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Model Predictive Control for Continuous Piecewise Affine Systems Using Optimistic Optimization

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***Abstract*— This paper considers model predictive control for continuous piecewise affine (PWA) systems. In general, this leads to a nonlinear, nonconvex optimization problem. We introduce an approach based on optimistic optimization to solve the resulting optimization problem. Optimistic optimization is based on recursive partitioning of the feasible set and is characterized by an efficient exploration strategy seeking for the optimal solution. The advantage of optimistic optimization is that one can guarantee bounds on the suboptimality with respect to the global optimum for a given computational budget. The 1-norm and ∞-norm objective functions often considered in model predictive control for continuous PWA systems are continuous PWA functions. We derive expressions for the core parameters required by optimistic optimization for the resulting optimization problem. By applying optimistic optimization, a sequence of control inputs is designed satisfying linear constraints. A bound on the suboptimality of the returned solution is also discussed. The performance of the proposed approach is illustrated with a case study on adaptive cruise control.**

I. INTRODUCTION

Piecewise affine (PWA) systems [1] are a subclass of hybrid systems, containing both continuous and discrete dynamics. PWA systems are defined by a polyhedral partition of the state and input space where each polyhedron is associated with an affine dynamical description. It has been proved [2] that continuous PWA systems are equivalent to other classes of hybrid systems, such as mixed logical dynamical systems and max-min-plus-scaling systems. Based on this equivalence between continuous PWA systems and mixed logical dynamical systems, the MPC problem for continuous PWA systems can be written as mixed integer linear programming (MILP) problems [3]. However, the efficiency of solving the resulting MILP problem is limited by the number of integer variables. The number of integer variables is proportional to the value of the prediction horizon and the number of polyhedral partitions of the considered PWA system. The complexity of current MILP algorithms increases in the worst case exponentially if the number of integer variables increases. On the other hand, from the equivalence between continuous PWA systems and max-min-plus-scaling systems, the corresponding MPC optimization problem can be solved by a sequence of linear programming (LP) problems [4]. Nevertheless, the complexity of that approach is determined by the number of LP problems to be solved, which may increase rapidly if the prediction horizon increases. Therefore, trying to find an efficient approach with guaranteed performance for solving the continuous PWA-MPC optimization problem is the motivation of this paper.

Optimistic optimization [5], [6] is a class of optimization algorithms based on recursively partitioning the feasible set. The regions that most likely contain the optimal solution are first refined. A sequence of feasible solutions are generated during the iterations and the best solution is returned at the end of the algorithm. The gap between the best value returned by the algorithm and the real global optimum can be made arbitrarily small as the computational budget increases. The rate of convergence of optimistic optimization is characterized using a measure of the problem complexity, called near-optimality dimension. Optimistic optimization can be applied to general optimization problem of nonlinear functions given evaluations of the function over general search spaces; in addition, the evaluations may be perturbed by noise [7]. In a previous paper [8], we have extended optimistic optimization to solve the model predictive control problem for max-plus linear systems. Optimistic optimization has also been used to solve the consensus problem in multi-agent systems [9]. Moreover, optimistic optimization has been adapted to planning resulting in a class of optimistic planning algorithms [10], [11], [12].

In this paper, we propose an approach based on optimistic optimization to solve the MPC problem for continuous PWA systems. At each time step, a sequence of control inputs is computed by using optimistic optimization to solve a nonlinear, nonconvex optimization problem subject to linear constraints. The feasible set is transformed into a hyperbox by applying the penalty function method. Considering a 1-norm and ∞-norm objective function, we design a dedicated semi-metric and the expressions for the parameters of the requirements for optimistic optimization. These requirements characterize the suboptimality of the solution. We show that the near-optimality dimension of the resulting optimization problem is zero, which results in the suboptimality bound of the returned solution decreasing exponentially in the computational budget. This implies that the MPC problem for continuous PWA systems is easy to solve by using optimistic optimization. Compared with the MILP method which provides the true optimum, the solution returned by optimistic optimization given a finite computational budget is near-optimal, but optimistic optimization can be computationally efficient when the number of polyhedral partitions of the PWA system is large.

This paper is organized as follows. In Section II, discrete-

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time PWA systems and the corresponding MPC problem are presented. In Section III, the background of optimistic optimization is introduced. In Section IV, the proposed approach is presented and the suboptimality is discussed. In Section V, the effectiveness of the proposed approach is illustrated with an adaptive cruise control case study.

II. PROBLEM STATEMENT

Consider the discrete-time PWA system

$$x(k+1) = A_i x(k) + B_i u(k) + g_i, \quad \text{for} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \in \Omega_i, \tag{1}$$

where A_i, B_i, and g_i are the system matrices and vectors for $i = 1, \ldots, N$. Each region Ω_i is a polyhedron given as $\Omega_i = \{F_i x(k) + G_i u(k) \leq h_i\}$ where F_i, G_i, and h_i are suitable matrices and vectors and $\{\Omega_i\}_{i=1}^N$ is a polyhedral partition of the state and input space.

As given in [3], the system (1) can be represented as

$$\begin{aligned} & x(k+1) = \sum_{i=1}^N z_i(k), \\ & z_i(k) \triangleq [A_i x(k) + B_i u(k) + g_i]\sigma_i(k), \\ & \sum_{i=1}^N \sigma_i(k) = 1, \\ & E_{1k} u(k) + E_{2k}\sigma(k) + E_{3k} z(k) \leq E_{4k} x(k) + E_{5k} \end{aligned} \tag{2}$$

where $\sigma_i(k) \in \{0,1\}$, $\sigma(k) = \begin{bmatrix} \sigma_1(k) & \cdots & \sigma_N(k) \end{bmatrix}^T$, $z(k) = \begin{bmatrix} z_1(k) & \cdots & z_N(k) \end{bmatrix}^T$, and $E_{1k}, \ldots, E_{5k}$ are linear constraint matrices at time step k. Systems in the form of (2) are a specific type of mixed logical dynamical systems.

Proposition 1: [13], [14] If $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous PWA function, then f can be represented in the max-min canonical form

$$f(w) = \max_{i \in \mathcal{I}} \min_{j \in \mathcal{J}_i} \{\alpha_{ij}^T w + \beta_{ij}\} \tag{3}$$

where $\mathcal{I}, \mathcal{J}_i$ are finite index sets and $\alpha_{ij} \in \mathbb{R}^n$, $\beta_{ij} \in \mathbb{R}$ for all i, j. For vector-valued functions, the above forms exist component-wise.

Systems that can be described as

$$x(k+1) = \mathcal{M}(x(k), u(k)), \tag{4}$$

where $\mathcal{M}$ is an expression of $x(k)$ and $u(k)$ in the form of (3) with $w = [x^T \ u^T]^T$ are called max-min-plus-scaling systems. By introducing additional auxiliary variables or extra constraints, the equivalence between (1) and (4) can be established (see [2] for details). If the system (1) is continuous (i.e. the right-hand side of (1) is continuous on the boundary of any two neighbouring regions), then a direct connection between (1) and (4) can be derived following Proposition 1 (see [4] for details).

Let N_p and N_c be the prediction horizon and the control horizon. Define the vectors $\tilde{x}(k) = \begin{bmatrix} x^T(k+1) & \cdots & x^T(k+N_\text{p}) \end{bmatrix}^T$, $\tilde{u}(k) = \begin{bmatrix} u^T(k) & \cdots & u^T(k+N_\text{c}-1) \end{bmatrix}^T$. At time step k, the MPC problem for the system (1) can be written as

$$\min_{\tilde{u}(k)} J(\tilde{u}(k)) \tag{5}$$

$$\begin{aligned} & \text{subject to the prediction model (1), (2) or (4),} \\ & u(k+s) = u(k+N_\text{c}-1) \text{ for } s = N_\text{c}, \ldots, N_\text{p}-1, \\ & x(k) \in \mathbb{X}, \ u(k) \in \mathbb{U}, \ \text{for all } k, \end{aligned}$$

where $\mathbb{X}$ and $\mathbb{U}$ are the feasible set of the states and the control inputs and correspond to the physical and operational constraints of the system. An optimal control sequence $\tilde{u}(k)$ is obtained by solving the problem (5); subsequently, only the first control input $u(k)$ is applied to the system. At the next time step, this process is repeated.

Let r be a given reference signal. Define

$$\Delta u(k) = u(k) - u(k-1). \tag{6}$$

In this paper, we consider the following objective function

$$J(\tilde{u}(k)) = \|\tilde{x}(k) - \tilde{r}(k)\|_p + \lambda \|\Delta\tilde{u}(k)\|_q \tag{7}$$

where $p, q \in \{1, \infty\}$, λ is a nonnegative scalar, and $\tilde{r}(k) = \begin{bmatrix} r^T(k+1) & \cdots & r^T(k+N_\text{p}) \end{bmatrix}^T$, $\Delta\tilde{u}(k) = \begin{bmatrix} \Delta u^T(k) & \cdots & \Delta u^T(k+N_\text{c}-1) \end{bmatrix}^T$.

Remark 2: If the system (2) is used as the prediction model, the PWA-MPC problem (5) can be recast into a mixed integer linear programming (MILP) problem following the procedures in [3] where the number of variables and constraints is proportional to the product nNN_p. However, in practice, the worst-case complexity of the MILP problem is exponential in nNN_p.

III. OPTIMISTIC OPTIMIZATION

In this section, we introduce optimistic optimization for the minimization of a function f over a set $\mathcal{U}$. The notations f and $\mathcal{U}$ remain generic and this section is based on [5].

The implementation of optimistic optimization is founded on a hierarchical partitioning of $\mathcal{U}$. For any integer $h \in \{0, 1, \ldots\}$, the set $\mathcal{U}$ is split into K^h cells with K a finite positive integer. This partition may be represented by a tree structure; thus, K is the number of branches at each node. Each cell is denoted as $U^{h,d}$, $d \in \{0, \ldots, K^h\}$, and corresponds to a node (h, d) in the tree (with h the depth and d the node index). The root node of the tree corresponds to the whole region $\mathcal{U}$ and is denoted as $U^{0,0}$. Expanding a node (h, d) corresponds to splitting the cell $U^{h,d}$ into K sub-cells $\{U^{h+1,d_i} | i = 1, \ldots, K\}$. Each cell $U^{h,d}$ is represented by a point $u^{h,d} \in U^{h,d}$ where f may be evaluated.

Definition 3 (Semi-metric): A semi-metric on a set $\mathcal{U}$ is a function $\ell : \mathcal{U} \times \mathcal{U} \to \mathbb{R}^+$ satisfying the following conditions for any $u, v \in \mathcal{U}$: 1) $\ell(u, v) = \ell(v, u) \geq 0$; 2) $\ell(u, v) = 0$ if and only if $u = v$.

Requirements for optimistic optimization. The following conditions need to be satisfied for avoiding degenerated partitions and for being able to characterize the suboptimality (see Remark 4 for details):

1. There exists a semi-metric ℓ defined on $\mathcal{U}$ such that for all $u \in \mathcal{U}$, $f(u) - f(u^*) \leq \ell(u, u^*)$, where $f(u^*) = \min_{u \in \mathcal{U}} f(u)$.
2. There exists a decreasing sequence $\{\delta(h)\}_{h=0}^\infty$ with $\delta(h) > 0$, such that for any $h \in \{0, 1, \ldots\}$, for any cell $U^{h,d}$ at depth h, we have $\sup_{u \in U^{h,d}} \ell(u, u^{h,d}) \leq \delta(h)$, where $\delta(h)$

Algorithm 1 Deterministic Optimistic Optimization

Given: computational budget $n_{\max}$, partitioning of $\mathcal{U}$
Initialize the tree $\mathcal{T} = \{(0,0)\}$ (root node)
for $t = 1$ to $n_{\max}$ **do**
  Select the leaf (h,d) with minimum $b^{h,d}$ value
  Expand this leaf by adding its K children to $\mathcal{T}$
end for
Return $u(n_{\max}) = \arg\max_{(h,d)\in\mathcal{T}} f(u^{h,d})$

is called the maximum diameter of the cells at depth h.
3. There exists a scalar $\nu > 0$ such that any cell $U^{h,d}$ at depth h contains an ℓ-ball of radius $\nu\delta(h)$ centered in $u^{h,d}$. Such an ℓ-ball is defined as $\mathcal{B} = \{u \in \mathcal{U} | \ell(u, u^{h,d}) \le \nu\delta(h)\}$.

Remark 4: The requirements guarantee bounds on the suboptimality with respect to the global optimum in relation to the computational budget (e.g. the number of evaluations of f). In particular, Requirement 1 regards the local smoothness of f with respect to the semi-metric ℓ near the optimum. Requirements 2-3 guarantee that the partitioning of the feasible set generates well-shaped cells that shrink with further partitioning. The decreasing sequence $\delta(h)$ corresponds to the maximum size of cells at each depth h. The scalar ν can be considered as the maximum ratio of the radius of the inscribed ball of any cell and the maximum distance between any two points in that cell.

The optimistic optimization algorithm is summarized in Algorithm 1. For each cell $U^{h,d}$, define $b^{h,d} = f(u^{h,d}) - \delta(h)$. From Requirements 1-2, for the cell $U^{h,d}$ containing an optimal solution u^*, we have $b^{h,d} \le f(u^{h,d}) - \ell(u, u^{h,d}) \le f(u^*)$, $\forall u \in U^{h,d}$. Hence, the value $b^{h,d}$ can be considered as a heuristic evaluation function for selecting the cell that most likely contains the optimal solution.

The performance of the optimistic optimization algorithm is influenced by the choice of the semi-metric ℓ (the estimation of the smoothness of f) and is characterized by the suboptimality of the returned solution given a finite computational budget $n_{\max}$. Let u^* be a global minimizer of f and let $\mathcal{U}_\varepsilon = \{u \in \mathcal{U} | f(u) - f(u^*) \le \varepsilon\}$ be the set of ε-near-optimal solutions.

Definition 5: [5] The local ν-near-optimality dimension is the smallest $\eta \ge 0$ such that for some $\varepsilon_0 > 0$, for any $\varepsilon \in (0, \varepsilon_0]$, there exists a $C > 0$ such that the maximal number of disjoint ℓ-balls of radius $\nu\varepsilon$ with center in $\mathcal{U}_\varepsilon$ is less than $C\varepsilon^{-\eta}$.

Proposition 6: [5] Assume that there exist $c > 0$ and $\gamma \in (0,1)$ such that $\delta(h) \le c\gamma^h$ for any h. Let $u^\natural$ be the solution returned after $n_{\max}$ iterations.

(i) If $\eta > 0$, then $f(u^\natural) - f(u^*) \le (\frac{C}{1-\gamma^\eta})^{1/\eta}(n_{\max})^{-1/\eta}$.

(ii) If $\eta = 0$, then $f(u^\natural) - f(u^*) \le c\gamma^{n_{\max}/C-1}$.

Remark 7: The near-optimality dimension actually characterizes the amount of the ε-near-optimal solutions of f with respect to the semi-metric ℓ around the global optimum. Proposition 6 gives bounds on the suboptimality of the returned solution. For $\eta > 0$, the suboptimality bound decreases in a power of the computational budget $n_{\max}$. The convergence speed of optimistic optimization is faster with smaller η. The best case is $\eta = 0$ which means that the suboptimality bound decreases exponentially with $n_{\max}$. Therefore, developing a semi-metric ℓ such that η is small is of great importance for optimistic optimization to be efficient.

IV. OPTIMISTIC OPTIMIZATION FOR THE CONTINUOUS PWA-MPC PROBLEM

In this section, we present the optimistic optimization approach for the PWA-MPC problem (5) provided that the PWA system (1) is continuous.

Recall the definitions of 1-norm and ∞-norm for vectors $\|x\|_1 = \sum_{i=1}^n |x_i|$, $\|x\|_\infty = \max_{i=1,\dots,n} |x_i|$, and $|x_i| = \max(x_i, -x_i)$. According to the equivalence between the system (1) and (4), the objective function (7) can be transformed into an expression in the form of (3).

Since the state vector $\tilde{x}(k)$ and the control input increments $\Delta\tilde{u}(k)$ can be eliminated using (4) and (6), the objection function (7) only has $\tilde{u}(k)$ as the independent variable:

$$J(\tilde{u}(k)) = \max_{i\in\mathcal{I}} \min_{j\in\mathcal{J}_i} \{\alpha_{ijk}^T \tilde{u}(k) + \beta_{ijk}\} \tag{8}$$

with $\alpha_{ijk} \in \mathbb{R}^{mN_\mathrm{c}}$, $\beta_{ijk} \in \mathbb{R}$. The parameter vectors α_{ijk} and the constant terms β_{ijk} can be computed from the known information at time step k (the system matrices and vectors A_i, B_i, and g_i in (1), the reference sequence $\tilde{r}$, the current state $x(k)$, and the previous control input $u(k-1)$). Besides, we consider the following constraints in the problem (5):

$$P_k\tilde{x}(k) + Q_k\tilde{u}(k) \le b_k, \tag{9}$$
$$x_{\min} \le x(k+s) \le x_{\max}, \tag{10}$$
$$u_{\min} \le u(k+s-1) \le u_{\max}, \ \ s = 1, \dots, N_\mathrm{p}, \tag{11}$$

with $P_k \in \mathbb{R}^{n_\mathrm{c}\times nN_\mathrm{p}}$, $Q_k \in \mathbb{R}^{n_\mathrm{c}\times mN_\mathrm{c}}$, $b_k \in \mathbb{R}^{n_\mathrm{c}}$, $x_{\min}, x_{\max} \in \mathbb{R}^n$, $u_{\min}, u_{\max} \in \mathbb{R}^m$.

The feasible set defined by constraints (9)-(11) is a polytope. In order to easily derive the Requirements 1-3 for optimistic optimization, we transform the problem into a problem with hyperbox constraints. Hence, we treat (9) and (10) as soft constraints and replace them by adding a penalty function to the objective function:

$$J_\mathrm{p}(\tilde{u}(k)) = \beta\cdot\max\big(0, \max_{i=1,\dots,n_\mathrm{c}} (P_{i,\cdot}\tilde{x}(k) + Q_{i,\cdot}\tilde{u}(k) - b_i),$$
$$\max_{s=1,\dots,N_\mathrm{p}} \max_{j=1,\dots,n} (x_j(k+s) - x_{\max,j}, x_{\min,j} - x_j(k+s))\big),$$

where β is the penalty coefficient; $P_{i,\cdot}$ and $Q_{i,\cdot}$ are the respective i-th rows of P_k and Q_k; b_i is the i-th element of b_k; $x_j(k+s)$ and $x_{\max,j}$ are the respective j-th elements of $x(k+s)$ and $x_{\max}$. So we have the new objective function

$$J_\mathrm{new}(\tilde{u}(k)) = J(\tilde{u}(k)) + J_\mathrm{p}(\tilde{u}(k)) \tag{12}$$

subject to the constraint (11). Consequently, the feasible set is actually an mN_c-dimensional hyperbox $\mathcal{U} = [u_{\min}, u_{\max}]^{N_\mathrm{c}}$. By performing scaling operations, the feasible set can be transformed into a hypercube $\mathcal{U}_c$. Note that the new objective function can also be written as

$$J_{\text{new}}(\tilde{u}) = \max_{i \in \mathcal{I}} \min_{j \in \mathcal{J}_i} \{\hat{\alpha}_{ij}^T \tilde{u} + \hat{\beta}_{ij}\} \tag{13}$$

with $\hat{\alpha}_{ij} \in \mathbb{R}^{mN_c}$, $\hat{\beta}_{ij} \in \mathbb{R}$. In the remaining part of this section the time counter k is omitted for sake of simplicity.

Now we design the semi-metric ℓ, the diameter $\delta(h)$, and the scalar ν that are dedicated to the continuous PWA-MPC problem (5) with the new objective function. These parameters are required for the implementation of optimistic optimization and for the characterization of the suboptimality of the returned solution.

Proposition 8: Define $\bar{\alpha} \triangleq \max_{i,j} \|\hat{\alpha}_{ij}\|_2$ where $\hat{\alpha}_{ij}$ are the parameter vectors in (13). Let L be the edge length of the hypercube $\mathcal{U}_c$. Let $\tilde{u}^*$ be a global optimizer of the objective function J_{new} subject to $\tilde{u} \in \mathcal{U}_c$. Recall the hierarchical partitioning framework presented in Section III, let the branching number $K = D^{mN_c}$ where mN_c is the dimension of the hypercube $\mathcal{U}_c$ and each edge of $\mathcal{U}_c$ is cut into D equal parts. Let $U^{h,d}$ be the cell at depth h with node index d and let $\tilde{u}^{h,d} \in U^{h,d}$ be the center of $U^{h,d}$.

(i) If we define

$$\ell(\tilde{u}, \tilde{v}) = \bar{\alpha}\|\tilde{u} - \tilde{v}\|_2, \tag{14}$$

for any $\tilde{u}, \tilde{v} \in \mathcal{U}_c$, then ℓ is a semi-metric defined on $\mathcal{U}_c$ such that for any $\tilde{u} \in \mathcal{U}_c$, we have

$$J_{\text{new}}(\tilde{u}) - J_{\text{new}}(\tilde{u}^*) \leq \ell(\tilde{u}, \tilde{u}^*). \tag{15}$$

(ii) If we define

$$\delta(h) = \frac{\bar{\alpha}}{2}(mN_c)^{1/2} L/D^h, \tag{16}$$

then for any cell $U^{h,d}$ at any depth h, we have

$$\sup_{\tilde{u} \in U^{h,d}} \ell(\tilde{u}, \tilde{u}^{h,d}) \leq \delta(h). \tag{17}$$

(iii) Select $0 < \rho \leq 1$. If we define $\nu = \rho(mN_c)^{-1/2}$, then any cell $U^{h,d}$ contains an ℓ-ball of radius $\nu\delta(h)$ centered in $\tilde{u}^{h,d}$.

Proof: (i) From Proposition 1, the objective function J_{new} is a continuous PWA function. The constant $\bar{\alpha} \triangleq \max_{i,j} \|\hat{\alpha}_{ij}\|_2$ is actually a Lipschitz constant for J_{new}. According to the Lipschitz continuity, we have $J_{\text{new}}(\tilde{u}) - J_{\text{new}}(\tilde{u}^*) \leq \bar{\alpha}\|\tilde{u} - \tilde{u}^*\|_2$ for any $\tilde{u} \in \mathcal{U}_c$. If we define the semi-metric as $\ell(\tilde{u}, \tilde{v}) = \bar{\alpha}\|\tilde{u} - \tilde{v}\|_2$, then the inequality (15) is satisfied. (This type of semi-metric is also developed for an arbitrary continuous PWA function in our submitted paper [15]).

(ii) Recall the hierarchical partitioning presented in Section III. The edge length of the hypercube $\mathcal{U}_c$ is L, so the maximum distance between any two points in $\mathcal{U}_c$ is $(mN_c)^{1/2}L$. The cell $U^{h,d}$ at depth h of the partitioning is also a hypercube and the edge length of $U^{h,d}$ is L/D^h. Because $\tilde{u}^{h,d}$ is the center of the cell $U^{h,d}$, for any $\tilde{u} \in U^{h,d}$, we have $\|\tilde{u} - \tilde{u}^{h,d}\|_2 \leq \frac{1}{2}(mN_c)^{1/2}L/D^h$. Define $\delta(h) = \frac{\bar{\alpha}}{2}(mN_c)^{1/2}L/D^h$. Therefore, for any $\tilde{u} \in U^{h,d}$, we have $\ell(\tilde{u}, \tilde{u}^{h,d}) = \bar{\alpha}\|\tilde{u} - \tilde{u}^{h,d}\|_2 \leq \delta(h)$.

(iii) An ℓ-ball of radius $\nu\delta(h)$ centered in $\tilde{u}^{h,d}$ can be written as $\mathfrak{B} = \{\tilde{u} \in \mathcal{U}_c | \ell(\tilde{u}, \tilde{u}^{h,d}) = \bar{\alpha}\|\tilde{u} - \tilde{u}^{h,d}\|_2 \leq \nu\delta(h)\}$. Note that $\mathcal{U}_c$ is a hypercube and so is the cell $U^{h,d}$. Thus, the center $u^{h,d}$ is also the center of the inscribed ball of $U^{h,d}$. Let $r(h)$ be the radius of the inscribed hyperball of $U^{h,d}$, so $r(h) = \frac{1}{2}L/D^h$. If we select $0 < \nu \leq \frac{\bar{\alpha}r(h)}{\delta(h)}$, then we have $\|\tilde{u} - \tilde{u}^{h,d}\|_2 \leq \frac{\nu\delta(h)}{\alpha} \leq r(h)$ for all $\tilde{u} \in \mathfrak{B}$. Hence, we have $\mathfrak{B} \subset U^{h,d}$. Note that $\frac{\bar{\alpha}r(h)}{\delta(h)} = (mN_c)^{-1/2}$. Thus if we select a scalar $0 < \rho \leq 1$ and choose $\nu = \rho(mN_c)^{-1/2}$, then $U^{h,d}$ contains an ℓ-ball of radius $\nu\delta(h)$ centered in $\tilde{u}^{h,d}$. ■

Up to now, we have derived the expressions for all core parameters of the requirements for optimistic optimization. At each time step k, we apply optimistic optimization to solve the MPC optimization problem (5) to obtain a sequence of control inputs. To discuss the suboptimality of the returned solution, we compute the local ν-near-optimality dimension for the objective function J_{new} over $\mathcal{U}_c$. Denote the set of ε-near-optimal solutions as $\mathcal{U}_\varepsilon = \{\tilde{u} \in \mathcal{U}_c | J_{\text{new}}(\tilde{u}) - J_{\text{new}}(\tilde{u}^*) \leq \varepsilon\}$.

Proposition 9: Let $\tilde{u}^*$ be a global optimizer of J_{new} subject to $\tilde{u} \in \mathcal{U}_c$ and let $\tilde{u}^\natural$ be the solution returned by optimistic optimization after $n_{\max}$ iterations. If $\tilde{u}^*$ is a strict local minimizer of J_{new}, then the local ν-near-optimality dimension is $\eta = 0$ and we have $J_{\text{new}}(\tilde{u}^\natural) - J_{\text{new}}(\tilde{u}^*) \leq \frac{\bar{\alpha}}{2}(mN_c)^{1/2}LD^{1-n_{\max}/C}$ with a constant $C > 0$.

Remark 10: Proposition 9 shows that with the semi-metric (14), for the continuous PWA-MPC problem with the objective function (13) subject to (11) the ν-near-optimality dimension is $\eta = 0$ when the optimizer is strict. This means that the optimization problem is simple, and the optimistic algorithm can solve it efficiently, converging quickly to the optimal solution.

V. CASE STUDY

In this section, we demonstrate the proposed approach with an adaptive cruise control problem for a road vehicle following a leader vehicle. We consider the setup introduced in [16]. As shown in Fig. 1, let $x(k)$ be the velocity of the follower vehicle at time step k. Let $r(k)$ be the velocity of the leader vehicle at time step k and be communicated to the follower vehicle as reference signals. A discrete-time model for the positive velocity of the follower vehicle is given in [16]. That model can be approximated by the following continuous PWA systems:

$$x(k+1) = A_i x(k) + B_i u(k) + g_i, \text{ if } x \in (p_{i-1}, p_i] \tag{18}$$

with $i = 1, 2$, $A_1 = 0.9883$, $B_1 = 4.598$, $g_1 = -0.0614$, $A_2 = 0.9655$, $B_2 = 4.5446$, $g_2 = 0.3711$, $p_0 = 0$, $p_1 = \frac{x_{\max}}{2}$ and $p_2 = x_{\max}$ where $x_{\max}$ is the maximum velocity and p_1 is the breakpoint for the least-squares fitting of the nonlinear friction. The control input $u(k)$ is the throttle/brake position at time step k.

Note that (18) is equivalent to the following max-min-plus-scaling system:

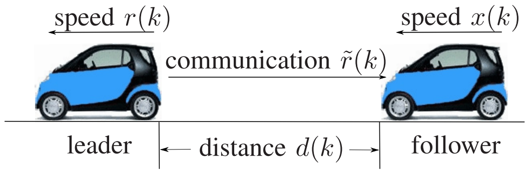

Fig. 1. Adaptive cruise control set-up considered in the case study

$$x(k+1) = \min\big(A_1 x(k) + B_1 u(k) + g_1, \\ A_2 x(k) + B_2 u(k) + g_2\big). \quad (19)$$

Let $d(k)$ be the distance between two vehicles at time step k, so $d(k+1) = d(k) + (r(k) - x(k))T$ with T the sampling time. Due to safety and human comfort requirements, we add constraints on $d(k)$, $x(k)$, $u(k)$ for any time step k

$$d_{\text{safe}} \leq d(k+s+1), \quad (20)$$

$$a_{\text{dec}} T \leq x(k+s+1) - x(k+s) \leq a_{\text{acc}} T, \quad (21)$$

$$-\tau \leq \Delta u(k+s-1) \leq \tau, \quad (22)$$

$$x_{\min} \leq x(k+s) \leq x_{\max}, \quad (23)$$

$$-u_{\max} \leq u(k+s-1) \leq u_{\max}, \ \ s = 1, \ldots, N_{\text{p}}, \quad (24)$$

where d_{safe} corresponds to the safe following distance to reduce the risk of collision, a_{acc} and a_{dec} are the allowable acceleration and deceleration for human comfort, $\Delta u(k) = u(k) - u(k-1)$, τ is the maximum brake variation, $x_{\max}$ and $x_{\min}$ are the maximum and minimum velocities, and $u_{\max}$ is the maximum brake. Set $d_{\text{safe}} = 10$ m for the constraint (20). The constraints (21)-(24) are specified by using the numerical values used in [16].

We consider the following objective function:

$$J(\tilde{u}(k)) = \|\tilde{x}(k) - \tilde{r}(k)\|_\infty + \lambda \|\Delta \tilde{u}(k)\|_1 \quad (25)$$

with the trade-off $\lambda = 0.05$ and $N_{\text{p}} = N_{\text{c}} = 2$. Based on (19), $\tilde{x}(k)$ and $\Delta\tilde{u}(k)$ in (25) can be substituted by $\tilde{u}(k)$. Moreover, the constraints (20)-(23) are replaced by adding a penalty function to the objective function. The penalty coefficient is $\beta = 10$. The new objective function can be rewritten in the form of (13) and the resulting feasible set is a hypercube $[-u_{\max}, u_{\max}]^{N_{\text{c}}}$.

At each time step k, the MPC optimization problem is respectively solved by using MILP method and the optimistic optimization approach. The corresponding MILP problem is solved by the `cplex` function (with the default settings) in the TOMLAB optimization environment in MATLAB. The optimistic optimization approach is implemented in MATLAB. The termination criteria of optimistic optimization (`oo`) are a combination of the computational budget and the depth limitation. Given the number of node expansions $t_{\max}$, the number of evaluations (computational budget) of the objective function is $n_{\max} = K t_{\max} + 1$ with $K = 2^{N_{\text{c}}}$ the branching number in the tree. In addition, the maximum depth of the resulting tree is limited as $h_{\max} = 10$. The algorithm will terminate and return the best solution if the computational budget is used or the maximum depth is reached.

Fig. 2 shows the simulation results of adaptive cruise control for the follower vehicle tracking different reference velocities over the simulation horizon $[1, 50]$. The constant reference velocity is 18.75 m/s and the varying reference velocity is given as $r(k) = 10e^{-0.05k}\sin(0.3k) + 18.75$. The number of node expansions in optimistic optimization is $t_{\max} = 10$. We can see that the trajectory of the velocity of the follower vehicle controlled by optimistic optimization can track both types of reference velocities (Fig. 2(1a) and 2(2a)). The distance between two vehicles stays in the range of safe distance. However, the variation of the control input is not smooth, especially for the case with constant reference. Now $t_{\max}$ is increased for optimistic optimization from 10 to 1000 and the simulation results are shown in Fig. 3. We can see that the trajectories of the velocity and the distance resulting from optimistic optimization track the trajectories resulting from `cplex` better than the case in Fig. 2. Moreover, the control inputs solved by optimistic optimization are smoother and quite close to the control inputs solved by `cplex`. The closed-loop cost over the simulation period of optimistic optimization with $t_{\max} = 1000$ is 96.92 for the varying reference signal; the relative error comparing with the cost of `cplex` is 0.3% (the relavive error is computed as $100|(\text{cost}_{\texttt{cplex}} - \text{cost}_{\texttt{oo}})/\text{cost}_{\texttt{cplex}}|$). The closed-loop costs of optimistic optimization given different computational budgets and the relative error comparing with `cplex` are listed in Table I. The relative error of closed-loop costs of optimistic optimization decreases if the computational budget increases. The average CPU times for optimistic optimization and `cplex` solving the optimization problem at each time step are also included in Table I. Optimistic optimization will be faster if we would transfer the MATLAB code into object code.

TABLE I

CPU TIMES, CLOSED-LOOP COSTS OVER THE SIMULATION PERIOD, AND THE RELATIVE ERROR OF `OO` AND `CPLEX`

	$t_{\max} = 10$	$t_{\max} = 100$	$t_{\max} = 1000$	`cplex`
CPU time (s)	0.001	0.01	0.1	0.004
Constant r	45.8	41.96	39.98	39.76
	15.19%	5.51%	0.53%	0
Varying r	102.62	100.22	96.92	96.62
	6.21%	3.72%	0.3%	0

VI. CONCLUSIONS

We have considered the model predictive control problem for continuous piecewise affine systems and max-min-plus-scaling systems, which in general leads to a nonlinear, nonconvex optimization problem. An approach based on optimistic optimization has been proposed to solve this problem. A 1-norm and ∞-norm objective function has been considered subject to a hyperbox feasible set. A dedicated semi-metric and other parameters required by optimistic optimization have been developed for the corresponding problem. A case study on adaptive cruise control has been implemented to illustrate the performance of the proposed approach.

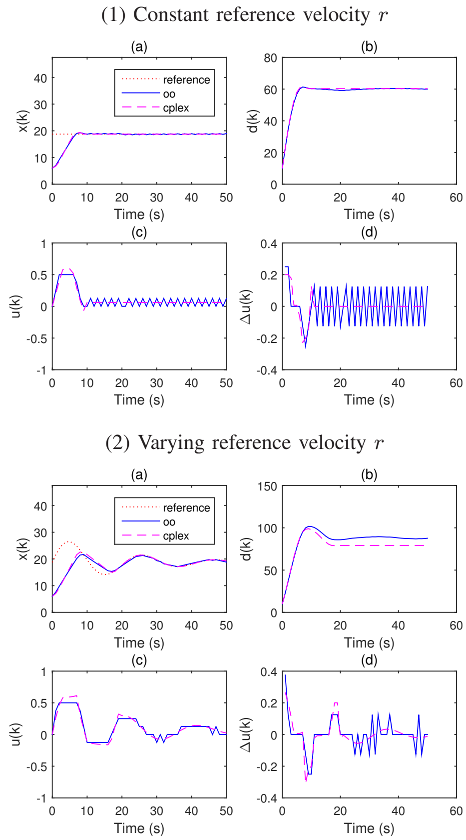

Fig. 2. Simulation results of `cplex` and optimistic optimization (`oo`) for constant and varying reference velocities ($t_{\max} = 10$ for `oo`): (a) Velocity of the follower vehicle; (b) Distance between two vehicles; (c) Control input; (d) Throttle/Brake variation

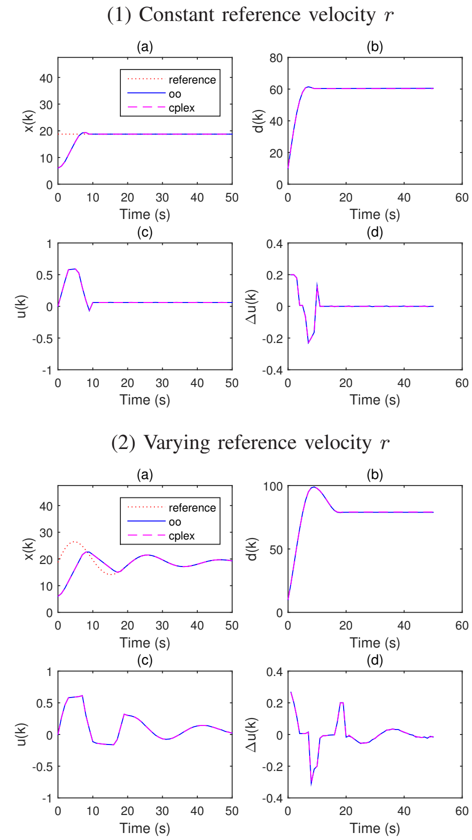

Fig. 3. Simulation results of `cplex` and optimistic optimization (`oo`) for constant and varying reference velocities ($t_{\max} = 1000$ for `oo`): (a) Velocity of the follower vehicle; (b) Distance between two vehicles; (c) Control input; (d) Throttle/Brake variation

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REFERENCES

[1] E. D. Sontag, "Nonlinear regulation: The piecewise linear approach," *IEEE Transactions on Automatic Control*, vol. 26, no. 2, pp. 346–358, 1981.

[2] W. P. M. H. Heemels, B. De Schutter, and A. Bemporad, "Equivalence of hybrid dynamical models," *Automatica*, vol. 37, no. 7, pp. 1085–1091, 2001.

[3] A. Bemporad and M. Morari, "Control of systems integrating logic, dynamics, and constraints," *Automatica*, vol. 35, no. 3, pp. 407–427, 1999.

[4] B. De Schutter and T. van den Boom, "MPC for continuous piecewise-affine systems," *Systems & Control Letters*, vol. 52, no. 3–4, pp. 179–192, July 2004.

[5] R. Munos, "Optimistic optimization of a deterministic function without the knowledge of its smoothness," in *25th Annual Conference on Neural Information Processing Systems*, Granada, Spain, Dec. 2011, pp. 783–791.

[6] ——, "From bandits to Monte-Carlo tree search: The optimistic principle applied to optimization and planning," *Foundations and Trends in Machine Learning*, vol. 7, no. 1, pp. 1–130, 2014.

[7] M. Valko, A. Carpentier, and R. Munos, "Stochastic simultaneous optimistic optimization," in *30th International Conference on Machine Learning*, vol. 28, Atlanta, Georgia, USA, 2013, pp. 19–27.

[8] J. Xu, B. De Schutter, and T. van den Boom, "Model predictive control for max-plus-linear systems via optimistic optimization," in *12th International Workshop on Discrete Event Systems*, Cachan, France, May 2014, pp. 111–116.

[9] L. Buşoniu and I.-C. Morărescu, "Consensus for black-box nonlinear agents using optimistic optimization," *Automatica*, vol. 50, no. 4, pp. 1201–1208, 2014.

[10] L. Buşoniu, R. Munos, and R. Babuska, "A review of optimistic planning in Markov decision processes," in *Reinforcement Learning and Adaptive Dynamic Programming for Feedback Control*. Wiley, 2012, pp. 494–516.

[11] L. Buşoniu, R. Postoyan, and J. Daafouz, "Near-optimal strategies for nonlinear networked control systems using optimistic planning," in *American Control Conference*, Washington DC, USA, June 2013, pp. 3020–3025.

[12] K. Máthé, L. Buşoniu, R. Munos, and B. De Schutter, "Optimistic planning with a limited number of action switches for near-optimal nonlinear control," in *53rd IEEE Conference on Decision and Control*, Los Angeles, California, USA, Dec. 2014, pp. 3518–3523.

[13] V. V. Gorokhovik and O. I. Zorko, "Piecewise affine functions and polyhedral sets," *Optimization*, vol. 31, no. 3, pp. 209–221, 1994.

[14] S. Ovchinnikov, "Max-min representation of piecewise linear functions," *Beiträge zur Algebra und Geometrie/Contributions to Algebra and Geometry*, vol. 43, no. 1, pp. 297–302, 2002.

[15] J. Xu, T. van den Boom, and B. De Schutter, "Optimistic optimization for continuous nonconvex piecewise affine functions," Delft Center for Systems and Control, Delft University of Technology, Delft, The Netherlands, Tech. Rep. 15-030, July 2015.

[16] D. Corona, I. Necoara, B. De Schutter, and T. van den Boom, "Robust hybrid MPC applied to the design of an adaptive cruise controller for a road vehicle," in *45th IEEE Conference on Decision and Control*, San Diego, California, Dec. 2006, pp. 1721–1726.