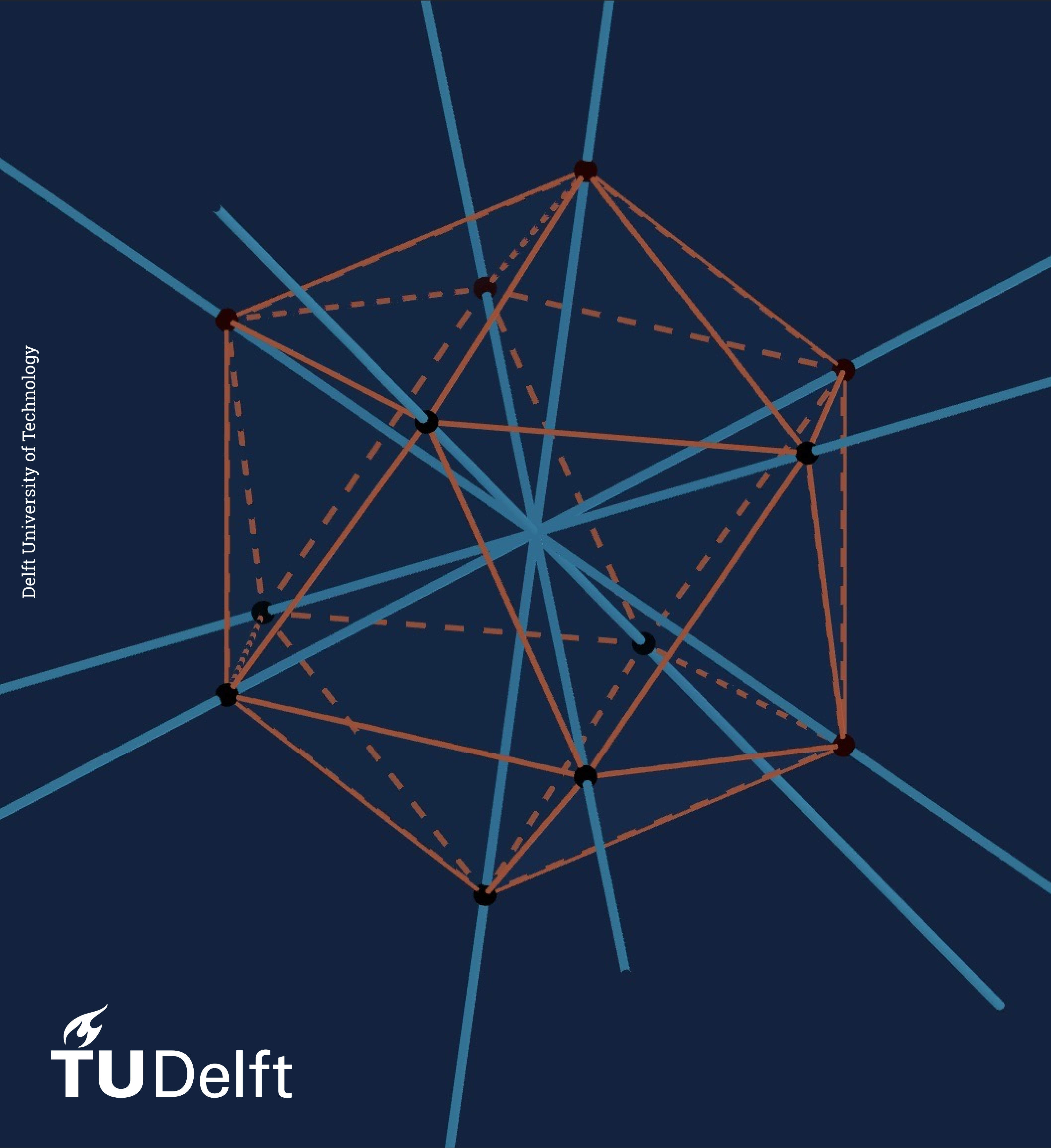
# Real equiangular lines

Giulia Montagna



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# Preface

*People don't do mathematics because it's useful.*
*They do it because it's interesting.*
—Paul Lockhart

From the beginning of my studies I have been attracted by the more theoretical aspect of mathematics. I like the mathematics purely for how nicely it all fits together. A natural question to ask then, is why I chose for *Applied* Mathematics in Delft. The answer is that secretly there are a lot of options to study theoretical mathematics, even here at a technical university. These can for example be found hidden in the Discrete Mathematics & Optimisation group of the faculty of Electrical Engineering, Mathematics and Computer Science.

The first time I read about equiangular lines was on an exam question which I was not able to solve. Luckily it wasn't something which kept me awake at night, but it did stay in my mind as an unresolved question. So, when Anurag proposed the topic for a thesis I did not have to think twice about it. Let us hope that by now I would be able to solve the exam problem.

While working on my Bachelor thesis people often asked what the practical applications of my subject are. A friend of mine had the perfect answer to this: 'The application of theoretical mathematics is simply more mathematics.' Now, during my Master thesis, I find myself in a similar situation. However, the topic of this thesis is a beautiful example of how the combination of multiple mathematical tools can lead to new insights. It gives proof of how valuable it can be to think outside of the box.

I would like to thank my supervisor Anurag Bishnoi for introducing me to the world of combinatorics and his guidance during the project. I also want to thank the Discrete Mathematics & Optimisation group for giving me the opportunity to present my findings during the seminar and mostly for actually coming to my talks. This journey would have been a lot more difficult if I would not have had my friends to study together with and share the ups and downs with that writing a thesis gives. Being one of the last two to finish, has made me appreciate the moments we were with a bigger group even more. Thank you especially to Zoë for being (not always) there all the way through. Another special thanks goes to Jasmijn, for all the breakfasts and great morning conversations. I want to thank my mother too, for taking care of me (read: feed me) during my Easter-writing-retreat. Lex, thank you for putting up with my (dis)ability of creating an overflowing schedule which often fails to include free time. Lastly, I want to take this opportunity to thank all the other people who have made my time as a student an unforgettable period of my life.

*Giulia Montagna*
*Delft, June 2023*

# Abstract

A set of lines passing through the origin in $\mathbb{R}^d$ is called equiangular if the angle between any two lines is the same. The question of finding the maximum number of such lines, $N(d)$ in any dimension $d$ is an extensively studied problem. Closely related, is the problem of finding the maximum number of lines, $N_\alpha(d)$, such that the common angle between the lines is $\arccos \alpha$. In recent years, many progress has been made on this problem. We review some of these breakthrough results and the techniques they use to approach this problem. The first main result is a linear upper bound on $N_\alpha(d)$ which is found using a completely novel approach with respect to techniques used in previous works. Another main result that we discuss solves the problem of finding $N_\alpha(d)$ for high enough dimensions. Some classic results from some of the first studies on equiangular lines are also discussed. Finally, some suggestions are given for possible further research.

# Contents

# 1
# Introduction

A set of lines $\mathcal{L}$ passing through the origin in $\mathbb{R}^d$ is called *equiangular* if the angle between any two lines $l_1, l_2 \in \mathcal{L}$ is the same. The very first results on equiangular lines date back to Haantjes [1] who showed the maximum number of equiangular lines in $\mathbb{R}^2$ is 3, in $\mathbb{R}^3$ is 6 and in $\mathbb{R}^4$ is also 6. The problem of finding the maximum number of equiangular lines in any dimension $d$ was formally stated for the first time by Van Lint and Seidel in [2]. They connect the problem to discrete mathematics by reformulating it in terms of matrices and constructing graphs from these matrices, making it a graph-theoretical problem. Godsil and Royle consider this problem a founding problem of algebraic graph theory [3, p. 249]. For any integer $d$, we will let $N(d)$ denote the maximum number of equiangular lines in $\mathbb{R}^d$. A closely related problem introduced by Lemmens and Seidel in [4] is that of finding the maximum number $N_\alpha(d)$ of equiangular lines in $\mathbb{R}^d$ with a fixed common angle $\arccos\alpha$.

Equiangular lines were first considered in terms of elliptic geometry [1], [2]. They are also connected to frame theory, in particular Grassmannian frames which are optimal if they are equiangular [5]. These frames in turn have applications in coding theory. Another important application of equiangular lines occurs in quantum theory. Maximal sets of complex equiangular lines are known in this field as Symmetric Informationally Complete Positive-Operator-Valued Measures, better known as SIC-POVM [6], [7].

For many years the results on the asymptotic behaviour of $N(d)$ and $N_\alpha(d)$ from Lemmens and Seidel in [4] were the best known. The results from this paper are still highly relevant to this day. A theorem due to Gerzon gives a general upper bound on $N(d)$ of $\binom{d+1}{2}$. This upper bound can only be reached in specific cases and is known to be attained only for a few dimensions. Another result in this paper is due to Neumann, which states that $N_\alpha(d)$ can only be larger than $2d$ if $1/\alpha$ is an odd integer. Since this result shows that large values of $N_\alpha(d)$ can only be reached in this specific case, there has been special interest in finding the maximum number of equiangular lines with fixed common angle $\arccos\alpha$, where $1/\alpha$ an odd integer. Lemmens and Seidel themselves already studied the first two cases, $\alpha = 1/3, 1/5$, and solved the problem for $\alpha = 1/3$. Neumaier later confirmed a conjecture of them concerning $\alpha = 1/5$ and said that the next interesting case, $\alpha = 1/7$, would require considerably stronger techniques [8].

In the past decade significant progress has been made regarding the problem of finding the maximum number of real equiangular lines in any dimension. The first breakthrough came from Bukh [9] who showed that $N_\alpha(d)$ grows linearly for every $\alpha$. Since then, many more improvements have followed and new contributions continue to be published. First of all, Balla, Dräxler, Keevash and Sudakov showed that $N_\alpha(d)$ reaches its maximum at $2d-2$ when $\alpha = 1/3$ and is at most $1.93d$ otherwise [10]. Jiang and Polyanskii further improve on this bound and give more evidence supporting a conjecture which states that $N_{1/(2k-1)} = \frac{k}{k-1}d + \mathcal{O}(1)$ [11]. This last conjecture has subsequently been proven to be true for large enough dimensions by Jiang, Tidor, Yao, Zhang and Zhao in [12]. All these results heavily rely on Ramsey theory which is why they are limited to large enough dimensions. Balla finds a way around this by using projections of matrices [13]. With this novel approach he shows a linear upper bound on $N_\alpha(d)$.

This thesis contains an overview of the most noteworthy recent contributions that have been made on the subject of equiangular lines. It can be used as an introduction into the current state of research concerning this topic. It covers some classic results on the subject and the most innovative recent results and techniques. These last results all have potential for further improvements, which will be suggested at the end of this thesis.

We start with a chapter containing the necessary linear algebraic an graph theoretical preliminaries. At the end of the chapter we show one of the first recent results connected to equiangular lines. This regards a theorem which bounds the multiplicities of eigenvalues of a bounded degree graph. It is a purely graph-theoretical theorem which will be of great importance later to prove one of the other main results discussed in this thesis. In Chapter 3 some classic results on equiangular lines will be treated. This chapter gives a good introduction into some techniques that can be used to analyse the behaviour of equiangular lines through the dimensions. Next, in Chapter 4 we give a linear upper bound on $N_\alpha(d)$ which holds for all dimensions. The results from this chapter are due to Balla in [13] and use a completely new approach which overcomes some limitations of previous works. Lastly, in Chapter 5 we discuss a result by Jiang, Tidor, Yao, Zhang and Zhao from [12] which solves the question of finding the maximum number of lines with a fixed angle in $\mathbb{R}^d$ for high enough dimensions. Chapter 4 is more technical and complex than Chapter 5. The two chapters can be read independently of each other. We conclude the thesis with multiple suggestions of further research that can be done related to the topic of equiangular lines.

# 2

# Preliminaries

In studying equiangular lines we will use linear algebraic and graph theoretical tools. In this chapter all concepts of these two subjects are introduced that will be needed throughout this thesis. We will start with a section on linear algebra, covering the basic notions, eigenvalues and some results on positive semidefinite matrices. The next section will go over all the necessary graph theory and spectral graph theory. We will conclude this section with a recent result on the multiplicity of the $j$-th eigenvalue of any connected graph, for $j > 1$.

## 2.1 Linear algebra

In this section some basic notions of linear algebra are discussed which will be needed throughout the whole thesis. Since the linear algebra will mostly be applied on adjacency matrices of graphs, our interest lies primarily in real symmetric matrices with non-negative entries. After covering the basic notions we introduce eigenvalues and some useful results on eigenvalues. Lastly we will review some properties of positive semidefinite matrices which will be needed later on. For an extensive introduction into linear algebra we refer to [14].

### 2.1.1 Basic notions

Let $V$ be a vector space over a field $\mathbb{F}$, where $\mathbb{F}$ is either the field of real numbers $\mathbb{R}$ or complex field $\mathbb{C}$. A function $\langle \cdot, \cdot \rangle : V \to \mathbb{F}$ is an *inner product* if for any $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V$ and scalars $\alpha, \beta \in \mathbb{F}$ the following properties hold:

(i) $\langle \boldsymbol{u}, \boldsymbol{u} \rangle \geq 0$ (Non-negativity)

(ii) $\langle \boldsymbol{u}, \boldsymbol{u} \rangle = 0$ if and only if $\boldsymbol{u} = \boldsymbol{0}$ (Non-degeneracy)

(iii) $\langle \alpha \boldsymbol{u} + \beta \boldsymbol{v}, \boldsymbol{w} \rangle = \alpha \langle \boldsymbol{u}, \boldsymbol{w} \rangle + \beta \langle \boldsymbol{v}, \boldsymbol{w} \rangle$ (Linearity)

(iv) $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \overline{\langle \boldsymbol{v}, \boldsymbol{u} \rangle}$. (Conjugate symmetry)

Note that if $\mathbb{F} = \mathbb{R}$, the conjugate symmetry is equivalent to normal symmetry, i.e.$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \langle \boldsymbol{v}, \boldsymbol{u} \rangle$.

Let $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)^\mathsf{T}, \boldsymbol{u} = (u_1, u_2, \ldots, u_n)^\mathsf{T}$ be any vectors in $\mathbb{R}^n$. The *standard* inner product $\langle \boldsymbol{v}, \boldsymbol{u} \rangle$ is defined by

$$\langle \boldsymbol{v}, \boldsymbol{u} \rangle = \boldsymbol{v}^\mathsf{T} \boldsymbol{u} = v_1 u_1 + v_2 u_2 + \cdots + v_n u_n.$$

It is easily verified that this function indeed satisfies the properties above and thus defines an inner product on $\mathbb{R}^n$. The *norm* $\|\boldsymbol{v}\|$ of $\boldsymbol{v}$ is given by $\|\boldsymbol{v}\| = \sqrt{\langle \boldsymbol{v}, \boldsymbol{v} \rangle}$. The following widely used inequality gives a relation between the inner product and norms of two vectors.

**Theorem 2.1** (Cauchy-Schwarz inequality)**.** *Let $\boldsymbol{v}, \boldsymbol{u}$ be two vectors in $\mathbb{R}^n$, then*

$$|\langle \boldsymbol{u}, \boldsymbol{v} \rangle| \leq \|\boldsymbol{u}\| \|\boldsymbol{v}\|.$$

An immediate consequence of the Cauchy-Schwarz inequality is the *triangle inequality* given by $\|\boldsymbol{u}+\boldsymbol{v}\| \leq \|\boldsymbol{u}\|+\|\boldsymbol{v}\|$. The two vectors $\boldsymbol{v}, \boldsymbol{u}$ are said to be *orthogonal* if their inner product is zero. A set of vectors $V = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_m\}$ in $\mathbb{R}^n$ is *linearly independent* if the equation $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \dots + c_m\boldsymbol{v}_m = 0$, for $c_1, c_2, \dots, c_m \in \mathbb{R}$ only has trivial solution, i.e. $c_1 = c_2 = \dots = c_m = 0$ is the only solution. If the set of vectors is not linearly independent, it is called *linearly dependent*. Any linearly independent set $V$ of $n$ vectors in any $n$-dimensional space *spans* the whole space. This means that any vector in the space can be written as a linear combination of vectors in $V$.

For any $m \times n$ matrix $A$ there are two important subspaces associated to it. The first is the *kernel* of the matrix, which is also called its null space, defined as

$$\operatorname{Ker} A = \{\boldsymbol{v} \in \mathbb{R}^n : A\boldsymbol{v} = 0\}.$$

The other space is the *range* of the matrix

$$\operatorname{Ran} A = \{A\boldsymbol{v} : \boldsymbol{v} \in \mathbb{R}^n\}.$$

The dimensions of these two spaces are known as the *nullity* and the *rank*, $\operatorname{rk} A$, of the matrix respectively. We say that an $m \times n$ matrix has *full* rank if its rank equals $\min(m, n)$. A fundamental result in linear algebra states that the rank of a matrix equals the rank of its transpose, i.e. $\operatorname{rk} A = \operatorname{rk} A^\intercal$. The *rank-nullity* theorem gives a useful relation between the rank and the nullity of a $m \times n$ matrix $A$. The theorem states that

$$\dim \operatorname{Ker} A + \operatorname{rk} A = n. \tag{2.1}$$

The following lemma gives an example of an application of this theorem.

**Lemma 2.2.** *Let $A$ be a real $m \times n$ matrix. Then* $\operatorname{rk} AA^\intercal = \operatorname{rk} A^\intercal A = \operatorname{rk} A$.

*Proof.* To prove the lemma we show that the null spaces of $A^\intercal A$ and $A$ are equal. Then from the rank-nullity theorem it follows that $\operatorname{rk} A^\intercal A = \operatorname{rk} A$, since $\dim \operatorname{Ker} A + \operatorname{rk} A = \dim \operatorname{Ker} A^\intercal A + \operatorname{rk} A^\intercal A = n$. Furthermore, from $\operatorname{rk} A = \operatorname{rk} A^\intercal$ it then follows that $\operatorname{rk} AA^\intercal = \operatorname{rk} A^\intercal A = \operatorname{rk} A^\intercal = \operatorname{rk} A$.

To show equality of the null spaces, first of all notice that for any $\boldsymbol{v}$ such that $A\boldsymbol{v} = 0$ we have $A^\intercal A\boldsymbol{v} = A^\intercal 0 = 0$. Thus $\operatorname{Ker} A \subseteq \operatorname{Ker} A^\intercal A$. Conversely, let $\boldsymbol{u}$ be such that $A^\intercal A\boldsymbol{u} = 0$. Then $\|Au\|^2 = \boldsymbol{u}^\intercal A^\intercal A\boldsymbol{u} = 0$. So we must have $A\boldsymbol{u} = 0$. This gives $\operatorname{Ker} A^\intercal A \subseteq \ker A$ which concludes the proof. □

Another useful property of the rank is its subadditivity. Let $A$ and $B$ be any two matrices of the same dimensions, then $\operatorname{rk}(A + B) \leq \operatorname{rk}(A) + \operatorname{rk}(B)$. Notice that from this it also follows that $\operatorname{rk}(A) - \operatorname{rk}(B) \leq \operatorname{rk}(A - B)$. Indeed, write $\operatorname{rk}(A) = \operatorname{rk}(A - B + B)$, then from the subadditivity we find

$$\operatorname{rk}(A) = \operatorname{rk}(A - B + B) \leq \operatorname{rk}(A - B) + \operatorname{rk}(B),$$

which after rearranging gives $\operatorname{rk}(A) - \operatorname{rk}(B) \leq \operatorname{rk}(A - B)$ as desired.

An $n \times n$ matrix is called a *square matrix*. The *trace* of a square matrix $A$, denoted $\operatorname{Tr}(A)$, is the sum of its diagonal entries. The trace is a linear mapping which means that for any two square matrices $A$ and $B$ and scalars $\alpha$ and $\beta$

$$\operatorname{Tr}(\alpha A + \beta B) = \alpha \operatorname{Tr}(A) + \beta \operatorname{Tr}(B).$$

Furthermore, for an $m \times n$ matrix $A$ and $n \times m$ matrix $B$

$$\operatorname{Tr}(AB) = \sum_{i=1}^{m} (AB)_{ii} = \sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}B_{ji} = \sum_{j=1}^{n}\sum_{i=1}^{m} B_{ji}A_{ij} = \sum_{j=1}^{n} (BA)_{jj} = \operatorname{Tr}(BA).$$

By substituting the matrix $A$ by a multiplication of any number of matrices in the equation above, we can deduce the cyclic property of the trace, which for any integer $n$ and matrices $A_1, A_2, \dots, A_n$ is given by

$$\operatorname{Tr}(A_1A_2 \dots A_n) = \operatorname{Tr}(A_nA_1A_2 \dots A_{n-1}) = \dots = \operatorname{Tr}(A_2A_3 \dots A_nA_1).$$

In particular, this shows that for any two vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$, which we can view as $n \times 1$ matrices, we have

$$\operatorname{Tr}(\boldsymbol{u}\boldsymbol{v}^\intercal) = \operatorname{Tr}(\boldsymbol{v}^\intercal\boldsymbol{u}) = \boldsymbol{v}^\intercal\boldsymbol{u} = \langle \boldsymbol{v}, \boldsymbol{u} \rangle. \tag{2.2}$$

Using the trace, we can define an inner product on the space of $m \times n$ matrices, $\mathcal{M}_{m\times n}$, by

$$\langle A, B\rangle_F = \operatorname{Tr}\left(\overline{B}^\mathsf{T} A\right) = \sum_{i,j} \overline{B}_{ij} A_{ij}$$

known as the *Frobenius inner product.* If the matrices are real, the inner product equals $\langle A, B\rangle_F = \operatorname{Tr}\left(B^\mathsf{T} A\right) = \sum_{i,j} B_{ij} A_{ij}$, and is also called the *trace inner product.* The subscript $F$ will be left out when it is clear from context that we are talking about the Frobenius inner product. Observe that if we view the matrices $A$ and $B$ as vectors in $\mathbb{R}^{mn}$, then we see that the trace inner product is actually equal to the standard inner product in $\mathbb{R}^{mn}$. Hence, it indeed satisfies the properties needed to be an inner product. The Frobenius inner product also has a corresponding norm, the *Frobenius norm*, defined as $\|A\|_F = \sqrt{\langle A, A\rangle_F}$, for any matrix $A \in \mathcal{M}_{m\times n}$.

From this point on all matrices considered will be square matrices. A *symmetric* matrix $A$ is a square matrix for which $A^\mathsf{T} = A$. For symmetric matrices $A$ and $B$ the trace inner product simplifies to $\langle A, B\rangle_F = \operatorname{Tr}(BA)$. Furthermore, by the non-negativity and non-degeneracy of an inner product for any symmetric matrix $A$, $\operatorname{Tr}(A^\mathsf{T} A) = \operatorname{Tr}(A^2) \geq 0$, with equality if and only if $A = \mathbf{0}$. We also note the following useful property of a symmetric matrix $A$ and vector $\boldsymbol{v} \in \mathbb{R}^n$:

$$\|A\boldsymbol{v}\|^2 = \langle A\boldsymbol{v}, A\boldsymbol{v}\rangle = \boldsymbol{v}^\mathsf{T} A^\mathsf{T} A\boldsymbol{v} = \boldsymbol{v}^\mathsf{T} A^2\boldsymbol{v} = \langle \boldsymbol{v}, A^2\boldsymbol{v}\rangle. \tag{2.3}$$

A square matrix $A$ is *invertible* if there exists a matrix denoted by $A^{-1}$ such that $A^{-1}A = AA^{-1} = I$, where $I$ is the identity matrix. The matrix $A^{-1}$ is called the *inverse* of $A$. Invertible matrices can be defined by multiple other properties. For any square matrix $A$, the following are equivalent:

(i) $A$ is invertible;
(ii) for any vector $\boldsymbol{b}$, $A\boldsymbol{v} = \boldsymbol{b}$ has a unique solution for $\boldsymbol{v} \in \mathbb{R}^n$;
(iii) the kernel of $A$ is trivial, i.e. $\operatorname{Ker}(A) = \{\mathbf{0}\}$;
(iv) $A$ has full rank;
(v) $\det A \neq 0$;
(vi) 0 is *not* an eigenvalue of $A$ (see Section 2.1.2).

### 2.1.2 Eigenvalues

Let $A$ be a square matrix and let $\lambda$ be a scalar such that there exists a non-zero vector $\boldsymbol{v}$ satisfying

$$A\boldsymbol{v} = \lambda\boldsymbol{v}.$$

A scalar $\lambda$ and vector $\boldsymbol{v}$ satisfying this property are respectively called an *eigenvalue* and *eigenvector* of the matrix $A$. Note that the definition states that an eigenvector can *never* be the zero vector, but the scalar 0 can be an eigen*value.* To find all eigenvectors corresponding to an eigenvalue $\lambda$ it suffices to solve the equation $A\boldsymbol{v} = \lambda\boldsymbol{v}$, which is equivalent to solving

$$(\lambda I - A)\boldsymbol{v} = 0.$$

This shows that finding all eigenvectors of a matrix $A$ corresponding to the eigenvalue $\lambda$ is equivalent to finding the nullspace of the matrix $\lambda I - A$. The nullspace $\operatorname{Ker}(\lambda I - A)$ is called the *eigenspace* of $A$ associated to the eigenvalue $\lambda$. The set of all eigenvalues of $A$ is called the *spectrum* of $A$ and denoted by $\sigma(A)$.

Since an eigenvector is a non-zero vector, a scalar $\lambda$ is an eigenvalue if and only if the nullspace of $\lambda I - A$ is non-trivial, i.e. it contains a non-zero vector. This means that the matrix $\lambda I - A$ is not invertible and thus $\det(\lambda I - A) = 0$. In the previous section we saw that a matrix is invertible if and only if it has a trivial kernel. The eigenspace of the eigenvalue 0 equals $\operatorname{Ker}(0 \cdot I - A) = \operatorname{Ker} A$, and so we see that a matrix is invertible if and only if 0 is not an eigenvalue.

The determinant of $\lambda I - A$ is a polynomial of degree $n$ of the variable $\lambda$ and is called the *characteristic polynomial* of $A$. The eigenvalues of the matrix $A$ thus coincide with all the roots of its characteristic polynomial. Let $p(x) = \det(xI - A)$ be the characteristic polynomial of the matrix $A$. The multiplicity of $\lambda$ as a root of $p(x)$ is called the *algebraic multiplicity* of the eigenvalue $\lambda$. The eigenvalue $\lambda$ also has a *geometric multiplicity* which is defined as the dimension of the eigenspace $\mathrm{Ker}(\lambda I - A)$. The algebraic multiplicity is used more often, this is why we will simply say 'multiplicity' when talking about the algebraic multiplicity of an eigenvalue. If $\lambda_1, \lambda_2, \ldots, \lambda_r$ are the distinct eigenvalues of a matrix $A$ with corresponding multiplicities $m_1, m_2, \ldots, m_r$, we write $\sigma(A) = \{\lambda_1^{m_1}, \lambda_2^{m_2}, \ldots, \lambda_r^{m_r}\}$.

The algebraic and geometric multiplicity of an eigenvalue can differ, but the geometric multiplicity never exceeds the algebraic multiplicity. The two multiplicities are equal for symmetric matrices. In particular, this leads to the following useful lemma concerning the nullity of a symmetric matrix. The lemma immediately follows from the observation that the kernel of any matrix equals the eigenspace of the eigenvalue 0.

**Lemma 2.3.** *The nullity of any symmetric matrix $A$ is equal to the multiplicity of the eigenvalue* 0.

From this lemma and the rank-nullity theorem it follows that any symmetric matrix $A$ has exactly $r = \mathrm{rk}\, A$ non-zero eigenvalues. Since $\mathrm{rk}\, A^\mathsf{T} A = \mathrm{rk}\, AA^\mathsf{T} = \mathrm{rk}\, A$, this implies that the matrices $A^\mathsf{T} A$ and $AA^\mathsf{T}$ also have $r$ non-zero eigenvalues. In particular, the two matrices even have the same non-zero eigenvalues with the same multiplicities. To see this, let $\lambda$ be a non-zero eigenvalue of $AA^\mathsf{T}$ with corresponding eigenvector $\boldsymbol{v}$. Then, $A^\mathsf{T} AA^\mathsf{T} \boldsymbol{u} = A^\mathsf{T} \lambda \boldsymbol{v} = \lambda A^\mathsf{T} \boldsymbol{v}$. So, $\lambda$ is an eigenvalue of $A^\mathsf{T} A$ with corresponding eigenvector $A^\mathsf{T} \boldsymbol{v}$. Similarly, we find that a non-zero eigenvalue $\mu$ of $A^\mathsf{T} A$ with corresponding eigenvector $\boldsymbol{u}$ is also an eigenvalue of $AA^\mathsf{T}$ with eigenvector $A\boldsymbol{u}$.

The eigenvalues of a matrix can be used to calculate it trace and determinant. Let $A$ be an $n \times n$ matrix and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be its eigenvalues counting multiplicities, then

- $\mathrm{Tr}\, A = \lambda_1 + \lambda_2 + \cdots + \lambda_n$, and
- $\det A = \lambda_1 \cdot \lambda_2 \cdot \cdots \cdot \lambda_n$.

This equation for the trace can be used to prove a lower bound on the rank of a symmetric matrix $A$ in terms of its trace and the trace of the squared matrix. The result follows immediately from the Cauchy-Schwarz inequality applied to its non-zero eigenvalues, of which there are exactly $\mathrm{rk}\, A$ by the previous lemma.

**Lemma 2.4.** *Let $A$ be a real symmetric matrix. Then*

$$\mathrm{rk}\, A \geq \frac{\mathrm{Tr}(A)^2}{\mathrm{Tr}\,(A^2)}.$$

*Proof.* Let $r = \mathrm{rk}\, A$ and notice that, by Lemma 2.3, $A$ has exactly $r$ non-zero eigenvalues, denoted by $\lambda_1, \lambda_2, \ldots, \lambda_r$. So, we have $\mathrm{Tr}\, A = \sum_{i=1}^r \lambda_i$ and $\mathrm{Tr}\left(A^2\right) = \sum_{i=1}^r \lambda_i^2$. Denote by $\boldsymbol{\lambda}$ the vector with entries $\lambda_1, \ldots, \lambda_r$. Then, by taking the inner product with the all-ones vector $\mathbf{1}$, the Cauchy-Schwarz inequality yields

$$\sum_{i=1}^r \lambda_i = \langle \mathbf{1}, \boldsymbol{\lambda} \rangle \leq \|\mathbf{1}\| \|\boldsymbol{\lambda}\| = \sqrt{r} \cdot \sqrt{\sum_{i=1}^r \lambda_i^2}.$$

Taking the square of both sides gives $\mathrm{Tr}(A)^2 \leq \mathrm{rk}\, A \cdot \mathrm{Tr}(A^2)$. Dividing both sides by $\mathrm{Tr}(A^2)$ gives the required inequality. □

Two matrices that will come up a lot are the $n \times n$ identity matrix, denoted $I_n$, and the $n \times n$ all-ones matrix, denoted $J_n$. When the dimensions are clear from context we will simply write $I$ and $J$. The identity matrix has eigenvalue $\lambda = 1$ with multiplicity $n$. The eigenvalues of the all-ones matrix are $\lambda_1 = n$ with multiplicity 1 and $\lambda_2 = 0$ with multiplicity $n - 1$. For a matrix $A$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ and a polynomial $p$, the eigenvalues of $p(A)$ are $\{p(\lambda_1), \ldots, p(\lambda_n)\}$.

The eigenvalues of symmetric matrix are always real. Furthermore, symmetric matrices have the useful property that they can always be decomposed using their eigenvalues and eigenvectors. These eigenvectors are also all orthogonal. These properties are given by the spectral decomposition theorem.

**Theorem 2.5** (Spectral decomposition theorem)**.** *Any real symmetric matrix $A$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ and corresponding eigenvectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ can be decomposed as*

$$A = \sum_{i=1}^n \lambda_i \boldsymbol{v}_i \boldsymbol{v}_i^\mathsf{T}. \tag{2.4}$$

*These eigenvectors form an orthonormal basis of $\mathbb{R}^n$.*

The decomposition of the matrix in the theorem above can also be written as $A = PDP^T$, where $D$ is a diagonal matrix with the eigenvalues $\lambda_1, \dots, \lambda_n$ on the diagonal and $P$ is an orthogonal matrix with the eigenvectors $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ as columns. If for any two symmetric matrices $A$ and $B$ there exists one orthogonal matrix $P$ such that $A = PD_AP^\top$ and $B = PD_BP^\top$, where $D_A$ and $D_B$ are both diagonal, we say that $A$ and $B$ are *simultaneously diagonalisable*. In this case, the eigenvalues of the sum $A + B$ are sums of the eigenvalues of $A$ and $B$, since $A + B = PD_AP^\top + PD_BP^\top = P(D_A + D_B)P^\top$.

For a symmetric matrix $A$ and a non-zero vector $\boldsymbol{v}$ we define the *Rayleigh quotient* as

$$\mathcal{R}(A, \boldsymbol{v}) = \frac{\boldsymbol{v}^\top A\boldsymbol{v}}{\boldsymbol{v}^\top \boldsymbol{v}} = \frac{\langle A\boldsymbol{v}, \boldsymbol{v}\rangle}{\langle \boldsymbol{v}, \boldsymbol{v}\rangle}.$$

The Rayleigh quotient gives a lower bound on the maximum eigenvalue, $\lambda_{\max}$, of the matrix $A$ and an upper bound on its minimum eigenvalue, $\lambda_{\min}$, as the following theorem shows.

**Theorem 2.6.** *For any symmetric $n \times n$ matrix $A$ and non-zero vector $\boldsymbol{u}$, $\lambda_{\min} \leq \mathcal{R}(A, \boldsymbol{u}) \leq \lambda_{\max}$.*

*Proof.* By the spectral decomposition theorem, there is an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of $A$. So, let $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ be an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors $\boldsymbol{v}_i$ of $A$ with corresponding eigenvalues $\lambda_i$ for $i = 1, \dots, n$.

The vector $\boldsymbol{u}$ can then be written as

$$\boldsymbol{u} = \sum_{i=1}^{n} c_i \boldsymbol{v}_i$$

for some constants $c_i$. Now we can write the inner product $\langle \boldsymbol{u}, \boldsymbol{u}\rangle$ as

$$\langle \boldsymbol{u}, \boldsymbol{u}\rangle = \left\langle \sum_{i=1}^{n} c_i \boldsymbol{v}_i, \sum_{i=1}^{n} c_i \boldsymbol{v}_i \right\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j \langle \boldsymbol{v}_i, \boldsymbol{v}_j\rangle = \sum_{i=1}^{n} c_i^2,$$

where the last step follows from $\langle \boldsymbol{v}_i, \boldsymbol{v}_j\rangle = 0$ if $i \neq j$ and $\langle \boldsymbol{v}_i, \boldsymbol{v}_i\rangle = 1$. For the inner product $\langle A\boldsymbol{u}, \boldsymbol{u}\rangle$ we find

$$\langle A\boldsymbol{u}, \boldsymbol{u}\rangle = \left\langle A\sum_{i=1}^{n} c_i \boldsymbol{v}_i, \sum_{i=1}^{n} c_i \boldsymbol{v}_i \right\rangle = \left\langle \sum_{i=1}^{n} c_i \lambda_i \boldsymbol{v}_i, \sum_{i=1}^{n} c_i \boldsymbol{v}_i \right\rangle = \sum_{i=1}^{n} \lambda_i c_i^2.$$

So the Rayleigh quotient is now given by

$$\mathcal{R}(A, \boldsymbol{u}) = \frac{\langle A\boldsymbol{u}, \boldsymbol{u}\rangle}{\langle \boldsymbol{u}, \boldsymbol{u}\rangle} = \frac{\sum_{i=1}^{n} \lambda_i c_i^2}{\sum_{i=1}^{n} c_i^2}.$$

Since $\lambda_{\min} \leq \lambda_i \leq \lambda_{\max}$ for all $i$, we conclude

$$\lambda_{\min} = \frac{\sum_{i=1}^{n} \lambda_{\min} c_i^2}{\sum_{i=1}^{n} c_i^2} \leq \frac{\sum_{i=1}^{n} \lambda_i c_i^2}{\sum_{i=1}^{n} c_i^2} \leq \frac{\sum_{i=1}^{n} \lambda_{\max} c_i^2}{\sum_{i=1}^{n} c_i^2} = \lambda_{\max}.$$

□

Only a specific type of real number can be an eigenvalue of a symmetric matrix. These numbers are the so called totally real algebraic integers. An *algebraic integer* $\lambda$ is the root of a monic polynomial with coefficients in $\mathbb{Z}$. The *minimal polynomial* of $\lambda$ is the lowest degree monic polynomial with $\lambda$ as its root. The *conjugates* of an algebraic integer $\lambda$ are the other roots of its minimal polynomial. Lastly, we say that $\lambda$ is *totally real* if all its conjugates are real.

**Lemma 2.7.** *If $\lambda$ is an eigenvalue of a symmetric matrix $A$ with integer entries, then it is a totally real algebraic integer.*

*Proof.* Let $A$ be a matrix with integer entries and $\lambda$ as an eigenvalue. Consider the characteristic polynomial $p(x) = \det(xI - A)$. The polynomial $p(x)$ is a monic polynomial with integer coefficients. This implies that all its roots are algebraic integers, from which follows that $\lambda$ is an algebraic integer. Furthermore, since the minimal polynomial of any root of $p(x)$ divides $p(x)$, all the conjugates of any root of $p(x)$ are also roots of $p(x)$. These conjugates are thus also eigenvalues of $A$. Since $A$ is symmetric, all its eigenvalues are real. So it follows, that all the conjugates of $\lambda$ are real, i.e. $\lambda$ is totally real. □

The converse of the lemma has also been shown to hold by Estes [15], meaning that every totally real algebraic integer occurs as the eigenvalue of a symmetric integer matrix.

*Remark* 2.8. The proof of the lemma shows us that all algebraic conjugates of any eigenvalue of the matrix $A$ are also eigenvalues of $A$. Let $p(x)$ be the characteristic polynomial of $A$ and let $\lambda$ be any eigenvalue of $A$. Then the multiplicity of its conjugates as a root of $p(x)$ equals the multiplicity of $\lambda$ as a root of $p(x)$. In other words, the algebraic conjugates of $\lambda$ have the same algebraic multiplicity as $\lambda$.

The maximum eigenvalue of a matrix is known as the *spectral radius* of the matrix. We will always write the eigenvalues of a matrix $A$ in non-increasing order $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$, so that $\lambda_1(A) = \lambda_{max}(A)$. In graph theory we mostly work with matrices that have no negative entries. We call such a matrix $A$ *non-negative* and write $A \geq 0$. For two matrices $A$ and $B$ we write $A \geq B$ if $A - B \geq 0$. The Perron-Frobenius theorem gives valuable results on the spectral radius of non-negative matrices. We will only state the theorem here and refer to [3] for more details. Before giving the theorem, we need to define the concept of an irreducible matrix. A real $n \times n$ matrix $A$ is *irreducible* if $(I + |A|)^{n-1}$ has all positive entries. An equivalent definition can be given in terms of graphs (see Section 2.2 for the necessary definitions). To the matrix $A$ we can associate a directed graph with vertex set $[n]$ and an arc from $i$ to $j$ if $A_{ij} \neq 0$. If this directed graph is strongly connected, then $A$ is irreducible. In particular this means that the adjacency matrix of any connected graph is irreducible.

**Theorem 2.9** (Perron-Frobenius)**.** *Let $A$ be an irreducible non-negative $n \times n$ matrix with spectral radius $\lambda_1$, then*

(i) *$\lambda_1$ has multiplicity one;*
(ii) *for any other eigenvalue $\lambda$ of $A$, $|\lambda| \leq \lambda_1$;*
(iii) *$A$ has an eigenvector $\boldsymbol{v}$ corresponding to $\lambda_1$ with positive entries;*
(iv) *if $B$ is a non-negative $n \times n$ matrix such that $A - B$ is non-negative, then $\lambda_1(B) \leq \lambda_1$.*

The second and last property of the theorem actually also hold when $A$ is non-negative but not irreducible. In this case the spectral radius of $A$ can have a multiplicity larger than one. For example, the $n \times n$ identity matrix $I$ is non-negative, but not irreducible. It has spectral radius $\lambda_1 = 1$ with multiplicity $n$. For a non-negative matrix $A$ a less strict version of the third property holds, namely that $A$ has an eigenvector corresponding to $\lambda_1$ with all non-negative entries.

When working with graphs, the concept of interlacing will help us to analyse the eigenvalues of subgraphs. Let $A$ be a symmetric $n \times n$ matrix, and $B$ a symmetric $m \times m$ matrix, where $m \leq n$. Denote the eigenvalues of $A$ by $\lambda_1, \ldots, \lambda_n$ and the eigenvalues of $B$ by $\mu_1, \ldots, \mu_m$. The eigenvalues of $B$ *interlace* those of $A$ if, for $i = 1, \ldots, m$,

$$\lambda_i \geq \mu_i \geq \lambda_{n-m+i}.$$

The following theorem states that the eigenvalues of principal submatrix of a symmetric matrix always interlace those of the matrix. A *principal submatrix* is obtained from a $n \times n$ matrix by removing the rows and columns indexed by a subset of $[n]$.

**Theorem 2.10.** *Let $B$ be a principal submatrix of a symmetric matrix $A$, then the eigenvalues of $B$ interlace the eigenvalues of $A$.*

### 2.1.3 Positive semidefinite matrices

A real symmetric $n \times n$ matrix $A$ is called *positive semidefinite*, denoted $A \succeq 0$, if for all $\boldsymbol{v} \in \mathbb{R}^n$, $\boldsymbol{v}^\mathsf{T} A \boldsymbol{v} \geq 0$. Positive semidefinite matrices can be characterised in many other equivalent ways. The following theorem gives the characterisations that are most often used.

**Theorem 2.11.** *For a real symmetric $n \times n$ matrix $A$, the following are equivalent:*

(i) *$A$ is positive semidefinite;*
(ii) *all eigenvalues of $A$ are non-negative;*
(iii) *there exists a matrix $L \in \mathbb{R}^{n \times k}$, $k \geq 1$, such that $A = LL^\mathsf{T}$, this is called the* Cholesky decomposition *of $A$;*

(iv) *there exist vectors* $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n \in \mathbb{R}^k$, *where* $k \geq 1$, *such that* $A_{ij} = \langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle$ *for all* $i, j \in [n]$.

The theorem shows that from any set of vectors $V = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ in $\mathbb{R}^k$ a positive semidefinite matrix $M_V$ can be constructed by taking as entries $(M_V)_{ij} = \langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle$ for all $i, j \in [n]$. We call this matrix the *Gram matrix* of $V$. If $A$ is the matrix with the vectors of $V$ as columns then notice that $M_V = A^\mathsf{T} A$. From Lemma 2.2 we can now easily deduce the rank of $M_V$ from the rank of $A$. Since $V$ is a set of vectors in $\mathbb{R}^k$, the matrix $A$ has rank at most $k$ and thus so does $M_V$.

Conversely, from a positive semidefinite matrix $A$, we can always find vectors as in (iv). These vectors can be easily be retrieved from its Cholesky decomposition. Denote the rows of $L$ by $\boldsymbol{v}_i \in \mathbb{R}^k$, $i \in [n]$. Then from the equality $A = LL^\mathsf{T}$ it immediately follows that the entries of $A$ are $A_{ij} = \langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle$. Algorithms exist to find the Cholesky decomposition of a matrix [16, Section 3.2]. Alternatively, it is possible to find the vectors through the spectral decomposition of the matrix [14, Theorem 7.2.7].

Positive semidefinite matrices have various useful properties. We will only discuss a few results here that we will need later on. For more details we refer to Chapter 7 of Horn and Johnson [14].

**Lemma 2.12.** *Let* $A$ *be a symmetric non-negative matrix such that* $A + rJ$ *is positive semidefinite for some* $r \in \mathbb{R}$. *Then* $A$ *has at most one negative eigenvalue.*

*Proof.* Denote the eigenvalues of the matrices $A, rJ$ and $A+rJ$ in non-decreasing order $\lambda_1(\cdot) \geq \cdots \geq \lambda_n(\cdot)$. We want to show that $A$ has at most one negative eigenvalue, which is equal to showing that $\lambda_{n-1}(A) \geq 0$. To show that this holds, we will use the following Weyl inequalities [14, Theorem 4.3.1]:

$$\lambda_i(A + rJ) \leq \lambda_{i-j+1}(A) + \lambda_j(rJ), \quad j \leq i.$$

Note that the eigenvalues of $rJ$ are $\lambda_1(rJ) = rn$ and $\lambda_i(rJ) = 0$, for all $i > 1$. Furthermore, since $A + rJ$ is positive semidefinite all its eigenvalues are non-negative and thus $\lambda_n(A + rJ) \geq 0$. Now take $i = n$ and $j = 2$ in the inequality above. Then we find

$$0 \leq \lambda_n(A + rJ) \leq \lambda_{n-1}(A) + \lambda_2(rJ) = \lambda_{n-1}(A) = 0,$$

and thus we conclude that indeed $\lambda_{n-1}(A) \geq 0$ as desired. □

The sum of any two positive semidefinite matrices is again positive semidefinite. The kernel of the sum is equal to the intersection of the kernels of the two matrices. To prove this, we first need the following lemma.

**Lemma 2.13.** *Let* $A$ *be a positive semidefinite matrix and let* $\boldsymbol{u}$ *be a vector. Then*

$$A\boldsymbol{u} = \boldsymbol{0} \Leftrightarrow \boldsymbol{u}^\mathsf{T} A \boldsymbol{u} = 0.$$

*Proof.* First suppose that $A\boldsymbol{u} = \boldsymbol{0}$. Then $\boldsymbol{u}^\mathsf{T} A\boldsymbol{u} = \boldsymbol{u}^\mathsf{T}\boldsymbol{0} = 0$.

Conversely, suppose that $\boldsymbol{u}^\mathsf{T} A\boldsymbol{u} = \boldsymbol{0}$. Since $A$ is a symmetric matrix there is an orthonormal basis $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ of eigenvectors of $A$. Let $\lambda_1, \dots, \lambda_n$ be the corresponding eigenvectors. Now we can write $\boldsymbol{u} = \sum_{i=1}^n c_i \boldsymbol{v}_i$ for some constants $c_i$. Then we have

$$A\boldsymbol{u} = A \sum_{i=1}^n c_i \boldsymbol{v}_i = \sum_{i=1}^n \lambda_i c_i \boldsymbol{v}_i$$

and

$$\boldsymbol{u}^\mathsf{T} A\boldsymbol{u} = \left( \sum_{i=1}^n c_i \boldsymbol{v}_i \right)^\mathsf{T} \sum_{i=1}^n \lambda_i c_i \boldsymbol{v}_i = \sum_{i=1}^n \lambda_i c_i^2 = 0,$$

since $A$ is positive semidefinite $\lambda_i \geq 0$ for all $i$. So for the last equality to hold we must have $c_i = 0$ for all $i$ for which $\lambda_i > 0$. This means that in the sum $\sum_{i=1}^n \lambda_i c_i \boldsymbol{v}_i$ either $c_i = 0$ or $\lambda_i = 0$ and thus we find

$$A\boldsymbol{u} = A \sum_{i=1}^n c_i \boldsymbol{v}_i = \sum_{i=1}^n \lambda_i c_i \boldsymbol{v}_i = \boldsymbol{0}$$

□

**Lemma 2.14.** *Let $A$ and $B$ be two positive semidefinite matrices. Then*

$$\operatorname{Ker}(A+B) = \operatorname{Ker} A \cap \operatorname{Ker} B.$$

*Proof.* First we show $\operatorname{Ker} A \cap \operatorname{Ker} B \subseteq \operatorname{Ker}(A+B)$. In order to do this let $\boldsymbol{v} \in \operatorname{Ker} A \cap \operatorname{Ker} B$. Then $A\boldsymbol{v} = B\boldsymbol{v} = \boldsymbol{0}$ and thus

$$(A+B)\boldsymbol{v} = A\boldsymbol{v} + B\boldsymbol{v} = \boldsymbol{0} + \boldsymbol{0} = \boldsymbol{0}.$$

This implies that $\boldsymbol{v} \in \operatorname{Ker}(A+B)$.

Now let $\boldsymbol{v} \in \operatorname{Ker}(A+B)$. Then using the previous lemma we have

$$0 = \boldsymbol{v}^\intercal(A+B)\boldsymbol{v} = \boldsymbol{v}^\intercal A\boldsymbol{v} + \boldsymbol{v}^\intercal B\boldsymbol{v},$$

since $A$ and $B$ are both positive semidefinite $\boldsymbol{v}^\intercal A\boldsymbol{v} \geq 0$ and $\boldsymbol{v}^\intercal B\boldsymbol{v} \geq 0$. So for the sum of both of them to be zero we must have $\boldsymbol{v}^\intercal A\boldsymbol{v} = \boldsymbol{v}^\intercal B\boldsymbol{v} = 0$. By the previous lemma we find $A\boldsymbol{v} = B\boldsymbol{v} = \boldsymbol{0}$ and thus $\boldsymbol{v} \in \operatorname{Ker} A \cap \operatorname{Ker} B$. ☐

## 2.2 Graph theory

Graphs are incredibly useful structures to model relations between objects. They will play a major role in our study of equiangular lines. In this section we will introduce all the graph-theoretical concepts that will be used throughout the thesis. We will start with some basic notions and then give an introduction into spectral graph theory. For more background material into these topics we refer to [17] and [18]. Lastly we will discuss a recent theorem on the multiplicity of the $j$-th eigenvalue of a graph, which will be of great importance in proving one of the main results on equiangular lines reviewed in this thesis.

### 2.2.1 Basic notions

A *graph* $G$ is an ordered pair of sets $(V, E)$ such that each element of $E$ is an unordered two-element subset of $V$. The elements of $V$ are called *vertices* and the elements of $E$ are called *edges*. An edge $\{u, v\}$ is written as $uv$. Note that if $uv$ is an edge, $vu$ defines the same edge. We say that two vertices $u, v$ are *adjacent*, or *neighbours*, if there exists an edge $uv$. In this case, the edge $uv$ is *incident* to $u$ and $v$. If all vertices of $G$ are pairwise adjacent, the graph $G$ is *complete*. All considered graphs will be simple, loopless graphs. This means that the graph has no edges from a vertex to itself and there can only be one edge between any two vertices.

A *subgraph* of $G$ is a graph $G' = (V', E')$ such that $V' \subset V$ and $E' \subset E$. If $V' = V$ we say that $G'$ is a *spanning* subgraph of $G$. For a subset $U \subset V$, the graph $G[U]$ is the graph with vertex set $U$ whose edges are precisely the edges of $G$ with both endpoints in $U$. We call $G[U]$ an *induced subgraph* of $G$. A *clique* is a complete subgraph, in other words, a clique is a subgraph such that there is an edge between any two vertices in the subgraph. A subgraph with no edges is an *independent set*. This can be seen as the opposite of a clique.

The degree $d(v)$ of a vertex $v$ is the number of edges incident to that vertex. This is also equal to its number of neighbours. The set of neighbours of a vertex $v$ is called its *neighbourhood* and denoted by $N(v)$. A vertex with no neighbours is called *isolated*. The maximum and minimum degree of the graph are denoted by $\Delta(G)$ and $\delta(G)$ respectively. The average degree of $G$ is the number $D(G) = \frac{1}{|V|}\sum_{v\in V} d(v)$. The number of edges of a graph is related to the degrees of the vertices by the equality $|E| = \frac{1}{2}\sum_{v\in V} d(v)$.

**Proposition 2.15.** *Let $G$ be a graph with average degree $d = d(G) > 0$. Then $G$ has a subgraph $H$ with minimum degree at least $d/2$.*

*Proof.* Notice that $\frac{d}{2} = \frac{|E|}{|V|}$ denotes the ratio of edges per vertex in the graph, since $|E| = \frac{1}{2}\sum_{v\in V} d(v) = \frac{1}{2}|V|d$.

Construct a sequence of induced subgraphs $G = G_0 \supset G_1 \supset G_2 \supset \dots$ where $G_{i+1} = G_i - v_i$ for a vertex $v_i \in G_i$ with $d(v_i) \leq d/2$. If such a vertex $v_i$ does not exist, set $G_i = H$. Let $V_i$ and $E_i$ denote the vertex and edge set of $G_i$ respectively. For all $i$ we have $|V_{i+1}| = |V_i| - 1$ and $|E_{i+1}| \geq |E_i| - \frac{d}{2}$. This gives

$$\frac{|E_1|}{|V_1|} \geq \frac{|E_0| - \frac{d}{2}}{|v_0| - 1} = \frac{|E_0| - \frac{|E_0|}{|V_0|}}{|V_0| - 1} = \frac{(|V_0| - 1)|E_0|}{|V_0|(|V_0| - 1)} = \frac{|E_0|}{|V_0|}$$

and hence

$$\frac{|E_2|}{|V_2|} \geq \frac{|E_1| - \frac{|E_0|}{|V_0|}}{|V_1| - 1} \geq \frac{|E_1| - \frac{|E_1|}{|V_1|}}{|V_1| - 1} = \frac{|E_1|}{|V_1|}.$$

Repeating this argument we find $\frac{|E_{i+1}|}{|V_{i+1}|} \geq \frac{|E_i|}{|V_i|} \geq \frac{|E_0|}{|V_0|}$, which implies that the ratio of edges per vertex does not decrease.

Now, since $\frac{|E_0|}{|V_0|} = \frac{d}{2} > 0$ and $\frac{|E_H|}{|V_H|} \geq \frac{|E_0|}{|V_0|}$, it follows that $H \neq \varnothing$. Furthermore, by construction $H$ does not have a vertex $v$ such that $d(v) \leq d/2$ and thus $\delta(H) \geq d/2$. □

To any graph $G$ several matrices can be associated. The most widely used matrix, is the *adjacency matrix*, denoted $A = A_G$. It is a square matrix indexed by $V(G)$ with entries

$$A_{uv} = \begin{cases} 1, & \text{if } uv \in E(G); \\ 0, & \text{otherwise.} \end{cases}$$

Notice that $A$ is a symmetric matrix with zeros on the diagonal since there can be no edge from a vertex to itself and for any edge $uv$, $vu$ denotes the same edge. Furthermore, the row and column indexed by $v$ sum up to the degree of $v$, for any $v \in V(G)$. So the diagonal entries of the matrix $A^2$ give the degrees of each vertex and hence, $\operatorname{Tr} A^2 = 2|E|$. In fact, the entries of $A^2$ denote the number of walks of length two from one vertex to another.

A *walk* in a graph $G = (V, E)$ is an ordered list of vertices $(v_0, v_1, \dots, v_k)$ where $v_{i-1}v_i \in E$ for all $1 \leq i \leq k$. The number of walks of length $k$ from one vertex to another are indicated by the entries of the matrix $A^k$. The walk is *closed* if $v_0 = v_k$. If all $v_i$ are distinct the walk becomes a *path*. A closed walk with distinct vertices except for the first and last is called a *cycle*. The graph is *connected* if for each pair of vertices $u, v \in V$ there is a path in $G$ from $u$ to $v$. A *tree* is a connected graph with no cycles. A (connected) *component* of a graph is a maximal connected subgraph, i.e. a connected subgraph that is not contained in a larger connected subgraph. The *distance* $d_G(u, v)$ in $G$ between any two vertices $u, v \in V$ is the length of the shortest path between the two vertices. If this path does not exist we set $d_G(u, v) = \infty$.

A different notion of a graph which we have already used in Section 2.1.2 to define an irreducible matrix, is that of a *directed graph*. In a directed graph $\Gamma = (V, E)$ the set $E$ is an *ordered* two element subset of $V$. The elements of $E$ are now called *arcs*. The definition of a path is analogous in a directed graph, in this case we speak of a *directed path*. Now we say that $\Gamma$ is *strongly connected* if for each pair of vertices $u, v \in V$ there is a directed path in $\Gamma$ from $u$ to $v$.

### 2.2.2 Spectral graph theory

When talking about the eigenvalues of the graph we mean the eigenvalues of its adjacency matrix. In the previous section we saw that the diagonal entries of the adjacency matrix are all zero implying that the trace of the matrix is also zero. Since the trace equals the sum of the eigenvalues, this means that the eigenvalues of any graph always sum to zero.

*Example* 2.16. Let $K_{1,m}$ be the star graph on $m+1$ vertices. This is the graph consisting of one vertex connected to $m$ other vertices. Let $v_1$ be the centre of the star. The adjacency matrix of $K_{1,m}$ is

$$A = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \dots & 0 \end{pmatrix},$$

with spectrum $\sigma(A) = \left\{\sqrt{m}^1, 0^{m-1}, -\sqrt{m}^1\right\}$. We can easily see that these eigenvalues indeed sum to zero.

The eigenvalues of a graph can be used to analyse certain properties of the graph. For example, a graph is bipartite if and only if it has a symmetric spectrum, which means that for each eigenvalue $\lambda$ of the graph $-\lambda$ is also an eigenvalue. A simple argument shows that the spectral radius of any graph can not exceed its maximum degree.

**Lemma 2.17.** *The spectral radius of a graph is bounded from above by its maximum degree.*

*Proof.* Let $G$ be a graph with maximum degree $\Delta$, adjacency matrix $A$ and spectral radius $\lambda_1$. Let $\boldsymbol{v}$ be an eigenvector associated to $\lambda_1$ and let $x$ be the vertex for which $\boldsymbol{v}_x$ has maximum value over all vertices. We may assume that $\boldsymbol{v}_x$ is positive since otherwise we can simply take $-\boldsymbol{v}$ as our associated eigenvector. Now we have

$$\lambda_1 \boldsymbol{v}_x = (A\boldsymbol{v})_x = \sum_{y \sim x} \boldsymbol{v}_y \leq \sum_{y \sim x} \boldsymbol{v}_x = d(x)\boldsymbol{v}_x.$$

So we see that $\lambda_1 \leq d(x) \leq \Delta$. □

As seen in Section 2.1.2, the adjacency matrix of any connected graph is irreducible. So by Perron-Frobenius (Theorem 2.9) it immediately follows that the spectral radius of a connected graph has multiplicity one. The spectrum of a graph that is not connected, depends on the spectra of its components.

**Lemma 2.18.** *The spectrum of a graph is the union of the spectra of its components.*

*Proof.* Let $G$ be a graph of $n$ vertices with $m$ components denoted by $G_1, \ldots, G_m$. For each $i \in [m]$, the component $G_i$ has $n_i$ vertices and adjacency matrix $A_i$. By definition of a component, there are no edges between any two $G_i$ and $G_j$ for $i \neq j$. The adjacency matrix of the graph $G$ can thus be written as

$$A_G = \begin{pmatrix} A_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & A_m \end{pmatrix},$$

where $\mathbf{0}$ denotes the zero matrix. The eigenvalues of $G$ are all $\lambda$ such that $\det(\lambda I - A_G) = 0$. Furthermore, since $A_G$ is a block diagonal matrix its determinant equals

$$\det(\lambda I - A_G) = \det(\lambda I_{n_1} - A_1) \cdot \det(\lambda I_{n_2} - A_2) \cdot \cdots \cdot \det(\lambda I_{n_m} - A_m).$$

It follows that for any eigenvalue $\lambda$ of $G$, there must be some $i \in [m]$ such that $\det(\lambda I_{n_i} - A_i) = 0$, implying that $\lambda$ is an eigenvalue of $G_i$. Conversely, if $\lambda$ is an eigenvalue of $G_i$ for some $i \in [m]$, then the right hand side of the equation is zero and thus $\det(\lambda I - A_G) = 0$. So $\lambda$ is also an eigenvalue of $G$ and the lemma follows. □

The proof of the lemma in fact shows that the characteristic polynomial of $A_G$ is the product of the characteristic polynomials of $A_1, \ldots, A_m$. Two useful properties follow from this lemma. The first is that adding isolated vertices to a graph only adds zeroes to its spectrum and thus doesn't change its non-zero eigenvalues. Secondly, let $H$ be a connected graph and let $G$ be the union of taking $n$ copies of $H$. Since $H$ is connected it has spectral radius $\lambda_1(H)$ with multiplicity one. By the above lemma the graph $G$ then has spectral radius $\lambda_1(G) = \lambda_1(H)$ with multiplicity $n$.

Knowing the spectrum of graph, we can deduce certain properties of its subgraphs by interlacing. Let $H$ be an induced subgraph of a graph $G$. Notice that the adjacency matrix of $H$ is a principal submatrix of $G$. It thus follows immediately from Theorem 2.10 that the eigenvalues of $H$ interlace those of $G$. In particular this leads to the following property.

**Lemma 2.19.** *Let $H$ be an induced subgraph of a graph $G$. Then*

$$\lambda_1(G) \geq \lambda_1(H) \geq \lambda_{\min}(H) \geq \lambda_{\min}(G).$$

The inequality $\lambda_1(H) \leq \lambda_1(G)$ actually also holds if $H$ is a subgraph that is not induced. To see this notice that if an edge of the graph is removed, the new adjacency matrix $A'$ is a non-negative matrix such that $A_G - A'$ is non-negative. So by Perron-Frobenius, $\lambda_1(A') \leq \lambda_1(A_G)$. This shows that the removal of edges can only decrease the spectral radius of the graph. Interlacing shows us that the removal of a vertex also cannot increase the spectral radius.

## 2.3 Graph eigenvalue multiplicities

If we denote the eigenvalues of a graph $G$ as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{|G|}$, we call $\lambda_j$ the *j-th eigenvalue* of $G$. One of the main results of Jiang, Tidor, Yao, Zhang and Zhao in [12] is a theorem that states that the $j$-th eigenvalue of a connected graph with bounded maximum degree has bounded multiplicity.

**Theorem 2.20** ([12]). *For every integer $j$ and $\Delta$, there is a constant $C = C(j, \Delta)$ so that* every *connected $n$-vertex graph with maximum degree at most $\Delta$ has $j$-th eigenvalue multiplicity at most $\frac{Cn}{\log \log n}$.*

Note that in the theorem, $\Delta$ is a constant that is independent of the number of vertices of the graph. The case $j = 2$ will be one of the main tools in proving their result on equiangular lines with a fixed angle. The proof is however analogous for any fixed $j$. The result is proven by using interlacing of the eigenvalues of a subgraph on a strategically constructed subgraph.

The theorem is a general graph theoretical result, which is why we state and prove it in this chapter. It will be used in Chapter 5 to prove a tight bound on the number of equiangular lines with a fixed angle. In this section we will give the proof of the theorem. First we will need to define a specific subgraph and subset of a graph. Then we discuss some lemmas that will be necessary to prove the theorem before turning to the full proof.

Let $G$ be a graph and $v$ be a vertex in the graphs. The *$r$-neighbourhood* of $v$ is the subgraph induced by all vertices that are at distance at most $r$ from $v$. We denote this graph by $G_r(v)$. By definition, $G_r(v)$ is a connected graph. For a subset $W \subseteq V$ we will write $G_r(W) = \bigcup_{w \in W} G_r(w)$. We call a subset $U$ of the vertices of $G$ an *$r$-net* if all vertices of $G$ are within distance $r$ of some vertex in $U$. Note that the $r$-net is a subset of $V(G)$ and hence not a graph itself. An upper bound exists on the minimum size of an $r$-net that can be found in a connected graph.

**Lemma 2.21.** *Every $n$-vertex connected graph has an $r$-net with size at most $\left\lceil \frac{n}{r+1} \right\rceil$, where $n$ and $r$ are positive integers.*

*Proof.* Let $n$ and $r$ be positive integers. First of all we note that it suffices to prove the lemma in the case that $G$ is a tree. To see this, notice that any $r$-net of a spanning tree $T$ of a graph $G$ is also an $r$-net of $G$ since the distance between any two vertices in $G$ is always smaller or equal to their distance in $T$. We now prove the lemma by constructing an $r$-net of size at most $\left\lceil \frac{n}{r+1} \right\rceil$ in a tree.

So suppose $G$ is an $n$-vertex tree and pick an arbitrary vertex $v$ of $G$. Let $u$ be a vertex at maximum distance $d$ from $v$. If $d \leq r$, then all other vertices of the graph are also at a distance of at most $r$ from $v$ and thus $\{v\}$ is an $r$-net of $G$.

Otherwise, if $d > r$, we will add vertices to the net $U$ until it is indeed an $r$-net. We start with $U = \{v\}$. Let $w$ be the vertex at distance $r$ from $u$ lying on the path from $v$ to $u$. Add $w$ to the net $U$ and remove it from the graph $G$. Since $G$ is a tree, the graph $G - w$ now has at least two components, including one containing $u$ and one containing $v$. Note that the vertices in the components that do *not* contain $v$ are at distance at most $r$ from $w$ in $G$. Indeed, if this would not be the case, there would be a vertex $u'$ at distance at least $r + 1$ from $w$. This would imply that $u'$ is at distance at least $d + 1$ from $v$ which is not possible since $d$ is the maximum distance from $v$ to any other vertex in $G$. Since $u$ is at distance $r$ from $w$, the component containing $u$ has at least $r$ elements.

So now we only need to look at the component of $v$ in $G - w$, which has at most $n - (r + 1)$ vertices. We repeat the above argument inside this component, adding a new vertex to the net and again removing at least $r + 1$ vertices from the graph. This can be repeated at most $\frac{n}{r+1}$ times and thus the $r$-net will have no more than $\left\lceil \frac{n}{r+1} \right\rceil$ vertices. $\square$

Figure 2.1 shows the first step of the proof of the lemma for a tree in which we want to find a 2-net. The left graph shows the tree with vertex $u$ at maximum distance from $v$ and the vertex $w$ at distance 2 from $u$ on the path to $v$. By removing vertex $w$ we obtain the graph on the right, which has three components. The vertices of the components not containing $v$ all lie at distance at most 2 from $w$ in the original graph.

The $r$-nets and $r$-neighbourhoods can give better insights into the spectrum of the graph as the following two lemmas will show. The first result tells us that removing an $r$-net from the graph decreases the spectral radius. Furthermore, the spectral radii of the $r$-neighbourhoods of a graph $G$ can be used

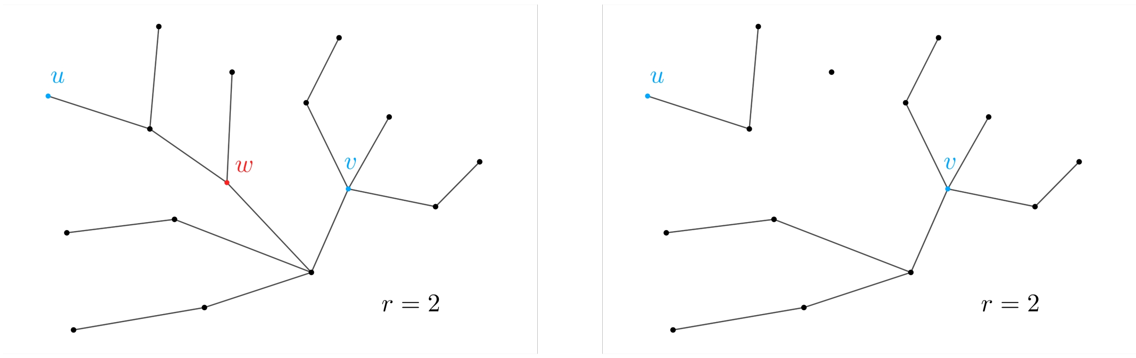


**Figure 2.1:** A tree before and after removing vertex $w$ to find a 2-net in it.

to bound its spectrum. This result follows by counting closed walks of length $2r$ in $G$ and the use of the Rayleigh quotient as a lower bound on the spectral radius.

**Lemma 2.22.** *Let $r$ be a positive integer and $G$ a graph. Let $H$ be the graph obtained from $G$ by removing an $r$-net of $G$. Then*

$$\lambda_1(H)^{2r} \leq \lambda_1(G)^{2r} - 1.$$

*Proof.* By Lemma 2.18 adding isolated vertices to a graph only adds zeroes to its spectrum. This implies that it suffices to show that the lemma holds for an $n$-vertex graph $G$ with no isolated vertices.

So let $G$ be a graph with no isolated vertices and let $H$ be the graph obtained from $G$ by removing an $r$-net. Add zeroes to the adjacency matrix $A_H$ of $H$ to extend it to an $n \times n$ matrix. We first show that $A_H^{2r} \leq A_G^{2r} - I$ entry-wise. Notice that the entries of $A_H^{2r}$ and $A_G^{2r}$ denote the number of walks of length $2r$ from one vertex to the other in $H$ and $G$ respectively. So, to show this inequality holds we will count the different types of walks in the graphs.

Note that since $H$ is a subgraph of $G$, all walks of length $2r$ in $H$ also exist in $G$. From this it immediately follows that $\left(A_H^{2r}\right)_{uv} \leq \left(A_G^{2r}\right)_{uv}$ for all distinct $u, v \in G$. We thus only need to show that for any vertex $v$ in $G$ there is at least one more walk of length $2r$ in $G$ than in $H$. Observe that in the graph $G$ there is a path from $v$ to a vertex in the $r$-net at distance at most $r$ from $v$. If this vertex is exactly at distance $r$, we can walk back along this path to find a walk of length $2r$. Otherwise, we can walk up and down between two vertices of this path a required number of times before turning back to $v$ to find a walk of length $2r$. This gives a walk of length $2r$ which is not available in the graph $H$, and thus $\left(A_H^{2r}\right)_{vv} \leq \left(A_G^{2r}\right)_{vv} - 1$. This shows that $A_H^{2r} \leq A_G^{2r} - I$ indeed holds.

The matrices $A_H^{2r}$ and $A_G^{2r} - I$ are both non-negative. By Perron-Frobenius it thus follows that

$$\lambda_1\left(A_H^{2r}\right) \leq \lambda_1\left(A_G^{2r} - I\right),$$

which implies the desired result $\lambda_1(H)^{2r} \leq \lambda_1(G)^{2r} - 1$. □

**Lemma 2.23.** *For every $n$-vertex graph $G$ and positive integer $r$,*

$$\sum_{i=1}^{n} \lambda_i(G)^{2r} \leq \sum_{v \in V(G)} \lambda_1\left(G_r(v)\right)^{2r}.$$

*Proof.* Let $G$ be an $n$-vertex graph and $r$ a positive integer. Recall that the sum of the eigenvalues of a matrix is equal to the trace of the matrix, so $\sum_{i=1}^{n} \lambda_i(G)^{2r} = \operatorname{Tr}\left(A_G^{2r}\right)$. The trace of $A_G^{2r}$ counts the number of closed walks of length $2r$ in $G$. For any $v \in V(G)$ all closed walks lie in the $r$-neighbourhood $G_r(v)$ of $v$, since a closed walk of length $2r$ can reach at most a vertex at distance $r$ from $v$ before returning. Let $\boldsymbol{e}_v$ be the vector with a 1 at entry $v$ at all other entries zero. The number of closed walks from $v$ are counted by $\boldsymbol{e}_v^\intercal A_{G_r(v)}^{2r} \boldsymbol{e}_v$, so $\operatorname{Tr}\left(A_G^{2r}\right) = \sum_{v \in V(G)} \boldsymbol{e}_v^\intercal A_{G_r(v)}^{2r} \boldsymbol{e}_v$. Since $\langle \boldsymbol{e}_v, \boldsymbol{e}_v \rangle = 1$, this quantity is equal to the Rayleigh quotient $\mathcal{R}\left(A_{G_r(v)}^{2r}, \boldsymbol{e}_v\right)$. By Theorem 2.6 it follows that

$$\boldsymbol{e}_v^\intercal A_{G_r(v)}^{2r} \boldsymbol{e}_v \leq \lambda_1\left(G_r(v)\right)^{2r}.$$

This yields $\sum_{i=1}^{n} \lambda_i(G)^{2r} \leq \sum_{v \in V(G)} \lambda_1\left(G_r(v)\right)^{2r}$ as required. □

To show the upper bound on the multiplicity of $\lambda_j$ we will create a subgraph by removing some strategically chosen vertices and then use interlacing of the eigenvalues. The first type of vertices we remove are all those whose $r$-neighbourhood have a large spectral radius, for some positive integer $r$. The size of the subset of these vertices is upper bounded.

**Lemma 2.24.** *Let $G$ be an $n$-vertex graph with maximum the degree $\Delta$ and denote by $\lambda = \lambda_j(G)$ the $j$-th eigenvalue of $G$. For any positive integer $r$, the set $W = \{v \in V(G) : \lambda_1(G_r(v)) > \lambda\}$ has at most $j\Delta^{2(r+1)}$ elements.*

*Proof.* Let $W_0$ be a maximal subset of $W$ such that the distance in $G$ between any two vertices of $W_0$ is at least $2(r+1)$. By the maximality of $W_0$, its $2(r+1)$-neighbourhood $G_{2(r+1)}(W_0)$ contains $W$. Indeed, suppose that this is not the case. Then there is a $w \in W$ which is not in $G_{2(r+1)}(W_0)$. This means that for all $u \in W_0$ the distance between $u$ and $w$ in $G$ is greater than $2(r+1)$. But then, $W_0 \cup \{w\}$ is a subset of $W$ larger than $W_0$ satisfying the same condition. This is not possible by the maximality of $W_0$. We thus have $W \subseteq G_{2(r+1)}(W_0)$. The $2(r+1)$ neighbourhood of any $w \in W_0$ has no more than $\Delta^{2(r+1)}$ vertices. This gives

$$|W| \le |G_{2(r+1)}(W_0)| \le |W_0|\Delta^{2(r+1)}.$$

So bounding the size of $W_0$ will give an upper bound on the size of $W$.

The $r$-neighbourhoods of any two vertices in $W_0$ are disjoint, so $G_r(W_0)$ has $|W_0|$ components. Since $W_0$ is a subset of $W$, each component of $G_r(W_0)$ has spectral radius larger than $\lambda$, by definition of $W$. This means that the $|W_0|$ first eigenvalues of $G_r(W_0)$ are all larger than $\lambda$, i.e.

$$\lambda_1\left(G_r(W_0)\right), \ldots, \lambda_{|W_0|}\left(G_r(W_0)\right) > \lambda = \lambda_j(G).$$

Since $G_r(W_0)$ is an induced subgraph of $G$ its eigenvalues interlace those of $G$, so the inequality

$$\lambda_i(G) \ge \lambda_i\left(G_r(W_0)\right)$$

holds for all $1 \le i \le |G_r(W_0)|$. It follows from the two inequalities that we must have $|W_0| < j$, which implies the upper bound $j\Delta^{2(r+1)}$ for $W$. $\square$

Next to the vertices from the above lemma, we will also remove an $r_1$-net from the graph for a certain positive integer $r_1 < r$. In the resulting graph we will first bound the multiplicity of $\lambda_j$ as an eigenvalue of this graph using the lemmas above. We then use interlacing to bound the multiplicity of the eigenvalue in the original graph. Recall that for a graph $G$ we write the eigenvalues of $G$ in non-increasing order $\lambda_1(G) \ge \lambda_2(G) \ge \cdots \ge \lambda_n(G)$.

*Proof of Theorem 2.20.* Let $G$ be a connected $n$-vertex graph with vertex set $V$ and maximum degree $\Delta$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $G$ and let $\lambda = \lambda_j(G)$ be the $j$-th eigenvalue. We want to show that the multiplicity $m_G(\lambda)$ of $\lambda$ in $G$ is of order $\mathcal{O}(n/\log\log n)$. We will first prove the theorem in the case that $\lambda \le 0$ and then prove it for $\lambda > 0$.

Suppose $\lambda \le 0$ and let $q$ be the number of distinct eigenvalues of $G$. The sum of the squared eigenvalues of $G$ satisfies

$$\sum_{i=1}^{n} \lambda_i^2 = m_G(\lambda_1)\lambda_1^2 + \cdots + m_G(\lambda)\lambda^2 + \cdots + m_G(\lambda_q)\lambda_q^2 \ge m_G(\lambda)\lambda^2,$$

since $\lambda_i^2 \ge 0$ for all $i \in [q]$. It thus suffices to bound the value of the sum. Since $\lambda \le 0$, for all $i \ge j$ the eigenvalue $\lambda_i$ is also at most zero. This implies

$$\left(\sum_{i=j}^{n} \lambda_i\right)^2 = \sum_{i=j}^{n} \lambda_i^2 + \sum_{\substack{i \ne k \\ i,k \ge j}} \lambda_i\lambda_k \ge \sum_{i=j}^{n} \lambda_i^2 \tag{2.5}$$

Furthermore, we note that since $0 = \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{j-1} \lambda_i + \sum_{i=j}^{n} \lambda_i$ the equality

$$\sum_{i=1}^{j-1} \lambda_i = -\sum_{i=j}^{n} \lambda_i \tag{2.6}$$

must hold. Now using these two identities and the fact that $0 \leq \lambda_i \leq \Delta$ for all $1 \leq i \leq j$, we find

$$\begin{aligned}
\sum_{i=1}^{n} \lambda_i^2 &\overset{(2.5)}{\leq} \sum_{i=1}^{j-1} \lambda_i^2 + \left(\sum_{i=j}^{n} \lambda_i\right)^2 \\
&\overset{(2.6)}{=} \sum_{i=1}^{j-1} \lambda_i^2 + \left(\sum_{i=1}^{j-1} \lambda_i\right)^2 \\
&\leq (j-1)\Delta^2 + (j-1)^2\Delta^2 \\
&\leq j^2\Delta^2.
\end{aligned}$$

It thus follows that $m_G(\lambda) \leq \left(\frac{j\Delta}{\lambda}\right)^2$ which is of order $\mathcal{O}(n/\log\log n)$ proving the theorem for $\lambda \leq 0$.

Now we turn to the case where $\lambda = \lambda_j > 0$. Let $c(\Delta, j)$ be a small enough constant and define $r_1 = \lfloor c \log\log n \rfloor$ and $r_2 = \lfloor c \log n \rfloor$. Denote the sum of these two constants as $r = r_1 + r_2$.

We start by constructing a subgraph $H$ of size at least $n - \mathcal{O}_{j,\Delta}(n/\log\log n)$ by removing specific vertices from the graph. Define the set $W = \{v \in V : \lambda_1(G_r(v)) > \lambda\}$, which by Lemma 2.24 has at most $j\Delta^{2(r+1)}$ elements. Let $U$ be an $r_1$-net in $G$ of size at most $\lceil n/(r_1+1) \rceil$, which exists by Lemma 2.21 since $G$ is connected. We define $H$ as the subgraph induced by $G \setminus (W \cup U)$. The two upper bounds on the sets $W$ and $U$ give

$$|W \cup U| \leq |W| + |U| \leq \left\lceil \frac{n}{r_1+1} \right\rceil + j\Delta^{2(r+1)} = \mathcal{O}_{j,\Delta}\left(\frac{n}{\log\log n}\right).$$

This leads to $|H| \geq n - \mathcal{O}_{j,\Delta}\left(\frac{n}{\log\log n}\right)$ as necessary.

The next step in the proof is to upper bound the multiplicity of $\lambda$ in $H$, denoted by $m_H(\lambda)$. We will first use Lemma 2.23 to bound $m_H(\lambda)$ using the spectral radii of the $2r$-neighbourhoods of vertices of $H$. Next, we will show that for any $v \in H$, the graph $H_{2r}(v)$ is obtained from $G_r(v)$ by removing an $r_1$-net. This will allow us to use Lemma 2.23 to further bound the multiplicity $m_H(\lambda)$.

Suppose $H$ has $k$ distinct eigenvalues denoted by $\mu_i$ and note that $\mu_i^{2r_2} \geq 0$ for all $i \in [k]$. We have

$$\begin{aligned}
\sum_{i=1}^{|H|} \mu_i^{2r_2} &= m_H(\mu_1) \cdot \mu_1^{2r_2} + \cdots + m_H(\lambda) \cdot \lambda^{2r_2} + \cdots + m_H(\mu_k) \cdot \mu_k^{2r_2} \\
&\geq m_H(\lambda) \cdot \lambda^{2r_2},
\end{aligned}$$

and by Lemma 2.23

$$\sum_{i=1}^{|H|} \mu_i^{2r_2} \leq \sum_{v \in V(H)} \lambda_1(H_{r_2}(v))^{2r_2}.$$

Hence, we find

$$m_H(\lambda) \cdot \lambda^{2r_2} \leq \sum_{v \in V(H)} \lambda_1(H_{r_2}(v))^{2r_2}. \tag{2.7}$$

It follows that to find an upper bound on $m_H(\lambda)$ we need to bound the spectral radius of $H_{r_2}(v)$ for any $v \in V(H)$.

To bound $\lambda_1(H_{r_2}(v))$ for any $v \in V(H)$ we will show that $H_{r_2}(v)$ is obtained from $G_r(v)$ by removing an $r_1$-net. Then we apply Lemma 2.22 to further bound $m_H(\lambda)$. First of all we show that $H_{r_2}(v)$ is a subset of $G_r(v)$. Then we prove that $G_r(v) \setminus H_{r_2}(v)$ is indeed an $r_1$-net of $G_r(v)$ by showing that for any $u \in G_r(v)$ there is a $w \in G_r(v) \setminus H_{r_2}(v)$ such that the distance in $G_r(v)$ between $u$ and $w$ is at most $r_1$.

Let $v$ be an arbitrarily chosen vertex of $V(H)$. Note that $H_{r_2}(v)$ is a subgraph of $H$ and hence contains no vertices from the sets $W$ and $U$. The graph $H_{r_2}(v)$ contains all vertices at distance at most $r_2$ from $v$ in $H$. Since $H \subseteq G$, this means that any vertex $u \in H_{r_2}(v)$ is also within distance $r_2$ of $v$ in $G$. In particular, it now follows from $r_2 \leq r$ that $u$ is within distance $r$ of $v$ in $G$, implying that $u \in G_r(v)$. This shows that $H_{r_2}(v) \subseteq G_r(v)$.

Figure 2.2 shows a sketch of the graph $G$ with the subsets $U$ and $W$ and the subgraphs $H_{r_2}(v)$ and $G_r(v)$ for some $v \in V(H)$. The red vertices in the picture represent the set $W$. The area outside of $U$

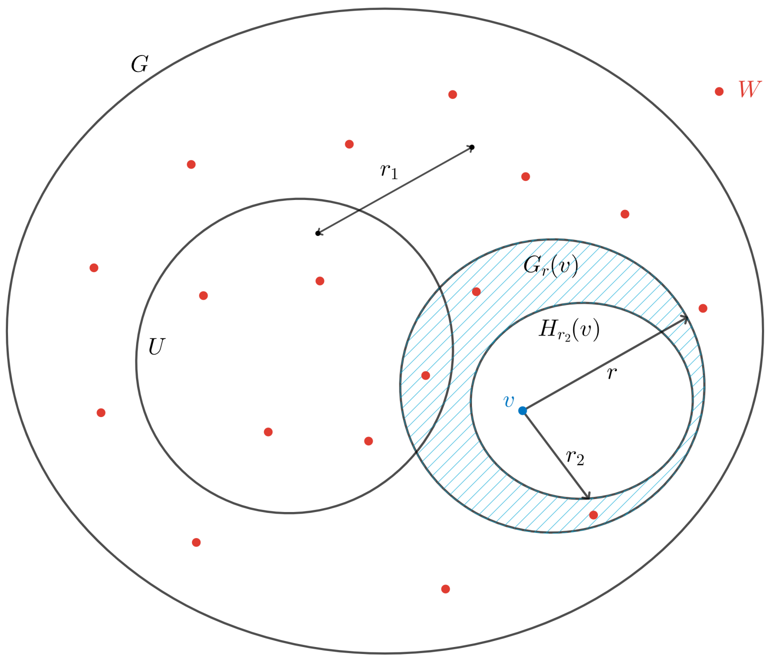


**Figure 2.2:** Sketch of the graph and subgraphs from the proof of Theorem 2.20

excluding $W$ is the graph $H$. For any vertex in $G$ there is a vertex in $U$ at distance at most $r_1$. The graph $H_{r_2}(v)$ is a subgraph of $H$ and hence contains no vertices from $U$ nor from $W$, i.e. no red vertices. The distance from $v$ to any other vertex in $H_{r_2}(v)$ is at most $r_2$ and its distance to any vertex in $G_r(v)$ is at most $r$. We now show that the marked area, i.e. $G_r(v) \setminus H_{r_2}(v)$, is ans $r_1$-net of $G_r(v)$.

Take $u \in G_r(v)$ arbitrarily. To show that $G_r(v) \setminus H_{r_2}(v)$ is an $r_1$-net of $G_r(v)$ we will prove that a vertex $x \in G_r(v) \setminus H_{r_2}(v)$ exists at distance at most $r_1$ from $u$. Note that if $u \notin H_{r_2}(v)$ the case is trivial, since we can take $x = u$. Hence, we only need to show such an $x$ exists if $u \in H_{r_2}(v)$. Since $H_{r_2}(v) \subseteq G_r(v)$ the distance of $u$ to $v$ in $G_r(v)$ is at most $r_2$. Let $x$ be a vertex in $U$ at distance at most $r_1$ from $u$ in $G$. Then

$$d_G(v, x) \leq d_G(v, u) + d_G(u, x) \leq r_1 + r_2 = r.$$

This implies that $x \in G_r(v)$. Furthermore, since $x$ is an element of $U$ it can not be an element of $H_{r_2}(v)$. This proves the existence of a vertex in $G_r(v) \setminus H_{r_2}(v)$ at distance at most $r_1$ from $u$ and thus $G_r(v) \setminus H_{r_2}(v)$ is indeed an $r_1$-net of $G_r(v)$.

We have now shown that $H_{r_2}(v)$ is obtained from $G_r(v)$ by removing an $r_1$-net, for any $v \in V(H)$. Applying Lemma 2.22 thus yields

$$\lambda_1(H_{r_2}(v))^{2r_1} \leq \lambda_1\left(G_r(v)\right)^{2r_1} - 1 \leq \lambda^{2r_1} - 1,$$

where the last step follows since $v \notin W$. Using this result and inequality (2.7) we now find

$$\begin{aligned} m_H(\lambda) \cdot \lambda^{2r_2} &\leq \sum_{v \in V(H)} \lambda_1(H_{r_2}(v))^{2r_2} \\ &= \sum_{v \in V(H)} \left(\lambda_1(H_{r_2}(v))^{2r_1}\right)^{\frac{r_2}{r_1}} \\ &\leq \sum_{v \in V(H)} \left(\lambda^{2r_1} - 1\right)^{\frac{r_2}{r_1}} \\ &\leq n\left(\lambda^{2r_1} - 1\right)^{\frac{r_2}{r_1}}. \end{aligned}$$

Dividing both sides by $\lambda^{2r_2}$ yields $m_H(\lambda) \leq \left(1 - \lambda^{-2r_1}\right)^{\frac{r_2}{r_1}} \leq e^{-\lambda^{-2r_1}\frac{r_2}{r_1}}$. For $c$ small enough this is at most $m := e^{-\sqrt{\log n}}$, which is of order $\mathcal{O}_{j,\Delta}(n/\log\log n)$. To see this, note that when $c$ becomes smaller,

$\lambda^{-2r_1} = \lambda^{-2c \log\log n}$ becomes larger. This makes $e^{-\lambda^{-2r_1}\frac{r_2}{r_1}}$ smaller, and for small enough $c$ this is indeed smaller than $e^{-\sqrt{\log n}}$. Figure 2.3 shows the two functions in the specific case where $\lambda^{2c} = \frac{3}{2}$. It also shows $\frac{n}{\log\log n}$, from which we easily see that indeed $e^{-\sqrt{\log n}} = \mathcal{O}_{j,\Delta}(n/\log\log n)$.

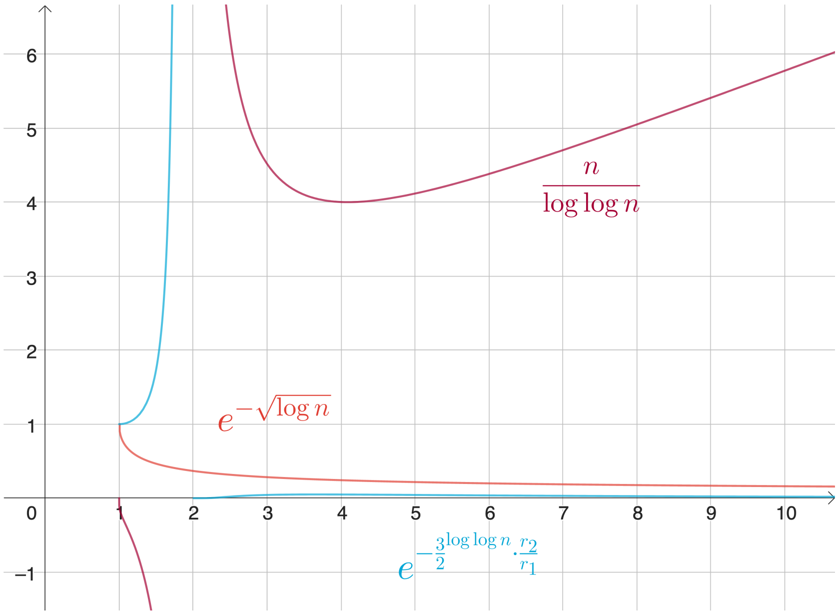


**Figure 2.3:** Growth of functions $e^{-\frac{3}{2}^{\log\log n}\frac{r_2}{r_1}}$, $e^{-\sqrt{\log n}}$ and $\frac{n}{\log\log n}$.

In the last step of this proof we will use interlacing to show that this implies that the multiplicity of $\lambda$ in $G$ is also at most $\frac{Cn}{\log\log n}$ for some constant $C = C(j, \Delta)$. Let $i$ be the first index such that $\mu_i = \lambda$. Then $\mu_{i-1} > \mu_i = \lambda$. By the interlacing of eigenvalues of induced subgraphs, Lemma 2.19, we have $\mu_{i-1} > \lambda = \lambda_j \geq \mu_j$, which implies that $i - 1 < j$. Next note that since the multiplicity of $\lambda$ in $H$ is at most $m$, $\lambda = \mu_i > \mu_{i+m+1}$. Let $m_G(\lambda)$ denote the multiplicity of $\lambda$ in $G$, so that $\lambda = \lambda_j = \lambda_{j+m_G(\lambda)}$. Again applying Lemma 2.19 we find

$$\lambda_{j+m_G(\lambda)} > \mu_{i+m+1} \geq \lambda_{n-|H|+i+m+1}.$$

Which implies

$$\begin{aligned} j + m_G(\lambda) &\leq n - |H| + i + m + 1 \\ &\leq n - n + \mathcal{O}_{j,\Delta}\left(\frac{n}{\log\log n}\right) + j + \mathcal{O}_{j,\Delta}\left(\frac{n}{\log\log n}\right) + 1 \\ &= j + \mathcal{O}_{j,\Delta}\left(\frac{n}{\log\log n}\right), \end{aligned}$$

and hence we find $m_G(\lambda) = \mathcal{O}_{j,\Delta}\left(\frac{n}{\log\log n}\right)$ as needed. □

The theorem does not work for disconnected graphs. This is because a disconnected graph can have many components with $\lambda_j$ as their spectral radius. Furthermore, the theorem also does not work without the assumption of the maximum degree. To see this, note that the complete graphs have $n$ vertices, degree $n-1$ and second eigenvalue $-1$ with multiplicity $n-1$.

In the study of equiangular lines, we are mostly interested in the case where $j = 2$. The theorem above was a new result and since then further research has been done into the multiplicity of the second eigenvalue. The upper bound has been improved in [19] to $\mathcal{O}\left(n/\log^{1/5-o(1)} n\right)$. The best known lower bound is due to Haiman, Schildkraut, Zhang, and Zhao in [20]. In this article a construction is given of graphs with bounded degree whose second eigenvalue have a multiplicity of order $n^{1/2-o(1)}$.

# 3

# Classic results

The problem of finding the maximum number of lines $N(d)$ in dimension $d$ such that the angle between any two lines is the same was first stated by Van Lint & Seidel [2]. A few years later Lemmens & Seidel [4] introduced the related problem of finding the maximum number of equiangular lines $N_\alpha(d)$ in dimension $d$ with fixed common angle $\arccos \alpha$. The results of these two papers lay a foundation into the topic of equiangular lines and give the first insights into the behaviours of $N(d)$ and $N_\alpha(d)$. In this chapter we will discuss three of these classic results. The first will be a theorem due to Neumann, which shows that $N_\alpha(d)$ can only be large if $\alpha^{-1}$ is an odd integer. The next two results are the *absolute* and *relative* bounds, which are both upper bounds. Neumanns theorem is used to classify the dimensions for which the absolute bound can hold with equality. Before discussing these results, we will rewrite the problem of finding a set of equiangular lines into finding a specific kind of matrix. To give an example of a proof using this notation, we will show an almost trivial lower bound on $N_\alpha(d)$.

Let $\mathcal{L} = \{l_1, l_2, \ldots, l_n\}$ be a set of equiangular lines in $\mathbb{R}^d$ with common angle $\arccos \alpha$, where $\alpha \in (0, 1)$. For any line $l_i \in L$, we can choose a unit vector $\boldsymbol{v}_i \in \mathbb{R}^d$ in the direction of the line, see Figure 3.1. Then, for all distinct $i, j \in [n]$, the inner product between any two unit vectors $\boldsymbol{v}_i$ and $\boldsymbol{v}_j$ satisfies $\langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle = \pm\alpha$. The set of resulting unit vectors $\mathcal{C} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ is a also known as a *spherical code*.

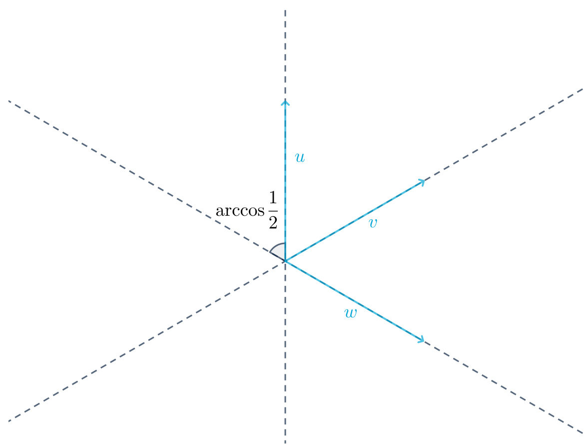


**Figure 3.1:** Unit vectors representing equiangular lines in $\mathbb{R}^2$.

**Definition 3.1.** A set of unit vectors $\mathcal{C}$ in $\mathbb{R}^n$ is a *spherical L-code* if $\langle \boldsymbol{v}, \boldsymbol{u} \rangle \in L$ for any pair of distinct vectors $\boldsymbol{v}, \boldsymbol{u} \in C$.

So we see that a set of $n$ equiangular lines with common angle $\arccos(\alpha)$ corresponds to a spherical $\{-\alpha, \alpha\}$-code, which we denote by $\mathcal{C}_\alpha$. We will say that $\mathcal{C}_\alpha$ *represents* the set of equiangular lines. By Theorem 2.11 a positive semidefinite matrix can be constructed from this set of vectors, which is known as the Gram matrix of $\mathcal{C} = \mathcal{C}_\alpha$, which we denote by $M_\mathcal{C}$. The entries of $M_\mathcal{C}$ are 1 on the diagonal and

$\pm\alpha$ everywhere else. It is a symmetric $n \times n$ matrix of rank at most $d$, since the vectors in $\mathcal{C}$ are all elements of $\mathbb{R}^d$.

*Example* 3.2. Let $\mathcal{C} = \{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ be the spherical $\{-\frac{1}{2}, \frac{1}{2}\}$-code representing the set of three equiangular lines in $\mathbb{R}^2$ in Figure 3.1. The inner products are $\langle \boldsymbol{u}, \boldsymbol{v}\rangle = \frac{1}{2}$, $\langle \boldsymbol{v}, \boldsymbol{w}\rangle = \frac{1}{2}$ and $\langle \boldsymbol{u}, \boldsymbol{v}\rangle = -\frac{1}{2}$. So the Gram matrix of $\mathcal{C}$ is

$$M_{\mathcal{C}} = \begin{array}{c c} & \begin{array}{ccc} u & v & w \end{array} \\ \begin{array}{c} u \\ v \\ w \end{array} & \begin{pmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix} \end{array}.$$

This matrix has eigenvalues $\lambda_1 = \lambda_2 = \frac{3}{2}$ and $\lambda_3 = 0$. These are all non-negative, so $M_{\mathcal{C}}$ is indeed positive semidefinite. The multiplicity of the eigenvalue 0 is one, and thus the rank of the matrix is $\operatorname{rk} M_{\mathcal{C}} = 3 - 1 = 2$ which is equal to the dimension $d = 2$ as required.

Conversely, any positive semidefinite matrix $M$ with diagonal entries 1 and off-diagonal entries $\pm\alpha$ of rank at most $d$ is the Gram matrix of a set of $n$ equiangular lines. By Theorem 2.11, vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \in R^d$ can be found such that $M_{ij} = \langle \boldsymbol{v}_i, \boldsymbol{v}_j\rangle$. These vectors give a spherical $\{-\alpha, \alpha\}$-code representing a set of $n$ equiangular lines. So to find a set of $n$ equiangular lines in $\mathbb{R}^d$ with common angle $\arccos\alpha$, it suffices to find an $n \times n$ matrix with all the properties of $M$. We will now prove a lower bound on $N_\alpha(d)$ by constructing such a matrix.

**Proposition 3.3.** *For $\alpha \in (0, 1)$ and integer $d$, $N_\alpha(d) \geq d$.*

*Proof.* Define the $d \times d$ matrix $M = (1 - \alpha)I + \alpha J$, where $\alpha \in (0, 1)$. This matrix has eigenvalue $1 + (d - 1)\alpha$ with multiplicity 1 and $1 - \alpha$ with multiplicity $d - 1$. Since 0 is not an eigenvalue, the matrix has full rank. Furthermore, all eigenvalues are positive, so $A$ is positive semidefinite. By Theorem 2.11 we can thus find vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_d \in \mathbb{R}^d$ such that $M_{ij} = \langle \boldsymbol{v}_i, \boldsymbol{v}_j\rangle$. These vectors correspond to a set of equiangular lines since for any distinct $i, j \in [d]$, $\langle \boldsymbol{v}_i, \boldsymbol{v}_j\rangle = \alpha$. Thus we have found a construction of $d$ equiangular lines in $\mathbb{R}^d$ with common angle $\alpha$, so $N_\alpha(d) \geq d$. ☐

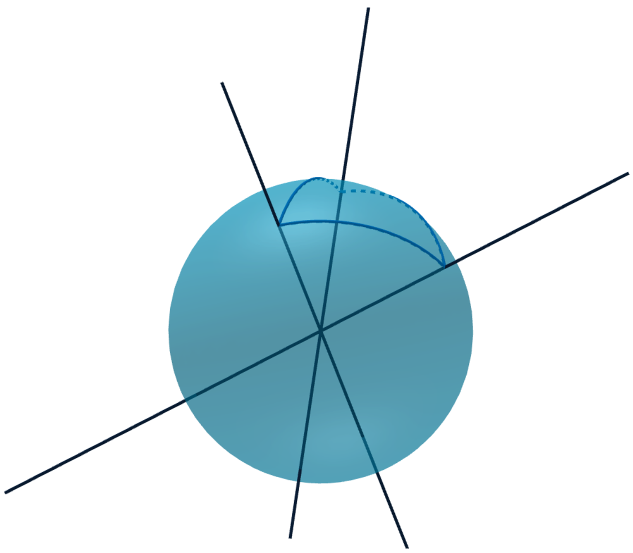

**Figure 3.2:** Three equiangular lines in $R^3$.

This lower bound is still the best general lower bound known to date. In Chapter 5 we will see a better lower bound which only holds for a specific class of values of $\alpha$. Furthermore, Schildkraut proved very recently that there are infinitely many $\alpha \in (0, 1)$ such that $N_\alpha(d) \geq n + \Omega(\log\log n)$ [21]. For all other values of $\alpha$, the above proposition is the best bound known.

It is natural to ask what the construction of $d$ equiangular lines in $\mathbb{R}^d$ actually looks like. This construction can be visualised in the following way. Take the unit sphere $S^{d-1}$ in $\mathbb{R}^d$ and embed a regular $d$-simplex onto this sphere. Draw lines connecting the centre of the sphere with each vertex of the simplex. This yields $d$ lines and the angle between any two of these is always the same. The angle between the lines can then be increased or decreased by making the simplex larger or smaller. In $\mathbb{R}^3$ this means that we take a regular triangle embedded onto the unit sphere $S^{d-1}$ and connect each vertex of the triangle to the center of the sphere. Figure 3.2 shows what such a construction looks like.

## 3.1 Neumanns theorem

In this section we prove a result by Neumann given in [4] which states that if $N_\alpha(d)$ is bigger than $2d$, then $\alpha^{-1}$ should be an odd integer. In particular this means that for all other values of $\alpha$ the number of equiangular lines in $\mathbb{R}^d$ with common angle $\arccos\alpha$ is upper bounded by $2d$. The result has led to an interest in the special case of $\alpha^{-1}$ being an odd integer, since it tells us that a high number of lines can only exist in this case. We follow the original proof from [4].

**Theorem 3.4** (Neumann)**.** *If $N_\alpha(d) > 2d$, then $1/\alpha$ is an odd integer.*

*Proof.* Let $\mathcal{C}$ be a $\{-\alpha, \alpha\}$-code representing a set of $n$ equiangular lines with common angle $\arccos\alpha$. Let $M_\mathcal{C}$ be the Gram matrix of the code $\mathcal{C}$. We know that this matrix is positive semidefinite and has rank at most $d$. By the rank-nullity theorem the kernel of $M_\mathcal{C}$ has dimension at least $n-d$. Since the matrix is positive semidefinite, this implies that it has smallest eigenvalue 0 with multiplicity $m \geq n-d$. We now construct the matrix

$$S = \frac{1}{\alpha}(M_\mathcal{C} - I),$$

which has smallest eigenvalue $-\frac{1}{\alpha}$ with multiplicity $m$. Observe that $S$ is an integer matrix, so $-\frac{1}{\alpha}$ is an algebraic integer by Lemma 2.7. Now, if $n > 2d$, then $m \geq n-d > \frac{1}{2}n$. Since $S$ is an $n \times n$ matrix it can not have more than one eigenvalue of multiplicity $m$. By Remark 2.8, all algebraic conjugates of $-\frac{1}{\alpha}$ are also eigenvalues of $A$ with the same multiplicity. Since, there can not be any more eigenvalues with multiplicity $m$, $-\frac{1}{\alpha}$ does not have any conjugates. This implies that $-\frac{1}{\alpha}$ is a rational algebraic integer. Any rational algebraic integer must be an integer, and thus $-\frac{1}{\alpha}$ is integer.

It now remains to show that $-\frac{1}{\alpha}$ is odd. In order to do this consider the matrix

$$B = \frac{1}{2}(J - I - S).$$

The eigenspace of $A$ corresponding to the eigenvalue $-\frac{1}{\alpha}$ has dimension $m$. The eigenspace of $J$ corresponding to the eigenvalue 0 has dimension $n-1$. Since $m > 1$ these two subspaces have a nontrivial intersection. That means that there exists a vector $\boldsymbol{v}$ which is an eigenvector of $A$ with eigenvalue $-\frac{1}{\alpha}$ and an eigenvector of $J$ with eigenvalue 0. This vector is also an eigenvector of $B$ with corresponding eigenvalue $\mu = \frac{1}{2}\left(-1 + \frac{1}{\alpha}\right)$. Indeed,

$$B\boldsymbol{v} = \frac{1}{2}(J - I - S)\boldsymbol{v} = \frac{1}{2}\left(0 - 1 - \left(-\frac{1}{\alpha}\right)\right)\boldsymbol{v} = \frac{1}{2}\left(-1 + \frac{1}{\alpha}\right)\boldsymbol{v}.$$

The matrix $B$ is again an integer matrix, so $\mu$ is an algebraic integer. Since $-\frac{1}{\alpha}$ is a rational algebraic integer, $\mu$ is also a rational algebraic integer and thus integer. For this to hold, $-\frac{1}{\alpha}$ must be odd. $\square$

The matrix $S = \frac{1}{\alpha}(M_\mathcal{C} - I)$ used in the proof, is a *Seidel matrix.* Seidel matrices are symmetric matrices with zeroes on the diagonal and all other entries $\pm 1$. They were first introduced by Van Lint and Seidel in [2]. Introductions into Seidel matrices can be found in Chapter 11 of [3] and Section 1.8.2 of [18].

Above we saw that finding a set of $n$ equiangular lines with common angle $\arccos\alpha$ in $\mathbb{R}^d$ corresponds to finding a positive semidefinite matrix $M$ of rank $d$ with diagonal entries 1 and off-diagonal entries $\pm\alpha$. Since the smallest eigenvalue of $M$ is zero with multiplicity at least $n-d$, the smallest eigenvalue of the Seidel matrix $S$ is $-1/\alpha$ with multiplicity at least $n-d$. It follows that to find a set of $n$ equiangular lines in $\mathbb{R}^d$, we must find an $n \times n$ Seidel matrix whose smallest eigenvalue $\lambda_{\min}$ has multiplicity at least $n-d$. The cosine of the angle between the lines is then $\pm\frac{1}{\lambda_{\min}}$. From the Seidel matrix we can find the Gram matrix corresponding to a set of equiangular lines with the equation $M = I - \frac{1}{\lambda_{\min}}S$.

## 3.2 The Absolute Bound

We now turn to the first upper bound on the number $N(d)$ of equiangular lines in $\mathbb{R}^d$. It is known as the *absolute bound* and is due to Gerzon [4, Theorem 3.5]. The name arises from the fact that the bound only depends on the dimension and not on the angle between the lines. The theorem says that there can be no more than $d(d+1)/2$ equiangular lines in $\mathbb{R}^d$. To prove the result we will follow the approach of Godsil and Royle in [3]. The proof uses projection matrices and their trace inner product.

Let $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ be a set of unit vectors corresponding to a set of equiangular lines with common angle $\arccos \alpha$. Define the matrix $X_i = \boldsymbol{v}_i \boldsymbol{v}_i^\top$. This matrix is a symmetric $d \times d$ matrix and $X_i^2 = X_i$. So $X_i$ is a symmetric projection matrix representing the projection onto the line $l_i$ spanned by the vector $\boldsymbol{v}_i$. These projection matrices $X_1, X_2, \ldots, X_n$ satisfy

$$X_i X_j = \boldsymbol{v}_i \boldsymbol{v}_i^\top \boldsymbol{v}_j \boldsymbol{v}_j^\top = \left(\boldsymbol{v}_i^\top \boldsymbol{v}_j\right) \boldsymbol{v}_i \boldsymbol{v}_j^\top$$

and hence, using the linearity of the trace and Equation 2.2, we find that the trace inner product is given by

$$\langle X_i, X_j \rangle_F = \mathrm{Tr}(X_i X_j) = \mathrm{Tr}\left(\left(\boldsymbol{v}_i^\top \boldsymbol{v}_j\right) \boldsymbol{v}_i \boldsymbol{v}_j^\top\right) = \left(\boldsymbol{v}_i^\top \boldsymbol{v}_j\right) \mathrm{Tr}\left(\boldsymbol{v}_i \boldsymbol{v}_j^\top\right) = \left(\boldsymbol{v}_i^\top \boldsymbol{v}_j\right)\left(\boldsymbol{v}_j^\top \boldsymbol{v}_i\right) = \langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle^2.$$

It follows that for any distinct $i, j \in [n]$

$$\langle X_i, X_j \rangle_F = \langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle^2 = \alpha^2.$$

If $i = j$ it follows from $X_i$ being a projection matrix and $\boldsymbol{v}_i$ being a unit vector that

$$\langle X_i, X_i \rangle_F = \mathrm{Tr}(X_i^2) = \mathrm{Tr}(X_i) = \mathrm{Tr}(\boldsymbol{v}_i \boldsymbol{v}_i^\top) = \langle \boldsymbol{v}_i, \boldsymbol{v}_i \rangle = 1.$$

To prove the theorem we will show that the matrices $X_1, X_2, \ldots, X_n$ defined above are linearly independent. Since they are symmetric matrices and the space of symmetric matrices has dimension $d(d+1)/2$ the theorem will follow. We will also show that the bound is tight only for a specific class of dimensions using Neumanns theorem. Two of the dimensions for which we know the bound is tight are $d = 2$ and $d = 3$. For these dimensions the problem was already solved by Haantjes [1] who showed that $N(2) = 3 = 2 \cdot (2+1)/2$ and $N(3) = 6 = 3 \cdot (3+1)/2$. Next to these two cases, the only other two known dimensions for which the absolute bound is reached are $d = 7$ and $d = 23$ [22, Table 1].

**Theorem 3.5** (Gerzon)**.** *We have $N(d) \leq \frac{d(d+1)}{2}$. If equality holds, then $d = 2$, $d = 3$ or $d + 2$ is the square of an odd integer.*

*Proof.* Let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ be unit vectors corresponding to a set of $n$ equiangular lines and let $\arccos \alpha$ be the common angle. Define the projections $X_1 = \boldsymbol{v}_1 \boldsymbol{v}_1^\top, \ldots, X_n = \boldsymbol{v}_n \boldsymbol{v}_n^\top$. As we have seen above that for all distinct $i, j \in [n]$, $\langle X_i, X_j \rangle = \alpha^2$ and $\langle X_i, X_i \rangle = 1$.

The space of symmetric $d \times d$ matrices has dimension $\binom{d+1}{2}$. We will prove the statement by showing that the matrices $X_1, \ldots, X_n$ are linearly independent.

Suppose $\sum_{i=1}^n c_i X_i = 0$ for some $c_i \in \mathbb{R}$, $i = 1, \ldots, n$. Then for any $j \in [n]$:

$$\begin{aligned}
0 &= \left\langle X_j, \sum_{i=1}^n c_i X_i \right\rangle \\
&= \sum_{i=1}^n c_i \,\mathrm{Tr}(X_i X_j) \\
&= c_j \cdot 1 + \alpha^2 \sum_{i \neq j}^n c_i \\
&= (1 - \alpha^2) c_j + \alpha^2 \sum_{i=1}^n c_i.
\end{aligned}$$

This holds for all $j \in [n]$ which implies that we must have $c_1 = \cdots = c_n =: c$. So the equation becomes

$$0 = (1 - \alpha^2) c + \alpha^2 n c = (1 + (n-1)\alpha^2) c.$$

For this to hold we must have $c = 0$, since $1 + (n-1)\alpha^2 > 0$. Thus it follows that the matrices are linearly independent and so $N(d) \leq \binom{d+1}{2}$.

Now suppose that equality holds. We have already seen above that for $d = 2$ and $d = 3$ equality holds, so we only need to show that $d + 2$ is the square of an odd integer. When equality holds, the matrices $X_1, \ldots, X_n$ form a basis of the space of symmetric $d \times d$ matrices. In particular, this means that there exist scalars $b_1, \ldots, b_n$ such that $I = \sum_{i=1}^n b_i X_i$. First we show that for this to hold we must have $b_i = d/n$ for all $i \in [n]$.

For any $j \in [n]$ we have

$$X_j = IX_j = \sum_{i=1}^n b_i X_i X_j.$$

Taking the trace gives

$$\begin{aligned} 1 = \operatorname{Tr}(X_j) &= \operatorname{Tr}\left(\sum_{i=1}^n b_i X_i X_j\right) \\ &= \sum_{i=1}^n b_i \operatorname{Tr}(X_i X_j) \\ &= (1-\alpha^2) b_j + \sum_{i=1}^n b_i \end{aligned} \tag{3.1}$$

Since this holds for all $j$, all $b_j$'s must be equal. Let $b$ denote this value. Then $I = b\sum_{i=1}^n X_i$ and by taking the trace again we find

$$d = \operatorname{Tr}(I) = b \sum_{i=1}^n \operatorname{Tr}(X_i) = bn.$$

By substituting $b = d/n$ into Equation 3.1 and simplifying, we find $d+2 = \frac{1}{\alpha^2}$. To prove the theorem it thus suffices to show that $1/\alpha$ is an odd integer. This immediately follows from Theorem 3.4, by noticing that for $d > 3$, $n = \frac{1}{2}d(d+1) > 2d$.

□

It follows from the theorem and the proof that this absolute maximum of equiangular lines can only be attained in very specific cases. To reach the absolute bound for a dimension $d > 3$, $d + 2$ must be the square of an odd integer and $1/\alpha$ must be an odd integer satisfying $d + 2 = 1/\alpha^2$. The first dimension which satisfies these conditions is $d = 7$. In this case $d + 2 = 9 = 3^2$ and $1/\alpha = 3$ which is indeed an odd integer. A construction of $d(d+1)/2 = 28$ equiangular lines in $\mathbb{R}^7$ with common angle $\arccos\left(\frac{1}{3}\right)$ indeed exists as shown by the following example. This construction was already given in a slightly different formulation by Van Lint and Seidel in [2].

*Example* 3.6. Define a unit vector in $\mathbb{R}^8$ by

$$\boldsymbol{v}_1^\mathsf{T} = \frac{1}{\sqrt{24}}(3, 3, -1, -1, -1, -1, -1, -1).$$

Let $\boldsymbol{v}_2, \boldsymbol{v}_3, \ldots, \boldsymbol{v}_{28}$ be all other vectors of this form with two entries 3 and all other entries $-1$. Notice that for all $i \in [28]$, $\langle \boldsymbol{v}_i, \mathbf{1} \rangle = \frac{1}{\sqrt{24}}(3 + 3 - 6) = 0$, so all vectors are orthogonal to $\mathbf{1}$. This means that the set of vectors lies in the 7-dimensional subspace of $\mathbb{R}^8$ orthogonal to $\mathbf{1}$, and hence the vectors correspond to 28 lines in $\mathbb{R}^7$. Furthermore, for any $i \neq j$, $\langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle = \pm 1/3$, indicating that the angle between any two vectors is $\arccos\left(\frac{1}{3}\right)$. So this yields the desired 28 lines in $\mathbb{R}^7$ with common angle $\arccos\left(\frac{1}{3}\right)$.

A construction of a tight bound is also known for $d = 23$, shown for the first time in [23]. In higher dimensions, no construction reaching the absolute bound is yet known. A construction of equiangular lines on the order of $d^2$ has been given by de Caen in [24] for an infinite number of dimensions. This is the first and only constructive lower bound on the order of $d^2$ known up to date.

## 3.3 The Relative Bound

The absolute bound gives an upper bound on the number of equiangular lines in $\mathbb{R}^d$ independent of the angle between the lines. An analogous version of the bound exist if we take the angle between the lines to be fixed. The resulting bound is called the *relative* bound, as opposed to the absolute bound. This result was first proven by Van Lint and Seidel in [2]. Their proof uses the trace of the Seidel matrix. Here we again use the approach from Godsil and Royle [3].

**Theorem 3.7.** *For any $\alpha \in (0,1)$, $N_\alpha(d) \leq d\frac{1-\alpha^2}{1-d\alpha^2}$ if $d < 1/\alpha^2$.*

*Proof.* Let $\mathcal{L}$ be a set of $n$ equiangular lines with common angle $\arccos\alpha$ and let $X_1,\ldots,X_n$ be the projections onto the lines of $\mathcal{L}$. Recall that we have $\langle X_i, X_j\rangle = \alpha^2$ and $\langle X_i, X_i\rangle = 1$ for all distinct $i,j \in [n]$. Put

$$Y = I - \frac{d}{n}\sum_{i=1}^{n} X_i.$$

Notice that $Y$ is a symmetric matrix, so by the non-negativity of the trace inner product $\mathrm{Tr}(Y^\mathsf{T} Y) = \mathrm{Tr}(Y^2) \geq 0$. We have $Y^2 = I - \frac{2d}{n}\sum_{i=1}^n X_i + \frac{d^2}{n^2}\left(\sum_{i=1}^n X_i\right)^2$, which gives

$$\mathrm{Tr}(Y^2) = d - 2d + \frac{d^2}{n^2}(1 + \alpha^2 n(n-1)) \geq 0.$$

Rearranging yields

$$d - d\alpha^2 \geq n(1 - d\alpha^2).$$

If $1 - d\alpha^2 > 0$, we can divide the inequality by it to obtain the desired result. □

From the proof it follows that equality can only hold in this bound if and only if $\langle Y, Y\rangle = \mathrm{Tr}(Y^2) = 0$. By the non-degeneracy of the inner product, this only happens whenever $Y = 0$, and thus $I = \frac{d}{n}\sum_{i=1}^n X_i$. In the previous section we saw that if equality holds in the absolute bound, then we must also have $I = \frac{d}{n}\sum_{i=1}^n X_i$. From this we conclude that if the absolute bound holds with equality, the relative bound is also met. That equality holds in the relative bound if it holds in the absolute bound can also be calculated using the identity $\alpha^2 = 1/(d+2)$. Substituting this into the relative bound we see that $d\frac{1-\alpha^2}{1-d\alpha^2} = \frac{d(d+1)}{2}$.

The converse statement is not true, which the following example will show. For this we note that we can view the Seidel matrix as a nonstandard adjacency matrix of a graph, where two distinct vertices $i$ and $j$ are adjacent if $S_{ij} = 1$ and non-adjacent if $S_{ij} = -1$. If $G$ is a graph, $S_G$ will denote the Seidel matrix of this graph.

*Example* 3.8. Let $G$ be the Petersen graph. Then the Seidel matrix $S_G$ of $G$ has eigenvalues 3 and $-3$, both with multiplicity 5. So the matrix

$$M = I + \frac{1}{3}S_G$$

has eigenvalues 2 and 0, again both with multiplicity 5. Since the eigenvalues are all non-negative, $M$ is a positive semidefinite matrix. Furthermore, the nullity of $M$ is 5 and so its rank is $\mathrm{rk}\, M = 10 - 5 = 5$. So $M$ is the Gram matrix of a set of unit vectors representing 10 equiangular lines in $\mathbb{R}^5$ with common angle $\arccos(1/3)$. If we plug these values into the relative bound, we find

$$d\frac{1-\alpha^2}{1-d\alpha^2} = 5\frac{1-\frac{1}{9}}{1-\frac{5}{9}} = 5\cdot\frac{8}{4} = 10 = d.$$

This shows that this example meets the relative bound with equality. On the other hand $\frac{d(d+1)}{2} = \frac{5\cdot 6}{2} = 15$, and hence the maximum of the absolute bound is not attained.

This example shows that equality in the relative bound only tells us that this is the maximum number of equiangular lines with that specific common angle. There could thus be an angle allowing a larger number of equiangular lines in the same dimension.

# 4

# Linear upper bound for all dimensions

For many years, no significant progress has been made in finding bounds on the maximum number of equiangular lines in $\mathbb{R}^d$. This was until Bukh showed in [9] that a linear upper bound can be found on $N_\alpha(d)$. This bound has been further improved by Balla, Dräxler, Keevash and Sudakov in [10] and later by Jiang and Polyanskii in [11]. The approach used in proving these bounds relies on Ramsey's theorem which is why they only hold in large enough dimensions. Recently, Balla used a completely different approach which overcomes the use of Ramsey's theorem and shows a linear upper bound on $N_\alpha(d)$ for all dimensions $d$ in [13]. Next to overcoming the use of Ramsey's theorem, the tools established in this article are also interesting because they can be generalised to complex space. This way, Balla also finds bounds on the number of complex equiangular lines. Since our focus lies on real equiangular lines we will not go into the details of these results.

The main tool used by Balla is a geometric inequality that holds for the Gram matrix of a spherical code representing a set of equiangular lines. Before stating the theorem we define what it means to apply a function to a matrix. For any $n \times n$ matrix $A$ and function $f : \mathbb{R} \to \mathbb{R}$ we will define the matrix $f(A)$ as $f(A)_{ij} = f(A_{ij})$ for all $i, j \in [n]$. This means that the function is applied to the matrix entry-wise. Similarly, we can apply the function to any vector $\boldsymbol{v} \in \mathbb{R}^n$ by defining the vector $f(\boldsymbol{v})$ as $f(\boldsymbol{v})_i = f(\boldsymbol{v}_i)$ for all $i \in [n]$.

**Theorem 4.1** ([13]). *Let $M$ be the Gram matrix of a spherical $\{-\alpha, \alpha\}$-code $\mathcal{C}$ corresponding to a set of $n$ equiangular lines with common angle* $\arccos \alpha$. *Define $f : \mathbb{R} \to \mathbb{R}$ as the function $f(x) = x^2$. For all $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$, we have*

$$\frac{1-\alpha^2}{2}\left(\langle \boldsymbol{v}, M\boldsymbol{v}\rangle \langle \boldsymbol{u}, M\boldsymbol{u}\rangle + \langle \boldsymbol{v}, M\boldsymbol{u}\rangle^2\right) + \frac{\alpha^2}{\alpha^2 n + 1 - \alpha^2}\langle M\boldsymbol{v}, M\boldsymbol{u}\rangle^2 \geq \langle f(M\boldsymbol{v}), f(M\boldsymbol{u})\rangle,$$

*with equality whenever $n = \binom{d+1}{2}$.*

By taking $\boldsymbol{u} = \boldsymbol{e}_i$, where $\boldsymbol{e}_i$ is a standard basis vector, the inequality of the theorem can be rewritten in terms of a single vector $\boldsymbol{v}$.

**Corollary 4.2.** *Let $M$ be the Gram matrix of a spherical $\{-\alpha, \alpha\}$-code $\mathcal{C}$ corresponding to a set of $n$ equiangular lines with common angle* $\arccos \alpha$. *For all $i \in [n]$ and $\boldsymbol{v} \in \mathbb{R}^n$, we have*

$$\frac{1-\alpha^2}{2\alpha^2}\left(\langle \boldsymbol{v}, M\boldsymbol{v}\rangle - (M\boldsymbol{v})_i^2\right) + \frac{1}{\alpha^2 n + 1 - \alpha^2}\left(M^2\boldsymbol{v}\right)_i^2 \geq \left\langle \boldsymbol{v}, M^2\boldsymbol{v}\right\rangle,$$

*with equality whenever $n = \binom{d+1}{2}$.*

The inequalities from Theorem 4.1 and its corollary can be used to find an upper bound on the second eigenvalue of the Gram matrix. Together with a bound on its spectral radius we can deduce the following theorem giving an upper bound of linear order.

**Theorem 4.3** ([13]). *For all $d \in \mathbb{N}$ and $\alpha \in (0,1)$,*

$$N_\alpha(d) \leq \frac{\sqrt{d}}{2\alpha^3} + \frac{(1+\alpha)d}{2\alpha}.$$

In Section 3.2 we have seen that if the absolute bound is met, the cosine of the angle between the lines must satisfy $\alpha = 1/\sqrt{d+2}$. Note that if we plug this value of $\alpha$ into the bound of the theorem above we find $N_\alpha(d) \leq (1+o(1))\frac{r^2}{2}$, which asymptotically reaches the absolute bound. For the values $\alpha = \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \frac{1}{11}$, this bound has been improved by De Laat, De Muinck Keizer and Machado in [25] using a semidefinite programming approach. They show that for these values of $\alpha$, $N_\alpha(d) \leq c_\alpha + \frac{(1+\alpha)d}{2\alpha}$ for all $d \geq d_\alpha$, where $c_\alpha$ and $d_\alpha$ are constants that only depend on $\alpha$. They also deduce exact values for these two constants.

Introducing some more tools will make it possible to use Theorem 4.1 to prove the following linear bound on $N_\alpha(d)$.

**Theorem 4.4** ([13])**.** *For all $d \in \mathbb{N}$ and $\alpha \in (0,1)$ we have*

$$N_\alpha(d) \leq \max\left(\frac{2}{\alpha^5} + \frac{2}{\alpha^3(1-\alpha)}, \left(2 + \frac{8\alpha^2}{(1-\alpha)^2}\right)(d+1)\right).$$

In this chapter we will prove these theorems. First we will prove the inequality from Theorem 4.1 using orthogonal projections of symmetric matrices. In Section 4.2 we will show upper bounds on the first and second eigenvalue of the Gram matrix with which we will prove Theorem 4.3. In Section 4.3 the *associated graph* of a spherical $\{-\alpha, \alpha\}$-code is introduced along with the concept of *switching*. This tool is then used in the last section together with Theorem 4.1 to bound the maximum degree of the graph. This will give us all the necessary ingredients to prove Theorem 4.4.

## 4.1 Matrix projections

In Section 3.2 we defined the matrices $X_i = \boldsymbol{v}_i\boldsymbol{v}_i^\intercal$ for $i \in [n]$, where the vectors $\boldsymbol{v}_i$ are unit vectors corresponding to a set of $n$ equiangular lines. These matrices are all elements of $\mathcal{S}^d$, the set of symmetric $d \times d$ matrices. The proof of Theorem 4.1 will use orthogonal projections of symmetric matrices with respect to the Frobenius inner product onto the span of $X_1, \ldots, X_n$. The Frobenius norm of a matrix can only decrease when projecting it. This observation will lead to the inequality from the theorem.

Before we can give the proof of the theorem we need to generalise the concept of orthogonal projections to general inner product spaces. As seen in Section 2.1 the Frobenius inner product defines an inner product on the space of $m \times n$ matrices $\mathcal{M}_{m\times n}$, hence making $\mathcal{M}_{m\times n}$ an inner product space. Since the space of symmetric matrices $\mathcal{S}^n$ is a subspace of $\mathcal{M}_{n\times n}$ the Frobenius inner product also makes $\mathcal{S}^d$ an inner product space. Let $L : U \to V$ be a linear map between any two inner product spaces $U$ and $V$. We define the *adjoint* linear map of $L$ as the map $L^* : V \to U$ satisfying $\langle L^*v, u\rangle_U = \langle v, Lu\rangle_V$ for all $u \in U$ and $v \in V$. Here $\langle\cdot,\cdot\rangle_U$ and $\langle\cdot,\cdot\rangle_V$ denote the inner products corresponding to $U$ and $V$ respectively. Any matrix $A \in \mathcal{S}^n$ defines a linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$. The adjoint matrix of $A$ is the matrix satisfying $\langle A^*\boldsymbol{u}, \boldsymbol{v}\rangle = \langle \boldsymbol{u}, A\boldsymbol{v}\rangle$, for any $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$. The matrix that satisfies this is the transpose of $A$ and so we see that the adjoint map is a generalisation of the transpose to any linear map between inner product spaces.

A linear map $P$ is a *projection* if $P^2 = P = P^*$. The orthogonal projection onto the range of a linear map $L$, is given by the map $P = L(L^*L)^{-1}L^*$. We can easily verify that this map is indeed a projection by calculating $P^2$ and $P^*$:

$$\begin{aligned}
P^2 &= L(L^*L)^{-1}L^*L(L^*L)^{-1}L^* = L(L^*L)^{-1}L^* = P \\
P^* &= \left(L(L^*L)^{-1}L^*\right)^* = L\left(\left(L^*L\right)^{-1}\right)^* L^* = L(L^*L)^{-1}L^* = P.
\end{aligned}$$

We will now use these type of projections onto a span of matrices to prove Theorem 4.1.

*Proof of Theorem 4.1.* Let $\mathcal{C} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ be spherical $\{-\alpha, \alpha\}$-code in $\mathbb{R}^d$, corresponding to a set of $n$ equiangular lines, with Gram matrix $M = M_{\mathcal{C}}$. For each $i \in [n]$ define the matrix $X_i = \boldsymbol{v}_i\boldsymbol{v}_i^\intercal$ and let $\mathcal{X} : \mathbb{R}^n \to \mathcal{S}^d$ be the linear map defined by $\mathcal{X}\boldsymbol{e}_i = X_i$. Let $\mathcal{X}^* : \mathcal{S}^d \to \mathbb{R}^n$ be the adjoint map of $\mathcal{X}$. Then $\mathcal{X}^*\mathcal{X}$ is a linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$ and can thus be represented by an $n \times n$ matrix. We will show that this matrix is $f(M)$.

For any $i \in [n]$, $(\mathcal{X}^*\mathcal{X})\boldsymbol{e}_i$ gives the $i$-th row of $\mathcal{X}^*\mathcal{X}$. Taking the inner product with $\boldsymbol{e}_j$, $j \in [n]$ yields the $j$-th entry of this row. So, the $ij$-th entry of $\mathcal{X}^*\mathcal{X}$ is given by

$$\langle(\mathcal{X}^*\mathcal{X})\boldsymbol{e}_i, \boldsymbol{e}_j\rangle = \langle\mathcal{X}^*X_i, \boldsymbol{e}_j\rangle = \langle X_i, \mathcal{X}\boldsymbol{e}_j\rangle_F = \langle X_i, X_j\rangle_F.$$

Recall from Section 3.2 that for $i, j \in [n]$ distinct, $\langle X_i, X_j\rangle = \alpha^2$ and $\langle X_i, X_i\rangle = 1$. It thus follows that the matrix $\mathcal{X}^*\mathcal{X}$ has diagonal entries 1 and all other entries $\alpha^2$, which is equal to the matrix $f(M)$. So, we have

$$\mathcal{X}^*\mathcal{X} = f(M) = (1-\alpha^2)I + \alpha^2 J.$$

This matrix has eigenvalues $1+(n-1)\alpha^2$ and $1-\alpha^2$, and since these are not equal to zero it is invertible and has full rank. The inverse of $f(M)$ is

$$\frac{1}{1-\alpha^2}\left(I - \frac{\alpha^2}{\alpha^2 n + 1 - \alpha^2}J\right). \tag{4.1}$$

We now define the orthogonal projection onto the span of $X_1, \dots, X_n$, $\mathcal{P} = \mathcal{X}(\mathcal{X}^*\mathcal{X})^{-1}\mathcal{X}^* : \mathcal{S}^d \to \mathcal{S}^d$. This map is well-defined since we have just shown that the inverse $(\mathcal{X}^*\mathcal{X})^{-1}$ indeed exists. It can easily be verified that $\mathcal{P}^2 = \mathcal{P} = \mathcal{P}^*$ and $\mathcal{P}\mathcal{X} = \mathcal{X}$, so $\mathcal{P}$ is indeed a projection onto $\mathcal{X}$.

Let $V$ be the matrix with the vectors $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ as columns, such that $V^\top V = M$. For any $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$ define the matrix

$$Y = \frac{1}{2}\left(V\boldsymbol{v}(V\boldsymbol{u})^\top + V\boldsymbol{u}(V\boldsymbol{v})^\top\right),$$

which is symmetric. Projecting $Y$ can only decrease its norm, so $\|Y\|_F^2 \geq \|\mathcal{P}Y\|_F^2$. The required inequality follows immediately from this one by computing $\|Y\|_F^2$ and $\|\mathcal{P}Y\|_F^2$.

We first compute $\|Y\|_F^2$, for which we will need to know the inner products $\left\langle V\boldsymbol{v}(V\boldsymbol{u})^\top, V\boldsymbol{v}(V\boldsymbol{u})^\top\right\rangle_F$ and $\left\langle V\boldsymbol{v}(V\boldsymbol{u})^\top, V\boldsymbol{u}(V\boldsymbol{v})^\top\right\rangle_F$. For the former we have

$$\begin{aligned}
\left\langle V\boldsymbol{v}(V\boldsymbol{u})^\top, V\boldsymbol{v}(V\boldsymbol{u})^\top\right\rangle_F &= \operatorname{Tr}\left(\left(V\boldsymbol{v}(V\boldsymbol{u})^\top\right)^\top V\boldsymbol{v}(V\boldsymbol{u})^\top\right)\\
&= \operatorname{Tr}\left(V\boldsymbol{u}(V\boldsymbol{v})^\top V\boldsymbol{v}(V\boldsymbol{u})^\top\right)\\
&= \operatorname{Tr}\left((V\boldsymbol{v})^\top V\boldsymbol{v}(V\boldsymbol{u})^\top V\boldsymbol{u}\right)\\
&= \langle V\boldsymbol{v}, V\boldsymbol{v}\rangle\langle V\boldsymbol{u}, V\boldsymbol{u}\rangle\\
&= \langle \boldsymbol{v}, M\boldsymbol{v}\rangle\langle \boldsymbol{u}, M\boldsymbol{u}\rangle.
\end{aligned}$$

Similarly, we find $\left\langle V\boldsymbol{v}(V\boldsymbol{u})^\top, V\boldsymbol{u}(V\boldsymbol{v})^\top\right\rangle_F = \operatorname{Tr}\left((V\boldsymbol{u})^\top V\boldsymbol{v}(V\boldsymbol{u})^\top V\boldsymbol{v}\right) = \langle \boldsymbol{v}, M\boldsymbol{u}\rangle^2$. Now the squared norm of $Y$ equals

$$\begin{aligned}
4\|Y\|_F^2 &= \left\langle V\boldsymbol{v}(V\boldsymbol{u})^\top + V\boldsymbol{u}(V\boldsymbol{v})^\top, V\boldsymbol{v}(V\boldsymbol{u})^\top + V\boldsymbol{u}(V\boldsymbol{v})^\top\right\rangle_F\\
&= \left\langle V\boldsymbol{v}(V\boldsymbol{u})^\top, V\boldsymbol{v}(V\boldsymbol{u})^\top\right\rangle_F + 2\left\langle V\boldsymbol{v}(V\boldsymbol{u})^\top, V\boldsymbol{u}(V\boldsymbol{v})^\top\right\rangle_F + \left\langle V\boldsymbol{u}(V\boldsymbol{v})^\top, V\boldsymbol{u}(V\boldsymbol{v})^\top\right\rangle_F\\
&= \langle \boldsymbol{v}, M\boldsymbol{v}\rangle\langle \boldsymbol{u}, M\boldsymbol{u}\rangle + 2\langle \boldsymbol{v}, M\boldsymbol{u}\rangle^2 + \langle \boldsymbol{u}, M\boldsymbol{u}\rangle\langle \boldsymbol{v}, M\boldsymbol{v}\rangle\\
&= 2\left(\langle \boldsymbol{v}, M\boldsymbol{v}\rangle\langle \boldsymbol{u}, M\boldsymbol{u}\rangle + \langle \boldsymbol{v}, M\boldsymbol{u}\rangle^2\right).
\end{aligned}$$

Next we compute $\|\mathcal{P}Y\|_F^2$, which is equal to

$$\begin{aligned}
\|\mathcal{P}Y\|_F^2 = \langle \mathcal{P}Y, \mathcal{P}Y\rangle_F &= (\mathcal{P}Y)^*\,\mathcal{P}Y = Y^*\mathcal{P}^*\mathcal{P}Y\\
&= Y^*\mathcal{P}Y = Y^*\mathcal{X}(\mathcal{X}^*\mathcal{X})^{-1}\mathcal{X}^*Y.
\end{aligned}$$

We thus need to calculate $\mathcal{X}^*Y$. Notice that $\mathcal{X}^*Y$ is a vector in $\mathbb{R}^n$. By taking the inner product with $\boldsymbol{e}_i$ for some $i \in [n]$ we find the $i$-th entry of this vector. To be able to compute this inner product we first need to calculate $\left\langle V\boldsymbol{v}(V\boldsymbol{u})^\top, \boldsymbol{v}_i\boldsymbol{v}_i^\top\right\rangle_F$. This inner product equals

$$\begin{aligned}
\left\langle V\boldsymbol{v}(V\boldsymbol{u})^\top, \boldsymbol{v}_i\boldsymbol{v}_i^\top\right\rangle_F &= \operatorname{Tr}\left(\boldsymbol{v}_i\boldsymbol{v}_i^\top V\boldsymbol{v}(V\boldsymbol{u})^\top\right)\\
&= \operatorname{Tr}\left((\boldsymbol{v}_i^\top V\boldsymbol{v}V\boldsymbol{u})^\top\boldsymbol{v}_i\right)\\
&= \langle V\boldsymbol{v}, \boldsymbol{v}_i\rangle\langle V\boldsymbol{u}, \boldsymbol{v}_i\rangle\\
&= (M\boldsymbol{v})_i(M\boldsymbol{u})_i,
\end{aligned}$$

where for the last step we notice that $\boldsymbol{v}_i^\top V$ is a vector with entries $\langle \boldsymbol{v}_i, \boldsymbol{v}_j\rangle$ for $j \in [n]$ which is in fact the $i$-th row of the Gram matrix $M$. Now let $\boldsymbol{w} \in \mathbb{R}^n$ be the vector with entries $\boldsymbol{w}_i = (M\boldsymbol{v})_i(M\boldsymbol{u})_i$ for any $i \in [n]$. Then, for the $i$-the entry of $\mathcal{X}^*Y$ we, find

$$\begin{aligned}
\langle \mathcal{X}^*Y, \boldsymbol{e}_i\rangle &= \langle Y, \mathcal{X}\boldsymbol{e}_i\rangle_F \\
&= \langle Y, X_i\rangle_F \\
&= \frac{1}{2}\left(\left\langle V\boldsymbol{v}(V\boldsymbol{u})^\mathsf{T}, \boldsymbol{v}_i\boldsymbol{v}_i^\mathsf{T}\right\rangle_F + \left\langle V\boldsymbol{u}(V\boldsymbol{v})^\mathsf{T}, \boldsymbol{v}_i\boldsymbol{v}_i^\mathsf{T}\right\rangle_F\right) \\
&= \frac{1}{2}\left((M\boldsymbol{v})_i(M\boldsymbol{u})_i + (M\boldsymbol{u})_i(M\boldsymbol{v})_i\right) \\
&= \boldsymbol{w}_i.
\end{aligned}$$

So, it now follows that $\mathcal{X}^*Y = \boldsymbol{w}$ and $Y^*\mathcal{X} = (\mathcal{X}^*Y)^* = \boldsymbol{w}^* = \boldsymbol{w}^\mathsf{T}$. Hence, we find

$$\begin{aligned}
\|\mathcal{P}Y\|_F^2 &= Y^*\mathcal{X}(\mathcal{X}^*\mathcal{X})^{-1}\mathcal{X}^*Y \\
&= \boldsymbol{w}^\mathsf{T}(\mathcal{X}^*\mathcal{X})^{-1}\boldsymbol{w} \\
&\overset{(4.1)}{=} \boldsymbol{w}^\mathsf{T}\frac{1}{1-\alpha^2}\left(I - \frac{\alpha^2}{\alpha^2 n + 1 - \alpha^2}J\right)\boldsymbol{w} \\
&= \frac{1}{1-\alpha^2}\left(\boldsymbol{w}^\mathsf{T}\boldsymbol{w} - \frac{\alpha^2}{\alpha^2 n + 1 - \alpha^2}\boldsymbol{w}^\mathsf{T}J\boldsymbol{w}\right) \\
&= \frac{1}{1-\alpha^2}\left(\sum_{i=1}^n ((M\boldsymbol{v})_i(M\boldsymbol{u})_i)^2 - \frac{\alpha^2}{\alpha^2 n + 1 - \alpha^2}\left(\sum_{i=1}^n (M\boldsymbol{v})_i(M\boldsymbol{u})_i\right)^2\right) \\
&= \frac{1}{1-\alpha^2}\left(\langle f(M\boldsymbol{v}), f(M\boldsymbol{u})\rangle - \frac{\alpha^2}{\alpha^2 n + 1 - \alpha^2}\langle M\boldsymbol{v}, M\boldsymbol{u}\rangle^2\right).
\end{aligned}$$

Now that we have computed $\|Y\|_F^2$ and $\|\mathcal{P}Y\|_F^2$, the inequality $\|Y\|_F^2 \geq \|\mathcal{P}Y\|_F^2$ becomes

$$\frac{1}{2}\left(\langle \boldsymbol{v}, M\boldsymbol{v}\rangle\langle \boldsymbol{u}, M\boldsymbol{u}\rangle + \langle \boldsymbol{v}, M\boldsymbol{u}\rangle^2\right) \geq \frac{1}{1-\alpha^2}\left(\langle f(M\boldsymbol{v}), f(M\boldsymbol{u})\rangle - \frac{\alpha^2}{\alpha^2 n + 1 - \alpha^2}\langle M\boldsymbol{v}, M\boldsymbol{u}\rangle^2\right),$$

which can be rearranged to the required inequality

$$\frac{1-\alpha^2}{2}\left(\langle \boldsymbol{v}, M\boldsymbol{v}\rangle\langle \boldsymbol{u}, M\boldsymbol{u}\rangle + \langle \boldsymbol{v}, M\boldsymbol{u}\rangle^2\right) + \frac{\alpha^2}{\alpha^2 n + 1 - \alpha^2}\langle M\boldsymbol{v}, M\boldsymbol{u}\rangle^2 \geq \langle f(M\boldsymbol{v}), f(M\boldsymbol{u})\rangle.$$

Finally, it is left to show that equality holds whenever $n = \binom{d+1}{2}$. Equality holds in $\|Y\|_F^2 \geq \|\mathcal{P}Y\|_F^2$ if and only if $\mathcal{P}$ is the identity map. Note that the matrix $\mathcal{X}^*\mathcal{X} = f(M)$ is invertible, which means that it has full rank. The rank of $\mathcal{X}^*\mathcal{X}$ is equal to the rank of $\mathcal{X}$, so $\operatorname{rk}\mathcal{X} = n$. The dimension of $\mathcal{S}^d$ is $\binom{d+1}{2}$. Hence, if $n = \binom{d+1}{2}$, the range of $\mathcal{X}$ is $\mathcal{S}^d$. This makes $\mathcal{P}$ a projection from $\mathcal{S}^d$ onto itself, which is the identity map. This shows that if $n = \binom{d+1}{2}$, then equality holds. □

Although the matrices $X_1, \ldots, X_n$ have already been used early on in the study of equiangular lines to prove results as the absolute and relative bound, this is the first result using projections onto the span of these matrices. In previous works, orthogonal projections have been used onto large subsets $S \subseteq \mathcal{C}$ such that all inner products between elements of $S$ are $\alpha$ [9], [10]. These large subsets are found using Ramsey theory, which limits the results to large values of $d$ relative to $\alpha$. The inequality from the theorem above makes it possible to find bounds on the number of equiangular lines without having to rely on Ramsey theory.

We conclude this section with the proof of Corollary 4.2 which is an immediate consequence of Theorem 4.1 and gives the equivalent of the inequality for only one vector $\boldsymbol{v} \in \mathbb{R}^n$.

*Proof of Corollary 4.2.* The inequality follows from Theorem 4.1 by taking $\boldsymbol{u} = \boldsymbol{e}_i$ and then simplifying. So let $f$ be defined as in the theorem. Recall that $M$ is symmetric, so $M^\mathsf{T} = M$. We first compute the necessary inner products, which are

- $\langle \boldsymbol{e}_i, M\boldsymbol{e}_i\rangle = M_{ii} = 1$,

- $\langle \boldsymbol{v}, M\boldsymbol{e}_i \rangle = (M\boldsymbol{e}_i)^\top \boldsymbol{v} = \boldsymbol{e}_i^\top M\boldsymbol{v} = (M\boldsymbol{v})_i$,
- $\langle M\boldsymbol{v}, M\boldsymbol{e}_i \rangle = (M\boldsymbol{e}_i)^\top M\boldsymbol{v} = \boldsymbol{e}_i^\top M^\top M\boldsymbol{v} = \left(M^2\boldsymbol{v}\right)_i$.

For the last inner product $\langle f(M\boldsymbol{v}), f(M\boldsymbol{e}_i)\rangle$, first note that $f(M\boldsymbol{e}_i) = \alpha^2\mathbf{1} + (1-\alpha^2)\boldsymbol{e}_i$, since all entries of $M\boldsymbol{e}_i$ are $\pm\alpha$ except the $i$-th entry which is 1. So, the inner product equals

$$\langle f(M\boldsymbol{v}), f(M\boldsymbol{e}_i)\rangle = \alpha^2\langle f(M\boldsymbol{v}), \mathbf{1}\rangle + (1-\alpha^2)\langle f(M\boldsymbol{v}), \boldsymbol{e}_i\rangle$$

Now observe that, for any vector $\boldsymbol{w} \in \mathbb{R}^n$, $\langle f(\boldsymbol{w}), \mathbf{1}\rangle = \sum_{j=1}^n \boldsymbol{w}_j^2 = \|\boldsymbol{w}\|^2$. Hence, $\langle f(M\boldsymbol{v})\rangle = \|M\boldsymbol{v}\|^2$ and by Equation 2.3 this is equal to $\langle \boldsymbol{v}, M^2\boldsymbol{v}\rangle$. Substituting this in the equation above yields

$$\begin{aligned}\langle f(M\boldsymbol{v}), f(M\boldsymbol{e}_i)\rangle &= \alpha^2\|M\boldsymbol{v}\|^2 + (1-\alpha^2)(M\boldsymbol{v})_i^2 \\ &= \alpha^2\langle \boldsymbol{v}, M^2\boldsymbol{v}\rangle + (1-\alpha^2)(M\boldsymbol{v})_i^2.\end{aligned}$$

Plugging this all into the inequality of Theorem 4.1 we conclude

$$\frac{1-\alpha^2}{2}\left(\langle \boldsymbol{v}, M\boldsymbol{v}\rangle + (M\boldsymbol{v})_i^2\right) + \frac{\alpha^2}{\alpha^2 n + 1 - \alpha^2}(M^2\boldsymbol{v})_i^2 \geq \alpha^2\langle \boldsymbol{v}, M^2\boldsymbol{v}\rangle + (1-\alpha^2)(M\boldsymbol{v})_i^2.$$

Subtracting $(1-\alpha^2)(M\boldsymbol{v})_i^2$ from both sides and dividing by $\alpha^2$ gives the desired result. Equality for $n = \binom{d+1}{2}$ also follows immediately from Theorem 4.1. $\square$

## 4.2 Bounds on the first and second eigenvalue

To prove Theorem 4.3 we will start by bounding the first and second eigenvalue of the Gram matrix $M$. Recall from Chapter 3 that a set of equiangular lines in $\mathbb{R}^d$ can be represented by a spherical code $\mathcal{C}$. If $\arccos\alpha$ is the corresponding angle between the lines, the Gram matrix $M = M_{\mathcal{C}}$ of $\mathcal{C}$ has diagonal entries 1 and off-diagonal entries $\pm\alpha$. The matrix $M$ is a symmetric $n \times n$ matrix of rank at most $d$. The first lemma of this section gives a bound on the spectral radius of $M$ which follows from the Cauchy-Schwarz inequality.

**Lemma 4.5.** *Let $M$ be the Gram matrix of a spherical $\{-\alpha,\alpha\}$-code representing a set of $n$ equiangular lines. The spectral radius $\lambda_1$ of $M$ satisfies*

$$\lambda_1 \leq 1 - \alpha + \alpha n.$$

*Proof.* Let $\boldsymbol{v}$ be a unit eigenvector corresponding to $\lambda_1$. Note that for a unit eigenvector $\boldsymbol{v}^\top M\boldsymbol{v} = \boldsymbol{v}^\top\lambda_1\boldsymbol{v} = \lambda_1\boldsymbol{v}^\top\boldsymbol{v} = \lambda_1$. Furthermore, the sum $\sum_{i=1}^n \boldsymbol{v}_i$ can be written as $\langle \boldsymbol{v}, \mathbf{1}\rangle$. Using the Cauchy-Schwarz inequality, we find

$$\begin{aligned}\lambda_1 = \boldsymbol{v}^\top M\boldsymbol{v} = \sum_{i,j\in[n]} M_{ij}\boldsymbol{v}_i\boldsymbol{v}_j \leq \sum_{i,j\in[n]} |M_{ij}||\boldsymbol{v}_i||\boldsymbol{v}_j| &= \sum_{i=1}^n |\boldsymbol{v}_i|^2 + \alpha\sum_{\substack{i\neq j\\ i,j\in[n]}} |\boldsymbol{v}_i||\boldsymbol{v}_j| \\ &= (1-\alpha)\sum_{i=1}^n |\boldsymbol{v}_i|^2 + \alpha\left(\sum_{i=1}^n |\boldsymbol{v}_i|\right)^2 \\ &\leq (1-\alpha)\sum_{i=1}^n |\boldsymbol{v}_i|^2 + \alpha\|\boldsymbol{v}\|^2\|\mathbf{1}\|^2 \\ &= 1 - \alpha + \alpha n.\end{aligned}$$

$\square$

Next, an upper bound on the second eigenvalue of the Gram matrix $M$ in terms of the first eigenvalue is shown. It will follow from Theorem 4.1 by letting $\boldsymbol{u}$ and $\boldsymbol{v}$ be unit eigenvectors corresponding to the first and second eigenvalue of the Gram matrix respectively.

**Lemma 4.6.** *Let $M$ be the Gram matrix of a spherical $\{-\alpha, \alpha\}$-code representing a set of $n$ equiangular lines, with first and second eigenvalues $\lambda_1$ and $\lambda_2$ respectively. If $\lambda_1 > \frac{1-\alpha^2}{2\alpha^2}$, then*

$$\lambda_2 \leq \frac{1-\alpha^2}{2} \frac{\frac{\lambda_1}{\alpha^2 n+1-\alpha^2} - \frac{1-\alpha^2}{2\alpha^2\lambda_1}}{1-\frac{1-\alpha^2}{2\alpha^2\lambda_1}},$$

*with equality whenever $n = \binom{d+1}{2}$.*

*Proof.* Let $\boldsymbol{v}$, $\boldsymbol{u}$ be unit eigenvectors corresponding to $\lambda_1$ and $\lambda_2$ respectively. Define $f$ as in Theorem 4.1 and apply the theorem to the unit eigenvectors. Since $\boldsymbol{v}$ and $\boldsymbol{u}$ are eigenvectors corresponding to two different eigenvalues of a symmetric matrix they are orthogonal. This means that both the inner products $\langle \boldsymbol{v}, M\boldsymbol{u}\rangle$ and $\langle M\boldsymbol{v}, M\boldsymbol{u}\rangle$ are equal to zero. Furthermore, note that $\langle \boldsymbol{v}, M\boldsymbol{v}\rangle = \boldsymbol{v}^\mathsf{T}\lambda_1\boldsymbol{v} = \lambda_1$ and similarly $\langle \boldsymbol{u}, M\boldsymbol{u}\rangle = \lambda_2$. Lastly, we have $\langle f(M\boldsymbol{v}), f(M\boldsymbol{u})\rangle = \langle f(\lambda_1\boldsymbol{v}), f(\lambda_2\boldsymbol{u})\rangle = \lambda_1^2\lambda_2^2\sum_{i=1}^n \boldsymbol{v}_i^2\boldsymbol{u}_i^2$. So by Theorem 4.1 we have

$$\frac{1-\alpha^2}{2}\lambda_1\lambda_2 \geq \lambda_1^2\lambda_2^2\sum_{i=1}^n \boldsymbol{v}_i^2\boldsymbol{u}_i^2.$$

Dividing by $\lambda_1\lambda_2$ and rearranging we find $\lambda_2 \leq \frac{1-\alpha^2}{2\lambda_1}\left(\sum_{i=1}^n \boldsymbol{v}_i^2\boldsymbol{u}_i^2\right)^{-1}$.

To find the desired inequality we thus have to lower bound the sum $\sum_{i=1}^n \boldsymbol{v}_i^2\boldsymbol{u}_i^2$ in terms of $\alpha$ and $\lambda_1$. We will do this by lower bounding $\boldsymbol{v}_i$ for all $i \in [n]$. Let $x$ denote this lower bound, then we have $\sum_{i=1}^n \boldsymbol{v}_i^2\boldsymbol{u}_i^2 \geq \sum_{i=1}^n x\boldsymbol{u}_i^2 = x\sum_{i=1}^n \boldsymbol{u}_i^2 = x$, so a lower bound on $\boldsymbol{v}_i$ implies a lower bound on the sum. Applying Corollary 4.2 to $\boldsymbol{v}$ we find for any $i \in [n]$

$$\frac{1-\alpha^2}{2\alpha^2}(\lambda_1 - \lambda_1^2\boldsymbol{v}_i^2) + \frac{1}{\alpha^2 n + 1 - \alpha^2}\lambda_1^4\boldsymbol{v}_i^2 \geq \lambda_1^2.$$

Dividing by $\lambda_1$ and rearranging the terms yields

$$\boldsymbol{v}_i^2\left(\frac{\lambda_1^2}{\alpha^2 n+1-\alpha^2} - \frac{1-\alpha^2}{2\alpha^2}\right) \geq 1 - \frac{1-\alpha^2}{2\alpha^2\lambda_1}.$$

Since we assumed $\lambda_1$ to be strictly larger than $\frac{1-\alpha^2}{2\alpha^2}$, the right hand side of this inequality is positive. The left hand side must thus also be positive. In particular, since $\boldsymbol{v}_i^2$ is positive, we must have that $\frac{\lambda_1^2}{\alpha^2 n+1-\alpha^2} - \frac{1-\alpha^2}{2\alpha^2}$ is positive. We can thus divide by this term, giving us the required lower bound on $\boldsymbol{v}_i^2$.

With this lower bound on $\boldsymbol{v}_i^2$ we conclude

$$\begin{aligned}\lambda_2 \leq \frac{1-\alpha^2}{2\lambda_1}\left(\sum_{i=1}^n \boldsymbol{v}_i^2\boldsymbol{u}_i^2\right)^{-1} &\leq \frac{1-\alpha^2}{2\lambda_1}\left(\frac{1-\frac{1-\alpha^2}{2\alpha^2\lambda_1}}{\frac{\lambda_1^2}{\alpha^2 n+1-\alpha^2} - \frac{1-\alpha^2}{2\alpha^2}}\right)^{-1} \\ &= \frac{1-\alpha^2}{2}\frac{\frac{\lambda_1}{\alpha^2 n+1-\alpha^2} - \frac{1-\alpha^2}{2\alpha^2\lambda_1}}{1-\frac{1-\alpha^2}{2\alpha^2\lambda_1}}\end{aligned}$$

□

*Remark* 4.7. Notice that we can use Lemma 4.5 to simplify the bound on the second eigenvalue of the Gram matrix. Combining this lemma with the fact that $\alpha^2 n+1-\alpha^2 = \alpha(\alpha n+1-\alpha)+1-\alpha > \alpha(\alpha n+1-\alpha)$, we see that

$$\frac{\lambda_1}{\alpha^2 n+1-\alpha^2} \leq \frac{\alpha n+1-\alpha}{\alpha(\alpha n+1-\alpha)} = \frac{1}{\alpha}.$$

With this, the upper bound of $\lambda_2$ can be simplified to

$$\begin{aligned}\lambda_2 &< \frac{1-\alpha^2}{2\left(1-\frac{1-\alpha^2}{2\alpha^2\lambda_1}\right)}\left(\frac{\lambda_1}{\alpha^2 n+1-\alpha^2} - \frac{1-\alpha^2}{2\alpha^2\lambda_1}\right) < \frac{1-\alpha^2}{2\left(1-\frac{1-\alpha^2}{2\alpha^2\lambda_1}\right)}\frac{\lambda_1}{\alpha^2 n+1-\alpha^2} \\ &< \frac{1-\alpha^2}{2\left(1-\frac{1-\alpha^2}{2\alpha^2\lambda_1}\right)}\frac{1}{\alpha} = \frac{1-\alpha^2}{2\alpha - \frac{1-\alpha^2}{\alpha\lambda_1}}.\end{aligned}$$

With these two lemmas we will be able to prove Theorem 4.3. Recall that we want to show that $n \leq \frac{\sqrt{d}}{2\alpha^3} + \frac{(1+\alpha)d}{2\alpha}$ for any set of $n$ equiangular lines in $\mathbb{R}^d$ with common angle $\arccos \alpha$. If the spectral radius $\lambda_1$ of the Gram matrix $M$ is large enough we will apply the above lemmas to the trace of $M$, which satisfies $n = \operatorname{Tr} M = \sum_{i=1}^{n} \lambda_i$. For small $\lambda_1$, we find the required bound by bounding the trace of the square of $M$ using the spectral radius.

*Proof of Theorem 4.3.* Let $\mathcal{L}$ be a set of $n$ equiangular lines in $\mathbb{R}^d$ with common angle $\arccos \alpha$, with $\alpha \in (0,1)$. Let $\mathcal{C}$ be a spherical $\{-\alpha, \alpha\}$-code representing $\mathcal{L}$ with corresponding Gram matrix $M = M_{\mathcal{C}}$. Denote the nonzero eigenvalues of $M$ as $\lambda_1, \lambda_2, \ldots, \lambda_d$ in non-increasing order. Since $M$ has rank at most $d$, its nullity is at least $n-d$ by the rank-nullity theorem. This implies that the multiplicity of the eigenvalue zero is at least $n-d$ and that $M$ has no more than $d$ nonzero eigenvalues.

Define $t = \alpha^2(1+\alpha)\sqrt{d}$ and note that

$$\frac{(1+t)\sqrt{d}}{2\alpha^3} = \frac{\sqrt{d} + \alpha^2(1+\alpha)d}{2\alpha^3} = \frac{\sqrt{d}}{2\alpha^3} + \frac{(1+\alpha)d}{2\alpha}.$$

Hence, in order to prove the theorem it suffices to show that $n \leq \frac{(1+t)\sqrt{d}}{2\alpha^3}$.

We will distinguish between two cases based on the value of the spectral radius of $M$, namely $\lambda_1 \leq \frac{1+t}{2\alpha^2}$ and $\lambda_1 > \frac{1+t}{2\alpha^2}$. So first assume that $\lambda_1 \leq \frac{1+t}{2\alpha^2}$. We will show that $n < \frac{\lambda_1}{\alpha}\sqrt{d}$, since then

$$n < \frac{\lambda_1}{\alpha}\sqrt{d} \leq \frac{\frac{1+t}{2\alpha^2}}{\alpha}\sqrt{d} = \frac{(1+t)\sqrt{d}}{2\alpha^3},$$

as required. The upper bound on $n$ follows by bounding the trace of the squared Gram matrix. This trace equals

$$\operatorname{Tr}\left(M^2\right) = \sum_{i,j=1}^{n} M_{ij}M_{ij} = \sum_{i=1}^{n} M_{ii} + \sum_{i \neq j} M_{ij}^2 = nn + n(n-1)\alpha^2 = n(\alpha^2 n + 1 - \alpha^2).$$

So, we find

$$\alpha^2 n^2 < n\left(\alpha^2 n + 1 - \alpha^2\right) = \operatorname{Tr}\left(M^2\right) = \sum_{i=1}^{d} \lambda_i^2 \leq \lambda_1^2 d.$$

Dividing by $\alpha^2$ and taking the square root yields the necessary inequality .

Now suppose that $\lambda_1 > \frac{1+t}{2\alpha^2}$. We will prove the theorem by showing that the following inequalities are valid:

$$\begin{aligned} n &< \frac{1+\alpha}{2\alpha - \frac{1-\alpha^2}{\alpha\lambda_1}}(d-1) + 1 && (4.2) \\ &< \frac{(1+\alpha)(1+t)}{2\alpha t}(d-1) + 1 && (4.3) \\ &< \frac{(1+\alpha)(1+t)}{2\alpha t} d && (4.4) \\ &= \frac{(1+\alpha)(1+t)}{2\alpha \cdot \alpha(1+\alpha)\sqrt{d}} d = \frac{1+t}{2\alpha^3}\sqrt{d}. \end{aligned}$$

First, let us prove inequality (4.2). For this we will use Lemmas 4.5 and 4.6. Note that this second lemma may be applied since we have $\lambda_1 > \frac{1+t}{2\alpha^2} > \frac{1-\alpha^2}{2\alpha^2}$. By Remark 4.7 the bound in Lemma 4.6 can be simplified to

$$\lambda_2 < \frac{1-\alpha^2}{2\left(1 - \frac{1-\alpha^2}{2\alpha^2\lambda_1}\right)}\frac{1}{\alpha} = \frac{1-\alpha^2}{2\alpha - \frac{1-\alpha^2}{\alpha\lambda_1}}$$

Combining this bound on $\lambda_2$ with Lemma 4.5, we find

$$\begin{aligned}(1-\alpha)(n-1) &= n - (\alpha n + 1 - \alpha) \leq n - \lambda_1 \\ &= \operatorname{Tr} M - \lambda_1 = \sum_{i=1}^{d} \lambda_i - \lambda_1 \\ &= \sum_{i=2}^{d} \lambda_i \leq (d-1)\lambda_2 \\ &\leq (d-1)\frac{1-\alpha^2}{2\alpha - \frac{1-\alpha^2}{\alpha\lambda_1}}.\end{aligned}$$

Dividing by $(1-\alpha)$ and adding 1 shows us that inequality (4.2) holds.

To prove inequality (4.3) we need to show that $2\alpha - \frac{1-\alpha^2}{\alpha\lambda_1} > 2\alpha\frac{t}{t+1}$. Using the assumption that $\lambda_1 > \frac{1+t}{2\alpha^2}$, we find

$$2\alpha - \frac{1-\alpha^2}{\alpha\lambda_1} > 2\alpha - \frac{(1-\alpha^2)2\alpha^2}{\alpha(1+t)} = 2\alpha\left(1 - \frac{1-\alpha^2}{2\alpha^2}\right) > 2\alpha\left(1 - \frac{1}{2\alpha^2}\right) = 2\alpha\frac{t}{t+1}.$$

Hence, inequality (4.3) also holds.

It is now only left to show that inequality (4.4) is also true. We need to show that $-\frac{(1+\alpha)(1+t)}{2\alpha t} + 1 < 0$ which after rearranging is equivalent to showing that $(1+\alpha)(1+t) > 2\alpha t = 2\alpha^3(1+\alpha)\sqrt{d}$. Notice that, $\alpha^k > \alpha^{k+1}$ for any $k \in N$, since $\alpha < 1$. Using this, we see that

$$\begin{aligned}(1+\alpha)(1+t) &= (1+\alpha)\left(1 + \alpha^2(1+\alpha)\sqrt{d}\right) \\ &= 1 + \alpha + \alpha^2(1+\alpha)\sqrt{d} + \alpha^3(1+\alpha)\sqrt{d} \\ &> \alpha^2(1+\alpha)\sqrt{d} + \alpha^3(1+\alpha)\sqrt{d} \\ &> 2\alpha^3(1+\alpha)\sqrt{d}.\end{aligned}$$

This proves the last inequality and with this we have proven the theorem. □

## 4.3 The associated graph

To be able to prove Theorem 4.4 we need some more tools. At the end of Section 3.3 we constructed a graph from the Seidel matrix of set of equiangular lines. In this section we formerly introduce this graph and redefine it in a slightly different way. We also introduce the concept of switching. A simple lemma is proven which gives a foundation for the results in the following section in which switching will be used to prove Theorem 4.4.

At the end of Section 3.3 we saw that we can view the Seidel matrix as a nonstandard adjacency matrix of a graph. If we let $\mathcal{C}$ be the spherical $\{-\alpha, \alpha\}$-code corresponding to a set of equiangular lines, the Seidel matrix is defined in terms of the Gram matrix $M$ of $\mathcal{C}$ as $S = \frac{1}{\alpha}(I - M)$. So we can also define this graph based on the Gram matrix, or equivalently, based on the spherical code $\mathcal{C}$.

**Definition 4.8.** Let $\mathcal{C} = \mathcal{C}_\alpha$ be a spherical $\{-\alpha, \alpha\}$-code. The *associated graph* $G_\mathcal{C}$ of $\mathcal{C}$ is the graph with vertex set $\mathcal{C}$ and an edge between any two vertices $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{C}$ if $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = -\alpha$.

Figure 4.1 shows the graph associated of a set of three equiangular lines in $\mathbb{R}^2$. The inner products between the vectors in the figure are $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \frac{1}{2}$, $\langle \boldsymbol{v}, \boldsymbol{w} \rangle = \frac{1}{2}$ and $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = -\frac{1}{2}$. This leads to a graph on the three vertices $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w}$ with one edge connecting $\boldsymbol{u}$ and $\boldsymbol{w}$ since only the inner product between those two vectors is negative.

The Gram matrix $M$ of the spherical code can be written in terms of the adjacency matrix $A = A_G$ of the associated graph $G = G_\mathcal{C}$ as follows:

$$M = (1-\alpha)I + \alpha J - 2\alpha A. \tag{4.5}$$

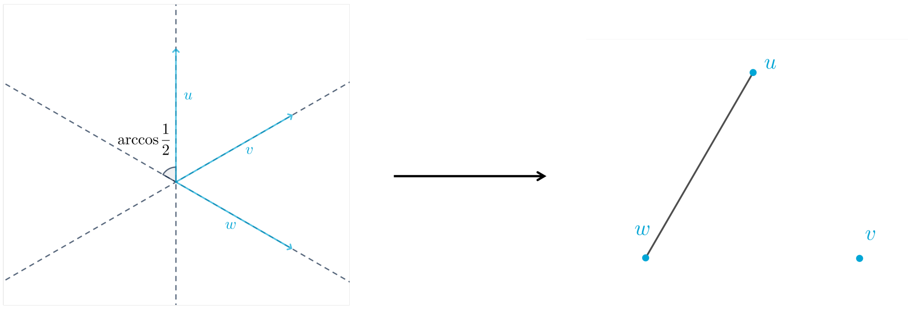


**Figure 4.1:** Associated graph

Indeed on the diagonal we find $M_{ii} = 1 - \alpha + \alpha = 1$, for any two adjacent vertices $i, j$ we have $M_{ij} = \alpha - 2\alpha \cdot 1 = -\alpha$ and for any two non-adjacent vertices $i, j$ we have $M_{ij} = \alpha - 2\alpha \cdot 0 = \alpha$. So the problem of finding a large set of equiangular lines in $\mathbb{R}^d$ becomes equivalent to finding a graph on as many vertices as possible such that the matrix $M$ is positive semidefinite and has rank $d$. Notice that in this context the construction of Proposition 3.3 can be recreated by taking the empty graph on $d$ vertices. In the following example we reconstruct the 28 lines in $\mathbb{R}^7$ from Example 3.6 with a graph.

*Example* 4.9. In Section 3.2 we gave a construction of 28 equiangular lines in $\mathbb{R}^7$ with common angle $\arccos\left(\frac{1}{3}\right)$. A construction of these lines can also be given in terms of graphs. Let $G = K(8, 2)$ be the Kneser graph on 28 vertices with adjacency matrix $A = A_G$. This is the graph on 28 vertices with as vertex set all two-element subsets of $\{1, 2, \ldots, 8\}$ and an edge between any two vertices if the sets are disjoint. The spectrum of $G$ is $\sigma(G) = \{15^1, 1^{20}, (-5)^7\}$. Now we show that the matrix

$$M = \left(1 - \frac{1}{3}\right) I + \frac{1}{3} J - 2 \cdot \frac{1}{3} A$$

is a positive semidefinite matrix of rank 7. The all-ones vector $\mathbf{1}$ is an eigenvector of $I$, $J$ and $A$, and hence also of $M$. It has corresponding eigenvalue

$$\left(1 - \frac{1}{3}\right) + \frac{28}{3} - \frac{2 \cdot 15}{3} = 0.$$

Since all other eigenvalues of $J$ are zero, the other eigenvalues of $M$ are $1 - \frac{1}{3} - \frac{2 \cdot 1}{3} = 0$ and $1 - \frac{1}{3} + \frac{2 \cdot 5}{3} = \frac{8}{3}$. This shows that all eigenvalues of $M$ are non-negative, hence $M$ is positive semidefinite. Furthermore, since the multiplicity of $-5$ as eigenvalue of $A$ is 7, the multiplicity of $\frac{8}{3}$ as eigenvalue of $M$ is also 7. The only other eigenvalue of $M$ is zero, which thus has multiplicity 21. Hence, the nullity of $M$ is also 21, implying that $\operatorname{rk} M = 7$. This shows that $M$ is the Gram matrix of a set of vectors in $\mathbb{R}^7$ with common angle $\arccos\left(\frac{1}{3}\right)$.

An important tool that can be used when working with the associated graph, is the concept of *switching*, which was already introduced by van Lint and Seidel in [2]. Let $\mathcal{C}$ be a spherical $\{-\alpha, \alpha\}$-code representing a set of equiangular lines $\mathcal{L}$ with common angle $\arccos \alpha$. Negating any vector $\boldsymbol{v} \in \mathcal{C}$ does not change the line it spans. So, if $\mathcal{C}'$ is a spherical code obtained from $\mathcal{C}$ by negating some vectors, $\mathcal{C}'$ also represents $\mathcal{L}$. This action of negating a vector in $\mathcal{C}$ is known as switching. In the associated graph, switching corresponds to inverting the adjacency of the vertex $\boldsymbol{v}$ since the inner product between $\boldsymbol{v}$ and any other vector $\boldsymbol{u} \in \mathcal{C}$ changes sign when $\boldsymbol{v}$ is negated. Figure 4.2 shows the associated graph of the spherical code $\mathcal{C} = \{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ representing three equiangular lines in $\mathbb{R}^3$ from Figure 3.1 and the graph obtained from switching $\boldsymbol{w}$, which is the associated graph of the code $\mathcal{C}' = \{\boldsymbol{u}, \boldsymbol{v}, -\boldsymbol{w}\}$.

Using a switching argument we can show that a spherical code representing a set of equiangular lines can always be chosen in such a way that the associated graph has an isolated vertex.

**Lemma 4.10.** *Let $\mathcal{L}$ be a set of equiangular lines in $\mathbb{R}^d$ with common angle* $\arccos \alpha$. *A spherical $\{-\alpha, \alpha\}$-code $\mathcal{C}$ representing $\mathcal{L}$ can be chosen such that the associated graph $G$ has an isolated vertex.*

*Proof.* Let $\mathcal{C}$ be a spherical $\{-\alpha, \alpha\}$-code representing $\mathcal{L}$ with associated graph $G$. Choose a vector $\boldsymbol{v} \in \mathcal{C}$ and negate all vectors $\boldsymbol{u}$ that in $G$ are adjacent to $\boldsymbol{v}$. The resulting code $\mathcal{C}'$ still represents $\mathcal{L}$ and in its associated graph $\boldsymbol{v}$ is an isolated vertex, since $\mathcal{C}'$ contains no vector whose inner product with $\boldsymbol{v}$ is negative. $\square$

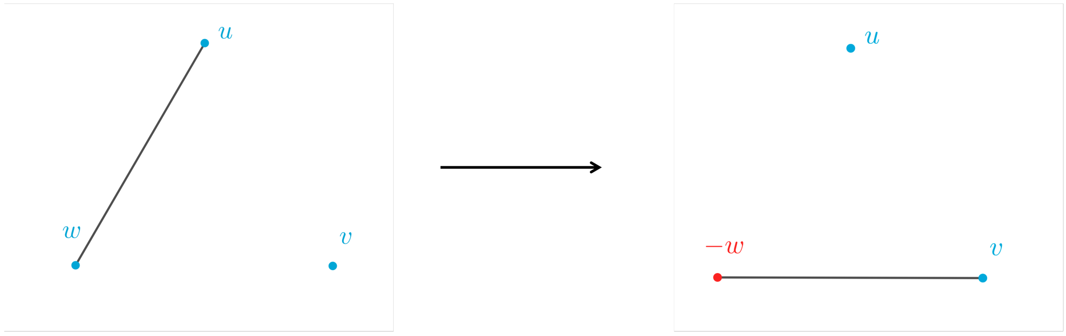


**Figure 4.2:** Switching

## 4.4 The maximum degree of the associated graph

We now continue onto the proof of Theorem 4.4, for which we will use the associated graph. This proof will use two main ingredients. The first, is the fact that we can choose the spherical code representing a set of equiangular lines in such a way that the associated graph has bounded maximum degree (Theorem 4.14). The second main tool is an upper bound on the number of lines that depends on the average degree of the associated graph (Lemma 4.15).

The bound on the maximum degree we show in this section is not the first known bound of this type. A bound on the maximum degree was already shown in [12]. The two results differ completely in their approach. The result in [12] relies on Ramsey's theorem and gives a bound of an order that is exponential in $1/\alpha$. However, the bound from [13] that we discuss here, follows from Theorem 4.1 and hence does not depend on Ramsey theory. The inequality from this theorem makes it possible to find a bound of order $\mathcal{O}(1/\alpha^4)$.

We start this section by stating some lemmas necessary to prove the upper bound on the maximum degree and then proving this bound. Then we give an upper bound on the number of equiangular lines in terms of the average degree of the associated graph. Subsequently, we derive an inequality which will help us to strengthen the bound on the maximum degree of the graph in order to prove Theorem 4.4.

First of all, we show a bound on the degree of the vertices of the graph $G$ associated to a spherical code that is chosen in such a way that $G$ has an isolated vertex. This bound follows immediately from Corollary 4.2 by taking $\boldsymbol{v} = \boldsymbol{e}_1$.

**Lemma 4.11.** *Let $G$ be the associated graph of a spherical $\{-\alpha, \alpha\}$-code $\mathcal{C}$ representing a set of equiangular lines, where $\mathcal{C}$ is chosen in such a way that $G$ has an isolated vertex. Let $\boldsymbol{v}_1$ be this isolated vertex. For all $i \geq 2$, the degree $d(\boldsymbol{v}_i)$ of the $i$-th vertex satisfies*

$$\left(n - 2d(\boldsymbol{v}_i) + \frac{2}{\alpha} - 2\right)^2 \geq \left(n + \frac{1}{\alpha^2} - 1\right)\left(n - \frac{1}{2}\left(\frac{1}{\alpha^4} - \frac{4}{\alpha^2} + 3\right)\right),$$

*with equality whenever $n = \binom{d+1}{2}$.*

*Proof.* The inequality follows by applying Corollary 4.2 with $\boldsymbol{v} = \boldsymbol{e}_1$ and $i \geq 2$. This gives

$$\frac{1-\alpha^2}{2\alpha^2}\left(\langle \boldsymbol{e}_1, M\boldsymbol{e}_1\rangle - (M\boldsymbol{e}_1)_i^2\right) + \frac{1}{\alpha^2 n + 1 - \alpha^2}\left(M^2\boldsymbol{e}_1\right)_i^2 \geq \left\langle \boldsymbol{e}_1, M^2\boldsymbol{e}_1\right\rangle. \tag{4.6}$$

Multiplying the matrix $M$ with the vector $\boldsymbol{e}_1$ gives the first column of $M$ which has entry 1 in the first position and $\alpha$ on all other positions, since $\boldsymbol{v}_1$ is an isolated vertex and thus its inner product with any other vector in $\mathcal{C}$ equals $\alpha$. So we have $M\boldsymbol{e}_1 = \alpha\mathbf{1} + (1-\alpha)\boldsymbol{e}_1$, which we us to calculate

$$\begin{aligned}
\langle \boldsymbol{e}_1, M\boldsymbol{e}_1\rangle &= 1\\
(M\boldsymbol{e}_1)_i &= (\alpha\mathbf{1} + (1-\alpha)\boldsymbol{e}_1)_i = \alpha,\\
\left(M^2\boldsymbol{e}_1\right)_i &= \left\langle M^2\boldsymbol{e}_1, \boldsymbol{e}_i\right\rangle = \langle M\boldsymbol{e}_1, M\boldsymbol{e}_i\rangle = \alpha\langle \mathbf{1}, M\boldsymbol{e}_i\rangle + (1-\alpha)\langle \boldsymbol{e}_1, M\boldsymbol{e}_i\rangle\\
&= \alpha\langle \mathbf{1}, M\boldsymbol{e}_i\rangle + \alpha(1-\alpha),\\
\langle \boldsymbol{e}_1, M^2\boldsymbol{e}_1\rangle &= \|M\boldsymbol{e}_1\|^2 = \langle \alpha\mathbf{1}, \alpha\mathbf{1}\rangle + 2\langle \alpha\mathbf{1}, (1-\alpha)\boldsymbol{e}_1\rangle + \langle (1-\alpha)\boldsymbol{e}_1, (1-\alpha)\boldsymbol{e}_1\rangle\\
&= \alpha^2 n + 1 - \alpha^2.
\end{aligned}$$

So Equation 4.6 now reads

$$\frac{1-\alpha^2}{2\alpha^2}\left(1-\alpha^2\right) + \frac{\alpha^2}{\alpha^2 n + 1 - \alpha^2}\left(\langle \mathbf{1}, M\boldsymbol{e}_i\rangle + (1-\alpha)\right)^2 \geq \alpha^2 n + 1 - \alpha^2,$$

which after rearranging becomes

$$\left(\langle \mathbf{1}, M\boldsymbol{e}_i\rangle + (1-\alpha)\right)^2 \geq \frac{\alpha^2 n + 1 - \alpha^2}{\alpha^2}\left(\alpha^2 n + 1 - \alpha^2 - \frac{\left(1-\alpha^2\right)^2}{2\alpha^2}\right). \tag{4.7}$$

To further simplify the left hand side of this inequality we need to calculate $\langle \mathbf{1}, M\boldsymbol{e}_i\rangle$. Similarly to the case with $\boldsymbol{e}_1$, the multiplication of $M$ with $\boldsymbol{e}_i$ yields the $i$-th column of $M$. This vector has entry 1 on the $i$-th position. Moreover, it has exactly $d(\boldsymbol{v}_i)$ entries $-\alpha$ and all other $n-1-d(\boldsymbol{v}_i)$ entries are $\alpha$. Since $\langle \mathbf{1}, M\boldsymbol{e}_i\rangle$ is the sum of all entries of $M\boldsymbol{e}_i$, we find

$$\langle \mathbf{1}, M\boldsymbol{e}_i\rangle = 1 - d(\boldsymbol{v}_i)\alpha + (n-1-d(\boldsymbol{v}_i))\,\alpha = 1 - \alpha + \alpha\,(n - 2d(\boldsymbol{v}_i)).$$

Substituting this into Equation 4.7 gives

$$\left(2 - 2\alpha + \alpha\,(n - 2d(\boldsymbol{v}_i))\right)^2 \geq \left(n + \frac{1}{\alpha^2} - 1\right)\left(\alpha^2 n + 1 - \alpha^2 - \frac{\left(1-\alpha^2\right)^2}{2\alpha^2}\right).$$

Dividing both sides by $\alpha^2$ yields

$$\begin{aligned}
\left(n - 2d(\boldsymbol{v}_i) + \frac{2}{\alpha} - 2\right)^2 &\geq \left(n + \frac{1}{\alpha^2} - 1\right)\left(\alpha^2 n + 1 - \alpha^2 - \frac{\left(1-\alpha^2\right)^2}{2\alpha^2}\right)\frac{1}{\alpha^2}\\
&= \left(n + \frac{1}{\alpha^2} - 1\right)\left(n - \frac{1}{2}\left(\frac{1}{\alpha^4} - \frac{4}{\alpha^2} + 3\right)\right)
\end{aligned}$$

as desired. □

We will now use this inequality to show that we can partition the associated graph with an isolated vertex into vertices of high and low degree for large enough $n$, so that it is possible to take the square of both sides of the inequality of the previous lemma.

**Lemma 4.12.** *Let $\mathcal{C}$ be a spherical code representing a set of equiangular lines with common angle* $\arccos\alpha$ *such that the associated graph $G$ has an isolated vertex. Suppose that $n > \frac{1}{2}\left(\frac{1}{\alpha^4} - \frac{4}{\alpha^2} + 3\right)$ and define the sets $H = \{\boldsymbol{v} \in \mathcal{C} : d(\boldsymbol{v}) > \frac{n}{2} + \frac{1}{\alpha} - 1\}$ and $L = \mathcal{C} \setminus H$. Then for all $\boldsymbol{v} \in H$ we have*

$$d(\boldsymbol{v}) > n - \frac{1}{4}\left(\frac{1}{\alpha^4} - \frac{4}{\alpha^2} + 3\right) + \frac{1}{\alpha} - 1,$$

*and for all $\boldsymbol{u} \in L$ we have*

$$d(\boldsymbol{u}) < \frac{1}{4\alpha^4} - \left(\frac{1}{\alpha} - \frac{1}{2}\right)^2.$$

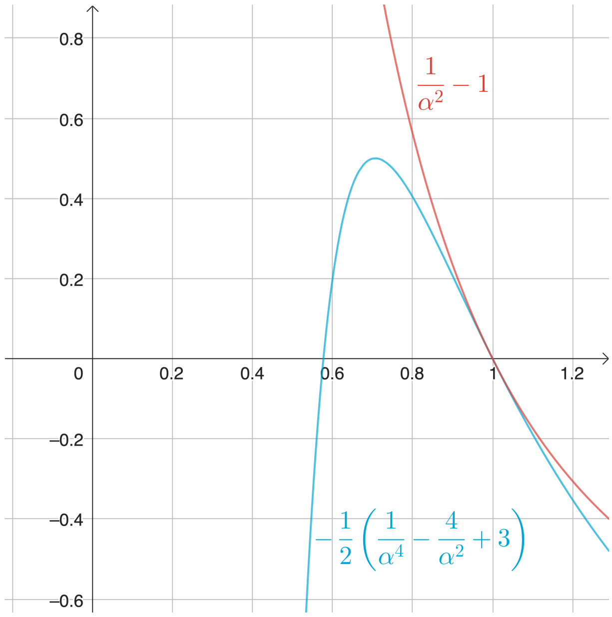


**Figure 4.3:** Plot of $-\frac{1}{2}\left(\frac{1}{\alpha^4} - \frac{4}{\alpha^2} + 3\right)$ and $\frac{1}{\alpha^2} - 1$.

*Proof.* Since $n > \frac{1}{2}\left(\frac{1}{\alpha^4} - \frac{4}{\alpha^2} + 3\right)$ we can take the square root on both sides of the inequality of Lemma 4.11 so that we have

$$\begin{aligned} \left| n - 2d(\boldsymbol{v}) + \frac{2}{\alpha} - 2 \right| &\geq \sqrt{\left(n + \frac{1}{\alpha^2} - 1\right)\left(n - \frac{1}{2}\left(\frac{1}{\alpha^4} - \frac{4}{\alpha^2} + 3\right)\right)} \\ &> n - \frac{1}{2}\left(\frac{1}{\alpha^4} - \frac{4}{\alpha^2} + 3\right), \end{aligned} \tag{4.8}$$

where the last inequality follows since $\frac{1}{\alpha^2} - 1 > -\frac{1}{2}\left(\frac{1}{\alpha^4} - \frac{4}{\alpha^2} + 3\right)$ for $\alpha \in (0,1)$, see Figure 4.3.

First let $\boldsymbol{v} \in H$. Then $n - 2d(\boldsymbol{u}) + \frac{2}{\alpha} - 2 < 0$, so the absolute value equals $\left|n - 2d(\boldsymbol{v}) + \frac{2}{\alpha} - 2\right| = -\left(n - 2d(\boldsymbol{u}) + \frac{2}{\alpha} - 2\right)$ and Equation 4.8 becomes

$$-\left(n - 2d(\boldsymbol{u}) + \frac{2}{\alpha} - 2\right) > n - \frac{1}{2}\left(\frac{1}{\alpha^4} - \frac{4}{\alpha^2} + 3\right).$$

which gives the desired result after some rearranging.

Now let $\boldsymbol{u} \in L$. In this case $n - 2d(\boldsymbol{u}) + \frac{2}{\alpha} - 2 > 0$ and so Equation 4.8 gives

$$\begin{aligned} d(\boldsymbol{u}) &< \frac{1}{4}\left(\frac{1}{\alpha^4} - \frac{4}{\alpha^2} + 3\right) + \frac{1}{\alpha} - 1 \\ &= \frac{1}{4\alpha^4} - \left(\frac{1}{\alpha} - \frac{1}{2}\right)^2, \end{aligned}$$

as required. □

We now show that the set $H$ of high degree vertices has a maximum number of elements. This observation will be a key ingredient in finding a spherical code whose associated graph has bounded degree. It ensures that after switching all vertices in $H$, the degrees of the vertices outside of $H$ can only increase by a bounded number of elements.

**Lemma 4.13.** *Let $\mathcal{C}$ be a spherical code representing a set of equiangular lines with common angle $\arccos\alpha$ such that the associated graph $G$ has an isolated vertex. Suppose that $n > \frac{1}{\alpha^4}$ and let $H = \{\boldsymbol{v} \in \mathcal{C} : d(\boldsymbol{v}) > \frac{n}{2} + \frac{1}{\alpha} - 1\}$ as in the previous lemma. Then*

$$|H| \leq \frac{1}{4\alpha^4 - \frac{3}{n}} < \frac{1}{\alpha^4}.$$

*Proof.* To prove the bound we will first give an upper and lower bound on the number $|E(H,L)|$ of edges between $H$ and $L$ in terms of $|H|$, $n$ and $\alpha$. Then we combine this with a bound on $|H|$ in terms of $n$ only. The upper bound will follow by combining the two resulting inequalities.

First of all, let us count $|E(H,L)|$ in two different ways to find an upper and lower bound on the number. For the upper bound on $|E(H,L)|$ observe that the number of edges with one vertex in $L$ can not exceed the sum of the degrees of the vertices in $L$. The degree of any vertex in $L$ is at most $\frac{1}{4\alpha^4}$ by the previous lemma. This gives

$$|E(H,L)| \leq \sum \boldsymbol{v} \in Ld(\boldsymbol{v}) \leq (n - |H|)\,\frac{1}{4\alpha^4}.$$

The sum of all degrees of the vertices in $H$ counts all the edges $E(H)$ with two vertices in $H$ twice. Subtracting all these edges from the sum leaves us with all edges with only one vertex in $E(H,L)$. Recall from the previous lemma that the degree of all vertices in $\boldsymbol{v}$ in $H$ satisfy $d(\boldsymbol{v}) \geq n - \frac{1}{4\alpha^4} + \frac{1}{\alpha} - 1 \geq n - \frac{1}{4\alpha^4}$. Furthermore, observe that $|E(H)|$ can be at most $\binom{|H|}{2}$ which is no larger than $|H|^2$. So, as a lower bound on $|E(H,L)|$, we find

$$|E(H,L)| = \sum \boldsymbol{v} \in Hd(\boldsymbol{v}) - 2|E(H)| \geq |H|\left(n - \frac{1}{4\alpha^4}\right) - |H|^2.$$

Combining the upper and lower bound on $|E(H,L)|$ that we found above, gives

$$|H|\left(n - \frac{1}{4\alpha^2}\right) - |H|^2 \leq (n - |H|)\,\frac{1}{4\alpha^4},$$

which can be simplified to

$$|H|\left(1 - \frac{|H|}{n}\right) \leq \frac{1}{4\alpha^4}. \tag{4.9}$$

We now show an upper bound on $|H|$ in terms of $n$ by using the Gram matrix $M$ of the code $\mathcal{C}$. Recall from the proof of Lemma 4.11 that for any $i \in [n]$, $\langle \mathbf{1}, M\boldsymbol{e}_i\rangle = 1 - \alpha + \alpha\,(n - 2d(\boldsymbol{v}_i))$, where $\boldsymbol{v}_i$ is the $i$-th vertex of $G$. Moreover, by our assumption on $n$, for any vertex $\boldsymbol{v} \in H$, its degree satisfies $d(\boldsymbol{v}) \geq n - \frac{1}{4\alpha^4} + \frac{1}{\alpha} - 1 \geq \frac{3}{4}n + \frac{1}{\alpha} - 1$ Using these two properties and the fact that $M$ is positive semidefinite we find

$$\begin{aligned} 0 \leq \mathbf{1}^{\mathsf{T}} M \mathbf{1} = \sum_{i=1}^{n} \langle \mathbf{1}, M\boldsymbol{e}_i\rangle &= n(\alpha n + 1 - \alpha) - 2\alpha \sum_{\boldsymbol{v}\in L} d(\boldsymbol{v}) - 2\alpha \sum_{\boldsymbol{v}\in H} d(\boldsymbol{v}) \\ &\leq n(\alpha n + 1 - \alpha) - 2|H|\left(\frac{3}{4}\alpha n + 1 - \alpha\right), \end{aligned}$$

which yields

$$|H| \leq \frac{n(\alpha n + 1 - \alpha)}{\frac{3}{2}\alpha n + 2(1-\alpha)} \leq \frac{2}{3}n. \tag{4.10}$$

From this last inequality it follows that $|H|\left(1 - \frac{|H|}{n}\right) \geq \frac{|H|}{3}$. Using Equation 4.9 we now find

$$|H| \leq \frac{3}{4\alpha^4}.$$

We use this improved bound on $|H|$ in Equation 4.9 to obtain

$$|H|\left(1 - \frac{3}{4\alpha^4 n}\right) \leq |H|\left(1 - \frac{|H|}{n}\right) \leq \frac{1}{4\alpha^4},$$

which can be rewritten as

$$|H| \leq \frac{1}{4\alpha^4\left(1 - \frac{3}{4\alpha^4 n}\right)} = \frac{1}{4\alpha^4 - \frac{3}{n}}.$$

Lastly, since we assumed that $n > \frac{1}{\alpha^4}$, we have $\frac{1}{4\alpha^4 - \frac{3}{n}} < \frac{1}{\alpha^4}$ as required. $\square$

Using the above lemmas we can now show that the spherical code $\mathcal{C}$ representing a set of equiangular lines can be chosen such that the associated degree has bounded maximum degree. In order to prove this a switching argument will be used. At first the code representing $\mathcal{L}$ will be chosen such that its associated graph has an isolated vertex. We then partition the vertices of the graph into high and low degree vertices. A new graph is created by negating all vectors in the set of high degree vertices. By this action all the vertices that had a high degree will now have a bounded degree. Furthermore, since the number of high degree vertices was bounded, the degree of all vertices with low degree can only increase by a bounded number.

**Theorem 4.14.** *Let $\mathcal{L}$ be a set of $n$ equiangular lines in $\mathbb{R}^d$ with common angle* $\arccos\alpha$. *If $n \geq 1/\alpha^4$, then there exists a spherical $\{-\alpha, \alpha\}$-code $\mathcal{C}$ representing $\mathcal{L}$ such that the associated graph $G_{\mathcal{C}}$ has maximum degree*

$$\Delta(G_{\mathcal{C}}) \leq \frac{1}{4\alpha^4} + \frac{1}{4\alpha^4 - \frac{3}{n}} < \frac{1}{4\alpha^4} + \frac{1}{\alpha^4}.$$

Note that the maximum degree of a graph is never more than its number of vertices. So, if $n < 1/\alpha^4$, then the maximum degree of the graph is also bounded since it can not be larger than $n$ itself. This means that in this case we also have $\Delta(G_{\mathcal{C}}) \leq \frac{1}{\alpha^4} \leq \frac{1}{4\alpha^4} + \frac{1}{\alpha^4}$.

*Proof.* By Lemma 4.10 there is a spherical $\{-\alpha, \alpha\}$-code $\mathcal{C}$ representing $\mathcal{L}$ such that its associated graph $G$ has an isolated vertex. So let $\mathcal{C}$ be such a code and define the sets $H = \{\boldsymbol{v} \in \mathcal{C} : d(\boldsymbol{v}) > \frac{n}{2} + \frac{1}{\alpha} - 1\}$ and $L = \mathcal{C} \setminus H$ of respectively high and low degree vertices as in Lemma 4.12. From the previous lemma we know that the set $H$ has at most $1/\left(4\alpha^4 - \frac{3}{n}\right)$ elements.

Construct the spherical $\{-\alpha, \alpha\}$-code $\mathcal{C}' = \{-\boldsymbol{v} : \boldsymbol{v} \in H\}$ by negating all vectors in the set $H$. This code also represents the set $\mathcal{L}$. Denote the associated graph of $\mathcal{C}'$ as $G'$. This switching action does not change the adjacency between any two vertices $\boldsymbol{v}$ and $\boldsymbol{u}$ in the set $L$. The edges inside $H$ are also not affected by this switching, since for any two vertices $\boldsymbol{v}$ and $\boldsymbol{u}$ we have $\langle -\boldsymbol{v}, -\boldsymbol{u}\rangle = \langle \boldsymbol{v}, \boldsymbol{u}\rangle$. Between the sets $H$ and $L$ the switching argument inverts the edges. So, the graph $G'$ is obtained from $G$ by inverting the adjacency between any vector $\boldsymbol{v} \in H$ and $\boldsymbol{u} \in L$.

We now show that the graph $G'$ has maximum degree at most $\frac{1}{4\alpha^4} + \frac{1}{4\alpha^4 - \frac{3}{n}}$ as required. We do this by first showing this upper bound holds for all vertices in $L$ and then showing it holds for all vertices in $H$.

So, let $\boldsymbol{v}$ be any vector in $L$. The degree of $\boldsymbol{v}$ in $G'$ can increase by at most $|H|$ with respect to its degree in the original graph $G$. Using Lemmas 4.12 and 4.13 it immediately follows that

$$d_{G'}(\boldsymbol{v}) \leq d_G(v) + |H| \leq \frac{1}{4\alpha^4} + \frac{1}{4\alpha^4 - \frac{3}{n}}.$$

Now we turn to the degree of the vertices in the set $H$. Let $\boldsymbol{v}$ be an arbitrarily chosen vector in $H$. For all vectors $\boldsymbol{u} \in H$ which are adjacent to $\boldsymbol{v}$ in $G$, the vector $-\boldsymbol{u}$ is adjacent to $-\boldsymbol{v}$ in the graph $G'$. This gives $|H \cap N_G(\boldsymbol{v})|$ neighbours. All other neighbours of $-\boldsymbol{v}$ must be vectors in $L$. All vectors $\boldsymbol{u} \in L$ which in $G$ were not neighbours of $\boldsymbol{v}$ are neighbours of $-\boldsymbol{v}$ in the new graph $G'$. These are exactly $|L \setminus N_G(\boldsymbol{v})|$ neighbours. It follows that the degree of $-\boldsymbol{v}$ in $G'$ is

$$d_{G'}(-\boldsymbol{v}) = |H \cap N_G(\boldsymbol{v})| + |L \setminus N_G(\boldsymbol{v})|.$$

The set $H \cap N_G(\boldsymbol{v})$ is a subset of $H$ and hence $|H \cap N_G(\boldsymbol{v})| \leq |H| \leq 1/\left(4\alpha^4 - \frac{3}{n}\right)$. By Lemma 4.12, $|N_G(\boldsymbol{v})| = d_G(\boldsymbol{v}) > n - \frac{1}{4\alpha^4}$. Combining this with $|L| < n$, gives $|L \setminus N_G(\boldsymbol{v})| \leq 1/(4\alpha^4)$. So, we conclude

$$d_{G'}(-\boldsymbol{v}) \leq \frac{1}{4\alpha^4} + \frac{1}{4\alpha^4 - \frac{3}{n}}.$$

We have now shown that the degree of any vertex in $G'$ is at most $\frac{1}{4\alpha^4} + \frac{1}{4\alpha^4 - \frac{3}{n}}$ and hence, this holds in particular for its maximum degree $\Delta(G')$.

As in the previous lemma, since $n \geq \frac{1}{\alpha^4}$, we have $\frac{1}{4\alpha^4 - \frac{3}{n}} \leq \frac{1}{\alpha^4}$ and thus it follows that

$$\Delta(G') \leq \frac{1}{4\alpha^4} + \frac{1}{4\alpha^4 - \frac{3}{n}} \leq \frac{1}{4\alpha^4} + \frac{1}{\alpha^4}.$$

□

The upper bound on the maximum degree of the graph will be the main tool in proving Theorem 4.4 together with the following lemma. This lemma bounds the maximum number of lines using the average degree of the associated graph. Since the average degree is no larger than the maximum degree, we can use the above bound together with this lemma to bound the number of lines. The proof of this lemma is completely based on Lemma 2.4.

**Lemma 4.15.** *Let $\mathcal{C}$ be a spherical $\{-\alpha, \alpha\}$-code representing a set of $n$ equiangular lines with common angle* $\arccos \alpha$. *Let $G$ be the associated graph of $\mathcal{C}$ with average degree $D = D(G)$. Then*

$$n \leq \left(1 + \left(\frac{2\alpha}{1-\alpha}\right)^2 D\right)(d+1).$$

*Proof.* Let $M$ be the Gram matrix of the code $\mathcal{C}$ and let $A = A_G$ be the adjacency matrix of the graph $G$. Recall from Equation 4.5 that we can write $M$ as $M = (1-\alpha)I + \alpha J - 2\alpha A$. Define the matrix $B = M - \alpha J = (1-\alpha)I - 2\alpha A$. We will use the inequality $\operatorname{Tr}(B)^2 \leq \operatorname{Tr}\left(B^2\right)\operatorname{rk}(B)$ from Lemma 2.4 to prove the required inequality.

The rank of the matrix $B$ is upper bounded by $\operatorname{rk} M + \operatorname{rk}(-\alpha J) \leq d+1$, by the subadditivity of the rank. For the trace of $B$ we have $\operatorname{Tr}(B) = (1-\alpha)\operatorname{Tr}(I) - 2\alpha\operatorname{Tr}(A) = (1-\alpha)n$. Note that for any $i \in [n]$ there exactly $d(\boldsymbol{v}_i)$ entries $j \in [n]$ such that $B_{ij} = -2\alpha$. Hence, the trace of the square of $B$ equals

$$\operatorname{Tr}\left(B^2\right) = \sum_{i,j=1}^{n} B_{ij}^2 = (1-\alpha)^2 n + 4\alpha^2 \sum_{i=1}^{n} d(\boldsymbol{v}_i) = \left((1-\alpha)^2 + 4\alpha^2 D\right) n.$$

Using Lemma 2.4 we now find

$$(1-\alpha)^2 n^2 \leq \left((1-\alpha)^2 + 4\alpha^2 D\right) n(d+1).$$

Dividing by $(1-\alpha)^2 n$ yields

$$n \leq \left(1 + \left(\frac{2\alpha}{1-\alpha}\right)^2 D\right)(d+1),$$

as required. $\square$

All we need now to upper bound the number of equiangular lines $n$ with a fixed common angle of $\arccos \alpha$ is an upper bound on the average degree $D$ of the associated graph $G$. Note that Theorem 4.14 already gives us an upper bound on the average degree for any $n \geq 1/\alpha^4$, since $D \leq \Delta(G)$. So, combining the bound from Theorem 4.14 and the above lemma already shows that

$$n \leq \left(1 + \frac{5}{\alpha^2(1-\alpha)^2}\right)(d+1).$$

This upper bound is not yet strong enough to prove Theorem 4.4. In order to prove this theorem we will need even sharper bounds on the maximum degree $\Delta$. To achieve these bounds we will derive an inequality using the maximum Rayleigh quotient of the adjacency matrix $A$ of the associated graph $G$ over all vectors orthogonal to $\mathbf{1}$, which we denote by

$$\rho(G) := \max_{\substack{\boldsymbol{v} \in \mathbb{R}^n \setminus \{0\} \\ \boldsymbol{v} \perp \mathbf{1}}} \mathcal{R}(A, \boldsymbol{v}) = \max_{\substack{\boldsymbol{v} \in \mathbb{R}^n \setminus \{0\} \\ \boldsymbol{v} \perp \mathbf{1}}} \frac{\boldsymbol{v}^\mathsf{T} A \boldsymbol{v}}{\boldsymbol{v}^\mathsf{T} \boldsymbol{v}}.$$

We will show an upper and lower bound on $\rho(G)$. Together, these two will give us an inequality which will help us find tighter upper bounds on $\Delta$.

**Lemma 4.16.** *Let $\mathcal{C}$ be a spherical $\{-\alpha, \alpha\}$-code in $\mathbb{R}^d$ corresponding to a set of $n$ equiangular lines with common angle* $\arccos \alpha$, *with associated graph $G = G_{\mathcal{C}}$. Then*

$$\rho(G) \leq \frac{1-\alpha}{2\alpha},$$

*with equality whenever $n \geq d+2$.*

*Proof.* Let $M = M_{\mathcal{C}}$ be the Gram matrix of $\mathcal{C}$ and $A = A_G$ the adjacency matrix of the graph $G$. Recall from Equation 4.5 that $M = (1-\alpha)I + \alpha J - 2\alpha A$. Let $\boldsymbol{v}$ be any nonzero vector in $\mathbb{R}^n$ such that $\boldsymbol{v} \perp \mathbf{1}$. Then $J\boldsymbol{v} = 0$ and so, since $M$ is positive semidefinite, we find

$$0 \leq \boldsymbol{v}^{\mathsf{T}} M \boldsymbol{v} = \boldsymbol{v}^{\mathsf{T}} \left((1-\alpha)I + \alpha J - 2\alpha A\right) \boldsymbol{v} = (1-\alpha)\boldsymbol{v}^{\mathsf{T}}\boldsymbol{v} - 2\alpha \boldsymbol{v}^{\mathsf{T}} A \boldsymbol{v}.$$

Rearranging this inequality yields

$$\frac{\boldsymbol{v}^{\mathsf{T}} A \boldsymbol{v}}{\boldsymbol{v}^{\mathsf{T}}\boldsymbol{v}} \leq \frac{1-\alpha}{2\alpha}.$$

Since we chose $\boldsymbol{v}$ arbitrarily, it follows that $\rho(G) \leq \frac{1-\alpha}{2\alpha}$.

Now suppose that $n \geq d+2$. The rank of $M$ is at most $d$ and hence $\dim \operatorname{Ker} M \geq 2$, by the rank-nullity theorem. Then $\operatorname{Ker} M$ intersects the subspace $U$ of all vectors orthogonal to $\mathbf{1}$. To see this note that $\dim(U) = n-1$ and $n \geq \dim(U) + \dim \operatorname{Ker} M - \dim(U \cap \operatorname{Ker} M) = n + 1 - \dim(U \cap \operatorname{Ker} M)$, which implies that $\dim(U \cap \operatorname{Ker} M) \geq 1$ and hence the intersection of $U$ and $\operatorname{Ker} M$ is nonempty. It follows that $\operatorname{Ker} M$ contains a nonzero vector $\boldsymbol{v}$ that is orthogonal to $\mathbf{1}$. So, we have

$$0 = \boldsymbol{v}^{\mathsf{T}} M \boldsymbol{v} = (1-\alpha)\boldsymbol{v}^{\mathsf{T}}\boldsymbol{v} - 2\alpha \boldsymbol{v}^{\mathsf{T}} A \boldsymbol{v},$$

which implies $\frac{\boldsymbol{v}^{\mathsf{T}} A \boldsymbol{v}}{\boldsymbol{v}^{\mathsf{T}}\boldsymbol{v}} = \frac{1-\alpha}{2\alpha}$. Since $\rho(G)$ is the maximum of $\mathcal{R}(A, \boldsymbol{u})$ over all $\boldsymbol{u}$ orthogonal to $\mathbf{1}$, it follows that $\rho(G) \geq \mathcal{R}(A, \boldsymbol{v})$. Hence, we must have equality. ☐

The upper bound on $\rho(G)$ only depends on the angle $\arccos \alpha$ between the lines. We now continue to show a lower bound on $\rho(G)$ which depends on its average and maximum degree and a subgraph of $G$. We first prove the general inequality for any subgraph of $G$. By then choosing a specific subgraph, the inequality will be simplified into an easier applicable form.

**Lemma 4.17.** *Let $G$ be a graph on $n$ vertices with maximum degree $\Delta = \Delta(G)$ and average degree $D = D(G)$. For any subgraph $H$ of $G$ we have*

$$\rho(G) \geq \lambda_1(H) - \frac{2\Delta - D}{n}|H|.$$

*Proof.* Let $A = A_G$ be the adjacency matrix of the graph $G$ with vertex set $V = \{v_1, \ldots, v_n\}$ and let $H$ be any subgraph of $G$ with spectral radius $\lambda_1(H)$. Let $\boldsymbol{v} \in \mathbb{R}^{|H|}$ be a unit eigenvector corresponding to $\lambda_1(H)$. Since the adjacency matrix $A_H$ of the subgraph $H$ is non-negative, the vector $\boldsymbol{v}$ can be chosen such that all its entries are non-negative. Extend $\boldsymbol{v}$ to a vector in $\mathbb{R}^n$ by adding zeroes to it such that $\boldsymbol{v}^{\mathsf{T}} A \boldsymbol{v} = \boldsymbol{v}^{\mathsf{T}} \lambda_1(H) \boldsymbol{v} = \lambda_1(H)$. Now define

$$\boldsymbol{u} = \boldsymbol{v} - \frac{\langle \boldsymbol{v}, \mathbf{1} \rangle}{n}\mathbf{1},$$

the projection of $\boldsymbol{v}$ onto the orthogonal complement of $\mathbf{1}$. Then, since $\rho(G)$ is the maximum of the Rayleigh quotient of $A$ over all vectors orthogonal to one, we have $\rho(G) \geq \mathcal{R}(A, \boldsymbol{v})$. In order to prove the lemma it thus suffices to show that

$$\mathcal{R}(A, \boldsymbol{u}) = \frac{\boldsymbol{u}^{\mathsf{T}} A \boldsymbol{u}}{\boldsymbol{u}^{\mathsf{T}} \boldsymbol{u}} \geq \lambda_1(H) - \frac{2\Delta - D}{n}|H|.$$

First, note that $\boldsymbol{u}^{\mathsf{T}}\boldsymbol{u} = \|\boldsymbol{v}\|^2 - \frac{\langle \boldsymbol{v}, \mathbf{1} \rangle}{n} \leq 1$ and hence

$$\mathcal{R}(A, \boldsymbol{u}) \geq \boldsymbol{u}^{\mathsf{T}} A \boldsymbol{u}.$$

We thus want to lower bound $\boldsymbol{u}^{\mathsf{T}} A \boldsymbol{u}$ and so we compute

$$\begin{aligned}
\boldsymbol{u}^{\mathsf{T}} A \boldsymbol{u} &= \boldsymbol{v}^{\mathsf{T}} A \boldsymbol{v} - 2\frac{\langle \boldsymbol{v}, \mathbf{1} \rangle}{n}\mathbf{1}^{\mathsf{T}} A \boldsymbol{v} + \frac{\langle \boldsymbol{v}, \mathbf{1} \rangle^2}{n^2}\mathbf{1}^{\mathsf{T}} A \mathbf{1} \\
&= \lambda_1(H) - 2\frac{\langle \boldsymbol{v}, \mathbf{1} \rangle}{n}\sum_{i=1}^{n} \boldsymbol{v}_i d(v_i) + \frac{\langle \boldsymbol{v}, \mathbf{1} \rangle^2}{n^2}\sum_{i=1}^{n} d(v_i) \\
&\geq \lambda_1(H) 2\frac{\langle \boldsymbol{v}, \mathbf{1} \rangle}{n}\sum_{i=1}^{n} \boldsymbol{v}_i + \frac{\langle \boldsymbol{v}, \mathbf{1} \rangle^2}{n^2} D \\
&= \lambda_1(H) - (2\Delta - D)\frac{\langle \boldsymbol{v}, \mathbf{1} \rangle^2}{n}.
\end{aligned}$$

The vector $\boldsymbol{v}$ has at most $|H|$ nonzero entries. If we let $\mathbf{1}_H$ denote the vector with entries one for all elements in $H$ and zero otherwise, we find $\langle \boldsymbol{v}, \mathbf{1} \rangle = \langle \boldsymbol{v}, \mathbf{1}_H \rangle \leq \|\boldsymbol{v}\| \|\mathbf{1}_H\| = \sqrt{|H|}$. So, we conclude

$$\begin{aligned} \rho(G) \geq \mathcal{R}(A, \boldsymbol{u}) \geq \boldsymbol{u}^\top A \boldsymbol{u} &\geq \lambda_1(H) - (2\Delta - D) \frac{\langle \boldsymbol{v}, \mathbf{1} \rangle^2}{n} \\ &\geq \lambda_1(H) - \frac{2\Delta - D}{n} |H|. \end{aligned}$$

□

By taking $H$ in the above lemma to be the star graph $K_{1,t}$ for any $t \leq \Delta$, we can simplify this lower bound.

**Corollary 4.18.** *Let $G$ be a graph on $n$ vertices with maximum degree $\Delta = \Delta(G)$. For all $t \in \mathbb{N}$ such that $t \leq \Delta$, we have*

$$\rho(G) \geq \sqrt{t} - \frac{2\Delta(t+1)}{n}.$$

*Proof.* Since $t$ is no larger than $\Delta$ we can always find a copy of the star graph $K_{1,t}$ as subgraph of $G$. Indeed, taking any vertex $v$ in $G$ of degree $\Delta$ together with $t$ of its neighbours yields such a subgraph. In Example 2.16 we saw that the spectral radius of $K_{1,t}$ is $\sqrt{t}$. It now immediately follows from the previous lemma that

$$\rho(G) \geq \lambda_1(K_{1,t}) - \frac{2\Delta}{n} |K_{1,t}| = \sqrt{t} - \frac{2\Delta(t+1)}{n}.$$

□

*Remark* 4.19. Notice that we can now combine the upper and lower bound on $\rho(G)$ to conclude that for any spherical $\{-\alpha, \alpha\}$-code $\mathcal{C}$ in $\mathbb{R}^d$ with $n$ elements and associated graph $G$ with maximum degree $\Delta$, we have

$$\frac{1-\alpha}{2\alpha} \geq \sqrt{t} - \frac{2\Delta(t+1)}{n}. \tag{4.11}$$

With this inequality we will be able to tighten the bound on $\Delta$ and prove Theorem 4.4. Recall that in order to prove this theorem we want to show that for a set of $n$ equiangular lines in $\mathbb{R}^d$ with common angle $\arccos \alpha$,

$$n \leq \max \left( \frac{2}{\alpha^5} + \frac{2}{\alpha^3(1-\alpha)}, \left( 2 + \frac{8\alpha^2}{(1-\alpha)^2} \right) (d+1) \right).$$

In the proof we first use inequality (4.11) to show that $n > \frac{2}{\alpha^5} + \frac{2}{\alpha^3(1-\alpha)}$ implies that $n > 4\Delta^{3/2}$. This lower bound $n$ will make it possible to use the inequality to give a tighter bound on $\Delta$, which, together with Lemma 4.15, will lead to the desired result.

*Proof of Theorem 4.4.* Let $\mathcal{L}$ be a set of $n$ equiangular lines in $\mathbb{R}^d$ with common angle $\arccos \alpha$. By Theorem 4.14, there exists a spherical $\{-\alpha, \alpha\}$-code $\mathcal{C}$ representing $\mathcal{L}$ such that its associated graph $G$ has maximum degree

$$\Delta = \Delta(G) \leq \frac{1}{4\alpha^4} + \frac{1}{4\alpha^4 - \frac{3}{n}}.$$

If $n \leq \frac{2}{\alpha^5} + \frac{2}{\alpha^3(1-\alpha)}$, then we are done. Hence, we assume $n > \frac{2}{\alpha^5} + \frac{2}{\alpha^3(1-\alpha)} > \frac{2}{\alpha^5}$. Note that in this case, we have

$$\begin{aligned} \Delta \leq \frac{1}{4\alpha^4} + \frac{1}{4\alpha^4 - \frac{3}{n}} &< \frac{1}{4\alpha^4} + \frac{1}{4\alpha^4\left(1 - \frac{3}{8}\alpha\right)} = \frac{1}{2\alpha^4} + \frac{\frac{3}{8}\alpha}{4\alpha^4\left(1 - \frac{3}{8}\alpha\right)} \\ &= \frac{1}{2\alpha^4} + \frac{3}{32\alpha^3 - 12\alpha^4} < \frac{1}{2\alpha^4} + \frac{3}{20\alpha^3}. \end{aligned} \tag{4.12}$$

We will first show that we must have $n > 4\Delta^{3/2}$. Then we will use this in inequality (4.11) from Remark 4.19 to tighten the bound on $\Delta$.

We show that $n$ must be bigger than $4\Delta^{3/2}$ by contradiction. So, suppose that $n \leq 4\Delta^{3/2}$. We will derive the inequality

$$n \leq 4\Delta \left( \frac{1}{\alpha} - 1 + \frac{2\alpha}{1-\alpha} \right), \tag{4.13}$$

which together with the bound on $\Delta$ from above will lead to a contradiction. Observe that if $n \leq 4\Delta\left(\frac{1}{\alpha}-1\right)$, then we are done. So, assume that $n > 4\Delta\left(\frac{1}{\alpha}-1\right)$. Define $t = \left\lceil \frac{n^2}{16\Delta^2} \right\rceil$ and note that $t \leq \Delta$. Using this $t$ in inequality (4.11) yields

$$\frac{1-\alpha}{2\alpha} \geq \sqrt{t} - \frac{2\Delta(t+1)}{n} \geq \frac{n}{4\Delta} - \frac{2\Delta\left(\frac{n^2}{16\Delta^2}+2\right)}{n} \geq \frac{n}{4\Delta} - \frac{n}{8\Delta} - \frac{4\Delta}{n} > \frac{n}{8\Delta} - \frac{\alpha}{1-\alpha},$$

which can be rearranged to inequality (4.13). Now if we use the bound on $\Delta$ from Equation 4.12 in (4.13), we find

$$\begin{aligned} n \leq 4\left(\frac{1}{2\alpha^4}+\frac{3}{20\alpha^3}\right)\left(\frac{1}{\alpha}-1+\frac{2\alpha}{1-\alpha}\right) &\leq \frac{2}{\alpha^5} + \left(\frac{1}{2\alpha^4}+\frac{3}{20\alpha^3}\right)\frac{2\alpha}{1-\alpha} \\ &= \frac{2}{\alpha^5} + \frac{1+\frac{3}{10}\alpha}{(1-\alpha)\alpha^3} < \frac{2}{\alpha^5} + \frac{2}{(1-\alpha)\alpha^3}. \end{aligned}$$

This contradicts our assumption $n > \frac{2}{\alpha^5} + \frac{2}{(1-\alpha)\alpha^3}$ and thus it follows that $n$ must be larger than $4\Delta^{3/2}$.

We have now shown that $n > 4\Delta^{3/2}$ indeed holds and using this in the inequality from Remark 4.19 with $t = \Delta$, we find

$$\frac{1-\alpha}{2\alpha} \geq \sqrt{\Delta} - \frac{2\Delta(\Delta+1)}{n} \geq \sqrt{\Delta} - \frac{2\Delta^2+2\Delta}{4\Delta^{3/2}} = \frac{1}{2}\sqrt{\Delta} - \frac{1}{\sqrt{\Delta}} \geq \frac{1}{2}\left(\sqrt{\Delta}-1\right).$$

In the last inequality above we use $\Delta \geq 1$. If this would not be the case, $G$ would be the empty graph, which by Proposition 3.3 corresponds to a set of $d$ equiangular lines in $\mathbb{R}^d$ and thus trivially satisfies the required result. By rearranging the above inequality it follows that $\Delta \leq 1/\alpha^2$.

With this new upper bound on $\Delta$ we will again apply inequality (4.11)with $t = \Delta$, now using our assumption $n > \frac{2}{\alpha^5} + \frac{2}{\alpha^3(1-\alpha)} > n > \frac{2}{\alpha^5} + \frac{2}{\alpha^3}$ as lower bound on $n$. This gives

$$\frac{1-\alpha}{2\alpha} \geq \sqrt{\Delta} - \frac{2\Delta(\Delta+1)}{n} \geq \sqrt{\Delta} - \frac{\frac{2}{\alpha^4}+\frac{2}{\alpha^2}}{\frac{2}{\alpha^5}+\frac{2}{\alpha^3}} = \sqrt{\Delta} - \alpha,$$

which implies $\Delta \leq \left(\frac{1-\alpha}{2\alpha} - \alpha\right)^2$. Finally, using this upper bound on $\Delta$ and the fact that $D(G) \leq \Delta$ we use Lemma 4.15 to conclude

$$\begin{aligned} n \leq \left(1+\left(\frac{2\alpha}{1-\alpha}\right)^2 D\right)(d+1) &\leq \left(1+\left(\frac{2\alpha}{1-\alpha}\right)^2\left(\frac{1-\alpha}{2\alpha}-\alpha\right)^2\right)(d+1) \\ &= \left(1+\left(1+\frac{2\alpha^2}{1-\alpha}\right)^2\right)(d+1) \\ &\leq \left(2+\frac{8\alpha^2}{(1-\alpha)^2}\right)(d+1). \end{aligned}$$

□

# 5

# Tight bound for high dimensions

As seen in the previous chapter, the number of equiangular lines in $\mathbb{R}^d$ with a fixed angle $\arccos \alpha$ grows linearly in the dimension. Due to Theorem 3.4, from early on in the study of equiangular lines there has been a special interest in the case where $1/\alpha$ is an odd integer. For $\alpha = 1/3$ the maximum number of lines $N_{1/3}(d)$ was already completely determined by Lemmens and Seidel in [4], who proved that $N_{\frac{1}{3}}(d) = 2d - 2$ for all $d \geq 15$. They also conjectured that $N_{1/5}(d) = \left\lfloor \frac{3}{2}(d-1) \right\rfloor$ for $d \geq 185$, which was first shown to be true for large enough $d$ by Neumaier in [8] and was completely solved only many years later in [26].

It was shown by Balla, Dräxler, Keevash and Sudakov in [10] that $N_\alpha(d)$ reaches its maximum at $\alpha = 1/3$ and that it can be at most $1.93d$ for all other values of $\alpha$. Building onto the ideas from this paper, Jiang and Polyanskii further improved these results in [11]. They showed that $N_{1/3}(d) = 2d + \mathcal{O}(1)$, $N_{1/5}(d) = \frac{3}{2}d + \mathcal{O}(1)$ as was already known and the new result $N_{1/(1+2\sqrt{2})} = \frac{3}{2}d + \mathcal{O}(1)$. For all other values of $\alpha$ they tightened the bound from [10], showing that $N_\alpha(d) \leq 1.49d + \mathcal{O}(1)$.

For all further values of $\alpha$ for which $1/\alpha$ is an odd integer not much was known. The results for $\alpha = 1/3, 1/5$ suggest a certain pattern which led to the conjecture $N_{\frac{1}{2k-1}} = kd/(k-1) + \mathcal{O}(1)$, for all integer $k \geq 2$ and $d \to \infty$, stated both by Bukh in [9] and Balla, Dräxler, Keevash and Sudakov in [10]. The results from Jiang and Polyanskii also give further evidence supporting the conjecture. The conjecture is completely solved in [12], as is the problem of determining the maximum number of equiangular lines with a fixed angle in high dimensions. In this chapter we will discuss this result.

First we need to introduce the notion of the *spectral radius order*, which was first used in the study on equiangular lines by Jiang and Polyanskii [11].

**Definition 5.1.** The *spectral radius order*, $k(\lambda)$, of a real number $\lambda > 0$ is the smallest integer $k$ such that there exists a $k$-vertex graph with spectral radius $\lambda$. If no such graph exists, $k(\lambda) = \infty$.

*Example* 5.2. Consider the two connected graphs on three vertices, the star graph $K_{1,2}$ and the complete graph $K_3$, shown in Figure 5.1. These two graphs have spectral radius $\sqrt{2}$ and 2 respectively. There are

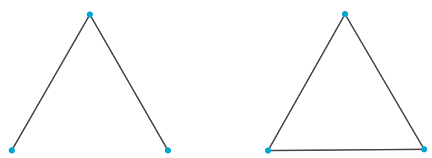

**Figure 5.1:** The connected graphs on three vertices

no smaller graphs with these values as spectral radius and so we have $k\left(\sqrt{2}\right) = k(2) = 3$.

*Remark* 5.3. The example shows that the spectral radius order of the integer 2 is $k(2) = 3$, which is satisfied by the complete graph. In general, for any integer $n$ the smallest graph with spectral radius $n$ is the complete graph on $n+1$ vertices and so for $\lambda = n$ the spectral radius order satisfies $k(\lambda) = n + 1$.

If the spectral radius order $k(\lambda)$ of any real number $\lambda$ exists, then $\lambda$ is an eigenvalue of a symmetric matrix. It thus follows from Lemma 2.7 that $\lambda$ must be a totally real algebraic integer. Furthermore, since $\lambda$ is the largest eigenvalue of a non-negative matrix, all its conjugates must in absolute value be smaller or equal to $\lambda$ itself. This means that the spectral radius order can not exist for any number that is not a totally real algebraic integer or for any totally real algebraic integer that is not largest amongst its conjugates. As an example of this last case, let $\lambda = 2\sqrt{3} - 1$. The minimal polynomial of $\lambda$ is $x^2 + 2x - 11$, which also has $-2\sqrt{3} - 1$ as a root. This root is larger in absolute value than $\lambda$ and hence $k(\lambda) = \infty$. Note that the condition of $\lambda$ being a totally real algebraic integer that is largest amongst its conjugates is only necessary and not sufficient. This means that such numbers can exist which do not have a spectral radius order.

The theorem that solves the problem of finding $N_\alpha(d)$ for high enough dimensions uses the spectral radius order of $(1-\alpha)/2\alpha$.

**Theorem 5.4** (Jiang, Tidor, Yao, Zhang and Zhao [12])**.** *Fix $\alpha \in (0,1)$. Let $\lambda = \frac{1-\alpha}{2\alpha}$ and $k = k(\lambda)$ be its spectral radius order. The maximum number $N_\alpha(d)$ of equiangular lines in $\mathbb{R}^d$ with common angle* $\arccos \alpha$ *satisfies*

(a) $N_\alpha(d) = \left\lfloor \frac{k(d-1)}{k-1} \right\rfloor$ *for all sufficiently large $d > d_0(\alpha)$ if $k < \infty$,*

(b) $N_\alpha(d) = d + o(d)$ *as $d \to \infty$ if $k = \infty$.*

As seen in Remark 5.3, the spectral radius order of any integer $n$ is $k(n) = n+1$. So, if we take $\alpha = 1/(2k-1)$ for any integer $k \geq 2$, we have $\lambda = k-1$ and hence $k(\lambda) = k$. This leads to the following corollary of the theorem, which extends the known results of $\alpha = 1/3$ and $\alpha = 1/5$ to any odd integer $1/\alpha$.

**Corollary 5.5.** *For every fixed integer $k \geq 2$ and all sufficiently large $d > d_0(k)$,*

$$N_{1/(2k-1)}(d) = \left\lfloor \frac{k(d-1)}{k-1} \right\rfloor.$$

To prove the theorem we will rewrite the problem of finding the maximum number of equiangular lines in $\mathbb{R}^d$ with common angle $\arccos \alpha$ in a slightly different way. Let $\mathcal{L}$ be a set of $n$ equiangular lines with common angle $\arccos \alpha$ and let $\mathcal{C}$ be an $\{-\alpha, \alpha\}$-code representing $\mathcal{L}$ with Gram matrix $M = M_{\mathcal{C}}$. Recall from Section 4.3 that we can associate a graph $G = G_{\mathcal{C}}$ to the spherical code $\mathcal{C}$ with vertex set $\mathcal{C}$ and an edge between two vertices $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{C}$ if $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = -\alpha$. By Equation 4.5 the Gram matrix can be written as $M = (1-\alpha)I + \alpha(J - 2A_G)$, where $A_G$ is the adjacency matrix of $G$. Furthermore, $M$ is a positive semidefinite matrix with rank at most $d$. Now let $\lambda$ be as in the theorem. Then dividing the Gram matrix by $2\alpha$ we find the matrix

$$N = \frac{1}{2\alpha} M = \lambda I - A_G + \frac{1}{2} J, \tag{5.1}$$

which is also positive semidefinite with rank at most $d$.

Conversely, if there is an $n$-vertex graph $G$ such that the matrix $N$ is positive semidefinite of rank at most $d$, then we can find vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ in $\mathbb{R}^d$ such that $(2\alpha N)_{ij} = M_{ij} = \langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle$, where again we let $\lambda = \frac{1-\alpha}{2\alpha}$. These vectors then correspond to a configuration of equiangular lines with common angle $\arccos \alpha$. From this we conclude that there is set of $n$ equiangular lines in $\mathbb{R}^d$ with common angle $\arccos \alpha$ if and only if there exists an $n$-vertex graph $G$ such that the matrix $N$ is positive semidefinite of rank at most $d$.

## 5.1 Lower bounds

In this section we establish the lower bounds of Theorem 5.4. In particular, we will show that

(a) $N_\alpha(d) \geq \left\lfloor \frac{k(d-1)}{k-1} \right\rfloor$ if $k < \infty$, and

(b) $N_\alpha(d) \geq d$ if $k = \infty$, where $\alpha \in (0,1)$, $\lambda = (1-\alpha)/2\alpha$ and $k = k(\lambda)$ is the spectral radius order of $\lambda$. The lower bounds actually hold for all dimensions $d$ as we will see.

Notice that we have already seen in Proposition 3.3 that $N_\alpha(d) \geq d$. This in particular also holds when the spectral radius order of $\lambda$ equals $k(\lambda) = \infty$, so this proves the lower bound of part (b) of Theorem 5.4.

For $k(\lambda) < \infty$, the following proposition gives a construction of a set of $\left\lfloor \frac{k(d-1)}{k-1} \right\rfloor$ equiangular lines. This lower bound has also been shown using a different approach by Jiang and Polyanskii in [11]. They relate a spherical $\{-\alpha,\alpha\}$-code to a $\{-1/\lambda,0\}$-code and then construct the required $\{-1/\lambda,0\}$-code. In the proof we give here, a graph is constructed on the required amount of vertices such that the matrix $N$ from Equation 5.1 is positive semidefinite of rank at most $d$.

**Proposition 5.6.** *Let $\alpha \in (0,1)$, $\lambda = \frac{1-\alpha}{2\alpha}$ and $d$ be a positive integer. If the spectral radius order $k = k(\lambda)$ of $\lambda$ satisfies $k < \infty$, then $N_\alpha(d) \geq \left\lfloor \frac{k(d-1)}{k-1} \right\rfloor$.*

*Proof.* To prove the lower bound we will construct a graph on $\left\lfloor \frac{k(d-1)}{k-1} \right\rfloor$ vertices such that the matrix $N$ is positive semidefinite of rank at most $d$.

Suppose $k = k(\lambda) < \infty$. Let $H$ be a $k$-vertex graph with spectral radius $\lambda_1(H) = \lambda$. Construct the graph $G$ by taking the disjoint union of $\left\lfloor \frac{d-1}{k-1} \right\rfloor$ copies of $H$ and adding $(d-1)-(k-1)\left\lfloor \frac{d-1}{k-1} \right\rfloor$ isolated vertices. The graph $G$ then has exactly $\left\lfloor \frac{k(d-1)}{k-1} \right\rfloor$ vertices.

The spectrum of $G$ is the union of the spectrum of its components (see Lemma 2.18). The addition of isolated vertices only adds zeroes to the spectrum. So, the spectral radius of $G$ is $\lambda_1(G) = \lambda$ with multiplicity $\left\lfloor \frac{d-1}{k-1} \right\rfloor$. The matrix $\lambda I - A_G$ thus has eigenvalue 0 with the same multiplicity. Since, $\lambda I - A_G$ is symmetric this means that $\dim \operatorname{Ker}(\lambda I - A_G) = \left\lfloor \frac{d-1}{k-1} \right\rfloor$. By the rank-nullity theorem we then find

$$\operatorname{rk}(\lambda I - A_G) = \left\lfloor \frac{k(d-1)}{k-1} \right\rfloor - \left\lfloor \frac{d-1}{k-1} \right\rfloor = d-1.$$

Since $J$ has rank 1 it follows by the subadditivity of the rank that

$$\operatorname{rk} N = \operatorname{rk}(\lambda I - A_G + \frac{1}{2}J) \leq d - 1 + 1 = d.$$

Furthermore, all other eigenvalues of $G$ are smaller than $\lambda$ and so we have $\lambda - \lambda_j(G) \geq 0$ for all $j > 1$. This implies that all eigenvalues $\lambda I - A_G$ are non-negative, so $\lambda I - A_G$ is positive semidefinite. The matrix $J$ is also positive semidefinite and so we conclude that $N$ is positive semidefinite.

We have thus found a matrix $G$ on $\left\lfloor \frac{k(d-1)}{k-1} \right\rfloor$ such that $N$ is positive semidefinite and has rank at most $d$, proving the proposition. $\square$

## 5.2 Upper bounds

Proving the upper bound of Theorem 5.4 requires more work than the proofs of the lower bounds. We start with a proposition that proves the upper bound when $\lambda = (1-\alpha)/2\alpha$, where $\arccos\alpha$ is the common angle of set of equiangular lines, is not a totally real algebraic integer.

**Proposition 5.7** ([11])**.** *Let $\lambda = \frac{1-\alpha}{2\alpha}$, where $\alpha \in (0,1)$. If $\lambda$ is not a totally real algebraic integer, then $N_\alpha(d) \leq d+1$.*

*Proof.* Let $\mathcal{C}$ be a $\{-\alpha,\alpha\}$-code in $\mathbb{R}^d$ with Gram matrix $M = M_{\mathcal{C}}$ and associated graph $G = G_{\mathcal{C}}$ with adjacency matrix $A = A_G$. The Gram matrix can be rewritten from Equation 4.5 as $M = (1-\alpha)\left(I - \frac{1}{\lambda}A\right) + \alpha J$ and has rank at most $d$. Since $\lambda$ is not a totally real algebraic integer, it can not be an eigenvalue of $A$ by Lemma 2.7. This implies that the matrix $I - \frac{1}{\lambda}A$ does not have 0 as an eigenvalue and is thus a full rank matrix. So we find

$$d \geq \operatorname{rk}(M) \geq \operatorname{rk}\left(I - \frac{1}{\lambda}A\right) - \operatorname{rk}(J) = |C| - 1,$$

which gives $N_\alpha(d) \leq d+1$ as desired. $\square$

*Remark* 5.8. Note that the proof of the proposition actually shows that if $\lambda$ is not an eigenvalue of the associated graph, then $N_\alpha(d) \leq d+1$.

By Lemma 2.7 we know that if $\lambda$ is not a totally real algebraic integer, then it can not be the eigenvalue of a graph. So, in this case the spectral radius order satisfies $k(\lambda) = \infty$. Hence, the above proposition already proves the upper bound in part (b) of Theorem 5.4 for specific values of $\alpha$. Note that the spectral radius order can also be infinite for certain numbers that are totally real algebraic integers. This case is not covered by the proposition, thus it does not yet prove part (b) of the theorem entirely.

To prove Theorem 5.4 we will use the bound on the second eigenvalue of a bounded degree graph from Theorem 2.20. In order to be able to apply this theorem we need the associated graph $G$ to have a bounded maximum degree. In the previous chapter we have seen that a spherical code $\mathcal{C}$ representing a set of equiangular lines can be chosen such that the associated graph has maximum degree at most $\frac{1}{4\alpha^4} + \frac{1}{\alpha^4}$. The original proof of Theorem 5.4 uses an upper bound of much larger order which is deduced using an argument based on Ramsey's theorem. We give a quick sketch of this proof here and refer to Appendix B for the full proof.

Let $G$ be the associated graph of a spherical $\{-\alpha, \alpha\}$-code with $n$ elements. Using Ramsey's theorem a large independent set $\mathcal{I}$ can be found in $G$ if $n$ is large enough. Next, a switching operation can be applied to make sure that no vertex is adjacent to more than half of the vertices in this independent set. We can then analyse how the vertices outside of the independent set attach to $\mathcal{I}$. First of all, one can show that for a subset $X$ of $\mathcal{I}$, the set of all neighbours of $X$ has a bounded amount of vertices depending only on $\alpha$. This implies that the set of all neighbours of the $\mathcal{I}$ is bounded and thus the degree of any vertex in $\mathcal{I}$ is also bounded . Furthermore, it can be shown that the set of all non-neighbours of $\mathcal{I}$ has bounded degree. Combining this with the bound on the number of elements in the set of all neighbours of $\mathcal{I}$, gives an upper bound on the degree of any vertex not in $\mathcal{I}$. Hence it follows that the degree of any vertex in the graph is upper bounded.

Before turning to the complete proof of Theorem 5.4 we discuss one last lemma to make the proof easier to read.

**Lemma 5.9.** *Fix $\alpha \in (0,1)$ and let $\lambda = \frac{1-\alpha}{2\alpha}$ Let $G$ be the associated graph of a spherical $\{-\alpha, \alpha\}$-code with spectral radius $\lambda_1(G) > \lambda$. Denote the connected components of $G$ by $C_1, \dots, C_t$ such that $\lambda_1(G) = \lambda_1(C_1)$. Then for all $i > 1$, $\lambda_1(C_i) < \lambda$.*

*Proof.* Take $i > 1$ arbitrarily. By definition both $C_1$ and $C_i$ are connected. By Perron-Frobenius, both graphs have an eigenvector corresponding to the spectral radius with non-negative entries. Let $\boldsymbol{u}'$ denote the eigenvector corresponding to $\lambda_1(C_1)$ with non-negative entries and let $\boldsymbol{v}'$ be the eigenvector corresponding to $\lambda_1(C_i)$ with non-negative entries. Now let $\boldsymbol{u}$ be the vector that is zero on all vertices not in $C_1$ and $\boldsymbol{u}'$ on the vertices of $C_1$. Similarly, let $\boldsymbol{v}$ be the vector that is zero on all vertices not in $C_i$ and $\boldsymbol{v}'$ on the vertices of $C_i$. Then $\mathbf{1}^\top \boldsymbol{u} = \mathbf{1}^\top \boldsymbol{u}' > 0$ and $\mathbf{1}^\top \boldsymbol{v} = \mathbf{1}^\top \boldsymbol{v}' > 0$. Choose a $c \neq 0$ such that the vector $\boldsymbol{w} = \boldsymbol{u} - c\boldsymbol{v}$ satisfies $\mathbf{1}^\top \boldsymbol{w} = 0$. Using the fact that $\lambda I - A_G + \frac{1}{2}J$ is positive semidefinite and that $J\boldsymbol{w} = 0$ we find

$$0 \leq \boldsymbol{w}^\top \left( \lambda I - A_G + \frac{1}{2} J \right) \boldsymbol{w} = \boldsymbol{w}^\top (\lambda I - A_G) \boldsymbol{w},$$

which implies

$$\boldsymbol{w}^\top A_G \boldsymbol{w} \leq \boldsymbol{w}^\top \lambda I \boldsymbol{w}. \tag{5.2}$$

Now we substitute $\boldsymbol{w}$ by $\boldsymbol{u} - c\boldsymbol{v}$ to expand both the left and right hand side of the equation. Note that $\boldsymbol{u}^\top \boldsymbol{v} = 0$. In this way the left hand side becomes

$$(\boldsymbol{u} - c\boldsymbol{v})^\top A_G (\boldsymbol{u} - c\boldsymbol{v}) = \boldsymbol{u}^\top A_G \boldsymbol{u} + c^2 \boldsymbol{v}^\top A_G \boldsymbol{v} = \lambda_1(C_1) \boldsymbol{u}^\top \boldsymbol{u} + c^2 \lambda_1(C_i) \boldsymbol{v}^\top \boldsymbol{v}.$$

Next the right hand side of Equation 5.2 is expanded to find

$$\lambda \boldsymbol{w}^\top \boldsymbol{w} = \lambda (\boldsymbol{u} - c\boldsymbol{v})^\top (\boldsymbol{u} - c\boldsymbol{v}) = \lambda \boldsymbol{u}^\top \boldsymbol{u} + \lambda c^2 \boldsymbol{v}^\top \boldsymbol{v}.$$

So Equation 5.2 now reads

$$\lambda_1(C_1) \boldsymbol{u}^\top \boldsymbol{u} + c^2 \lambda_1(C_i) \boldsymbol{v}^\top \boldsymbol{v} \leq \lambda \boldsymbol{u}^\top \boldsymbol{u} + \lambda c^2 \boldsymbol{v}^\top \boldsymbol{v}.$$

By assumption $\lambda_1(C_1) > \lambda$. For the inequality to hold, we thus must have $\lambda_1(C_i) < \lambda$. Since $i$ was chosen arbitrary, this holds for all $i > 1$, hence proving the lemma. $\square$

We now turn to the full proof of Theorem 5.4. Since we have already established the lower bounds, we will now show the upper bounds. In order to do this we will start with an arbitrary set of $n$ equiangular lines in $\mathbb{R}^d$ with common angle $\arccos \alpha$. Let $G$ be a graph associated to a spherical code representing this set and let $\lambda = (1-\alpha)/2\alpha$. We will bound the number of lines $n$ using the rank-nullity theorem. Since

$$\begin{aligned} n &= \operatorname{rk}\left(\lambda I - A_G + \frac{1}{2}J\right) + \dim \operatorname{Ker}\left(\lambda I - A_G + \frac{1}{2}J\right) \\ &= \operatorname{rk}(\lambda I - A_G) + \dim \operatorname{Ker}(\lambda I - A_G), \end{aligned}$$

finding an upper bound on both the rank and the nullity of the matrix $\lambda I - A_G$, will give an upper bound on $n$. Upper bounding these values will be done through a case distinction depending on the relation of $\lambda$ to the associated graph $G$. When $\lambda$ is not an eigenvalue of $G$, we have already shown in Proposition 5.7 that Theorem 5.4 indeed holds. If $\lambda$ is the spectral radius of $G$, $\lambda I - A_G$ is positive semidefinite and we will use Lemma 2.14 to bound its rank. The dimension of the kernel will be bounded by bounding the number of components of $A_G$ which have spectral radius $\lambda$. The last case will be when $\lambda$ is an eigenvalue of the graph but not its spectral radius. The rank of $\lambda I - A_G$ will then simply be bounded using the subadditivity of the rank. In bounding the nullity we will use Lemma 5.9 to show that $\lambda$ is the second eigenvalue of the first component of the graph. It will then follow that the nullity of $\lambda I - A_G$ equals the nullity of $\lambda I - A_{C_1}$. Using the upper bound on the multiplicity of the second eigenvalue of $C_1$ from Theorem 2.20 will give an upper bound on the nullity of $\lambda I - A_G$.

*Proof of Theorem 5.4.* Fix $\alpha \in (0,1)$ and let $\lambda = \frac{1-\alpha}{2\alpha}$ with spectral radius order $k = k(\lambda)$. The lower bounds of the theorem follow from Propositions 3.3 and 5.6. So, it remains to show that

(a) $N_\alpha(d) \leq \left\lfloor \frac{k(d-1)}{k-1} \right\rfloor$ for all sufficiently large $d > d_0(\alpha)$ if $k < \infty$, and
(b) $N_\alpha(d) \leq d + o(d)$ as $d \to \infty$ if $k = \infty$.

To show the upper bounds, consider a set of $n$ equiangular lines in $\mathbb{R}^d$ with common angle $\arccos \alpha$. By Theorem 4.14 there is a spherical $\{-\alpha, \alpha\}$-code $\mathcal{C}$ representing this set of equiangular lines such that the associated graph $G$ has maximum degree at most $\Delta := \frac{1}{4\alpha^4} + \frac{1}{\alpha^4}$. Note that since our graph $G$ comes from a set of equiangular lines, the matrix $N = \lambda I - A_G + \frac{1}{2}J$ is positive semidefinite and has rank at most $d$.

In this proof we will distinguish between three different cases:

1. $\lambda \notin \sigma(G)$,
2. $\lambda_1(G) = \lambda$, and
3. $\lambda_1(G) > \lambda, \lambda \in \sigma(G)$.

We will show that in all three cases the theorem holds. Notice that the conclusion from the first case in particular holds when $k = \infty$ and thus we will show that in this case $N_\alpha(d)$ is upper bounded by $d + o(d)$. The second case implies that $k < \infty$ by definition, which means we will show that $N_\alpha(d) \leq \left\lfloor \frac{k(d-1)}{k-1} \right\rfloor$. The last case holds both when $k < \infty$ and $k = \infty$. We will distinguish between these two at the end of the proof and show that the theorem also holds in this case.

For the first case, suppose that $\lambda \notin \sigma(G)$. Then it immediately follows from Remark 5.8 that $N_\alpha(d) \leq d + 1$.

For the next two cases, let $C_1, \ldots, C_t$ denote the connected components of $G$ in such a way that $\lambda_1(G) = \lambda_1(C_1)$. Recall that $n = \operatorname{rk}(\lambda I - A_G) + \dim \operatorname{Ker}(\lambda I - A_G)$, by the rank-nullity theorem. In both cases the theorem will be shown by upper bounding both the rank and nullity of $\lambda I - A_G$.

Now suppose that $\lambda_1(G) = \lambda$. To show that $N_\alpha(d) \leq \left\lfloor \frac{k(d-1)}{k-1} \right\rfloor$, we first bound the rank of $\lambda I - A_G$. Since all eigenvalues of $A_G$ are smaller or equal to $\lambda$, the matrix $\lambda I - A_G$ has only non-negative eigenvalues and is thus positive semidefinite. The matrix $J$ is also positive semidefinite, so by Lemma 2.14

$$\operatorname{Ker} N = \operatorname{Ker}(\lambda I - A_G) \cap \operatorname{Ker}\left(\frac{1}{2}J\right).$$

The adjacency matrix of $G$ has non-negative entries. It follows by Perron-Frobenius that $\lambda_1(G)$ has a corresponding eigenvector $\boldsymbol{v}_1$ with all non-negative entries. This vector lies in the kernel of $\lambda I - A_G$, but is not an element of the kernel of $\frac{1}{2}J$. This implies that the intersection of the kernels of the two graphs is at least one element smaller than the kernel of $\lambda I - A_G$, so

$$\dim \operatorname{Ker} N \leq \dim \operatorname{Ker}(\lambda I - A_G) - 1.$$

The rank-nullity theorem now yields

$$\begin{aligned}\operatorname{rk}(\lambda I - A_G) + \dim \operatorname{Ker}(\lambda I - A_G) &= \operatorname{rk} N + \dim \operatorname{Ker} N \\ &\leq \operatorname{rk} N + \dim \operatorname{Ker}(\lambda I - A_G) - 1,\end{aligned}$$

so $\operatorname{rk}(\lambda I - A_G) \leq \operatorname{rk} N - 1 \leq d - 1$.

Next, we bound the nullity of $\lambda I - A_G$. Together with the bound on the rank, this will give us an upper bound on $n$. Let $C_1, \ldots, \mathcal{C}_j$ be the components of $G$ with spectral radius $\lambda$. Then each of these components has at least $k$ elements by the definition of $k$. This implies that $n \geq k \cdot j$. Furthermore, since each component is connected, the multiplicity of $\lambda$ as an eigenvalue of $C_i$ is 1 for each $i \in [j]$ by Perron-Frobenius. It follows that the multiplicity of $\lambda$ in $G$ is equal to $j$ and hence $\dim \operatorname{Ker}(\lambda I - A_G) = j$. We thus need to bound $j$ to bound the nullity of $\lambda I - A_G$. By the rank-nullity theorem we find

$$\operatorname{rk}(\lambda I - A_G) = N - \dim \operatorname{Ker}(\lambda I - A_G) \geq kj - j = (k-1)j.$$

Combining this lower bound of the rank of $\lambda I - A_G$ with the upper bound on the rank found above we get $(k-1)j \leq d-1$ and so $j \leq \frac{d-1}{k-1}$.

Combining the upper bound on the rank and the upper bound on the nullity of $\lambda I - A_G$ we find

$$n = \operatorname{rk}(\lambda I - A_G) + \dim \operatorname{Ker}(\lambda I - A_G) \leq d - 1 + \frac{d-1}{k-1} = \frac{k(d-1)}{k-1}$$

as desired. This proves the theorem for the case where $\lambda_1(G) = \lambda$.

It is now left to prove the theorem when $\lambda_1(G) \geq \lambda$ and $\lambda$ is an eigenvalue of $G$. In this case the spectral radius order $k(\lambda)$ of $\lambda$ can be both smaller or equal to infinity. We will show that the theorem holds in both cases. Again, we first give an upper bound on the rank of $\lambda I - A_G$, which satisfies

$$\operatorname{rk}(\lambda I - A_G) \leq \operatorname{rk}\left(\lambda I - A_G + \frac{1}{2}J\right) + 1 \leq d + 1.$$

The next step is to bound the nullity of $\lambda I - A_G$. We will do this by showing that $\dim \operatorname{Ker}(\lambda I - A_G) = \dim \operatorname{Ker}(\lambda I - A_{C_1})$. Then the nullity of $\lambda I - A_G$ equals the multiplicity of $\lambda$ in $C_1$.

First of all note that $\lambda I - A_G$ has at most one negative eigenvalue. Indeed, since $N \succeq 0$ and $N = (\lambda I - A_G) + \frac{1}{2}$ it follows from Lemma 2.12 that $\lambda I - A_G$ has no more than one negative eigenvalue. This implies that the second eigenvalue of $G$ cannot be larger than $\lambda$. Furthermore, from Lemma 5.9 we know that $\lambda_1(C_i) < \lambda$ for all $i > 1$. Since we assumed that $\lambda$ is indeed an eigenvalue of $G$ it follows that we must have $\lambda_2(G) = \lambda_2(C_1) = \lambda$.

From $\lambda_1(C_i) < \lambda$ it also follows that $\lambda \notin \sigma(C_i)$ and hence $\lambda I - A_{C_i}$ has trivial kernel for all $i > 1$. So, we conclude that the nullity of $\lambda I - A_G$ is indeed equal to the nullity of $\lambda I - A_{C_1}$ which in turn is equal to the multiplicity of $\lambda$ in $C_1$. Since $C_1$ is a connected graph with maximum degree $\Delta$ by Theorem 2.20 the multiplicity of $\lambda$ is $\mathcal{O}_\Delta\left(\frac{|C_1|}{\log\log|C_1|}\right)$. Hence,

$$\dim \operatorname{Ker}(\lambda I - A_G) = \mathcal{O}_\Delta\left(\frac{|C_1|}{\log\log|C_1|}\right) = \mathcal{O}_\Delta\left(\frac{n}{\log\log n}\right).$$

Combining the upper bound on the rank and the upper bound on the nullity of $\lambda I - A_G$ we find

$$n = \operatorname{rk}(\lambda I - A_G) + \dim \operatorname{Ker}(\lambda I - A_G) \leq d + 1 + \mathcal{O}_\Delta\left(\frac{n}{\log\log n}\right).$$

This implies (see Appendix A)

$$n \leq \mathcal{O}_\Delta\left(\frac{d}{\log\log d}\right) + d = d + o(d).$$

If $k < \infty$, then this is smaller than $\left\lfloor \frac{k(d-1)}{k-1} \right\rfloor$ for large enough $d$.

We have now shown that the theorem holds in all three cases, thus concluding the proof. $\square$

From the proof we see that if a spherical code $\mathcal{C}$ representing the set of equiangular lines exist such that its associated graph has spectral radius $\lambda$, then the first part of the theorem holds for all dimensions. Only when no such graph exists, the results hold for large dimensions only. In this case, it is not yet known how large $d$ needs to be exactly for the theorem to hold.

If $k = \infty$, the theorem does not yet give an exact value for $N_\alpha(d)$. From Proposition 5.7 we know that if $\lambda$ is not a totally real algebraic integer, then $N_\alpha(d) \leq d+1$, which together with Proposition 3.3 gives $d \leq N_\alpha(d) \leq d+1$. A result by Jiang and Polyanskii [11] shows that if $\lambda$ is a totally real algebraic integer that is not the largest among its conjugates, then $d \leq N_\alpha(d) \leq d+2$. These two results bring us a lot closer to an exact solution for $N_\alpha(d)$ in these specific cases.

What happens when $\lambda$ is a totally real algebraic integer which is largest among its conjugates that has infinite spectral radius order, is however not known. The above two results suggest a conjecture which states that $N_\alpha(d) = d + \mathcal{O}(1)$ if $k(\lambda) = \infty$. However, only very recently Schildkraut disproves this by showing that an infinite amount of $\alpha \in (0,1)$ exist such that $d+\Omega(\log\log d) \leq N_\alpha(d) \leq d+\mathcal{O}(d/\log\log d)$ [21].

# 6
# Conclusion

The last decade has seen remarkable progress in the study on equiangular lines. In this thesis we have given an overview of the current state of research on equiangular lines including some classic results and the most significant contributions made in recent years. Even after all this progress, there still are many open problems left. In this chapter we give some concluding remarks and discuss possible topics for further research in this field.

In Chapter 3 we introduced equiangular lines through some classic results, including the absolute bound. The proof of this bound gives conditions that have to be satisfied for a construction that reaches the bound. This significantly limits the number of dimensions that can possibly reach the absolute bound. However, de Caen has shown that for an infinite number of dimensions a construction of lines on the order of $d^2$ exists [24]. No other constructions are known up to now of this order. It could be of interest to investigate what happens if we start by assuming $\Omega(d^2)$ equiangular. Can some conditions be deduced on this construction that must be satisfied?

A crucial result needed to find the linear upper bound from Balla [13] in Chapter 4 is Lemma 4.15, which gives a bound on the number $N_\alpha(d)$ in terms of the average degree of the associated graph. The upper bounds on $N_\alpha(d)$ are deduced from this lemma by using upper bounds on the maximum degree of the associated graph, since a bound on the maximum degree also gives a bound on the average degree. This does however motivate the question if improvements could be made by bounding the average degree of the associated graph. Is it possible to find upper bounds on the average degree instead of the maximum degree of the associated graph? To find such a bound, one will need to construct an associated graph in such way that the high and low degree vertices balance each other out.

We concluded Chapter 2 with a theorem bounding the multiplicity of the $j$-th eigenvalue of a connected bounded degree graph. This bound plays a key role in the proof of Theorem 5.4 which gives strict bounds for the number of equiangular lines with a fixed angle in high dimensions. Following these results, more research has been done into the multiplicity of the second eigenvalue of a graph. In the context of equiangular lines we are this is the multiplicity we are most interested in. Currently, the upper bound of this multiplicity has been improved to $\mathcal{O}\left(n/\log^{1/5-o(1)} n\right)$ in [19] and a construction is known of graphs with second eigenvalue multiplicity of $\Omega\left(n^{1/2-o(1)}\right)$ [20]. The gap between these two bounds is still very big.

The bound on the multiplicity of the second eigenvalue is connected to the dimension for which part (a) of Theorem 5.4 holds. While this theorem essentially solves the problem of finding the maximum number of equiangular lines with a fixed angle, it is still an open question as to the exact value of the dimension for which their results hold. The uncertainty concerning this value arises from the bound on the multiplicity of the second eigenvalue. Improvements on this bound can thus lead to stricter results and more knowledge on the dimension $d$ for which the results hold. A first step could be to analyse what happens when using the bound $\mathcal{O}\left(n/\log^{1/5-o(1)} n\right)$ from [19].

Another important tool in the results of [12] is the spectral radius order. This parameter was first introduced by Jiang and Polyanskii in [11]. Not much is yet known about this graph parameter, which is why we can not yet exactly classify the cases which fall into part (a) or (b) of Theorem 5.4. We know that if for any $\lambda$ the spectral radius order $k(\lambda)$ is not equal to infinity, then $\lambda$ must be a totally real algebraic integer which is largest amongst its conjugates. This condition is however only necessary and not sufficient. A number $\lambda$ can exist satisfying these conditions which has spectral radius order $k(\lambda) = \infty$. Can we find sufficient conditions for any number to be the spectral radius of some graph? Or equivalently, can we find conditions for which a totally real algebraic integer that is largest amongst its conjugates has infinite spectral radius order?

Regarding lower bounds on the number of equiangular lines, there are probably still improvements possible. The recent result by Schildkraut [21] shows an unexpected behaviour of $N_\alpha(d)$ for many values of $\alpha$. It disproves a conjecture which stated that if $\lambda = (1-\alpha)/2\alpha$ has spectral radius order $k(\lambda) = \infty$, then $N_\alpha(d) = d + \mathcal{O}(1)$. This conjecture does hold for values of $\alpha$ for which $\lambda$ is not a totally real algebraic integer or $\lambda$ is a totally real algebraic integer that is not the largest amongst its conjugates. Shildkrauts result shows that the case $k(\lambda) = \infty$ is more complicated than expected and needs more case distinctions to come closer to an exact solution. For the specific values of $\alpha$ covered by Schildkraut the gap between the lower and upper bound is still very large and thus suggests room for improvement. Furthermore, we now know that for all values of $\lambda$ that aren't a totally real algebraic integer we have $d \leq N_\alpha(d) \leq d + 1$ (Proposition 5.7). For values of $\lambda$ that are a totally real algebraic integer that is not the largest amongst its conjugates we have $d \leq N_\alpha(d) \leq d + 2$ [11]. Lastly, there are an infinite number of $\alpha$ for which $d + \Omega(\log \log d) \leq N_\alpha(d) \leq d + \mathcal{O}(d/\log \log d)$. There still are many values of $\alpha$ for which $k(\lambda) = \infty$ which are not covered by any of these results and for which we thus do not know exactly how $N_\alpha(d)$ grows and if $d$ is the best lower bound or not.

# References


[1] J. Haantjes, "Equilateral point-sets in elliptic two- and three-dimensional spaces," *Nieuw. Arch. Wisk.*, vol. 22, pp. 355–362, 1948.

[2] J.H. van Lint and J.J. Seidel, "Equilateral point sets in elliptic geometry," *Proceedings of the Koninklijke Nederlandse Akademie van Wetenschappen: Series A: Mathematical Sciences*, vol. 69, no. 3, pp. 335–348, 1966.

[3] C. Godsil and G. Royle, *Algebraic Graph Theory*. New York, NY: Springer New York, 2001, vol. 207, ISBN: 978-0-387-95220-8. DOI: 10.1007/978-1-4613-0163-9.

[4] P. W. H. Lemmens and J. J. Seidel, "Equiangular Lines," *Journal of Algebra*, vol. 24, pp. 494–512, 1973.

[5] T. Strohmer and R. W. Heath, "Grassmannian frames with applications to coding and communication," *Applied and Computational Harmonic Analysis*, vol. 14, no. 3, pp. 257–275, May 2003, ISSN: 10635203. DOI: 10.1016/S1063-5203(03)00023-X.

[6] C. Fuchs, M. Hoang, and B. Stacey, "The SIC Question: History and State of Play," *Axioms*, vol. 6, no. 4, p. 21, Jul. 2017, ISSN: 2075-1680. DOI: 10.3390/axioms6030021.

[7] B. C. Stacey, *A First Course in the Sporadic SICs*. Cham: Springer International Publishing, 2021, vol. 41, ISBN: 978-3-030-76103-5. DOI: 10.1007/978-3-030-76104-2.

[8] A. Neumaier, "Graph representations, two-distance sets, and equiangular lines," *Linear Algebra and its Applications*, vol. 114-115, pp. 141–156, Mar. 1989, ISSN: 00243795. DOI: 10.1016/0024-3795(89)90456-4.

[9] B. Bukh, "Bounds on equiangular lines and on related spherical codes," Aug. 2015. DOI: 10.48550/arXiv.1508.00136.

[10] I. Balla, F. Dräxler, P. Keevash, and B. Sudakov, "Equiangular Lines and Spherical Codes in Euclidean Space," Jun. 2016. DOI: 10.48550/arXiv.1606.06620.

[11] Z. Jiang and A. Polyanskii, "Forbidden subgraphs for graphs of bounded spectral radius, with applications to equiangular lines," Aug. 2017. DOI: 10.1007/s11856-020-1983-2.

[12] Z. Jiang, J. Tidor, Y. Yao, S. Zhang, and Y. Zhao, "Equiangular lines with a fixed angle," *Annals of Mathematics*, vol. 194, no. 3, Feb. 2021. DOI: 10.4007/annals.2021.194.3.3.

[13] I. Balla, "Equiangular lines and regular graphs," Oct. 2021. DOI: 10.48550/arXiv.2110.15842.

[14] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge University Press, Dec. 1985, ISBN: 9780521386326. DOI: 10.1017/CBO9780511810817.

[15] D. R. Estes, "Eigenvalues of Symmetric Integer Matrices," *Journal of Number Theory*, vol. 42, pp. 292–296, 1992. DOI: 10.1016/0022-314X(92)90094-6.

[16] J. E. Gentle, *Numerical Linear Algebra for Applications in Statistics*. New York, NY: Springer New York, 1998, ISBN: 978-1-4612-6842-0. DOI: 10.1007/978-1-4612-0623-1.

[17] R. Diestel, *Graph Theory*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2017, vol. 173, ISBN: 978-3-662-53621-6. DOI: 10.1007/978-3-662-53622-3.

[18] A. E. Brouwer and W. H. Haemers, *Spectra of Graphs*. New York, NY: Springer New York, 2012, ISBN: 978-1-4614-1938-9. DOI: 10.1007/978-1-4614-1939-6.

[19] T. McKenzie, P. M. R. Rasmussen, and N. Srivastava, "Support of closed walks and second eigenvalue multiplicity of graphs," *Proceedings of the 53rd Annual ACM SIGACT Symposium on Theory of Computing*, pp. 396–407, Jun. 2021. DOI: 10.1145/3406325.3451129.

[20] M. Haiman, C. Schildkraut, S. Zhang, and Y. Zhao, "Graphs with high second eigenvalue multiplicity," *Bulletin of the London Mathematical Society*, vol. 54, no. 5, pp. 1630–1652, Oct. 2022, ISSN: 0024-6093. DOI: 10.1112/blms.12647.

[21] C. Schildkraut, "Equiangular lines and large multiplicity of fixed second eigenvalue," Feb. 2023. DOI: 10.48550/arXiv.2302.12230.

[22] W.-J. Kao and W.-H. Yu, "Four-point semidefinite bound for equiangular lines," Mar. 2022. DOI: 10.48550/arXiv.2203.05828.

[23] J. Goethals and J. Seidel, "The regular two-graph on 276 vertices," *Discrete Mathematics*, vol. 12, no. 2, pp. 143–158, 1975, ISSN: 0012365X. DOI: 10.1016/0012-365X(75)90029-1.

[24] D. De Caen, "Large Equiangular Sets of Lines in Euclidean Space," *The Electronic Journal of Combinatorics*, vol. 7, no. 1, Nov. 2000, ISSN: 1077-8926. DOI: 10.37236/1533.

[25] D. de Laat, W. de Muinck Keizer, and F. C. Machado, "The Lasserre hierarchy for equiangular lines with a fixed angle," Jan. 2023. DOI: 10.48550/arXiv.2211.16471.

[26] M.-Y. Cao, J. H, Koolen, Y.-C. R. Lin, and W.-H. Yu, "The Lemmens-Seidel conjecture and forbidden subgraphs," Mar. 2020. DOI: 10.48550/arXiv.2003.07511.

[27] S. Jukna, *Extremal Combinatorics*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2011, ISBN: 978-3-642-17363-9. DOI: 10.1007/978-3-642-17364-6.

# A

# Asymptotic notation

The bounds in this thesis make use of asymptotic notations. This notation is used to denote the asymptotic growth rate of a function. Let $f$ and $g$ be two functions such that $g(x)$ is positive for all large enough values of $x$. The different relevant asymptotic notations are defined as

- $f(x) = \mathcal{O}(g(x))$ if there exists a constant $C > 0$ and $x_0 \in \mathbb{N}$ such that $|f(x)| \leq Cg(x)$ for all $x \geq x_0$,
- $f(x) = \Omega(g(x))$ if $g(x) = \mathcal{O}(f(x))$ or in other words, there is a constant $c > 0$ and $x_0 \in N$ such that $|f(x)| \geq cg(x)$ for all $x \geq x_0$, and
- $f(x) = o(g(x))$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$.

The first two notations can also be defined using limits. With this notation we have $f(x) = \mathcal{O}(g(x))$ if $\limsup_{x\to\infty} \frac{f(x)}{g(x)} < \infty$, and $f(x) = \Omega(g(x))$ if $\liminf_{x\to\infty} \frac{f(x)}{g(x)} > 0$. The definitions slightly abuse the notation of the 'is equal to' sign,'='. The notation $f(x) = \mathcal{O}(g(x))$ means that $f$ grows like $g$, and so it is read as $f(x)$ 'is' $\mathcal{O}(g(x))$ and not $f(x)$ 'is equal to' $\mathcal{O}(g(x))$. This means that the '=' sign is only read one way. If $f(x) = \mathcal{O}(g(x)$ is true, $\mathcal{O}(g(x)) = f(x)$ is however *not* true.

Intuitively, $f(x) = \mathcal{O}(g(x))$ means that $f(x)$ grows at most as fast as $g(x)$. This means that in any function the term that grows the fastest will always dominate the others. If, for example, we let $f(x) = 4\log x + \sqrt{x} + 3x^2$, then $x^2$ is the fastest growing term and hence $f(x) = \mathcal{O}\left(x^2\right)$. The notation $f(x) = o(g(x))$ means that $g(x)$ grows a lot faster than $f(x)$. Any function $f(x)$ that is $o(g(x))$ is also $\mathcal{O}(g(x))$ but not the other way around.

At the end of the proof of Theorem 5.4 we claim that if $n \leq \mathcal{O}_\Delta\left(\frac{n}{\log\log n}\right) + d$ then it follows that $n \leq \mathcal{O}_\Delta\left(\frac{d}{\log\log d}\right) + d$ and furthermore that $\mathcal{O}_\Delta\left(\frac{d}{\log\log d}\right) + d = d + o(d)$. This last step is easily shown to hold, since

$$\limsup_{d\to\infty} \frac{\frac{d}{\log\log d}}{d} = \limsup_{d\to\infty} \frac{1}{\log\log d} = 0.$$

Proving the first step requires a bit more work.

**Lemma A.1.** *Let $\mathcal{L}$ be a set of $n$ equiangular lines in $\mathbb{R}^d$ and choose the associated graph such that it has maximum degree at most $\Delta$. If $n \leq \mathcal{O}_\Delta\left(\frac{n}{\log\log n}\right) + d$ then $n \leq \mathcal{O}_\Delta\left(\frac{d}{\log\log d}\right) + d$.*

*Proof.* By proposition 3.3 we have $d \leq n$. Suppose $n \leq C\frac{n}{\log\log n} + d$ for some constant $C$. Thus we have the following two inequalities

$$n \geq d \tag{A.1}$$

$$d \geq n - C\frac{n}{\log\log n}. \tag{A.2}$$

To prove the statement we show that there is a $\Gamma > C$ such that $n \leq \Gamma\frac{d}{\log\log d} + d$.

By A.2, $\Gamma\frac{d}{\log\log d}+d\geq\Gamma\frac{n-C\frac{n}{\log\log n}}{\log\log d}+n-C\frac{n}{\log\log n}$. We need to show that

$$\Gamma\frac{n-C\frac{n}{\log\log n}}{\log\log d}+n-C\frac{n}{\log\log n}-n\geq 0.$$

From inequality A.1 it follows that $\frac{1}{\log\log n}\leq\frac{1}{\log\log d}$. This gives

$$\begin{aligned}\Gamma\frac{n-C\frac{n}{\log\log n}}{\log\log d}-C\frac{n}{\log\log n}&\geq\Gamma\frac{n-C\frac{n}{\log\log n}}{\log\log n}-C\frac{n}{\log\log n}\\&\geq(\Gamma-C)\frac{n}{\log\log n}-C\frac{n}{\log\log n^2}\\&\geq 0,\end{aligned}$$

for $n$ large enough.

$\square$

# B

# Ramsey theory

Ramsey theory is a branch of the study of combinatorics that looks at the structures that arise in combinatorial objects as these become larger. The questions that arise in Ramsey theory usually ask how large or small some structure must be for a specific property to hold. For example, how many edges can a graph have such that it does not contain a cycle? Or, how many people can we invite to a party such that there is always a group of three people who all know each other or a group of three mutual strangers? This question is actually equivalent to the following graph-theoretic problem. If we colour the edges of any complete graph with two colours, how large can such a graph be without containing a monochromatic triangle? And of course, the problem this thesis is about, what is the largest number of lines in $\mathbb{R}^n$ that pass through the origin such that any two lines share the same angle?

Ramsey's theorem gives the foundation of this branch of combinatorics. In its original form, the theorem is about colourings of large subsets. We state the theorem here in its graph-theoretical form and refer to [27] for more details into this subject.

**Theorem B.1** (Ramsey's theorem)**.** *For any two integers $s, t \geq 1$ there exists a number $R = R(s,t)$ such that any graph on at least $R$ vertices either contains a clique of size $s$ or an independent set of size $t$.*

Note that $R(s,t) = R(s,t)$, since we can simply invert the terms 'clique' and 'independent set' in the theorem above.

The results on equiangular lines from [9]–[11] and [12] all rely on this theorem to find a certain subgraph in the graph associated to a set of equiangular lines. In this appendix we will show how this theorem can be used to prove that a spherical code representing a set of equiangular lines can be chosen in such a way that its associated graph has bounded maximum degree.

## B.1 The maximum degree using Ramsey

In Chapter 4 we have seen that a spherical code representing a set of equiangular lines can be chosen in such a way that its associated graph has bounded maximum degree (Theorem 4.14). In [12] already showed a bound on the maximum degree with a proof that relies on Ramsey's theorem. The bound they found is of a much larger order than the new bound given by Balla in [13]. We will now give the proof from [12] to show how Ramsey's theorem is used in the context of equiangular lines. The theorem we will prove is the following.

**Theorem B.2** ([12])**.** *For a set of equiangular lines, the unit vectors representing each line can be chosen in such a way that the associated graph $G$ has bounded degree $\Delta(\alpha)$.*

To prove the theorem we will use a switching operation that will ensure that the obtained graph will have bounded degree. First, a large independent set in the graph will be found using Ramsey's theorem. Then a switching argument will be used in such a way that no vertex outside the independent set will be adjacent to more than half the vertices of the independent set. By partitioning the vertices outside the independent set and analyzing how these subsets attach to the independent set, the degree of all vertices can be bounded.

To find the large independent set using Ramsey, we first need the following lemma which gives a limitation on the size of a clique in the graph associated to a set of equiangular lines.

**Lemma B.3.** *Let $G$ be the associated graph of a spherical $\{-\alpha, \alpha\}$-code, where $\alpha \in (0,1)$. Then any clique in $G$ has size at most $\frac{1}{\alpha} + 1$.*

*Proof.* Let $K$ be any clique in $G$. We have

$$0 \leq \left\| \sum_{\boldsymbol{v} \in K} \boldsymbol{v} \right\|^2 = \sum_{\boldsymbol{v} \in K} \|\boldsymbol{v}\|^2 + \sum_{\substack{\boldsymbol{v},\boldsymbol{u} \in K \\ \boldsymbol{v} \neq \boldsymbol{u}}} \langle \boldsymbol{v}, \boldsymbol{u} \rangle = |K| - \alpha |K|(|K| - 1).$$

Rearranging gives $|K| \leq \frac{1}{\alpha} + 1$ as needed. □

Since the clique of the associated graph can not be larger than $\alpha^{-1} + 1$, by Ramsey's theorem we will be able to find a large independent set in the graph. Once we have found this independent set, we will partition the vertices outside of this set. To make this partition we will use the following notation.

**Definition B.4.** Let $G$ be a graph with vertex set $V$ and let $X \subset Y \subset V$. We define $C_Y(X)$ as the set of vertices in $V \setminus Y$ that are adjacent to all vertices in $X$ and no other vertices of $Y$ (see Figure B.1).

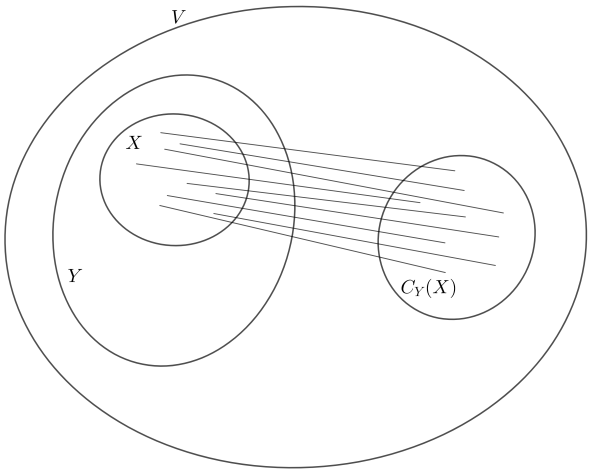


**Figure B.1:** The set $C_Y(X)$

*Remark* B.5. Note that in this notation, $C_Y(\varnothing)$ denotes the set of all vertices that have no neighbour in $Y$. Furthermore, observe that $C_Y(X)$ is defined as a *set* and not a subgraph. Since it is a subset of the vertices of $G$ it does, however, induce a subgraph of $G$.

The vertices outside of the independent set will be partitioned depending on their neighbours in the independent set. The following lemma will enable us to bound the degrees of all vertices in the graph. We will skip its proof, for which we refer to [12].

**Lemma B.6.** *Let $G$ be the associated graph of a spherical $\{-\alpha, \alpha\}$-code, $\alpha \in (0,1)$, and let $\lambda = \frac{1-\alpha}{2\alpha}$. There exist positive integers $M_1, M_2$ depending only on $\alpha$ such that if $\mathcal{I}$ is an independent set of $G$ with at least $M_1$ vertices, then*

(a) *the subgraph induced by $C_{\mathcal{I}}(\varnothing)$ has maximum degree at most $\lceil \lambda^2 \rceil$, and*
(b) *$|C_{\mathcal{I}}(X)| \leq M_2$ for every nonempty proper subset $X$ of $\mathcal{I}$.*

It almost immediately follows from this proof that all vertices in the independent set $\mathcal{I}$ have bounded degree. To see this, notice that the neighbours of any vertex $v \in \mathcal{I}$ are all elements of $C_{\mathcal{I}}(X)$ for all proper subset $X$ of $\mathcal{I}$ containing $v$. Since all these sets $C_{\mathcal{I}}(X)$ are bounded, it follows that the number of neighbours of $v$ is also bounded. Proving that the vertices outside the independent set are bounded requires a bit more work as we will now show in the proof of Theorem B.2.

*Proof of theorem B.2.* Let $\mathcal{L}$ be a set of $n$ equiangular lines with common angle $\arccos\alpha$ and let $\lambda = \frac{1-\alpha}{2\alpha}$. Let $G$ be the associated graph of $\mathcal{L}$ with vertex set $V = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$.

Define integers $M_1, M_2$ as in the lemma above. By Ramsey's theorem there exists an integer $R = R\left(\lceil\alpha^{-1}\rceil + 2, 2M_1\right)$ such that if $|V| > R$, then $G$ either has a clique of size $\lceil\alpha^{-1}\rceil + 2$ or an independent set of size $2M_1$. Note that if $|V| < R$, then we can simply choose $\Delta \geq R$ and the theorem holds.

We may thus assume $|V| > R$. By lemma B.3 $G$ can not contain a clique of size $\lceil\alpha^{-1}\rceil + 2$, so it must have an independent set of size $2M_1$. Denote this independent set by $\mathcal{I}$.

We now modify our set of vectors $V$ with the following switching operation. For any $\boldsymbol{v}_i \notin \mathcal{I}$ that is adjacent to more than half of the vertices in $\mathcal{I}$, replace $\boldsymbol{v}_i$ with $-\boldsymbol{v}_i$, as shown in Figure B.2. In the figure $N_{\mathcal{I}}(\boldsymbol{v}_i)$ denotes the set of neighbours of $\boldsymbol{v}_i$ in $\mathcal{I}$. Negating a vector in the set $V$ inverts the adjacency of this vector as a vertex of $G$. Thus, after applying this switching operation there are no vertices outside of $\mathcal{I}$ which are adjacent to more than $M_1$ vertices of $\mathcal{I}$. In particular this means that for all $X \subseteq \mathcal{I}$, with $|X| > M_1$, the set $C_{\mathcal{I}}(X)$ is empty.

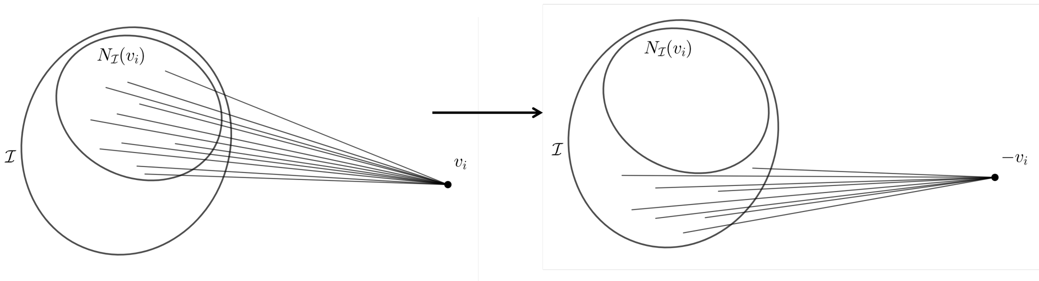


**Figure B.2:** Switching adjacency of $\boldsymbol{v}_i$

The set $V \setminus \mathcal{I}$ can now be partitioned into disjoint subsets $C_{\mathcal{I}}(X)$, where $X$ ranges over all subsets of $\mathcal{I}$ with at most $M_1$ elements. By part (b) of the above lemma all $C_{\mathcal{I}}(X)$ have at most $M_2$ vertices. Now let $A = C_{\mathcal{I}}(\varnothing)$ be the set of vertices in $V \setminus \mathcal{I}$ non-adjacent to any vertex in $\mathcal{I}$. Then

$$|V \setminus A| = |\mathcal{I}| + \sum_{X \neq \varnothing} |C_{\mathcal{I}}(X)| \leq 2M_1 + 2^{2M_1} M_2 =: M.$$

From this we find that the degree of any vertex $v$ in $\mathcal{I}$ can be at most $M$, since it can only have neighbours in $V \setminus A$.

To prove the theorem, it remains to bound the degree of the vertices not in $\mathcal{I}$. So let $v \notin \mathcal{I}$ and let $Y$ be the set of *non*-neighbours of $v$ in $\mathcal{I}$. Since after the switching operation no vertex outside of $\mathcal{I}$ has more than $M_1$ neighbours in $\mathcal{I}$, the set $Y$ contains at least $M_1$ vertices. Note that since $Y$ is a subset of $\mathcal{I}$, it is also an independent set. Consider the subgraph induced by $C_Y(\varnothing)$. The set $C_Y(\varnothing)$ contains all vertices that are not adjacent to any vertex of $Y$ and thus contains the vertex $v$. By part (a) of the above lemma, $C_Y(\varnothing)$ has maximum degree $\lceil\lambda^2\rceil$, which means that the degree of $v$ in $G[C_Y(\varnothing)]$ is at most $\lceil\lambda^2\rceil$. Furthermore, notice that all vertices in $A$ have no neighbours in $\mathcal{I}$ and thus in particular have no neighbours in $Y$. This implies that $A \subseteq C_Y(\varnothing)$ and thus to find an upper bound on the degree of $v$ it suffices to bound its degree into $V \setminus A$. This is at most $M$, since $V \setminus A$ has no more than $M$ elements. It follows that the degree of $v$ in $G$ is at most $\Delta = \lceil\lambda^2\rceil + M$.

So we have now found that $d(v) \leq M \leq \Delta$ for all $v \in \mathcal{I}$ and that $d(v) \leq \Delta$ for all $v \in V \setminus \mathcal{I}$. Since $\Delta$ is a constant that only depends on $\alpha$, this proves the theorem. ☐