following theorem, which is Theorem 6.17 in [24]. Here we have added a fourth condition, as is done in the proof in [24], which neatly ties into the theory of Kalinichenko [12].

Theorem 2.16. *For an H-strongly measurable function $\Phi : [0,T] \to \mathscr{L}(H,X)$ the following are equivalent:*

1. *Φ is stochastically integrable with respect to W_H;*
2. *$\Phi^* x^* \in L^2(0,T;H)$ for all $x^* \in X^*$, and there exists an X-valued random variable $\mathcal{I}_T$ such that for all $x^* \in X^*$, almost surely we have*

$$\langle \mathcal{I}_T, x^* \rangle = \int_0^T \Phi^* x^*\ dW_H.$$

3. $\Phi^* x^* \in L^2(0,T;H)$ *for all* $x^* \in X^*$, *and there exists an operator* $R_\Phi \in \gamma(L^2(0,T;H),X)$ *such that for all* $f \in L^2(0,T;H)$ *and* $x^* \in X^*$ *we have*

$$\langle Rf, x^*\rangle = \int_0^T \langle \Phi(t)f(t), x^*\rangle \, dt.$$

4. $\Phi^* x^* \in L^2(0,T;H)$ *for all* $x^* \in X^*$ *and there exists a* γ*-radonifying operator* $\widetilde{R}$ *from a Hilbert space* $\widetilde{H}$ *to* X *such that for all* $x^* \in X^*$

$$\|\Phi^* x^*\|^2_{L^2(0,T;H)} \leq \|\widetilde{R}^* x^*\|^2_{\widetilde{H}}.$$

If these equivalent conditions are satisfied, the random variable $\mathcal{I}_T$ *and the operator* R_Φ *are uniquely determined with* $\mathcal{I}_T = \int_0^T \Phi \, dW_H$ *almost surely and*

$$\mathbb{E}\Big\| \int_0^T \Phi(t) \, dW_H(t) \Big\|_X^p = \|R_\Phi\|^p_{\gamma_p(L^2(0,T;H),X)}.$$

We note here that (4) implies the existence of an X-valued Gaussian random variable g such that for all $x^* \in X^*$,

$$\int_0^T \|\Phi(t)^* x^*\|_H^2 \, dt \leq \mathbb{E}\langle g, x^*\rangle^2. \tag{2.4}$$

Here the covariance of g is given by $\widetilde{Q} := \widetilde{R}\widetilde{R}^* : X^* \to X$. On the other hand, if there exists a Gaussian $g : \Omega \to X$ with covariance $Q : X^* \to X$ such that Equation (2.4) holds, we can simply take $\widetilde{H} = H_Q$ with $\widetilde{R} = i : H_Q \to X$. In this case (4) follows.

2.5 Stochastic integration in UMD spaces

The definition of the stochastic integral can be extended naturally for stochastic finite rank simple functions $\Phi : [0,T] \times \Omega \to \mathcal{L}(H,X)$. Let $\Phi := 1_{(s,t]\times F} h \otimes x$, with $F \in \mathcal{F}_s$. We define

$$\int_0^T \Phi(t) \, dW_H(t) := 1_F W_H(1_{(s,t]} h)x. \tag{2.5}$$

Unfortunately trying to adapt the above proofs to the setting where $\Phi : [0,T] \times \Omega \to \mathcal{L}(H,X)$ is stochastic, is hopeless. The proof of the theorems in the previous subsection rely heavily on the fact that for functions of the form $\Phi := 1_{(s,t]} h \otimes x$, where $h \in H$ and $x \in X$, the integral

$$\int_0^T 1_{(s,t]}(h \otimes x) \, dW_H := W_H(1_{(s,t]} h)x$$

is an X-valued Gaussian random variable. If we take $\Phi = 1_{(s,t]\times F} h \otimes x$, with $F \in \mathcal{F}_s$, as above, the stochastic integral will have the form of 2.5. But as we can see, this is not Gaussian anymore. What we preferably want to do, is evaluate the stochastic integral for each $\omega \in \Omega$ independently, in this sense we want to look at each stochastic integral of $\Phi(\cdot, \widetilde{\omega}) : [0,T] \to \mathscr{L}(H,X)$ for any fixed $\widetilde{\omega} \in \Omega$. In this case, we go from a stochastic process to a deterministic one, and our previous theory would be directly applicable.

For this type of integration to make sense, we need the notion of a UMD space, where UMD stands for "unconditional martingale differences". To make this precise, we first need the definition of a Banach valued martingale. Let I be a partially ordered set. A family of σ-algebras $\{\mathcal{F}_i : i \in I\}$ is called a filtration if $\mathcal{F}_i \subset \mathcal{F}_j$ for all $i \leq j$. A family $\{M_i : i \in I\}$ of X-valued random variables is called *adapted to the filtration* $\{\mathcal{F}_i : i \in I\}$ if M_i is strongly $\mathcal{F}_i$-measurable for all $i \in I$.

Definition 2.17. *A family $(M_i)_{i\in I}$ of X-valued random variables is called an X-valued martingale with respect to a filtration $(\mathcal{F}_i)_{i\in}$ if it is adapted to $(\mathcal{F}_i)_{i\in I}$ and*

$$\mathbb{E}[M_j|\mathcal{F}_i] = M_i,$$

whenever $i \leq j$. If, in addition the above, we have for some $1 < p < \infty$ that $\mathbb{E}\|M_i\|^p < \infty$ for all $i \in I$, then we call M an L^p-martingale.

Now let $(M_n)_{n=1}^N$ be an X-valued martingale. The sequence $(d_n)_{n=1}^N$ defined by

$$d_n := M_n - M_{n-1},$$

where we take $M_0 = 0$, is called the martingale difference sequence associated with $(M_n)_{n=1}^N$. We now state the definition of a UMD space:

Definition 2.18. *A Banach space X is called a UMD space if for some (equivalently, for all) $1 < p < \infty$, there exists a constant $\beta_{p,X} > 0$ such that for all martingale difference sequences $(d_n)_{n=1}^N$ in $L^p(\Omega; X)$ and every $\{-1, 1\}$-valued sequence $(\varepsilon_n)_{n=1}^N$ we have*

$$\left(\mathbb{E}\Big\|\sum_{n=1}^N \varepsilon_n d_n\Big\|_X^p\right)^{1/p} \leq \beta_{p,X}\left(\mathbb{E}\Big\|\sum_{n=1}^N d_n\Big\|_X^p\right)^{1/p}.$$

Examples of UMD spaces include all Hilbert spaces, and $L^p(S)$ spaces for $1 < p < \infty$ and $(S, \mathcal{A}, \mu)$ a σ-finite measure space.

The usefulness of the UMD assumption becomes clear with the next proposition. Before we state it, we let $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ be an independent copy of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. On this second probability space we define another H-cylindrical Brownian motion $\widetilde{W}_H$ adapted to the filtration $\{\widetilde{\mathcal{F}}_t\}_{t\geq 0}$. Now let $\Phi : [0, T] \times \Omega \to H \otimes X$ have the following form

$$\Phi(t, \omega) = \sum_{n=1}^N \sum_{m=1}^M 1_{(t_{n-1}, t_n] \times F_{mn}}(t, \omega) \sum_{k=1}^K h_k \otimes x_{kmn},$$

where $0 = t_0 < \ldots < t_N = T$, the sets F_{mn} are disjoint and in $\mathcal{F}_{t_{n-1}}$ for each $n \geq 1$, and the vectors $(h_k)_{k\geq 1}$ are orthonormal in H. We define the *decoupled integral* of Φ with respect to $\widetilde{W}_H$ in the following way:

$$\int_0^T \Phi(t, \omega)\, d\widetilde{W}_H(t, \widetilde{\omega}) = \sum_{n=1}^N \sum_{m=1}^M 1_{F_{mn}}(\omega) \sum_{k=1}^K (\widetilde{W}_H(t_n, \widetilde{\omega})h_k - \widetilde{W}_H(t_{n-1}, \widetilde{\omega})h_k)x_{kmn}.$$

This defines an element of $L^p(\Omega; L^p(\widetilde{\Omega}; X))$, and this decoupling essentially turns the stochastic integral of an adapted process into the stochastic integral of a deterministic one. The definition of a UMD space has been carefully chosen so this integral can be estimated by the integral with respect to W_H, as is shown in the following proposition.

Proposition 2.19. *Let H be a separable Hilbert space and fix $p \in (1, \infty)$. The following assertions are equivalent:*

1. *X is a UMD space;*
2. *For every finite rank step function $\Phi : [0,T] \times \Omega \to \mathscr{L}(H,X)$ we have*

$$\beta_{p,X}^{-p} \mathbb{E}\widetilde{\mathbb{E}} \Big\| \int_0^T \Phi(t)\; d\widetilde{W}_H(t) \Big\|_X^p \leq \mathbb{E} \Big\| \int_0^T \Phi(t)\; dW_H(t) \Big\|_X^p \leq \beta_{p,X}^p \mathbb{E}\widetilde{\mathbb{E}} \Big\| \int_0^T \Phi(t)\; d\widetilde{W}_H(t) \Big\|_X^p.$$

We can now use the above proposition to obtain a similar Itō isometry as before, but this time for stochastic processes under the assumption that X is UMD. Assume that the function $\Phi : [0,T] \times \Omega \to H \otimes X$ is a finite rank step function such that the associated operator $R_\Phi(\omega) : L^2(0,T;H) \to X$, defined by

$$R_\Phi(\omega) f = \int_0^T \Phi(t,\omega) f(t)\; dt, \quad f \in L^2(0,T;H),$$

is in $\gamma(L^2(0,T;H),X)$ for all $\omega \in \Omega$. Moreover we assume that for some $1 < p < \infty$ we have $R_\Phi \in L^p(\Omega; \gamma(L^2(0,T;H),X))$. We now define the random variable $I^{W_H}(R_\Phi)$ by

$$I^{W_H}(R_\Phi) = \int_0^T \Phi(t)\; dW_H(t).$$

We have $I^{W_H}(R_\Phi) \in L_0^p(\Omega, \mathcal{F}_T; X)$, the subspace of $L^p(\Omega; X)$ consisting of all mean-zero, $\mathcal{F}_T$-measurable random variables. Now write $L_{\mathcal{F}}^p(\Omega; \gamma(L^2(0,T;H),X))$ for the closure of the finite rank, adapted simple functions in $L^p(\Omega; \gamma(L^2(0,T;H),X))$. We have the following theorem, which is one of the main results of [25]:

Theorem 2.20. *Let X be a UMD space and $1 < p < \infty$. The mapping $I^{W_H}(R_\Phi)$ has a unique extension to a bounded operator*

$$I^{W_H} : L_{\mathcal{F}}^p(\Omega; \gamma(L^2(0,T;H),X)) \to L_0^p(\Omega, \mathcal{F}_T; X).$$

This operator is an isomorphism onto its range and we have the two-sided estimate:

$$\beta_{p,X}^{-p} \|R_\Phi\|_{L^p(\Omega;\gamma(L^2(0,T;H),X))}^p \leq \|I^{W_H}(R_\Phi)\|_{L^p(\Omega;X)} \leq \beta_{p,X}^p \|R_\Phi\|_{L^p(\Omega;\gamma(L^2(0,T;H),X))}^p.$$

If $\{\mathcal{F}_t\}_{t\geq 0}$ is the augmented Brownian filtration, that is, the filtration generated by $W_H(t)$, then we have an isomorphism of Banach spaces

$$I^{W_H} : L_{\mathcal{F}}^p(\Omega; \gamma(L^2(0,T;H),X)) \eqsim L_0^p(\Omega, \mathcal{F}_T; X).$$

Finally, we state the main theorem for stochastic integration in UMD spaces:

Theorem 2.21. *Let X be a UMD space and $1 < p < \infty$. Assume that $\Phi : [0,T] \times \Omega \to \mathscr{L}(H,X)$ is H-strongly measurable and for all $x^* \in X^*$, we have $\Phi^* x^* \in L^2(0,T;H)$. Then the following assertions are equivalent:*

1. *There exists a sequence $(\Phi_n)_{n\geq 1}$ of elementary adapted processes such that:*
   (a) *for all $h \in H$ and $x^* \in X^*$ we have $\lim_{n\to\infty} \langle \Phi_n h, x^* \rangle = \langle \Phi h, x^* \rangle$ in measure on $[0,T] \times \Omega$.*
   (b) *there exists a strongly measurable random variable $\eta \in L^p(\Omega; X)$ such that*

$$\eta = \lim_{n\to\infty} \int_0^T \Phi_n(t)\; dW_H(t).$$

2. *There exists a strongly measurable random variable $\eta \in L^p(\Omega; X)$ such that for all $x^* \in X^*$ we have*
$$\langle \eta, x^* \rangle = \int_0^T \Phi(t)^* x^* \; dW_H(t).$$

3. *Φ represents an element $R_\Phi \in L^p(\Omega; \gamma(L^2(0,T;H), X))$.*

4. *For almost all $\omega \in \Omega$, the function $\Phi_\omega := \Phi(\cdot, \omega)$ is stochastically integrable with respect to an independent H-cylindrical Brownian motion $\widetilde{W}_H$, and*
$$\omega \mapsto \int_0^T \Phi(t,\omega) \; d\widetilde{W}_H(t) \in L^p(\Omega; L^p(\widetilde{\Omega}; X)).$$

The first two assertions are equivalent even in the non-UMD setting, provided the process Φ is adapted to the augmented filtration of W_H. We will need this later on. The implication (1)$\Rightarrow$(2) is obvious. We will here prove the implication (2)$\Rightarrow$(1). For $K \geq 1$ we let $\mathcal{F}_T^{(K)}$ be the σ-algebra generated by the $W_H(T)h_k$, for $1 \leq k \leq K$. Now let $\eta \in L^p(\Omega, \mathcal{F}_T; X)$ be as above, i.e. for all $x^* \in X^*$ we have
$$\langle \eta, x^* \rangle = \int_0^T \Phi(t)^* x^* \; dW_H(t).$$
Now fix $\varepsilon > 0$. By the martingale convergence theorem, and since the augmented filtration of W_H is defined to be the smallest σ-algebra consisting of all $\mathcal{F}_T^{(K)}$ for $K \geq 1$, we know
$$\lim_{K\to\infty} \mathbb{E}[\eta | \mathcal{F}_T^{(K)}] = \mathbb{E}[\eta|\mathcal{F}_T] = \eta,$$
where the convergence holds both almost surely and in $L^p(\Omega; X)$. Thus we can find $M \geq 1$ such that
$$\left\| \mathbb{E}[\eta|\mathcal{F}_T^{(M)}] - \eta \right\|_{L^p(\Omega;X)}^p < \frac{\varepsilon}{2}.$$
Now by density and since η has zero mean, we can find a mean zero simple random variable $\eta^{(M)} \in L^p(\Omega, \mathcal{F}_T^{(M)}; X)$ such that
$$\left\| \eta^{(M)} - \mathbb{E}[\eta|\mathcal{F}_T^{(M)}] \right\|_{L^p(\Omega;X)}^p < \frac{\varepsilon}{2}.$$
Putting the above together we obtain $\|\eta - \eta^{(M)}\|_{L^p(\Omega;X)} < \varepsilon$. Since $\eta^{(M)}$ is a simple function, it takes values only in a finite dimensional subspace of $W \subset X$. By the martingale representation theorem, we can then write for some adapted $\Phi^{(M)} : (0,T) \to \mathscr{L}(\mathrm{span}\{h_1, \ldots, h_k\}, W)$,
$$\eta^{(M)} = \sum_{k=1}^{M} \int_0^T \Phi^{(M)}(t) h_k \; dW_H(t) h_k = \int_0^T \Phi^{(M)}(t) \; dW_H(t).$$
In the last equality we have extended $\Phi^{(M)}$ to all of H by setting it zero on $\mathrm{span}\{h_1, \ldots, h_k\}^\perp$. Note that $\Phi^{(M)}$ need not be elementary, and in this case we can approximate $\Phi^{(M)}$ by a simple function $\widetilde{\Phi}^{(M)}$ such that
$$\Big\| \int_0^T \Phi^{(M)}(t) \; dW_H(t) - \int_0^T \widetilde{\Phi}^{(M)}(t) \; dW_H(t) \Big\|_{L^p(\Omega;X)}^p < \varepsilon.$$

We now have

$$\left\|\eta - \int_0^T \widetilde{\Phi}^{(M)}(t)\ dW_H(t)\right\|_{L^p(\Omega;X)}^p \leq \|\eta - \eta^{(M)}\|_{L^p(\Omega;X)}^p + \left\|\eta^{(M)} - \int_0^T \widetilde{\Phi}^{(M)}(t)\ dW_H(t)\right\|_{L^p(\Omega;X)}^p < 2\varepsilon.$$

This proves the (b) part of Assumption 1 in Theorem 2.21. Now the (a) part of the assumption follows by the Burkholder-Davis-Gundy inequalities.

Chapter 3

Kalinichenko's Lemma 1 and 2

In this section we will carefully reproduce the proofs of Lemma 1 and 2 in [12], and in some cases repair the proof or add details. Before we begin this section, we need the notion of a Gaussian process.

3.1 Gaussian processes on a metric space

Let (T, d) be a metric space. We define a Gaussian process on T in the following way:

Definition 3.1. *A collection $(g(t))_{t \in T}$ of Gaussian random variables is called a Gaussian process if for all $N \geq 1$ and $t_1, \ldots, t_N \in T$ we have that $(g(t_1), \ldots, g(t_N))$ is a Gaussian random variable in $\mathbb{R}^N$.*

In the proofs of [12], it is often useful to go back and forth between viewing g as a continuous Gaussian process on T and a $C(T)$-valued Gaussian random variable. This is permitted due to the following proposition.

Proposition 3.2. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{g(t) : t \in T\}$ be a collection of Gaussian random variables. Then g is a continuous Gaussian process on T if and only if g is a $C(T)$-valued Gaussian random variable.*

Proof. Assume g is a $C(T)$-valued Gaussian random variable. By the Gaussianity of g, we know that g has some covariance operator $K : C(T)^* \to C(T)$. Then, for each $t \in T$, the random variable $\langle g, \delta_t \rangle$ is a real valued Gaussian with mean zero and variance $\langle K\delta_t, \delta_t \rangle$. In particular, for any selection of points $t_1, \ldots, t_N$, the vector $(\langle g, \delta_{t_1} \rangle, \ldots, \langle g, \delta_{t_N} \rangle)$ is a Gaussian vector in $\mathbb{R}^N$: for all $1 \leq i, j \leq N$ we have $\mathbb{E}[\langle g, \delta_{t_i} \rangle \langle g, \delta_{t_j} \rangle] = \langle K\delta_{t_i}, \delta_{t_j} \rangle$, so its covariance matrix is given by $(\langle K\delta_{t_i}, \delta_{t_j} \rangle)_{1 \leq i,j \leq N}$. Now since

$$(g(t_1), \ldots, g(t_N)) = (\langle g, \delta_{t_1} \rangle, \ldots, \langle g, \delta_{t_N} \rangle),$$

g is a Gaussian process.

On the other hand, assume that g is a continuous Gaussian process on T. We first show that $g : \Omega \to C(T)$ is strongly $\mathcal{F}$-measurable by using the Pettis measurability theorem. Note that the set $F := \operatorname{span}\{\delta_t : t \in T\}$ is norming for $C(T)$. Moreover, since g is a continuous process, it takes values in $C(T)$, and thus is separably valued.

Fix $\omega \in \Omega$. Since g is continuous on T, we have $g(\cdot, \omega) \in C(T)$, thus it makes sense to look at $\langle g, \delta_t \rangle$. Now since $g(t) = \langle g, \delta_t \rangle$ is $\mathcal{F}$-measurable, the conditions of Pettis' measurability theorem are satisfied and we can conclude that g is strongly $\mathcal{F}$-measurable.

To prove the Gaussianity, we note that F is weak*-dense in $C(T)^* = M(T)$. For any $\mu \in M(T)$ we can find a sequence $(\mu_n)_{n\geq 1}$ in F such that $\langle f, \mu_n\rangle \to \langle f, \mu\rangle$ for all $f \in C(T)$. Thus $\langle g(\cdot, \omega), \mu_n\rangle \to \langle g(\cdot, \omega), \mu\rangle$ for all $\omega \in \Omega$, so $\langle g, \mu_n\rangle$ converges almost surely to $\langle g, \mu\rangle$. Since each $\langle g, \mu_n\rangle$ is Gaussian by the assumption, $\langle g, \mu\rangle$ is Gaussian for all $\mu \in M(T)$. Thus g is a $C(T)$-valued Gaussian random variable. □

We will also need the following lemma:

Lemma 3.3. *Let (T, d) be a compact metric space and $\{g(t) : t \in T\}$ a continuous Gaussian process. Then when considered as a function $g : (T, d) \to L^2(\Omega)$ is uniformly continuous. In other words, for each $\varepsilon > 0$, there exists $\delta > 0$ such that $d(s, t) < \delta$ implies $(\mathbb{E}|g(s)-g(t)|)^{1/2} < \varepsilon$.*

Proof. To show uniform continuity, we show that g is continuous in each point $(s \in T$. To this end, let $\{s_n\}$ be a sequence in T converging to s. Then $d(s_n, s) \to 0$. We show that $d_g(s_n, s) = \|g(s_n) - g(s)\|_{L^2(\Omega)} \to 0$. We know for almost all fixed $\omega \in \Omega$ we have by the continuity of g,

$$\lim_{n\to\infty} |g(s_n, \omega) - g(s, \omega)|^2 = 0.$$

Moreover, we have pointwise for $\omega \in \Omega$,

$$|g(s_n, \omega) - g(s, \omega)|^2 \leq \sup_{t\in T} |2g(t, \omega)|^2 = 4\left(\sup_{t\in T} |g(t, \omega)|\right)^2 = 4\|g(\cdot, \omega)\|^2_{C(T)}.$$

By Fernique's theorem, we have $\mathbb{E}\|g\|^2_{C(T)} < \infty$, so we can use the dominated convergence theorem. We have

$$\lim_{n\to\infty} \mathbb{E}|g(s_n) - g(s)|^2 = \mathbb{E}\left[\lim_{n\to\infty} |g(s_n, \omega) - g(s, \omega)|^2\right] = 0.$$

Now the uniform continuity follows from the pointwise continuity and the fact that (T, d) is compact. □

3.2 Kalinichenko's Lemma 1

Let (T, d) be a compact metric space, $(\Omega, \mathscr{F}, \mathbb{P})$ a probability space and $g : \Omega \to C(T)$ a Gaussian random variable. That is, g is a continuous Gaussian process indexed by T. Note that as a function $g : \Omega \times T \to \mathbb{R}$, we have that g is jointly measurable, since it is measurable with respect to $\mathscr{F}$ and continuous on T.

Now consider another metric space (X, ρ) and some measurable $x : \Omega \times T \to X$. We call the process $\{x(t) : t \in T\}$ subgaussian with respect to $\{g(t) : t \in T\}$ if the inequality

$$\mathbb{P}(\rho(x(s), x(t)) > u) \leq K \exp\left(-\frac{u^2}{2d_g(s, t)^2}\right), \tag{3.1}$$

holds for all $s, t \in T$. Here we define $d_g : T \times T \to \mathbb{R}_{\geq 0}$ as the pseudometric given by

$$d_g(s, t) := \left(\mathbb{E}|g(s) - g(t)|^2\right)^{1/2}.$$

Lemma 3.4 (Lemma 1 in Kalinichenko [12])**.** *Let $\{x(t) : t \in T\}$ be subgaussian with respect to $\{g(t) : t \in T\}$, i.e. (3.1) holds. Then x has a continuous version and for every $\varepsilon > 0$ there is $\delta > 0$ depending only on ε and g such that for all $p \geq 1$*

$$\left(\mathbb{E} \sup_{d(s,t)<\delta} \rho(x(s), x(t))^p\right)^{1/p} \leq K^{1/p}\sqrt{p}\varepsilon.$$

Moreover, for some universal constant L,

$$\left(\mathbb{E}\sup_{s,t\in T}\rho(x(s),x(t))^p\right)^{1/p}\le LK^{1/p}\sqrt{p}\cdot\mathbb{E}\sup_{t\in T}g(t).$$

Before proving this lemma, we first need a key definition and theorem from Talagrand [34].

Definition 3.5 (Definition 2.2.17 in Talagrand [34])**.** *Given a set T and a sequence of partitions $(\mathcal{A}_n)$ of T, we call $(\mathcal{A}_n)$ an admissible sequence if $\#\mathcal{A}_n\le 2^{2^n}$ and $(\mathcal{A}_n)$ is increasing, meaning that for every $A\in\mathcal{A}_{n+1}$, there exists a $B\in\mathcal{A}_n$ such that $A\subseteq B$.*

Let $\mathcal{A}:=\{\mathcal{A}_n\}_{n=0}^\infty$ be an increasing sequence of partitions of T, then every set of $\mathcal{A}_{n+1}$ is contained in a set of $\mathcal{A}_n$, such that $\mathcal{A}_n=\{A_n^1,\dots,A_n^{k_n}\}$ with $k_n=1,\dots,2^{2^n}$ for all $n\ge 1$, and $k_0=1$. For any $t\in T$ and $n\ge 0$, let $A_n(t)$ be the unique element of $\mathcal{A}_n$ containing t. Using the above definition, we can define the quantity

$$\gamma_{\mathcal{A}}(T,d_g):=\sup_{t\in T}\sum_{n\ge 0}2^{n/2}\sup_{s,r\in A_n(t)}d_g(s,r).$$

We can now state the following theorem:

Theorem 3.6 (Theorem 2.4.1 in Talagrand [34])**.** *Consider a real-valued Gaussian process $\{g(t):t\in T\}$, that is, a collection of jointly Gaussian random variables indexed by T. Assume that $\mathbb{E}\sup_{t\in T}g(t)<\infty$ and equip T with the pseudometric d_g. Then there exists some universal constant $L>0$ such that*

$$\frac{1}{L}\inf_{\mathcal{A}}\gamma_{\mathcal{A}}(T,d_g)\le\mathbb{E}\sup_{t\in T}g(t)\le L\inf_{\mathcal{A}}\gamma_{\mathcal{A}}(T,d_g).$$

The infimum here is taken over all admissible sequences.

Note that the supremum of $g(t)$ over all of T might not be measurable. Talagrand fixes this by defining

$$\mathbb{E}\sup_{t\in T}g(t):=\sup\{\mathbb{E}\sup_{t\in\widetilde{T}}g(t):\widetilde{T}\subset T\text{ is finite}\}.$$

Whenever $\sup_{t\in T}g(t)$ is measurable, so in particular when g is continuous on (T,d') with d' any metric on T, then the above agrees with our usual definition of $\mathbb{E}\sup_{t\in T}g(t)$.

Proof of Lemma 3.4. We first show that $\{x(t):t\in T\}$ has a continuous version. To this end, we first note that since g is continuous on (T,d), we actually have that g is a $C(T)$-valued Gaussian, so by Fernique, $\mathbb{E}\|g\|_{C(T)}<\infty$. Thus we have

$$\mathbb{E}\|g\|_{C(T)}=\mathbb{E}\sup_{t\in T}|g(t)|\ge\mathbb{E}\sup_{t\in T}g(t).$$

Here the sup is measurable due to the continuity of g. We are now in a position to use Theorem 3.6 to obtain an admissible sequence $(\mathcal{A}_n)_{n\ge 0}$ such that for each $N\in\mathbb{N}$, the following quantity is finite:

$$\gamma_{\mathcal{A}}(T,d_g):=\sup_{t\in T}\sum_{n\ge 0}2^{n/2}\sup_{s,r\in A_n(t)}d_g(s,r).$$

We now denote the tail of the above by

$$\gamma_{\mathcal{A}}(N):=\sup_{t\in T}\sum_{n\ge N}2^{n/2}\sup_{s,r\in A_n(t)}d_g(s,r).$$

Fix a finite set $\widetilde{T} \subset T$ and for every $n \geq 0$ we construct T_n by taking one element in each $\widetilde{T} \cap A_n^j$, whenever this intersection is non-empty. Fix also $u > 4$ and $n \geq 1$ we define the event

$$B_n := \{\exists s \in T_{n-1}, \exists t \in T_n : \rho(x(s), x(t)) > u2^{n/2} d_g(s,t)\},$$

and set, for any $N \in \mathbb{N}$, $B(N) := \bigcup_{n \geq N} B_n$. Note that these events are indeed measurable, since each B_n is the finite union of measurable events, and B is a countable union. We have for each $n \geq 1$

$$\begin{aligned}
\mathbb{P}(B_n) &= \mathbb{P}\left(\bigcup_{s \in T_{n-1}} \bigcup_{t \in T_n} \{\rho(x(s), x(t)) > u2^{n/2} d_g(s,t)\}\right) \\
&\leq \sum_{s \in T_{n-1}} \sum_{t \in T_n} \mathbb{P}\left(\rho(x(s), x(t)) > u2^{n/2} d_g(s,t)\right) \\
&\leq \sum_{s \in T_{n-1}} \sum_{t \in T_n} \exp\left(-\frac{u^2 d_g(s,t)^2 2^n}{2 d_g(s,t)^2}\right) \\
&\leq 2^{2^{n-1}} \cdot 2^{2^n} \cdot \exp(-u^2 2^{n-1}).
\end{aligned}$$

Now we know $2^{2^{n-1}} \cdot 2^{2^n} \leq 2^{2^{n+1}}$. Moreover, by the estimate in Equation (3.1), we have

$$\mathbb{P}(B_n) \leq 2^{2^{n+1}} K \exp(-u^2 2^{n-1}) = K \exp\left(-\frac{2^n(u^2-4)}{2}\right)\left(\frac{2}{e}\right)^{2^{n+1}} \leq K \exp\left(-\frac{u^2}{4}\right)\left(\frac{2}{e}\right)^{2^{n+1}}.$$

Then,

$$\begin{aligned}
\mathbb{P}(B(N)) = \mathbb{P}\left(\bigcup_{n \geq N} B_n\right) \leq \sum_{n \geq N} \mathbb{P}(B_n) &= K \exp\left(-\frac{u^2}{4}\right) \sum_{n \geq N} \left(\frac{2}{e}\right)^{2^{n+1}} \\
&\leq K \exp\left(-\frac{u^2}{4}\right) \sum_{n \geq 2^{N+1}} \left(\frac{2}{e}\right)^n \\
&\leq K \exp\left(-\frac{u^2}{4}\right)\left(\frac{2}{e}\right)^{2^{N+1}} \sum_{n \geq 0} \left(\frac{2}{e}\right)^n \\
&= K \frac{\left(\frac{2}{e}\right)^{2^{N+1}}}{1 - \frac{2}{e}} \exp\left(-\frac{u^2}{4}\right).
\end{aligned}$$

In a similar way we define for any $N \in \mathbb{N}$ the set

$$C(N) := \{\exists s, t \in T_N : \rho(x(s), x(t)) > u2^{N/2} d_g(s,t)\},$$

and we immediately see similarly as for B_n and inequality (3.1) that

$$\mathbb{P}(C(N)) \leq 2^{2^{N+1}} \mathbb{P}\left(\rho(x(s), x(t)) > u2^{N/2} d_g(s,t)\right) \leq K \exp\left(-\frac{u^2}{4}\right)\left(\frac{2}{e}\right)^{2^{N+1}}.$$

Define moreover the (measurable) event

$$D := \{\exists s, t \in \widetilde{T} : \rho(x(s), x(t)) \neq 0, d_g(s,t) = 0\}.$$

By (3.1) we have $\mathbb{P}(D)=0$. If we denote $F(N) := B(N) \cup C(N) \cup D$, then

$$\begin{aligned}\mathbb{P}(F(N)) &\leq \mathbb{P}(B(N)) + \mathbb{P}(C(N)) + \mathbb{P}(D) \\ &\leq K\frac{\left(\frac{2}{e}\right)^{2^{N+1}}}{1-\frac{2}{e}} \exp\left(-\frac{u^2}{4}\right) + K\exp\left(-\frac{u^2}{4}\right)\left(\frac{2}{e}\right)^{2^{N+1}} \leq \widetilde{K}\exp\left(-\frac{u^2}{4}\right),\end{aligned}$$

where we explicitly choose $\widetilde{K} \geq K \geq 1$. We now show that on $\complement F(N)$, for any $s,t \in \widetilde{T}$, the distance between $x(s)$ and $x(t)$ can be bounded by $\gamma_{\mathcal{A}}(N)$. Recall the construction of T_n as above, where we chose one element in each $\widetilde{T} \cap A_n^j$. Let $\pi_n(t)$ be the unique element of T_n in $A_n(t)$, if this exists. Since the following uniform convergence holds on (T, d_g),

$$\sum_{n\geq 1} 2^{n/2} \sup_{s,r\in A_n(t)} d_g(s,r) < \infty,$$

the terms converge uniformly to zero as $n \to \infty$. This can be seen in the following way. For any $N \geq 1$ we have

$$\begin{aligned}\sup_{t\in T}\left|2^{N/2}\sup_{s,r\in A_N(t)} d_g(s,r)\right| &= \sup_{t\in T}\left|\sum_{n\geq N} 2^{n/2}\sup_{s,r\in A_n(t)} d_g(s,r) - \sum_{n\geq N+1} 2^{n/2}\sup_{s,r\in A_n(t)} d_g(s,r)\right| \\ &\leq \sup_{t\in T}\left(\left|\sum_{n\geq N} 2^{n/2}\sup_{s,r\in A_n(t)} d_g(s,r)\right| + \left|\sum_{n\geq N+1} 2^{n/2}\sup_{s,r\in A_n(t)} d_g(s,r)\right|\right) \\ &\leq \sup_{t\in T}\sum_{n\geq N} 2^{n/2}\sup_{s,r\in A_n(t)} d_g(s,r) + \sup_{t\in T}\sum_{n\geq N+1} 2^{n/2}\sup_{s,r\in A_n(t)} d_g(s,r) \\ &= \gamma_{\mathcal{A}}(N) + \gamma_{\mathcal{A}}(N+1) \to 0.\end{aligned}$$

In the second inequality we have omitted the absolute values, since we are summing only non-negative terms. The convergence in the end holds since the quantity $\gamma_{\mathcal{A}}(T, d_g)$ is finite, so its tails converge to zero. It immediately follows that also

$$\sup_{t\in T}\left[\sup_{s,r\in A_n(t)} d_g(s,r)\right] \to 0,$$

as $n \to \infty$. Since $\widetilde{T}$ is finite, there is some $M \in \mathbb{N}$ such that for all $t \in \widetilde{T}$, we have $d_g(\pi_n(t), t) = 0$ for all $n \geq M$. Since we are on the complement of D, it follows that $\rho(x(\pi_n(t)), x(t)) = 0$ as well for all $n \geq M$. Thus, for any $N \geq 1$,

$$\rho(x(s), x(t)) \leq \rho(x(s), x(\pi_N(s))) + \rho(x(\pi_N(s)), x(\pi_N(t))) + \rho(x(\pi_N(t)), x(t)).$$

Expanding the first term on the RHS gives, for all $N \geq 1$,

$$\begin{aligned}\rho(x(s), x(\pi_N(s))) &\leq \rho(x(s), x(\pi_{N+1}(s))) + \rho(x(\pi_{N+1}(s)), x(\pi_N(s))) \\ &\leq \rho(x(s), x(\pi_{N+2}(s))) + \rho(x(\pi_{N+2}(s)), x(\pi_{N+1}(s))) + \rho(x(\pi_{N+1}(s)), x(\pi_N(s))).\end{aligned}$$

Continuing like this, we have for $s,t \in \widetilde{T}$

$$\rho(x(s), x(\pi_N(s))) \leq \rho(x(s), x(\pi_M(s))) + \sum_{n=N}^{M-1} \rho(x(\pi_{n+1}(s)), x(\pi_n(s))). \tag{3.2}$$

Now, for all $n \geq M$ we have $\rho(x(s), x(\pi_n(s))) = 0$, so for all $n \geq M$,

$$\rho(x(\pi_n(s)), x(\pi_{n+1}(s))) \leq \rho(x(\pi_n(s)), x(s)) + \rho(x(s), x(\pi_{n+1}(s))) = 0 + 0 = 0.$$

Hence we can actually replace the finite sum in Equation (3.2) by the infinite sum. Moreover, since $\rho(x(s), x(\pi_M(s))) = 0$, Equation (3.2) becomes

$$\rho(x(s), x(\pi_N(s))) \leq \sum_{n \geq N} \rho(x(\pi_{n+1}(s)), x(\pi_n(s))).$$

On the next page there is a sketch of what is happening here.

Doing the same thing for $\rho(x(\pi_N(t)), x(t))$, we can write

$$\begin{aligned}
\rho(x(s), x(t)) &\leq \sum_{n \geq N} \rho(x(\pi_{n+1}(s)), x(\pi_n(s))) + \rho(x(\pi_N(s)), x(\pi_N(t))) + \sum_{n \geq N} \rho(x(\pi_{n+1}(t)), x(\pi_n(t))) \\
&\leq u\Bigg(\sum_{n \geq N} 2^{n/2} d_g(\pi_{n+1}(s), \pi_n(s)) + 2^{N/2} d_g(\pi_N(s), \pi_N(t)) \\
&+ \sum_{n \geq N} 2^{n/2} d_g(\pi_{n+1}(t), \pi_n(t)) \Bigg) =: u(\mathrm{I} + \mathrm{II} + \mathrm{III}).
\end{aligned}$$

Here the second inequality holds specifically because we are on $\complement F(N)$. We have for all $t \in \widetilde{T}$

$$d_g(\pi_N(t), t) \leq \sup_{s,r \in A_N(t)} d_g(s, r) = 2^{-N/2} 2^{N/2} \sup_{s,r \in A_N(t)} d_g(s, r) \leq 2^{-N/2} \gamma_{\mathcal{A}}(N),$$

so

$$\begin{aligned}
(\mathrm{II}) = 2^{N/2} d_g(\pi_N(s), \pi_N(t)) &\leq 2^{N/2}(d_g(\pi_N(s), s) + d_g(s, t) + d_g(t, \pi_N(t))) \\
&\leq 2^{N/2}(d_g(s, t) + 2 \cdot 2^{-N/2} \gamma_{\mathcal{A}}(N)) = 2^{N/2} d_g(s, t) + 2\gamma_{\mathcal{A}}(N).
\end{aligned}$$

Moreover, for each $t \in \widetilde{T}$ we have (recall $A_{n+1}(t) \subset A_n(t)$),

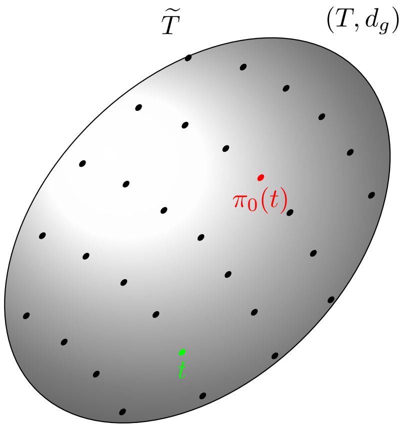
$$\begin{aligned}
d_g(\pi_{n+1}(t), \pi_n(t)) &\leq d_g(\pi_{n+1}(t), t) + d_g(t, \pi_n(t)) \\
&\leq \sup_{s,r \in A_{n+1}(t)} d_g(s, r) + \sup_{s,r \in A_n(t)} d_g(s, r) \leq 2 \sup_{s,r \in A_n(t)} d_g(s, r).
\end{aligned}$$

So it follows that for all $N \geq 1$

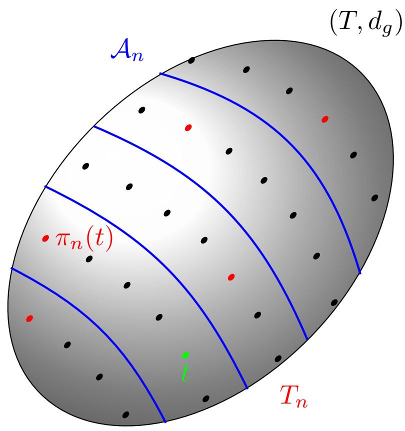
$$(\mathrm{III}) = \sum_{n \geq N} 2^{n/2} d_g(\pi_{n+1}(t), \pi_n(t)) \leq 2 \sum_{n \geq N} 2^{n/2} \sup_{s,r \in A_n(t)} d_g(s, r) = 2\gamma_{\mathcal{A}}(N).$$

Similarly we have (I) $\leq 2\gamma_{\mathcal{A}}(N)$. Putting all of the above together, we obtain the following bound for all $N \geq 1$, and $s, t \in \widetilde{T}$,

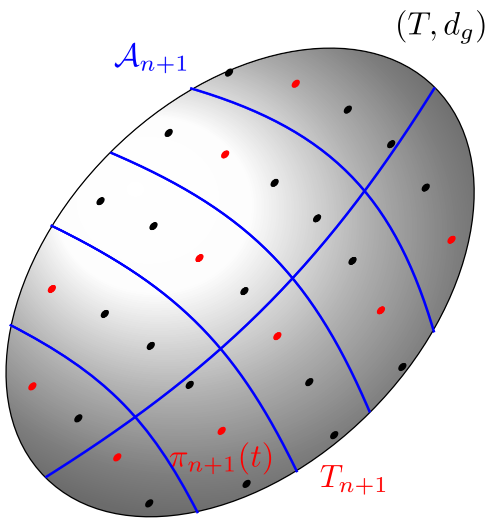
$$\rho(x(s), x(t)) \leq u\Big(\overbrace{2^{N/2} d_g(s, t) + 2\gamma_{\mathcal{A}}(N)}^{\mathrm{II}} + \overbrace{2\gamma_{\mathcal{A}}(N)}^{\mathrm{I}} + \overbrace{2\gamma_{\mathcal{A}}(N)}^{\mathrm{III}} \Big) = u\left(2^{N/2} d_g(s, t) + 6\gamma_{\mathcal{A}}(N)\right)).$$



(a) We approximate t (green) with the points $\pi_n(t)$ in $\widetilde{T} \cap A_n(t)$. When $n = 0$ we just have $A_n(t) = T$.



(b) The blue lines represent the borders of the sets of the partition $\mathcal{A}_n$.



(c) As can be seen in the above picture, we always eventually have $\pi_n(t) = t$ for some large n.

Now, from Lemma 3.3 we know that g is uniformly continuous when considered as function from (T,d) to $L^2(\Omega)$, hence, we can find a $\delta(N) > 0$ small enough such that $d(s,t) < \delta(N)$ implies $d_g(s,t) < 2^{-N/2}\gamma_{\mathcal{A}}(N)$. Now by setting $\beta(N) := 7\gamma_{\mathcal{A}}(N)$ and $v := \beta(N)u$, we have on $\complement F(N)$

$$\sup_{s,t\in\widetilde{T},d(s,t)<\delta(N)} \rho(x(s),x(t)) \leq v. \tag{3.3}$$

Using this fact, we can write

$$\mathbb{P}\left(\sup_{s,t\in\widetilde{T},d(s,t)<\delta(N)} \rho(x(s),x(t)) > v\right) \leq \mathbb{P}(F(N)) \leq \widetilde{K}\exp\left(-\frac{v^2}{4\beta(N)^2}\right). \tag{3.4}$$

Recall that we had chosen $u > 4$, so $v > 4\beta(N)$. Since $\delta(N), \widetilde{K}$ and $\beta(N)$ are independent of u, it follows from 3.4 that

$$\mathbb{P}\left(\sup_{s,t\in\widetilde{T},d(s,t)<\delta(N)} \rho(x(s),x(t)) > v\right) \leq \widetilde{K}\exp\left(-\frac{v^2}{4\beta(N)^2}\right), \quad \forall v \geq 4\beta(N).$$

We will also need the case where $v \leq 4\beta(N)$, in which case it holds that

$$\mathbb{P}\left(\sup_{s,t\in\widetilde{T},d(s,t)<\delta(N)} \rho(x(s),x(t)) > v\right) \leq 1 \leq \widetilde{K}\exp\left(\frac{(4\beta(N))^2}{4\beta(N)^2}\right)\exp\left(-\frac{v^2}{4\beta(N)^2}\right).$$

Thus for all $v \geq 0$ we can write

$$\mathbb{P}\left(\sup_{s,t\in\widetilde{T},d(s,t)<\delta(N)} \rho(x(s),x(t)) > v\right) \leq \widetilde{K}e^4\exp\left(-\frac{v^2}{4\beta(N)^2}\right).$$

We now want to extend this to a countable and dense subset $S \subset T$. To this end, we construct this set S of T in the following way: since T is compact in the d-metric, it is totally bounded. For each $n \geq 1$, we can cover T with a finite amount of balls with radius $\frac{1}{n}$ in the d-metric. Now let A_n be the centers of these balls and let $S_n := \bigcup_{k=1}^n A_k$. Then set $S := \bigcup_{n\geq 1} S_n$, so that S is countable and dense. We have by continuity of measures and since $\{S_n\}_{n\geq 1}$ is an increasing sequence of sets,

$$\begin{aligned}
\mathbb{P}\left(\sup_{s,t\in S,d(s,t)<\delta(N)} \rho(x(s),x(t)) > v\right) &= \mathbb{P}\left(\lim_{n\to\infty}\sup_{s,t\in S_n,d(s,t)<\delta(N)} \rho(x(s),x(t)) > v\right) \\
&= \lim_{n\to\infty}\mathbb{P}\left(\sup_{s,t\in S_n,d(s,t)<\delta(N)} \rho(x(s),x(t)) > v\right) \\
&\leq \widetilde{K}e^4\exp\left(-\frac{v^2}{4\beta(N)^2}\right) = \widetilde{K}e^4\exp\left(-\frac{v^2}{4(7\gamma_{\mathcal{A}}(N))^2}\right).
\end{aligned}$$

We can now choose $N \geq 1$ such that $\gamma_{\mathcal{A}}(N) < \varepsilon$. Then by choosing the appropriate $\delta(N) > 0$ as described above,

$$\mathbb{P}\left(\sup_{s,t\in S,d(s,t)<\delta(N)} \rho(x(s),x(t)) > v\right) \leq \widetilde{K}e^4\exp\left(-\frac{v^2}{4(7\varepsilon)^2}\right), \quad \forall v \geq 0. \tag{3.5}$$

This implies that for all $n \in \mathbb{N}$, we can choose $\delta_n > 0$ such that

$$\mathbb{P}(E_n) := \mathbb{P}\left(\sup_{s,t\in S, d(s,t)<\delta_n} \rho(x(s), x(t)) > 2^{-n}\right) \leq 2^{-n}.$$

Since $\sum_{n\geq 1} \mathbb{P}(E_n) < \infty$ we obtain by the Borel-Cantelli lemma that

$$\mathbb{P}\left(\left\{\sup_{s,t\in S, d(s,t)<\delta_n} \rho(x(s), x(t)) > 2^{-n}\right\} \text{ i.o.}\right) = 0.$$

This means that for large n, then for $s, t \in S$ we have $d(s,t) < \delta_n$ implies $\rho(x(s), x(t)) \leq 2^{-n}$. But this is exactly the definition of uniform continuity on S. We can thus continuously extend x to a process x' on the whole T. This process is a modification of the original process $\{x(t) : t \in T\}$, this can be seen in the following way. Let $t \in T$ and $\{t_n\}$ be a sequence in T converging to t. We have for any $\varepsilon > 0$,

$$\mathbb{P}(\rho(x'(t), x(t)) > \varepsilon) \leq \mathbb{P}(\rho(x'(t), x(t_n)) > \varepsilon/2) + \mathbb{P}(\rho(x(t_n), x(t)) > \varepsilon/2).$$

By definition of x', $x(t_n)$ converges to $x'(t)$ almost surely, so it also converges in probability. Thus, for any $\eta > 0$ we can choose n large enough so $\mathbb{P}(\rho(x'(t), x(t_n)) > \varepsilon/2) < \eta$. On the other hand, x is almost surely uniformly continuous, so almost surely $x(t_n) \to x(t)$. This converges also holds in probability, so we can choose an $n' \geq n$ such that $\mathbb{P}(\rho(x(t_n), x(t)) > \varepsilon/2) < \eta$. Then $\mathbb{P}(\rho(x'(t), x(t)) > \varepsilon) < \eta$ for all $\eta > 0$, so

$$\mathbb{P}(\rho(x'(t), x(t)) = 0) = 1, \quad \forall t \in T.$$

We will from now on denote x' simply by x again, and we have the estimate for $\delta := \delta(N)$ (by continuity of x and density of S in T)

$$\mathbb{P}\left(\sup_{s,t\in T, d(s,t)<\delta} \rho(x(s), x(t)) > v\right) \leq \widetilde{K} e^4 \exp\left(-\frac{v^2}{4(7\varepsilon)^2}\right).$$

Note that the above event is measurable, since by our construction of the modification of x, this event is exactly the event as in Equation (3.5). Define the modulus of continuity

$$w(\delta) := \sup_{s,t\in T, d(s,t)<\delta} \rho(x(s), x(t)) = \sup_{s,t\in S, d(s,t)<\delta} \rho(x(s), x(t)),$$

where we switch to a countable, dense subset to deal with any measurability issues. For convenience we now set $\xi := 7\varepsilon$. Then, integrating by parts,

$$\begin{aligned} \mathbb{E}[w(\delta)^p] &= \int_0^\infty \mathbb{P}(w(\delta) > v) p v^{p-1}\, dv \\ &\leq \widetilde{K} e^4 p \int_0^\infty \exp\left(-\frac{v^2}{4\xi^2}\right) v^{p-1}\, dv. \end{aligned}$$

We now substitute $z = \frac{v^2}{4\xi^2}$. Note that $\frac{dz}{dv} = \frac{v}{2\xi^2}$. Then

$$\begin{aligned} \int_0^\infty \exp\left(-\frac{v^2}{4\xi^2}\right) v^{p-1}\, dv &= \int_0^\infty \exp\left(-\frac{v^2}{4\xi^2}\right) v^{p-2} (2\xi^2) \left(\frac{v}{2\xi^2} dv\right) \\ &= \int_0^\infty e^{-z} (2\xi)^{p-2} z^{p/2-1} (2\xi^2)\, dz = 2^{p-1} \xi^p \Gamma(p/2). \end{aligned}$$

Plugging this back into the previous equation and using the estimate $(p\Gamma(p/2))^{1/p} \lesssim \sqrt{p}$, we obtain

$$(\mathbb{E}[w(\delta)^p])^{1/p} \lesssim \widetilde{K}^{1/p} p^{1/p} \xi (\Gamma(p/2))^{1/p} \lesssim \widetilde{K}^{1/p} \sqrt{p} \xi \lesssim \widetilde{K}^{1/p} \sqrt{p} \varepsilon.$$

This finishes the proof of the first assertion. For the second part, we have first of all

$$d_g(s,t) \leq \sup_{r,q \in T} d_g(r,q) \leq \gamma_{\mathcal{A}}(0).$$

Consequently, for any $\delta > 0$, in particular δ larger than the diameter of T, we have by Equation (3.3) on $\complement F(N)$,

$$\sup_{s,t \in T} \rho(x(s), x(t)) \leq 7\gamma_{\mathcal{A}}(0).$$

Note that taking $\delta > 0$ large and $N = 0$ does not change the proof of the first part, so we can write, similarly to Equation (3.5),

$$\mathbb{P}\left(\sup_{s,t \in S} \rho(x(s), x(t)) > v \right) \leq \widetilde{K} e^4 \exp\left(-\frac{v^2}{4(7\gamma_{\mathcal{A}}(0))^2} \right).$$

In the same way as before we obtain the estimate for the expectation,

$$\mathbb{E}\left(\sup_{s,t \in T} \rho(x(s), x(t))^p \right)^{1/p} \lesssim K^{1/p} \sqrt{p} \gamma_{\mathcal{A}}(0).$$

Since the LHS does not depend on our choice of $\mathcal{A}$, we can take the infimum:

$$\mathbb{E}\left(\sup_{s,t \in T} \rho(x(s), x(t))^p \right)^{1/p} \lesssim K^{1/p} \sqrt{p} \inf_{\mathcal{A}} \gamma_{\mathcal{A}}(0).$$

But by Theorem 3.6 we obtain an L such that $\inf_{\mathcal{A}} \gamma_{\mathcal{A}}(0) \leq L \mathbb{E} \sup_{t \in T} g(t)$, so

$$\mathbb{E}\left(\sup_{s,t \in T} \rho(x(s), x(t))^p \right)^{1/p} \lesssim L K^{1/p} \sqrt{p} \mathbb{E} \sup_{t \in T} g(t).$$

□

We will use this result to prove Lemma 3.7, which is Lemma 2 in [12].

3.3 Kalinichenko's Lemma 2

Lemma 3.7. *Let X be a separable Banach space. For $T > 0$ consider a collection of random variables $\{x_t(\phi) : \phi \in X^*, t \in [0,T]\}$, continuous in t for every fixed ϕ and linear in ϕ in the sense that*

$$x_t(\alpha\phi + \beta\psi) = \alpha x_t(\phi) + \beta x_t(\psi) \ \text{a.s.}, \quad \phi, \psi \in X^*, \ t \in [0,T], \ \alpha, \beta \in \mathbb{R}. \tag{3.6}$$

Suppose that for an X-valued centered Gaussian random variable g and a constant $K \geq 1$ the inequality

$$\mathbb{P}(\|x(\phi)\|_{C[0,T]} > u) \leq K \exp\left(-\frac{u^2}{2\mathbb{E}\langle g, \phi \rangle^2} \right)$$

holds for all $\phi \in X^*$. Then there exists an X-valued continuous process $\{y_t, t \in [0,T]\}$ such that

$$\langle y_t, \phi \rangle = x_t(\phi), \quad \forall \phi \in X^*, \forall t \in [0,T], \tag{3.7}$$

and

$$\left(\mathbb{E} \sup_{t \in [0,T]} \|y_t\|_X^p \right)^{1/p} \lesssim K^{1/p} \sqrt{p} \mathbb{E} \|g\|_X. \tag{3.8}$$

Before we prove the statement, we first introduce the following Theorem from Kallenberg [14], in which its proof can also be found.

Given another metric space (S, ρ), and S-valued, continuous processes X, and $X_1, X_2, \dots$ on T. Write $X_n \xrightarrow{uld} X$ for the convergence in distribution in $C(T,S)$ with respect to the locally uniform topology. We moreover define finite dimensional convergence in distribution $X_n \xrightarrow{fd} X$ by

$$(X^n_{t_1}, \dots, X^n_{t_m}) \xrightarrow{d} (X_{t_1}, \dots, X_{t_m}), \quad t_1, \dots, t_m \in T, \quad m \in \mathbb{N}.$$

Now let $\mathcal{K}_T$ be the collection of compact subsets of T. For $K \in \mathcal{K}_T$ and $h > 0$ we define the local modus of continuity for a function $x \in C(T,S)$

$$w_K(x,h) = \sup\{\rho(x_s, x_t) : s,t \in K, d(s,t) \le h\}, \quad h > 0, \quad K \in \mathcal{K}_T.$$

We can now state the following theorem:

Theorem 3.8 (Corollary 23.5 in Kallenberg [14])**.** *Let $X, X_1, X_2, \dots$ be continuous S-valued processes on T, where S and T are separable, complete metric spaces and T is locally compact with a dense subset $T' \in T$. Then $X_n \xrightarrow{uld} X$ iff*

1. *$X_n \xrightarrow{fd} X$ on T';*
2. *$\lim_{h \to 0} \limsup_{n \to \infty} \mathbb{E}[w_K(X_n, h) \wedge 1] = 0$.*

Using the above theorem, we are now ready to prove Lemma 3.7:

Proof of Lemma 3.7. The Banach-Mazur theorem states that X is isometrically isomorphic to a closed subspace $\widetilde{X} \subset C[0,1]$. In other words, there exists an isometric isomorphism $\iota : X \to \widetilde{X}$. Consider $\iota \circ g : \Omega \to \widetilde{X}$ and write $\widetilde{g} := \iota \circ g$. Then we can view $\widetilde{g}$ simply as a $C[0,1]$-valued Gaussian random variable, and look at the continuous process $\{\widetilde{g}(s) : s \in [0,1]\}$. The Gaussianity follows from the Ideal property for γ-radonifying operators.

Since $\widetilde{X} \subset C[0,1]$, each functional on $C[0,1]$ defines a functional on $\widetilde{X}$ through restriction, therefore the functionals δ_s, defined by the relation $\delta_s(f) := f(s)$, can be viewed as elements in $\widetilde{X}^*$.

Instead of x_t we will consider the process $\{\widetilde{x}_t(\phi) : \phi \in \widetilde{X}^*, t \in [0,T]\}$. Note that the continuity in t and the pointwise a.s. linearity holds for $\widetilde{x}$ as in the statement of the theorem, moreover the following inequality holds, similar to Equation (3.6),

$$\begin{aligned} \mathbb{P}(\|\widetilde{x}(\phi)\|_{C[0,T]} > u) = \mathbb{P}(\|x(\iota^* \phi)\|_{C[0,T]} > u) &\le K \exp\left(-\frac{u^2}{2\mathbb{E}\langle g, \iota^*\phi \rangle^2} \right) \\ &= K \exp\left(-\frac{u^2}{2\mathbb{E}\langle \iota \circ g, \phi \rangle^2} \right) \\ &= K \exp\left(-\frac{u^2}{2\mathbb{E}\langle \widetilde{g}, \phi \rangle^2} \right). \end{aligned}$$

Now for any $s \in [0,1]$, we denote $\widetilde{x}_t(s) := \widetilde{x}_t(\delta_s)$. We have by the linearity that $\widetilde{x}_t(s) - \widetilde{x}_t(s') = \widetilde{x}_t(\delta_s - \delta_{s'})$ almost surely, for all $t \in [0,T]$. In fact this holds for all $t \in [0,T]$ almost surely: this is because $\widetilde{x}_t(s) - \widetilde{x}_t(s')$ and $\widetilde{x}_t(\delta_s - \delta_{s'})$ are modifications of each other, but since they are both continuous in t by our assumption on x, they are indistinguishable. We have now, by the above inequality,

$$\begin{aligned}
\mathbb{P}(\|\widetilde{x}(s) - \widetilde{x}(s')\|_{C[0,T]} > u) &= \mathbb{P}(\|\widetilde{x}(\delta_s - \delta_{s'})\|_{C[0,T]} > u) \\
&\leq K \exp\left(-\frac{u^2}{2\mathbb{E}|\widetilde{g}(s) - \widetilde{g}(s')|^2}\right).
\end{aligned}$$

This implies that, when considered as a stochastic process $[0,1] \to C[0,T]$, then $\widetilde{x}$ is subgaussian with respect to the Gaussian process $\widetilde{g} : \Omega \times [0,1] \to \mathbb{R}$. We are in the setting of Lemma 1 and Equation (3.1), so in fact $\widetilde{x} : [0,1] \to C[0,T]$ has a continuous version, which we will denote by $\widetilde{y}_t(s)$. For this process, we can also use the second bound given in Lemma 3.4, for $p \geq 1$,

$$\left(\mathbb{E} \sup_{s,s' \in [0,1]} \|\widetilde{y}(s) - \widetilde{y}(s')\|^p_{C[0,T]}\right)^{1/p} \lesssim K^{1/p}\sqrt{p} \cdot \mathbb{E} \sup_{t \in [0,1]} \widetilde{g}(t), \tag{3.9}$$

where again the sup's are measurable in both cases, since $\widetilde{y}$ and g are continuous. We will now prove that $\widetilde{y}_t$ almost surely takes values in $\widetilde{X}$. We do this by a reproducing kernel Hilbert space construction. Since $\widetilde{g} : \Omega \to \widetilde{X}$ is Gaussian, we consider its covariance operator $Q : \widetilde{X}^* \to \widetilde{X}$, and H_Q, the reproducing kernel Hilbert space of $\widetilde{X}$ with respect to Q with inner product $[\cdot,\cdot]$. Now since H_Q is a separable Hilbert space, there exists some orthonormal basis $(h_k)_{k\geq 1}$ for H_Q, which is contained in the dense subspace $Q\widetilde{X}^*$. Denote $Qe_k = h_k$ where $(e_k)_{k\geq 1}$ is a sequence in $\widetilde{X}^*$. We define the finite rank projection operator $P_n : \widetilde{X} \to \widetilde{X}$ by

$$P_n x := \sum_{i=1}^n \langle x, e_i\rangle Qe_i, \quad x \in \widetilde{X}.$$

The adjoint of this operator is given by $P_n^* : \widetilde{X}^* \to \widetilde{X}^*$ with

$$P_n^*\phi = \sum_{i=1}^n \langle Qe_i, \phi\rangle e_i, \quad \phi \in \widetilde{X}^*.$$

Moreover,

$$QP_n^*\phi = \sum_{i=1}^n \langle Qe_i, \phi\rangle Qe_i = \sum_{i=1}^n [Q\phi, Qe_i]Qe_i, \quad \phi \in \widetilde{X}^*.$$

So in fact QP_n^* is just the projection of $Q\phi$ on the first n basis vectors in H_Q. We have by the above, $\|QP_n^*\phi\|_{H_Q} \leq \|Q\phi\|_{H_Q}$. We now consider for $n \geq 1$ processes $\widetilde{y}^n : [0,T] \times \Omega \to H$ given by

$$\widetilde{y}_t^n := \sum_{i=1}^n \widetilde{x}_t(e_i)Qe_i.$$

Note that these processes are measurable, and in fact they are continuous for each $n \geq 1$: each $\widetilde{x}_t(e_i)$ is measurable, and $\widetilde{y}_t^n$ is just a linear combination of these. We have

$$\langle \phi, \widetilde{y}_t^n\rangle = \sum_{i=1}^n \widetilde{x}_t(e_i)\langle \phi, Qe_i\rangle = \widetilde{x}_t\left(\sum_{i=1}^n \langle \phi, Qe_i\rangle e_i\right) = \widetilde{x}_t(P_n^*\phi).$$

By our RKHS construction,

$$\mathbb{E}\langle g, P_n^*\phi\rangle^2 = [QP_n^*\phi, QP_n^*\phi] = \|QP_n^*\phi\|_{H_Q}^2 \leq \|Q\phi\|_{H_Q}^2 = \mathbb{E}\langle g, \phi\rangle^2.$$

Now,

$$\begin{aligned}
\mathbb{P}(\|\widetilde{y}^n(s) - \widetilde{y}^n(s')\|_{C[0,T]} > u) &= \mathbb{P}(\|\widetilde{x}(P_n^*(\delta_s - \delta_{s'}))\|_{C[0,T]} > u) \\
&\leq K \exp\left(-\frac{u^2}{2\mathbb{E}\langle g, P_n^*(\delta_s - \delta_{s'})\rangle^2}\right) \\
&\leq K \exp\left(-\frac{u^2}{2\mathbb{E}\langle g, \delta_s - \delta_{s'}\rangle^2}\right) \\
&= K \exp\left(-\frac{u^2}{2\mathbb{E}|g(s) - g(s')|^2}\right),
\end{aligned}$$

where the RHS does not depend on n. We can now again use Lemma 3.4. Note that for each $n \geq$ and each $t \in [0, T]$, $\widetilde{y}^n(s) := \widetilde{y}_t^n(\delta_s)$ is already continuous in $s \in [0, 1]$ since it takes values in $\widetilde{X} \subset C[0, 1]$. Lemma 3.4 however also gives us a $\delta > 0$ such that

$$\mathbb{E}\left[\sup_{|s-s'|<\delta} \|\widetilde{y}^n(s) - \widetilde{y}^n(s')\|_{C[0,T]}\right] \leq \varepsilon, \quad \forall n \geq 1. \tag{3.10}$$

Now for any $s \in [0, 1]$ we have

$$\begin{aligned}
\mathbb{P}(\|\widetilde{y}(s) - \widetilde{y}^n(s)\|_{C[0,T]} > u) &= \mathbb{P}(\|x(\delta_s - P_n^*\delta_s)\|_{C[0,T]} > u) \\
&\leq K \exp\left(-\frac{u^2}{2\mathbb{E}\langle g, \delta_s - P_n^*\delta_s\rangle^2}\right) \\
&\leq K \exp\left(-\frac{u^2}{2\|\delta_s - P_n^*\delta_s\|_{H_Q}^2}\right) \to 0.
\end{aligned}$$

The convergence here is in n, since $\|\delta_s - P_n^*\delta_s\|_{H_Q} \to 0$ as $n \to \infty$. This means that $\widetilde{y}^n(s) \to \widetilde{y}(s)$ in probability. We will now use Corollary 23.5 from Kallenberg [14] to show convergence in distribution of $\widetilde{y}^n \to \widetilde{y}$ in $C([0, 1], C[0, T])$. Note first of all that $\widetilde{y} \in C([0, 1], C[0, T])$ and for any $n \geq 1$, $\widetilde{y}^n \in C([0, 1], C[0, T])$. Since for each $s \in [0, 1]$, $\widetilde{y}^n(s) \to \widetilde{y}(s)$ in probability, we have for any finite collection of points $\{s_1, \ldots, s_k\}$ that

$$(\widetilde{y}^n(s_1), \ldots, \widetilde{y}^n(s_k)) \to (\widetilde{y}(s_1), \ldots, \widetilde{y}(s_k)), \quad \text{as } n \to \infty$$

in probability. Therefore the convergence also holds in distribution. The first assumption from Corollary 23.5 is therefore satisfied. For the second assumption, note that

$$\mathbb{E}\left[\sup_{|s-s'|<\delta} \|\widetilde{y}^n(s) - \widetilde{y}^n(s')\|_{C[0,T]} \wedge 1\right] \leq \mathbb{E}\left[\sup_{|s-s'|<\delta} \|\widetilde{y}^n(s) - \widetilde{y}^n(s')\|_{C[0,T]}\right].$$

The latter converges to zero as $\delta \to 0$ by Equation (3.10), where we obtained a bound uniformly in $n \geq 1$. So the second assumption of 23.5 holds as well. Then $\widetilde{y}^n \to \widetilde{y}$ in $C([0, 1], C[0, T])$ in distribution. Since $C([0, 1], C[0, T]) \approx C([0, T], C[0, 1])$, the convergence in distribution also holds in this space. Since $\widetilde{y}^n \in C([0, T], \widetilde{X})$ almost surely for all $n \geq 1$, we have by the convergence in distribution,

$$1 = \lim_{n\to\infty} \mathbb{P}(\widetilde{y}^n \in C([0, T], \widetilde{X})) \leq \mathbb{P}(\widetilde{y} \in C([0, T], \widetilde{X})).$$

Thus $\widetilde{y}$ almost surely has paths in $C([0,T],\widetilde{X})$. The next step in the proof is now to show that $\widetilde{y}$ satisfies Equation (3.8). To this end, we have

$$\mathbb{P}(\|\widetilde{y}(0)\|_{C[0,T]} > v) = \mathbb{P}(\|x(\delta_0)\|_{C[0,T]} > v).$$

Write $\beta := \sqrt{\mathbb{E}|\widetilde{g}(0)|^2}$. Integrating by parts again gives

$$\begin{aligned}
\mathbb{E}\|y(0)\|^p_{C[0,T]} &= \int_0^\infty \mathbb{P}(\|y(0)\|_{C[0,T]} > v) p v^{p-1}\, dv \\
&\leq Kp\int_0^\infty \exp\left(-\frac{v^2}{2\beta^2}\right) v^{p-1}\, dv \\
&\leq Kp\int_0^\infty \exp\left(-\frac{v^2}{2\beta^2}\right) (2\beta^2)^{p/2-1}\left(\frac{v^2}{2\beta^2}\right)^{p/2-1} \beta^2\frac{v}{\beta^2}\, dv.
\end{aligned}$$

Now substituting $u = v^2/(2\beta^2)$, we find $du = \frac{v}{\beta^2}dv$ and we get that the above is equal to

$$Kp2^{p/2-1}\beta^p\int_0^\infty \exp(-u)u^{p/2-1}\, du = Kp2^{p/2-1}\beta^p\Gamma(p/2).$$

We can now take p-th roots to obtain, where we again note the estimate $p^{1/p}\Gamma(p/2) \leq \sqrt{p}$, resulting in

$$(\mathbb{E}\|\widetilde{y}(0)\|^p_{C[0,T]})^{1/p} \leq K^{1/p}\sqrt{p}\beta.$$

Since $\widetilde{g}(0)$ is Gaussian, we can write by Kahane-Khintchine,

$$\beta = \sqrt{\mathbb{E}\widetilde{g}(0)^2} \lesssim \mathbb{E}|\widetilde{g}(0)| \leq \mathbb{E}\|\widetilde{g}\|_{C[0,1]}.$$

Combined with Equation (3.9), we get

$$\begin{aligned}
\left\|\sup_{t\in[0,T]}\|\widetilde{y}_t\|_{C[0,1]}\right\|_p &= \left\|\sup_{s\in[0,1]}\|\widetilde{y}(s)\|_{C[0,T]}\right\|_p \\
&\leq \left\|\sup_{s,s'\in[0,1]}\|\widetilde{y}(s)-\widetilde{y}(s')\|_{C[0,T]}\right\|_p + \left\|\|\widetilde{y}(0)\|_{C[0,T]}\right\|_p \\
&\leq K^{1/p}\sqrt{p}\mathbb{E}\|\widetilde{g}\|_{C[0,1]}.
\end{aligned}$$

It remains to prove that for all $\phi \in \widetilde{X}^*$,

$$\langle\widetilde{y}_t, \phi\rangle = \widetilde{x}_t(\phi), \quad \forall t \in [0,T].$$

By construction, this already holds for $\phi = \delta_s$, $s \in [0,1]$, and thus for all functionals in the linear span L of $\{\delta_s, s \in [0,1]\}$. Now let $\widetilde{\phi} \in \widetilde{X}^*$. We can extend this by the Hahn-Banach extension theorem to a functional ϕ on all of $C[0,1]$. Now, each functional on $C[0,1]$ is a signed measure, which we can approximate with functionals in L in the weak* sense. We can do this in the following way. For a measure μ, we define the sequence

$$\mu_n(A) = \sum_{j=0}^{2^n-1}\mu([j2^n,(j+1)2^n))\delta_{j2^n}(A) + \mu(\{1\})\delta_1(A).$$

For every $f \in C[0,1]$ we have

$$\int_0^1 f(s)\, d\mu_n(s) = \sum_{j=0}^{2^n-1} f(j2^n)\mu([j2^n,(j+1)2^n)) + f(1)\mu(\{1\}) \to \int_0^1 f(s)\, ds,$$

since this is just a Riemann integral of a uniformly continuous function.

To this end, let $\{\phi_n\} \subset L$ be such an approximating sequence for ϕ. We denote by $\widetilde{\phi}_n$ the restrictions of ϕ_n to $\widetilde{X} \subset C[0,1]$. We have

$$\begin{aligned} \mathbb{P}(\|\widetilde{x}(\phi) - \langle \widetilde{y}_t, \widetilde{\phi}_n \rangle\|_{C[0,T]} > u) &= \mathbb{P}(\|\widetilde{x}(\phi - \widetilde{\phi}_n)\|_{C[0,T]} > u) \\ &\leq K \exp\left(-\frac{u^2}{2\mathbb{E}\langle g, \phi - \widetilde{\phi}_n \rangle^2} \right). \end{aligned}$$

Since we have $\langle f, \phi_n \rangle \to \langle f, \phi \rangle$ for every $f \in C[0,1]$, we have

$$\sup_{n\geq 1} \langle f, \phi_n \rangle < \infty, \quad \forall f \in C[0,1],$$

so the family $\{\phi_n\}_{n\geq 1}$ is uniformly norm-bounded in $C[0,1]^*$ by the uniform boundedness principle. Now, on all of Ω, we have $\langle \widetilde{g}, \phi - \phi_n \rangle^2 \leq \|\phi - \phi_n\|^2_{C[0,1]^*} \|\widetilde{g}\|^2_{C[0,1]}$, so by the DCT:

$$\mathbb{E}\langle \widetilde{g}, \phi - \phi_n \rangle^2 \to 0.$$

This means that $\langle \widetilde{y}_t, \phi_n \rangle \to \widetilde{x}_t(\phi)$ in probability. By passing to a subsequence we obtain almost sure convergence. Since also $\langle \widetilde{y}_t, \widetilde{\phi}_n \rangle \to \langle \widetilde{y}_t, \phi \rangle$, it follows that $\langle \widetilde{y}_t, \phi \rangle = \widetilde{x}_t(\phi)$.

Now recall that $\widetilde{g} = \iota \circ g$ and $\widetilde{x}_t(\widetilde{\phi}) = x_t(\iota^* \widetilde{\phi})$ with $\widetilde{\phi} \in \widetilde{X}^*$ and $\iota : X \to \widetilde{X}$ and $\iota^* : \widetilde{X}^* \to X^*$ isometric isomorphisms. Note that the dual spaces are isometrically isomorphic as well. Recall that $\widetilde{y} \in C([0,T], \widetilde{X})$. Define $y := \iota^{-1} \circ \widetilde{y} : [0,T] \to X$. Since ι^{-1} is an isometric isomorphism, $y \in C([0,T], X)$. Moreover, for any $\phi \in X^*$ we have some $\widetilde{\phi} \in \widetilde{X}^*$ such that $\phi = \iota\widetilde{\phi}$. Now,

$$x_t(\phi) = x_t(\iota^* \widetilde{\phi}) = \widetilde{x}_t(\widetilde{\phi}) = \langle \widetilde{y}_t, \widetilde{\phi} \rangle = \langle \iota(y_t), \widetilde{\phi} \rangle = \langle y_t, \iota^* \widetilde{\phi} \rangle = \langle y_t, \phi \rangle.$$

Moreover, since $\|y_t\|_X = \|\widetilde{y}_t\|_{C[0,1]}$ for all $t \in [0,T]$ and $\|g\|_X = \|\widetilde{g}\|_{C[0,1]}$, we have

$$\left(\mathbb{E} \sup_{t\in[0,T]} \|y_t\|_X^p \right)^{1/p} \lesssim K^{1/p} \sqrt{p} \mathbb{E}\|g\|_X.$$

This concludes the proof. $\square$

Chapter 4

Stochastic Integration

In this section, the ideas of the previous section will be applied to stochastic integration in separable Banach spaces. In particular, we will define stochastic integration in non-UMD Banach spaces. The first subsection will be entirely based on [12]. The comparison between the setting of Kalinichenko and the case where X is a Hilbert space is new. Moreover, Corollary 4.4 is a new result and an extension of Theorem 3.1 from [27].

4.1 General separable Banach spaces

We start in the usual setting, where X is a separable Banach space, H a separable Hilbert space and W_H an H-cylindrical Brownian motion. The definition for stochastic integrability is given in [12] as follows:

Definition 4.1. *Let $\Phi : [0,T] \times \Omega \to \mathscr{L}(H,X)$ be an adapted process. We say that Φ is stochastically integrable with respect to $W_H(t)$ if there exists a continuous X-valued process $\{\mathcal{I}_t\}_{t\geq 0}$ such that*

$$\langle \mathcal{I}_t, x^* \rangle = \int_0^t \Phi(s)^* x^* \; dW_H(s) \quad a.s.$$

for all $x^* \in X^*$, $t \in [0,T]$. In this case the process $\mathcal{I}_t$ is called the stochastic integral of Φ.

Note that this definition is different than the classical definition of stochastic integrability, where we approximate with finite rank step functions. However, in the preliminaries section we have seen that these two definitions are equivalent.

We are now ready to state and prove the main theorem of [12]:

Theorem 4.2. *Let X be a separable Banach space, and H a separable Hilbert space. Moreover we let W_H be an H-cylindrical Brownian motion and $\Phi : [0,T] \times \Omega \to \mathscr{L}(H,X)$ an adapted process. Now assume the existence of a centered Gaussian random variable $g : \Omega \to X$ such that for all $x^* \in X^*$*

$$\int_0^T \|\Phi(t)^* x^*\|_H^2 \; dt \leq \mathbb{E}\langle g, x^* \rangle^2 \quad a.s.$$

Then the process Φ is stochastically integrable on $[0,T]$ and the following inequality holds

$$\left(\mathbb{E}\Big\| \int_0^T \Phi(t) \; dW_H(t) \Big\|_X^p \right)^{1/p} \lesssim \sqrt{p} \; \mathbb{E}\|g\|_X.$$

Proof. For this proof, we write the H-cylindrical Brownian motion as an infinite sum of one dimensional Brownian motions. To this end, let $\{W^{(k)}\}_{k\geq 1}$ be a collection of independent Brownian motions on Ω. We write for all $h \in H$,

$$W_H(t)h = \sum_{k=1}^{\infty}(h, h_k)_H W^{(k)}(t).$$

For any x^* we now define the (real valued) stochastic integral $\mathcal{I}_t(x^*)$ by

$$\mathcal{I}_t(x^*) = \int_0^t \Phi(s)^* x^* \; dW_H(s) = \sum_{k=1}^{\infty} \int_0^t (\Phi(s)^* x^*, h_k)_H \; dW^{(k)}(s). \tag{4.1}$$

Note that for all $k \geq 1$, we have

$$\mathbb{E}\int_0^t (\Phi(s)^* x^*, h_k)_H^2 \; dt \leq \mathbb{E}\int_0^T \|\Phi(t)^* x^*\|_H^2 \; dt \leq \mathbb{E}\langle g, x^*\rangle^,$$

by the assumption. So each $(\Phi(s)^* x^*, h_k)_H$ is stochastically integrable with respect to the $W^{(k)}$. Now since

$$\mathbb{E}\sum_{k=1}^{\infty}\int_0^t (\Phi(s)^* x^*, h_k)_H^2 \; dt = \mathbb{E}\int_0^T \|\Phi(t)^* x^*\|_H^2 \; dt < \infty,$$

we have that $\mathcal{I}_t(x^*)$ is well-defined and exists for all $x^* \in X^*$. Moreover we also know that it is a real valued martingale with quadratic variation

$$\mathbb{E}\int_0^T \|\Phi(t)^* x^*\|_H^2 \; dt.$$

We have by the exponential inequality for martingales (Theorem 20.17 in [14])

$$\begin{aligned}
\mathbb{P}(\|\mathcal{I}(x^*)\|_{C[0,T]} > u) &= \mathbb{P}(\|(\mathbb{E}\langle g, x^*\rangle^2)^{-1/2}\mathcal{I}(x^*)\|_{C[0,T]} > (\mathbb{E}\langle g, x^*\rangle^2)^{-1/2}u) \\
&\leq \exp\left(-\frac{u^2}{2\mathbb{E}\langle g, x^*\rangle^2}\right).
\end{aligned}$$

Thus we are in the setting of Lemma 3.7. We can use this to obtain an X-valued continuous process y such that

$$\langle y_t, x^*\rangle = \mathcal{I}_t(x^*), \tag{4.2}$$

and the following inequality holds

$$\mathbb{E}\left(\sup_{t\in[0,T]}\|y_t\|_X^p\right)^{1/p} \lesssim K^{1/p}\sqrt{p}\;\mathbb{E}\|g\|_X.$$

But now by Equation 4.2 we know for all $t \in [0,T]$,

$$y_t = \int_0^t \Phi(s) \; dW_H(s).$$

This finishes the proof. $\square$

We can apply the above theory in the Abstract Wiener Space setting. Let g be an X-valued centered Gaussian random variable. Associated with g is its covariance operator $Q : X^* \to X$, and the reproducing kernel Hilbert space H_Q. We let $i : H_Q \to X$ be the embedding. Then (X, H_Q, i) is an Abstract Wiener Space.

Now let Φ take values only in $\mathscr{L}(H_Q)$, and assume that for some $M > 0$,

$$\int_0^T \|\Phi(t)\|^2_{\mathscr{L}(H_Q)}\, dt \leq M \quad \text{a.s.}$$

Then $i \circ \Phi : [0,T] \times \Omega \to \mathscr{L}(H_Q, X)$, and

$$\int_0^T \|\Phi(t)^* i^* x^*\|^2_{H_Q}\, dt \leq \int_0^T \|\Phi(t)\|^2_{\mathscr{L}(H_Q)} \|i^* x^*\|^2_{H_Q}\, dt \leq M \|i^* x^*\|^2_{H_Q} = M \mathbb{E}\langle g, x^*\rangle^2$$

We can then apply Theorem 4.2 to obtain stochastic integrability of $i \circ \Phi$:

Corollary 4.3. *Let (X, H_Q, i) be an Abstract Wiener space as above. Let $\Phi : [0,T] \times \Omega \to \mathscr{L}(H_Q)$ be an adapted process. Assume that*

$$\sup_{\omega \in \Omega} \int_0^T \|\Phi(t,\omega)\|^2_{\mathscr{L}(H_Q)}\, dt < \infty.$$

Then, if $i : H_Q \to X$ is the natural embedding, the function $i \circ \Phi : [0,T] \times \Omega \to \mathscr{L}(H_Q, X)$ is stochastically integrable and

$$\left(\mathbb{E}\Big\| \int_0^T i \circ \Phi(t)\, dW_{H_Q}(t)\Big\|^p_X\right)^{1/p} \leq \sqrt{p} \left(\sup_{\omega \in \Omega} \int_0^T \|\Phi(t,\omega)\|^2_{\mathscr{L}(H_Q)}\, dt\right)^{1/2} \mathbb{E}\|g\|_X.$$

Moreover, we can extend Theorem 3.1 from [27] to the case where X is non-UMD, as long as one of the functions is deterministic. The result can be formulated as follows:

Corollary 4.4. *Let $\Phi : [0,T] \times \Omega \to \mathscr{L}(H,X)$ be an adapted process and $\Psi : [0,T] \to \mathscr{L}(H,X)$ be H-strongly measurable and assume that Ψ is stochastically integrable with respect to W_H. If for all $x^* \in X^*$ we have*

$$\int_0^T \|\Phi(t,\omega)^* x^*\|^2_H\, dt \leq \int_0^T \|\Psi(t)^* x^*\|^2_H\, dt$$

for almost all $\omega \in \Omega$. Then Φ is stochastically integrable with respect to W_H and we have

$$\mathbb{E}\Big\| \int_0^T \Phi(t)\, dW_H(t)\Big\|^p_X \lesssim \sqrt{p}^p\, \mathbb{E}\Big\| \int_0^T \Psi(t)\, dW_H(t)\Big\|^p_X.$$

Proof. We know from Theorem 6.17 in [24] that if $\Psi : [0,T] \to \mathscr{L}(H,X)$ is stochastically integrable with respect to W_H, then there exists an X-valued Gaussian G such that for all $x^* \in X^*$

$$\langle G, x^*\rangle = \int_0^T \Psi(t)^* x^*\, dW_H(t).$$

Moreover, we have for all $x^* \in X^*$

$$\mathbb{E}\langle G, x^*\rangle^2 = \mathbb{E}\Big| \int_0^T \Psi(t)^* x^*\, dW_H(t)\Big|^2 = \int_0^T \|\Psi(t)^* x^*\|^2_H\, dt.$$

Thus we can apply 4.2, since almost surely for all $x^* \in X^*$,

$$\int_0^T \|\Phi(t,\omega)^* x^*\|_H^2 \, dt \leq \mathbb{E}\langle G, x^* \rangle^2.$$

It follows that Φ is stochastically integrable and we have

$$\begin{aligned}
\mathbb{E}\Big\| \int_0^T \Phi(t)\ dW_H(t) \Big\|_X^p &\lesssim \left(\sqrt{p}\ \mathbb{E}\Big\| \int_0^T \Psi(t)\ dW_H(t) \Big\|_X \right)^p \\
&\lesssim \sqrt{p}^p\ \mathbb{E}\Big\| \int_0^T \Psi(t)\ dW_H(t) \Big\|_X^p.
\end{aligned}$$

□

4.2 The Hilbert space case

In this subsection we will consider the case where X is a Hilbert space instead of a Banach space. We will show that Theorem 4.2 can be proven without any advanced techniques, and even without the theory of γ-radonifying operators. Later, in Section 6, we will extend this result to UMD$^-$ spaces.

To this end, let X and H be separable Hilbert spaces, and W_H an H-cylindrical Brownian motion. By Riesz we can identify X^* with X and H^* with H. We assume that the stochastic process $\sigma : [0,T] \times \Omega \to \mathscr{L}(H,X)$ is adapted, and that there exists an X-valued Gaussian $G : \Omega \to X$ such that for almost surely for all $x \in X$,

$$\int_0^T \|\sigma_t^* x\|_H^2 \, dt \leq \mathbb{E}[G, x]_X^2. \tag{4.3}$$

We show that σ is stochastically integrable with respect to W_H. To this end, we note that the covariance operator $Q : X \to X$ of G is trace class. This can be seen in the following way. Let $(e_n)_{n\geq 1}$ be an orthonormal basis of X. Then

$$\int_X \|x\|_X^2 \, d\mu(x) = \sum_{n\geq 1} \int_X [x, e_n]_X^2 \, d\mu(x) = \sum_{n\geq 1} [e_n, Qe_n]_X = \mathrm{tr}(Q).$$

Now since Q is trace class on X, we know $Q^{1/2}$ is Hilbert-Schmidt on X. Using this, we can rewrite Equation 4.3 to

$$\|\sigma_t^* x\|_{L^2(0,T;H)}^2 \leq [x, Qx]_X = \|Q^{1/2} x\|_X^2, \quad \forall x \in X, \quad \text{a.s.}$$

By the above, we have almost surely for every basis $(e_n)_{n\geq 1}$ of X that

$$\sum_{n\geq 1} \|\sigma^* e_n\|_{L^2(0,T;H)}^2 \leq \sum_{n\geq 1} \|Q^{1/2} e_n\|_X^2 < \infty. \tag{4.4}$$

If we denote by $\mathscr{L}_2(H_1, H_2)$ the space of Hilbert-Schmidt operastors between two Hilbert spaces H_1 and H_2, then $\sigma^* \in \mathscr{L}_2(X, L^2(0,T;H))$, so $\sigma \in \mathscr{L}_2(L^2(0,T;H), X)$. In order to use the theory in Da Prato-Zabczyck [6] however, we need that $\sigma \in L^2(0,T;\mathscr{L}_2(H,X))$. We have the following theorem:

Theorem 4.5. *Let X and H be separable Hilbert spaces and $0 < T < \infty$. Then we have the following isomorphisms of Hilbert spaces:*

$$\mathscr{L}_2(L^2(0,T;H),X) \approx L^2(0,T;\mathscr{L}_2(H,X)).$$

Proof. We recall that for any two separable Hilbert spaces H_1 and H_2, each Hilbert-Schmidt operator $A \in \mathscr{L}_2(H_1, H_2)$ can be approximated in the $\mathscr{L}_2$-norm by operators in $H_1 \otimes H_2$. Thus the space $L^2(0,T) \otimes (H \otimes X)$ is dense in $L^2(0,T;\mathscr{L}_2(H,X))$ in its norm, and $(L^2(0,T) \otimes H) \otimes X$ is dense in $\mathscr{L}_2(L^2(0,T;H),X)$ in its own norm. Moreover we have

$$L^2(0,T) \otimes (H \otimes X) \approx (L^2(0,T) \otimes H) \otimes X.$$

Let now $(h_n)_{n\geq 1}$ be an orthonormal basis of H and $(f_k)_{k\geq 1}$ an orthonormal basis of $L^2(0,T)$. Let $A \in L^2(0,T) \otimes H \otimes X$ be of the form

$$A = \sum_{k=1}^{K} \sum_{n=1}^{N} f_k \otimes h_n \otimes x_{kn}.$$

We have

$$\begin{aligned}
\|A\|^2_{L^2(0,T;\mathscr{L}_2(H,X))} &= \int_0^T \|A(t)\|^2_{\mathscr{L}_2(H,X)}\, dt = \int_0^T \sum_{\ell\geq 1} \|A(t)h_\ell\|_X^2\, dt \\
&= \sum_{\ell\geq 1} \int_0^T \|A(t)h_\ell\|_X^2\, dt = \sum_{\ell\geq 1} \int_0^T \Big\| \sum_{k=1}^{K}\sum_{n=1}^{N} f_k(t)(h_n, h_\ell)_H x_{kn} \Big\|_X^2\, dt \\
&= \sum_{n=1}^{N} \int_0^T \Big\| \sum_{k=1}^{K} f_k(t) x_{kn} \Big\|_X^2\, dt = \sum_{n=1}^{N} \int_0^T \sum_{k,k'=1}^{K} f_k(t) f_{k'}(t) [x_{kn}, x_{k'n}]_X\, dt \\
&= \sum_{n=1}^{N} \sum_{k,k'=1}^{K} [x_{kn}, x_{k'n}]_X \int_0^T f_k(t) f_{k'}(t)\, dt = \sum_{n=1}^{N}\sum_{k=1}^{K} \|x_{kn}\|_X^2.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\|A\|^2_{\mathscr{L}_2(L^2(0,T;H),X)} &= \sum_{\ell_1\geq 1}\sum_{\ell_2\geq 1} \|A(f_{\ell_1} \otimes h_{\ell_2})\|_X^2 \\
&= \sum_{\ell_1,\ell_2\geq 1} \Big\| \sum_{k=1}^{K}\sum_{n=1}^{N} (f_{\ell_1}, f_k)_{L^2(0,T)} (h_{\ell_2}, h_n)_H x_{kn} \Big\|_X^2 = \sum_{k=1}^{K}\sum_{n=1}^{N} \|x_{kn}\|_X^2.
\end{aligned}$$

Since for any $A \in L^2(0,T) \otimes H \otimes X$ we have $\|A\|_{L^2(0,T);\mathscr{L}_2(H,X)} = \|A\|_{\mathcal{L}_2(L^2(0,T;H),X)}$. But because $L^2(0,T) \otimes H \otimes X$ is dense in both spaces, we conclude that

$$\mathscr{L}_2(L^2(0,T;H),X) \approx L^2(0,T;\mathscr{L}_2(H,X)).$$

□

Equation 4.3, together with the above theorem, implies that $\sigma \in L^2(0,T;\mathscr{L}_2(H,X))$. Moreover, we have by Equation 4.4 some $M > 0$, which depends on Q, such that for almost all $\omega \in \Omega$

$$\begin{aligned}
\int_0^T \|\sigma_t(\omega)\|^2_{\mathscr{L}_2(H,X)}\, dt &= \|\sigma(\omega)\|^2_{L^2(0,T;\mathscr{L}_2(H,X))} = \|\sigma(\omega)\|^2_{\mathscr{L}_2(L^2(0,T;H),X)} \\
&= \|\sigma(\omega)^*\|^2_{\mathscr{L}_2(X,L^2(0,T;H))} \leq \sum_{n\geq 1} \|Q^{1/2}\|^2_{\mathscr{L}_2(X)} =: M < \infty.
\end{aligned}$$

Then it immediately follows that

$$\mathbb{P}\Big(\int_0^T \|\sigma_t(\omega)\|^2_{\mathscr{L}_2(H,X)} \, dt < \infty \Big) = 1.$$

Thus the conditions for Proposition 4.22 in [6] are satisfied and σ is stochastically integrable with respect to W_H.

Chapter 5

Applications to Stochastic Differential Equations

5.1 Weak solutions

We are in the usual setting, where $\{W_H(t) : t \geq 0\}$ is a cylindrical Brownian motion on a separable Hilbert space H and a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. Moreover we let X be a separable Banach space. In this chapter we will primarily deal stochastic differential equations of the form

$$\begin{cases} dx_t &= b(t, x_t)\ dt + \sigma(t, x_t)\ dW_H(t), \quad 0 < t \leq T \\ x_0 &= Z \end{cases}, \tag{5.1}$$

where Z is a random variable with probability measure μ_0 and b and σ are two adapted $\mathscr{L}(H, X)$-valued stochastic processes. Before we go into the proofs of solving an equation of the above form, we need to define what exactly we mean by "solving". We distinguish between *strong solutions* and *weak solutions*.

Definition 5.1. *We call an adapted X-valued process x_t a* strong solution *to Equation 5.1 if it satisfies the following equation for all $t \in [0, T]$:*

$$x_t = \int_0^t b(s, x_s)\ ds + \int_0^t \sigma(s, x_s)\ dW_H(s).$$

In this chapter we will find *weak solutions* of 5.1 under certain conditions on b and σ. We give the definition:

Definition 5.2. *We call an adapted X-valued process $\widetilde{x}_t$ a* weak solution *to Equation 5.1 if there exists a filtered probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \{\widetilde{\mathcal{F}}\}_{t\geq 0}, \widetilde{\mathbb{P}})$ and an H-cylindrical Brownian motion $\widetilde{W}_H$ on $\widetilde{\Omega}$ such that*

$$\widetilde{x}_t = Z + \int_0^t b(s, \widetilde{x}_s)\ ds + \int_0^t \sigma(s, \widetilde{x}_s)\ d\widetilde{W}_H(s), \quad t \in [0, T].$$

Note that the concept of weak solutions for SDEs is entirely different than the concept of weak solutions for PDEs in Sobolev spaces. Moreover, existence of a strong solution implies existence of a weak solution, but the other way around is not true. Taking $X = \mathbb{R}$, an example is given by Theorem 5.4. Before we prove this, we need the notion of quadratic variation:

Definition 5.3. *Assume Y_t is a stochastic process on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. The quadratic variation is defined by*

$$[Y]_t = \lim_{\mathrm{mesh}(P)\to 0} \sum_{n=0}^{N-1} (Y_{t_{n+1}} - Y_{t_n})^2,$$

where P ranges over all partitions of $[0,t]$ with $P = \{t_0, t_1, \ldots, t_N\}$ and the limit is taken in probability.

We can now construct the following example of an SDE with a weak solution but no strong solution, which is called Tanake's equation.

Theorem 5.4. *Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space and W a Brownian motion on Ω. Then the SDE*

$$dX_t = -\mathrm{sgn}(X_t)\ dW_t \tag{5.2}$$

has a weak solution but no strong solution.

Proof. We first show that 5.2 has a weak solution. To this end, let $\widetilde{W}_t$ be a Brownian motion on some probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \{\widetilde{\mathcal{F}}\}_{t\geq 0}, \widetilde{\mathbb{P}})$. We assume $\widetilde{W}_t$ is adapted to $\{\widetilde{\mathcal{F}}_t\}_{t\geq 0}$. Set

$$Y_t := -\int_0^s \mathrm{sgn}(\widetilde{W}_s)\ d\widetilde{W}_s.$$

We have

$$[Y]_t = \int_0^t (-\mathrm{sgn}(\widetilde{W}_t))^2\ d[\widetilde{W}]_s = \int_0^t 1\ d[\widetilde{W}]_s = t,$$

thus by Levy's characterization of the Brownian motion, Y_t is also a Brownian motion on $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \{\widetilde{\mathcal{F}}\}_{t\geq 0}, \widetilde{\mathbb{P}})$. Because

$$dY_t = -\mathrm{sgn}(\widetilde{W}_t)\ d\widetilde{W}_t,$$

we have

$$d\widetilde{W}_t = -\mathrm{sgn}(\widetilde{W}_t)\ dY_t.$$

Thus $\widetilde{W}_t$ is a weak solution to 5.2. Next we show that no strong solution to 5.2 exists through a proof by contradiction.

Suppose that X_t satisfies 5.2. By Levy's characterization, we again see that X_t must be a Brownian motion, this time on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. Define the local time process L_t by

$$L_t = \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} |\{s \in [0,t] : B_s \in (-\varepsilon, \varepsilon)\}|.$$

By Tanaka's formula we have

$$|X_t| = \int_0^t \mathrm{sgn}(X_s)\ dX_s + L_t.$$

The stochastic differential equation 5.2 now implies that

$$|W_t| = |X_t| - L_t.$$

But since

$$L_t = \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} |\{s \in [0,t] : B_s \in (-\varepsilon, \varepsilon)\}| = \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} |\{s \in [0,t] : |B_s| \in (0, \varepsilon)\}|,$$

W_t is a measurable function only depending on $|X_t|$. Thus W_t is adapted with respect to the filtration $\mathcal{F}^{|X|}$ generated by $|X_t|$, so we have $\mathcal{F}_t \subseteq \mathcal{F}_t^{|X|}$ for all $t \geq 0$. Now if X_t is a strong solution to 5.2, then it is adapted to $\{\mathcal{F}_t\}_{t\geq 0}$, and thus adapted to $\{\mathcal{F}_t^{|X|}\}_{t\geq 0}$. However, this is not possible, since $\operatorname{sgn}(X_s)$ is not for all $s > 0$ measurable on $\mathcal{F}_s^{|X|}$, since $\{\mathcal{F}^{|X|}\}_{t\geq 0}$ carries no information on the sign of X_t. □

5.2 Tightness

We have the following theorem, which is Theorem 3 in Kalinichenko:

Theorem 5.5. *Let I be an index set and $\sigma^\alpha : \Omega \times [0,T] \to \mathscr{L}(H,X)$, and $b^\alpha : \Omega \times [0,T] \to X$ be predictable processes for each $\alpha \in I$. We assume that there is a Gaussian random variable g on X with the property that*

$$\begin{aligned} \|(\sigma_t^\alpha)^* x^*\|_H^2 &\leq \mathbb{E}\langle g, x^*\rangle^2 \\ \langle b_t^\alpha, x^*\rangle^2 &\leq \mathbb{E}\langle g, x^*\rangle^2, \end{aligned}$$

almost surely for every $x^* \in X^*$ and $t \geq 0$. Define for each $\alpha \in I$,

$$x_T^\alpha = \int_0^T b_t^\alpha \, dt + \int_0^T \sigma_t^\alpha \, dW_H(t). \tag{5.3}$$

Then the family $\{x^\alpha\}_{\alpha\in I}$ of $C([0,\infty), X)$-valued random variables is relatively compact in distribution.

Proof. Note first of all that x_T^α is well-defined and an element of X for all $T \geq 0$ and $\alpha \in I$. We have for all $\alpha \in I$

$$\int_0^T \|(\sigma_t^\alpha)^* x^*\|_H^2 \, dt \leq T \cdot \mathbb{E}\langle g, x^*\rangle^2.$$

So the conditions of Lemma 3.4 hold, thus the stochastic integral of σ_t^α exists in X. We have for each $\alpha \in I, t \geq 0$, and $\omega \in \Omega$,

$$\|b_t^\alpha(\omega)\|_X = \sup_{x^*:\|x^*\|\leq 1} |\langle b_t^\alpha(\omega), x^*\rangle| \leq \sup_{x^*:\|x^*\|\leq 1} \sqrt{\mathbb{E}\langle g, x^*\rangle^2} \leq \sqrt{\mathbb{E}\|g\|_X^2}.$$

So the Bochner integral of b_t^α with respect to t exists in X almost surely. Thus $\{x_t^\alpha\}_{t\geq 0}$ as defined in Equation (5.3) exists and is continuous. Moreover, $\{x_t^\alpha\}_{t\geq 0}$ is an X-valued semimartingale for each $\alpha \in I$, and it has a continuous version. Fix $T \geq 0$. By the Banach-Mazur theorem, we can view X as a closed subspace of $C[0,1]$. In this way, x^α is a $C([0,T], C[0,1])$-valued random variable. Since $C([0,T], C[0,1]) = C([0,1], C[0,T])$, we can alternatively view x^α as a $C([0,1], C[0,T])$-valued random variable. We write $x_t^\alpha(s)$ for $s \in [0,1]$ and $t \in [0,T]$. Since we embedded X in $C[0,1]$, we view g as a $C[0,1]$-valued Gaussian random variable. Thus it makes sense to define the metric d_g on $[0,1]$ in the usual way, i.e.

$$d_g(s,t) = (\mathbb{E}|g(s) - g(t)|^2)^{1/2}.$$

Now, for any $s \in [0,1]$ the δ_s is an element in $C[0,1]^*$, and since $x_t^\alpha \in C[0,1]$ for all $t \in [0,1]$, it makes sense to write, for fixed $s, s' \in [0,1]$,

$$\begin{aligned} \langle x_t^\alpha, \delta_s - \delta_{s'}\rangle &= \Big\langle \int_0^t b_\tau^\alpha \, d\tau + \int_0^t \sigma_\tau^\alpha \, dW_H(\tau), \delta_s - \delta_{s'} \Big\rangle \\ &= \int_0^t \langle b_\tau^\alpha, \delta_s - \delta_{s'}\rangle \, d\tau + \Big\langle \int_0^t \sigma_\tau^\alpha \, dW_H(\tau), \delta_s - \delta_{s'} \Big\rangle. \end{aligned}$$

We have almost surely for $u \geq 1$

$$\int_0^t |\langle b_\tau^\alpha, \delta_s - \delta_{s'}\rangle| \; d\tau \leq t\sqrt{\mathbb{E}\langle g, \delta_s - \delta_{s'}\rangle^2} \leq Td_g(s,s') \leq Tud_g(s,s').$$

Moreover, by Theorem 5.6 the term

$$\Big\langle \int_0^t \sigma_\tau^\alpha \; dW_H(\tau), \delta_s - \delta_{s'} \Big\rangle$$

is a real valued martingale, with quadratic variation given by

$$\int_0^t \|(\sigma_\tau^\alpha)^*(\delta_s - \delta_{s'})\|_H^2 \; d\tau.$$

We have the following estimate,

$$\Big[\Big\langle \int_0^\cdot \sigma_\tau^\alpha \; dW_H(\tau), \delta_s - \delta_{s'} \Big\rangle\Big]_T = \int_0^T \|(\sigma_t^\alpha)^*(\delta_s - \delta_{s'})\|_H^2 \; dt \leq T\mathbb{E}\langle g, \delta_s - \delta_{s'}\rangle^2 = Td_g(s,s'). \quad (5.4)$$

Using this, we can apply the exponential inequality for martingales which is Theorem 20.17 in [14], we have for $u \geq 1$

$$\begin{aligned}
&\mathbb{P}\left(\|\langle x^\alpha, \delta_s - \delta_{s'}\|_{C[0,T]} \geq 2Tud_g(s,s')\right) \\
&\qquad \leq \mathbb{P}\left(\sup_{t\in[0,T]} \int_0^t |\langle b_\tau^\alpha, \delta_s - \delta_{s'}\rangle| \; d\tau + \left|\Big\langle \int_0^t \sigma_\tau^\alpha \; dW_H(\tau), \delta_s - \delta_{s'}\Big\rangle\right| \geq 2Tud_g(s,s')\right) \\
&\qquad\qquad \leq \mathbb{P}\left(Tud_g(s,s') + \sup_{t\in[0,T]} \left|\Big\langle \int_0^t \sigma_\tau^\alpha \; dW_H(\tau), \delta_s - \delta_{s'}\Big\rangle\right| \geq 2Tud_g(s,s')\right) \\
&\quad = \mathbb{P}\left(\sup_{t\in[0,T]} \frac{1}{Td_g(s,s')} \left|\Big\langle \int_0^t \sigma_\tau^\alpha \; dW_H(\tau), \delta_s - \delta_{s'}\Big\rangle\right| \geq u\right) \leq \exp(-u^2/2) \leq 2\exp(-u^2/2).
\end{aligned}$$

On the other hand, when $u \geq 1$,

$$\mathbb{P}\left(\|\langle x^\alpha, \delta_s - \delta_{s'}\rangle\|_{C[0,T]} \geq 2Tud_g(s,s')\right) \leq 1 \leq 2\exp(-u^2/2).$$

Thus $x^\alpha : ([0,1], d) \to C[0,T]$ is subgaussian with respect to $2Tg : ([0,1], d) \to \mathbb{R}$ for all $\alpha \in I$. Fix $\varepsilon > 0$. We can now use Lemma 1 to find for every $n \geq 1$ a $\delta_n > 0$ such that

$$\mathbb{E}\left(\sup_{d(s,s')<\delta_n} \|x^\alpha(s) - x^\alpha(s')\|_{C[0,T]}\right) \leq \varepsilon 2^{-2n-2},$$

for all $\alpha \in I$. Note that this bound holds uniformly in α since by Lemma 1, the δ_n does not depend on the process x^α itself but only on ε, the process g, and 2^{-n}. Now by Chebyshev's inequality,

$$\mathbb{P}\left(\sup_{d(s,s')<\delta_n} \|x^\alpha(s) - x^\alpha(s')\|_{C[0,T]} \geq 2^{-n}\right) \leq 2^n\mathbb{E}\left(\sup_{d(s,s')<\delta_n} \|x^\alpha(s) - x^\alpha(s')\|_{C[0,T]}\right) \leq \varepsilon 2^{-n-2}. \quad (5.5)$$

Fix $s \in [0,1]$. Since for all $\alpha \in I$, the process $\{x_t^\alpha(s)\}_{t\geq 0}$ is a real valued martingale, we also have that for any $t \geq 0$, the shifted process $\{x_{\tau+t}^\alpha(s) - x_t^\alpha(s)\}_{\tau\geq 0}$ is a martingale with quadratic variation given by, for $t' > t$,

$$\begin{aligned}
[x_{\cdot+t}^\alpha(s) - x_t^\alpha(s)]_{t'-t} &= \left[\left\langle \int_t^{\cdot+t} \sigma_\tau^\alpha \, dW_H(\tau), \delta_s \right\rangle + \left\langle \int_t^{\cdot+t} b_\tau^\alpha \, dt, \delta_s \right\rangle\right]_{t'-t} \\
&= \left[\left\langle \int_t^{\cdot+t} \sigma_\tau^\alpha \, dW_H(\tau), \delta_s \right\rangle\right]_{t'-t} \\
&= \left[\left\langle \int_0^{\cdot+t} \sigma_\tau^\alpha \, dW_H(\tau), \delta_s \right\rangle\right]_{t'-t} - \left[\left\langle \int_0^{\cdot} \sigma_\tau^\alpha \, dW_H(\tau), \delta_s \right\rangle\right]_t \leq (t'-t)\mathbb{E}g(s)^2.
\end{aligned}$$

The last inequality here follows in a similar way to Equation (5.4), where we substract the two quadratic variations to obtain the $(t'-t)$ term instead of T. We can now use the Burkholder inequality in the following way, for $t' > t$:

$$\begin{aligned}
\mathbb{E}|x_{t'}^\alpha(s) - x_t^\alpha(s)|^4 \leq \mathbb{E}\left(\sup_{\tau\in[0,t'-t]} |x_{\tau+t}^\alpha(s) - x_t^\alpha(s)|\right)^4 &\leq \mathbb{E}\left(x_{\cdot+t}^\alpha(s) - x_t^\alpha(s)]_{t'-t}\right)^2 \\
&\leq |t'-t|^2(\mathbb{E}g(s)^2)^2.
\end{aligned}$$

Note that for each $s \in [0,1]$ and $\alpha \in I$, we have almost surely $x_0^\alpha(s) = 0$, thus trivially $\{x_0^\alpha(s)\}_{\alpha\in I}$ is tight in X, and by the above estimate

$$\sup_{\alpha\in I} \mathbb{E}|x_{t'}^\alpha(s) - x_t^\alpha(s)|^4 \leq C|t'-t|^2.$$

Thus the conditions for Theorem 23.7 in [14] are satisfied with $a = 4$ and $b < 2$. Hence the family $\{x^\alpha(s)\}_{\alpha\in I}$ of $C[0,T]$-valued random variables is tight for each $s \in [0,1]$. Now choose a dense, countable subset $D = \{s_n\}_{n\geq 1}$ of $[0,1]$. By the tightness, for every $n \geq 1$ we can find a compact set $K_n \subset C[0,T]$ such that

$$\inf_{\alpha\in I} \mathbb{P}(x^\alpha(s_n) \in K_n) > 1 - \varepsilon 2^{-n-2}. \tag{5.6}$$

Now define for all $n \geq 1$ the following subsets of $C([0,1], C[0,T])$;

$$F_n := \left\{\sup_{d(s,s')<\delta_n} \|x(s) - x(s')\|_{C[0,T]} \leq 2^{-n}\right\} \cap \{x(s_n) \in K_n\}.$$

Note that by Equation (5.5) and (5.6), the above sets are non-empty for all $n \geq 1$, moreover we have the inclusion $F_{n+1} \subseteq F_n$. Set

$$F := \bigcap_{n\geq 1} F_n.$$

We use Arzela-Ascoli to show that this is compact. Obviously, F is equicontinuous. Moreover, $\{x(s) : x \in F\}$ is pre-compact for $s \in [0,1]$ fixed. This can be seen in following way. Let $s \in D$. Then $s = s_N$ for some $N \in \mathbb{N}$. By definition of F, we have $\{x(s_N) : x \in F\} \subset K_N \subset C[0,T]$. Thus $\{x(s) : x \in F\}$ is pre-compact for all $s \in D$.

Now let $s \in [0,1] \setminus D$ and suppose $(s_k)_{k\geq 1}$ is a sequence in D such that $d(s, s_k) < \delta_k$ for all $k \geq 1$. We have for all $x \in F$

$$\|x(s_k) - x(s)\|_{C[0,T]} \leq 2^{-k}.$$

Thus for all $k \geq 1$

$$\{x(s) : x \in F\} \subset \{x(s_k) : x \in F\} + B_{C[0,T]}(0, 2^{-k}).$$

Thus $\{x(s) : x \in F\}$ is totally bounded, to see this take $\xi > 0$. Then let $K \in \mathbb{N}$ be such that $2^{-K} < \xi$. Since we know $\{x(s) : x \in F\}$ is a subset of $\{x(s_K) : x \in F\} + B_{C[0,T]}(0, 2^{-K})$, and $\{x(s_K) : x \in F\}$ is totally bounded, we can find a net of n points $\{x^1(s_K), x^2(s_K), \dots, x^n(s_K)\}$ such that their balls of radius ξ cover $\{x(s_K) : x \in F\}$. Since $2^{-K} < \xi$, we have

$$\{x(s) : x \in F\} \subset \{x(s_K) : x \in F\} + B_{C[0,T]}(0, 2^{-K}) \subset \bigcup_{j=1}^{n} B(x^j(s_K), 2\xi).$$

Then $\{x(s) : x \in F\}$ is totally bounded. Since $C[0,T]$ is complete, $\overline{\{x(s) : x \in F\}}$ is complete. We can thus conclude that $\{x(s) : x \in F\}$ is pre-compact in $C[0,T]$ for all $s \in [0,1]$.

Once we show F is closed, we can use Arzela-Ascoli. To this end, we define for any $n \geq 1$ the set $F'_n \subset C([0,1], C[0,T])$ by

$$F'_n := \Big\{ x \in C([0,1], C[0,T]) : \sup_{d(s,s') \leq \delta_n} \|x(s) - x(s')\|_{C[0,T]} \leq 2^{-n} \Big\}.$$

Then F'_n is closed. This can be seen in the following way. Let $\{y_k\}_{k\geq 1}$ be a sequence in F'_n that converges to some $y \in C([0,1], C[0,T])$. We show $y \in F'_n$. For all $s, s' \in [0,1]$ we have

$$\|y(s) - y(s')\|_{C[0,T]} \leq \|y(s) - y_k(s)\|_{C[0,T]} + \|y_k(s) - y_k(s')\|_{C[0,T]} + \|y_k(s') - y(s')\|_{C[0,T]}.$$

Let again $\xi > 0$. We can find $k \geq 1$ such that $\|y - y_k\|_{C([0,1],C[0,T])} < \xi$. By plugging in this estimate in the above, and taking the supremum over all $s, s' \in [0,1]$ such that $d(s,s') < \delta_n$, we find

$$\sup_{d(s,s') \leq \delta_n} \|y(s) - y(s')\|_{C[0,T]} \leq 2\xi + \sup_{d(s,s') \leq \delta_n} \|y_k(s) - y_k(s')\|_{C[0,T]} \leq 2\xi + 2^{-n}.$$

But since we can do this for all $\xi > 0$, it follows that F'_n is closed for all $n \geq 1$. Moreover, we can define for each $n \geq 1$ the evaluation map

$$\mathrm{ev}_{s_n}(x) : C([0,1], C[0,T]) \to C[0,T], \quad \mathrm{ev}_{s_n}(x) = x(s_n) \in C[0,T].$$

Since this map is continuous, K_n is compact, and thus closed, for each $n \geq 1$ by definition, and $\{x(s_n) \in K_n\} = \mathrm{ev}_{s_n}^{-1}(K_n)$, we have that $\{x(s_n) \in K_n\}$ is closed. Thus for each $n \geq 1$, the set $F'_n \cap \{x(s_n) \in K_n\}$ is closed in $C([0,1], C[0,T])$. So their intersection F is also closed in $C([0,1], C[0,T])$. All three conditions for Arzela-Ascoli are now verified, namely F is equicontinuous, $\{x(s) : x \in F\}$ is pre-compact for every $s \in [0,1]$, and F is closed. Thus F is compact in $C([0,1], C[0,T])$.

Now we can explicitly compute

$$\begin{aligned}
\mathbb{P}(x^\alpha \notin F) = \mathbb{P}(x^\alpha \in \complement F) &\leq \sum_{n\geq 1} \mathbb{P}\left(\sup_{d(s,s') < \delta_n} \|x^\alpha(s) - x^\alpha(s')\|_{C[0,T]} \geq 2^{-n} \right) + \sum_{n\geq 1} \mathbb{P}(x^\alpha(s_n) \notin K_n) \\
&\leq \varepsilon \sum_{n\geq 1} 2^{-n-2} + \varepsilon \sum_{n\geq 1} 2^{-n-2} < \varepsilon.
\end{aligned}$$

Now by Prohorov's theorem, the relative compactness in distribution of the functions $\{x^\alpha\}_{\alpha\in I}$ in $C([0,T] \times [0,1])$ follows. By Theorem 16.6 in [14] the relative compactness in distribution of $\{x^\alpha\}_{\alpha \in I}$ in $C([0,\infty) \times [0,1])$ follows. $\square$

5.3 Existence of weak solutions

Before we state the main theorem of this subsection, Theorem 5.7, we first need the following result, which we will frequently use in the proof of 5.7:

Lemma 5.6. *Let $1 < p < \infty$ and $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space. If the process $\Phi : \Omega \times (0,T) \to \mathscr{L}(H,X)$ is L^p-stochastically integrable, then for each $x^* \in X^*$,*

$$\langle Z_t, x^* \rangle := \Big\langle \int_0^t \Phi(s)\ dW_H(s), x^* \Big\rangle$$

is a real-valued martingale with quadratic variation

$$[\langle Z, x^* \rangle]_t = \int_0^t \|\Phi(s)^* x^*\|_H^2\ ds.$$

Proof. We know from Theorem 13.5 in [24] that the integral of Φ with respect to W_H is a martingale in X. Before showing $\langle Z, x^* \rangle$ is a real valued martingale, let $f : \Omega \to X$ be a simple function, so

$$f = \sum_{n=1}^{N} 1_{A_n} x_n,$$

with $A_n \in \mathcal{F}$ disjoint. Now let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra of $\mathcal{F}$. We have for any $x^* \in X^*$,

$$\mathbb{E}[\langle f, x^* \rangle | \mathcal{G}] = \mathbb{E}\Big[\Big\langle \sum_{n=1}^{N} 1_{A_n} x_n, x^* \Big\rangle | \mathcal{G}\Big] = \Big\langle \sum_{n=1}^{N} x_n \mathbb{E}[1_{A_n} | \mathcal{G}], x^* \Big\rangle = \langle \mathbb{E}[f|\mathcal{G}], x^* \rangle.$$

Now for all $p > 1$, we can extend this to general $f \in L^p(\Omega; X)$ by a density argument. Continuing now with $\langle Z, x^* \rangle$, we have for all $t \geq 0$ that Z_t is strongly $\mathcal{F}$-measurable. Hence each $\langle Z_t, x^* \rangle$ is measurable. Moreover, since $Z_t \in L^p(\Omega; X)$ for some $p > 1$, we also know $Z_t \in L^1(\Omega; X)$. Now for any $s < t$ we have by the martingale property of Z,

$$\mathbb{E}[\langle Z_t, x^* \rangle | \mathcal{F}_s] = \langle \mathbb{E}[Z_t | \mathcal{F}_s], x^* \rangle = \langle Z_s, x^* \rangle.$$

Thus, by the above, $\langle Z, x^* \rangle$ is a martingale. To compute the quadratic variation, we consider a simple adapted process Φ, given by

$$\Phi = \sum_{n=1}^{N} 1_{(t_{n-1}, t_n]} \sum_{m=1}^{M} 1_{F_{mn}} \otimes \sum_{j=1}^{k} h_j \otimes x_{jmn}.$$

We use the sum representation of the W_H, i.e. for each $h \in H$,

$$W_H(t)h = \sum_{j=1}^{\infty} W^{(j)}(t)[h_j, h],$$

with $W^{(j)}$ independent 1-dimensional Brownian motions. Now,

$$\int_0^T \Phi^* x^*\ dW_H = \sum_{n=1}^{N} \sum_{m=1}^{M} \sum_{j=1}^{k} 1_{F_{mn}} \langle x_{jmn}, x^* \rangle (W^{(j)}(t_n) - W^{(j)}(t_{n-1})).$$

Thus we can compute the integral element-wise, in the following sense:

$$\int_0^T \Phi^* x^* \, dW_H = \sum_{j=1}^k \int_0^T (\Phi^* x^*)_j(t) \, dW^{(j)}(t),$$

where the $(\Phi^* x^*)_j : \Omega \times (0,T) \to \mathbb{R}$ are given by the formula

$$(\Phi^* x^*)_j(t) := [\Phi^*(t)x^*, h_j] = \sum_{n=1}^N \sum_{m=1}^M 1_{F_{mn}} 1_{(t_{n-1},t_n]} \langle x_{jmn}, x^* \rangle.$$

The quadratic variation can now simply be computed by the Ito isometry on $\mathbb{R}$, we have for all $1 \leq j \leq k$,

$$\Big[\int_0^{\cdot} (\Phi^* x^*)_j(t) \, dW^{(j)}(t) \Big]_T = \int_0^T |(\Phi^* x^*)_j(t)|^2 \, dt = \int_0^T [\Phi(t)^* x^*, h_j]^2 \, dt.$$

Summing over j, and using the fact that the Brownian motions $W^{(j)}$ are independent, we can take the sum out of the covariation to find

$$\begin{aligned} \Big[\int_0^{\cdot} \Phi^* x^* \, dW_H \Big]_T = \sum_{j=1}^k \Big[\int_0^{\cdot} (\Phi^* x^*)_j(t) \, dW^{(j)}(t) \Big]_T &= \sum_{j=1}^k \int_0^T [\Phi(t)^* x^*, h_j]^2 \, dt \\ &= \int_0^T \|\Phi(t)^* x^*\|_H^2 \, dt. \end{aligned}$$

By a density argument, we can now extend this to general adapted L^p-stochastically integrable processes $\Phi : \Omega \times (0,T) \to \mathscr{L}(H,X)$, and the conclusion follows. □

We can now state and prove the folllowing theorem:

Theorem 5.7. *Let X and H be a separable Banach and Hilbert space respectively. Consider functions $\sigma : [0,\infty) \times X \to \mathscr{L}(H,X)$ and $b : [0,\infty) \times X \to X$ satisfying the following*

1. *The map $(t,x) \mapsto \langle \sigma(t,x)\sigma(t,x)^* x^*, y^* \rangle$ is continuous for all $x^*, y^* \in X^*$.*
2. *The map $(t,x) \mapsto \langle b(t,x), x^* \rangle$ is continuous for any $x^* \in X^*$.*
3. *There is an X-valued zero mean Gaussian random variable g such that*

$$\begin{aligned} \|\sigma(t,x)^* x^*\|_H^2 &\leq \mathbb{E}\langle g, x^* \rangle^2 \\ \langle b(t,x), x^* \rangle^2 &\leq \mathbb{E}\langle g, x^* \rangle^2, \end{aligned}$$

   for all $(t,x) \in [0,\infty) \times X$ and $x^* \in X^*$.

Let μ_0 be a probability measure on X. Then on some filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, there exists an X-valued continuous $\mathcal{F}_t$-adapted process x_t and a cylindrical Brownian motion $\{W_H(t)h : t \geq 0, h \in H\}$ such that

$$x_t = x_0 + \int_0^t \sigma(s,x_s) \, dW_H(s) + \int_0^t b(s,x_s) \, ds \quad a.s., \quad t \geq 0, \tag{5.7}$$

and x_0 has distribution μ_0.

Proof. Our approach is as follows. We try to solve Equation (5.7) by using an Euler approximation. In this way, we circumvent the dependence of σ on x_t in (5.7), and instead work with the discretized version of x_t, so we can use what we already know about (5.3). To this end, choose a sequence of partitions $\pi_n = \{0 = t_0^n < t_1^n < \ldots < t_k^n < \ldots\}$ of $\mathbb{R}$ with mesh tending to zero. We let W_H be an H-cylindrical Brownian motion on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. For n fixed we define

$$x_t^n := x_{t_i^n}^n + \int_{t_i^n}^t \sigma(s, x_{t_i^n}^n)\ dW_H(s) + \int_{t_i^n}^t b(s, x_{t_i^n}^n)\ ds, \quad t \in (t_i^n, t_{i+1}^n].$$

This is actually well-defined by the above assumptions and Theorem 4.2. Define the following processes:

$$\widetilde{\sigma}^n(t) := \sum_{i=1}^{\infty} 1_{(t_{i-1}, t_i]}\sigma(t, x_{t_{i-1}}^n), \quad \widetilde{b}^n(t) := \sum_{i=1}^{\infty} 1_{(t_{i-1}, t_i]} b(t, x_{t_{i-1}}^n), \quad t \geq 0.$$

Note that for each $t \geq 0$, the processes $\widetilde{\sigma}^n$ and $\widetilde{b}^n$ are $\mathcal{F}_t$-measurable, and predictable. By the assumptions on σ and b, we also have

$$\begin{aligned} \|\widetilde{\sigma}^n(t, x_{t_i^n}^n)^* x^*\|_H^2 &= \|\sigma(t, x_{t_i^n}^n)^* x^*\|_H^2 \leq \mathbb{E}\langle g, x^*\rangle^2 \\ \langle \widetilde{b}^n(t, x_{t_i^n}^n), x^*\rangle^2 &= \langle b(t, x_{t_i^n}^n), x^*\rangle^2 \leq \mathbb{E}\langle g, x^*\rangle^2. \end{aligned}$$

By Theorem 4.2 it makes sense to write the stochastic integral of σ. Now by our definition of x_t^n, we have

$$x_t^n = x_0 + \int_0^t \widetilde{\sigma}^n(s)\ dW_H(s) + \int_0^t \widetilde{b}^n(s)\ ds.$$

Thus by the assumptions on σ and b, we can apply Theorem 5.5 to the above, to obtain that the sequence $\{x^n\}_{n\geq 1}$ is relatively compact in distribution in $C([0,\infty), X)$. Hence there is some subsequence $\{x^{n_k}\}_{k\geq 1}$ that converges in $C([0,\infty), X)$ to some process $\{x_t : t \geq 0\}$ in distribution. Let μ_{n_k} and μ be the probability measures on $C([0,\infty), X)$ associated with respectively x^{n_k} and x. Since $C([0,\infty), X)$ is separable, we can use Skorohod's representation theorem to conclude that there are random variables $\widetilde{x}^{n_k}$ and $\widetilde{x}$ with laws μ_{n_k} and μ, such that $\widetilde{x}^{n_k} \to \widetilde{x}$ almost surely.

It follows that for all $t \geq 0$ we have $\widetilde{x}_{t_i^{n_k}}^{n_k} \to \widetilde{x}_t$, where $t_i^{n_k}$ is the element of the n-th partition closest below t. From now on we will not distinguish between x^{n_k} and $\widetilde{x}^{n_k}$ and x and $\widetilde{x}$. Now fix $x^* \in X^*$. By Theorem 4.2 it holds that we can find a $C_T > 0$ such that

$$\Big\| \int_0^t \widetilde{\sigma}^n(s)\ dW_H(s)\Big\|_{L^1(\Omega;X)} \leq C_T, \tag{5.8}$$

for all $n \geq 1$ and $t \in (0, T)$. Moreover, for all $x^* \in X^*$ and $k \geq 1$,

$$\langle x_t^{n_k}, x^*\rangle - \langle x_0^{n_k}, x^*\rangle - \int_0^t \langle \widetilde{b}^{n_k}(s), x^*\rangle\ ds = \Big\langle \int_0^t \widetilde{\sigma}^{n_k}(s)\ dW_H(s), x^*\Big\rangle.$$

Since the RHS is an L^1-martingale by 5.8, so is the LHS. Taking the almost sure limit as $n \to \infty$, we obtain again an L^1-martingale. Hence the process $\langle x_t, x^*\rangle$ is a semimartingale. Denote its martingale part by $m_t(x^*)$ and its bounded variation part by $c_t(x^*)$. We have by the above and the dominated convergence theorem that

$$c_t(x^*) = \lim_{k\to\infty} \int_0^t \langle \widetilde{b}^{n_k}(s), x^*\rangle\ ds = \int_0^t \Big\langle \lim_{k\to\infty} \widetilde{b}^{n_k}(s), x^*\Big\rangle\ ds.$$

But since we have the convergence $x_t^{n_k} \to x_t$ almost surely as $k \to \infty$ and b is continuous, it holds that

$$\lim_{k\to\infty} \widetilde{b}^{n_k}(s) = \lim_{k\to\infty} \widetilde{b}(s, x_{\lfloor s \rfloor}^{n_k}) = b(s, x_s),$$

almost surely, where we denote $\lfloor s \rfloor$ as the element in the n_k-th partition closest below s. Thus

$$c_t(x^*) = \int_0^t \langle b(s, x_s), x^* \rangle \ ds.$$

Similarly we have $\widetilde{\sigma}^{n_k}(s) \to \sigma(s, x_s)$ almost surely for all $s \geq 0$. We now have, again by the dominated convergence theorem and Theorem 5.6,

$$[m.(x^*), m.(y^*)]_t = \int_0^t (\sigma(s, x_s)^* x^*, \sigma(s, x_s)^* y^*)_H \ ds$$

almost surely. We can now extend the stochastic base to $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \{\widetilde{\mathcal{F}}_t\}_{t\geq 0}, \widetilde{\mathbb{P}})$ such that $\widetilde{\Omega}$ supports another H-cylindrical Brownian motion W_H', independent of $\langle x_t, x^* \rangle$ for any $x^* \in X^*$ in the following way. Let $(\Omega', \mathcal{F}', \{\mathcal{F}_t'\}_{t\geq 0}, \mathbb{P}')$ be another probability space with an H-cylindrical Brownian motion W_H' living on it. We define the space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \{\widetilde{\mathcal{F}}_t\}_{t\geq 0}, \widetilde{\mathbb{P}})$ in the following way

$$\widetilde{\Omega} := \Omega \times \Omega', \quad \widetilde{\mathcal{F}} := \sigma(\mathcal{F} \times \mathcal{F}'), \quad \mathbb{P} := \mathbb{P} \otimes \mathbb{P}'.$$

We define, with abuse of notation, the following stochastic processes on $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \{\widetilde{\mathcal{F}}_t\}_{t\geq 0}, \widetilde{\mathbb{P}})$:

$$\langle x_t, x^* \rangle(\omega, \omega') := \langle x_t, x^* \rangle(\omega), \quad W_H(t, \omega, \omega') := W_H(t, \omega), \quad W_H'(t, \omega, \omega') := W_H'(t, \omega').$$

Note that indeed $\{\langle x_t, x^* \rangle : x^* \in X^*\}$ and W_H are independent of W_H'. We can then use Theorem 2 from [31] to find an H-cylindrical Brownian motion $\widetilde{W}_H$ on $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \{\widetilde{\mathcal{F}}_t\}_{t\geq 0}, \widetilde{\mathbb{P}})$, such that $\widetilde{W}_H$ is independent of m_t and

$$m_t(x^*) = \Big\langle \int_0^t \sigma(s, x_s) \ d\widetilde{W}_H(s), x^* \Big\rangle.$$

Here we have defined σ on $\widetilde{\Omega}$ in the same way as above, $\sigma(t, x, \omega, \omega') = \sigma(t, x, \omega)$. Hence

$$\langle x_t, x^* \rangle = \int_0^t \langle b(s, x_s), x^* \rangle \ ds + \Big\langle \int_0^t \sigma(s, x_s) \ d\widetilde{W}_H(s), x^* \Big\rangle,$$

for all $x^* \in X^*$ which finishes the proof. $\square$

5.4 Weak solutions for the stochastic abstract Cauchy problem

In the above subsection, we have found weak solutions to the problem

$$\begin{cases} dx_t &= b(t, x_t) \ dt + \sigma(t, x_t) \ dW_H(t), \quad t > 0 \\ x_0 &= x \end{cases}.$$

However, usually we are interested in solving the stochastic abstract Cauchy problem which can be written in the following way

$$\begin{cases} dU(t) &= AU(t) \ dt + B \ dW_H(t), \quad t > 0 \\ U(0) &= u_0 \end{cases}. \tag{5.9}$$

Here $A : X \to X$ is an unbounded operator which generates a strongly continuous semigroup $(S(t))_{t\geq 0}$ on X, and B can either be a bounded operator or an $\mathscr{L}(H,X)$-valued process.

The problem (5.9) has been extensively studied: in the case where X is a general separable Banach space and $B : H \to X$ is a fixed, deterministic operator, the problem was studied in [5] and then in [28], where in the latter the approach through γ-radonifying operators was first used.

In an even more general case, where B is allowed to depend on t and on $U(t)$, and under the additional assumption that X is a UMD space, the problem (5.9) has been shown by Van Neerven, Veraar, and Weis in [26] to have a unique mild solution in a suitable function space.

In this subsection, we will consider (5.9) when $B : (0,T) \times \Omega \to \mathscr{L}(H,X)$ is a stochastic operator valued process, and X is a general separable Banach space X. We will use Theorem 4.2 (which is Theorem 1 in [12]).

Before we do this, we first repeat the theory from [28]. We will use, as is done in [28], a different notion of *weak solution* than above.

Definition 5.8. *An X-valued process $\{U(t)\}_{t\geq 0}$ is called a weak solution of 5.9 if it is weakly progressively measurable and for all $x^* \in D(A^*)$ the following two conditions are satisfied*

1. *Almost surely, the paths $t \mapsto \langle U(t), A^*x^* \rangle$ are integrable;*
2. *For all $t \in [0,T]$ we have, almost surely,*

$$\langle U(t), x^* \rangle = \langle u_0, x^* \rangle + \int_0^T \langle U(t), A^*x^* \rangle \; ds + W_H(t)B^*x^*.$$

Note that this notion of weak solution is stronger than what we had in the subsection before, since here our solution can not be constructed with just any arbitrary $\widetilde{W}_H$, but rather the W_H from the equation itself.

This definition of weak solution leads us to the following theorem, which is Theorem 7.1 in [28].

Theorem 5.9. *The following assertions are equivalent:*

1. *The problem 5.9 has a weak solution $\{U(t)\}_{t\geq 0}$;*
2. *The function $t \mapsto S(t)B$ is stochastically integrable on $(0,T)$ with respect to the H-cylindrical Brownian motion W_H;*
3. *The operator $R \in \mathscr{L}(X^*, X)$ defined by*

$$Rx^* := \int_0^T S(t)BB^*S(t)^*x^* \; dt$$

   is a Gaussian covariance operator;

4. *The operator $I \in \mathscr{L}(L^2(0,T;H), X)$ defined by*

$$If := \int_0^T S(t)Bf(t) \; dt$$

   is γ-radonifying from $L^2(0,T;H)$ to X.

In this case, for every $t \in [0,T]$ the function $s \mapsto S(t-s)B$ is stochastically integrable with respect to W_H and we have

$$U(t) = S(t)u_0 + \int_0^t S(t-s)B \; dW_H(s)$$

almost surely.

We will from now on assume B to be a continuous stochastic process, that is

$$B : [0,T] \times \Omega \to \mathscr{L}(H,X)$$

is progressively measurable and adapted. We thus consider the following Cauchy problem

$$\begin{cases} dU(t,\omega) &= AU(t,\omega)\; dt + B(t,\omega)\; dW_H(t,\omega) \\ U(0,\omega) &= u_0(\omega) \end{cases}. \tag{5.10}$$

We finally state and prove the following theorem. A generalization of Theorem 5.9:

Theorem 5.10. *Consider the problem 5.10 in a general separable Banach space X. Assume that there exists a $\widetilde{B} : (0,T) \to \mathscr{L}(H,X)$ such that $\widetilde{B}$ is stochastically integrable with respect to W_H and the following two assumptions hold:*

1. *We have*
$$\|B(t,\omega)^* x^*\|_H \leq \|\widetilde{B}(t)^* x^*\|_H, \quad \forall x^* \in X^*.$$
2. *The operator $Q_T : X^* \to X$, defined for all $x^* \in X^*$ by*
$$Q_T x^* := \int_0^T S(T-t)\widetilde{B}(t)\widetilde{B}(t)^* S(T-t)^* x^* \; dt,$$
is a Gaussian covariance operator.

Then the problem 5.10 has a weak solution $\{U(t)\}_{t\geq 0}$ given by

$$U(t,\omega) = S(t)u_0 + \int_0^t S(t-s)B(t,\omega)\; dW_H(s). \tag{5.11}$$

Proof. The proof that 5.11 actually gives a weak solution to 5.10 can be copied mutatis mutandis from the proof of Theorem 7.1 in [28]. We still need to prove that $s \mapsto S(t-s)B(s,\omega)$ is stochastically integrable with respect to the H-cylindrical Brownian motion W_H for all $t \in [0,T]$, under assumption (1) and (2). Note first of all that $s \mapsto S(t-s)B(s,\omega)$ is strongly continuous. Moreover, we have for all $\omega \in \Omega$ and $t \in [0,T]$

$$\int_0^t \|B(s,\omega)^* S(t-s)^* x^*\|_H^2 \; ds \leq \int_0^T \|B(s,\omega)^* S(T-s)^* x^*\|_H^2 \; ds.$$

By (1)

$$\int_0^T \|B(s,\omega)^* S(T-s)^* x^*\|_H^2 \; ds \leq \int_0^T \|\widetilde{B}(s)^* S(T-s)^* x^*\|_H^2 \; ds.$$

Now by assumption (2) there exists an X-valued Gaussian G_T with covariance operator Q_T. We have

$$\begin{aligned}
\int_0^T \|\widetilde{B}(s)^* S(T-s)^* x^*\|_H^2 \, ds &= \int_0^T (\widetilde{B}(s)^* S(T-s)^* x^*, \widetilde{B}(s)^* S(T-s)^* x^*)_H \, ds \\
&= \int_0^T \langle S(T-s)\widetilde{B}(s)\widetilde{B}(s)^* S(T-s)^* x^*, x^* \rangle \, ds \\
&= \Big\langle \int_0^T S(T-s)\widetilde{B}(s)\widetilde{B}(s)^* S(T-s)^* x^* \, ds, x^* \Big\rangle \\
&= \langle Q_T x^*, x^* \rangle = \mathbb{E}\langle G_T, x^* \rangle^2.
\end{aligned}$$

We can now use 4.2 to conclude the stochastic integrability of $s \mapsto S(t-s)B(s,\omega)$, thus 5.11 is well-defined for almost all $(t,\omega) \in [0,T] \times \Omega$. This finishes the proof. $\square$

Chapter 6

Kalinichenko in the UMD$^-$-setting

As it turns out, the proof of Kalinichenko's main theorem is much more immediate when we assume our Banach space X to be UMD$^-$, and in the case that our integrator is deterministic, the result was already known. In both cases, no advanced chaining techniques are necessary.

We start with the more complicated case, in which X is UMD$^-$ and Φ is an $\mathscr{L}(H,X)$-valued stochastic process.

First we define a UMD$^-$ space as in [21].

Definition 6.1. *Let $(r_n)_{n\geq 1}$ be a Rademacher sequence and let X be a Banach space. We call X a UMD$^-$ space if for some (equivalently, for all) $1 < p < \infty$ there exists a constant β_p such that for all finite X-valued martingale difference sequences $(d_n)_{n\geq 1}$ independent of $(r_n)_{n\geq 1}$ we have*

$$\mathbb{E}\Big\| \sum_{n=1}^{N} d_n \Big\|^p \leq \beta_p^p \mathbb{E}\Big\| \sum_{n=1}^{N} r_n d_n \Big\|^p.$$

Now let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a probability space and H a separable Hilbert space with an H-cylindrical Brownian motion W_H. Assume $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \{\widetilde{\mathcal{F}}_t\}_{t\geq 0}, \widetilde{\mathbb{P}})$ is an independent copy of this probability space with H-cylindrical Brownian motion $\widetilde{W}_H$. Then, in the UMD$^-$ setting there is the following one-sided decoupling lemma, which is similar to Lemma 3.4 from [25]:

Lemma 6.2. *Let H be a separable Hilbert space and $1 < p < \infty$. Then if X is a UMD$^-$ space, there exists some constant $\beta_{p,X}$ such that the following inequality holds for every elementary adapted process $\Phi : [0,T] \times \Omega \to \mathscr{L}(H,X)$*

$$\mathbb{E}\Big\| \int_0^T \Phi(t) \; dW_H(t) \Big\|^p \leq \beta_{p,X}^p \mathbb{E}\widetilde{\mathbb{E}}\Big\| \int_0^T \Phi(t) \; d\widetilde{W}_H(t) \Big\|^p.$$

Here we will not prove the lemma, as its proof is identical to the proof of Lemma 3.4 in [25], with the only difference being the one-sided UMD property. Similarly, we can prove a version of Theorem 3.5 from the same article for UMD$^-$ spaces using the above. For elementary adapted $\Phi \in L^p(\Omega; \gamma(L^2(0,T;H), X))$ we define the random variable $I^{W_H}(\Phi) \in L^p(\Omega; X)$ by

$$I^{W_H}(\Phi) = \int_0^T \Phi(t) \; dW_H(t).$$

We have $I^{W_H}(\Phi) \in L_0^p(\Omega, \mathcal{F}_T; X)$, the subspace of all mean-zero $\mathcal{F}_T$-measurable random variables in $L^p(\Omega; X)$. In the next theorem we will extend I^{W_H} to a bounded operator from

$L^p_{\mathcal{F}}(\Omega;\gamma(L^2(0,T;H),X))$ to $L^p_0(\Omega,\mathcal{F}_T;X)$. Here $L^p_{\mathcal{F}}(\Omega;\gamma(L^2(0,T;H),X))$ denotes the closure of elementary adapted elements in $L^p(\Omega;\gamma(L^2(0,T;H),X))$.

Theorem 6.3. *Let X be a UMD$^-$ space and fix $1<p<\infty$. The mapping $\Phi\mapsto I^{W_H}(\Phi)$ has a unique extension to a bounded operator*

$$I^{W_H}:L^p_{\mathcal{F}}(\Omega;\gamma(L^2(0,T;H),X))\to L^p_0(\Omega,\mathcal{F}_T;X).$$

Proof. Let $\Phi\in L^p(\Omega;\gamma(L^2(0,T;H),X))$ be elementary and adapted. We have by Lemma 6.2 and the Ito isometry (Theorem 5.1 in [23])

$$\begin{aligned}\mathbb{E}\|I^{W_H}(\Phi)\|^p=\mathbb{E}\Big\|\int_0^T\Phi(t)\;dW_H(t)\Big\|^p &\le \beta^p_{p,X}\mathbb{E}\Big\|\int_0^T\Phi(t)\;d\widetilde{W}_H(t)\Big\|^p_{L^p(\widetilde{\Omega};X)}\\ &= \beta^p_{p,X}\mathbb{E}\|\Phi\|^p_{\gamma^p(L^2(0,T;H),X)}.\end{aligned}$$

Since we have $\|\cdot\|_{\gamma^p(L^2(0,T;H),X)}\le k_{p,2}\|\cdot\|_{\gamma(L^2(0,T;H),X)}$ by the Kahane-Khintchine inequalities, it follows that

$$\|I^{W_H}(\Phi)\|^p_{L^p(\Omega;X)}=\mathbb{E}\|I^{W_H}(\Phi)\|^p\le\beta^p_{p,X}k^p_{p,2}\mathbb{E}\|\Phi\|^p_{\gamma(L^2(0,T;H),X)}=\beta^p_{p,X}k^p_{p,2}\|\Phi\|^p_{L^p(\Omega;\gamma(L^2(0,T;H),X))}.$$

Thus the operator I^{W_H} extends uniquely to a bounded operator from $L^p_{\mathcal{F}}(\Omega;\gamma(L^2(0,T;H),X))$ into $L^p_0(\Omega,\mathcal{F}_T;X)$. $\square$

We can now start proving a version of Theorem 1 from [12].

Theorem 6.4. *Let X be a UMD$^-$ space and H a separable Hilbert space. We moreover define W_H to be an H-cylindrical Brownian motion on a filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\ge0},\mathbb{P})$. Let $\Phi:[0,T]\times\Omega\to\mathcal{L}(X,H)$ be a predictable process such that for some X-valued Gaussian random variable we have*

$$\int_0^T\|\Phi_t^*x^*\|_H^2\;dt\le\mathbb{E}\langle g,x^*\rangle^2\quad a.s,\quad x^*\in X^*. \tag{6.1}$$

Then Φ_t is stochastically integrable on $[0,T]$ with respect to W_H and we have the following inequality

$$\left(\mathbb{E}\Big\|\int_0^T\Phi(t)\;dW_H(t)\Big\|^p\right)^{1/p}\le K_p\mathbb{E}\|g\|_X.$$

Proof. Let $Q:X^*\to X$ be the covariance operator of g. We start by showing that the process $\Phi:[0,T]\times\Omega\to\mathcal{L}(X,H)$ actually is an element of $\gamma(L^2(0,T;H),X)$ for each $\omega\in\Omega$. To this end, we use Theorem 9.4.1 from [9] with $H_1=L^2(0,T;H)$ and H_2 the reproducing kernel Hilbert space H_Q of g. We know that the embedding $i:H_Q\to X$ is compact and $Q=ii^*$. Thus, by Theorem 5.16 from [24], $i\in\gamma(H_Q,X)$.

Moreover we have, by assumption, for almost all $\omega\in\Omega$,

$$\|\Phi^*(\omega)x^*\|^2_{L^2(0,T;H)}=\int_0^T\|\Phi_t^*(\omega)x^*\|_H^2\;dt\le\mathbb{E}\langle g,x^*\rangle^2=\|Qx^*\|^2_{H_Q}.$$

As maps in $\mathcal{L}(X^*,H_Q)$, we have the identity $i^*=Q$, and we know $i:H_Q\to X$ is γ-radonifying, so by 9.4.1 in [9], we have $\Phi(\omega)\in\gamma(L^2(0,T;H),X)$ for almost every $\omega\in\Omega$. We also have the almost sure estimate

$$\|\Phi(\omega)\|^2_{\gamma(L^2(0,T;H),X)}\le\|i\|^2_{\gamma(H_Q,X)}=\mathbb{E}\|g\|_X^2.$$

By now taking $p/2$ powers and the expectation of the above we have

$$\|\Phi\|^p_{L^p(\Omega;\gamma(L^2(0,T;H),X))} = \mathbb{E}\|\Phi\|^p_{\gamma(L^2(0,T;H),X)} \leq \|g\|^p_{L^2(\Omega;X)}.$$

Before we can use Theorem 6.3, we still need to show that Φ is $\mathcal{F}$-strongly adapted. Since Φ is predictable, we have for all $t \in [0,T]$, $f \in H$ and $x^* \in X^*$

$$\langle \Phi(1_{[0,t]}f), x^*\rangle = [1_{[0,t]}f, \Phi^* x^*]_{L^2(0,T;H)} = \int_0^t [f, \Phi^* x^*]_H \, dt.$$

Since Φ is a predictable process, so is $[f, \Phi^* x^*]_H$, thus the above integral is in particular $\mathcal{F}_t$-measurable. From Proposition 2.10 in [25] it follows that Φ is strongly adapted to $\mathcal{F}$. We are now in the setting of Theorem 6.3. We have

$$\mathbb{E}\|I^{W_H}(\Phi)\|^p \leq \beta^p_{p,X} k^p_{p,2} \|\Phi\|^p_{L^p(\Omega;\gamma(L^2(0,T;H),X))} \leq \beta^p_{p,X} k_{p,2} \|g\|^p_{L^2(\Omega;X)}.$$

We can now once again use the Kahane-Khintchine inequality to estimate $\|g\|^p_{L^2(\Omega;X)} \leq k^p_{2,1}\|g\|^p_{L^1(\Omega;X)} = K(\mathbb{E}\|g\|_X)^p$. In conclusion:

$$\left(\mathbb{E}\Big\| \int_0^T \Phi(t) \, dW_H(t) \Big\|^p\right)^{1/p} \leq \beta_{p,X} k_{p,2} k_{2,1} \mathbb{E}\|g\|_X.$$

□

As we have seen from the above proof, the existence of an X-valued Gaussian random variable g such that (6.1) holds, implies that $\Phi \in L^\infty(\Omega; \gamma(L^2(0,T;H),X)$ and that Φ is stochastically integrable. On the other hand, if $\Phi \in \gamma(L^2(0,T;H),X)$ is deterministic, the existence of such a g already follows. Indeed, assume $\Phi \in \gamma(L^2(0,T;H),X)$. Then by Theorem 5.16 in [24], the operator $Q := \Phi\Phi^* : X^* \to X$ is a covariance operator associated with a Gaussian random variable $g : \Omega \to X$. Let again H_Q be its reproducing kernel Hilbert space. We have for all $x^* \in X^*$,

$$\mathbb{E}\langle g, x^*\rangle^2 = \|Qx^*\|^2_{H_Q} = (Qx^*, x^*)_X = (\Phi^* x^*, \Phi^* x^*)_{L^2(0,T;H)} = \int_0^T \|\Phi^* x^*\|^2_H \, dt.$$

We thus have the following corollary, which sums up the equivalence between Kalinichenko's condition and theory of radonifying operators:

Corollary 6.5. *Let X be a separable (not necessarily UMD$^-$) Banach space, and $\Phi : [0,T] \to \mathscr{L}(H,X)$ be a measurable process. Then Φ satisfies the conditions of Kalinichenko, that is, there exists an X-valued Gaussian g such that for all $x^* \in X^*$*

$$\int_0^T \|\Phi^* x^*\|^2_H \, dt \leq \mathbb{E}\langle g, x^*\rangle^2,$$

if and only if $\Phi \in \gamma(L^2(0,T;H),X)$.

In general however (the non-deterministic case), such g need not exist for all processes $\Phi \in L^\infty(\Omega; \gamma(L^2(0,T;H),X)$. We take a look at the following example.

Example 6.6. *Let $\Omega = [0,1]$ with the Lebesgue measure. We set $X = c_0$ and write $H_1 = L^2(0,T;H)$. Let $(h_n)_{n\geq 1}$ be an ONB of H and set $g_n := (2^n T^{-1})^{1/2} 1_{(2^{-n}T, 2^{-n+1}T]} \otimes h_n$ for every $n \geq 1$. Note that $(g_n)_{n\geq 1}$ is an orthonormal system in H_1. Let $(e_n)_{n\geq 1}$ be the canonical basis for c_0. We define Φ in the following way:*

$$\Phi(\omega) = \sum_{n\geq 1} \Phi_n 1_{A_n}(\omega), \quad \omega \in \Omega,$$

with each $\Phi_n = g_n \otimes e_n$, and where $(A_n)_{n\geq 1}$ is a partition of Ω. By starting the filtration with $\mathcal{F}_0 := \sigma(A_n : n \geq 1)$, this process is adapted. Essentially, at time $t = 0$ we roll an 'infinite dice' to determine which process Φ_n we will take as a path. Note that for each $n \geq 1$, we have $\|\Phi_n\|^2_{\gamma(H_1,X)} = 1$ and thus $\Phi \in L^\infty(\Omega; \gamma(H_1, X))$. Fix $\varepsilon > 0$. Now let g be any X-valued Gaussian with covariance $Q : X^* \to X$ and let H_Q denote its reproducing kernel Hilbert space. Let $(h'_n)_{n\geq 1}$ denote the ONB of H_Q and choose any $\Psi \in \gamma(H_Q, X)$. Then there exists $\widetilde{\Psi} \in \gamma(H_Q, X)$ such that $\|\widetilde{\Psi} - \Psi\|^2_{\gamma(H_Q,X)} < \varepsilon$ of the form

$$\widetilde{\Psi} = \sum_{k=0}^{K} h'_k \otimes x_k.$$

Since each $x_k \in c_0$, there is an $N \in \mathbb{N}$ such that $|x_k^{(n)}| \leq \varepsilon/(K+1)$ for all $n \geq N$, for all $k \in \{0, 1, \ldots, K\}$. With some abuse of notation, we denote the canonical basis of $(c_0)^* = \ell^1$ by $(e_n^*)_{n\geq 1}$. Now,

$$\|\widetilde{\Psi}^* e_N^*\|^2_{H_Q} = \Big\| \sum_{k=0}^{K} (x_k, e_N^*) h'_k \Big\|^2_{H_Q} \leq \Big\| \sum_{k=0}^{K} \frac{\varepsilon}{K+1} h'_k \Big\|^2_{H_Q} = \varepsilon^2.$$

Where this last equality is due to Plancherel. Now,

$$\begin{aligned} \|\Psi^* e_N^*\|_{H_Q} &\leq \|\widetilde{\Psi}^* e_N^*\|_{H_Q} + \|(\Psi - \widetilde{\Psi})^* e_N^*\|_{H_Q} \\ &\leq \varepsilon + \|\Psi - \widetilde{\Psi}\|_{\mathcal{L}(H_Q,X)} \|e_N^*\|_{X^*} \\ &\leq \varepsilon + \|\Psi - \widetilde{\Psi}\|_{\gamma(H_Q,X)} \|e_N^*\|_{X^*} < 2\varepsilon. \end{aligned}$$

On the other hand, if we take $\omega \in A_N$ we have $\Phi(\omega) = g_N \otimes e_N$, so

$$\|\Phi(\omega)^* e_N^*\|^2_{H_1} = \|(e_N, e_N^*) g_N\|^2_{H_1} = 1.$$

In conclusion, for each $\Psi \in \gamma(H_Q, X)$, we can find an $x^* \in X^*$ and $N \in \mathbb{N}$ such that for all $\omega \in (2^{-N}, 2^{-N+1}]$,

$$\|\Psi^* x^*\|^2_{H_Q} < \|\Phi(\omega)^* x^*\|^2_{H_1}.$$

In particular, if we let $i \in \gamma(H_Q, X)$ be the embedding $i : H_Q \to X$, we have

$$\mathbb{E}\langle g, x^*\rangle^2 = \|Qx^*\|^2_{H_Q} = \|i^* x^*\|^2_{H_Q} < \|\Phi(\omega)^* x^*\|^2_{H_1} = \int_0^T \|\Phi(\omega)^* x^*\|^2_H \, dt.$$

So for each X-valued Gaussian g, there exists some $x^* \in X^*$ such that on a set $A \subset \Omega$ with $\mathbb{P}(A) > 0$, Equation (6.1) does not hold for our Φ.

Despite all of this, our process Φ is stochastically integrable anyway. To prove this, we let $\{W_H(t) : t \in [0,T]\}$ be an H-cylindrical Brownian motion, which we construct in the following way. Let $W^{(n)}$ be independent Brownian motions on Ω. We set for any $h' \in H$,

$$W_H(t)h' = \sum_{n\geq 1} W^{(n)}(t)[h', h_n].$$

Using the definition of stochastic integral, we find for each $\omega \in \Omega$,

$$\begin{aligned}\int_0^T \Phi(\omega)\ dW_H(\omega) &= \sum_{n\geq 1} 1_{A_n}(\omega) \int_0^T \Phi_n dW_H(\omega) \\ &= \sum_{n\geq 1} 1_{A_n}(\omega)(W_H g_n)e_n \\ &= \sum_{n\geq 1} (2^n T^{-1})^{1/2} 1_{A_n}(\omega)(W^{(n)}(2^{-n}T) - W^{(n)}(2^{-n+1}T))e_n.\end{aligned}$$

Note that for each $n \geq 1$, we have that $X_n := (2^n T^{-1})^{1/2}(W^{(n)}(2^{-n}T) - W^{(n)}(2^{-n+1}T))$ is a standard Gaussian. Since the $W^{(n)}$ are all independent Brownian motions, the X_n are independent as well. So the above integral exists and takes values in c_0, and for $p \geq 1$,

$$\mathbb{E}\Big\| \int_0^T \Phi\ dW_H \Big\|_{c_0}^p = \mathbb{E}\Big[\mathbb{E}\Big[\Big\| \int_0^T \Phi\ dW_H \Big\|_{c_0}^p |\mathcal{F}_0\Big]\Big].$$

Computing now the conditional expectation gives

$$\mathbb{E}\Big[\Big\| \int_0^T \Phi\ dW_H \Big\|_{c_0}^p |\mathcal{F}_0\Big] = \mathbb{E}\Big[\Big\| \sum_{n\geq 1} 1_{A_n}(\omega) X_n e_n \Big\|_{c_0}^p |\mathcal{F}_0\Big] \leq \sum_{n\geq 1} 1_{A_n}(\omega)\mathbb{E}[\|X_n e_n\|_{c_0}^p |\mathcal{F}_0].$$

Note that since each X_n is independent of $\mathcal{F}_0$, we have for each $n \geq 1$,

$$\mathbb{E}[\|X_n e_n\|_{c_0}^p |\mathcal{F}_0] = \mathbb{E}[|X_n|^p |\mathcal{F}_0] = \mathbb{E}|X_n|^p = \mathbb{E}|N(0,1)|^p.$$

In conclusion,

$$\mathbb{E}\Big[\mathbb{E}\Big[\Big\| \int_0^T \Phi\ dW_H \Big\|_{c_0}^p |\mathcal{F}_0\Big]\Big] \leq \mathbb{E}\Big[\sum_{n\geq 1} 1_{A_n}(\omega)\mathbb{E}|N(0,1)|^p\Big] = \mathbb{E}|N(0,1)|^p.$$

Chapter 7

Kalinichenko in the martingale type 2 setting

The attentive reader may have noticed that in this thesis, we have not discussed the setting where X is a Banach space with martingale type 2 at all, and we have even omitted the martingale type 2 spaces from the preliminaries section. The reason for this is that, unlike in the UMD($^-$) situation, there is no natural way to compare the conditions from Kalinichenko's article [12] with the setting where X is a martingale type 2 space. Before we explain this more rigorously, we need to set up some definitions and theorems.

7.1 Type p and martingale type p spaces

In this section we will mainly use theory from the survey [23] and the 2005 article by Jan van Neerven and Lutz Weis [29]. We let $(r_n)_{n\geq 1}$ be a sequence of independent Rademacher random variables, that is, $\mathbb{P}(r_n = 1) = \mathbb{P}(r_n = -1) = 1/2$ for all $n \geq 1$.

Definition 7.1. *Let $p \in [1, 2]$. A Banach space X has type p if there exists a constant $\tau > 0$ such that for all finite sequences $(x_n)_{n=1}^N$ in X we have*

$$\mathbb{E}\Big\| \sum_{n=1}^{N} r_n x_n \Big\|_X^p \leq \tau^p \sum_{n=1}^{N} \|x_n\|_X^p.$$

On the other hand, we say that X has cotype $q \in [2, \infty]$ if there exists a constant $c > 0$ such that for all finite sequences $(x_n)_{n=1}^N$ in X we have

$$\sum_{n=1}^{N} \|x_n\|^q \leq c^q \mathbb{E}\Big\| \sum_{n=1}^{N} r_n x_n \Big\|_X^q.$$

We denote the least admissible constants in the above with $\tau_{p,X}$ and $c_{p,X}$ respectively. We also define martingale type p spaces:

Definition 7.2. *Let $p \in [1, 2]$. A Banach space X is has martingale type p if there exists a constant $\mu > 0$ such that for all finite X-valued martingale difference sequences $(d_n)_{n=1}^N$ we have*

$$\mathbb{E}\Big\| \sum_{n=1}^{N} d_n \Big\|_X^p \leq \mu^p \sum_{n=1}^{N} \mathbb{E}\|d_n\|_X^p.$$

We denote the least admissible constant with $\mu_{p,X}$. Some examples of martingale type 2 spaces are Hilbert spaces, and $L^p(S)$ spaces for $p \geq 2$. In fact $L^p(S)$ has martingale type $p \wedge 2$. Moreover, we can immediately see that every Banach space with martingale type p also has type p, since the sequence $(r_n d_n)_{n=1}^N$ can be viewed as a martingale difference sequence. As it turns out, martingale type 2 spaces are the appropriate setting for integrating stochastic processes $\Phi : (0,T) \times \Omega \to \gamma(H,X)$ with

$$\mathbb{E}\int_0^T \|\Phi(t)\|_{\gamma(H,X)}^2 \, dt < \infty.$$

We have the following theorem, which is Theorem 4.6 in the survey [23]:

Theorem 7.3. *Let X be a Banach space with martingale type 2 and assume we have a progressively measurable stochastic process $\Phi \in L^p(\Omega; L^p(0,T;\gamma(H,X)))$. Then Φ is stochastically integrable with respect to any H-cylindrical Brownian motion W_H and*

$$\mathbb{E}\Big\| \int_0^T \Phi(t) \, dW_H(t) \Big\|_X^p \leq \mu_{p,X}^p \mathbb{E}\int_0^T \|\Phi(t)\|_{\gamma(H,X)}^p \, dt.$$

Thus, if X is a martingale type 2 space, the appropriate space for our Φ to live in would be $L^p(\Omega; L^2(0,T;\gamma(H,X)))$. This in contrast to the case when X is a UMD space, where we require $\Phi \in L^p(\Omega; \gamma(L^2(0,T;H),X))$. The assumption on Φ in the UMD case is weaker than the assumption in the martingale type 2 case, meaning every function $\Psi \in L^2(0,T;\gamma(H,X))$ defines an operator in $\gamma(L^2(0,T;H),X)$. We can see this in the following way: let $\Psi \in L^2(0,T;\gamma(H,X))$ be of the form $\Psi = 1_{[a,b]} \otimes R$ with $R \in \gamma(H,X)$ and $[a,b] \subset (0,T)$, then we can define the operator I_Ψ by

$$I_\Psi f := \int_0^T R(f, 1_{[a,b]})_{L^2(0,T)} \, dt, \quad f \in L^2(0,T;H).$$

Since R is γ-radonifying, then so is I_Ψ by the ideal property. We can then extend by linearity. Note that this argument works for general Banach spaces. It turns out that in the type 2 case, the above mapping is actually a continuous embedding. The following theorem is from [29]:

Theorem 7.4. *Let X be a Banach space with type 2. Then the mapping*

$$I : L^2(0,T;\gamma(H,X)) \hookrightarrow \gamma(L^2(0,T;H),X)$$

given by $I : \Psi \mapsto I_\Psi$ has a unique extension to a continuous embedding with norm $\|I\| \leq \tau_{2,X}$.

The above theorem has another version when X has cotype 2, where we can look at I^{-1}.

Theorem 7.5. *Let X be a Banach space with cotype 2. Then*

$$I^{-1} : \gamma(L^2(0,T;H),X) \hookrightarrow L^2(0,T;\gamma(H,X))$$

has a unique extension to a continuous embedding with norm $\|I^{-1}\| \leq c_{2,X}$.

We repeat the conditions for stochastic integration from Kalinichenko [12]. We need some X-valued Gaussian g such that for all $x^* \in X^*$

$$\int_0^T \|\Phi(t)^* x^*\|_H^2 \, dt \leq \mathbb{E}\langle g, x^*\rangle^2, \quad \text{a.s.} \tag{7.1}$$

As we have seen in the previous section, this condition implies that $\Phi \in L^\infty(\Omega; \gamma(L^2(0,T;H),X))$, however the other way around does not hold, as was showcased with a counterexample.

From now on assume that X is a martingale type 2 space. Take any $\phi \in L^2(0,T;\gamma(H,X))$ and let ξ be a real valued standard Gaussian random variable. Set $\Phi = \xi \otimes \phi$. Then

$$\|\Phi\|^2_{L^2(\Omega;L^2(0,T;\gamma(H,X)))} = \mathbb{E}\|\xi \otimes \phi\|^2_{L^2(0,T;\gamma(H,X))} = \|\phi\|^2_{L^2(0,T;\gamma(H,X))} < \infty.$$

Now since X is a martingale type 2 space, by Theorem 7.3, we have that Φ is stochastically integrable. However, this conclusion can not be drawn from Kalinichenko. Note that by Theorem 7.4 we have $\Phi(\cdot,\omega) \in \gamma(L^2(0,T;H),X)$, but by the Gaussianity of ξ, we also have $\Phi \notin L^\infty(\Omega;\gamma(L^2(0,T;H),X))$. Thus Equation 7.1 does not hold for any X-valued Gaussian.

On the other hand if X does not have cotype 2, we can choose a $\psi \in \gamma(L^2(0,T;H),X)$ such that $\psi \notin L^2(0,T;\gamma(H,X))$, and thus does not satisfy the conditions of Theorem 7.3. If $\zeta \in L^\infty(\Omega)$ is any non-constant random variable, then we can set $\Psi = \zeta \otimes \psi$. We have

$$\int_0^T \|\Psi(t)^* x^*\|_H^2 \, dt \leq \int_0^T \|\zeta\|^2_{L^\infty(\Omega)} \|\psi^*(t)x^*\|_H^2 \, dt = \|\zeta\|^2_{L^\infty(\Omega)} \|\psi^* x^*\|^2_{L^2(0,T;H)}.$$

Since we assumed $\psi : L^2(0,T;H) \to X$ is γ-radonifying, the operator $Q = \psi\psi^*$ is a covariance operator of some X-valued Gaussian G. Then

$$\int_0^T \|\Psi(t)^* x^*\|_H^2 \, dt \leq \mathbb{E}\langle \|\zeta\|_{L^\infty(\Omega)} G, x^*\rangle^2.$$

Thus the conditions in Kalinichenko are satisfied.

7.2 Comparing UMD and martingale type p spaces

In this section we will briefly go over the differences between UMD spaces and martingale type 2 spaces, which are the two main settings for stochastic integration in Banach spaces. The class of UMD spaces and the martingale type 2 class are different, for example the space $L^{3/2}(S)$ is UMD, since all L^p-spaces for $1 < p < \infty$ are UMD, but it has martingale type $3/2$. Finding an example the other way around, of a Banach space which is martingale type 2 but not UMD, still seems to be an open problem. A proof of existence for a type 2 space that is not UMD was given by R.C. James in 1978 [10]. In fact, he constructed an example of a non-reflexive Banach space having type 2; since UMD space are reflexive, this proves the existence of a non-UMD type 2 space. Then in 1983, Bourgain proved in [2] that for each $p \in (1,2)$ there exists a martingale type p space that is not UMD. We will briefly go over the theorem in this section. The definitions here are taken from Lindenstrauss and Tzafriri [17], and Theorem 7.8 is from [2].

Bourgain works with Banach lattices, which is a special type of Banach space:

Definition 7.6. *A partially ordered Banach space X over the real number field is called a Banach lattice provided*

1. $x \leq y$ *implies* $x + z \leq y + z$ *for all* $x, y, z \in X$*;*
2. *for all* $x \in X$ *such that* $x \geq 0$ *and* $a \in \mathbb{R}_+$ *we have* $ax \geq 0$*;*
3. *for all* $x, y \in X$ *there exist* $x \wedge y \in X$ *and* $x \vee y \in X$*;*
4. $|x| \leq |y|$ *implies* $\|x\| \leq \|y\|$*, where we have defined* $|x| = x \vee (-x)$*.*

An example of a Banach lattice would be $C[0,1]$, with $f \leq g$ if $f(x) \leq g(x)$ for all $x \in [0,1]$. We say that $x, y \in X$ are disjoint if $|x| \wedge |y| = 0$.

Definition 7.7. *Let $1 < p < \infty$. We say that a Banach lattice X satisfies an upper, respectively lower, p-estimate if there exists an $M > 0$ such that for every choice of pairwise disjoint elements $\{x_j\}_{j=1}^n$ in X we have*

$$\Big\| \sum_{j=1}^n x_j \Big\| \leq M \Big(\sum_{j=1}^n \|x_j\|^p \Big)^{1/p},$$

respectively,

$$\Big\| \sum_{j=1}^n x_j \Big\| \geq M^{-1} \Big(\sum_{j=1}^n \|x_j\|^p \Big)^{1/p}.$$

We are now ready to state the following result by Bourgain [2]:

Theorem 7.8. *For $1 < p < q < \infty$ there exists a Banach lattice X such that X satisfies an upper $p-$ and lower q-estimate, but is not UMD.*

We say that a Banach space has p-smoothness for $p \in [1, 2]$ if for some $D > 0$

$$\|x + y\|^p + \|x - y\|^p \leq 2\|x\|^p + 2D^p\|y\|^p.$$

Now the following theorem, which is Theorem 1.f.10 in [17] says that the upper p-estimate implies p-smoothness:

Theorem 7.9. *Let $1 < p < 2 < q$ and assume X is a Banach lattice with upper p-estimate and lower q-estimate. Then there exists a norm $\| \cdot \|'$ on X which is equivalent to the original norm of X such that $(X, \| \cdot \|')$ with the original order is a p-smooth Banach space. Moreover X has type p and cotype q.*

It is a result by Pisier [32] (see also [22]) that for any $p \in [1, 2]$ a Banach space has p-smoothness if and only if it is a martingale type p space, up to equivalent norms. Hence by 7.8, there exists for all $1 < p < q < \infty$ a non-UMD Banach lattice X such that X has upper p-estimate and lower q-estimate. Now by 7.9, if $1 < p < 2 < q$, we have that this X has an equivalent norm $\| \cdot \|'$ such that $(X, \| \cdot \|')$ is p-smooth. By [32], we have that there exists a norm $\| \cdot \|''$ equivalent to $\| \cdot \|'$ (and thus to $\| \cdot \|$) such that $(X, \| \cdot \|'')$ has martingale type p. Now since equivalent norms preserve the UMD property, $(X, \| \cdot \|'')$ is a Banach space with martingale type p, for some $p \in (1, 2)$, but it is not UMD.

However, the above approach can not be used to prove existence of a space with martingale type 2 which is not UMD. Whether such a space actually exists still seems to be an open problem.

Chapter 8

Conclusion

In this thesis we have set up the theory to read and understand the proof for stochastic integration in general separable Banach spaces, by Kalinichenko [12]. Although mostly a literature study, at times we have fixed some inaccuracies in the proofs of [12]. We have also generalized a few results from the known theory to non-UMD cases, or extended some results from deterministic to stochastic cases. We extended Theorem 3.1 from [27] to the case where one of the functions is stochastic in the non-UMD case in Corollary 4.4. Moreover, we considered the stochastic abstract Cauchy problem in non-UMD spaces and extended Theorem 7.1 from [28] to the stochastic case.

The thesis started out with a section on the current setting in which stochastic integration in Banach spaces is usually done, where we look at γ-radonifying operators and our Banach space X is assumed to be a UMD space. The advantage of this theory is the minimal and natural assumptions we need on our stochastic process Φ. However, the theory has its limits, for example when trying to do stochastic integration in the space of continuous functions $C(\mathcal{X})$, with $(\mathcal{X}, d)$ any metric space. In this case the theory set up for UMD spaces falls short, since $C(\mathcal{X})$ is not a UMD space, and it does not have martingale type 2 either.

This is where the theory of [12] can be used. Under a very strict assumption on the integrating process, we achieve stochastic integrability in general separable Banach spaces. The conditions, stated in Theorem 4.2, can be roughly compared to the radonifying assumption from [25]. In fact we have seen the following chain of one-sided implications in Section 6 (where we write $\Phi \in \mathrm{Kal}$ if Φ satisfies the assumption from Theorem 4.2):

$$\Phi \in \gamma(L^2(0,T;H),X) \implies \Phi \in \mathrm{Kal} \implies \Phi \in L^\infty(\Omega;\gamma(L^2(0,T;H),X)).$$

The question is thus whether we are willing to exchange an assumption on our Banach space X for an assumption on our process Φ. In practice, the assumptions from Kalinichenko [12] are difficult to work with, and can be simplified significantly in the Hilbert space case, as can be seen in 4, and in the UMD$^-$ case in 6. From where we stand now, the only situation in which it is easy to confirm the condition is when $\Phi \in \gamma(L^2(0,T;H),X)$, and this is precisely when we have the Itō isometry in general Banach spaces already.

To solve stochastic differential equations in non-UMD Banach spaces, we can now use Theorem 2 from [12], or more generally Theorem 5.10, the last one being a direct extension of Theorem 7.1 from [28]. Note that the complicated conditions from Kalinichenko still work through to the conditions of this theorem, and that it is therefore mostly an academic result.

After this, we looked at Theorem 4.2 in the context of the UMD setting and the martingale type 2 setting. We have seen in that in both cases, the proof simplifies. We have also seen that Theorem 4.2 does not necessarily generalize any of these theories, as we can find examples of processes in both the UMD and the martingale type 2 case which do not satisfy the conditions of 4.2, but do satisfy the conditions for stochastic integrability in their respective spaces.

Even though Kalinichenko has brilliantly used the techniques from Talagrand [34] to propose a new way of doing stochastic integration in general separable Banach spaces, it only further seems to solidify that the theory from Van Neerven, Veraar and Weis [25] is indeed the correct way to do stochastic integration in Banach spaces.

Bibliography

[1] Bachelier, Louis. *"Théorie de la spéculation."* Annales scientifiques de l'École normale supérieure. Vol. 17. 1900.

[2] Bourgain, Jean. *"Some remarks on Banach spaces in which martingale difference sequences are unconditional."* Arkiv för Matematik 21.1 (1983): 163-168.

[3] Brzeźniak, Zdzisław. *"On stochastic convolution in Banach spaces and applications."* Stochastics: An International Journal of Probability and Stochastic Processes 61.3-4 (1997): 245-295.

[4] Brzeźniak, Zdzisław. *"Stochastic partial differential equations in M-type 2 Banach spaces."* Potential Analysis 4.1 (1995): 1-45.

[5] Brzezniak, Zdzislaw, and Jan van Neervan. *"Stochastic convolution in separable Banach spaces and the stochastic linear Cauchy problem."* University of Warwick. Mathematics Institute, 1999.

[6] Da Prato, Giuseppe, and Jerzy Zabczyk. *Stochastic equations in infinite dimensions.* Cambridge university press, 2014.

[7] Einstein, Albert. *"On the motion of small particles suspended in liquids at rest required by the molecular-kinetic theory of heat."* Annalen der physik 17.549-560 (1905): 208.

[8] Garling, David John Haldane. *"Brownian motion and UMD-spaces."* Probability and Banach spaces. Springer, Berlin, Heidelberg, 1986. 36-49.

[9] Hytönen, Tuomas, Jan Van Neerven, Mark Veraar, and Lutz Weis. *Analysis in Banach Spaces: Volume II: Probabilistic Methods and Operator Theory.* Vol. 67. Springer, 2018.

[10] James, R. C. *"Nonreflexive spaces of type 2."* Israel Journal of Mathematics 30.1 (1978): 1-13.

[11] Jarrow, Robert, and Philip Protter. *"A short history of stochastic integration and mathematical finance: The early years, 1880–1970."* A festschrift for Herman Rubin. Institute of Mathematical Statistics, 2004. 75-91.

[12] Kalinichenko, Artem Alexandrovich. *"An approach to stochastic integration in general separable Banach spaces."* Potential Analysis 50.4 (2019): 591-608.

[13] Kalinichenko, Artem Alexandrovich. *"Construction of diffusions on current groups."* Sbornik: Mathematics 209.1 (2018): 71.

[14] Kallenberg, Olav. *Foundations of modern probability.* Vol. 2. New York: springer, 1997.

[15] Kiyoshi Itō. *Differential equations determining Markov processes.* Zenkoku Shijo Sug- aku Danwakai, 1077:1352–1400, 1942.

[16] Kiyoshi Itō. *Stochastic integral.* Proceedings of the Imperial Academy. Tokyo, 20:519–524, 1944. ISSN 0369-9846.

[17] Lindenstrauss, Joram, and Lior Tzafriri. *"Classical Banach spaces II: function spaces."* Vol. 97. Springer Science & Business Media, 2013.

[18] McConnell, Terry R. *"Decoupling and stochastic integration in UMD Banach spaces."* Probab. Math. Statist 10.2 (1989): 283-295.

[19] Montgomery-Smith, Stephen. *"Concrete representation of martingales."* Electronic Journal of Probability 3 (1998): 1-15.

[20] Van Neerven, Jan, and Markus Riedle. *"A Semigroup Approach to Stochastic Delay Equations in Spaces of Continuous Functions."* Semigroup Forum. Vol. 74. No. 2. 2007.

[21] Van Neerven, Jan, and Mark C. Veraar. *"26 On the Stochastic Fubini Theorem in Infinite Dimensions."* Stochastic Partial Differential Equations and Applications-VII (2005): 323.

[22] van Neerven, Jan, and Mark Veraar. *"Maximal inequalities for stochastic convolutions in 2-smooth Banach spaces and applications to stochastic evolution equations."* Philosophical Transactions of the Royal Society A 378.2185 (2020): 20190622.

[23] Van Neerven, Jan, Mark Veraar, and Lutz Weis. *"Stochastic integration in Banach spaces–a survey." Stochastic analysis: a series of lectures.* Birkhäuser, Basel, 2015. 297-332.

[24] Van Neerven, Jan. *"Stochastic evolutions equations—Internet Seminar."* (2007).

[25] van Neerven, Jan, Mark C. Veraar, and Lutz Weis. *"Stochastic integration in UMD Banach spaces."* The Annals of Probability 35.4 (2007): 1438-1478.

[26] van Neerven, Jan, Mark C. Veraar, and Lutz Weis. *"Stochastic evolution equations in UMD Banach spaces."* Journal of Functional Analysis 255.4 (2008): 940-993.

[27] Van Neerven, Jan, Mark Veraar, and Lutz Weis. *"Conditions for stochastic integrability in UMD Banach spaces."* Banach spaces and their applications in analysis. de Gruyter, 2011. 125-146.

[28] Van Neerven, Jan, and Lutz Weis. *"Stochastic integration of functions with values in a Banach space."* Studia Math 166.2 (2005): 131-170.

[29] van Neerven, Jan, and Lutz Weis. *"Weak limits and integrals of Gaussian covariances in Banach spaces."* Probability and mathematical statistics-Wroclaw University 25.1 (2005): 55.

[30] Neidhardt, A.L. *"Stochastic integrals in 2-uniformly smooth Banach spaces."* Ph.D. thesis, University of Wisconsin, 1978.

[31] Ondreját, Martin. *"Brownian representations of cylindrical local martingales, martingale problem and strong Markov property of weak solutions of SPDEs in Banach spaces."* Czechoslovak Mathematical Journal 55.4 (2005): 1003-1039.

[32] Pisier, Gilles. *"Martingales with values in uniformly convex spaces."* Israel Journal of Mathematics 20.3 (1975): 326-350.

[33] Rosiński, Jan, and Zdzisław Suchanecki. *"On the space of vector-valued functions integrable with respect to the white noise."* Colloquium Mathematicum. Vol. 1. No. 43. 1980.

[34] Talagrand, Michel. *Upper and lower bounds for stochastic processes: modern methods and classical problems.* Vol. 60. Springer Science & Business Media, 2014.

[35] Yor, Marc. *"Sur les integrales stochastiques a valeurs dans un espace de Banach."* Annales de l'IHP Probabilités et statistiques. Vol. 10. No. 1. 1974.