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Gerrit Vos

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MULTIPLIERS AND TRANSFERENCE ON NONCOMMUTATIVE L_p-SPACES

AND THE RELATIVE HAAGERUP PROPERTY

Proefschrift

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SUMMARY

This thesis deals with multipliers and transference on noncommutative L_p-spaces. It falls within the realm of noncommutative harmonic analysis, i.e. harmonic analysis on noncommutative spaces. Such spaces appear in several areas of mathematics, such as non-abelian groups, quantum groups, noncommutative geometry and the general theory of operator algebras. Already the pioneering work of von Neumann, Gelfand and Dixmier showed that there are close connections between operator theory and representation theory of locally groups. These results formed the foundations for the theory of noncommutative harmonic analysis.

More recently, the L_p-theory of noncommutative spaces have been studied, opening the door to many interesting generalisations of the classical theory. Among those is one of the most fundamental questions of harmonic analysis: which symbols give rise to bounded Fourier multipliers on L_p? These Fourier multipliers are an example of the multipliers on noncommutative spaces studied in this thesis. However, noncommutative spaces also yield other interesting kinds of multipliers, such as Schur multipliers. They are defined on bounded operators on a Hilbert space. It turns out that within the setting of locally compact groups, the notions of Fourier and Schur multipliers are intimately related, as was first proven by Bożejko and Fendler in [BF84]. This is called a transference result between Fourier and Schur multipliers. Transference results such as this will be another recurring theme in this thesis.

After the introduction and before going into the main results, we first give an exposition of the required background material in Chapter 2. In particular, we give a detailed account of the theory of noncommutative L_p-spaces for non-tracial weights, including references for all results or providing proofs where there is a lack of good references. We also treat in detail the theory of locally compact groups, their group von Neumann algebras and corresponding noncommutative L_p-spaces. Finally, we give the definitions of Fourier and Schur multipliers and provide some of the backgroud theory.

In Chapter 3, we revisit the definition of linear Fourier multipliers on noncommutative L_p-spaces of (non-unimodular) locally compact groups. We then discuss how to define multilinear Fourier multipliers in this setting; the choice of definition turns out to be subtle. The main result of this chapter is a transference result between multilinear Fourier and Schur multipliers on noncommutative L_p-spaces. One implication uses an adaptation of an advanced result from [CJKM23] which is in turn based on involved estimates from [CPPR15]. Parts of these estimates have to be generalised to arbitrary von Neumann algebras using Haagerup reduction. The other implication holds only for amenable groups (just as in the linear case). As a corollary, we get a multilinear De Leeuw-type restriction result for non-unimodular groups.

In Chapter 4 we prove non-boundedness results for the bilinear Hilbert transform in the case of $p = 1$. This shows that a result of [AU20], [DLMV22], which is a Schatten-valued version a dimension-independent version of the Lacey-Thiele result [LT99], cannot be extended to the case $p = 1$. This result relies on transference between Fourier and Schur multipliers, although not on the result proven in Chapter 3. We also give a similar result for Calderon-Zygmund operators.

In Chapter 5, we turn our attention to semigroup BMO spaces of von Neumann algebras. These BMO spaces are constructed from a quantum Markov semigroup on the von Neumann algebra. We extend (one of) the definition(s) by Junge and Mei [JM12] to the non-tracial case, in a slightly different way than Caspers [Cas19]. We prove a version of the Fefferman-Stein duality, but only for the column and row BMO spaces, and the Hardy spaces we construct are abstract in nature. We also prove that our BMO space serves as an endpoint for interpolation. As preparation for the proof of the Fefferman-Stein duality, we need to generalise some theory of L_p-modules, which we do in the first two sections.

With these BMO results under our belt, we prove L_p-boundedness of so-called Fourier-Schur multipliers in Chapter 6. These Fourier-Schur multipliers are an analogue of Schur multipliers on quantum groups. They should be considered as 'Schur multipliers in the Fourier domain', where the Fourier domain is the space of matrix coefficients of finite dimensional unitary corepresentations of the quantum group. We consider the quantum group $\mathbb{G} = SU_q(2)$ and construct a quantum Markov semigroup on it, using again a transference trick. We then prove that if a symbol $m \in \ell_\infty(\mathbb{Z})$ yields a bounded Fourier multiplier $L_\infty(\mathbb{T}) \to \mathrm{BMO}(\mathbb{T})$, then this induces a bounded Fourier-Schur multiplier on $L_p(\mathbb{G})$ for $1 < p < \infty$. In order to prove the L_∞-BMO endpoint estimate, the Fefferman-Stein duality results proven in Chapter 5 are crucial.

In a final chapter which is somewhat isolated from the rest of the thesis, we undertake a systematic study of the relative Haagerup property for σ-finite von Neumann algebras. The setting is a unital inclusion $\mathcal{N} \subseteq \mathcal{M}$ equipped with a normal faithful conditional expectation $\mathbb{E}_{\mathcal{N}} : \mathcal{M} \to \mathcal{N}$. We give our definition, and show that it does not depend on the choice of normal faithful state preserved by $\mathbb{E}_{\mathcal{N}}$. If $\mathcal{N}$ is finite, then the relative Haagerup property does not even depend on the choice of conditional expectation. It is therefore an intrinsic property of the inclusion $\mathcal{N} \subseteq \mathcal{M}$. This is the main result of the chapter. We also show that in this case, one can weaken or strengthen some of the conditions in the definition. Finally, we give several examples in cases where $\mathcal{M} = \mathcal{B}(\mathcal{H})$, $\mathcal{N}$ is finite dimensional, or $\mathcal{M}$ is the amalgamated free product of two von Neumann algebras over a common subalgebra $\mathcal{N}$.

Samenvatting

Dit proefschrift gaat over multiplicatoren en transferentie op niet-commutatieve L_p-ruimtes. Het valt binnen het vakgebied van niet-commutatieve harmonische analyse, oftewel harmonische analyse op niet-commutatieve ruimtes. Zulke ruimtes spelen een rol in verschillende gebieden van de wiskunde, zoals niet-abelse groepen, kwamtumgroepen, niet-commutatieve geometrie en de algemene theorie van operator-algebra's. Het pionierswerk van Gelfand, von Neumann en Dixmier legde al nauwe connecties bloot tussen operatortheorie en de representatietheorie van lokaal compacte groepen. Deze resultaten vormden het fundament voor de theorie van niet-commutatieve harmonische analyse.

Later is ook de L_p-theorie van niet-commutatieve ruimtes bestudeerd. Dit baande de weg voor het generaliseren van allerlei interesante stellingen en problemen uit de klassieke theorie. Dit gold ook voor een van de meest fundamentele vragen van de harmonische analyse: welke symbolen geven begrensde Fourier-multiplicatoren op L_p? Zulke Fourier-multiplicatoren zijn een voorbeeld van de multiplicatoren die we in dit proefschrift bestuderen. Op niet-commutatieve ruimtes bestaan echter ook andere interessante multiplicatoren, zoals Schur-multiplicatoren; deze zijn gedefinieerd voor begrensde operatoren op Hilbert ruimtes. Binnen de context van lokaal compacte groepen blijken Fourier en Schur-multiplicatoren sterk gerelateerd te zijn; dit werd voor het eerst bewezen door Bożejko en Fendler in [BF84]. Dit wordt een transferentieresultaat tussen Fourier en Schur-multiplicatoren genoemd. Zulke transferentieresultaten zijn een terugkerend thema in dit proefschrift.

Na de introductie en voordat we ingaan op de hoofdresultaten, geven we eerst een expositie van het benodigde achtergrondmateriaal in Hoofdstuk 2. In het bijzonder gaan we uitgebreid in op de theorie van niet-commutatieve L_p-ruimtes voor niet-traciale gewichten. Hierbij geven we bronnen voor alle resultaten of een bewijs waar een goede bron ontbreekt. Ook geven we een gedetailleerd overzicht van de theorie van lokaal compacte groepen, groeps-von Neumann-algebra's en hun niet-commutatieve L_p-ruimtes. Tenslotte geven we de definities van Fourier- en Schur-multiplicatoren en noemen we een paar resultaten over deze multiplicatoren en de connecties tussen beide.

In Hoofdstuk 3 beginnen we met het opnieuw bekijken van de definitie van lineaire Fourier-multiplicatoren op niet-commutatieve L_p-ruimtes van (niet-unimodulaire) lokaal compacte groepen. Dit is de opmaat voor een discussie over de definitie van multilineaire Fourier-multiplicatoren in deze setting. De keuze voor deze definitie blijkt subtiel te zijn. Het hoofdresultaat van dit hoofdstuk is een transferentieresultaat tussen multilineaire Fourier- en Schur-multiplicatoren op niet-commutatieve L_p-ruimtes. De ene implicatie gebruikt een aangepaste versie van een geavanceerd resultaat uit [CJKM23] dat op zijn beurt weer gebaseerd is op ingewikkelde afschattingen uit [CPPR15]. Een deel van deze afschattingen moet gegeneraliseerd worden naar algemene von Neumann-algebra's; hiervoor gebruiken we Haagerupreductie. De andere implicatie geldt alleen voor amenabele (of middelbare) groepen. Als gevolg van dit resultaat verkrijgen we een De Leeuw-restrictieresultaat voor niet-unimodulaire groepen.

In Hoofdstuk 5 bekijken we semigroep BMO ruimtes van von Neumann-algebra's. Deze BMO ruimtes worden geconstrueerd door middel van een kwantum Markovsemigroep op de von Neumann-algebra. We breiden een (van de) definitie(s) van Junge en Mei [JM12] uit naar het niet-traciale geval, op een net wat andere manier dan Caspers [Cas19]. We bewijzen een versie van de Fefferman-Stein-dualiteit, maar alleen voor de kolom- en rij-BMO ruimtes; bovendien is de Hardyruimte die we construeren abstract van aard. Verder bewijzen we dat onze BMO ruimte gebruikt kan worden als eindpunt voor interpolatie. Als voorbereiding op het bewijs van de Fefferman-Stein-dualiteit moeten we eerst wat theorie van L_p-modules generaliseren; dit doen we in de eerste twee secties.

Met deze BMO resultaten op zak, bewijzen we in Hoofdstuk 6 L_p-begrensdheid van zogenoemde Fourier-Schur-multiplicatoren. Dit zijn een soort Schur-multiplicatoren op kwantumgroepen; ze kunnen worden beschouwd als 'Schur-multiplicatoren op het Fourier-domein'. Het Fourier-domein is hierbij de ruimte van matrixcoefficienten van eindig-dimensionale unitaire corepresentaties van de kwantumgroep. We beschouwen de kwantumgroep $\mathbb{G} = SU_q(2)$ en construeren daarop een kwantum Markovsemigroep; hierbij maken we opnieuw gebruik van een transferentiemethode. Het hoofdresultaat stelt dat als een symbool $m \in \ell_\infty(\mathbb{Z})$ een begrensde Fourier-multiplicator $L_\infty(\mathbb{T}) \to \mathrm{BMO}(\mathbb{T})$ geeft, dan induceert die ook een begrensde Fourier-Schur-multiplicator op $L_p(\mathbb{G})$ voor $1 < p < \infty$. De Fefferman-Stein-dualiteit bewezen in Hoofdstuk 5 is cruciaal bij de L_∞-BMO afschatting.

Het laatste hoofdstuk is ietwat geïsoleerd ten opzichte van de rest van het proefschrift. Hierin ondernemen we een systematische studie van de relatieve Haagerupeigenschap voor σ-eindige von Neumann-algebra's. We beschouwen een unitale inclusie van von Neumann-algebra's $\mathcal{N} \subseteq \mathcal{M}$, samen met een normale trouwe conditionele verwachting $\mathbb{E}_{\mathcal{N}} : \mathcal{M} \to \mathcal{N}$. We geven onze definitie van de relatieve Haagerupeigenschap, en laten zien dat deze niet afhangt van de keuze van een normale trouwe toestand die bewaard wordt door $\mathbb{E}_{\mathcal{N}}$. Als $\mathcal{N}$ eindig is, hangt de relatieve Haagerupeigenschap zelfs niet af van de keuze van conditionele verwachting; het is dan een intrinsieke eigenschap van de inclusie $\mathcal{N} \subseteq \mathcal{M}$. Dit is het hoofdresultaat van dit hoofdstuk. We laten ook zien dat je in dit geval sommige van de voorwaarden in de definitie kunt verzwakken of versterken. Ten slotte geven we een aantal voorbeelden: we bekijken de gevallen $\mathcal{M} = \mathcal{B}(\mathcal{H})$, $\mathcal{N}$ is eindig-dimensionaal, of $\mathcal{M}$ is het geamalgameerde vrije product van twee von Neumann-algebra's over een gezamenlijke subalgebra $\mathcal{N}$.

1

Introduction

This introduction aims to give context to the material of the thesis. The style of writing will gradually become more advanced. The intent is for the beginning to be readable for any mathematics PhD student; this means more or less that the background required is no more than that of a Bachelor in Mathematics. We start by taking the reader along with the story of 'quantum mathematics'. We then give a brief, low-level overview of selected topics in harmonic analysis before combining the two to come to the main topic of this thesis: noncommutative harmonic analysis. After that, we highlight several parts of the theory, each leading up to the contents of one or two of the chapters. We will not go into much detail here; a more elaborate overview of the contents of each chapter is included in the beginning of the chapter.

Quantum mathematics

Quantum mathematics could loosely be described as the study of noncommutative objects. This field of study was inspired by the noncommutative nature of quantum physical phenomena, and the need for a mathematical framework to explain these phenomena. In particular, the Heisenberg uncertainty principle implies, from a mathematical point of view, that the operations of measuring position and momentum are mutually 'noncommutative'.

Mathematics students quickly stumble upon noncommutative objects when learning about matrix multiplication: for $n \times n$ matrices A and B, the products AB and BA do not coincide in general. Behind this simple fact is a rich mathematical theory for the beginning mathematician to explore. In quantum physical terms, carrying out a measurement corresponds to applying a self-adjoint matrix A on some vector $\psi \in \mathbb{C}^n$, called a *state*. The result of the measurement is a random eigenvalue $\lambda \in \mathbb{R}$ of A, and the probability distribution is determined by the state ψ. Moreover, the measurement 'collapses' the state ψ, meaning that the state becomes an eigenvector corresponding to the eigenvalue λ. In this formalism, the noncommutativity of matrix multiplication implies that the order in which we apply measurements makes a difference.

To explain quantum physical phenomena, it is not always enough to consider finite dimensional objects. Indeed, let P and Q be the position and momentum operators respectively; so these are the operators we apply to our physical state to measure the position and momentum. The Heisenberg commutation relation underlying the Heisenberg uncertainty principle states that

$$QP - PQ = i\hbar I$$

where $\hbar$ is the reduced Planck constant, and I is the identity operator. This equation implies that P and Q cannot be finite dimensional matrices; indeed, in that case, the left hand side would have trace 0 while the right hand side would not. This means that we need to consider infinite dimensional states on a *Hilbert space* $\mathcal{H}$ and (possibly unbounded) *linear operators* T on $\mathcal{H}$. Such linear operators (or just operators) take the place of functions, and a great amount of effort has been made for the past 90 years or so to generalise results from the classical 'commutative' theory of functions to the noncommutative realm of operators.

Let us first restrict our attention to the space of bounded operators on a Hilbert space $\mathcal{H}$. It will be denoted by $\mathcal{B}(\mathcal{H})$. Such spaces have a rich mathematical structure. We still have all the structure from finite dimensional matrix spaces; i.e. the vector space operations, multiplication, the taking of adjoints (called *involution*). In other words, $\mathcal{B}(\mathcal{H})$ forms a $*$-algebra. Moreover, we get several topological structures; the norm topology, the strong topology, the weak topology, and the list goes on. These will be discussed in Section 2.2.1. As opposed to the finite dimensional case, these topologies are all fundamentally different, leading to a significant enrichment (and complication) of the theory.

A subspace of $\mathcal{B}(\mathcal{H})$ that is closed under multiplication is called an *operator algebra*. It is this object that grants its name to the *theory of operator algebras*, of which this thesis is a part. Although the concept of an operator algebra is rather broad, the theory of operator algebras usually focuses on two specific types of operator algebras: C^*-algebras and von Neumann algebras. The latter algebras are named after John von Neumann, whom one could consider the founder of the theory of operator algebras. C^*-algebras are operator algebras that are closed under the involution and in norm. Von Neumann algebras are C^*-algebras that are moreover closed in any of the topologies mentioned in the previous paragraph. In this thesis, we will focus almost exclusively on von Neumann algebras. We will always be explicit about the Hilbert space $\mathcal{H}$ that the C^*- or von Neumann algebra is represented on, although it is possible to build an abstract theory without fixing the representation.

C^*-algebras and von Neumann algebras are said to be noncommutative analogues of topological spaces and measure spaces, respectively. This statement is justified by considering commutative C^*- and von Neumann algebras. Indeed, it can be proven that every unital commutative C^*-algebra is isomorphic to $C(X)$, the space of continuous functions on some compact Hausdorff topological space X. Commutative von Neumann algebras acting on a separable Hilbert space are moreover isomorphic to $L_\infty(X,\mu)$, where (X,μ) is a σ-finite measure space. In fact, X is again a Hausdorff compact space and μ a regular Borel measure. Here $L_\infty(X,\mu)$ acts on $L_2(X,\mu)$ by multiplication. This gives an intuitive reason why von Neumann algebras are the main object of interest in *noncommutative harmonic analysis*, the main topic of this thesis.

We will treat some general theory of von Neumann algebras in Section 2.2, and give references there containing more extensive treatment. Of particular importance is the special case of *group C^*-algebras* or *group von Neumann algebras* corresponding to a locally compact group. We will treat those in Section 2.6.

Classical harmonic analysis

Harmonic analysis is a wide area that is not easily described in one sentence, but at its basis it is more or less the theory that studies functions using their Fourier expansions. In other words, the idea is to to decompose complicated functions into simpler 'harmonic' functions. It has a wide range of uses in partial differential equations and many other areas of mathematics. We will focus here on a few parts of harmonic analysis that are relevant for the thesis.

One part of harmonic analysis is what is usually called Fourier analysis; i.e. the analysis of a function f via its Fourier transform $\mathscr{F}(f)=\hat{f}$. Related to the Fourier transform is the notion of Fourier multipliers. If ϕ is a bounded function on $\mathbb{R}^d$, then the *Fourier multiplier* T_ϕ with symbol ϕ is the operator given by multiplication with ϕ in the Fourier domain, i.e.

$$T_\phi(f)=\mathscr{F}^{-1}(\phi\hat{f}).$$

If ϕ is the Fourier transform of some integrable function ψ, then the Fourier multiplier is nothing but left convolution with ψ. In turn, convolution operators are exactly those operators that commute with translations, and these have important applications in the study of partial differential equations.

We recall that the space $L_p(\mathbb{R}^d)$ contains all functions $f:\mathbb{R}^d\to\mathbb{C}$ for which $|f|^p$ is integrable. One of the most important problems in harmonic analysis is to find conditions on symbols to define bounded Fourier multipliers on L_p-spaces. For $p=2$, this is true for all bounded symbols. If the Fourier multiplier is bounded on $L_p(\mathbb{R}^d)$, then it is bounded also on $L_q(\mathbb{R}^d)$ where $\frac{1}{p}+\frac{1}{q}=1$, with the same norm. The celebrated Mikhlin multiplier theorem gives sufficient conditions for $1<p<\infty$ in terms of the decay of the partial derivatives of the symbol ϕ. For these and more results in this direction, we refer to [Gra14a].

Some tools that have been used to great results in this endeavour are complex interpolation and BMO spaces. In its most basic form, complex interpolation tells us that if an operator T is bounded on $L_{p_1}(\mathbb{R}^d)$ and $L_{p_2}(\mathbb{R}^d)$ where $1\le p_1\le p_2\le\infty$, then it is bounded on $L_p(\mathbb{R}^d)$ for $p_1\le p\le p_2$. Its norm is then bounded by the Riesz-Torin formula:

$$\|T:L_p(\mathbb{R}^d)\to L_p(\mathbb{R}^d)\|\le\|T:L_{p_1}(\mathbb{R}^d)\to L_{p_1}(\mathbb{R}^d)\|^{1-\theta}\|T:L_{p_2}(\mathbb{R}^d)\to L_{p_2}(\mathbb{R}^d)\|^{\theta}$$

where θ is such that $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$. The 'endpoint' values p_1, p_2 are usually taken to be either 1, 2 or ∞. However, the cases $p = 1, \infty$ can sometimes be the exceptions where an operator is not bounded. If this happens, one can sometimes use *BMO spaces* as a replacement for $L_\infty(\mathbb{R}^d)$.

A function of *bounded mean oscillation* is a function that 'does not oscillate too wildly'. Let us describe this more precisely. For any cube $Q \subseteq \mathbb{R}^d$, the mean of a (locally integrable) function f on $\mathbb{R}^d$ is denoted by $f_Q := \frac{1}{|Q|}\int_Q f\,ds$. The oscillation, or distance from the average, in a point $s \in Q$ is given by $|f(s) - f_Q|$. For the mean oscillation, we take the square of this term and average over Q: $\frac{1}{|Q|}\int_Q |f(s) - f_Q|^2\,ds$. Finally, we say that a function has *bounded mean oscillation* if

$$\|f\|_{\mathrm{BMO}} := \left(\sup_Q \frac{1}{|Q|}\int_Q |f(s) - f_Q|^2\,ds\right)^{1/2} < \infty$$

where the supremum is taken over all cubes $Q \subseteq \mathbb{R}^d$. The space of all functions with bounded BMO-norm is denoted by $\mathrm{BMO}(\mathbb{R}^d)$. However, $\|\cdot\|_{\mathrm{BMO}}$ is only a seminorm, as $\|f\|_{\mathrm{BMO}} = 0$ only implies that f is constant. Hence, in the space $\mathrm{BMO}(\mathbb{R}^d)$, a usual convention is to identify functions whose difference is constant.

Clearly, one has $L_\infty(\mathbb{R}^d) \subseteq \mathrm{BMO}(\mathbb{R}^d)$ with $\|f\|_{\mathrm{BMO}} \leq 2\|f\|_\infty$. This inclusion is indeed strict, as the unbounded function $f(x) = \log(|x|)$ is in $\mathrm{BMO}(\mathbb{R}^d)$. As we alluded to earlier, one has the following interpolation result: if an operator is bounded on $L_p(\mathbb{R}^d)$ and bounded as an operator $L_\infty(\mathbb{R}^d) \to \mathrm{BMO}(\mathbb{R}^d)$, then it is bounded on $L_q(\mathbb{R}^d)$ for $p \leq q < \infty$ with

$$\|T : L_q(\mathbb{R}^d) \to L_q(\mathbb{R}^d)\| \leq \|T : L_p(\mathbb{R}^d) \to L_p(\mathbb{R}^d)\|^{p/q}\|T : L_\infty(\mathbb{R}^d) \to \mathrm{BMO}(\mathbb{R}^d)\|^{1-p/q}.$$

As an example, Calderon-Zygmund operators are bounded from $L_\infty(\mathbb{R}^d)$ to $\mathrm{BMO}(\mathbb{R}^d)$. Many are not bounded on $L_\infty(\mathbb{R}^d)$ though, including the Hilbert and Riesz transforms. One way to prove this is via a duality argument. This brings us to another important property of BMO spaces: the Fefferman-Stein duality. This duality states that BMO is the Banach dual of the so-called Hardy space H^1. We will not give a definition here, but it is a concretely defined space that is an object of intensive study in its own right. The duality argument mentioned above consists of first proving boundedness from H^1 to $L_1(\mathbb{R}^d)$, and then taking adjoints. For more about BMO and Hardy spaces, we refer to [Gra14b].

NONCOMMUTATIVE HARMONIC ANALYSIS

With the knowledge of the previous two sections, we can now introduce the main topic of this thesis: a 'quantised' or noncommutative version of harmonic analysis. As mentioned in the first section, the goal is to build a theory that replaces functions by operators, but retains as much of the original theory as possible. The first issue at hand, then, is how to define L_p-spaces of operators. This is where von Neumann algebras, the noncommutative variant of measure spaces, come into play. They take the role of the L_∞-space, and are used as a basis to construct a family of L_p-spaces with properties very similar to the classical ones. What makes von Neumann algebras so suitable in this context is the existence of a good analogue for integration over a measure: a so-called weight. These are positive functionals on the von Neumann algebra that are allowed to take infinite values. Not all weights are suitable to build a satisfying L_p-theory, but a sufficiently 'nice' weight (namely a normal, faithful and semifinite one) always exists. Moreover, it turns out that the resulting L_p-spaces do not depend on the choice of such a 'nice' weight.

Given a von Neumann algebra $\mathscr{M}$ (and a 'nice' weight), let us denote the corresponding L_p-spaces by $L_p(\mathscr{M})$ for now. Classically, L_p-functions need not be bounded; similarly, operators in $L_p(\mathscr{M})$ will be unbounded operators in general. The simplest case occurs when the von Neumann algebra admits a 'nice' weight that is tracial; i.e. when $\mathscr{M}$ is *semifinite*. In this case, the L_p-spaces overlap, and all are contained in the set of operators affiliated to $\mathscr{M}$ (see Section 2.2.4 for a definition). This is the case that is the most similar to L_p-spaces on $\mathbb{R}^d$. If our trace is moreover finite, i.e. an actual positive functional, then the von Neumann algebra is called *finite* and we have $L_p(\mathscr{M}) \subseteq L_q(\mathscr{M})$ for $p \geq q$. This corresponds to the case of a compact measure space. If on the other hand $\mathscr{M} = \mathscr{B}(\mathscr{H})$, or $\mathscr{M} = \bigoplus_{i\in I} \mathscr{B}(\mathscr{H}_i)$, then $L_p(\mathscr{M}) \supseteq L_q(\mathscr{M})$ for $p \geq q$. This corresponds to discrete measure spaces.

The theory of noncommutative L_p-spaces with respect to a trace was developed already in the 50's in [Dix53], [Kun58]; see also [Nel74] and [Yea75]. The basic idea is quite simple: we extend the trace to positive self-adjoint affiliated operators through the spectral decomposition, and simply replace integration by the trace:

$$\|x\|_p = \tau(|x|^p)^{1/p}; \qquad L_p(\mathscr{M}) = \{x : x \text{ affiliated with } \mathscr{M}, \tau(|x|^p) < \infty\}.$$

For a long time, there was no alternative for non-tracial weights. The difficulty is that, if you try using the above definition, then the p-norm will not satisfy the triangle inequality. One can even construct a counterexample for this with $2x2$ matrices for $p = 1$: take matrices x and y such that $|x + y| > |x| + |y|$ (e.g. $x = I$, the identity matrix, and $y = e_{21}$, the matrix with a 1 in the lower left corner) and find a corresponding positive functional φ for which $\varphi(|x + y| - |x| - |y|) > 0$.

Haagerup [Haa79a] was the first to propose a construction of noncommutative L_p-spaces with respect to a 'nice' weight. His solution was to consider the crossed product, and use the trace that comes with it. Later, Hilsum [Hil81] created a different construction of operators that act on the original Hilbert space using Connes' theory of spatial derivatives [Con76], and it is this construction that will be used in most of the thesis. Later, Kosaki [Kos84], Terp [Ter82] and Izumi [Izu97] showed that these L_p-spaces allow interpolation arguments. In fact, complex interpolation gives a third way to construct these L_p-spaces. If the weight considered is a finite weight, i.e. a positive functional, the construction simplifies somewhat. We usually assume that the positive functional is normalised (meaning that its value in the identity is 1), i.e. it is a *state*. If a von Neumann algebra admits such a 'nice' state, it is called σ-finite.

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L_p-spaces corresponding to a non-tracial weight are significantly more complicated than their tracial counterparts; both in their definition and in their use. Nowadays, new results in noncommutative harmonic analysis are still often written only in the semifinite or finite cases. A large part of this thesis will be dedicated to generalising results from the finite or semifinite to the σ-finite or general case. For that reason, we will give an extensive overview of the latter two methods and corresponding results in Section 2.4.

In the next two sections, we will look at noncommutative variants of two other topics discussed in the previous section on classical harmonic analysis: Fourier multipliers and BMO spaces.

FOURIER AND SCHUR MULTIPLIERS

Before, we defined Fourier multipliers on $\mathbb{R}^d$, as we have a Fourier transform handy there. The Fourier transform can be similarly defined for any abelian locally compact group G. It makes use of the fact that an abelian locally compact group has a *dual group* $\hat{G}$, which consists of all group homomorphisms $G \to \mathbb{T}$ mapping to the unit circle $\mathbb{T}$ of $\mathbb{C}$. Such homomorphisms are called *characters*. The Fourier transform sends functions on G to functions on the dual group $\hat{G}$ of characters on G. Instead of the Lebesgue measure, we use the *left Haar measure* of G in order to define the spaces $L_p(G)$. This is a measure that is invariant under left translations (and satisfies other 'nice' properties), and every locally compact groups has a unique one. Now, the Fourier transform maps $L_1(G)$ to $C_0(\hat{G})$, and it defines an isometric isomorphism $L_2(G) \cong L_2(\hat{G})$.

Just as in the case $G = \mathbb{R}^d$, we can define a Fourier multiplier by multiplying with some symbol ϕ in the Fourier domain. The symbol will now be a function in $L_\infty(\hat{G})$. Many of the basic properties holding on $\mathbb{R}^d$ also hold on G. We note that there is a different way to view Fourier multipliers which does not involve the dual group. This involves the *left regular representation*, which sends a function f to the convolution operator $\lambda(f) : g \mapsto f * g$. It also involves the *group von Neumann algebra* of G, denoted by $\mathcal{L}G$. This is the von Neumann algebra generated by the image of $L_1(G)$ in $\mathcal{B}(L_2(G))$ under the left regular representation. As it turns out, upon conjugation with the Fourier transform, the group von Neumann algebra $\mathcal{L}G$ gets transformed into $L_\infty(\hat{G})$; i.e. $\mathcal{F}\mathcal{L}G\mathcal{F}^{-1} = L_\infty(\hat{G})$. Moreover, under this transformation, an element $\lambda(f)$ gets transformed into $\hat{f}$. So, if we reverse the roles of G and $\hat{G}$, then we can define the Fourier multiplier with symbol $\phi \in L_\infty(G)$ on $\mathcal{L}G$ by $\lambda(f) \mapsto \lambda(\phi f)$. Of course, this assignment might not extend to a bounded operator on $\mathcal{L}G$; it does if and only if the Fourier multiplier in the 'original' sense is bounded on $L_\infty(\hat{G})$.

When a locally compact group is not abelian, there is no dual group or Fourier transform available. But in that case, the map $\lambda(f) \mapsto \lambda(\phi f)$ above still makes sense. This is now taken as the definition of the Fourier multiplier. The noncommutative L_p-spaces $L_p(\mathcal{L}G)$ now take the place of $L_p(\hat{G})$, and we may wonder when the symbol ϕ induces a bounded Fourier multiplier on $L_p(\mathcal{L}G)$. For $p = \infty$, the answer is known; it was proven by Eymard [Eym64] that the map $\lambda(f) \mapsto \lambda(\phi f)$ extends to a bounded map on $\mathcal{L}G$ precisely when the symbol ϕ is in the multiplier algebra $M(A(G))$. Here $A(G)$ is Eymard's *Fourier algebra*. The same symbols work for $p = 1$. For $p = 2$, all symbols in $L_\infty(G)$ work, just as in the abelian case.

Another interesting question is when the corresponding Fourier multipliers are *completely bounded* (see Section 2.5). This is where *Schur multipliers* come into play. Schur multiplication is what one could define as 'naive matrix multiplication'; it is what an elementary schooler would probably do if you gave them two squares of numbers and asked them to somehow multiply these. Namely, he would carry out a pointwise multiplication of each of the entries. So, if x is a $n \times n$ matrix, a Schur multiplier with symbol x is defined by

$$M_x : y \mapsto (x_{ij} y_{ij})_{i,j}.$$

One can also define Schur multipliers on 'continuous matrices'. With this, we mean elements from $S_2(L_2(X))$, the Hilbert-Schmidt class operators on functions on a measure space X. These elements can be written as integral operators over some kernel in $L_2(X \times X)$, and it is this kernel that takes the role of the matrix.

Let us now come back to our question about completely bounded Fourier multipliers. Bożejko and Fendler [BF84] proved that a symbol ϕ defines a completely bounded Fourier multiplier on $\mathcal{L}G$ precisely when the 'diagonal symbol' $(s,t) \mapsto \phi(st^{-1})$ defines a bounded Schur multiplier on $\mathcal{B}(\mathcal{H})$. This is called a transference result between Fourier and Schur multipliers. Similar transference results hold for $p < \infty$: this was proven for discrete groups by Neuwirth and Ricard [NR11], and in the general case by Caspers and de la Salle [CS15a]. However, for the Schur to Fourier direction, the group needs to be amenable. We will give some intuition behind the relation between Fourier and Schur multipliers and an overview of these results in Section 2.6.5.

The relation between Fourier and Schur multipliers has been an important tool to prove several multiplier results. For instance, bounding the norm of Fourier multipliers by that of Schur multipliers played a crucial role in [PRS22]. The converse transference was used in [Pis98] to give examples of bounded multipliers on L_p-spaces that are not completely bounded. Similar transference techniques were used in [CGPT23] to prove Hörmander-Mikhlin criteria for the boundedness of Schur multipliers, and in [LS11] to find examples of non-commutative L_p-spaces without the completely bounded approximation property.

In Chapter 3, we focus our attention on multilinear multipliers. Multilinear Schur multipliers were defined in [JTT09]. Such multipliers and the related notion of multiple operator integrals have been used to prove several interesting results such as the resolution of Koplienko's conjecture on higher order spectral shift functions in [PSS13]. Todorov and Turowska [TT10] defined a multidimensional Fourier algebra and proved a transference result for multiplicatively bounded multilinear Fourier and Schur multipliers. A bilinear transference result for L_p-spaces of discrete groups was proven 'along the way' in [CJKM23, Proof of Theorem 7.2], in order to provide examples of L_p-multipliers for semidirect products of groups. In Chapter 3 we finish the picture by proving transfer-

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ence results for general Hölder combinations of L_p-spaces.

BOUNDEDNESS OF VECTOR-VALUED MULTIPLIERS

In Chapter 4, we will give some applications of transference between Fourier and Schur multipliers. The setting will be a bit different however, namely we consider bilinear Fourier multipliers on matrix-valued functions on $\mathbb{R}$. Hence, this chapter should be seen not as an immediate application of the results of Chapter 3, but rather as a collection of separate interesting results which happen to use similar techniques.

We consider two boundedness results on multilinear mappings on $L_p(\mathbb{R}, S_q^m)$. The first, by Lacey and Thiele [LT99], proves that the bilinear Hilbert transform is bounded as a map $L_{p_1}(\mathbb{R}, S_{q_1}^m) \times L_{p_2}(\mathbb{R}, S_{q_2}^m) \to L_p(\mathbb{R}, S_q^m)$, with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, as long as $\frac{2}{3} < p < \infty$ and $1 < p_1, p_2 < \infty$. Moreover, for $1 < p < \infty$ and under extra conditions on the values of q, q_1, q_2, [AU20] and [DLMV22] independently proved that this operator is bounded uniformly in m. Hence, the result can be extended to certain Banach-valued function spaces.

The Hilbert transform is generally one of the first examples that one considers when analysing boundedness properties of multipliers or, more generally, singular integral operators. Another class of singular integral operators is the class of Calderón-Zygmund operators. These are operators given by integration against a kernel, where the kernel has to satisfy certain properties. The paper [DLMV20] considers multilinear Calderón-Zygmund operators on scalar-valued functions. These can be 'extended' to act on vector-valued functions. [DLMV20] proves that a multilinear Calderon-Zygmund operator on scalar-valued functions extends to a bounded operator on functions with values in a so-called *UMD* Banach space. This result holds only for $p > 1$ in the range space. This is the second boundedness result we consider.

In Chapter 4, we prove that both of these results cannot be extended to $p = 1$ in the range space. The idea is to consider functions with values in $m \times m$-matrices, and prove that the associated operators are bounded from below by $C\log(m)$. We do this first for the bilinear Hilbert transform. Then we give an example of a bilinear Calderon-Zygmund operator (or really just a Fourier multiplier) that is bounded as a map $L_{p_1} \times L_{p_2} \to L_p$ for $\frac{1}{2} < p < \infty$, but whose bound on $m \times m$-matrix-valued functions for $p = 1$ is bounded below by $C\log(m)$. Both proofs use transference to Schur multipliers.

NONCOMMUTATIVE BMO SPACES

Let us now go back to the setting of a general von Neumann algebra. The first instances of noncommutative BMO spaces were defined in [PX97]. This paper uses noncommutative martingales to define noncommutative Hardy and BMO spaces, and proves the noncommutative analogue of the Fefferman-Stein duality $(H^1)^* = \mathrm{BMO}$. We will refer to these BMO spaces as *martingale BMO spaces*. This work was continued by several authors. [Pop00] proved a noncommutative analogue of the classical boundedness results on the Riesz transform; [Mus03] studied complex interpolation for martingale BMO spaces. [JM07] proved the noncommutative analogue of the John-Nirenberg theorem, an important result describing integrability properties of BMO functions, and in [JP14] the methods from [PX97] were extended for continuous martingales.

Martingale BMO spaces are however not the BMO spaces we will be considering in this paper. The disadvantage is that they require the existence of a filtration of the von Neumann algebra. For some applications this structure is insufficient, see e.g. [JMP14], [Mei17], [Cas19], [CJSZ20]. We will focus instead on *semigroup BMO spaces*, introduced by Tao Mei in [Mei08]. It uses the classical idea of replacing averages over cubes by diffusion semigroups, which goes back (at least) to [Var85], [SV74]. Much more recently an analysis of duality and comparison of several such BMO-spaces was carried out in [DY05a], [DY05b]. In the noncommutative case, Tao Mei used instead *Markov semigroups*. These are families of unital completely positive maps on the von Neumann algebra satisfying certain extra conditions. As opposed to the classical case, we have separate *column* and *row* BMO spaces, that are each others adjoints. The column BMO norm of an element $x \in L_2(\mathcal{M})$ with respect to a Markov semigroup $\Phi := (\Phi_t)_{t\geq 0}$ is defined by

$$\|x\|^2_{\mathrm{BMO}^c_\Phi} = \sup_{t\geq 0} \|\Phi_t(|x - \Phi_t(x)|^2)\|_\infty,$$

where the Markov maps Φ_t extend naturally to $L_2(\mathcal{M})$ and $L_1(\mathcal{M})$. Again, this is only a seminorm. To get a normed space $\mathrm{BMO}^c(\mathcal{M},\Phi)$, we will have to divide out some nondegenerate part. The space $\mathrm{BMO}^c(\mathcal{M},\Phi)$ is then defined by taking all equivalence classes in $L_2(\mathcal{M})$ for which the column BMO norm is finite. The row norm is defined as the column norm of the adjoint, and the row BMO space is defined as equivalence classes with finite row BMO norm. Finally, the BMO norm is the maximum of the column and row norms and the BMO space is the intersection of the column and row spaces.

The theory of semigroup BMO spaces was further developed in [JM12]. This paper considers several variants of the BMO norms and proves interpolation results, using the results for martingale BMO spaces. These BMO spaces have been studied by several authors; we highlight the recent paper [JMPX21] where a very general Calderon-Zygmund theory was achieved via a 'metric' BMO space. It should be noted that these papers only consider finite von Neumann algebras. [Cas19] studies BMO spaces for σ-finite von Neumann algebras and generalises the interpolation result from the finite case. However, in [Cas19] BMO is defined by only considering x in $\mathcal{M}$ and then taking an abstract completion with respect to the BMO norm. This 'smaller BMO space' has the benefit that basic properties like the triangle inequality and completeness follow rather easily.

In Section 6.4, we introduce semigroup BMO spaces for σ-finite von Neumann algebras. Here we stay closer to the 'larger BMO space' of L_2-elements with finite BMO-norm as defined above, and show that the triangle inequality and completeness still hold. We do this by proving a Fefferman-Stein duality result. Our predual will be of an abstract nature, unlike the concretely defined Hardy spaces whose dual are the martingale BMO spaces. We will also only prove that the interpolation results from [Cas19] still hold for our 'larger' BMO spaces.

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MULTIPLIERS ON QUANTUM GROUPS

We have described how von Neumann algebras play a role in the Fourier analysis of non-abelian locally compact groups. If we go yet one step further in the generalisation/quantisation process, we get the concept of a *quantum group*. For a quantum group $\mathbb{G}$, there is no longer an underlying group but *only* an 'algebra of functions'. This can be either a Hopf algebra (when one wants to work purely algebraically) or a C^*-or von Neumann algebra (when one wants to work topologically as well). The latter two are denoted by $C(\mathbb{G})$ and $L_\infty(\mathbb{G})$ respectively, and will be our main source of interest. These algebras have some additional structure that describes the 'group laws'. For instance, it has a *comultiplication* $\Delta : C(\mathbb{G}) \to C(\mathbb{G}) \otimes C(\mathbb{G})$ that describes the 'group multiplication'. If $\mathbb{G} = G$ is an actual locally compact group, then the comultiplication is given by $\Delta f(s,t) = f(st)$.

We will focus on the case of *compact quantum groups* as developed by Woronowicz [Wor98]. There is also a notion of locally compact quantum groups, but this theory is outside the scope of this thesis. For a compact quantum group, one can define the concept of unitary representation. Specifically, we will need the concept of a finite dimensional representation. If G is a compact group, then a finite dimensional (strongly continuous, unitary) representation is a strongly continuous group homomorphism $u : G \to \mathcal{U}(\mathbb{C}^n)$. We can also identify u as an element $(u_{ij})_{i,j} \in M_n(C(G))$. In this identification, the group homomorphism property means $u_{ij}(pq) = \sum_k u_{ik}(p)u_{kj}(q)$. Generalising this picture, a finite dimensional representation of a quantum group Γ is an element $u \in M_n(C(\mathbb{G}))$ such that $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$.

In Section 6.2 we will define so-called *Fourier-Schur* multipliers on compact quantum groups. These can be viewed as Schur multipliers on finite dimensional representations. The main result of Chapter 6 is an L_p-boundedness results for Fourier-Schur multipliers on the quantum group $SU_q(2)$. The proof uses our interpolation result on BMO spaces as well as our construction of the preduals $h_1^r(\mathcal{M},\Phi)$, $h_1^c(\mathcal{M},\Phi)$. This chapter should therefore be viewed as an application of the results of Chapter 5.

In Chapter 6, we will only consider the quantum group $SU_q(2)$. These are so-called *q-deformations* of the compact group $SU(2)$. Also, at the very end of Chapter 7, we will very briefly consider the free orthogonal quantum group O_F^+. Hence, we will not need the general theory of (compact/locally compact) quantum groups very much. We will only give a short recap of the definitions in Section 6.1.2. For an accessible treatment of compact quantum groups, we refer the reader to the lecture notes [MV98]. For a more comprehensive treatment about quantum groups in general, the reader may consult the book [Tim08]

HAAGERUP PROPERTY

Chapter 7 is a bit different from the rest of the thesis, as it doesn't really have anything to do with noncommutative harmonic analysis. Indeed, the noncommutative L_p-spaces will not appear here. Instead, it deals with approximation properties of groups and their generalisations to von Neumann algebras.

The prototypical example of an approximation property of a group is *amenability*. Let us limit ourselves to discrete groups for ease of exposition. One of the many equivalent definitions of amenability for a discrete group G is that there exists a net of positive definite, finitely supported functions $(\phi_i)_{i\in I}$ on G such that $\phi_i \to 1$ pointwise. Two weaker versions of this property have been studied. The first relaxes the finite support condition: we say that G has the *Haagerup property* (or *HAP*) if there exists a net of positive definite functions $(\phi_i)_{i\in I}$ vanishing at ∞ such that $\phi_i \to 1$. Here, a function ϕ vanishing at ∞ means that for every $\varepsilon > 0$, the set $\{s \in G : |\phi_i(s)| > \varepsilon\}$ is finite. This is the property we are interested in. The second property is called *weak amenability*, and it relaxes the positive definite condition. We will not give the definition here since we will not use this property; see [BO08a, Section 12.3] for more information. As an example, the free group $\mathbb{F}_n$ has the Haagerup property and is weakly amenable but not amenable ([Haa79c]).

Similar definitions can be made for von Neumann algebras. A von Neumann algebra is said to be *amenable* (or *semidiscrete*) if there exists a net $(\Phi_i)_{i\in I}$ of finite rank, completely positive normal maps $\mathcal{M} \to \mathcal{M}$ such that $\sup_i \|\Phi_i\| < \infty$ and $\Phi_i(x) \to x$ strongly for all $x \in \mathcal{M}$. A discrete group G is amenable if and only if the group von Neumann algebra $\mathcal{L}G$ is amenable. Here too, we can relax the finite rank condition. Let φ be a normal faithful state on a von Neumann algebra $\mathcal{M}$. Then $\mathcal{M}$ has the *Haagerup property* (or *HAP*) if there exists a net $(\Phi_i)_{i\in I}$ of normal maps $\mathcal{M} \to \mathcal{M}$ such that

1. $\Phi_i(x) \to x$ strongly for each $x \in \mathcal{M}$
2. For each $i \in I$, Φ_i is completely positive
3. $\sup_{i\in I} \|\Phi_i\| < \infty$
4. $\varphi \circ \Phi \leq \varphi$
5. $\Phi_i^{(2)}$ is compact.

The reader is encouraged to compare this definition with Definition 7.2.2. Here $\Phi_i^{(2)}$ is the 'L_2-implementation' of Φ_i, see Section 7.1. The first three conditions overlap with those of amenability, while the latter two should be seen as a relaxation of the finite rank property. We note that a priori, this definition seems to depend on the choice of normal faithful state. However, by [CS15b, Theorem 5.6], the HAP is independent of such a choice. Note that the Haagerup property is actually defined for normal faithful semifinite weights in [CS15b], but we will restrict our attention to the case of states.

Let us give some background to these approximation properties. The Haagerup property is the property that Haagerup proved for free groups in [Haa79c]. Haagerup then used this property to prove other approximation properties for free groups. The Haagerup property is often used as a sort of opposite to Kazhdan's property (T): a group has both properties if and only if it is compact. We refer to [BO08a, Chapter 12]. Another famous result is that groups with HAP satisfy the Baum-Connes conjecture [HK01]. It was Choda [Cho83] who gave the above definition of HAP for von Neumann algebras with a normal faithful trace. Jolissaint [Jol02] proved that this definition does not depend on the choice of trace. Moreover Bannon and Fang [BF11] showed that some of the conditions in the definition can be weakened.

For several years the study focused on finite von Neumann algebras, mainly as the motivating examples came from discrete groups. This changed with the articles [Bra12a], [DFY14], which established the Haagerup property for the von Neumann algebras of certain discrete quantum groups, and the paper [DFSW16], which introduced and studied the analogous property for quantum groups themselves. A definition of the Haagerup property for general von Neumann algebras was then given by Caspers and Skalski in the aforementioned paper [CS15b], and a different equivalent definition was given simultaneously by Okayasu and Tomatsu in [OT15].

In Chapter 7, we consider the notion of *relative Haagerup property* (or *rHAP*). Such relative properties have been studied in several group-theoretic and operator algebraic contexts. For example relative Property (T) is key to showing that $\mathbb{Z}^2 \rtimes SL_2(\mathbb{Z})$ does not have the Haagerup property. In the context of tracial von Neumann algebras, the relative Haagerup property appeared first in [Boc93] in the study of Jones' towers associated with irreducible finite index subfactors. It was later applied in [Pop06] as a key tool to obtain deep structural results about algebras admitting a certain type of 'Cartan inclusion'. The case of Cartan subalgebras was also the first in which a definition of a relative Haagerup property was proposed beyond finite von Neumann algebras [Ued06], [Ana13]. Notably the latter developments took place even before the usual Haagerup property for arbitrary von Neumann algebras was well understood.

In Chapter 7 we propose a definition of the relative Haagerup property for general σ-finite von Neumann algebras. More precisely, we consider a unital inclusion $\mathscr{N} \subseteq \mathscr{M}$ equipped with a faithful normal conditional expectation $\mathbb{E}_{\mathscr{N}} : \mathscr{M} \to \mathscr{N}$. Again we first define it in terms of a fixed faithful normal state (preserved by $\mathbb{E}_{\mathscr{N}}$) but then quickly show that it depends only on the conditional expectation in question. Much more can be said in the case where $\mathscr{N}$ is assumed to be finite; in this case the rHAP does not even depend on the choice of conditional expectation. It is therefore an intrinsic property of the inclusion $\mathscr{N} \subseteq \mathscr{M}$. We also show that in this case, one can weaken or strengthen some of the conditions in the definition. Finally, we give several examples in cases where $\mathscr{M} = \mathscr{B}(\mathscr{H})$, $\mathscr{N}$ is finite dimensional, or $\mathscr{M}$ is the amalgamated free product of two von Neumann algebras over a common subalgebra $\mathscr{N}$.

2

Preliminaries & Notation

In this chapter, we will give an overview of the material needed for this thesis. We have found the existing literature lacking in certain aspects of the theory; in particular there seems to be no good expository text which treats embeddings of Connes-Hilsum L_p-spaces for normal semifinite faithful weights. Similarly, we have not found satisfactory expositions of Connes-Hilsum L_p-spaces of non-unimodular locally compact groups, although the appearance of Terp's final article [Ter17] has partially filled that gap. Due to these facts, we have chosen to give a rather extensive picture of these topics, including even results that are not strictly necessary for the thesis. We have made an effort to include proofs or references to sources containing clear proofs for all claims that we make. This means that Sections 2.4 and 2.6 are perhaps longer than strictly necessary. However, the hope is that these sections might be helpful for people that are not as familiar with noncommutative L_p-spaces of general von Neumann algebras.

2.1. Notation and general preliminaries

Let us first fix some general notation. We shall use the convention $\mathbb{N} = \{1, 2, \ldots\}$. With an isomorphism (of Banach spaces), we shall mean a linear bijection that is bounded and whose inverse is also bounded. We write $\cong$ when the isomorphism is isometric.

We use the following notation for tensor products:

- $A \otimes B$ for the algebraic tensor product of vector spaces.
- $\mathscr{M} \bar{\otimes} \mathscr{N}$ for the von Neumann algebra tensor product.
- $\mathscr{A} \otimes_{\min} \mathscr{B}$ for the minimal tensor product of C^*-algebras.
- $\mathscr{H} \otimes_2 \mathscr{K}$ for the Hilbert space tensor product.

The following standard result shall be used several times in this paper. The proof follows directly from the definitions.

Proposition 2.1.1 (See [Con90]). *Let X, Y be Banach spaces and $T: X \to Y$ a bounded linear map. Then $T^*: Y^* \to X^*$ is weak-$*$/weak-$*$ continuous.*

Let us recall the notion of completely positive maps here. Let $\mathscr{H}$ be a Hilbert space and $V \subseteq \mathscr{B}(\mathscr{H})$ be an *operator system*, i.e. a norm closed, $*$-closed subspace containing the identity $1_{\mathscr{B}(\mathscr{H})}$. Let $M_n(V) := M_n(\mathbb{C}) \otimes V$ be the space of $n \times n$-matrices with coefficients in V. Then $M_n(V)$ has a natural ordering given by the positive cone $M_n(V)^+ := M_n(V) \cap \mathscr{B}(\mathscr{H}^n)^+$. Positive maps on $M_n(V)$ are then those maps on $M_n(V)$ sending $M_n(V)^+$ to itself. Now let V, W be operator systems. A map $\Phi: V \to W$ is called *completely positive* if the maps

$$\Phi^{(m)}: M_n(V) \to M_n(W), \qquad (x_{ij})_{i,j} \mapsto (\Phi(x_{ij}))_{i,j}$$

are positive for all $m \geq 1$. More restrictively, Φ is called n-positive if $\Phi^{(m)}$ is positive for all $1 \leq m \leq n$. We refer to [ER00, Chapter 5] for more information about operator systems and completely positive maps. In this thesis, the only operator systems we will encounter are von Neumann algebras.

2.2. PRELIMINARIES ON VON NEUMANN ALGEBRAS

For general von Neumann algebra theory we refer to [Mur90] or Takesaki's books [Tak02], [Tak03a], [Tak03b].

2.2.1. OPERATOR TOPOLOGIES

As mentioned in the introduction, von Neumann algebras $\mathscr{M}$ are $*$-subalgebras of $\mathscr{B}(\mathscr{H})$ that are closed under some operator topology. Since these topologies are all weaker than the norm topology, every von Neumann algebra is a C^*-algebra. There are several topologies that work equally well to define von Neumann algebras. The most common topologies are:

- The strong topology. This is the locally convex topology on $\mathscr{B}(\mathscr{H})$ determined by the seminorms $x \mapsto \|x\xi\|$, for $\xi \in \mathscr{H}$. A net $(x_\lambda)_\lambda$ in $\mathscr{B}(\mathscr{H})$ converges to x in the strong topology whenever $x_\lambda \xi \to x\xi$ in $\mathscr{H}$ for all $\xi \in \mathscr{H}$.

- The weak topology. This is the locally convex topology on $\mathscr{B}(\mathscr{H})$ determined by the seminorms $x \mapsto |\langle x\xi, \eta\rangle|$ for $\xi, \eta \in \mathscr{H}$. A net $(x_\lambda)_\lambda$ in $\mathscr{M}$ converges to x in the weak topology whenever $\langle x_\lambda \xi, \eta\rangle \to \langle x\xi, \eta\rangle$ in $\mathbb{C}$ for all $\xi, \eta \in \mathscr{H}$.

- The σ-weak topology. This is the locally convex topology on $\mathscr{B}(\mathscr{H})$ determined by the seminorms $x \mapsto |\sum_n \langle x\xi_n, \eta_n\rangle|$ for sequences $\xi_n, \eta_n \in \mathscr{H}$ such that $\sum_n \|\xi_n\|^2 < \infty$, $\sum_n \|\eta_n\|^2 < \infty$. A net $(x_\lambda)_\lambda$ in $\mathscr{M}$ converges to x in the σ-weak topology whenever $\sum_{n=1}^\infty \langle x_\lambda \xi_n, \eta_n\rangle \to \sum_{n=1}^\infty \langle x\xi_n, \eta_n\rangle$ for all sequences ξ_n, η_n as above. As it turns out, every von Neumann algebra has a unique predual $\mathscr{M}_*$, and the σ-weak topology coincides with the resulting weak-$*$ topology on $\mathscr{M}$.

There exist also the σ-strong, the strong-$*$ and the σ-strong-$*$ topologies, but these will rarely be used in the thesis. Note that the $*$ in the strong-$*$ topology refers to the fact that the involution is continuous, so this does not have the same meaning as the $*$ in the weak-$*$ topology! For more information on all these topologies, we refer to [Tak02, Section II.2]. We recall here some important facts, that will be used throught the thesis without reference. Firstly, the various topologies are related as follows:

$$\begin{array}{ccccccc} \text{Norm} & \prec & \sigma\text{-strong-}* & \prec & \sigma\text{-strong} & \prec & \sigma\text{-weak} \\ & & \curlywedge & & \curlywedge & & \curlywedge \\ & & \text{strong-}* & \prec & \text{strong} & \prec & \text{weak} \end{array}$$

Here, "$\prec$" means that the left hand side is finer, i.e. stronger than the right hand side. Next, we have the following facts:

- The weakly continuous and strongly continuous functionals on $\mathscr{M}$ are the same. Similarly, the σ-weakly and σ-strongly continuous functionals are the same.
- The weak and σ-weak topology coincide on the unit ball of $\mathscr{M}$. Similarly, the strong and σ-strong topologies coincide on the unit ball of $\mathscr{M}$.
- The unit ball is compact under the weak and σ-weak topologies.
- A convex subset C of $\mathscr{B}(\mathscr{H})$ is strongly closed if and only if it is weakly closed. Similarly, it is σ-strongly closed if and only if it is σ-weakly closed. This is in particular the case for linear subspaces.

A bounded operator $T : \mathscr{M} \to \mathscr{M}$ is called *normal* if it is σ-weak/σ-weak continuous. This terminology should not be confused with normal *weights* as defined in the next section. Categorically speaking, normal bounded operators are the 'morphisms' of von Neumann algebras.

2.2.2. WEIGHTS

Weights are the noncommutative analogues of measures, and they are the cornerstone of noncommutative integration theory.

Definition 2.2.1. A *weight* on a von Neumann algebra $\mathscr{M}$ is a map $\varphi : \mathscr{M}_+ \to [0,\infty]$ which preserves addition and scalar multiplication with positive scalars. Associated to a weight φ are the following sets:

$$\begin{aligned} \mathfrak{p}_\varphi &= \{x \in \mathscr{M}_+ : \varphi(x) < \infty\} \\ \mathfrak{n}_\varphi &= \{x \in \mathscr{M} : x^* x \in \mathfrak{p}_\varphi\} \\ \mathfrak{m}_\varphi &= \left\{ \sum_{i=1}^n x_i^* y_i : x_1, \dots, x_n, y_1, \dots, y_n \in n_\varphi \right\}. \end{aligned}$$

By [Tak03a, Lemma VII.1.2], $\mathfrak{n}_\varphi$ is a left ideal of $\mathscr{M}$ and every element of $\mathfrak{m}_\varphi$ can be written as a linear combination of four elements in $\mathfrak{p}_\varphi$. Hence a weight can be linearly extended to the set m_φ, satisfying also $\varphi(x^*) = \overline{\varphi(x)}$. Additionally, this implies that $\mathfrak{p}_\varphi \subseteq \mathfrak{m}_\varphi \subseteq \mathfrak{n}_\varphi$ and that $\mathfrak{m}_\varphi$ is a $*$-subalgebra. A weight φ is called

i) *normal* if $\sup_\lambda \varphi(x_\lambda) = \varphi(\sup_\lambda x_\lambda)$ for every bounded increasing net $(x_\lambda)_\lambda$ in $\mathscr{M}_+$. This can be viewed as an analogue of the monotone convergence theorem.

ii) *faithful* if $\varphi(x) \neq 0$ for any $0 \neq x \in \mathcal{M}_+$.

iii) *semifinite* if $\mathfrak{p}_\varphi$ generates $\mathcal{M}$ as a von Neumann algebra; equivalently, if $\mathfrak{n}_\varphi$ is σ-strongly dense in $\mathcal{M}$.

iv) *tracial* if $\varphi(x^*x) = \varphi(xx^*)$ for any $x \in \mathcal{M}$. Tracial weights are usually denoted by τ.

For a normal semifinite weight φ, one can construct a so-called *semi-cyclic* representation $(\pi_\varphi, \mathcal{H}_\varphi, \eta_\varphi)$ by applying the GNS construction on $\mathfrak{n}_\varphi$. More precisely, define

$$\mathcal{N}_\varphi = \{x \in \mathfrak{n}_\varphi : \varphi(x^*x) = 0\}$$

and consider the quotient map $\eta_\varphi : \mathfrak{n}_\varphi \to \mathfrak{n}_\varphi / \mathcal{N}_\varphi$. We endow $\mathfrak{n}_\varphi / \mathcal{N}_\varphi$ with the sesquilinear form

$$\langle \eta_\varphi(x), \eta_\varphi(y) \rangle = \varphi(y^*x)$$

and define $\mathcal{H}_\varphi$ to be the Hilbert space completion of $\mathfrak{n}_\varphi / \mathcal{N}_\varphi$ with respect to this inner product. For $x \in \mathcal{M}$, consider the operator given by

$$\pi_\varphi(x) : \mathfrak{n}_\varphi / \mathcal{N}_\varphi \to \mathfrak{n}_\varphi / \mathcal{N}_\varphi, \quad \eta_\varphi(y) \mapsto \eta_\varphi(xy).$$

By the inequality $x^*a^*ax \leq \|a\|^2 x^*x$, this map is well-defined and bounded, hence it extends to $\mathcal{H}_\varphi$. This defines a normal $*$-representation $\pi_\varphi : \mathcal{M} \to \mathcal{B}(\mathcal{H}_\varphi)$. The assignment $x \cdot \xi = \pi_\varphi(x)\xi$ gives a left action of $\mathcal{M}$ on $\mathcal{H}_\varphi$. In practice we often leave out π_φ in the notation, writing just $x\xi$. We remark that if φ is not bounded, then $\mathfrak{n}_\varphi$ does not contain the unit, hence we do not have a canonical cyclic and separating vector in $\mathcal{H}_\varphi$.

If φ is a moreover faithful, then the semi-cyclic representation $(\pi_\varphi, \mathcal{H}_\varphi, \eta_\varphi)$ is faithful, and the map η_φ is injective. In this thesis, we will generally only consider normal faithful semifinite weights, as this allows for the construction of a canonical family of L_p-spaces. Moreover, these conditions are natural when considering weights as analogues of measures, as the following example shows.

Example 2.2.2. Let (X,μ) be a σ-finite measure space, and set $\mathcal{M} := L_\infty(X,\mu)$ to be the commutative von Neumann algebra of (equivalence classes of) essentially bounded functions on X. Set φ to be the weight on $L_\infty(X,\mu)$ given by $\varphi(f) = \int_X f\, d\mu$. We show that φ is normal, faithful and semifinite. By standard integration theory, if $f \geq 0$, then $\int_X f\, d\mu = 0$ if and only if $f = 0$ a.e., hence φ is faithful. Next, note that $\mathfrak{n}_\varphi = L_\infty(X,\mu) \cap L_2(X,\mu)$. It is straightforward to show that this is weak-$*$ dense in $L_\infty(X,\mu)$. Hence (since its σ-strong closure is convex) it is also σ-strong dense. Thus, φ is semifinite.

For normality, we cannot just use the monotone convergence theorem since it does not hold for nets. Instead, we use a different characterisation of normality (see [Tak03a, Theorem VII.1.11 (iv)]): φ is normal if and only if

$$\varphi(x) = \sup\{\omega(x) : \omega \in \mathcal{M}_*^+, \omega \leq \varphi\}, \qquad x \in \mathcal{M}_+.$$

We show that this condition holds. Let $f \in L_\infty(X,\mu)$. First assume that $\int_X f\, d\mu = \infty$, i.e. $f \notin \mathfrak{p}_\varphi$. Since X is σ-finite, we can find measurable sets A_n with finite measure such that $\int_X 1_{A_n} f\, d\mu > n$. Note here that $1_{A_n} \in L_1(X,\mu)^+ \cong \mathcal{M}_*^+$. Now assume that $\int_X f\, d\mu < \infty$ and let $\varepsilon > 0$. Using again σ-finiteness, we find a measurable set A with $\mu(A) < \infty$ such that $\int_X 1_{X \setminus A} f\, d\mu < \varepsilon$. This shows normality.

We will see another example of a normal faithful semifinite weight, the Plancherel weight, in section 2.6. As it turns out, a weight with these three properties always exists.

Proposition 2.2.3. *[Tak03a, Theorem VII.2.7] Every von Neumann algebra $\mathcal{M}$ admits a normal faithful semifinite weight.*

In the remainder of the thesis, we will use the abbreviation *nfs weight* for a normal faithful semifinite weight.

2.2.3. FINITE, SEMIFINITE AND σ-FINITE VON NEUMANN ALGEBRAS

There are several ways to classify von Neumann algebras in several subcategories. We will distinguish between von Neumann algebras of four types: finite, semifinite, σ-finite and general von Neumann algebras. This distinction depends on the existence of nfs weights with stronger properties.

Definition 2.2.4. A von Neumann algebra is called

i) *semifinite* if it has a nfs tracial weight τ.
ii) *σ-finite* if it has a normal faithful state φ; i.e. a normal faithful functional with $\varphi(1_{\mathcal{M}}) = 1$.
iii) *finite* if it has a normal faithful tracial state τ.

A different characterisation of a σ-finite von Neumann algebras is that it has at most countably many orthogonal non-zero projections (see [Tak02, Proposition II.3.19]). This explains the term 'σ-finite'.

The Tomita-Takesaki theory described in the Section 2.3 is trivial when working with a tracial weight. Moreover, the L_p-theory described in Section 2.4 is relatively straightforward in this case. However, even if a von Neumann algebra is semifinite (resp. finite), we sometimes want to work with a more canonical non-tracial weight (resp. state). Therefore Tomita-Takesaki theory can still be useful even when working with semifinite von Neumann algebras.

2.2.4. UNBOUNDED OPERATORS

We will recall some theory of unbounded operators that we will need in the next section. An *unbounded operator* on a Hilbert space $\mathcal{H}$ is a linear map $x : D(T) \to \mathcal{H}$, where $D(x)$ is some linear subspace of $\mathcal{H}$ called the *domain* of x. Sums and products are defined through $D(x_1 + x_2) = D(x_1) \cap D(x_2)$, $D(x_1 x_2) = \{\xi \in D(x_2) : x_2\xi \in D(x_1)\}$ and the obvious operations. We say that $x_1 \subseteq x_2$ if $D(x_1) \subseteq D(x_2)$ and $x_1\xi = x_2\xi$ for $\xi \in D(x_1)$. An unbounded operator x is said to be *densely defined* if $D(x)$ is dense in $\mathcal{H}$; *closed* if the graph $G(x) = \{(\xi, x\xi) : \xi \in D(T)\}$ is closed in $\mathcal{H} \times \mathcal{H}$. An unbounded operator x is *closable* if the closure $\overline{G(x)}$ is the graph of some operator. In that case, the operator with graph $\overline{G(X)}$ is called the *closure* of x and denoted by $[x]$.

If an operator x is densely defined, we set

$$D(x^*) = \{\eta \in \mathcal{H} : \text{the map } D(x) \to \mathbb{C},\ \xi \mapsto \langle x\xi, \eta\rangle \text{ is bounded}\}.$$

For each $\eta \in D(x^*)$, there exists by the Riesz representation theorem an element $x^*\eta \in \mathcal{H}$ such that $\langle x\xi, \eta\rangle = \langle \xi, x^*\eta\rangle$ for all $\xi \in D(x)$. The unbounded operator $x^* : D(x^*) \to \mathcal{H}$ thus defined is called the *adjoint* of x. The adjoint x^* is automatically a closed operator. It is densely defined iff x is closable, and in that case $x^{**} = [x]$. We also have the following elementary inclusions for densely defined operators x_1, x_2, which we will use without reference:

- If $x_1 + x_2$ is densely defined, then $x_1^* + x_2^* \subseteq (x_1 + x_2)^*$.
- If $x_2 x_1$ is densely defined, then $x_1^* x_2^* \subseteq (x_2 x_1)^*$.
- If $x_1 \subseteq x_2$, then $x_2^* \subseteq x_1^*$.

We do prove the following lemma, although it is also elementary, as we will use the results a lot.

Lemma 2.2.5. *Let y be a closed operator on $\mathcal{H}$ and $x \in \mathcal{B}(\mathcal{H})$. Then yx is closed. If moreover y is densely defined, then $(xy)^* = y^* x^*$.*

Proof. Take $\xi_n \in D(yx)$ such that $(\xi_n, yx\xi_n) \to (\xi, \eta)$ for some $\xi, \eta \in \mathcal{H}$. Then $x\xi_n \to x\xi$ as x is bounded. Hence, since y is closed, we have $x\xi \in D(y)$ and $yx\xi_n \to yx\xi$. This means that $\xi \in D(yx)$ and $\eta = yx\xi$, hence $(\xi, \eta) \in G(yx)$ and thus yx is closed.

For the second part, we need only show that $D((xy)^*) \subseteq D(y^* x^*)$. So let $\eta \in D((xy)^*)$. Then there is some $C_\eta > 0$ such that $|\langle y\xi, x^*\eta\rangle| = |\langle xy\xi, \eta\rangle| \le C_\eta \|\xi\|$ for $\xi \in D(y)$. This means that $x^*\eta \in D(y^*)$, i.e. $\eta \in D(y^* x^*)$. □

An operator x is called *symmetric* if $x \subseteq x^*$; it is called *self-adjoint* if $x = x^*$; it is called *positive* if $\langle x\xi, \xi\rangle \ge 0$ for all $\xi \in D(x)$. A self-adjoint operator satisfies the spectral theorem, i.e. there is a projection-valued measure χ^x on $\mathbb{R}$, mapping Borel sets to the corresponding eigenspaces, such that $x = \int_{\mathbb{R}} \lambda d\chi^x$.

Now let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$. An unbounded operator x on $\mathcal{H}$ with polar decomposition $x = u|x|$ is *affiliated* with $\mathcal{M}$ if $u \in \mathcal{M}$ and $\chi_E^x \in \mathcal{M}$ for every bounded set $E \subseteq \sigma(x)$; equivalently, if x commutes with every unitary operator in $\mathcal{M}'$. The set of all closed, densely operators affiliated with $\mathcal{M}$ will be denoted by $\bar{\mathcal{M}}$.

2.3. Tomita-Takesaki theory and spatial derivatives

2.3.1. Tomita-Takesaki theory

Let φ be a nfs weight on a von Neumann algebra $\mathcal{M}$. Tomita-Takesaki theory is the analysis of the structure of the semi-cyclic representation $(\pi_\varphi, \mathcal{H}_\varphi, \eta_\varphi)$ obtained through the adjoint mapping. We give here a rather short overview of the terms we need, since the details of the underlying theory are not very relevant to this thesis.

Since φ is faithful, η_φ is nothing but the identity mapping. Hence we may consider $\mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$ as a subset of $\mathcal{H}_\varphi$, on which the antilinear map $S_0 : x \mapsto x^*$ is well-defined. $\mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$ is a so-called *left Hilbert algebra* (cf. [Tak03a, Chapter VI, Theorem VII.2.6]), although we will avoid using that terminology whenever possible.

As proved in the mentioned reference, the map S_0 is closable, and we define its closure by S. The antilinear operator S is densely defined; thus, we can define its antilinear adjoint S^*, satisfying $\langle S^* f, g\rangle = \langle Sg, f\rangle$. We write $S = J_\varphi \Delta_\varphi^{1/2}$ to be the polar decomposition of S. The operator J_φ is called the *modular conjugation*. It is an antilinear isometry satisyfing $J_\varphi^2 = I$. The operator Δ_φ is called the *modular operator* with respect to φ. It is a positive self-adjoint operator satisfying $\Delta_\varphi = S^* S$ and $\Delta_\varphi^{-1} = SS^*$, i.e.

$$\langle \Delta_\varphi f, g\rangle = \langle Sg, Sf\rangle; \qquad \langle \Delta_\varphi^{-1} f, g\rangle = \langle S^* g, S^* f\rangle.$$

By the main theorem of Tomita-Takesaki theory [Tak03a, Theorem VI.1.19], we have

$$J_\varphi \mathcal{M} J_\varphi = \mathcal{M}'; \qquad \Delta_\varphi^{it} \mathcal{M} \Delta_\varphi^{-it} = \mathcal{M}, \quad t \in \mathbb{R}.$$

Thanks to this result, we can define an automorphism group $\sigma_t^\varphi(x) = \Delta_\varphi^{it} x \Delta_\varphi^{-it}$, known as the *modular automorphism group*. By [Tak03a, Theorem VIII.1.2], it satisfies $\varphi \circ \sigma_t^\varphi = \varphi$. Moreover, the relation $J_\varphi \mathcal{M} J_\varphi = \mathcal{M}'$ allows us to define a right multiplication action of $\mathcal{M}$ on $\mathcal{H}_\varphi$ by setting $\xi \cdot x := J_\varphi x^* J_\varphi \xi$. When the weight φ is clear, we will leave out the reference to φ and just write J, Δ and σ_t. As we will see in the next section, there are also other ways to implement the modular automorphism group.

The *centralizer* of a von Neumann algebra with respect to a nfs weight φ is given by

$$\mathcal{M}^\varphi = \{x \in \mathcal{M} : \sigma_t^\varphi(x) = x \ \forall t \in \mathbb{R}\}.$$

If φ is a normal faithful state (meaning that $\mathcal{M}$ is σ-finite), then by [Tak03a, Theorem VIII.2.6] we have the equivalent characterisation

$$\mathcal{M}^\varphi := \{x \in \mathcal{M} : \varphi(xy) = \varphi(yx) \ \forall y \in M\}.$$

There is an analogous, but slightly more complicated characterisation for nfs weights. However we will only use the centralizer in the state case.

Of special importance are those $x \in \mathcal{M}$ for which the mapping $t \mapsto \sigma_t^\varphi(x)$ extends analytically to an entire function $z \mapsto \sigma_z^\varphi(x)$. These elements are called *analytic elements*, and the set of analytic elements with respect to φ is denoted by $\mathcal{M}_a^\varphi$. By [Tak03a, Lemma VIII.2.3], $\mathcal{M}_a^\varphi$ is a σ-weakly dense $*$-subalgebra in $\mathcal{M}$, and σ_z^φ satisfies the expected arithmetic properties; let us only emphasize the property $\sigma_z^\varphi(x^*) = \sigma_{\bar{z}}^\varphi(x)^*$. Analytic elements satisfy the following commutation relation. We have not been able to find a proof in the literature, hence we provide it here for convenience of the reader.

Proposition 2.3.1. *Let $x \in \mathcal{M}_a^\varphi$ and $a \in \mathbb{R}$. Then*

$$x\Delta_\varphi^a \subseteq \Delta_\varphi^a \sigma_{it}(x).$$

Proof. Let us write $\Delta := \Delta_\varphi$. Take $\xi \in D(x\Delta^a) = D(\Delta^a)$. We claim that, for each $\eta \in D(\Delta^a)$,

$$\langle \sigma_{ia}(x)\xi, \Delta^a \eta\rangle = \langle x\Delta^a \xi, \eta\rangle.$$

2

From the claim, we conclude that the map $\eta \mapsto \langle \sigma_{ia}(x)\xi, \Delta^a \eta\rangle$, $\eta \in D(\Delta^a)$ is bounded, and hence $\sigma_{ia}(x)\xi \in D((\Delta^a)^*) = D(\Delta^a)$. Taking Δ^a to the other side, we moreover find $\Delta^a \sigma_{ia}(x)\xi = x\Delta^a \xi$, which finishes the proof.

Let us now prove the claim. Denote $\mathbf{D}_a = \{z \in \mathbb{C} : -a < \mathrm{Im}(z) < 0\}$ and set $\mathscr{A}_{\mathbf{D}_a}(\mathscr{H})$ to be all bounded and continuous functions $f : \overline{\mathbf{D}_a} \to \mathscr{H}$ that are analytic on $\mathbf{D}_a$. Now, the function $z \mapsto \Delta^{iz}\xi$ is in $\mathscr{A}_{\mathbf{D}_a}(\mathscr{H})$ by (the proof of) [Tak03a, Lemma VI.2.3]. Since x is a bounded operator, the function $z \mapsto x\Delta^{iz}\xi$ is also in $\mathscr{A}_{\mathbf{D}_a}(\mathscr{H})$. Hence, for all $\eta \in \mathscr{H}$, the function $z \mapsto \langle x\Delta^{iz}\xi, \eta\rangle$ is in $\mathscr{A}_{\mathbf{D}_a}(\mathbb{C})$ (this is a standard result for Banach-valued analytic functions).

Now fix some $\eta \in D(\Delta^a)$. As above, the function $z \mapsto \Delta^{iz}\eta$ is in $\mathscr{A}_{\mathbf{D}_a}(\mathscr{H})$. By a straightforward noncommutative variant of the product rule, the function $z \mapsto \sigma_{-\bar{z}}(x^*)\Delta^{iz}\eta$ is also in $\mathscr{A}_{\mathbf{D}_a}(\mathscr{H})$. Hence, the function $z \mapsto \langle \sigma_{-z}(x)\xi, \Delta^{iz}\eta\rangle = \langle \xi, \sigma_{-\bar{z}}(x^*)\Delta^{iz}\eta\rangle$ is in $\mathscr{A}_{\mathbf{D}_a}(\mathbb{C})$. We have now proved that the functions

$$z \mapsto \langle x\Delta^{iz}\xi, \eta\rangle; \qquad z \mapsto \langle \sigma_{-z}(x)\xi, \Delta^{iz}\eta\rangle$$

are both in $\mathscr{A}_{\mathbf{D}_a}(\mathbb{C})$. But for $t \in \mathbb{R}$ we have

$$\langle \sigma_{-t}(x)\xi, \Delta^{it}\eta\rangle = \langle \Delta^{it}\sigma_{-t}(x)\xi, \eta\rangle = \langle x\Delta^{it}\xi, \eta\rangle.$$

Hence the functions coincide on $\mathbb{R}$; but then they must coincide on $\overline{\mathbf{D}_a}$. Filling in $z = -ia$ proves the claim. □

A further useful subset of $\mathscr{M}_a^\varphi$ is the *Tomita algebra* $\mathscr{T}_\varphi$. It is defined as

$$\mathscr{T}_\varphi := \{x \in \mathscr{M}_a^\varphi : \sigma_z^\varphi(x) \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*, z \in \mathbb{C}\}.$$

Note that 'our' Tomita algebra $\mathscr{T}_\varphi$ is defined differently from the Tomita algebra a_0 as defined in [Tak03a, p. 99]; but by [Tak03a, Equation (5)] we have $a_0 \subseteq \mathscr{T}_\varphi$. Hence $\mathscr{T}_\varphi$ is also a σ-weakly dense $*$-subalgebra of $\mathscr{M}$, see [Tak03a, Theorem VI.2.2]. Moreover, $\eta_\varphi(\mathscr{T}_\varphi)$ is dense in $\mathscr{H}_\varphi$. By taking a suitable approximation of the unity, one can prove the same statements for $\mathscr{T}_\varphi^2$ (see [Ter82, Lemma 9], [Cas13, Appendix A]).

Finally, let us recall the notion of standard form. A quadruple $(M, \mathscr{H}, J, P)$ of a von Neumann algebra $\mathscr{M}$ on a Hilbert space $\mathscr{H}$, an antilinear isometric involution $J : \mathscr{H} \to \mathscr{H}$ and a selfdual cone P in $\mathscr{H}$ is called a *standard form* of $\mathscr{M}$ if

1. $J\mathscr{M}J = \mathscr{M}'$,
2. $JxJ = x^*$ for all $x \in Z(M)$, the center of $\mathscr{M}$,
3. $J\xi = \xi$ for all $\xi \in P$
4. $xJxJ(P) \subseteq P$ for all $x \in \mathscr{M}$.

By [Haa75, Theorem 1.6], the quadruple $(\mathscr{M}, \mathscr{H}_\varphi, J_\varphi, \mathscr{H}_\varphi^+)$ is a standard form, where

$$\mathscr{H}_\varphi^+ := \overline{\{x(J_\varphi x J_\varphi) \mid x \in \mathscr{M}\}} \subseteq \mathscr{H}_\varphi.$$

The main result about standard forms is that they are unique in a rather strong sense:

Proposition 2.3.2. *[Haa75, Theorem 2.3] Let $(\mathcal{M},\mathcal{H},J,P)$ and $(\tilde{\mathcal{M}},\tilde{\mathcal{H}},\tilde{J},\tilde{P})$ be two standard forms and assume that there is a $*$-isomorphism $\Phi:\mathcal{M}\to\tilde{\mathcal{M}}$. Then there exists a unique unitary $u:\mathcal{H}\to\tilde{\mathcal{H}}$ such that $\Phi(x)=uxu^*$ for all $x\in\mathcal{M}$ and $\tilde{J}=uJu^*$, $\tilde{P}=u(P)$.*

2.3.2. SPATIAL DERIVATIVES

In this section, we fix a von Neumann algebra $\mathcal{M}$ represented on a Hilbert space $\mathcal{H}$, and let $\mathcal{M}'$ be the corresponding commutant. The spatial derivative $\frac{d\varphi}{d\psi}$ is a self-adjoint positive (unbounded) operator on $\mathcal{H}$ corresponding to a pair of weights φ on $\mathcal{M}$ and ψ on $\mathcal{M}'$. It functions as a noncommutative analogue of the Radon-Nikodym derivative of measures, and it is an important tool in the construction and analysis of noncommutative L_p-spaces. Spatial derivatives were originally introduced by Connes in [Con80], but we will mostly refer to the more detailed works [Ter81] and [Tak03a, Section IX.3]. We will give here the basic construction, without going into too much detail.

Let us fix a nfs weight ψ on $\mathcal{M}'$, and let $(\pi_\psi,\mathcal{H}_\psi,\eta_\psi)$ be the corresponding semi-cyclic representation. For $\xi\in\mathcal{H}$, we define the operator

$$R^\psi(\xi):\eta_\psi(\mathfrak{n}_\psi)\to\mathcal{H},\qquad R^\psi(\xi)\eta_\psi(y)=y\xi.$$

We say that ξ is ψ-bounded if $R^\psi(\xi)$ extends to a bounded operator $\mathcal{H}_\psi\to\mathcal{H}$. We denote by $D(\mathcal{H},\psi)$ the set of ψ-bounded vectors in $\mathcal{H}$. Note that when $\mathcal{H}=\mathcal{H}_\psi$, $R^\psi(\xi)$ is simply a right multiplication operator.

Now let $\xi\in D(\mathcal{H},\psi)$, $y\in\mathcal{M}'$ and $z\in\mathfrak{n}_\psi$. Then

$$yR^\psi(\xi)\eta_\psi(z)=yz\xi=R^\psi(\xi)\eta_\psi(yz)=R^\psi(\xi)\pi_\psi(y)\eta_\psi(z),$$

hence $yR^\psi(\xi)=R^\psi(\xi)\pi_\psi(y)$ for all $y\in\mathcal{M}'$. By taking adjoints, we also find $\pi_\psi(y)R^\psi(\xi)^*=R^\psi(\xi)^*y$ for all $y\in\mathcal{M}'$. Hence, the operator $R^\psi(\xi)R^\psi(\xi)^*$ commutes with $\mathcal{M}'$ and thus it is in $\mathcal{M}$.

Now let φ be a normal semifinite weight on $\mathcal{M}$. By [Tak03a, Theorem IX.3.8], there exists a unique positive self-adjoint operator $\frac{d\varphi}{d\psi}$ such that

$$\varphi(R^\psi(\xi)R^\psi(\xi)^*)=\left\|\left(\frac{d\varphi}{d\psi}\right)^{1/2}\xi\right\|^2,\qquad\forall\,\xi\in D(\mathcal{H},\psi):\ R^\psi(\xi)R^\psi(\xi)^*\in\mathfrak{p}_\varphi.\tag{2.3.1}$$

Definition 2.3.3. The operator $\frac{d\varphi}{d\psi}$ determined by 2.3.1 is called the *spatial derivative* with respect to φ and ψ.

Example 2.3.4. Let $\mathcal{M}=\mathcal{B}(\mathcal{H})$. Then $\mathcal{M}'=\mathbb{C}1_{\mathcal{H}}$, where $1_{\mathcal{H}}$ denotes the identity operator on $\mathcal{H}$. Hence the only non-zero semifinite weights on $\mathcal{M}'$ are the positive functionals $\psi(\lambda 1_{\mathcal{H}})=a\lambda$, $a>0$. Now take $\psi(\lambda 1_{\mathcal{H}})=\lambda$. We have $\mathcal{H}_\psi=\mathbb{C}1_{\mathcal{H}}$ and $R^\psi(\xi)\lambda 1_{\mathcal{H}}=\lambda\xi$. Hence, $\|R^\psi(\xi)\|=\|\xi\|_{\mathcal{H}}$, and $D(\mathcal{H},\psi)=\mathcal{H}$. Through a straightforward calculation, one finds that $R^\psi(\xi)^*\eta=\langle\eta,\xi\rangle_{\mathcal{H}}1_{\mathcal{H}}$ and $R^\psi(\xi)R^\psi(\xi)^*$ is the rank 1 operator $\eta\mapsto\langle\eta,\xi\rangle_{\mathcal{H}}\xi$.

Now let $\varphi \in \mathscr{B}(\mathscr{H})_*^+$. Let $d_\varphi \in S_1(\mathscr{H})^+$ be the corresponding operator such that $\varphi(x) = \operatorname{tr}(d_\varphi x)$ for $x \in \mathscr{B}(\mathscr{H})$. We will show that $\frac{d\varphi}{d\psi} = d_\varphi$. Let $\xi \in \mathscr{H}$ and assume that $\|\xi\| = 1$. Let $(\xi_i)_i$ be an orthonormal basis for $\mathscr{H}$ where $\xi_1 = \xi$. Then we have

$$\varphi(R^\psi(\xi)R^\psi(\xi)^*) = \operatorname{tr}(d_\varphi R^\psi(\xi)R^\psi(\xi)^*) = \sum_i \langle d_\varphi R^\psi(\xi)R^\psi(\xi)^*\xi_i, \xi_i\rangle = \langle d_\varphi \xi, \xi\rangle$$

by definition of $R^\psi(\xi)R^\psi(\xi)^*$. Hence, we find $\frac{d\varphi}{d\psi} = d_\varphi$.

Example 2.3.5. For this example, we look at matrix amplifications of a von Neumann algebra. Let $\mathscr{M}$ be represented on $\mathscr{H}$, let ψ be some nfs weight on $\mathscr{M}'$, and let $n \geq 1$. Then $M_n(\mathscr{M})$ is naturally represented on $\mathscr{H}^n$, and in this case $M_n(\mathscr{M})' = 1 \otimes \mathscr{M}'$. We endow $M_n(\mathscr{M})'$ with the nfs weight $\psi^n := \operatorname{Tr} \otimes \psi$ where the trace is normalised, in other words, $\psi^n(1 \otimes y) = \psi(y)$. Now $\eta_{\psi^n}(1 \otimes y) = 1 \otimes \eta_\psi(y)$, $\mathfrak{n}_{\psi^n} = 1 \otimes \mathfrak{n}_\psi$ and $\mathscr{H}_{\psi^n} = 1 \otimes \mathscr{H}_\psi$ with $\langle 1 \otimes \xi, 1 \otimes \eta\rangle_{\mathscr{H}_{\psi^n}} = \langle x, y\rangle_{\mathscr{H}_\psi}$. It follows by a straightforward check that $D(\mathscr{H}^n, \psi^n) = D(\mathscr{H}, \psi)^n$ and $R^{\psi^n}(\xi)(1 \otimes \zeta) = (R^{\psi^n}(\xi_i)\zeta)_i$ for $\xi = (\xi_i)_i \in D(\mathscr{H}^n, \psi^n)$ and $1 \otimes \zeta \in \mathscr{H}_{\psi^n}$. Moreover, one finds that $R^{\psi^n}(\xi)^*\eta = 1 \otimes \sum_{j=1}^n R^\psi(\xi_j)^*\eta_j$ for $\eta \in \mathscr{H}^n$. Hence,

$$R^{\psi^n}(\xi)R^{\psi^n}(\xi)^*\eta = \left(\sum_{j=1}^n R^\psi(\xi_i)R^\psi(\xi_j)^*\eta_j\right)_i = (R^\psi(\xi_i)R^\psi(\xi_j)^*)_{i,j}\eta.$$

We will continue this calculation in the proof of Proposition 2.5.4.

We will calculate the spatial derivative in another concrete case in Proposition 2.6.7. Next, we will state some of the properties of spatial derivatives, namely the ones that we will need in thesis. For the following, we refer to [Ter81, Theorem III.14]:

$$\frac{d(\varphi_1 + \varphi_2)}{d\psi} = \frac{d\varphi_1}{d\psi} + \frac{d\varphi_2}{d\psi} \tag{2.3.2}$$

and

$$\frac{d(x \cdot \varphi \cdot x^*)}{d\psi} = x \cdot \frac{d\varphi}{d\psi} \cdot x^*, \qquad x \in \mathscr{M}. \tag{2.3.3}$$

where $x \cdot \varphi \cdot x^*(y) = \varphi(x^* y x)$. Here φ, φ_1, φ_2 are normal semifinite weights on $\mathscr{M}$. Moreover, it turns out that spatial derivatives implement the modular automorphism group: if φ is a nfs weight on $\mathscr{M}$, then

$$\sigma_t^\varphi(x) = \left(\frac{d\varphi}{d\psi}\right)^{it} x \left(\frac{d\varphi}{d\psi}\right)^{-it}, \qquad x \in \mathscr{M}, t \in \mathbb{R}.$$

We refer to [Tak03a, Theorem IX.3.8]. This fact will be used repeatedly without reference. We also get the following commutation relation for analytic elements:

$$x\left(\frac{d\varphi}{d\psi}\right)^t \subseteq \left(\frac{d\varphi}{d\psi}\right)^t \sigma_{it}^\varphi(x), \qquad x \in \mathscr{M}_a, t \in \mathbb{R}. \tag{2.3.4}$$

The proof is the same as that of Proposition 2.3.1, after replacing all instance of Δ by $\frac{d\varphi}{d\psi}$.

Remark 2.3.6. Let φ be some nfs weight on $\mathscr{M}$ and assume that $\mathscr{H} = \mathscr{H}_\varphi$. Then there is some weight ψ on $\mathscr{M}'$ such that the modular operator Δ_φ is equal to the spatial derivative $\frac{d\varphi}{d\psi}$; this ψ is the so-called *opposite weight* of φ. We refer to [Con80, Proof of Theorem 9].

2.4. NONCOMMUTATIVE L_p-SPACES CORRESPONDING TO VON NEUMANN ALGEBRAS

As we have mentioned before, the theory of von Neumann algebras is considered to be a noncommutative analogue of measure theory. Hence, it makes sense to attempt to construct a scale of Banach spaces that satisfy general properties of classical L_p-spaces. Indeed, the fact that a von Neumann algebra admits a predual and a standard form already gives us plausible candidates for L_1- and L_2- spaces. We will see that indeed, the L_1- and L_2-spaces below are isomorphic to these.

In most cases, the noncommutative L_p-space is constructed as a space of closed densely defined unbounded operators. However, this creates difficulties when defining sums and products of elements in L_p. Indeed, the sum or product of closed densely defined operators need not be closed again. A subtle part of the theory deals with the problem of showing that sums and products of elements are closable; once this is done, one simply *defines* addition and multiplication by taking closures of the corresponding sum and product operators. Such sums and products are called the *strong sum* resp. *strong product*, and are denoted by $[x+y]$ and $[xy]$.

We will start by considering the relatively simple case of noncommutative L_p-spaces with respect to a trace τ. In this case, the L_p-spaces are contained in $\bar{\mathcal{M}}$, the space of closed densely defined affiliated operators. If we consider instead a nfs weight φ, then the construction of a suitable scale of L_p-spaces becomes significantly more complicated. For instance, the triangle inequality is no longer true if we would just take the tracial definition, as mentioned in the introduction. Haagerup [Haa79a] was the first to propose a construction of noncommutative L_p-spaces with respect to a nfs weight. His solution was to consider the crossed product, and use the trace that comes with it. Later, Hilsum [Hil81] defined L_p-spaces of unbounded operators on the Hilbert space that the von Neumann algebra is represented on, using Connes' spatial derivatives [Con80] which we define in Section 2.3.2. This is called the Connes-Hilsum construction. However, he was only able to prove that sums of elements in L_p are closable after first constructing an isometric isomorphism between his L_p-spaces and that of Haagerup. Later still, Kosaki [Kos84] defined L_p-spaces for normal faithful states by using complex interpolation. He proved that these L_p-spaces are isomorphic to the ones considered before, thus giving access to interpolation methods to prove boundedness results of operators on L_p. Terp [Ter82] and Izumi [Izu97] generalised this construction to general nfs weights; we will call this the Kosaki-Terp-Izumi construction.

We will give the Connes-Hilsum and Kosaki-Terp-Izumi constructions here. We will not use the Haagerup construction. However, it should be noted that this construction is essential in proving properties of the other constructions; it is the foundation which made the other constructions possible.

2

2.4.1. THE TRACIAL CASE

We give the construction of the tracial case mostly to give some intuition. The idea is to take the tracial weight as a replacement for integration over some measure. Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, and let τ be a nfs tracial weight on $\mathcal{M}$. We can extend τ to $\bar{\mathcal{M}}$ as follows: if x is a positive self-adjoint operator affiliated to $\mathcal{M}$, then we set

$$\tau(x) = \sup_{n\in\mathbb{N}} \tau(\chi^x_{[0,n]}x)$$

where $\chi^x_{[0,n]}$ is the spectral projection of x. The definition of the L_p-space is now remarkably simple.

Definition 2.4.1. Fix $1 \leq p < \infty$. The noncommutative L_p-space with respect to τ is defined as

$$L_p(\mathcal{M},\tau) := \{x \in \bar{\mathcal{M}} : \tau(|x|^p) < \infty\}.$$

For $x \in L_p(\mathcal{M},\tau)$, we set

$$\|x\|_p = \tau(|x|^p)^{1/p}.$$

We simply set $L_\infty(\mathcal{M}) = \mathcal{M}$.

It is possible to define a space of τ-*measurable operators* (an analogue of the measurable functions of a measure space) together with a so-called measure topology, in which every L_p-space continuously embeds as a closed subspace. Using the properties of τ-measurable operators, one can prove that sums and products of elements in $L_p(\mathcal{M},\tau)$ are closable. With respect to strong sum and product, the L_p-norms satisfy the Minkowski and Hölder inequalities. This makes $(L_p(\mathcal{M},\tau), \|\cdot\|_p)$ into a Banach space with respect to strong sums. The set $\{x \in \mathcal{M} : \tau(|x|) < \infty\}$ is dense in $L_p(\mathcal{M},\tau)$ for each $1 \leq p < \infty$, and the L_p-spaces satisfy the usual duality relations. For these facts, we refer to [Nel74].

2.4.2. THE CONNES-HILSUM CONSTRUCTION

In this section, we consider a nfs weight ψ on $\mathcal{M}'$. We will start by introducing the notion of γ-homogeneous operators and defining an 'integral' for (-1)-homogeneous operators.

Let $\gamma \in \mathbb{R}$. A closed densely defined operator x on $\mathcal{H}$ is called γ-homogeneous with respect to ψ if, for all $y \in (\mathcal{M}')^\psi_a$ (i.e. analytic elements with respect to ψ),

$$yx \subseteq x\sigma^\psi_{i\gamma}(y). \tag{2.4.1}$$

We denote by $\bar{\mathcal{M}}_\gamma$ the set of all γ-homogeneous (hence closed, densely defined) operators. In [Con80] a different but equivalent definition is used (see [Ter82, p. 339]).

Lemma 2.4.2. *i) If $x_1 \in \bar{\mathcal{M}}_{\gamma_1}, x_2 \in \bar{\mathcal{M}}_{\gamma_2}$ such that x_1x_2 is closable, then $[x_1x_2]$ is a $\gamma_1 + \gamma_2$-homogeneous operator.*

ii) If x is γ-homogeneous, then x^* is also γ-homogeneous.

Proof. For part i), it is easy to see that x_1x_2 satisfies (2.4.1). But then $[x_1x_2]$ must satisfy (2.4.1) as well. For part ii), take $y \in (\mathcal{M}')^{\psi}_a$. Then also $\sigma^{\psi}_{-i\gamma}(y^*) \in (\mathcal{M}')^{\psi}_a$, hence $\sigma^{\psi}_{-i\gamma}(y^*)x \subseteq xy^*$. By taking adjoints and applying Lemma 2.2.5, we get

$$yx^* \subseteq (xy^*)^* \subseteq (\sigma^{\psi}_{i\gamma}(y^*)x)^* = x^*\sigma^{\psi}_{i\gamma}(y).$$

This shows that x^* is γ-homogeneous. □

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Note that the 0-homogeneous operators are precisely operators affiliated with $\mathcal{M}$, i.e. $\bar{\mathcal{M}}_0 = \bar{\mathcal{M}}$. By [Tak03a, Theorem IX.3.11] (using the equivalent notion of γ-homogeneity), the self-adjoint positive (-1)-homogeneous operators are precisely the $\frac{d\varphi}{d\psi}$ for normal semifinite weights φ on $\mathcal{M}$. This implies that if x is a $(-1/p)$-homogeneous operator, then we must have $|x|^p = \frac{d\varphi}{d\psi}$ for some normal semifinite weight φ on $\mathcal{M}$. We will later define the noncommutative L_p-space to be a subspace of the $(-1/p)$-homogeneous operators.

Now let x be a self-adjoint positive (-1)-homogeneous operator. Then we define the *integral with respect to* ψ as

$$\int x d\psi = \varphi(1),$$

where φ is the (unique) normal semifinite weight on $\mathcal{M}$ such that $x = \frac{d\varphi}{d\psi}$. Remark the similarity of this formula to the property of the Radon-Nikodym derivative.

Definition 2.4.3. Let ψ be a nfs weight on $\mathcal{M}'$ and $1 \leq p < \infty$. The noncommutative L_p-space with respect to ψ is defined as

$$L_p(\mathcal{M},\psi) := \{x \in \bar{\mathcal{M}}_{-1/p} : \int |x|^p d\psi < \infty\}.$$

For $x \in L_p(\mathcal{M},\psi)$ we set

$$\|x\|_p = \left(\int |x|^p d\psi\right)^{1/p}.$$

For $p = \infty$, we set $L_\infty(\mathcal{M},\psi) = \mathcal{M}$.

Let us now fix a nfs weight ψ on $\mathcal{M}'$. The following lemma is a slight generalisation of [Hil81, Theorem 4 (1)], and we will need it several times:

Lemma 2.4.4. *Let $x \in L_p(\mathcal{M},\psi)$ and $y \in \bar{\mathcal{M}}_{-1/p}$ such that $x \subseteq y$. Then $y = x \in L_p(\mathcal{M},\psi)$.*

Proof. Let u_0 and T be as defined in [Ter81, IV.(7) and IV.(8)]. By [Ter81, Corollary IV.6], $u_0^*(x \otimes T^{1/p})u_0$ is in the Haagerup L_p-space; in particular, it is τ-measurable (see [Ter81, Definition I.14]) on the crossed product $\mathcal{N} := \mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$. By [Ter81, Corollary IV.7], $u_0^*(y \otimes T^{1/p})u_0 \in \bar{\mathcal{N}}$. Moreover, $u_0^*(x \otimes T^{1/p})u_0 \subseteq u_0^*(y \otimes T^{1/p})u_0$, hence it follows from the definition that $u_0^*(y \otimes T^{1/p})u_0$ is also τ-measurable. But then, by [Ter81, Corollary I.15], we have $u_0^*(x \otimes T^{1/p})u_0 = u_0^*(y \otimes T^{1/p})u_0$. Hence by [Ter81, Corollary IV.7], $x = y$. □

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We summarise some properties of $L_p(\mathcal{M},\psi)$ in the following proposition. They can be found in [Ter81, Section IV]; the proofs all use the passage to the Haagerup construction. In particular for part ii) this seems to be the only viable path to a proof.

Proposition 2.4.5. *Let $x, y \in L_p(\mathcal{M},\psi)$ and $z \in L_q(\mathcal{M},\psi)$, where $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$ and $1 \leq p,q,r \leq \infty$. Then we have*

i) $x^* \in L_p(\mathcal{M},\psi)$ *and* $\|x^*\|_p = \|x\|_p$.

ii) $x+y$ *is densely defined and closable, and* $[x+y] \in L_p(\mathcal{M},\psi)$ *with* $\|[x+y]\|_p \leq \|x\|_p + \|y\|_p$. $L_p(\mathcal{M},\psi)$ *is a Banach space with respect to strong sums.*

iii) xz *is densely defined and closable,* $[xz] \in L_r(\mathcal{M},\psi)$ *and the (generalized) Hölder's inequality holds:* $\|[xz]\|_r \leq \|x\|_p\|z\|_q$.

iv) The integral can be linearly extended to all of $L_1(\mathcal{M},\psi)$, *and if* $r=1$, *then* $\int [xz]d\psi = \int [zx]d\psi$.

v) If $r=1$ *and* $q>1$, *then* $L_q(\mathcal{M},\psi)$ *is isometrically isomorphic to the dual space of* $L_p(\mathcal{M},\psi)$ *through the pairing* $\langle a,b\rangle_{p,q} = \int [ab]d\psi$ *for* $a \in L_p(\mathcal{M},\psi)$, $b \in L_q(\mathcal{M},\psi)$.

Remark 2.4.6. The first part of Proposition 2.4.5 iv) is implicitly used, but not proven in [Ter81]. This fact uses parts i) and ii) and Lemma 2.4.4 (or [Hil81, Theorem 4 (1)]). Indeed, the operator $a+a^*$ is a priori only symmetric; but since it is in $L_1(\mathcal{M},\psi)$ and its adjoint is in $L_1(\mathcal{M},\psi)$, by Lemma 2.4.4 we must have $(a+a^*)^* = a+a^*$. This shows that every element is the sum of two self-adjoint elements. Now let $x \in L_1(\mathcal{M},\psi)$ be self-adjoint with polar decomposition $x = u|x| = |x|u$. Then $u = p-q$, for two projections $p,q \in \mathcal{M}$ with orthogonal ranges (namely $p = \chi^x_{(-\infty,0)}$, $q = \chi^x_{[0,\infty)}$). Hence, we have $px = pu|x| = p|x| \geq 0$ and $qx = -q|x| \leq 0$. Hence $x = px+qx$ expresses x as a linear combination of two positive operators.

Notation 2.4.7. Part iv) of Proposition 2.4.5 tells us that the integral satisfies a tracial property on the scale $\{L_p(\mathcal{M},\psi) : 1 \leq p \leq \infty\}$ (with respect to the strong product). This justifies the following notation, which we will use throughout the thesis for ease of writing:

$$Tr(x) := \int x d\psi, \qquad x \in L_1(\mathcal{M},\psi).$$

In all situations where this is used, the nfs weight ψ on $\mathcal{M}'$ will be fixed, so the lack of reference to ψ will not lead to any confusion.

We now prove a concrete expression for the identification $L_1(\mathcal{M},\psi)^+ \cong \mathcal{M}_*^+$.

Proposition 2.4.8. *Let* $\phi \in \mathcal{M}_*^+$. *Then we have*

$$\mathrm{Tr}\left(\frac{d\phi}{d\psi}x\right) = \phi(x), \qquad x \in \mathcal{M}.$$

In other words, if we define for $x \in L_1(\mathcal{M},\psi)$ *the corresponding element in* $\mathcal{M}_*$ *by* $\varphi_x(y) = \mathrm{Tr}(xy)$, *then* $\varphi_{d\phi/d\psi} = \phi$.

Proof. Note first that $\frac{d\phi}{d\psi}x$ is automatically closed by Lemma 2.2.5, so that the statement makes sense. Now assume that x is positive and write $x = y^*y$ for $y \in \mathcal{M}$. Then, by Proposition 2.4.5 iv) and (2.3.3) we have

$$\mathrm{Tr}\left(\frac{d\phi}{d\psi}x\right) = \mathrm{Tr}\left(y\frac{d\phi}{d\psi}y^*\right) = \mathrm{Tr}\left(\frac{d(y\cdot\phi\cdot y^*)}{d\psi}\right) = \phi(y^*y) = \phi(x).$$

For general $x \in \mathcal{M}$, the result follows by taking linear combinations. □

Remark 2.4.9. Another fact that follows through the identification with Haagerup L_p-spaces is that the noncommutative L_p-spaces do not depend on the choice of nfs weight ψ. More precisely, if $\tilde{\psi}$ is another nfs weight on $\mathcal{M}'$, then there exists an isometric isomorphism

$$L_p(\mathcal{M},\psi) \xrightarrow{\sim} L_p(\mathcal{M},\tilde{\psi}), \qquad 1 \le p \le \infty.$$

Remark 2.4.10. If ψ is non-tracial and $p \neq q$, then $L_p(\mathcal{M},\psi) \cap L_q(\mathcal{M},\psi) = \{0\}$; more generally, we have $\bar{\mathcal{M}}_{\gamma_1} \cap \bar{\mathcal{M}}_{\gamma_2} = \{0\}$ if $\gamma_1 \neq \gamma_2$.

Remark 2.4.11. $L_p(\mathcal{M},\psi)$ may also be defined in the same way for $0 < p < 1$. It is not a normed space though. All we shall need in the thesis, in particular in the construction of L_p-modules in Chapter 5, are the following properties for $\frac{1}{2} \le p < 1$. Let q, r such that $\frac{1}{q} + \frac{1}{r} = \frac{1}{p}$. Then a product of elements in $L_q(\mathcal{M},\psi)$ and $L_r(\mathcal{M},\psi)$ is in $L_p(\mathcal{M},\psi)$. Moreover, the square root of a positive element in $L_p(\mathcal{M},\psi)$ is in $L_{2p}(\mathcal{M},\psi)$.

Our next goal is to construct embeddings from a suitable subset of $\mathcal{M}$ into $L_p(\mathcal{M},\psi)$ for $1 \le p < \infty$. We fix now a nfs weight φ on $\mathcal{M}$ and write $D_\varphi := \frac{d\varphi}{d\psi}$ and $L_p(\mathcal{M}) := L_p(\mathcal{M},\psi)$ for notational convenience. The proof of the next proposition is an adaptation of [Cas13, Proposition 2.21 (1)].

Proposition 2.4.12. *Let $x \in \mathcal{T}_\varphi^2$, $\theta \in [0,1]$ and $1 \le p < \infty$. Then $D_\varphi^{\frac{1-\theta}{p}} x D_\varphi^{\frac{\theta}{p}}$ is closable and $\left[D_\varphi^{\frac{1-\theta}{p}} x D_\varphi^{\frac{\theta}{p}}\right] \in L_p(\mathcal{M})$.*

Proof. From [Ter82, Theorem 26], we know that for $p \ge 2$, $xD_\varphi^{1/p}$ is closable and $[xD_\varphi^{1/p}] \in L_p(\mathcal{M},\psi)$. So by Lemma 2.2.5 and Proposition 2.4.5 i), we also have $D_\varphi^{1/p}x = (x^*D_\varphi^{1/p})^* = [x^*D_\varphi^{1/p}]^* \in L_p(\mathcal{M},\psi)$ for $p \ge 2$.

Now write $x = ab$, $a, b \in \mathcal{T}_\varphi$. Assume that $\theta \ge \frac{1}{2}$. Let us use the notation '$\cdot$' for the strong product. By using (2.3.4) in the first inequality and Proposition 2.4.5 iii) in the last,

$$\begin{aligned} D_\varphi^{\frac{1-\theta}{p}} x D_\varphi^{\frac{\theta}{p}} &\subseteq D_\varphi^{\frac{1}{2p}} \sigma_{i\frac{\theta-1/2}{p}}(ab) D_\varphi^{\frac{1}{2p}} \subseteq D_\varphi^{\frac{1}{2p}} \sigma_{i\frac{\theta-1/2}{p}}(a) \cdot \left[\sigma_{i\frac{\theta-1/2}{p}}(b) D_\varphi^{\frac{1}{2p}}\right] \\ &\in L_{2p}(\mathcal{M}) \cdot L_{2p}(\mathcal{M}) \subseteq L_p(\mathcal{M}). \end{aligned}$$

Hence,

$$\left(D_\varphi^{\frac{1-\theta}{p}} x D_\varphi^{\frac{\sigma}{p}}\right)^* \supseteq \left(D_\varphi^{\frac{1}{2p}} \sigma_{i\frac{\theta-1/2}{p}}(a) \cdot \left[\theta_{i\frac{\theta-1/2}{p}}(b) D_\varphi^{\frac{1}{2p}}\right]\right)^* \in L_p(\mathcal{M}).$$

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Thus, $\left(D_\varphi^{\frac{1-\theta}{p}} x D_\varphi^{\frac{\theta}{p}}\right)^*$ is densely defined, so $D_\varphi^{\frac{1-\theta}{p}} x D_\varphi^{\frac{\theta}{p}}$ is closable. Then $\left[D_\varphi^{\frac{1-\theta}{p}} x D_\varphi^{\frac{\theta}{p}}\right]$ and its adjoint must be $(-1/p)$-homogeneous by Lemma 2.4.2. Now by Lemma 2.4.4, we get $\left(D_\varphi^{\frac{1-\theta}{p}} x D_\varphi^{\frac{\theta}{p}}\right)^* \in L_p(\mathcal{M},\psi)$. Hence by Proposition 2.4.5 i), $\left[D_\varphi^{\frac{1-\theta}{p}} x D_\varphi^{\frac{\theta}{p}}\right] = \left(D_\varphi^{\frac{1-\theta}{p}} x D_\varphi^{\frac{\theta}{p}}\right)^{**} \in L_p(\mathcal{M})$. Now let $\theta < \frac{1}{2}$. Then,

$$\left(D_\varphi^{\frac{1-\theta}{p}} x D_\varphi^{\frac{\theta}{p}}\right)^* \supseteq \left(x D_\varphi^{\frac{\theta}{p}}\right)^* D_\varphi^{\frac{1-\theta}{p}} = D_\varphi^{\frac{\theta}{p}} x^* D_\varphi^{\frac{1-\theta}{p}}.$$

By what we have just proved, the right hand side is closable and we have

$$\left(D_\varphi^{\frac{1-\theta}{p}} x D_\varphi^{\frac{\theta}{p}}\right)^* \supseteq \left[D_\varphi^{\frac{\theta}{p}} x^* D_\varphi^{\frac{1-\theta}{p}}\right] \in L_p(\mathcal{M}).$$

Hence by the same arguments as before, we conclude that $D_\varphi^{\frac{1-\theta}{p}} x D_\varphi^{\frac{\theta}{p}}$ is closable and $\left[D_\varphi^{\frac{1-\theta}{p}} x D_\varphi^{\theta/p}\right] \in L_p(\mathcal{M})$. □

We recall that D_φ^a is injective and self-adjoint for $a \in [0,1]$. Since $\ker(x^*) = \operatorname{ran}(x)^\perp$ for unbounded operators x, we find that D_φ^a also has dense range. This proves the injectivity statement in the following definition:

Definition 2.4.13. For $1 \le p < \infty$ and $\theta \in [0,1]$ we define the injective embeddings

$$\kappa_p^\theta : \mathcal{T}_\varphi^2 \to L_p(\mathcal{M},\psi); \qquad x \mapsto \left[D_\varphi^{\frac{1-\theta}{p}} x D_\varphi^{\frac{\theta}{p}}\right].$$

As a corollary of Lemma 2.4.4, or directly from [Hil81, Theorem 4 (1)], we have the following strengthening of (2.3.4):

$$\kappa_p^\theta(x) = \kappa_p^{\theta'}(\sigma^\varphi_{i\frac{\theta-\theta'}{p}}(x)), \qquad x \in \mathcal{T}_\varphi^2,\ \theta,\theta' \in [0,1]. \tag{2.4.2}$$

In some cases, the assignment of κ_p^θ can be extended to larger spaces.

Proposition 2.4.14. *i) Let $2 \le p < \infty$ and $\theta = 1$. Then for $x \in \mathfrak{n}_\varphi$, $x D_\varphi^{\frac{1}{p}}$ is closable and $\left[x D_\varphi^{\frac{1}{p}}\right] \in L_p(\mathcal{M},\psi)$. Hence, we can extend κ_p^1 to $\mathfrak{n}_\varphi$.*

ii) Let $1 \le p < \infty$ and $\theta = \frac{1}{2}$, or let $2 \le p < \infty$ and $\theta \in [0,1]$. Then for $a,b \in \mathfrak{n}_\varphi$, the element $D_\varphi^{\frac{1-\theta}{p}} a^* \left[b D_\varphi^{\frac{\theta}{p}}\right]$ is closable and its closure is in $L_p(\mathcal{M},\psi)$. Hence, by linearity, we can extend κ_p^θ to $\mathfrak{m}_\varphi$.

Proof. Part i) is proven in [Ter82, Theorem 23 & 26]. Now let $x = a^* b$, $a,b \in \mathfrak{n}_\varphi$. Then by Proposition 2.4.5 i), iii) and part i), $D_\varphi^{\frac{1-\theta}{p}} a^* \left[b D_\varphi^{\frac{\theta}{p}}\right]$ is closable and its closure is in $L_p(\mathcal{M},\psi)$. This proves part ii). □

We will see in the next subsection that the images of the embeddings are dense for $p \le 2$, see Proposition 2.4.25. For $p \ge 2$ and $\theta = 1$, this can be proven directly.

Proposition 2.4.15. *[Ter82, Theorem 23] We have*

$$\|\kappa_2^1(x)\|_{L_2(\mathcal{M},\psi)} = \|\eta_\varphi(x)\|_{\mathcal{H}_\varphi}, \qquad x \in \mathfrak{n}_\varphi.$$

Moreover, $\kappa_2^1(\mathfrak{n}_\varphi)$ is dense in $L_2(\mathcal{M},\psi)$; hence, the map $\eta_\varphi(x) \mapsto \kappa_2^1(x)$ extends to a unitary $\mathcal{H}_\varphi \to L_2(\mathcal{M},\psi)$.

Proposition 2.4.16. *[Ter82, Theorem 26] Let $2 \le p < \infty$. Then $\kappa_p^1(\mathfrak{n}_\varphi)$ is dense in $L_p(\mathcal{M},\psi)$.*

Let us now consider the extension of operators on $\mathcal{M}$ to the L_p-spaces. The next proposition is a special case of [HJX10, Remark 5.6].

Proposition 2.4.17. *Let $T : \mathcal{M} \to \mathcal{M}$ be a positive φ-preserving map. Then T extends to a bounded positive map $T^{(p)} : L_p(\mathcal{M},\psi) \to L_p(\mathcal{M},\psi)$ for $1 \le p < \infty$ satisfying*

$$T^{(p)}(\kappa_p^{\frac{1}{2}}(x)) = \kappa_p^{\frac{1}{2}}(T(x)), \qquad x \in \mathfrak{m}_\varphi.$$

Remark 2.4.18. A map T satisfying the conditions of Proposition 2.4.17 is automatically normal; this is a special case of [HJX10, Proposition 5.4].

Notation 2.4.19. Justified by Proposition 2.4.5 ii), iii) and Proposition 2.4.12 we will simply denote $x+y$, xy and $D_\varphi^{\frac{1-\theta}{p}} z D_\varphi^{\frac{\theta}{p}}$ for the strong sum and strong product for the rest of the thesis, whenever we are in one of the cases described above.

In this notation, we can use the previous results to deduce, for $x, y \in \mathcal{M}$ and $1 \le p < \infty$, $2 \le q < \infty$:

$$\kappa_p^\theta(x)^* = \kappa_p^{1-\theta}(x^*), \qquad \kappa_q^0(x)\kappa_q^1(y) = \kappa_{q/2}^{\frac{1}{2}}(xy). \tag{2.4.3}$$

We leave the details to the reader.

2.4.3. INTERPOLATION OF NONCOMMUTATIVE L_p-SPACES AND THE KOSAKI-TERP-IZUMI CONSTRUCTION

Let us first recall some theory on compatible couples and interpolation. A *compatible couple* is a pair of Banach spaces (A_0, A_1) together with continuous inclusions $i_0 : A_0 \to A$, $i_1 : A_1 \to A$ into some Hausdorff topological vector space A. We will identify A_j and $i_j(A_j)$ and suppress the embeddings i_0, i_1 in our notation. Given a compatible couple (A_0, A_1), one can define a sum space $A_0 + A_1 \subseteq A$ and an intersection space $A_0 \cap A_1 \subseteq A$. These are Banach spaces when endowed with the norms

$$\|x\|_{A_0+A_1} := \inf\{\|a\|_{A_0} + \|b\|_{A_1} : x = a+b\}; \qquad \|x\|_{A_0\cap A_1} := \max\{\|x\|_{A_0}, \|x\|_{A_1}\}.$$

The *complex interpolation functor* yields Banach spaces $[A_0, A_1]_{[\eta]} \subseteq A_0 + A_1$ for $\eta \in [0,1]$ called interpolation spaces. We refer to [BL76] or [Cas13] for the construction. We have a contractive, dense inclusion $A_0 \cap A_1 \subseteq [A_0, A_1]_{[\eta]}$ for $0 \le \eta \le 1$ ([BL76, Theorem

4.2.2]).

Let (B_0, B_1) be another compatible couple. Two maps $T_i : A_i \to B_i$, $i = 0,1$ are called *compatible morphisms* if T_0 and T_1 agree on $A_0 \cap A_1$. In that case, there exists a unique extension $T : A_0 + A_1 \to B_0 + B_1$. This extension restricts to a map $T_\eta : [A_0, A_1]_{[\eta]} \to [B_0, B_1]_{[\eta]}$, which satisfies the Riesz-Torin inequality:

$$\|T_\eta\| \leq \|T_0\|^{1-\eta} \|T_1\|^{\eta}. \tag{2.4.4}$$

Let us now fix a von Neumann algebra $\mathscr{M}$ with nfs weight φ. The goal is to embed the spaces $\mathscr{M}$ and $\mathscr{M}_*$ into some common Hausdorff topological vector space (in our case, a Banach space) such that $(\mathscr{M}, \mathscr{M}_*)$ becomes a compatible couple. It turns out that there are several ways to do this; indeed, every $z \in \mathbb{C}$ gives a different compatible couple structure. The values $z \in [-\frac{1}{2}, \frac{1}{2}]$ correspond to the values of $\theta \in [0,1]$ from the previous chapter, as we will see later.

We first define an 'intersection', i.e. a map from a subspace of $\mathscr{M}$ to $\mathscr{M}_*$. For $z \in \mathbb{C}$ (called the *interpolation parameter*), let

$$L_{(z)} = \{x \in \mathscr{M} : \exists \varphi_x^{(z)} \in \mathscr{M}_* \text{ s.t. } \forall a, b \in \mathscr{T}_\varphi : \varphi_x^{(z)}(a^* b) = \langle \pi_\varphi(x) J \Delta^{\bar{z}} \eta_\varphi(a), J\Delta^{-z} \eta_\varphi(b)\rangle\}.$$

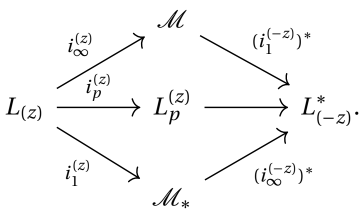
By [Izu97, Proposition 2.3], we have $\mathscr{T}_\varphi^2 \subseteq L_{(z)}$. This means that $L_{(z)}$ is in particular σ-weakly dense in $\mathscr{M}$. We now endow $L_{(z)}$ with the intersection norm

$$\|x\|_{L_{(z)}} = \max\{\|x\|_{\mathscr{M}}, \|\varphi_x^{(z)}\|_{\mathscr{M}_*}\}.$$

With this norm, $L_{(z)}$ is a Banach space. Next, we define contractive injections

$$i_\infty^{(z)} : L_{(z)} \to \mathscr{M},\ x \mapsto x; \qquad i_1^{(z)} : L_{(z)} \to \mathscr{M}_*,\ x \mapsto \varphi_x^{(z)}.$$

Here injectivity of $i_1^{(z)}$ follows from the density of $\eta_\varphi(\mathscr{T}_\varphi)$ in $\mathscr{H}_\varphi$. Considering the (restriction of) the adjoint maps for interpolation parameter $-z$, we get the outer arrows in the following diagram:



By [Izu97, Theorem 2.5], the outer arrows commute; this implies in particular the injectivity of $(i_1^{(-z)})^*$, since injectivitiy of the bottom arrows is clear. This yields a compatible couple $(\mathscr{M}, \mathscr{M}_*)$. By [Izu97, Corollary 2.13], we have

$$(i_1^{(-z)})^*(\mathscr{M}) \cap (i_\infty^{(-z)})^*(\mathscr{M}_*) = (i_\infty^{(-z)})^* \circ i_1^{(z)}(L_{(z)}).$$

In other words, if one suppresses again the embeddings in the notation, we have $\mathscr{M} \cap \mathscr{M}_* = L_{(z)}$.

Definition 2.4.20. For $z \in \mathbb{C}$ and $1 \leq p < \infty$, we define

$$L_p^{(z)}(\mathscr{M}) := [\mathscr{M}, \mathscr{M}_*]_{[1/p]} \subseteq L_{(-z)}^*.$$

For $p = \infty$, we set $L_\infty^{(z)}(\mathscr{M}) = (i_{(-z)}^1)^*(\mathscr{M})$. Moreover, we set $i_p^{(z)} : L_{(z)} \to L_p^{(z)}(\mathscr{M})$ to be the canonical embedding of the intersection space in the interpolation spaces.

We note that for $p = 1$, we get $L_1^{(z)}(\mathscr{M}) = (i_\infty^{(-z)})^*(\mathscr{M}_*)$ by density of $i_1^{(z)}(L_{(z)})$ in $\mathscr{M}_*$; see [Izu97, Proposition 2.4] and [BL76, Theorem 4.2.2].

As it turns out, the Banach spaces $L_p^{(z)}(\mathscr{M})$ are isomorphic for different values of z; however, it should be stressed that the spaces $L_{(z)}$ are still different in general.

Theorem 2.4.21. *[Izu97, Theorem 3.8] Let $1 < p < \infty$ and $z = t + si, z' = t' + s'i$ for $t, t', s, s' \in \mathbb{R}$. Then there exists an isometric isomorphism*

$$U_{p,(z',z)} : L_p^{(z)}(\mathscr{M}) \to L_p^{(z')}(\mathscr{M})$$

such that for $x \in \mathscr{T}_\varphi^2$:

$$U_{p,(z',z)}(i_p^{(z)}(x)) = i_p^{(z')}(\sigma_{i\frac{t'-t}{p}-(s'-s)}(x)). \tag{2.4.5}$$

We now focus on the case $z = t \in [-\frac{1}{2}, \frac{1}{2}]$. The previous theorem is summarised in the following commutative diagram:

$$\begin{array}{ccc}
\mathscr{T}_\varphi^2 & \xrightarrow{\;i_p^{(t)}\;} & L_p^{(t)}(\mathscr{M}) \\
\Big\downarrow \sigma_{i\frac{t'-t}{p}} & & \Big\downarrow U_{p,(t',t)} \\
\mathscr{T}_\varphi^2 & \xrightarrow{\;i_p^{(t')}\;} & L_p^{(t')}(\mathscr{M})
\end{array}$$

We construct isomorphisms between the Connes-Hilsum construction and the above Kosaki-Terp-Izumi construction. We will need the following lemma.

Lemma 2.4.22. *For $x \in L_{(t)}$ and $a, b \in \mathscr{T}_\varphi$, we have*

$$\varphi_x^{(t)}(a^* b) = \varphi(\sigma_{i(\frac{1}{2}+z)}(b) x \sigma_{i(\frac{1}{2}-t)}(a^*)). \tag{2.4.6}$$

If moreover $x \in L_{(t)} \cap \mathfrak{m}_\varphi$, then for $y \in \mathscr{T}_\varphi^2$,

$$\varphi_x^{(t)}(y) = \varphi(\sigma_{i(\frac{1}{2}+t)}(y) x). \tag{2.4.7}$$

Proof. Let $x \in L_{(t)}$ and $a, b \in \mathscr{T}_\varphi$. By [Tak03a, VIII.(5)],

$$\begin{aligned}
\varphi_x^{(t)}(a^* b) &= \langle \pi_\varphi(x) J \Delta^{\frac{1}{2}} \eta_\varphi(\sigma_{i(\frac{1}{2}-t)}(a)), J\Delta^{\frac{1}{2}} \eta_\varphi(\sigma_{i(\frac{1}{2}+t)}(b)) \rangle \\
&= \langle \pi_\varphi(x) S \eta_\varphi(\sigma_{i(\frac{1}{2}-t)}(a)), S\eta_\varphi(\sigma_{i(\frac{1}{2}+t)}(b)) \rangle \\
&= \langle \eta_\varphi(x \sigma_{i(t-\frac{1}{2})}(a^*)), \eta_\varphi(\sigma_{-i(\frac{1}{2}+t)}(b^*)) \rangle \\
&= \varphi(\sigma_{i(\frac{1}{2}+t)}(b) x \sigma_{i(t-\frac{1}{2})}(a^*)).
\end{aligned}$$

This proves (2.4.6). Now assume that moreover $x \in \mathfrak{m}_\varphi$. Then we can apply [Tak03a, Lemma VIII.2.5 (ii)] to obtain from the previous calculation:

$$\varphi_x^{(t)}(a^*b) = \varphi(\sigma_{i(\frac{1}{2}+t)}(b)x\sigma_{i(t-\frac{1}{2})}(a^*)) = \varphi(\sigma_{i(\frac{1}{2}+t)}(a^*b)x).$$

This proves (2.4.7). □

Proposition 2.4.23. *For $x \in L_1(\mathcal{M},\psi)$, set $\phi_x : y \mapsto \mathrm{Tr}(xy)$ to be the corresponding functional in $\mathcal{M}_*$. Then for $x \in \mathcal{T}_\varphi^2$ and $\theta \in [0,1]$, we have*

$$\phi_{\kappa_1^\theta(x)} = \varphi_x^{(\frac{1}{2}-\theta)}.$$

Proof. Let $y \in \mathcal{T}_\varphi^2$. Using subsequently Proposition 2.4.5 iv) twice, (2.4.2), again Proposition 2.4.5 iv), Proposition 2.4.15 and (2.4.7), we get:

$$\begin{aligned}
\phi_{\kappa_1^\theta(x)}(y) &= \mathrm{Tr}(D_\varphi^{1-\theta}xD_\varphi^\theta \cdot y) = \mathrm{Tr}(D_\varphi^\theta y \cdot D_\varphi^{1-\theta}x) = \mathrm{Tr}(x \cdot D_\varphi^\theta y D_\varphi^{1-\theta}) = \mathrm{Tr}(xD_\varphi\sigma_{i(1-\theta)}(y)) \\
&= \mathrm{Tr}(D_\varphi^{\frac{1}{2}}\sigma_{i(1-\theta)}(y) \cdot xD_\varphi^{\frac{1}{2}}) = \langle xD_\varphi^{\frac{1}{2}}, \sigma_{i(1-\theta)}(y)^* D_\varphi^{\frac{1}{2}}\rangle_{L_2(\mathcal{M})} \\
&= \langle \eta_\varphi(x), \eta_\varphi(\sigma_{i(1-\theta)}(y)^*)\rangle = \varphi(\sigma_{i(1-\theta)}(y)x) = \varphi_x^{(\frac{1}{2}-\theta)}(y).
\end{aligned}$$

By σ-weak density of $\mathcal{T}_\varphi^2$ in $\mathcal{M}$, the result follows. □

Proposition 2.4.24. *Let $1 \le p < \infty$ and $t \in [-\frac{1}{2}, \frac{1}{2}]$. Set $\theta = \frac{1}{2} - t$. There exists an isometric isomorphism $\Phi_p^{(t)} : L_p^{(t)}(\mathcal{M}) \to L_p(\mathcal{M},\psi)$ such that, for $x \in \mathcal{T}_\varphi^2$,*

$$\Phi_p^{(t)} : i_p^{(t)}(x) \mapsto k_p^\theta(x). \tag{2.4.8}$$

Proof. For $p = 1$, we set $\Phi_p^{(t)} : \phi_x \mapsto x$. By Proposition 2.4.23, we have for $x \in \mathcal{T}_\varphi^2$:

$$\Phi_p^{(t)}(i_1^{(t)}(x)) = \Phi_p^{(t)}(\varphi_x^{(t)}) = \Phi_p^{(t)}(\phi_{\kappa_1^\theta(x)}) = \kappa_1^\theta(x).$$

For $p > 1$, [Cas13, Proposition 2.21 (2)] gives an isometric isomorphism $\Phi_p : L_p^{(-\frac{1}{2})}(\mathcal{M}) \to L_p(\mathcal{M},\psi)$ satisfying $i_p^{(-\frac{1}{2})}(x) \mapsto \kappa_p^1(x)$ for $x \in \mathcal{T}_\varphi^2$. Now we can compose with the isomorphism from Theorem 2.4.21 to get an isometric isomorphism

$$\Phi_p^{(t)} := \Phi_p \circ U_{p,(-\frac{1}{2},t)} : L_p^{(t)}(\mathcal{M}) \to L_p(\mathcal{M},\psi).$$

By applying (2.4.5) and (2.4.2) we get, for $x \in \mathcal{T}_\varphi$,

$$\Phi_p(U_{p,(-\frac{1}{2},t)}(i_p^{(t)}(x))) = \Phi_p(i_p^{(-\frac{1}{2})}(\sigma_{i\frac{-1/2-t}{p}}(x))) = \kappa_p^1(\sigma_{i\frac{\theta-1}{p}}(x)) = \kappa_p^\theta(x).$$

□

Let $\theta \in [0,1]$ and set $t = \frac{1}{2} - \theta$. Using Proposition 2.4.24, we can construct an interpolation structure corresponding to θ in the Connes-Hilsum picture. First, for $1 \le p < \infty$, we can extend κ_p^θ to $L_{(t)}$ by setting

$$\kappa_p^\theta(y) = \Phi_p^{(t)}(i_p^{(t)}(y)).$$

The maps κ_p^θ then describe the inclusion of the intersection space $L_{(t)}$ in the interpolation spaces $L_p(\mathscr{L}G)$. The θ-embeddings $\mathscr{M} \hookrightarrow L_{(-t)}^*$, $L_1(\mathscr{M},\psi) \hookrightarrow L_{(-t)}^*$ are described as follows in the Connes-Hilsum picture. If $y \in L_1(\mathscr{M},\psi)$, it acts on $L_{(-t)}$ simply by restriction:

$$\langle y, x\rangle_{L_{(-t)}^*, L_{(-t)}} = \phi_y(x) = \mathrm{Tr}(yx).$$

If $y \in \mathscr{M}$, it acts on $L_{(-t)}$ as follows:

$$\langle y, x\rangle_{L_{(-t)}^*, L_{(-t)}} = \langle y, i_1^{(-t)}(x)\rangle = \mathrm{Tr}(y\kappa_1^{1-\theta}(x)).$$

With these embeddings, $(\mathscr{M}, L_1(\mathscr{M},\psi))$ becomes a compatible couple. We will denote this compatible couple structure by $(\mathscr{M}, L_1(\mathscr{M},\psi))_\theta$. Its interpolation spaces are isomorphic to $L_p(\mathscr{M},\psi)$, and the embeddings mapping the intersection $L_{(t)}$ into the L_p-spaces are given by κ_p^θ.

Let us now focus on density results. By general interpolation theory, the spaces $\kappa_p^\theta(L_{(t)})$ are dense in $L_p(\mathscr{M},\psi)$ for any $1 \le p < \infty$, $\theta \in [0,1]$. For $1 \le p \le 2$, we have the following stronger result:

Proposition 2.4.25. *For any $1 \le p \le 2$ and $\theta \in [0,1]$, $\kappa_p^\theta(\mathscr{T}_\varphi^2)$ is dense in $L_p(\mathscr{M},\psi)$.*

Proof. We first claim that it suffices to prove the Proposition for $\theta = 1$. Indeed, it is clear from the definition that $\mathscr{T}_\varphi$ (and thus $\mathscr{T}_\varphi^2$) is invariant under any σ_z, $z \in \mathbb{C}$. Hence, by (2.4.2), we have

$$\kappa_p^\theta(\mathscr{T}_\varphi^2) = \kappa_p^1(\sigma_{i\frac{\theta-1}{p}}(\mathscr{T}_\varphi^2)) = \kappa_p^1(\mathscr{T}_\varphi^2).$$

This proves the claim. Now by [Cas13, Proposition 3.4], the set $(i_\infty^{(\frac{1}{2})})^* \circ i_1^{(-\frac{1}{2})}(\mathscr{T}_\varphi^2)$ is dense in $L_1^{(-\frac{1}{2})}(\mathscr{M}) \cap L_2^{(-\frac{1}{2})}(\mathscr{M})$ in the intersection norm. Since intersection spaces are dense in interpolation spaces and $\|\cdot\|_p \le \max\{\|\cdot\|_1, \|\cdot\|_2\}$ for $1 \le p \le 2$, we also conclude that $i_p^{(-\frac{1}{2})}(\mathscr{T}_\varphi^2)$ is dense in $L_p^{(-\frac{1}{2})}(\mathscr{M})$ for $1 \le p \le 2$. Hence, by Proposition 2.4.24, $\kappa_p^1(\mathscr{T}_\varphi^2)$ is dense $L_p(\mathscr{M},\psi)$ for $1 \le p \le 2$. □

2.4.4. SPECIALISATION FOR σ-FINITE VON NEUMANN ALGEBRAS

We now consider the special case where $\mathscr{M}$ is a σ-finite von Neumann algebra with normal faithful state φ. In that case the theory simplifies somewhat, and we get some nice additional properties. We fix again a nfs weight ψ on $\mathscr{M}'$ and write $D_\varphi := \frac{d\varphi}{d\psi}$. The GNS Hilbert space $\mathscr{H}_\varphi$ now contains a cyclic vector, which we denote by Ω_φ. The map η_φ now takes the form $x \mapsto x\Omega_\varphi$.

Perhaps the most important property that becomes true in the σ-finite case is that we have $D_\varphi \in L_1(\mathcal{M},\psi)$; indeed,

$$\|D_\varphi\|_1 = \mathrm{Tr}(D_\varphi) = \int \frac{d\varphi}{d\psi}\, d\psi = \varphi(1) = 1.$$

It is now clear that the embeddings κ_p^θ are contractive and defined on all of $\mathcal{M}$. We can now also apply the properties from Proposition 2.4.5 on D_φ. For instance, part iv) combined with Proposition 2.4.8 immediately yields that any embedding κ_1^θ is 'state-preserving':

Proposition 2.4.26. *Let $x \in \mathcal{M}$. Then for any $\theta \in [0,1]$ we have*

$$\mathrm{Tr}\left(\kappa_1^\theta(x)\right) = \varphi(x).$$

Another application is the following density result. Note that the proof in the reference given is written in the Haagerup construction, but the exact same proof works in the Connes-Hilsum construction when replacing the density operators D with the spatial derivative D_φ.

Proposition 2.4.27. *[JX03, Lemma 1.1] For any $1 \le p < \infty$ and any $\theta \in [0,1]$, the set $\kappa_p^\theta(\mathcal{M}_a^\varphi)$ (and hence $\kappa_p^\theta(\mathcal{M})$) is dense in $L_p(\mathcal{M},\psi)$.*

Next, we state the following strenghtening of Proposition 2.4.17. This is a special case of [HJX10, Proposition 5.5].

Proposition 2.4.28. *Let $T : \mathcal{M} \to \mathcal{M}$ be a positive φ-preserving map such that $T \circ \sigma_t = \sigma_t \circ T, t \in \mathbb{R}$. Then T extends to a positive bounded map $T^{(p)} : L_p(\mathcal{M},\psi) \to L_p(\mathcal{M},\psi)$ for $1 \le p < \infty$ satisfying*

$$T^{(p)}(\kappa_p^\theta(x)) = \kappa_p^\theta(T(x)), \quad x \in \mathcal{M},$$

which is independent of the choice of $\theta \in [0,1]$. Additionally, $T^{(1)}$ is trace-preserving. If T is unital, then $T^{(p)}$ is contractive.

Proof. We prove only the statement that $T^{(1)}$ is trace-preserving. Consider first $x = x' D_\varphi \in L_1(\mathcal{M},\psi)$ for $x' \in \mathcal{M}$. By Proposition 2.4.26 we have

$$\mathrm{Tr}(T^{(1)}(x)) = \mathrm{Tr}(T(x')D_\varphi) = \varphi(T(x')) = \varphi(x') = \mathrm{Tr}(x).$$

For general $x \in L_1(\mathcal{M},\psi)$ the statement follows by approximation. □

In the current situation we furthermore have $\mathcal{T}_\varphi = \mathcal{M}_a^\varphi$. Also, since $\mathfrak{m}_\varphi = \mathcal{M}$, we can now apply [Tak03a, Lemma VIII.2.5 (ii)] to obtain that the map $z \mapsto \varphi(\sigma_z(x))$ is analytic for $x \in \mathcal{M}_a^\varphi$. But this map is constant on $\mathbb{R}$, hence it is constant everywhere, and we find

$$\varphi(\sigma_z(x)) = \varphi(x), \qquad x \in \mathcal{M}_a^\varphi,\ z \in \mathbb{C}. \tag{2.4.9}$$

As for interpolation, let $t \in [-\frac{1}{2}, \frac{1}{2}]$. By (2.4.7) and (2.4.9), we have for $x, y \in \mathcal{T}_\varphi^2$:

$$\varphi_x^{(t)}(y) = \varphi(\sigma_{i(t+\frac{1}{2})}(y)x) = \varphi(y\sigma_{-i(t+\frac{1}{2})}(x))$$

On the other hand, by [Kos84, Theorem 2.5], the following is true for any $x \in \mathcal{M}$: the map $t \mapsto \varphi(\cdot\sigma_t(x))$ extends to a bounded $\mathcal{M}_*$-valued function on the strip $-1 \le \operatorname{Im} z \le 0$, analytic on the interior. This means that $L_{(t)} = \mathcal{M}$ for any $t \in [-\frac{1}{2}, \frac{1}{2}]$, and the Izumi embeddings coincide with that of Kosaki in [Kos84, Part II]. As we have $\mathcal{M} \hookrightarrow \mathcal{M}_*$ in this case, we can apply the reiteration theorem (see e.g. [BL76, Theorem 4.6.1]) to conclude, for $1 \le p_1, p_2 \le \infty$,

$$[L_{p_1}(\mathcal{M},\psi), L_{p_2}(\mathcal{M},\psi)]_\theta \cong L_p(\mathcal{M},\psi), \qquad \theta \in [0,1],\ \frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}. \tag{2.4.10}$$

Next, we consider a very useful topological property of σ-finite von Neumann algebras.

Proposition 2.4.29. *Set the GNS topology on $\mathcal{M}$ to be the topology inherited from the inclusion $\eta_\varphi : \mathcal{M} \hookrightarrow \mathcal{H}_\varphi$. The GNS topology coincides on bounded sets with the strong (equivalently σ-strong) topology.*

Proof. By [Tak02, Corollary III.3.10], the σ-strong topology on $\mathcal{M}$ does not depend on the representation. Thus, the same is true for the strong topology on bounded sets. Hence, we may assume that $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H}_\varphi)$. Now if $(x_i)_{i\in I}$ is a net in M converging strongly to x, then $\|\eta_\varphi(x_i - x)\|_{\mathcal{H}_\varphi} = \|(x_i - x)\Omega_\varphi\|_{\mathcal{H}_\varphi} \to 0$. Conversely, suppose that $(x_i)_{i\in I}$ is a bounded net in M such that $\|\eta_\varphi(x_i - x)\|_{\mathcal{H}_\varphi} = \|(x_i - x)\Omega_\varphi\| \to 0$. Then for $a \in \mathcal{M}_a^\varphi = \mathcal{T}_\varphi$, we have by [Tak03a, Lemma VIII.3.18 (ii)] that

$$\|(x_i - x)a\Omega_\varphi\| \le \|\sigma_{i/2}^\varphi(a)\| \|(x_i - x)\Omega_\varphi\| \to 0.$$

Recall from Section 2.3 that such elements $a\Omega_\varphi$, $a \in M$ are dense in $\mathcal{H}_\varphi$. Since $(x_i)_{i\in I}$ is bounded, we conclude by a 2ε-estimate that $x_i \to x$ strongly. □

By Proposition 2.4.15, the GNS topology coincides with the topology induced by κ_2^1. Combining this observation with Proposition 2.4.29 and [JS05, Lemma 2.3], we get the following:

Proposition 2.4.30. *Let $x_\lambda \in \mathcal{M}$ be a bounded net converging to 0 in the strong topology. Then for any $1 \le p < \infty$ and $x \in L_p(\mathcal{M},\psi)$:*

$$\|a_\lambda x\|_p \to 0.$$

We can 'extend' the domain of the embeddings κ_p^θ to the L_p-spaces as well: we define for $\frac{1}{2} \le q \le p \le \infty$:

$$\kappa_{p,q}^\theta : L_p(\mathcal{M},\psi) \to L_q(\mathcal{M},\psi), \quad x \mapsto D_\varphi^{\frac{1-\theta}{q} - \frac{1-\theta}{p}} x D_\varphi^{\frac{\theta}{q} - \frac{\theta}{p}}.$$

See Remark 2.4.11 for the case $q < 1$. It is immediate by Proposition 2.4.27 that the embeddings have dense range for $q \ge 1$. Moreover, we can extend (2.4.3) as follows: for $x, y \in L_q(\mathcal{M})$ and $1 \le p \le q \le \infty$ we have

$$\kappa_{q,p}^{(z)}(x)^* = \kappa_{q,p}^{(-z)}(x^*), \qquad \kappa_{q,p}^{(-1)}(x)\kappa_{q,p}^{(1)}(y) = \kappa_{q/2,p/2}^{(0)}(xy). \tag{2.4.11}$$

2

2.5. OPERATOR SPACES

2.5.1. GENERAL OPERATOR SPACE THEORY

Let us recall here some theory on operator spaces and completely bounded maps. We refer to [ER00] and [Pis03] for details. A *concrete* operator space E on a Hilbert space $\mathscr{H}$ is a linear subspace of $\mathscr{B}(\mathscr{H})$. E carries natural matrix norms on the matrix spaces $M_m(E)$, namely the one on $M_m(\mathscr{B}(\mathscr{H})) \cong \mathscr{B}(\mathscr{H}^n)$. An *abstract* operator space is a linear space E together with matrix norms $\|\cdot\|_m$ on $M_m(E)$ satisfying

1. $\|v \oplus w\|_{n+m} = \max\{\|v\|_n, \|w\|_m\}, \qquad v \in M_n(E),\ w \in M_m(E);$

2. $\|\alpha v \beta\|_n \leq \|\alpha\|\|v\|_m\|\beta\|, \qquad \alpha \in M_{n,m}(\mathbb{C}),\ \beta \in M_{m,n}(\mathbb{C}),\ v \in M_m(E).$

An assignment of matrix norms to a linear space E satisfying the above conditions is said to be an *operator space structure* on E. Every abstract operator space can be represented as a concrete operator space, although it is often preferable to work in the abstract setting. This is true for instance in the case of L_p-spaces.

From now on, we will use the letter m to denote the matrix size and reserve the letter n for the amount of variables of a multilinear operator in the next subsection. Let E be an (abstract) operator space and let E^* be its dual space. There exists a natural operator space structure on E^* through the identification $M_m(E^*) \cong \mathscr{B}(E, M_m(\mathbb{C}))$. Now let $T : E \to E$ be an operator, and $m \geq 1$ an integer. The m'th matrix amplification of T is defined as

$$T^{(m)} = 1_{M_m(\mathbb{C})} \otimes T : M_m(E) \to M_m(E)$$

T is said to be *completely bounded* if $\|T\|_{cb} := \sup_m \|T^{(m)}\| < \infty$. We say that T is *completely contractive* if $\|T\|_{cb} \leq 1$ and that T is *completely isometric* if $\|T^{(m)}(v)\|_m = \|v\|_m$ for all $m \geq 1$, $v \in M_m(E)$. If F is another operator space and there exists a completely isometric isomorphism $T : E \to F$, then E and F are said to be *completely isomorphic*.

We can again use interpolation arguments for completely bounded maps. Indeed, if (E_0, E_1) is a compatible couple of operator spaces, then there is a canonical operator space structure on $E_0 \cap E_1$, $E_0 + E_1$ and $(E_0, E_1)_{[\eta]}$ for $0 \leq \eta \leq 1$. Moreover, if $T_0 : E_0 \to E_0$ and $T_1 : E_1 \to E_1$ are completely bounded and compatible morphisms, then the maps T_η on $(E_0, E_1)_{[\eta]}$ are again completely bounded with

$$\|T_\eta\|_{cb} \leq \|T_0\|_{cb}^{1-\theta} \|T_1\|_{cb}^{\theta}. \tag{2.5.1}$$

We refer to [Pis96, Section 2] for the details.

2.5.2. OPERATOR SPACE STRUCTURE OF L_p-SPACES

Now let $\mathscr{M}$ be a von Neumann algebra on $\mathscr{H}$ and $t \in [-\frac{1}{2}, \frac{1}{2}]$. Set $L_p^{(t)}(\mathscr{M})$ to be the Kosaki-Terp-Izumi L_p-space. Let us now describe the operator space structure of the spaces $L_p^{(t)}(\mathscr{M})$ (and hence of the Connes-Hilsum L_p-spaces). Firstly, $\mathscr{M}$ has a natural (concrete) operator space structure, being a subspace of $\mathscr{B}(\mathscr{H})$. Next, $\mathscr{M}_*$ has a natural operator space structure from its inclusion in the dual $\mathscr{M}^*$. However, to correctly define the operator space structure on L_p-spaces, we need to consider the opposite operator space structure here; see [Pis03, p. 139]. This gives us the cases $p = 1, \infty$. Hence, for $1 < p < \infty$, we can endow $L_p^{(t)}(\mathcal{M})$ with the operator space structure obtained via interpolation.

Now set $L_p(\mathcal{M},\psi)$ to be the Connes-Hilsum L_p-space for some nfs weight ψ on $\mathcal{M}'$. Then the isomorphisms $\Phi_p^{(t)}$ from Proposition 2.4.24 also give an operator space structure on $L_p(\mathcal{M},\psi)$. By (a special case of) [Fid99, Theorem 6], all operator space structures obtained in this way are completely isomorphic. This gives a canonical operator space structure on $L_p(\mathcal{M},\psi)$.

We now construct a different set of matrix norms on $L_p(\mathcal{M},\psi)$, which do not give an operator space structure (they do not satisfy the axioms), but for which we will nonetheless define a notion of complete boundedness. This is the definition used in [CS15a]. The idea is to embed the matrix spaces $M_m(L_p(\mathcal{M},\psi))$ into $L_p(M_m(\mathcal{M}),\psi^m)$ (see Example 2.3.5).

Proposition 2.5.1. *Let $x \in M_m(L_p(\mathcal{M},\psi))$ and consider x as an unbounded operator on $\mathcal{H}^m$ with domain $D(x) = \prod_{j=1}^m \bigcap_{i=1}^m D(x_{ij})$. Then x is closable, and $[x] \in L_p(M_m(\mathcal{M}),\psi^m)$.*

Proof. We first consider a matrix unit $x = e_{ij} \otimes y$, $y \in L_p(\mathcal{M},\psi)$. This operator is easily seen to be closed and densely defined. From the fact that $\sigma_t^{\psi^m} = 1 \otimes \sigma_t^{\psi}$ and $(M_n(\mathcal{M})')_\psi^a = 1 \otimes (\mathcal{M}')_\psi^a$, it also follows readily that x is $(-1/p)$-homogeneous. Now let $\phi \in \mathcal{M}_*$ be such that $|y|^p = \frac{d\phi}{d\psi}$, and set $\phi_i = e_{ii} \otimes \phi$. Recall from Example 2.3.5 that $D(\mathcal{H}^m,\psi^m) = D(\mathcal{H},\psi)^m$ and, for $\xi \in D(\mathcal{H}^m,\psi^m)$, we have $R^{\psi^m}(\xi)R^{\psi^m}(\xi)^* = (R^{\psi}(\xi_i)R^{\psi}(\xi_j)^*)_{i,j}$. Now let $\xi \in D(\mathcal{H}^m,\psi^m)$. Then, by Example 2.3.5 and the definition of the spatial derivative,

$$\begin{aligned}\phi_i(R^{\psi^m}(\xi)R^{\psi^m}(\xi)^*) &= \phi(R^{\psi}(\xi_i)R^{\psi}(\xi_i)^*) = \||y|^{p/2}\xi_i\|_{\mathcal{H}}^2 \\ &= \|(e_{ii}\otimes|y|^{p/2})\xi\|_{\mathcal{H}^m}^2 = \||x|^{p/2}\xi\|_{\mathcal{H}^m}^2.\end{aligned}$$

This means that $|x|^p = \frac{d\phi_i}{d\psi}$, and hence $x \in L_p(M_m(\mathcal{M}),\psi^m)$ with $\|x\|_p = \|y\|_p$. This proves the result for matrix units. The general case follows by taking linear combinations and using Proposition 2.4.5 ii). ☐

Definition 2.5.2. We set $S_p^m \otimes L_p(\mathcal{M},\psi)$ to be the space $M_n(L_p(\mathcal{M},\psi))$ equipped with the norm

$$\|x\|_{S_p^m\otimes L_p(\mathcal{M},\psi)} := \|[x]\|_{L_p(M_m(\mathcal{M}),\psi^m)}.$$

We say that an operator $T : L_p(\mathcal{M},\psi) \to L_p(\mathcal{M},\psi)$ is *p-completely bounded* if

$$\|T\|_{p-cb} := \sup_{m\geq 1} \|T^{(m)} : S_p^m \otimes L_p(\mathcal{M},\psi) \to S_p^m \otimes L_p(\mathcal{M},\psi)\| < \infty.$$

Remark 2.5.3. Definition 2.5.2 should be compared to Pisier's construction of vector-valued L_p-spaces. Indeed, if $\mathcal{M}$ is semifinite, then by [Pis98, Equation (3.6)', Lemma 3.3] we have $S_p^m \otimes L_p(\mathcal{M},\psi) \cong S_p^m[L_p(\mathcal{M},\psi)]$, where the latter are as constructed in [Pis98]. Hence it follows from [Pis98, Lemma 1.7] that our notion of p-completely bounded maps coincides with the notion of completely bounded maps defined by the natural operator space structure on $L_p(\mathcal{M},\psi)$.

In the proof of Proposition 2.5.1, we have seen that $\|e_{ij}\otimes x\|_{S_p^m\otimes L_p(\mathcal{M},\psi)} = \|x\|_p$. Next, we show the more general property that the norm on $S_m^p\otimes L_p(\mathcal{M},\psi)$ is a 'cross norm' in the following sense:

Proposition 2.5.4. *Let $x\in L_p(\mathcal{M},\psi)$ and $\alpha\in M_m(\mathbb{C})$ for $m\geq 1$. Then*

$$\|\alpha\otimes x\|_{S_p^m\otimes L_p(\mathcal{M},\psi)} = \|\alpha\|_{S_m^p}\|x\|_p.$$

Proof. Let $\phi\in\mathcal{M}_*^+$ and $\bar\phi\in M_m(\mathcal{M})_*^+$ be such that

$$|x|^p = \frac{d\phi}{d\psi};\qquad [|\alpha\otimes x|^p] = [|\alpha|^p\otimes|x|^p] = \frac{d\bar\phi}{d\psi^m}.$$

Now it suffices to prove that $\bar\phi(1_m\otimes 1_{\mathcal{M}}) = \|\alpha\|_{S_m^p}\phi(1_{\mathcal{M}})$. We can write $\bar\phi = (\bar\phi_{ij})_{i,j}$ for $\bar\phi_{ij}\in\mathcal{M}_*$. Then, for $\xi\in D(\mathcal{H}^m,\psi^m)$, we have by Example 2.3.5,

$$\bar\phi(R^{\psi^m}(\xi)R^{\psi^m}(\xi)^*) = \sum_{i,j}\bar\phi_{ij}(R^{\psi}(\xi_i)R^{\psi}(\xi_j)^*).$$

Set $\beta = |\alpha|^{p/2}$. Then on the other hand, we have

$$\|(|\alpha|^{p/2}\otimes|x|^{p/2})\xi\|^2_{\mathcal{H}^m} = \sum_k\|\sum_i\beta_{k,i}|x|^{p/2}\xi_i\|^2_{\mathcal{H}} = \sum_{i,j}\sum_k\langle\beta_{k,i}|x|^{p/2}\xi_i,\beta_{k,j}|x|^{p/2}\xi_j\rangle.$$

We find from the definition of the spatial derivative (and the fact that we may choose $\xi\in D(\mathcal{H}^m,\psi^m)$ freely):

$$\begin{aligned}\bar\phi_{i,i}(R^{\psi}(\xi_i)R^{\psi}(\xi_i)^*) &= \sum_k\langle\beta_{k,i}|x|^{p/2}\xi_i,\beta_{k,i}|x|^{p/2}\xi_i\rangle\\ &= \sum_k|\beta_{k,i}|^2\left\|\left(\frac{d\phi}{d\psi}\right)^{\frac12}\xi_i\right\|^2_{\mathcal{H}} = \sum_k|\beta_{k,i}|^2\phi(R^{\psi}(\xi_i)R^{\psi}(\xi_i)^*).\end{aligned}$$

Hence

$$\bar\phi(1_m\otimes 1_{\mathcal{M}}) = \sum_i\bar\phi_{i,i}(1_{\mathcal{M}}) = \sum_{i,k}|\beta_{k,i}|^2\phi(1_{\mathcal{M}}) = \|\beta\|^2_{S_2^m}\|x\|_p^p = \|\alpha\|^p_{S_p^m}\|x\|_p^p.$$

□

We close this section by mentioning the following result for later use.

Proposition 2.5.5. *[CH85, Lemma 1.5] Let $\mathcal{M},\mathcal{N}$ be von Neumann algebras and $T:\mathcal{M}\to\mathcal{M}$ be a normal completely bounded map. Then the map $1_{\mathcal{N}}\otimes T$ extends to a normal operator on $\mathcal{N}\bar\otimes\mathcal{M}$.*

2.5.3. MULTIPLICATIVELY BOUNDED MAPS

In this section, we consider an analogue of complete boundedness for multilinear maps. Let $E_1,\dots,E_n,E$ be operator spaces and $T : E_1 \times \dots \times E_n \to E$ be a multilinear map. First recall that T is said to be bounded if the norm

$$\|T\| = \sup_{x_i \in E_i} \frac{\|T(x_1,\dots,x_n)\|}{\|x_1\| \cdot \ldots \cdot \|x_n\|}$$

is finite. For $m \geq 1$, the *multiplicative amplification* $T^{(m)} : M_m(E_1) \times \ldots M_m(E_n) \to M_m(E)$ of T is defined as

$$T^{(m)}(\alpha_1 \otimes x_1,\dots,\alpha_n \otimes x_n) = \alpha_1 \ldots \alpha_n \otimes T(x_1,\dots,x_n), \qquad \alpha_i \in M_m(\mathbb{C}), x_i \in E_i$$

and extended linearly. The map T is said to be *multiplicatively bounded* if

$$\|T\|_{mb} := \sup_{m \geq 1} \|T^{(m)}\| < \infty.$$

Now let $\mathcal{M}$ be a von Neumann algebra and let $L_p(\mathcal{M},\psi)$ be the Connes-Hilsum L_p-spaces for some nfs weight ψ on $\mathcal{M}'$. We denote $L_p(\mathcal{M}) := L_p(\mathcal{M},\psi)$ for notational convenience. One could use the natural operator space stucture on $L_p(\mathcal{M})$ to define a multiplicatively bounded norm on L_p-spaces. However, this definition is not the correct one for our transference results of Chapter 3. Instead, we use a multilinear generalisation of p-complete boundedness, which was introduced by Caspers, Janssens, Krishnaswamy-Usha and Miaskiwskyi [CJKM23]. We note that this time, there does not seem to be a case for which this coincides with 'normal' multiplicative boundedness, as [Pis98, Lemma 1.7] does not generalise to the multilinear case.

Definition 2.5.6. Let $1 \leq p_1,\dots,p_n,p \leq \infty$ with $p^{-1} = \sum_{i=1}^n p_i^{-1}$. We define a map $T : L_{p_1}(\mathcal{M}) \times \ldots \times L_{p_n}(\mathcal{M}) \to L_p(\mathcal{M})$ to be $(p_1,\dots,p_n)$-multiplicatively bounded if

$$\|T\|_{(p_1,\dots,p_n)-mb} := \sup_{m \geq 1} \|T^{(m)} : S^m_{p_1} \otimes L_{p_1}(\mathcal{M}) \times \ldots \times S^m_{p_n} \otimes L_{p_n}(\mathcal{M}) \to S^m_p \otimes L_p(\mathcal{M})\| < \infty.$$

Remark 2.5.7. If $\mathcal{M}$ is semifinite, then $S^m_p \otimes L_p(\mathcal{M}) \cong S^m_p[L_p(\mathcal{M})]$ as mentioned in Remark 2.5.3. So in this case, Definition 2.5.6 coincides with the definition of $(p_1,\dots,p_n)$-multiplicative boundedness from [CJKM23] and [CKV23].

Remark 2.5.8. Even in the semifinite case, it is unclear if this definition of $(p_1,\dots,p_n)$-multiplicative boundedness corresponds to complete boundedness of some linear map on some appropriate tensor product of the E_i's. In the special case the range space is $\mathbb{C}$ and $n = 2$ such a tensor product has been constructed in [Xu06, Remark 2.7]. However, this tensor norm does not seem to admit a natural operator space structure, nor does it seem to work in the multilinear case.

2.6. LOCALLY COMPACT GROUPS AND THEIR NONCOMMUTATIVE L_p-SPACES

In this section we outline the theory of locally compact groups and the associated group von Neumann algebras. We will consider the Plancherel weight and the associated noncommutative L_p-spaces. We also introduce Fourier and Schur multipliers and state some transference results.

2.6.1. LOCALLY COMPACT GROUPS AND FUNCTION SPACES

Recall that a topological group is a group with a topology under which the group operations are continuous. A *locally compact group* is a topological group whose topology is locally compact and Hausdorff. A *left Haar measure* μ on a locally compact group G is a nonzero Radon measure satisfying $\mu(sE) = \mu(E)$ for every $s \in G$ and Borel set $E \subseteq G$. It turns out that every locally compact group has a left Haar measure, which is unique up to scalar multiplication with positive scalars; see [Fol16, Theorems 2.10 and 2.20].

We now fix a locally compact group G and a left Haar measure μ on G. Then the measure $\mu_x(E) := \mu(Ex)$ is also a left Haar measure. Hence there is a scalar $\Delta(x) > 0$ such that $\mu_x = \Delta(x)\mu$. This yields a function $\Delta : G \to (0,\infty)$ that we call the modular function.

Proposition 2.6.1. *[Fol16, Propositions 2.24 and 2.31]*
Δ is a continuous group homomorphism $G \to ((0,\infty),\times)$. It satisfies

$$\int_G f(s^{-1})\Delta(s^{-1})d\mu(s) = \int_G f(s)d\mu(s) = \Delta(t)\int_G f(st)d\mu(s), \qquad t \in G, f \in L_1(G).$$

We will frequently use these properties without reference. In the sequel, we will just write ds for integration against the left Haar measure.

An important subclass of the locally compact groups are the *unimodular groups*, those groups where $\Delta = 1$. In this case, the left Haar measure μ will also be right invariant, in the sense that $\mu(Es) = \mu(E)$ for every $s \in G$ and Borel $E \subseteq G$. Therefore, we will refer to the chosen left Haar measure of a unimodular group as just the Haar measure. Many concrete groups are unimodular; for instance, abelian groups, discrete groups and compact groups are unimodular. Indeed, discrete groups have the counting measure as Haar measure which is trivially left and right invariant. If G is compact, then $\Delta(G)$ is a compact subgroup of $((0,\infty),\times)$, hence $\Delta(G) = \{1\}$. Other examples of unimodular groups are connected semisimple Lie groups and connected nilpotent Lie groups.

For $1 \le p \le \infty$, we will denote by $L_p(G)$ the classical function spaces corresponding to the left Haar measure μ. For functions $f, g \in L_1(G)$ we will define the convolution and involution operations as follows:

$$(f * g)(t) = \int_G f(s)g(s^{-1}t)ds,$$
$$f^*(t) = \Delta(t^{-1})\overline{f(t^{-1})}.$$

With convolution as multiplication and the given involution, $L_1(G)$ becomes a Banach $*$-algebra. Convolution can be defined for other L_p spaces by the same formula. We have the following non-unimodular version of the Young inequalities:

Proposition 2.6.2. *[Ter17, Lemma 1.1] Let $1 \le p_1, p_2, p \le \infty$, with $\frac{1}{p_1} + \frac{1}{p_2} = 1 + \frac{1}{p}$. Take q_1 such that $\frac{1}{p_1} + \frac{1}{q_1} = 1$. Then for $f \in L_{p_1}(G)$ and $g \in L_{p_2}(G)$, the convolution $f * \Delta^{\frac{1}{q_1}} g$ exists and is contained in $L_p(G)$, with*

$$\|f * \Delta^{\frac{1}{q_1}} g\|_p \le \|f\|_{p_1} \|g\|_{p_2}.$$

Let $C_c(G)$ be the space of continuous compactly supported functions of G. Multiplication with Δ is a homomorphism with respect to convolution products in $C_c(G)$: if $f = g * h$, $g, h \in C_c(G)$ and $z \in \mathbb{C}$, we have

$$\begin{aligned}(\Delta^z f)(s) &= \Delta^z(s)\int_G g(t)h(t^{-1}s)dt \\ &= \int_G \Delta^z(t)g(t)\Delta^z(t^{-1}s)h(t^{-1}s)dt \\ &= ((\Delta^z g) * (\Delta^z h))(s)\end{aligned} \tag{2.6.1}$$

Next, we recall some facts about abelian groups G from [Fol16, Secction 4]. In this case, the set of characters $\hat{G}$ forms another locally compact abelian group, called the *Pontryagin dual*. We have $\hat{\hat{G}} = G$. For $f \in L_1(G)$, the *Fourier transform* $\mathscr{F} : L_1(G) \to C_0(\hat{G}), f \mapsto \hat{f}$ is defined as

$$\hat{f}(\xi) = \int_G f(s)\xi(s)ds.$$

It is a norm-decreasing $*$-homomorphism with dense range. By the Plancherel theorem, the Fourier transform defines a unitary $L_2(G) \to L_2(\hat{G})$.

2.6.2. THE GROUP C^*-ALGEBRA AND VON NEUMANN ALGEBRA

The *left regular representation* on G is the map $\lambda : G \to \mathscr{U}(L_2(G))$, $s \mapsto \lambda_s$ where the latter is defined by

$$(\lambda_s g)(t) = g(s^{-1}t), \qquad g \in L_2(G),\ t \in G.$$

For $f \in L_1(G)$, we define $\lambda(f) \in \mathscr{B}(L_2(G))$ by

$$(\lambda(f)g)(t) = \int_G f(s)(\lambda_s g)(t)ds = (f * g)(t).$$

Through this assignment, λ defines a $*$-representation of the Banach $*$-algebra $L_1(G)$. A straightforward calculation yields the following commutation formulae:

$$\Delta^z \lambda_s = \Delta^z(s)\lambda_s \Delta^z, \qquad z \in \mathbb{C}, s \in G \tag{2.6.2}$$

and

$$\Delta^z \lambda(f) = \lambda(\Delta^z f)\Delta^z, \qquad z \in \mathbb{C}, f \in C_c(G). \tag{2.6.3}$$

We similarly define the *right regular representation* on G by

$$(\rho_s g)(t) = \Delta^{\frac{1}{2}}(s)g(ts), \qquad g \in L_2(G),\ t \in G.$$

Set $\check{f}(s) := f(s^{-1})$. Now ρ induces another $*$-representation on $L_1(G)$ given by

$$(\rho(f)g)(t) = \int_G f(s)(\rho_s g)(t)ds = \int_G f(s)\Delta^{\frac{1}{2}}(s)g(ts)ds = (g * \Delta^{-\frac{1}{2}}\check{f})(t).$$

We leave the calculations to the reader.

2

The *reduced group C^*-algebra* $C^*_\lambda(G)$ is defined as the norm closure of $\lambda(L_1(G))$ within $\mathscr{B}(L_2(G))$. Note that, unless G is discrete, $\lambda_s \notin C^*_\lambda(G)$ for $s \in G$. In particular, $C^*_\lambda(G)$ is non-unital. The *(left) group von Neumann algebra* $\mathscr{L}G$ is defined as the σ-weak closure of $C^*_\lambda(G)$ within $\mathscr{B}(L_2(G))$. Equivalently, it is given by

$$\mathscr{L}G = \lambda(L_1(G))'' = \overline{\lambda(L_1(G))}^{\sigma-w} = \lambda(G)'' = \overline{\operatorname{Span}\lambda(G)}^{\sigma-w}.$$

We refer to [Tak03a, Proposition VII.3.1] for the equality of the second and fourth space. Similarly, we can define a right group von Neumann algebra by

$$\mathscr{R}G = \rho(L_1(G))'' = \rho(G)'' = (\mathscr{L}G)'.$$

We refer again to [Tak03a, Proposition VII.3.1] for the last two equalities.

In analogy with the abelian case, we have the following:

Proposition 2.6.3. *The left regular representation λ is a norm-decreasing, injective *-homomorphism $L_1(G) \to C^*_\lambda(G)$ with dense range. Consequently, $\lambda(L_1(G))$ is σ-weakly dense in $\mathscr{L}G$.*

Proof. It is straightforward to check that $\lambda(f)\lambda(g) = \lambda(f * g)$ and $\lambda(f)^* = \lambda(f^*)$. λ is norm-decreasing by Young's inequality. Now assume $f, g \in L_1(G)$ are such that $\lambda(f) = \lambda(g)$. Let $\varphi_U \in C_c(G)$ be an approximate identity for $L_1(G)$; see [Fol16, Proposition 2.44] and the discussion thereafter. Then

$$f = \lim_U f * \phi_U = \lim_U g * \phi_U = g$$

where the limits are in the L_1-norm. Finally, the range of λ is dense by definition of $C^*_\lambda(G)$. □

Corollary 2.6.4. *$\lambda(C_c(G))$ is σ-weakly dense in $\mathscr{L}G$.*

Proof. Since $C_c(G)$ is norm dense in $L_1(G)$, it follows from Proposition 2.6.3 that $\lambda(C_c(G))$ is norm dense in $C^*_\lambda(G)$, hence σ-weakly dense in $\mathscr{L}G$. □

Let us now consider the *Fourier algebra* $A(G)$. It can be defined in several equivalent ways (see [KL18, Proposition 2.3.3]). We will use the following: for a function $g : G \to \mathbb{C}$, denote $g^\dagger(s) := \overline{g(s^{-1})}$. Then we define

$$A(G) := \{f * g^\dagger : f, g \in L_2(G)\} \subseteq C_0(G).$$

We refer to [Fol16, Proposition 2.41] for the last inclusion. The fact that this is an algebra is non-trivial, we refer to [Eym64, p. 218] or [KL18, Theorem 2.4.3]. The main result about this algebra is the following:

Theorem 2.6.5. *[KL18, Theorem 2.3.9] We have $A(G) \cong \mathscr{L}G_*$, where the pairing is given by*

$$\langle \phi, \lambda(f) \rangle = \int_G \phi(s) f(s) ds.$$

For $\phi \in A(G)$, we denote again by ϕ the corresponding functional in $\mathscr{L}G_*$: it satisfies $\phi(\lambda_s) = \phi(s)$ for $s \in G$. We will come back to the Fourier algebra in Section 2.6.5.

2.6.3. THE PLANCHEREL WEIGHT AND CORRESPONDING L_p-SPACES

We say that a measurable function $f: G \to \mathbb{C}$ is *left bounded* if the mapping $g \mapsto f * g$ extends from $C_c(G)$ to a bounded operator on $L_2(G)$. In that case, we denote this mapping again by $\lambda(f)$. Similarly, we say that f is *right bounded* if the mapping $g \mapsto g * f$ extends boundedly from $C_c(G)$ to $L_2(G)$, and write $\lambda'(f)$ for the resulting bounded operator. Note that for $f \in C_c(G)$, we have $\rho(f) = \lambda'(\Delta^{-\frac{1}{2}} \check{f})$.

Definition 2.6.6. The *Plancherel weight* φ on $\mathscr{L}G$ is defined for $x \in \mathscr{L}G$ as

$$\varphi(x^*x) = \begin{cases} \|f\|_2^2 & \text{if } x = \lambda(f) \text{ for some } f \in L_2(G) \\ \infty & \text{else.} \end{cases}$$

We have a Plancherel weight ψ on $\mathscr{R}G = (\mathscr{L}G)'$ defined for $y \in \mathscr{R}G^+$ as $\psi(y) = \varphi(JyJ)$, or in other words,

$$\psi(y^*y) = \begin{cases} \|f\|_2^2 & \text{if } x = \lambda'(f) \text{ for some } f \in L_2(G) \\ \infty & \text{else.} \end{cases}$$

These weights are nfs weights; this follows from [Tak03a, Theorem VII.2.5, Theorem VII.3.4]. We see from the definition that $\mathfrak{n}_\varphi = \{\lambda(f) : f \in L_2(G) \text{ left bounded}\}$. Moreover, for $x = \lambda(f) \in \mathfrak{n}_\varphi$, we have

$$\|\eta_\varphi(x)\|_{\mathscr{H}_\varphi} = \varphi(x^*x)^{1/2} = \|f\|_2.$$

Hence, the map $\eta_\varphi(\mathfrak{n}_\varphi) \to L_2(G)$ given by $\eta_\varphi(\lambda(f)) \mapsto f$ is isometric, so it extends to an isometric map $\mathscr{H}_\varphi \to L_2(G)$. Now since $C_c(G)$ is dense in $L_2(G)$ and $\lambda(C_c(G)) \subseteq \mathfrak{n}_\varphi$, this map is surjective and we get a unitary $\mathscr{H}_\varphi \xrightarrow{\sim} L_2(G)$. Identifying these spaces, the various objects from Tomita-Takesaki theory take the following form:

$$\eta_\varphi : \lambda(f) \mapsto f, \qquad S : f \mapsto f^*, \qquad (\Delta_\varphi f)(s) = \Delta(s) f(s), \qquad (Jf)(s) = \Delta^{\frac{1}{2}}(s^{-1})\overline{f(s^{-1})}.$$

The latter two facts follow from straightforward calculations. In a similar way, we find that $\mathscr{H}_\psi \cong L_2(G)$ and that under this identification, we have $\mathfrak{n}_\psi = \{\lambda'(f) : f \in L_2(G) \text{ right bounded}\}$ and $\eta_\psi : \lambda'(f) \mapsto f$.

We will now calculate the spatial derivative $\frac{d\varphi}{d\psi}$, where we identify $\mathscr{H}_\psi$ and $L_2(G)$.

Proposition 2.6.7. *We have* $\frac{d\varphi}{d\psi} = \Delta_\varphi$.

Proof. For $\xi \in C_c(G)$, we have

$$R^\psi(\xi) f = R^\psi(\xi)\eta_\psi(\lambda'(f)) = \lambda'(f)\xi = \lambda(\xi) f.$$

Hence, by density, we have $R^\psi(\xi) = \lambda(\xi)$ and $D(L_2(G), \psi) = \mathfrak{n}_\varphi$. If $\xi \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$, then also $\lambda(\xi)\lambda(\xi)^* \in \mathfrak{p}_\varphi$. Thus the spatial derivative $\frac{d\varphi}{d\psi}$ is the unique positive self-adjoint operator satisfying

$$\varphi(\lambda(\xi)\lambda(\xi)^*) = \left\| \left(\frac{d\varphi}{d\psi}\right)^{1/2} \xi \right\|^2, \qquad \xi \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*.$$

Now Δ_φ is a positive self-adjoint operator and we have for $\xi \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$:

$$\begin{aligned}\varphi(\lambda(\xi)\lambda(\xi)^*) = \|\xi^*\|_2^2 &= \int_G |\xi(s^{-1})|^2 \Delta(s^{-2})\,ds \\ &= \int_G |\xi(s)|^2 \Delta(s)\,ds = \int_G |(\Delta_\varphi^{\frac{1}{2}}\xi)(s)|^2\,ds = \|\Delta_\varphi^{\frac{1}{2}}\xi\|_2^2.\end{aligned}$$

This shows that indeed $\frac{d\varphi}{d\psi} = \Delta_\varphi$. □

The domain of Δ_φ is exactly $\{f \in L_2(G) : \int_G \Delta^2(s)|f(s)|^2 ds < \infty\}$. Usually we will only apply Δ_φ to continuous compactly supported functions, so that no technical complications can arise. In what follows, we will generally drop the subscript and just write Δ for both the modular operator and the modular function.

Let us now consider the L_p-spaces corresponding to $\mathscr{L}G$. We define $L_p(\mathscr{L}G) := L_p(\mathscr{L}G, \psi)$ to be the Connes-Hilsum L_p-spaces with respect to the Plancherel weight ψ on $(\mathscr{L}G)'$. By combining the unitary $L_2(G) \cong \mathscr{H}_\varphi$ with the one from Proposition 2.4.15, we get a unitary

$$L_2(G) \cong L_2(\mathscr{L}G), \qquad f \mapsto \lambda(f)\Delta^{\frac{1}{2}}. \tag{2.6.4}$$

Moreover we have $L_1(\mathscr{L}G) \cong A(G)$ through Theorem 2.6.5. As mentioned after this theorem, we again denote by ϕ the corresponding functional in $\mathscr{L}G_*$ given by $\phi(\lambda_s) = \phi(s)$, $s \in G$. Now for $\phi \in A(G)_+ \cong (\mathscr{L}G)_*^+$, Proposition 2.4.8 tells us that the identification $A(G) \cong L_1(\mathscr{L}G)$ is given by $\phi \mapsto D_\phi$. Let us push this a bit further still. Recall the notation $\check{f}(s) = f(s^{-1})$. Let $\phi \in A(G)_+$ be such that $\check{\phi} \in L_1(G)$. Then by [Ter17, Corollary 5.14], we have $D_\varphi = \lambda(\check{\phi})\Delta$. So in this case we can identify $\phi \in A(G)$ with $\lambda(\check{\phi})\Delta \in L_1(\mathscr{L}G)$.

For $2 < p < \infty$, we have the following variant of the Hausdorff-Young theorem. Here we will consider $\lambda(f)$ for general f as an unbounded operator on $L_2(G)$.

Theorem 2.6.8. *[Ter17, Theorem 4.5] Let $1 \leq q \leq 2 \leq p < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $f \in L_q(G)$. Then $\lambda(f)\Delta^{1/p} \in L_p(\mathscr{L}G)$ and the resulting mapping $f \mapsto \lambda(f)\Delta^{1/p}$ is linear, norm-decreasing, injective and has dense range.*

Remark 2.6.9. The operator $\lambda(f)\Delta^{\frac{1}{p}}$ mentioned above was not proved to be closable in Section 2.4.2. However, this operator is in fact already closed, as can be seen by use of Proposition 2.6.2.

Now define

$$C_c(G) \star C_c(G) := \mathrm{Span}\{f * g : f, g \in C_c(G)\}; \qquad L := \lambda(C_c(G) \star C_c(G)).$$

Note that $L \subseteq \lambda(C_c(G)) \subseteq \mathscr{T}_\varphi$ by (2.6.3), with $\sigma_z^\varphi(\lambda(f)) = \lambda(\Delta^z f)$ (and in fact $L \subseteq \mathscr{T}_\varphi^2$). The following corollary of Theorem 2.6.8 will be crucial in Chapter 3.

Corollary 2.6.10. *Let $\theta \in [0,1]$. Then $\kappa_p^\theta(\lambda(C_c(G)))$ is dense in $L_p(\mathscr{L}G)$ for $2 \leq p < \infty$ and $\kappa_p^\theta(L)$ (and hence $\kappa_p^\theta(\mathscr{T}_\varphi^2)$) is dense in $L_p(\mathscr{L}G)$ for $1 \leq p < \infty$. Moreover, L is dense in $\mathscr{L}G$ in the σ-weak topology.*

Proof. Note that by (2.6.1) we have $\Delta^z(C_c(G) \star C_c(G)) = C_c(G) \star C_c(G)$. By (2.6.3), this implies

$$\kappa_p^\theta(L) = \kappa_p^{\theta'}(L), \qquad \theta, \theta' \in [0,1]. \tag{2.6.5}$$

So it suffices to prove the result for any choice of $\theta \in [0,1]$. Now the first part follows from Theorem 2.6.8 and the fact that $C_c(G)$ is dense in each $L_q(G)$.

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Now let $1 \le p < \infty$ and $x \in L_p(\mathscr{L}G)$ with polar decomposition $x = u|x|$. Then, by the first part, there are sequences $(x_n), (y_n)$ in $\lambda(C_c(G))$ such that $\Delta^{\frac{1}{2p}} x_n \to u|x|^{\frac{1}{2}}$ and $y_n \Delta^{\frac{1}{2p}} \to |x|^{\frac{1}{2}}$. By Hölder's inequality and a 2ε-argument, we then find that $\kappa_p^{\frac{1}{2}}(xy) = \Delta^{\frac{1}{2p}} x_n \cdot y_n \Delta^{\frac{1}{2p}} \to x$. For the final part, one can take a continuous approximate identity as in [Fol16, p59] to see that $C_c(G) \star C_c(G)$ is dense in $L_1(G)$. By Proposition 2.6.3, L is then norm dense in $C_\lambda^*(G)$ and hence σ-weakly dense in $\mathscr{L}G$. □

Remark 2.6.11. The weight ψ considered here is slightly different than the weight considered in, for example, [CS15a]. There, the following weight is considered instead:

$$\tilde{\psi}(y^* y) = \begin{cases} \|f\|_2^2 & \text{if } x = \rho(f) \text{ for some } f \in L_2(G) \\ \infty & \text{else.} \end{cases}$$

But since both weights are nfs (in fact they are equal on $\{x^* x : x \in \rho(C_c(G))\}$), we have $\mathscr{H}_\psi \cong \mathscr{H}_{\tilde{\psi}}$ and $L_p(\mathscr{M}, \psi) \cong L_p(\mathscr{M}, \tilde{\psi})$. Moreover, it follows from a straightforward calculation that $R^{\tilde{\psi}}(\xi) R^{\tilde{\psi}}(\xi)^* = \lambda(\xi)\lambda(\xi)^*$, and hence $\frac{d\varphi}{d\tilde{\psi}} = \Delta_\varphi$ as well. This means that $L_p(\mathscr{M}, \psi)$ and $L_p(\mathscr{M}, \tilde{\psi})$ have a common dense subset $\kappa_p^\theta(L)$.

Let us close this section by mentioning that multiplication with λ_s preserves the L_p-norm:

$$\|\lambda_s x \lambda_t\|_{L_p(\mathscr{L}G)} = \|x\|_{L_p(\mathscr{L}G)}, \qquad s, t \in G. \tag{2.6.6}$$

Left multiplication is easy since $|\lambda_s x|^2 = |x|^2$ and hence $|\lambda_s x| = |x|$. Right multiplication then follows by taking adjoints and using Proposition 2.4.5 i).

2.6.4. SCHATTEN CLASSES AND SCHUR MULTIPLIERS

For this section, we temporarily leave the domain of locally compact groups. We start by stating some properties of Schatten classes for general Hilbert spaces $\mathscr{H}$, and then specialise to the case $\mathscr{H} = L_2(X)$ for an arbitrary measure space (X, μ). This is the most natural context in which to define Schur multipliers.

For a general Hilbert space $\mathscr{H}$, we denote by $S_p(\mathscr{H})$ the standard Schatten class on $\mathscr{H}$. Recall that $S_p(\mathscr{H}) \subseteq S_q(\mathscr{H})$ for $p \le q$. Moreover, we have $S_p(\mathscr{H}) = L_p(\mathscr{B}(\mathscr{H}), \mathrm{Tr})$ for $1 \le p < \infty$ where the latter is defined as in Section 2.4.1. Hence, we have an operator space structure on $S_p(\mathscr{H})$ as defined in Section 2.5.2. Also, this allows us to use interpolation. Note that as $S_1(\mathscr{H}) \subseteq S_\infty(\mathscr{H})$, $(S_\infty(\mathscr{H}), S_1(\mathscr{H}))$ is also a compatible couple. We claim that

$$[S_\infty(\mathscr{H}), S_1(\mathscr{H})]_{[1/p]} = [\mathscr{B}(\mathscr{H}), S_1(\mathscr{H})]_{[1/p]} = S_p(\mathscr{H}), \qquad 1 \le p \le \infty. \tag{2.6.7}$$

Indeed, by [BL76, Theorem 4.2.2 (b), (c)], the space $[\mathcal{B}(\mathcal{H}), S_1(\mathcal{H})]_{[0]}$ is precisely the closure of $S_1(\mathcal{H}) = S_1(\mathcal{H}) \cap \mathcal{B}(\mathcal{H})$ in $\mathcal{B}(\mathcal{H})$, which is $S_\infty(\mathcal{H})$. Hence, the claim follows by [BL76, Theorem 4.2.2 (b)].

To define Schur multipliers, we will consider only the case $\mathcal{H} = L_2(X)$ where (X, μ) is an arbitrary measure space. If X is a finite set, then clearly $S_p(\ell_2(X)) \cong S_p^m$, where $m = |X|$. In this case, the Schur multiplier with symbol $x \in \ell_\infty(X^2)$ is given by 'pointwise matrix multiplication', i.e.

$$M_x : y \mapsto (x_{ij} y_{ij})_{i,j \in X}.$$

This map is clearly bounded on $S_2(\ell_2(X))$. When considering the Schur multiplier as an operator on $\mathcal{B}(\mathbb{C}^m)$, it depends on the choice of basis. However, we will always specify the index set X and define the Schur multiplier with respect to the standard basis $\{\delta_i : i \in X\}$, so that no ambiguity is possible.

Now let X be a general measure space. Motivated by the matrix example, we can isometrically identify $S_2(L_2(X))$ with the space of kernels $L_2(X \times X)$. Through this identification, a kernel $A \in L_2(X \times X)$ corresponds to the operator

$$(A\xi)(s) = \int_X A(s,t)\xi(t)dt; \qquad \xi \in L_2(X),\ s \in X.$$

This should be seen as a continuous version of matrix multiplication. We will make no distinction between an operator A and its kernel. Let us state some further consequences of this identification before going to the definition of Schur multipliers. Let $1 \le p \le 2 \le q \le \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. The dual pairing of $S_p(L_2(X))$ and $S_q(L_2(X))$ can be expressed in terms of kernels, at least for a dense subset of $S_q(L_2(X))$. Indeed,

$$\langle A, B \rangle_{p,q} = \int_{X^{\times 2}} A(s,t)B(t,s)dtds, \qquad A \in S_p(L_2(X)), B \in S_2(L_2(X)). \tag{2.6.8}$$

We refer to [LS11, Section 1.2] for more details. Next, we can define complex conjugate and transpose operations on $S_2(L_2(X))$ via its identification with $L_2(X \times X)$, i.e. $\overline{A}(s,t) = \overline{A(s,t)}$ and $A^T(s,t) = A(t,s) = \overline{A}^*(s,t)$. These operations can also be defined on $S_p(L_2(X))$ for $1 \le p \le \infty$, by continuous extension for $p > 2$. We claim that

$$\|A^*\|_{S_p(\mathcal{H})} = \|A\|_{S_p(\mathcal{H})}; \qquad \|\overline{A}\|_{S_p(\mathcal{H})} = \|A\|_{S_p(\mathcal{H})}; \qquad \|A^T\|_{S_p(\mathcal{H})} = \|A\|_{S_p(\mathcal{H})}. \tag{2.6.9}$$

The first equality is well-known (and also follows from Proposition 2.4.5). The second equality is clear for $p = 2$ and $p = \infty$; now it follows for general $1 \le p < \infty$ by

$$\|\overline{A}\|_{S_p(\mathcal{H})}^p = \||\overline{A}|^{p/2}\|_{S_2(\mathcal{H})}^2 = \|\overline{|A|^{p/2}}\|_{S_2(\mathcal{H})}^2 = \||A|^{p/2}\|_{S_2(\mathcal{H})}^2 = \|A\|_{S_p(\mathcal{H})}.$$

In the second step, we use that $|\overline{A}|^{p/2} = \overline{|A|^{p/2}}$ holds since $P(\overline{A}) = \overline{P(A)}$ holds for polynomials P. Finally, the third equality follows by the first two.

Definition 2.6.12. Let $\psi \in L_\infty(X \times X)$. The *Schur multiplier* with symbol ψ is defined as

$$M_\psi : S_2(L_2(X)) \to S_2(L_2(X)), \qquad S_\psi(A)(s,t) = \psi(s,t)A(s,t).$$

Now let $1 \leq p \leq \infty$. Assume that $1 \leq p < 2$ (resp. $2 < p \leq \infty$). Then we say that ψ is a p-Schur symbol if S_ψ restricts (resp. extends) to a bounded map $S_p(L_2(X)) \to S_p(L_2(X))$. Similarly, we say that ψ is a p-cb Schur symbol if S_ψ is completely bounded on $S_p(L_2(X))$. Note that for $p = \infty$, we are considering the space $S_\infty(L_2(X))$ and not $\mathscr{B}(L_2(X))$, although we will see in Lemma 2.6.13 below that it doesn't matter.

Lemma 2.6.13. *Let $\psi \in L_\infty(X \times X)$. Then*

i) Assume that ψ is a p-Schur symbol (resp. p-cb Schur symbol) and let $\frac{1}{p} + \frac{1}{q} = 1$. Then ψ is also a q-Schur symbol (resp. q-cb Schur symbol) with

$$\|M_\psi : S_p(L_2(X)) \to S_p(L_2(X))\| = \|M_\psi : S_q(L_2(X)) \to S_q(L_2(X))\| \tag{2.6.10}$$

and a similar equality holds for the completely bounded norms. Also, ψ is a r-Schur symbol (resp. r-cb Schur symbol) for r between p and q.

ii) The following statements are equivalent:

(a) ψ is a ∞-Schur symbol;

(b) M_ψ extends to a bounded normal operator on $\mathscr{B}(L_2(X))$;

(c) M_ψ extends to a bounded operator on $\mathscr{B}(L_2(X))$.

iii) For any $1 \leq p \leq \infty$, ψ is a p-Schur symbol (resp. p-cb Schur symbol) if and only if for every σ-finite subset $X_0 \subseteq X$, $\psi|_{X_0 \times X_0}$ is a p-Schur symbol (resp. p-cb Schur symbol).

iv) If ψ is a ∞-Schur symbol, then it is automatically a ∞-cb Schur symbol with $\|M_\psi : \mathscr{B}(L_2(X)) \to \mathscr{B}(L_2(X))\|_{cb} = \|M_\psi : \mathscr{B}(L_2(X)) \to \mathscr{B}(L_2(X))\|$. Hence the statements from ii) are equivalent to the completely bounded versions.

Proof. Let us first prove (2.6.10) for $1 < p \leq \infty$. We note that the adjoint map M_ψ^* on $S_q(L_2(X))$ satisfies $M_\psi^* = M_{\psi^T}$, where $\psi^T(s,t) = \psi(t,s)$. This means that ψ^T is a q-Schur symbol with the same norm. By applying (2.6.9), we find

$$\begin{aligned}\|M_\psi A\|_{S_q(L_2(X)} &= \|(M_{\psi^T} A^T)^T\|_{S_q(L_2(X))} = \|M_{\psi^T} A^T\|_{S_q(L_2(X))} \\ &\leq \|M_{\psi^T} : S_q(L_2(X)) \to S_q(L_2(X))\| \|A\|_{S_q(L_2(X))}.\end{aligned}$$

This shows that ψ is also a q-Schur symbol and

$$\|M_\psi : S_q(L_2(X)) \to S_q(L_2(X))\| \leq \|M_{\psi^T} : S_q(L_2(X)) \to S_q(L_2(X))\|.$$

By symmetry, the other inequality also holds. For the completely bounded case, we apply the result on the space $X_N = X \times \{1, \ldots, N\}$ and function $\psi_N((s,i)) = \psi(s)$. We use the isometric identifications

$$S_q(L_2(X_N)) \cong S_q(L_2(X)^N) \cong S_q^N \otimes S_q(L_2(X))$$

and the fact that under these identifications, we have $M_{\psi_N}((A_{ij})_{i,j}) = M_\psi^{(N)}((A_{ij})_{i,j})$ for $A_{ij} \in S_q(L_2(X))$. We leave the details to the reader.

Let us now prove ii). Assume that ψ is an ∞-Schur symbol. Then the double adjoint M_ψ^{**} is a normal bounded extension of M_ψ to $\mathscr{B}(L_2(X)) \to \mathscr{B}(L_2(X))$. This proves (a) $\Rightarrow$ (b). The implication (b) $\Rightarrow$ (c) is trivial. Now assume that M_ψ extends to a bounded map on $\mathscr{B}(L_2(X))$. Then for $A \in \mathscr{B}(L_2(X))$ and $B \in S_1(L_2(X))$, we have

$$\langle A, M_\psi^*(B)\rangle_{\mathscr{B}(L_2(X)),\mathscr{B}(L_2(X))^*} = \langle M_\psi(A), B\rangle_{\mathscr{B}(L_2(X)),S_1(L_2(X))} = \langle A, M_{\psi^T}(B)\rangle.$$

Hence $M_\psi^*(B) = M_{\psi^T}(B) \in S_2(L_2(X))$. Since $\|M_\psi^*(B)\|_{S_1(L_2(X))} = \|M_\psi^*(B)\|_{\mathscr{B}(L_2(X))^*} < \infty$, we moreover have $M_\psi^*(B) \in S_1(L_2(X))$. This means that M_ψ^* is an extension of the map $M_{\psi^T} : S_1(L_2(X)) \to S_1(L_2(X))$, which is therefore bounded. Hence ψ is a 1-Schur symbol. Thus, by interpolation (and the proof of (2.6.7)), we find that M_ψ defines a bounded map on $S_\infty(L_2(X))$. This proves (c) $\Rightarrow$ (a).

Now let us finish the proof of i). If $p = 1$, then the adjoint map M_{ψ^T} is a bounded normal map on $\mathscr{B}(\Lambda_2(X))$, hence it restricts to a bounded map on $S_\infty(L_2(X))$. The result for the bounded and completely bounded cases now follow by the same arguments as before. The final statement follows by (2.6.7) (using (2.5.1) for the completely bounded case).

Part iii) follows exactly as in [CS15a, Proof of Theorem 3.1]; the continuity assumption on ψ is not used here. Then the completely bounded case follows as in the proof of part i).

Finally, part iv) follows for σ-finite X by [LS11, Theorem 1.7] and a straightforward continuous analogue of [Pis01, Proof of Theorem 5.1 (ii) $\Rightarrow$ (iv)]. The case of general measure spaces now follows by part iii). □

Remark 2.6.14. For the case $p = 1$ (or $p = \infty$), there is a characterisation for symbols of p-Schur multipliers called the Grothendieck characterisation, which was used in the final part of the proof above. See [Pis01, Theorem 5.1] for a good overview of the discrete case. For the continuous case, the provided reference [LS11, Theorem 1.7] contains a nice proof, although the result was known earlier. However, we will not use the Grothendieck characterisation further in this thesis, therefore we do not give the precise statement.

Let us now define multilinear Schur multipliers. Again, we first consider the matrix case. Let X be a finite set and let $x \in \ell_\infty(X^{\times n+1})$. Then the multilinear Schur multiplier with symbol x is defined as

$$(y^1,\dots,y^n) \mapsto \left(\sum_{i_1,\dots,i_{n-1}\in X} x_{i_0,\dots,i_n} y^1_{i_0 i_1} y^2_{i_1 i_2}\cdots y^n_{i_{n-1} i_n}\right)_{i_0,i_n\in X}.$$

If x is the constant 1 function, then the above is just matrix multiplication. For general measure spaces, we again take a continuous analogue of this definition.

Definition 2.6.15. Let $\psi \in L_\infty(X^{\times n+1})$. The *multilinear Schur multiplier* with symbol ψ

is defined as

$$M_\psi : S_2(L_2(X)) \times \ldots \times S_2(L_2(X)) \to S_2(L_2(X)),$$
$$M_\psi(A_1,\ldots,A_n)(t_0,t_n) = \int_{X^{\times n-1}} \psi(t_0,\ldots,t_n)A_1(t_0,t_1)A_2(t_1,t_2)\ldots A_n(t_{n-1},t_n)dt_1\ldots dt_{n-1}.$$

The fact that this takes values in $S_2(L_2(X))$ follows from Cauchy-Schwarz, as we show here in the case of $n=2$:

$$\begin{aligned}&\iint_{X^2}\left|\int_X \phi(r,s,t)A(r,s)B(s,t)ds\right|^2 drdt\\ \leq&\|\phi\|_\infty^2\int_X\int_X\left(\int_X |A(r,s)|^2ds\right)dr\left(\int_X|B(s,t)|^2ds\right)dt\\ =&\|\phi\|_\infty^2\|A\|_2^2\|B\|_2^2.\end{aligned}$$

The case of higher order n is similar to [PSST17, Lemma 2.1]. Now let $1 \leq p, p_1,\ldots,p_n \leq \infty$, with $p^{-1} = \sum_{i=1}^n p_i^{-1}$. Restrict M_ψ in the i-th input to $S_2(L_2(X)) \cap S_{p_i}(L_2(X))$. Assume that this restriction maps to $S_p(L_2(X))$ and has a bounded extension to $S_{p_1}(L_2(X)) \times \ldots \times S_{p_n}(L_2(X))$. Then we say that ψ is a $(p_1,\ldots,p_n)$-Schur symbol. The extension of M_ψ is again denoted by M_ψ. If M_ψ is $(p_1,\ldots,p_n)$-multiplicatively bounded, we say that ψ is a $(p_1,\ldots,p_n)$-mb Schur symbol.

Next, we note that the norms of multilinear Schur multipliers are determined by the restriction of the symbol to finite sets. This is the multilinear version of [LS11, Theorem 1.19] and [CS15a, Theorem 3.1]. It will be the starting point for the proof of Theorem 3.3.1.

Theorem 2.6.16. *Let μ be a Radon measure on a locally compact space X, and $\psi : X^{n+1} \to \mathbb{C}$ a continuous function. Let $K > 0$. The following are equivalent for $1 \leq p_1,\ldots,p_n, p \leq \infty$:*

(i) ψ is the symbol of a bounded Schur multiplier $S_{p_1}(L_2(X))\times\ldots\times S_{p_n}(L_2(X)) \to S_p(L_2(X))$ with norm less than K.

(ii) For every σ-finite measurable subset X_0 in X, the restriction $\psi|_{X_0^{\times n+1}}$ is the symbol of a bounded Schur multiplier $S_{p_1}(L_2(X_0)) \times \ldots \times S_{p_n}(L_2(X_0)) \to S_p(L_2(X_0))$ with norm less than K.

(iii) For any finite subset $F = \{s_1,\ldots,s_N\} \subset X$ belonging to the support of μ, the restriction $\psi|_{F^{\times(n+1)}}$ is the symbol of a bounded Schur multiplier $S_{p_1}(\ell_2(F)) \times \ldots \times S_{p_2}(\ell_2(F)) \to S_p(\ell_2(F))$ with norm less than K.

The same equivalence is true for the $(p_1,\ldots,p_n)-mb$ norms.

Proof. $(i) \Rightarrow (ii)$ is trivial. The implication $(ii) \Rightarrow (i)$ remains exactly the same as in [CS15a, Theorem 3.1] except for the fact that we have to take $x_i \in S_{p_i}(L_2(X))$ and take into account the support projections of $x_1, x_1^*,\ldots,x_n,x_n^*$ when choosing X_0. Again, continuity of ψ is not used here. The equivalence $(ii) \Leftrightarrow (iii)$ is mutatis mutandis the same as in [LS11, Theorem 1.19]. For the $(p_1,\ldots,p_n)-mb$ norms, we apply the theorem on the space $X_N = X \times \{1,\ldots,N\}$ and function $\psi_N((s_0,i_0),\ldots,(s_n,i_n)) = \psi(s_0,\ldots,s_n)$ and use the isometric identifications that we already used in the proof of Lemma 2.6.13. □

2.6.5. FOURIER MULTIPLIERS AND TRANSFERENCE RESULTS

Let us now return to the setting of a locally compact group G. Recall that if G is abelian, then the Fourier multiplier with symbol ϕ is defined as $T_\phi(f) = \mathscr{F}^{-1}(\phi\hat{f})$. As we have already seen in Proposition 2.6.3, the role of the Fourier transform on $L_1(G)$ is taken by the left regular representation λ. However, we now wish to see G as the 'frequency side'; i.e. it takes the place of the dual group $\hat{G}$ from the abelian case. Hence, we interpret λ instead as the inverse Fourier transform. This yields the following definition:

$$T_\phi : \lambda(f) \mapsto \lambda(\phi f).$$

We say that ϕ is a ∞-Fourier symbol when this map extends to a normal map on $\mathscr{L}G$. The following classical result says that the set of continuous ∞-Fourier symbols coincides with $M(A(G))$, the multiplier algebra of $A(G)$. Note that functions in $M(A(G))$ are automatically continuous, since $A(G)$ contains only continuous functions and vanishes nowhere.

Proposition 2.6.17. *Let $\phi \in L_\infty(G)$. The following are equivalent:*

1. $\phi \in M(A(G))$
2. *There exists a (unique) normal operator T_ϕ on $\mathscr{L}G$ such that $T_\phi\lambda_s = \phi(s)\lambda_s$ for $s \in G$.*
3. *ϕ is a bounded continuous function on G, and there exists some $C > 0$ such that*

$$\|\lambda(\phi f)\| \leq C\|\lambda(f)\|, \qquad f \in L_1(G).$$

Proof. This statement is just [CH85, Proposition 1.2] if we add the global condition that ϕ is continuous. However, this assumption is only necessary in the third statement, and it is automatically satisfied in the first as mentioned above. So the only non-trivial part is to show that the implication 2. $\Rightarrow$ 1. holds without assuming continuity in 2., i.e. to show that every ∞-Fourier symbol is continuous. By normality of T_ϕ, its adjoint restricts to an operator on the predual $A(G)$. This 'pre-adjoint' satisfies, for $f \in A(G)$ and $s \in G$,

$$((T_\phi)_* f)(s) = \langle (T_\phi)_* f, \lambda_s \rangle = \langle f, \phi(s)\lambda_s \rangle = \phi(s) f(s).$$

Hence $\phi f = (T_\phi)_* f \in A(G)$. This means that $\phi \in M(A(G))$. □

We define by $M_{cb}A(G)$ the subset of $M(A(G))$ of (continuous) symbols ϕ for which T_ϕ is completely bounded on $\mathscr{L}G$. It is endowed with the norm $\|\phi\|_{M_{cb}A(G)} = \|T_\phi : \mathscr{L}G \to \mathscr{L}G\|_{cb}$.

Let us now look at the relation between Fourier and Schur multipliers. We start with a motivating example.

Example 2.6.18. Let $G = \mathbb{Z}$. Elements of $B(\ell_2(\mathbb{Z}))$ can be expressed as matrices; in this way, an element λ_n is a matrix which is supported down a single diagonal. A general element in $\mathscr{L}\mathbb{Z}$ looks as follows:

$$\sum_{i\in\mathbb{Z}} x_i\lambda_i = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & x_0 & x_{-1} & \ddots & \ddots \\ \ddots & x_1 & x_0 & x_{-1} & \ddots \\ \ddots & \ddots & x_1 & x_0 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Now let $\phi \in L_\infty(\mathbb{Z})$, and define $\widetilde{\phi} \in L_\infty(\mathbb{Z}^2)$ by $\widetilde{\phi}(i,j) = \phi(i-j)$. Then we see that

$$M_{\widetilde{\phi}}\left(\sum_{i\in\mathbb{Z}} x_i\lambda_i\right) = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \phi(0)x_0 & \phi(-1)x_{-1} & \ddots & \ddots \\ \ddots & \phi(1)x_1 & \phi(0)x_0 & \phi(-1)x_{-1} & \ddots \\ \ddots & \ddots & \phi(1)x_1 & \phi(0)x_0 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} = \sum_{i\in\mathbb{Z}} \phi(i)x_i\lambda_i = T_\phi\left(\sum_{i\in\mathbb{Z}} x_i\lambda_i\right).$$

Hence, we have $T_\phi = M_{\widetilde{\phi}}|_{\mathscr{L}Z}$. Assuming that $M_{\widetilde{\phi}}$ is bounded on $\mathscr{B}(\ell_2(\mathbb{Z}))$, we find that T_ϕ is bounded on $\mathscr{L}\mathbb{Z}$ and

$$\|T_\phi : \mathscr{L}\mathbb{Z} \to \mathscr{L}\mathbb{Z}\| \le \|M_{\widetilde{\phi}} : \mathscr{B}(\ell_2(\zeta)) \to \mathscr{B}(\ell_2(\zeta))\|.$$

We get the same inequality for completely bounded norms by taking matrix amplifications.

As it turns out, the inequality from the previous example holds for general locally compact groups. Moreover, the inequality becomes an equality when considering completely bounded norms. We call this a transference result between Fourier and Schur multipliers. Proofs for the following result can also be found in [Jol92] or [Pis01, Theorem 6.4].

Proposition 2.6.19 ([BF84])**.** *Let $\phi \in L_\infty(G)$ and define $\widetilde{\phi} \in L_\infty(G\times G)$ by $\widetilde{\phi}(s,t) = \phi(st^{-1})$. Then $M_{\widetilde{\phi}}$ is bounded on $B(L_2(G))$ iff $\phi \in M_{cb}A(G)$. In this case*

$$\|M_{\widetilde{\phi}} : B(L_2(G)) \to B(L_2(G))\|_{cb} = \|T_\phi : \mathscr{L}G \to \mathscr{L}G\|_{cb}.$$

If $\phi \in L_\infty(G)$ is such that $M_{\widetilde{\phi}}$ is bounded on $\mathscr{B}(L_2(G))$, then ϕ is sometimes called a *Herz-Schur multiplier*. The space of all Herz-Schur multipliers is denoted by $B_2(G)$, and it is endowed with the norm $\|\varphi\|_{B_2(G)} = \|M_{\widetilde{\phi}} : B(L_2(G)) \to B(L_2(G))\|_{cb}$. Hence, Proposition 2.6.19 says in short that $M_{cb}A(G) = B_2(G)$ with equality of norms.

Remark 2.6.20. Proposition 2.6.19 also implies that a Schur multiplier is bounded on $\mathscr{B}(L_2(G))$ if and only if it is completely bounded. Of course, we already knew this from Lemma 2.6.13.

2

MULTILINEAR FOURIER MULTIPLIERS

Let us now look at the multilinear case. We have the following definition from [TT10]:

$$A^n(G) := \{f \in L_\infty(G^{\times n}) : \text{ there is a normal m.b. map } \Phi : (\mathscr{L}G)^{\times n} \to \mathbb{C} \text{ s.t. } f(s_1, \ldots, s_n) = \Phi(\lambda_{s_1}, \ldots, \lambda_{s_n})\}$$

From the paragraph below (2.6.4), $A^1(G)$ coincides with $A(G)$. Note that by the normality condition, functions in $A^n(G)$ are continuous. For $f \in A(G)$, we define $\theta_n(f) \in A^n(G)$ by

$$\theta_n(f)(s_1, \ldots, s_n) = f(s_1 \ldots s_n).$$

Now we define $M_n A(G)$ to be those $\phi \in L_\infty(G^{\times n})$ for which $\phi\theta_n(f) \in A^n(G)$ for all $f \in A(G)$. Once more, functions in $M_n A(G)$ have to be continuous.

We say that a symbol $\phi \in L_\infty(G^{\times n})$ is a $(\infty, \ldots, \infty)$-mb Fourier symbol if the map

$$(\lambda_{s_1}, \ldots, \lambda_{s_n}) \mapsto \phi(s_1, \ldots, s_n)\lambda_{s_1 \ldots s_n}$$

extends to a multiplicatively bounded normal map $(\mathscr{L}G)^{\times n} \to \mathscr{L}G$. The space of such symbols ϕ is denoted by $M_n^{cb}A(G)$. By [TT10, Proposition 5.4], $M_n^{cb}A(G) \subseteq M_n A(G)$, and hence $(\infty, \ldots, \infty)$-mb Fourier symbols are continuous.

We have a transference result in the multilinear case as well:

Proposition 2.6.21. *[TT10] Let $\phi \in L_\infty(G^{\times n})$ and define $\widetilde{\phi} \in L_\infty(G^{\times n+1})$ by*

$$\widetilde{\phi}(s_0, \ldots, s_n) = \phi(s_0 s_1^{-1}, s_1 s_2^{-1}, \ldots, s_{n-1} s_n^{-1}).$$

Then $M_{\widetilde{\phi}}$ is multiplicatively bounded as an operator $S_\infty(L_2(G))^{\times n} \to S_\infty(L_2(G))$ iff $\phi \in M_n^{cb}A(G)$. In this case, we have

$$\|T_\phi\|_{mb} = \|M_{\widetilde{\phi}}\|_{mb}.$$

Let us give a proof of the "if" direction that is slightly different from [TT10] by using the transference techniques from Theorem 3.3.1, which simplify in the current setup.

Proof of "⇐". Assume that T_ϕ is multiplicatively bounded. Let $F \subseteq G$ finite with $|F| = N$. Let $p_s \in B(\ell_2(F))$ be the projection on the one dimensional space spanned by the delta function δ_s. Let $\widetilde{\phi}_F := \widetilde{\phi}|_{F^{\times n+1}}$. By Theorem 2.6.16 (using that ϕ is continuous), it suffices to prove that

$$M_{\widetilde{\phi}_F} : B(\ell_2(F))^{\times n} \to B(\ell_2(F))$$

and its matrix amplifications are bounded by $\|T_\phi\|_{mb}$. Define the unitary $U = \sum_{s \in F} p_s \otimes \lambda_s \in B(\ell_2(F)) \otimes \mathscr{L}G$ and the isometry

$$\pi : B(\ell_2(F)) \to B(\ell_2(F)) \otimes \mathscr{L}G, \quad \pi(x) = U(x \otimes \mathrm{id})U^*.$$

Note that π satisfies $\pi(E_{st}) = E_{st} \otimes \lambda_{st^{-1}}$. For $s_0, \ldots, s_n \in F$:

$$\begin{aligned}\pi(M_{\widetilde{\phi}_F}(E_{s_0 s_1}, E_{s_1 s_2}, \ldots, E_{s_{n-1} s_n})) &= \pi(\widetilde{\phi}(s_0, \ldots, s_n)E_{s_0 s_n}) \\ &= \phi(s_0 s_1^{-1}, \ldots, s_{n-1} s_n^{-1})E_{s_0 s_n} \otimes \lambda_{s_0 s_n^{-1}},\end{aligned}$$

while

$$\begin{aligned} T_\phi^{(N)}(\pi(E_{s_0s_1}),\dots,\pi(E_{s_{n-1}s_n})) &= T_\phi^{(N)}(E_{s_0s_1}\otimes\lambda_{s_0s_1}^{-1},\dots,E_{s_{n-1}s_n}\otimes\lambda_{s_{n-1}s_n^{-1}}) \\ &= E_{s_0s_n}\otimes T_\phi(\lambda_{s_0s_1^{-1}},\dots,\lambda_{s_{n-1}s_n^{-1}}) \\ &= \phi(s_0s_1^{-1},\dots,s_{n-1}s_n^{-1})E_{s_0s_n}\otimes\lambda_{s_0s_n^{-1}}. \end{aligned}$$

It follows that $T_\phi^{(N)}\circ\pi^{\times n}=\pi\circ M_{\tilde{\phi}_F}$. This implies that

$$\|M_{\tilde{\phi}_F}\| = \|\pi\circ M_{\tilde{\phi}_F}\| = \|T_\phi^{(N)}\circ\pi^{\times n}\| \le \|T_\phi^{(N)}\| \le \|T_\phi\|_{mb}.$$

By taking matrix amplifications, we prove similarly that $\|M_{\tilde{\phi}_F}\|_{mb}\le\|T_\phi\|_{mb}$. □

Remark 2.6.22. A multilinear map on the product of some operator spaces is multiplicatively bounded iff its linearization is completely bounded as a map on the corresponding Haagerup tensor product. However, as [JTT09, Lemma 3.3] shows, for Schur multipliers $M_{\tilde{\phi}}$ on $S_\infty(L_2(G))^{\times n}$, just boundedness on the Haagerup tensor product is sufficient to guarantee that $M_{\tilde{\phi}}$ is multiplicatively bounded. Note that even in the linear case, when $p<\infty$, it is unknown whether a bounded Schur multiplier on $S_p(L_2(\mathbb{R}))$ is necessarily completely bounded unless ϕ has continuous symbol (we refer to [Pis98, Conjecture 8.1.12], [LS11, Theorem 1.19], [CW19]).

LINEAR FOURIER MULTIPLIERS ON L_p

We now give the construction of linear Fourier multipliers on $L_p(\mathscr{L}G)$ from [CS15a]. The definition is given with the help of a certain intertwining property with Schur multipliers.

Let $1\le p<\infty$. Then we have $L_p(\mathscr{L}G)\cap S_p(L_2(G))=\{0\}$ in general, so we cannot directly link Fourier and Schur multipliers as in the case $p=\infty$. We use the following trick to circumvent this difficulty. We will also use this trick in Section 3.4. Let $F\subset G$ be a relatively compact set with positive measure and let $P_F: L_2(G)\to L_2(F)$ be the orthogonal projection. Then, for $1\le p\le\infty$, and $x\in L_{2p}(\mathscr{L}G)$, xP_F defines an operator in $S_{2p}(L_2(G))$ (see [CS15a, Proposition 3.1]). For $x\in L_p(\mathscr{L}G)$ with polar decomposition $x=u|x|$, we will abusively denote by P_FxP_F the operator $(|x|^{1/2}u^*P_F)^*|x|^{1/2}P_F$, which lies in $S_p(L_2(G))$ whenever $x\in L_p(\mathscr{L}G)$. By [CS15a, Theorem 5.1], we have

$$\|P_FxP_F\|_{S_p(L_2(G))}\le\mu(F)^{1/p}\|x\|_{L_p(\mathscr{L}G)}. \tag{2.6.11}$$

Moreover, by [CS15a, Theorem 6.2 (ii), (iv)], we have for $x\in L_p(\mathscr{L}G)$

$$\text{If } P_FxP_F=0\ \forall F\subseteq G \text{ relatively compact, then } x=0. \tag{2.6.12}$$

We can now define Fourier multipliers on $L_p(\mathscr{L}G)$.

Proposition 2.6.23. *[CS15a, Definition-Proposition 3.5] Let $\phi\in M_{cb}A(G)$ and $1\le p\le\infty$. There is a unique p-completely bounded map $T_\phi^p: L_p(\mathscr{L}G)\to L_p(\mathscr{L}G)$ that satisfies*

$$P_FT_\phi^p(x)P_F = M_{\tilde{\phi}}(P_FxP_F).$$

It has completely bounded norm less than $\|\phi\|_{M_{cb}A(G)}$.

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For $\phi \in M_{cb}A(G)$, the map T^p_ϕ from Proposition 2.6.23 will be called the Fourier multiplier on $L_p(\mathcal{L}G)$.

Remark 2.6.24. We note that Proposition 2.6.23 defines Fourier multipliers on $L_p(\mathcal{L}G)$ only for $\phi \in M_{cb}A(G)$. However, with the same proof, we can also define Fourier multipliers T^p_ϕ on $L_p(\mathcal{L}G)$ for $\phi \in M(A(G))$ (this uses the fact that the map $\lambda(f) \mapsto \lambda(\check{f})$ on $\mathcal{L}G$ is isometric; see for instance [KL18, text above Definition 2.3.10]). These multipliers are then only bounded, not completely bounded. We note that trivially, $A(G) \subseteq M(A(G))$, and hence the space $A(G)$ suffices to provide us with plenty of Fourier multipliers on $L_p(\mathcal{L}G)$; we will use this fact in the proof of Theorem 3.3.1.

We now state the corresponding transference result.

Proposition 2.6.25. *[CS15a, Theorem 4.2 & 5.2] Let $\phi \in M_{cb}A(G)$. Then $\tilde{\phi}$ is a p-cb Schur symbol and*

$$\| M_{\tilde{\phi}} : S_p(L_2(G)) \to S_p(L_2(G)) \|_{cb} \le \| T_\phi : L_p(\mathcal{L}G) \to L_p(\mathcal{L}G) \|_{cb}.$$

If G is amenable, then the opposite inequality also holds.

The transference theorems in Chapter 3 applied to the case $n = 1$ give a strenghtening of this result. Namely, they work also for general p-Fourier symbols, not just symbols in $M_{cb}A(G)$.

Now let $f \in C_c(G) \star C_c(G)$, $1 \le p \le \infty$ and $\theta \in [0,1]$. Then $P_F \kappa^\theta_p(\lambda(f)) P_F \in S_p(L_2(G))$ as mentioned above. It is given by integration against a kernel, which we calculate now for later use. We write $a := \frac{1-\theta}{p}$ and $b := \frac{\theta}{p}$ so that $\kappa^\theta_p(\lambda(f)) = \Delta^a \lambda(f) \Delta^b$. Then,

$$\begin{aligned}
(P_F \Delta^a \lambda(f) \Delta^b P_F g)(s) &= 1_F(s) \Delta^a(s) \int_G f(t) \Delta^b(t^{-1}s) 1_F(t^{-1}s) g(t^{-1}s)\, dt \\
&= 1_F(s) \Delta^a(s) \int_G f(st) \Delta^b(t^{-1}) 1_F(t^{-1}) g(t^{-1})\, dt \\
&= 1_F(s) \Delta^a(s) \int_G f(st^{-1}) \Delta^b(t) 1_F(t) \Delta(t^{-1}) g(t)\, dt.
\end{aligned}$$

Hence the kernel of $P_F \kappa^\theta_p(\lambda(f)) P_F$ is given by

$$(s,t) \mapsto 1_F(s) \Delta^a(s) f(st^{-1}) \Delta^{b-1}(t) 1_F(t). \tag{2.6.13}$$

Clearly, this function is compactly supported and bounded, hence it is in $L_2(X \times X)$. This means in particular that $P_F \kappa^\theta_p(\lambda(f)) P_F \in S_2(L_2(G))$.

MULTILINEAR FOURIER MULTIPLIERS ON L_p

Finally, we consider multilinear Fourier multipliers on $L_p(\mathcal{L}G)$. We will do this only for unimodular groups G for now. Let $p_1, \ldots, p_n$ be such that $1 \le p_1, \ldots, p_n \le \infty$. Unlike in the linear case, the choice of how to define the case $p_i = \infty$ is important.

Definition 2.6.26. Assume that G is unimodular. Let $\phi \in L_\infty(G^{\times n})$ and $1 \leq p_1, \ldots, p_n, p \leq \infty$ with $p^{-1} = \sum_{i=1}^n p_i^{-1}$. Consider the map $T_\phi : L^{\times n} \to \mathscr{L}G$ defined by

$$T_\phi(\lambda(f_1), \ldots, \lambda(f_n)) = \int_{G^{\times n}} \phi(t_1, \ldots, t_n) f_1(t_1) \ldots f_n(t_n) \lambda_{t_1 \ldots t_n} dt_1 \ldots dt_n$$

for $f_i \in C_c(G) \star C_c(G)$. If this map takes values in $L_p(\mathscr{L}G)$ and extends continuously to $L_{p_1}(\mathscr{L}G) \times \ldots \times L_{p_n}(\mathscr{L}G)$, then we say that ϕ is a $(p_1, \ldots, p_n)$-Fourier symbol. The extension is again denoted by T_ϕ. In case $p_i = \infty$, we replace $L_{p_i}(\mathscr{L}G)$ by $C^*_\lambda(G)$ in the i'th coordinate. If the extension is $(p_1, \ldots, p_n)$-multiplicatively bounded, then we say that ϕ is a $(p_1, \ldots, p_n)$-mb Fourier symbol.

Note that this definition works due to the fact that $L \subseteq L_p(\mathscr{L}G)$. This is not the case for non-unimodular groups. We will give a definition for the non-unimodular case in Section 3.2.

Remark 2.6.27. We note that a priori, the set of $(\infty, \ldots, \infty)$-mb Fourier symbols is smaller than $M_n^{cb} A(G)$. However, these sets are actually the same. This follows for instance from a combination of our results in Chapter 3 and [TT10, Theorem 5.5]. One does not need the complicated machinery of Section 3.3 however. It follows already from the proof of [TT10, Theorem 5.5], or from the proof of Proposition 2.6.21, that it suffices to require that T_ϕ is multiplicatively bounded on $(C^*_\lambda(G))^{\times n}$.

The purpose of Chapter 3 is to prove a transference result for multilinear Fourier multipliers on $L_p(\mathscr{L}G)$, extending Proposition 2.6.25.

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L_p-TRANSFERENCE BETWEEN MULTILINEAR FOURIER AND SCHUR MULTIPLIERS FOR GENERAL LOCALLY COMPACT GROUPS

This chapter is based on the following articles:

1. **Martijn Caspers, Amudhan Krishnaswamy-Usha, Gerrit Vos**, *Multilinear transference of Fourier and Schur multipliers acting on non-commutative L_p-spaces*, Canadian Journal of Mathematics, **75**(6):1986-2006 (2023).

2. **Gerrit Vos**, *Transference of multilinear Fourier and Schur multipliers acting on non-commutative L_p-spaces for non-unimodular groups.*
Preprint: 2023.arXiv:2308.16595

In this chapter, G will be an arbitrary locally compact group. We prove here a multilinear transference result between Fourier and Schur multipliers, extending the results from [CS15a] and [CJKM23]. This chapter is based on [CKV23], except for the final section, and [Vos23]. [CKV23] contains the main ideas for the transference results of Theorem 3.3.1 and Corollary 3.4.2. However, this paper only deals with unimodular groups. [Vos23] contains the entirety of Section 3.2 and generalises the results of [CKV23] for non-unimodular groups. We will follow the proofs of the unpublished manuscript, occasionally indicating where the proof simplifies in the unimodular case. We have also added a separate section on linear Fourier multipliers, whereas in [Vos23] this was incorporated in the preliminaries.

The main difficulty for non-unimodular groups comes from the non-traciality of the Plancherel weight. This means that we have to deal with spatial derivatives, which by

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Proposition 2.6.7 is just multiplication with the modular function Δ. In particular, this raises the question how the multilinear Fourier multiplier should be defined for $p<\infty$. It turns out that, in order to prove transference results, one needs to use the definition that 'leaves the Δ's in place'. More precisely, for $f_i \in C_c(G)\star C_c(G)$ and $x_i = \Delta^{\frac{1}{2p_i}}\lambda(f_i)\Delta^{\frac{1}{2p_i}}$, the Fourier multiplier is defined as

$$T_\phi(x_1,\ldots,x_n) = \int_{G^{\times n}} \phi(s_1,\ldots,s_n) f_1(s_1)\ldots f_n(s_n)\Delta^{\frac{1}{2p_1}}\lambda_{s_1}\Delta^{\frac{1}{2p_1}}\ldots\Delta^{\frac{1}{2p_n}}\lambda_{s_n}\Delta^{\frac{1}{2p_n}}\,ds_1\ldots ds_n.$$

A major drawback of this definition is that it is not suitable for interpolation results when $n>2$, unless the 'intermediate' p_i's are all equal to ∞. All this will be discussed in Section 3.2. Our first main result gives the multilinear transference from Fourier multipliers as defined above to Schur multipliers. This is Theorem 3.3.1. The definition of the $(p_1,\ldots,p_n)$-multiplicative norm was given in Section 2.5.3.

Theorem A. *Let G be a locally compact first countable group and let $1\le p\le\infty$, $1<p_1,\ldots,p_n\le\infty$ be such that $p^{-1}=\sum_{i=1}^n p_i^{-1}$. Let $\phi\in C_b(G^{\times n})$ and define $\widetilde{\phi}\in C_b(G^{\times n+1})$ by*

$$\widetilde{\phi}(s_0,\ldots,s_n) = \phi(s_0s_1^{-1}, s_1s_2^{-1},\ldots,s_{n-1}s_n^{-1}), \qquad s_i\in G.$$

If ϕ is the symbol of a $(p_1,\ldots,p_n)$-multiplicatively bounded Fourier multiplier T_ϕ of G, then $\widetilde{\phi}$ is the symbol of a $(p_1,\ldots,p_n)$-multiplicatively bounded Schur multiplier $M_{\widetilde{\phi}}$ of G. Moreover,

$$\begin{aligned}&\|M_{\widetilde{\phi}}: S_{p_1}(L_2(G))\times\ldots\times S_{p_n}(L_2(G))\to S_p(L_2(G))\|_{(p_1,\ldots,p_n)-mb}\\ &\qquad\le \|T_\phi : L_{p_1}(\mathcal{L}G)\times\ldots\times L_{p_n}(\mathcal{L}G)\to L_p(\mathcal{L}G)\|_{(p_1,\ldots,p_n)-mb}.\end{aligned}$$

The proof is mostly an adaptation of the proof of [CJKM23, Lemma 4.6]. As there are changes in several places, we have chosen to include the proof in full detail here. To tackle the non-unimodular case, we will need a generalisation of the reduction lemmas [CJKM23, Lemmas 4.3 and 4.4]. We do this already in Section 3.2. Note that [CJKM23, Lemma 4.4] does not give any details for the proof, even though it is not that trivial even in the unimodular case. In Lemma 3.2.8 we give an elegant induction argument which fills this gap. Also, we need an extension of the intertwining result [CJKM23, Proposition 3.9] for non-unimodular groups, which we state in Proposition 3.3.3. We will sketch the proof in a separate technical section at the end. We also note that in [CKV23, Theorem 3.1], the group was required to be second countable, but in the proof actually only first countability was needed.

For amenable groups, we also have the converse transference result. In fact, one no longer needs a continuous symbol, nor the first countability condition on the group. This is Corollary 3.4.2.

Theorem B. *Let G be an amenable locally compact group and $1\le p,p_1,\ldots,p_n\le\infty$ be such that $\frac{1}{p}=\sum_{i=1}^n\frac{1}{p_i}$. Let $\phi\in L_\infty(G^{\times n})$ and define $\widetilde{\phi}$ as in Theorem A. If $\widetilde{\phi}$ is the symbol of a $(p_1,\ldots,p_n)$-bounded (resp. multiplicatively bounded) Schur multiplier then ϕ is the symbol of a $(p_1,\ldots,p_n)$-bounded (resp. multiplicatively bounded) Fourier multiplier. Moreover,*

$$\|T_\phi\|_{(p_1,\ldots,p_n)}\le\|M_{\widetilde{\phi}}\|_{(p_1,\ldots,p_n)}, \qquad \|T_\phi\|_{(p_1,\ldots,p_n)-mb}\le\|M_{\widetilde{\phi}}\|_{(p_1,\ldots,p_n)-mb}.$$

Again, the proof is similar to [CKV23], but with additional technical complications. We also abstain from using ultraproduct techniques since they were not actually necessary for the proof. It should be noted that if $p_i = \infty$ for some $1 \leq i \leq n$, then our methods only yield the above boundedness results of the multilinear Fourier multiplier on $C^*_\lambda(G)$ in the i'th input (and conversely, boundedness on $C^*_\lambda(G)$ is all we need for the converse direction in Theorem A). Of course, if $p_1 = \ldots = p_n = p = \infty$, then the result from [TT10] guarantees that the Fourier multiplier is indeed bounded on $(\mathcal{L}G)^{\times n}$.

Let us describe the structure of the chapter. First we briefly look again at the linear case in Section 3.1 as preparation for the multilinear case. In Section 3.2, we discuss possible definitions of the multilinear Fourier multiplier, and explain why the definition as stated above is the correct one for transference. We also prove some properties of the multilinear Fourier multiplier that we will need later. In Sections 3.3 and 3.4, we prove the transference from Fourier to Schur (Theorem A) and transference from Schur to Fourier (Theorem B) respectively. In Section 3.5, we sketch the proof of Proposition 3.3.3 using Haagerup reduction. This section is rather technical and not essential to understand the bigger picture.

3.1. LINEAR FOURIER MULTIPLIERS

Before considering the multilinear case, let us first give an alternative, more concrete definition of linear Fourier multipliers which will be more in line with the multilinear definition given later. This definition is broader than the one in Section 2.6.5, since it also allows for symbols that only define bounded Fourier multipliers for a specific p.

Recall from Section 2.6.5 that λ takes the role of the inverse Fourier transform on $L_1(G)$ in order to define Fourier multipliers on $\mathcal{L}G$. In the case $p = 2$, the Plancherel identity (2.6.4) gives a natural Fourier transformation $f \mapsto \lambda(f)\Delta^{1/2}$. As such, the Fourier transform with symbol $\phi \in L_\infty(G)$ is given by

$$T_\phi : L_2(\mathcal{L}G) \to L_2(\mathcal{L}G), \qquad \lambda(f)\Delta^{\frac{1}{2}} \mapsto \lambda(\phi f)\Delta^{\frac{1}{2}}, \quad f \in L_2(G).$$

By (2.6.4), this map is well-defined and bounded:

$$\|\lambda(\phi f)\Delta^{\frac{1}{2}}\|_2 = \|\phi f\|_2 \leq \|\phi\|_\infty \|f\|_2 = \|\phi\|_\infty \|\lambda(f)\Delta^{\frac{1}{2}}\|_2.$$

For $2 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, one can consider more generally the map $L_q(G) \to L_p(\mathcal{L}G)$, $f \mapsto \lambda(f)\Delta^{1/p}$ as the noncommutative analogue of the Fourier transform on $L_q(G)$, see Theorem 2.6.8. For $1 \leq p < 2$ we use the same mapping, but we will a priori take $\overline{\mathcal{L}G}_{(-1/p)}$, the space of closed densely defined $(-1/p)$-homogeneous operators on $L_2(G)$, as codomain. This yields the following definition (recall that $L := \lambda(C_c(G) * C_c(G))$):

Definition 3.1.1. Let $\phi \in L_\infty(G)$ and $1 \leq p < \infty$ (the case $p = \infty$ was already defined in Section 2.6.5). We set

$$T_\phi : \kappa^1_p(L) \to \overline{\mathcal{L}G}_{(-1/p)}, \qquad \lambda(f)\Delta^{1/p} \mapsto \lambda(\phi f)\Delta^{1/p}, \quad f \in C_c(G) \star C_c(G), \tag{3.1.1}$$

When the map T_ϕ maps to $L_p(\mathcal{L}G)$ and extends continuously to a bounded map on $L_p(\mathcal{L}G)$, we say that ϕ *is a p-Fourier symbol.* When the extension is moreover p-completely bounded on $L_p(\mathcal{L}G)$ (see Section 2.5.2), we say ϕ *is a p-cb Fourier symbol.*

It follows from Lemma 2.4.2 that the map (3.1.1) is well-defined. Note that by Remark 2.4.10, the lack of reference to p in the notation cannot lead to any confusion. Recall that by Corollary 2.6.10, $\kappa_p^1(L)$ is dense in $L_p(\mathcal{L}G)$, and hence the extension to $L_p(\mathcal{L}G)$, if it exists, is unique.

Remark 3.1.2. For $p \geq 2$, $T_\phi(x)$ always maps $\kappa_p^\theta(L)$ to $L_p(\mathcal{L}G)$. Indeed, let $f \in C_c(G) \star C_c(G)$. Then ϕf is a bounded, compactly supported function, hence in $L_q(\mathbb{R})$. So by Theorem 2.6.8, $T_\phi(f) = \lambda(\phi f)\Delta^{1/p} \in L_p(\mathcal{L}G)$.

Remark 3.1.3. One can define analogous Fourier multipliers for any $\theta \in [0,1]$, and it turns out that these are all equal. Indeed, let $1 \leq p < \infty$ and $\theta \in [0,1]$ and define

$$T_\phi^\theta(\kappa_p^\theta(\lambda(f))) = \kappa_p^\theta(\lambda(\phi f)), \qquad f \in C_c(G) \star C_c(G).$$

Now let $f \in C_c(G) \star C_c(G)$ and take $g = \Delta^{\frac{1-\theta}{p}} f$; then by (2.6.3),

$$T_\phi^\theta(\kappa_p^\theta(\lambda(f))) = \Delta^{\frac{1-\theta}{p}}\lambda(\phi f)\Delta^{\frac{\theta}{p}} = \lambda(\Delta^{\frac{1-\theta}{p}}\phi f)\Delta^{\frac{1}{p}} = \lambda(\phi g)\Delta^{\frac{1}{p}} = T_\phi(\kappa_p^1(\lambda(g))).$$

Also, we have $\kappa_p^\theta(\lambda(f)) = \kappa_p^1(\lambda(g))$ (by the above formula for $\phi = 1$); hence $T_\phi^\theta = T_\phi$ on $\kappa_p^\theta(L) = \kappa_p^1(L)$.

Let us now prove that our definition coincides with the one from Proposition 2.6.23 (or Remark 2.6.24) for $\phi \in M(A(G))$. We will denote by T_ϕ^p the Fourier multiplier from Remark 2.6.24.

Proposition 3.1.4. *Let $\phi \in M(A(G))$ and $1 \leq p \leq \infty$. Then for $x \in L$, we have*

$$T_\phi^p(\kappa_p^1(x)) = T_\phi(\kappa_p^1(x)).$$

This means that ϕ is a p-Fourier symbol and T_ϕ coincides with T_ϕ^p on $L_p(\mathcal{L}G)$. Similarly, if $\phi \in M_{cb}A(G)$, then ϕ is a p-cb Fourier symbol.

Proof. For $p = \infty$ there is nothing to prove. For $p = 1$, we use (2.6.12). In fact, we will show that for $f \in C_c(G) \star C_c(G)$,

$$P_F T_\phi(\lambda(f)\Delta)P_F = M_{\tilde{\phi}}(P_F\lambda(f)\Delta P_F) = P_F T_\phi^1(\lambda(f)\Delta)P_F.$$

The equality on the right is true by Proposition 2.6.23. By (2.6.13), the kernel of the left most term is given by

$$(s,t) \mapsto 1_F(s)\phi(st^{-1})f(st^{-1})1_F(t)$$

which coincides with $M_{\tilde{\phi}}(P_F\lambda(f)\Delta P_F)$ by again (2.6.13). This proves the case $p = 1$.

For $1 < p < \infty$, T_ϕ^p is defined through interpolation. Hence we know its values on $\kappa_p^1(L)$, as it is contained in the intersection space (see the text after Proposition 2.4.24):

$$T_\phi^p(\kappa_p^1(x)) = \kappa_p^1(T_\phi^\infty(x)) = T_\phi(\kappa_p^1(x)), \qquad x \in L.$$

This finishes the proof. □

Remark 3.1.5. We note that the proof of the fact that T_ϕ^∞ and T_ϕ^1 are compatible morphisms is non-trivial, and uses (2.6.12). This argument does not extend to the multilinear case; see also Remark 3.2.4.

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Proposition 3.1.4 in particular tells us that

$$P_F T_\phi(x) P_F = M_\phi(P_F x P_F), \qquad x \in L_p(\mathscr{L}G),\ \phi \in M(A(G)). \tag{3.1.2}$$

3.2. MULTILINEAR FOURIER MULTIPLIERS

Let $1 \leq p_1, \ldots, p_n, p \leq \infty$ with $p^{-1} = \sum_{i=1}^n p_i^{-1}$ and $\phi \in L_\infty(G^{\times n})$. In this section we explore what a suitable definition for the Fourier multiplier $T_\phi : L_{p_1}(\mathscr{L}G) \times \ldots L_{p_n}(\mathscr{L}G) \to L_p(\mathscr{L}G)$ might be. Our first requirement is that it must coincide with the linear definition for $n = 1$, i.e. it must satisfy (3.1.1).

Secondly, we would like the definition to be compatible with interpolation arguments. More precisely, if T_ϕ is bounded as a map $\mathscr{L}G \times \ldots \times \mathscr{L}G \to \mathscr{L}G$ and as a map $L_{p_1}(\mathscr{L}G) \times \ldots \times L_{p_n}(\mathscr{L}G) \to L_p(\mathscr{L}G)$, then it should also be bounded as map $L_{\frac{p_1}{\nu}}(\mathscr{L}G) \times \ldots \times L_{\frac{p_n}{\nu}}(\mathscr{L}G) \to L_{\frac{p}{\nu}}(\mathscr{L}G)$ for all $0 < \nu < 1$. This means that the definition must be 'compatible' with the definition on $(\mathscr{L}G)^{\times n}$, in the sense that in each input, the maps T_ϕ must coincide on the intersection space of some compatible couple (with respect to some θ). This tells us what the Fourier multiplier should look like on the dense subsets $\kappa_{p_i}^\theta(L)$:

Definition 3.2.1 ("Wrong definition"). Let $\theta_1, \ldots, \theta_n, \theta \in [0,1]$ and $x_i = \kappa_{p_i}^{\theta_i}(\lambda(f_i))$ for $i = 1, \ldots, n$, where $f_i \in C_c(G) * C_c(G)$. We set

$$T_{\phi,\mathrm{int}}^{\theta_1,\ldots,\theta_n,\theta}(x_1, \ldots, x_n) = \kappa_p^\theta(T_\phi(\lambda(f_1), \ldots, \lambda(f_n))). \tag{3.2.1}$$

Definition 3.2.1 might seem reasonable at first glance; it coincides with the linear definition for $n = 1$, and it is the only option if we want interpolation results. However, there are several problems with Definition 3.2.1. Firstly, the definition depends on the choice of embeddings, which is not an issue in the linear case by Remark 3.1.3. Secondly, there are several properties of multilinear Fourier multipliers on unimodular groups which do not carry over. This includes for instance [CJKM23, Lemma 4.3 and Lemma 4.4], which are crucial in the proof of the transference from Fourier to Schur multipliers. Moreover, if we want to prove an approximate intertwining property as in (3.4.1), Corollary 3.4.5 tells us that the definition of the Fourier multiplier has to 'preserve products of linear multipliers', in the sense that

$$T_\phi(x_1, \ldots, x_n) = T_{\phi_1}(x_1) \ldots T_{\phi_n}(x_n)$$

whenever $\phi(s_1,\dots,s_n) = \phi_1(s_1)\dots\phi_n(s_n)$. Defition 3.2.1 does not do this. This means that there is essentially no hope of proving the transference from Schur to Fourier multipliers either.

The above requirement on the preservation of products leads us to consider instead the following definition. Let $\theta_i \in [0,1]$ and set $a_i = \frac{1-\theta_i}{p_i}$ and $b_i = \frac{\theta_i}{p_i}$, so that $\kappa_{p_i}^{\theta_i}(x) = \Delta^{a_i} x \Delta^{b_i}$. Now for $f_i \in C_c(G) \star C_c(G)$, we formally define the Fourier multiplier corresponding to $\theta_1,\dots,\theta_n$ by

$$T_{\phi,(\theta_1,\dots,\theta_n)}(\kappa_{p_1}^{\theta_1}(\lambda(f_1)),\dots,\kappa_{p_n}^{\theta_n}(\lambda(f_n))) = \int_{G^{\times n}} \phi(t_1,\dots,t_n) f_1(t_1)\dots f_n(t_n) \times \Delta^{a_1}\lambda_{t_1}\Delta^{b_1+a_2}\lambda_{t_2}\dots\Delta^{b_{n-1}+a_n}\lambda_{t_n}\Delta^{b_n}\, dt_1\dots dt_n. \tag{3.2.2}$$

A priori, it is not clear how to define the integral in (3.2.2). After all, the integrand

$$H(t_1,\dots,t_n) := \phi(t_1,\dots,t_n) f_1(t_1)\dots f_n(t_n)\Delta^{a_1}\lambda_{t_1}\Delta^{b_1+a_2}\lambda_{t_2}\dots\Delta^{b_{n-1}+a_n}\lambda_{t_n}\Delta^{b_n}$$

is a function that has unbounded operators as values. However, on closer inspection, the 'unbounded part' of this operator doesn't really depend on the integration variables. Indeed, using the commutation formula (2.6.2), we can write

$$\begin{aligned} &H(t_1,\dots,t_n)\\ &= \phi(t_1,\dots,t_n) f_1(t_1)\dots f_n(t_n)\Delta^{a_1}(t_1)\Delta^{a_1+a_2+b_1}(t_2)\dots\Delta^{\sum_{i=1}^n a_i + \sum_{i=1}^{n-1} b_i}(t_n)\lambda_{t_1\dots t_n}\Delta^{1/p}\\ &= \phi(t_1,\dots,t_n)(\Delta^{\beta_1} f_1)(t_1)\dots(\Delta^{\beta_n} f_n)(t_n)\lambda_{t_1\dots t_n}\cdot\Delta^{1/p}, \qquad \beta_j = \sum_{i=1}^{j} a_i + \sum_{i=1}^{j-1} b_i. \end{aligned}$$

Note here that the functions $\Delta^{\beta_i} f_i$ are still in $C_c(G)\star C_c(G)$ by (2.6.1). Hence, a more rigorous way to define the Fourier multiplier is

$$T_{\phi,(\theta_1,\dots,\theta_n)}(\kappa_{p_1}^{\theta_1}(\lambda(f_1)),\dots,\kappa_{p_n}^{\theta_n}(\lambda(f_n))) = T_\phi(\lambda(\Delta^{\beta_1} f_1),\dots,\lambda(\Delta^{\beta_n} f_n))\Delta^{1/p}.$$

However, we will keep the notation from (3.2.2). The integral is justified through the above arguments. The latter expression also makes clear that (3.2.2) takes values in $\overline{\mathcal{L}G}_{(-1/p)}$; see Lemma 2.4.2. Just as in the linear case, it is not clear that it takes values in $L_p(\mathcal{L}G)$ in general; this will be part of the assumptions.

It turns out that the operator $T_{\phi,(\theta_1,\dots,\theta_n)}$ in (3.2.2) does not depend on the choice of θ_i's:

Proposition 3.2.2. *Let $1 \le p_1,\dots,p_n, p < \infty$ and $\theta_1,\dots,\theta_n \in [0,1]$. The maps $T_{\phi,(\theta_1,\dots,\theta_n)}$ and $T_{\phi,(0,\dots,0)}$ coincide on the space $\kappa_{p_1}^0(L)\times\dots\times\kappa_{p_n}^0(L)$. Consequently, if one of the maps has image in $L_p(\mathcal{L}G)$ and extends continuously to $L_{p_1}(\mathcal{L}G)\times\dots\times L_{p_n}(\mathcal{L}G)$, then the other does as well and these extensions are equal.*

Proof. The proof is similar to that of Remark 3.1.3. Recall that by (2.6.5), $\kappa_{p_i}^{\theta_i}(L) = \kappa_{p_i}^0(L)$ for $i = 1\dots,n$. For any such i, take a_i, b_i as above, i.e. so that $\kappa_{p_i}^{\theta}(x) = \Delta^{a_i} x\Delta^{b_i}$. Let

$f_i \in C_c(G) \star C_c(G)$ and set $g_i = \Delta^{-b_i} f_i$, so that $x_i := \kappa^{\theta_i}_{p_i}(\lambda(f_i)) = \kappa^0_{p_i}(\lambda(g_i))$. By (2.6.2) we find

$$\begin{aligned} T_{\phi,(\theta_1,\dots,\theta_n)}(x_1,\dots,x_n) =& \int_{G^{\times n}} \phi(t_1,\dots,t_n) f_1(t_1)\dots f_n(t_n) \times \\ & \Delta^{a_1}\lambda_{t_1}\Delta^{b_1+a_2}\lambda_{t_2}\dots\Delta^{b_{n-1}+a_n}\lambda_{t_n}\Delta^{b_n} dt_1\dots dt_n \\ =& \int_{G^{\times n}} \phi(t_1,\dots,t_n)\Delta^{-b_1}(t_1) f_1(t_1)\Delta^{-b_2}(t_2) f_2(t_2)\dots\Delta^{-b_n}(t_n) f_n(t_n) \times \\ & \Delta^{1/p_1}\lambda_{t_1}\Delta^{1/p_2}\dots\Delta^{1/p_n}\lambda_{t_n} dt_1\dots dt_n \\ =& T_{\phi,(0,\dots,0)}(\Delta^{1/p_1}\lambda(g_1),\dots,\Delta^{1/p_n}\lambda(g_n)) = T_{\phi,(0,\dots,0)}(x_1,\dots,x_n). \end{aligned}$$

□

With this issue out of the way, we can now formally define Fourier multipliers independent of the choice of θ_i's:

Definition 3.2.3 ("Correct definition"). Let $1 \le p_1,\dots,p_n, p \le \infty$ with $p^{-1} = \sum_{i=1}^n p_i^{-1}$. Also let $\phi \in L_\infty(G^{\times n})$. For $i = 1,\dots,n$, take any $a_i, b_i \in [0,1]$ such that $a_i + b_i = p_i^{-1}$. If the map

$$T_\phi : \kappa^0_{p_1}(L) \times \dots \times \kappa^0_{p_n}(L) \to \overline{\mathscr{L}G}_{(-1/p)}$$

which is given for $x_i = \Delta^{a_i}\lambda(f_i)\Delta^{b_i}$ with $f_i \in C_c(G) \star C_c(G)$ by

$$\begin{aligned} T_\phi(x_1,\dots,x_n) = & \int_{G^{\times n}} \phi(t_1,\dots,t_n) f_1(t_1)\dots f_n(t_n) \times \\ & \Delta^{a_1}\lambda_{t_1}\Delta^{b_1+a_2}\lambda_{t_2}\dots\Delta^{b_{n-1}+a_n}\lambda_{t_n}\Delta^{b_n} dt_1\dots dt_n, \end{aligned} \tag{3.2.3}$$

takes values in $L_p(\mathscr{L}G)$ and extends boundedly to $L_{p_1}(\mathscr{L}G) \times \dots \times L_{p_n}(\mathscr{L}G)$ in the norm topology (in case $p_i = \infty$ for some i, we use the space $C^*_\lambda(G)$ instead of $L_\infty(\mathscr{L}G) = \mathscr{L}G$ in the i'th leg) then we say that *ϕ is a $(p_1,\dots,p_n)$-Fourier symbol*. We denote the extension by T_ϕ, or $T_\phi^{p_1,\dots,p_n}$ when we wish to emphasize the domain of the operator. This is especially useful when writing an operator norm, since writing out the full domain and codomain generally makes equations too long. If T_ϕ is $(p_1,\dots,p_n)$-multiplicatively bounded, then we say that *ϕ is a $(p_1,\dots,p_n)$-mb Fourier symbol*.

Clearly, for $n = 1$, Definition 3.2.3 reduces to (3.1.1). It does not give the problems that Definition 3.2.1 does; as we saw already, it does not depend on the choice of embeddings. Moreover, the properties [CJKM23, Lemma 4.3 and 4.4] do carry over, as we show in Lemmas 3.2.6 and 3.2.8. Finally, it preserves products in the following more general way: if ϕ is such that there exist $m < n$ and $\phi_1 : G^{\times m} \to \mathbb{C}$, $\phi_2 : G^{\times n-m} \to \mathbb{C}$ such that

$$\phi(s_1,\dots,s_n) = \phi_1(s_1,\dots,s_m)\phi_2(s_{m+1},\dots,s_n),$$

then

$$T_\phi(x_1,\dots,x_n) = T_{\phi_1}(x_1,\dots,x_m)T_{\phi_2}(x_{m+1},\dots,x_n). \tag{3.2.4}$$

However, we have to give up interpolation results in general. The only instances where interpolation might work is when the L_p-spaces 'in the middle' are all equal to $C^*_\lambda(G)$.

Indeed, in that case we can take $\theta_1 = 0$, $\theta_n = 1$, $\theta = \frac{p}{p_n}$, so that the Fourier multiplier T_ϕ from Definition 3.2.3 also satisfies (3.2.1). We note that for $n > 2$ and $p_i < \infty$ for some $2 \le i \le n-1$, (3.2.3) is not of the form (3.2.1) for any $\theta_1, \ldots, \theta_n, \theta$, and hence T_ϕ cannot be a compatible morphism for any 'usual' compatible couple structures on $(\mathcal{L}G, L_{p_i}(\mathcal{L}G))_{\theta_i}$.

3

Remark 3.2.4. Although (3.2.1) is a necessary condition for the Fourier multiplier to allow interpolation, we have not been able to prove that it is a sufficient condition. The issue is that to prove that the mapping for $(p_1, \ldots, p_n)$ is compatible with the one for $(\infty, \ldots, \infty)$, we have to prove that they coincide on the entire intersection space $L_p(\mathcal{L}G) \cap \mathcal{L}G$ (within $L^*_{(t)}$). However, we do not know whether L is dense in this space in the intersection norm. In fact, for $p > 2$, we do not even know if $\mathcal{T}^2_\varphi$ is dense in the intersection norm.

Remark 3.2.5. We could have just taken (the extension of) the map $T_\phi^{\frac{1}{2}, \ldots, \frac{1}{2}}$ as the definition of our Fourier multiplier. This would have allowed us to skip Proposition 3.2.2, and all the proofs further on in this paper would still work by approximating only with elements in the central embedding. However, the more general definition allows some flexibility to choose convenient embeddings for notation or to avoid some technicalities (in particular in Lemma 3.2.6).

Let us now prove some properties of the multilinear Fourier multiplier for later use. Lemmas 3.2.6, 3.2.7 and 3.2.8 are used in the proof of Theorem 3.3.1. Here Lemma 3.2.6 generalises [CJKM23, Lemma 4.3] and Lemma 3.2.8 generalises [CJKM23, Lemma 4.4]. Since the proofs of these two lemmas require careful bookkeeping with modular functions, we will give the full details. The proof of [CJKM23, Lemma 4.4] was omitted, but it is not that trivial; our argument fills that gap.

Lemma 3.2.6. *Let $1 \le p_j, p \le \infty$ and fix some $1 \le i \le n$. Suppose that $\phi : G^{\times n} \to \mathbb{C}$ is bounded and measurable and set for $r, t, r' \in G$:*

$$\bar{\phi}(s_1, \ldots, s_n; r, t, r') := \phi(r s_1, \ldots, s_i t, t^{-1} s_{i+1}, \ldots, s_n r').$$

Then ϕ is a $(p_1, \ldots, p_n)$-Fourier symbol (resp. $(p_1, \ldots, p_n)$-mb Fourier symbol) iff $\bar{\phi}(\cdot; r, t, r')$ is a $(p_1, \ldots, p_n)$-Fourier symbol (resp. $(p_1, \ldots, p_n)$-mb Fourier symbol). In that case, for $x_j \in L_{p_j}(\mathcal{L}G)$,

$$T_{\bar{\phi}(\cdot;r,t,r')}(x_1, \ldots, x_n) = \lambda_r^* T_\phi\big(\lambda_r x_1, x_2, \ldots, x_i \lambda_t, \lambda_t^* x_{i+1}, \ldots, x_n \lambda_{r'}\big)\lambda_{r'}^*. \tag{3.2.5}$$

Further, we have

$$\| T_\phi^{p_1, \ldots, p_n} \| = \| T_{\bar{\phi}(\cdot;r,t,r')}^{p_1, \ldots, p_n} \|$$

and $(r, t, r') \mapsto T_{\bar{\phi}(\cdot;r,t,r')}$ is strongly continuous. In the multiplicatively bounded case, we have for any $N \ge 1$

$$\| (T_\phi^{p_1, \ldots, p_n})^{(N)} \| = \| (T_{\bar{\phi}(\cdot;r,t,r')}^{p_1, \ldots, p_n})^{(N)} \|$$

as maps $S^N_{p_1}[L_{p_1}(\mathcal{L}G)] \times \ldots \times S^N_{p_n}[L_{p_n}(\mathcal{L}G)] \to S^N_p[L_p(\mathcal{L}G)]$, and $(r, t, r') \mapsto T^{(N)}_{\bar{\phi}(\cdot;r,t,r')}$ is strongly continuous.

Proof. It is straightforward to check that for $s \in G$, $f \in C_c(G)$, we have $\lambda_s \lambda(f) = \lambda(\lambda_s(f)) = \lambda(f(s^{-1}\cdot))$; moreover, we have

$$\lambda(f)\lambda_s = \int_G f(t)\lambda_{ts} dt = \Delta(s^{-1})\int_G f(ts^{-1})\lambda_t dt = \Delta(s^{-1})\lambda(f(\cdot s^{-1})). \tag{3.2.6}$$

We will only make a choice for some of the embeddings and leave the rest open; this is notationally more convenient. Let $x_j = \Delta^{a_j}\lambda(f_j)\Delta^{b_j} \in L_{p_j}(\mathscr{L}G)$, with $f_j \in C_c(G) \star C_c(G)$ and $a_1 = b_i = a_{i+1} = b_n = 0$ (hence $b_1 = \frac{1}{p_1}$, $a_i = \frac{1}{p_i}$, etc). We compute

$$\begin{aligned}
&T_{\tilde{\phi}(\cdot;r,t,r')}(x_1,\dots,x_n)\\
=&\int_{G^{\times n}} \tilde{\phi}(s_1,\dots,s_n;r,t,r') f_1(s_1)\dots f_n(s_n)\lambda_{s_1}\Delta^{b_1+a_2}\lambda_{s_2}\dots\Delta^{b_{n-1}+a_n}\lambda_{s_n} ds_1\dots ds_n\\
=&\int_{G^{\times n}} \phi(s_1,\dots,s_n) f_1(r^{-1}s_1)\dots\Delta(t)^{-1} f_i(s_i t^{-1}) f_{i+1}(ts_{i+1})\dots\Delta(r')^{-1} f_n(s_n(r')^{-1})\times\\
&\lambda_{r^{-1}s_1}\Delta^{b_1+a_2}\dots\Delta^{b_{i-1}+a_i}\lambda_{s_i s_{i+1}}\Delta^{b_{i+1}+a_{i+2}}\dots\Delta^{b_{n-1}+a_n}\lambda_{s_n(r')^{-1}} ds_1\dots ds_n\\
=&\lambda_r^* T_\phi\big(\widetilde{x_1}, x_2,\dots,\widetilde{x_i},\widetilde{x_{i+1}},\dots,\widetilde{x_n}\big)\lambda_{r'}^*.
\end{aligned}$$

Here

$$\begin{aligned}
\widetilde{x_1} &:= \lambda(f_1(r^{-1}\cdot))\Delta^{b_1}; & \widetilde{x_i} &:= \Delta^{-1}(t)\Delta^{a_i}\lambda(f_i(\cdot t^{-1}));\\
\widetilde{x_{i+1}} &:= \lambda(f_{i+1}(t\cdot))\Delta^{b_{i+1}}; & \widetilde{x_n} &:= \Delta^{-1}(r')\Delta^{a_n}\lambda(f_n(\cdot(r')^{-1})).
\end{aligned}$$

By (3.2.6) we can write

$$\widetilde{x_n} = \Delta^{a_n}\lambda(f_n)\lambda_{r'} = x_n\lambda_{r'}$$

and similarly

$$\widetilde{x_1} = \lambda_r x_1; \qquad \widetilde{x_i} = x_i\lambda_t; \qquad \widetilde{x_{i+1}} = \lambda_t^* x_{i+1}.$$

Combining everything together we conclude

$$T_{\tilde{\phi}(\cdot;r,t,r')}(x_1,\dots,x_n) = \lambda_r^* T_\phi\big(\lambda_r x_1, x_2,\dots,x_i\lambda_t,\lambda_t^* x_{i+1},\dots,x_n\lambda_{r'}\big)\lambda_{r'}^*.$$

By (2.6.6), we have

$$\begin{aligned}
\|T_{\tilde{\phi}(\cdot;r,t,r')}(x_1,\dots,x_n)\|_p &= \|T_\phi\big(\lambda_r x_1, x_2,\dots,x_i\lambda_t,\lambda_t^* x_{i+1},\dots,x_n\lambda_{r'}\big)\|_p\\
&\le \|T_\phi^{p_1,\dots,p_n}\|\|x_1\|_{p_1}\dots\|x_n\|_{p_n}.
\end{aligned}$$

Hence, on the dense subsets of elements x_j as above we have $\|T_{\tilde{\phi}(\cdot;r,t,r')}\| \le \|T_\phi\|$. If we set $\psi = \tilde{\phi}(\cdot;r,t,r')$, then $\tilde{\psi}(\cdot;r^{-1},t^{-1},(r')^{-1}) = \phi$. Hence, applying the above result to $\tilde{\psi}(\cdot;r^{-1},t^{-1},(r')^{-1})$ yields the reverse inequality. By density, we conclude that the first three statements of the lemma hold. By [JS05, Lemma 2.3], the (left or right) multiplication with λ_s, $s \in G$ is strongly continuous in s. This implies the strong continuity of $(r,t,r') \mapsto T_{\tilde{\phi}(\cdot;r,t,r')}$.

Now assume ϕ is a $(p_1,\ldots,p_n)$-mb Fourier symbol and let $N \geq 1$. Denote by ι_N the $N \times N$-identity matrix. Then by writing out the definitions and using (3.2.5) we find, for $x_i \in S^N_{p_i} \otimes L_{p_i}(\mathcal{L}G)$,

$$\begin{aligned} &T^{(N)}_{\tilde{\phi}(\cdot;r,t,r')}(x_1,\ldots,x_n) \\ =&(\iota_N \otimes \lambda^*_r)T^{(N)}_{\phi}((\iota_N \otimes \lambda_r)x_1,\ldots,x_i(\iota_N \otimes \lambda_t),(\iota_N \otimes \lambda^*_t)x_{i+1},\ldots,x^n(\iota_N \otimes \lambda_{r'}))(\iota_N \otimes \lambda^*_{r'}). \end{aligned}$$

Hence, by a complete/matrix amplified version of the above arguments, we deduce the last two statements. □

Lemma 3.2.7. *Let $1 \leq p_j, p \leq \infty$ and fix some $1 \leq i \leq n$. Suppose that $\phi : G^{\times n} \to \mathbb{C}$ is a $(p_1,\ldots,p_n)$-Fourier symbol and $\phi_i : G \to \mathbb{C}$ is a p_i-Fourier symbol. Set*

$$\bar{\phi}(s_1,\ldots,s_n) = \phi(s_1,\ldots,s_n)\phi_i(s_i).$$

Then $\bar{\phi}$ is a $(p_1,\ldots,p_n)$-Fourier symbol and for $x_j \in L_{p_j}(\mathcal{L}G)$,

$$T_{\bar{\phi}}(x_1,\ldots,x_n) = T_{\phi}(x_1,\ldots,x_{i-1},T_{\phi_i}(x_i),x_{i+1},\ldots,x_n). \tag{3.2.7}$$

In particular,

$$\| T^{p_1,\ldots,p_n}_{\bar{\phi}} \| \leq \| T^{p_1,\ldots,p_n}_{\phi} \| \| T_{\phi_i} : L_{p_i}(\mathcal{L}G) \to L_{p_i}(\mathcal{L}G) \|. \tag{3.2.8}$$

Proof. For $x_j \in \kappa^0_{p_j}(L)$ (or any other embedding), it follows directly from writing out the definitions that (3.2.7) holds (cf. (3.1.1)). By density, (3.2.7) holds for general $x_j \in L_{p_j}(\mathcal{L}G)$ which implies (3.2.8), so $T^{p_1,\ldots,p_n}_{\bar{\phi}}$ is bounded. □

Lemma 3.2.8. *Let $1 \leq p_1,\ldots,p_n \leq \infty$. Let $q_j^{-1} = \sum_{i=j}^n p_i^{-1}$ and suppose that $\phi_j : G \to \mathbb{C}$ is a q_j-Fourier symbol for $1 \leq j \leq n$. Set*

$$\bar{\phi}(s_1,\ldots,s_n) = \phi_1(s_1 \ldots s_n)\phi_2(s_2 \ldots s_n)\ldots\phi_n(s_n).$$

Then $\bar{\phi}$ is a $(p_1,\ldots p_n)$-Fourier symbol and for $x_i \in L_{p_i}(\mathcal{L}G)$ we have

$$T_{\bar{\phi}}(x_1,\ldots,x_n) = T_{\phi_1}(x_1 T_{\phi_2}(x_2 \ldots T_{\phi_n}(x_n)\ldots)). \tag{3.2.9}$$

Proof. We first show (3.2.9) on the dense subset $\kappa^0_{p_i}(L) \times \ldots \times \kappa^0_{p_n}(L)$. The lemma then follows from boundedness of the T_{ϕ_i} together with Hölders inequality.

We make the slightly stronger claim that for any $\phi_2,\ldots,\phi_n$ as in the assumptions and any $x_i \in \kappa^0_{p_i}(L)$, there exists a compactly supported function $g : G \to \mathbb{C}$ such that for all ϕ_1 as in the assumptions,

$$T_{\bar{\phi}}(x_1,\ldots,x_n) = \Delta^{\frac{1}{q_1}}\lambda(\phi_1 g) = T_{\phi_1}(x_1 T_{\phi_2}(x_2 \ldots T_{\phi_n}(x_n)\ldots)). \tag{3.2.10}$$

We will prove (3.2.10) with induction on n. We will need this intermediate step in order to expand the outer Fourier multiplier in the right-hand side of (3.2.9).

The case $n = 1$ follows directly from (3.1.1). Now assume that (3.2.10) holds for any choice of $\phi_1, \dots, \phi_{n-1}$ as above and $x_1, \dots, x_{n-1}$ with $x_i \in \kappa^0_{p_i'}(L)$. Fix functions $\phi_1, \dots, \phi_n$ as in the assumptions and $x_1, \dots, x_n$ so that $x_i = \Delta^{\frac{1}{p_i}} \lambda(f_i)$ for $f_i \in C_c(G) \star C_c(G)$. Take g compactly supported such that, for any q_2-Fourier symbol $\psi : G \to \mathbb{C}$,

$$T_\psi(x_2 T_{\phi_3}(x_3 \dots T_{\phi_n}(x_n) \dots)) = \Delta^{\frac{1}{q_2}} \lambda(\psi g) = T_{\bar{\psi}}(x_2, \dots, x_n) \tag{3.2.11}$$

where $\bar{\psi}(s_2, \dots, s_n) = \psi(s_2 \dots s_n)\phi_3(s_3 \dots s_n) \dots \phi_n(s_n)$. We calculate

$$\begin{aligned} T_{\phi_1}(x_1 T_{\phi_2}(x_2 \dots T_{\phi_n}(x_n) \dots)) &\overset{(3.2.11)}{=} T_{\phi_1}(\Delta^{\frac{1}{p_1}} \lambda(f_1) \Delta^{\frac{1}{q_2}} \lambda(\phi_2 g)) \\ &\overset{(2.6.3)}{=} T_{\phi_1}(\Delta^{\frac{1}{q_1}} \lambda((\Delta^{-\frac{1}{q_2}} f_1) * (\phi_2 g))) \\ &\overset{(3.1.1)}{=} \Delta^{\frac{1}{q_1}} \lambda(\phi_1((\Delta^{-\frac{1}{q_2}} f_1) * (\phi_2 g))). \end{aligned}$$

This shows the second equality from (3.2.10). Continuing the previous equation,

$$\begin{aligned} T_{\phi_1}(x_1 T_{\phi_2}(x_2 \dots T_{\phi_n}(x_n) \dots)) &= \int_G \phi_1(t) \left(\int_G (\Delta^{-\frac{1}{q_2}} f_1)(s_1)(\phi_2 g)(s_1^{-1} t) ds_1 \right) \Delta^{\frac{1}{q_1}} \lambda_t dt \\ &= \int_G \int_G \phi_1(s_1 t)(\Delta^{-\frac{1}{q_2}} f_1)(s_1)(\phi_2 g)(t) \Delta^{\frac{1}{q_1}} \lambda_{s_1 t} dt ds_1 \\ &\overset{(2.6.2)}{=} \int_G f_1(s_1) \Delta^{\frac{1}{p_1}} \lambda_{s_1} \int_G \phi_1(s_1 t) \phi_2(t) g(t) \Delta^{\frac{1}{q_2}} \lambda_t dt ds_1 \\ &= \int_G f_1(s_1) \Delta^{\frac{1}{p_1}} \lambda_{s_1} \Delta^{\frac{1}{q_2}} \lambda(\phi_1(s_1 \cdot) \phi_2 g) ds_1. \end{aligned}$$

Applying (3.2.11) again but now with $\phi_1(s_1 \cdot)\phi_2$ in place of ψ, we get:

$$\begin{aligned} &T_{\phi_1}(x_1 T_{\phi_2}(x_2 \dots T_{\phi_n}(x_n) \dots)) \\ &= \int_G f_1(s_1) \Delta^{\frac{1}{p_1}} \lambda_{s_1} \int_{G^{\times n-1}} \phi_1(s_1 s_2 \dots s_n) \phi_2(s_2 \dots s_n) \phi_3(s_3 \dots s_n) \dots \phi_n(s_n) \times \\ &\qquad f_2(s_2) \dots f_n(s_n) \Delta^{\frac{1}{p_2}} \lambda_{s_2} \dots \Delta^{\frac{1}{p_n}} \lambda_{s_n} ds_2 \dots ds_n ds_1 \\ &= T_{\bar{\phi}}(x_1, \dots, x_n). \end{aligned}$$

This finishes the proof. □

Finally, we calculate a convenient form for the kernel of a corner of the Fourier multiplier for use in Theorem 3.4.1.

Lemma 3.2.9. *Let $F \subseteq G$ compact and $x_i = \Delta^{a_i} \lambda(f_i) \Delta^{b_i} \in L_{p_i}(\mathcal{L}G)$ as above for $f_i \in C_c(G) \star C_c(G)$. Then the kernel of $P_F T_\phi(x_1, \dots, x_n) P_F$ is given by*

$$\begin{aligned} (t_0, t_n) \mapsto 1_F(t_0) 1_F(t_n) \int_{G^{\times n-1}} &\phi(t_0 t_1^{-1}, \dots, t_{n-1} t_n^{-1}) f_1(t_0 t_1^{-1}) \dots f_n(t_{n-1} t_n^{-1}) \times \\ &\Delta^{a_1}(t_0) \Delta^{b_1 + a_2}(t_1) \dots \Delta^{b_n}(t_n) \Delta((t_1 \dots t_n)^{-1}) dt_1 \dots dt_{n-1}. \end{aligned}$$

Proof. Let $g \in C_c(G) \star C_c(G)$ and $t_0 \in G$. Then the function $P_F T_\phi^{\theta_1,\ldots,\theta_n}(x_1,\ldots,x_n) P_F g$ is given by

$$
\begin{aligned}
t_0 \mapsto 1_F(t_0) \int_{G^{\times n}} & \phi(t_1,\ldots,t_n) f_1(t_1)\ldots f_n(t_n) \times \\
& (\Delta^{a_1}\lambda_{t_1}\Delta^{b_1+a_2}\lambda_{t_2}\ldots\Delta^{b_{n-1}+a_n}\lambda_{t_n}\Delta^{b_n}P_F g)(t_0)\, dt_1\ldots dt_n \\
= 1_F(t_0) \int_{G^{\times n}} & \phi(t_1,\ldots,t_n) f_1(t_1)\ldots f_n(t_n)\Delta^{a_1}(t_0)\Delta^{b_1+a_2}(t_1^{-1}t_0) \times \ldots \\
& \Delta^{b_{n-1}+a_n}(t_{n-1}^{-1}\ldots t_1^{-1}t_0)(\Delta^{b_n}1_F g)(t_n^{-1}\ldots t_1^{-1}t_0)\, dt_1\ldots dt_n \\
= 1_F(t_0) \int_{G^{\times n}} & \phi(t_0t_1, t_2,\ldots,t_n) f_1(t_0t_1) f_2(t_2)\ldots f_n(t_n)\Delta^{a_1}(t_0)\Delta^{b_1+a_2}(t_1^{-1}) \times \\
& \Delta^{b_2+a_3}(t_2^{-1}t_1^{-1})\ldots\Delta^{b_{n-1}+a_n}(t_{n-1}^{-1}\ldots t_1^{-1})(\Delta^{b_n}1_F g)(t_n^{-1}\ldots t_1^{-1})\, dt_1\ldots dt_n \\
= 1_F(t_0) \int_{G^{\times n}} & \phi(t_0t_1^{-1}, t_2,\ldots,t_n) f_1(t_0t_1^{-1}) f_2(t_2)\ldots f_n(t_n)\Delta^{a_1}(t_0)\Delta^{b_1+a_2}(t_1) \times \\
& \Delta^{b_2+a_3}(t_2^{-1}t_1)\ldots\Delta^{b_{n-1}+a_n}(t_{n-1}^{-1}\ldots t_2^{-1}t_1)(\Delta^{b_n}1_F g)(t_n^{-1}\ldots t_2^{-1}t_1)\Delta(t_1^{-1})\, dt_1\ldots dt_n \\
= \ldots & \\
= 1_F(t_0) \int_{G^{\times n}} & \phi(t_0t_1^{-1},\ldots,t_{n-1}t_n^{-1}) f_1(t_0t_1^{-1})\ldots f_n(t_{n-1}t_n^{-1})\Delta^{a_1}(t_0)\Delta^{b_1+a_2}(t_1) \times \ldots \\
& \Delta^{b_{n-1}+a_n}(t_{n-1})(\Delta^{b_n}1_F g)(t_n)\Delta((t_1\ldots t_n)^{-1})\, dt_1\ldots dt_n.
\end{aligned}
$$

It follows that the kernel has the required form. □

3.3. FOURIER TO SCHUR TRANSFERENCE

Let G be a locally compact first countable group. In this section we prove the transference from Fourier to Schur multipliers for such groups. We will not indicate any simplifications for the unimodular case, since the extra complications are only in the proof of Proposition 3.3.3 and the final part of Lemma 3.3.2.

An important ingredient will be the following 'split' coordinate-wise convolution: fix functions $\varphi_k \in C_c(G) \star C_c(G) \subseteq A(G)$ such that $\|\varphi_k\|_1 = 1$ and the supports of φ_k form a decreasing neighbourhood basis of $\{e\}$. In other words, (φ_k) is an approximate unit for the Banach $*$-algebra $L_1(G)$. Note that we use the first countability of G here. Now, given a bounded function $\phi: G^{\times n} \to \mathbb{C}$ and some fixed $1 \leq i \leq n$, we define

$$
\phi_{t_1,\ldots,t_n}(s_1,\ldots,s_n) = \phi(t_1^{-1}s_1t_2,\ldots,t_{i-1}^{-1}s_{i-1}t_i, t_i^{-1}s_i, s_{i+1}t_{i+1}^{-1}, t_{i+1}s_{i+2}t_{i+2}^{-1},\ldots,t_{n-1}s_nt_n^{-1})
$$

and

$$
\begin{aligned}
\phi_k(s_1,\ldots,s_n) &:= \int_{G^{\times n}} \phi_{t_1,\ldots,t_n}(s_1,\ldots,s_n)\left(\prod_{j=1}^n \varphi_k(t_j)\right) dt_1\ldots dt_n \\
&= \int_{G^{\times n}} \phi_{t_1,\ldots,t_n}(e,\ldots,e)\left(\prod_{j=1}^i \varphi_k(s_j\ldots s_it_j)\right)\left(\prod_{j=i+1}^n \varphi_k(t_js_{i+1}\ldots s_j)\Delta(s_{i+1}\ldots s_j)\right) dt_1\ldots dt_n.
\end{aligned}
\tag{3.3.1}
$$

The last term of (3.3.1) in combination with Lemma 3.2.8 will allow us to reduce the problem to the linear case. The necessity of the 'split' between indices i and $i+1$ will be explained later.

Theorem 3.3.1. *Let G be a locally compact first countable group and let $1 \le p \le \infty$, $1 < p_1, \dots, p_{n-1} \le \infty$ be such that $p^{-1} = \sum_{i=1}^n p_i^{-1}$. Let $\phi \in C_b(G^{\times n})$ and define $\widetilde{\phi} \in C_b(G^{\times n+1})$ by*

$$\widetilde{\phi}(s_0, \dots, s_n) = \phi(s_0 s_1^{-1}, s_1 s_2^{-1}, \dots, s_{n-1} s_n^{-1}), \qquad s_i \in G.$$

If ϕ is a $(p_1, \dots, p_n)$-mb Fourier symbol, then $\widetilde{\phi}$ is a $(p_1, \dots, p_n)$-mb Schur symbol. Moreover,

$$\begin{aligned} &\| M_{\widetilde{\phi}} : S_{p_1}(L_2(G)) \times \ldots \times S_{p_n}(L_2(G)) \to S_p(L_2(G)) \|_{(p_1,\dots,p_n)-mb} \\ &\quad \le \| T_\phi : L_{p_1}(\mathcal{L}G) \times \ldots \times L_{p_n}(\mathcal{L}G) \to L_p(\mathcal{L}G) \|_{(p_1,\dots,p_n)-mb}. \end{aligned}$$

Proof. Let $F \subseteq G$ finite with $|F| = N$. By Theorem 2.6.16, it suffices to show that the norm of

$$M_{\widetilde{\phi}} : S_{p_1}(\ell_2(F)) \times \ldots S_{p_n}(\ell_2(F)) \to S_p(\ell_2(F))$$

and its matrix amplifications are bounded.

For $s \in F$, let $p_s \in B(\ell_2(F))$ be the orthogonal projection onto the span of δ_s. Define the unitary $U = \sum_{s \in F} p_s \otimes \lambda_s \in B(\ell_2(F)) \otimes \mathcal{L}G$. In the case $p = \infty$, the Fourier to Schur transference is proven through the transference identity

$$T_\phi^{(N)}(U(a_1 \otimes 1)U^*, \dots, U(a_n \otimes 1)U^*) = U(M_{\widetilde{\phi}}(a_1, \dots, a_n) \otimes 1)U^*, \qquad a_i \in B(\ell_2(F)).$$

The idea is to do something similar in the case $p < \infty$. However, the unit does not embed in $L_p(\mathcal{L}G)$, so we need to use some approximation of the unit instead. We construct this as follows: let $\mathcal{V} = (V_i)_{i \in \mathbb{N}}$ be a decreasing symmetric neighbourhood basis of the identity (this is possible because G is first countable). For $V \in \mathcal{V}$ we define the operator

$$k_V = \| 1_V \Delta^{-1/4} \|_2^{-1} \lambda(1_V \Delta^{-1/4}) \Delta^{1/2} \in L_2(\mathcal{L}G)$$

which is proven to be self-adjoint in [CPPR15, Section 8.3]. Let $k_V = u_V h_V$ be its polar decomposition. Then we have $h_V^{2/p} \in L_p(\mathcal{L}G)$ and, by (2.6.4), $\| h_V^{2/p} \|_p = 1$. Now for any $V \in \mathcal{V}$ we have, by Proposition 2.5.4,

$$\| M_{\widetilde{\phi}}(a_1, \dots, a_n) \|_{S_p^N} = \| M_{\widetilde{\phi}}(a_1, \dots, a_n) \otimes h_V^{\frac{2}{p}} \|_{S_p^N \otimes L_p(\mathcal{L}G)}, \qquad a_i \in B(\ell_2(F)). \tag{3.3.2}$$

Next, fix an $i \in \{1, \dots, n\}$ such that $\bar{p}_1 := \left(\sum_{l=1}^i p_l^{-1} \right)^{-1} > 1$ and $\bar{p}_2 := \left(\sum_{l=i+1}^n p_l^{-1} \right)^{-1} > 1$. This is possible by our assumption that $p_1, \dots, p_n > 1$. We now define the functions $\phi_{t_1,\dots,t_n}$ and ϕ_k as in (3.3.1) for the chosen i.

The condition $\bar{p}_1, \bar{p}_2 > 1$ is necessary for the use of Proposition 3.3.3 at the end of the proof of Lemma 3.3.2; this also explains why we need the 'split' in the pointwise convolutions. If $p > 1$, then one can take $i = n$ in which case the proof of Lemma 3.3.2 simplifies somewhat. Note that in [CJKM23] and [CKV23], the convolutions were defined for

$i = n-1$. In the latter paper, this creates a problem in case $p_n = \infty, p = 1$; this problem is resolved by splitting instead at some i chosen as above.

Let $a_1, \ldots, a_n \in B(\ell_2(F))$. By continuity of ϕ, we have that $\phi_k \to \phi$ pointwise. Indeed, for $\varepsilon > 0$, we can take K such that for $t_1, \ldots, t_n \in \operatorname{supp} \varphi_K$, $|\phi_{t_1,\ldots,t_n}(s_1, \ldots, s_n) - \phi(s_1, \ldots, s_n)| < \varepsilon$. Then for $k > K$, we get $|\phi_k(s_1, \ldots, s_n) - \phi(s_1, \ldots, s_n)| < \varepsilon$. Since we are working in finite dimensions, this implies

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$$M_{\widetilde{\phi_k}}(a_1, \ldots, a_n) \to M_{\tilde{\phi}}(a_1, \ldots, a_n)$$

in S_p^N. Together with (3.3.2) we find

$$\begin{aligned}
\|M_{\tilde{\phi}}(a_1, \ldots, a_n)\|_{S_p^N} &= \lim_k \limsup_{V \in \mathcal{V}} \|M_{\widetilde{\phi_k}}(a_1, \ldots, a_n) \otimes h_V^{\frac{2}{p}}\|_{S_p^N \otimes L_p(\mathcal{L}G)} \\
&= \lim_k \limsup_{V \in \mathcal{V}} \|U(M_{\widetilde{\phi_k}}(a_1, \ldots, a_n) \otimes h_V^{\frac{2}{p}})U^*\|_{S_p^N \otimes L_p(\mathcal{L}G)} \\
&\le \limsup_k \limsup_{V \in \mathcal{V}} \|T_{\phi_k}^{(N)}(U(a_1 \otimes h_V^{\frac{2}{p_1}})U^*, \ldots, U(a_n \otimes h_V^{\frac{2}{p_n}})U^*)\|_{S_p^N \otimes L_p(\mathcal{L}G)} \\
&\quad + \limsup_k \limsup_{V \in \mathcal{V}} \|T_{\phi_k}^{(N)}(U(a_1 \otimes h_V^{\frac{2}{p_1}})U^*, \ldots, U(a_n \otimes h_V^{\frac{2}{p_n}})U^*) \\
&\qquad\qquad - U(M_{\widetilde{\phi_k}}(a_1, \ldots, a_n) \otimes h_V^{\frac{2}{p}})U^*\|_{S_p^N \otimes L_p(\mathcal{L}G)} \\
&:= A + B.
\end{aligned}$$

First, we have

$$\begin{aligned}
A &\le \limsup_k \limsup_{V \in \mathcal{V}} \|T_{\phi_k}^{(N)}\| \|a_1 \otimes h_V^{\frac{2}{p_1}}\|_{S_{p_1}^N \otimes L_{p_1}(\mathcal{L}G)} \cdots \|a_n \otimes h_V^{\frac{2}{p_n}}\|_{S_{p_n}^N \otimes L_{p_n}(\mathcal{L}G)} \\
&= \limsup_k \|T_{\phi_k}^{(N)}\| \|a_1\|_{S_{p_1}^N} \cdots \|a_n\|_{S_{p_n}^N}.
\end{aligned}$$

By repeated use of Lemma 3.2.6 (in particular we can use Fubini because of the strong continuity property) we find

$$\|T_{\phi_k}^{(N)}\| \le \int_{G^{\times n}} \|T_\phi^{(N)}\| \left(\prod_{i=1}^n |\varphi_k(t_j)| \right) dt_1 \ldots dt_n = \|T_\phi^{(N)}\| \|\varphi_k\|_1^n = \|T_\phi^{(N)}\| \le \|T_\phi\|_{(p_1, \ldots, p_n)-mb}.$$

and hence

$$A \le \|T_\phi\|_{(p_1, \ldots, p_n)-mb} \|a_1\|_{S_{p_1}^N} \cdots \|a_n\|_{S_{p_n}^N}.$$

It remains to show that $B = 0$. By the triangle inequality, it suffices to show this for $a_i = E_{r_{i-1}, r_i}$, $r_0, \ldots, r_n \in F$ (for other combinations of matrix units, the term below becomes 0).

In that case we get, by applying Lemma 3.2.6 in the second equality:

$$
\begin{aligned}
&T^{(N)}_{\phi_k}(U(E_{r_0,r_1}\otimes h_V^{\frac{2}{p_1}})U^*,\dots,U(E_{r_{n-1},r_n}\otimes h_V^{\frac{2}{p_n}})U^*)-U(M_{\widetilde{\phi_k}}(E_{r_0,r_1},\dots,E_{r_{n-1},r_n})\otimes h_V^{\frac{2}{p}})U^*\\
&=E_{r_0,r_n}\otimes\left(T_{\phi_k}(\lambda_{r_0}h_V^{\frac{2}{p_1}}\lambda^*_{r_1},\dots,\lambda_{r_{n-1}}h_V^{\frac{2}{p_n}}\lambda^*_{r_n})-\phi_k(r_0r_1^{-1},\dots,r_{n-1}r_n^{-1})\lambda_{r_0}h_V^{\frac{2}{p}}\lambda^*_{r_n}\right)\\
&=E_{r_0,r_n}\otimes\lambda_{r_0}\left(T_{\phi_k(r_0\cdot r_1^{-1},\dots,r_{n-1}\cdot r_n^{-1})}(h_V^{\frac{2}{p_1}},\dots,h_V^{\frac{2}{p_n}})-\phi_k(r_0r_1^{-1},\dots,r_{n-1}r_n^{-1})h_V^{\frac{2}{p}}\right)\lambda^*_{r_n}.
\end{aligned}
$$

Hence,

$$
B=\limsup_k\limsup_{V\in\mathcal{V}}\left\|T_{\phi_k(r_0\cdot r_1^{-1},\dots,r_{n-1}\cdot r_n^{-1})}(h_V^{\frac{2}{p_1}},\dots,h_V^{\frac{2}{p_n}})-\phi_k(r_0r_1^{-1},\dots,r_{n-1}r_n^{-1})h_V^{\frac{2}{p}}\right\|_p. \tag{3.3.3}
$$

The limit over k exists and is 0; we postpone the proof to Lemma 3.3.2 below.

For the multiplicatively bounded estimate, we prove using similar methods that, for $K\geq 1$ and $a_1,\dots,a_n\in M_K(B(\ell_2(F)))$,

$$
\|M^{(K)}_{\widetilde{\phi}}(a_1,\dots,a_n)\|_{S_p^{KN}}\leq\|T^{(KN)}_{\phi}\|\|a_1\|_{S_p^{KN}}\dots\|a_n\|_{S_p^{KN}}.
$$

Here we use $1_{M_K}\otimes U$ in place of U. Moreover, by the triangle inequality it suffices to prove the estimate for B for $a_i=E_{j_{i-1},j_i}\otimes E_{r_{i-1},r_i}$, with $1\leq j_i\leq K$ and $r_i\in F$; the expression for B then reduces to (3.3.3) again. □

The following Lemma is similar to [CJKM23, Lemma 4.6]. In our case we have $x_j=1$ which allows us to avoid the SAIN condition used in that paper; on the other hand, we work with translated functions and our result works for non-unimodular groups. Already in [CKV23, Theorem 3.1] it was explained how to adapt the proof of [CJKM23, Lemma 4.6] for the translated functions. However this paper only considered unimodular groups. Here we spell out the full proof for convenience of the reader.

Lemma 3.3.2. *In the proof of Theorem 3.3.1, we have that*

$$
\lim_k\limsup_{V\in\mathcal{V}}\left\|T_{\phi_k(r_0\cdot r_1^{-1},\dots,r_{n-1}\cdot r_n^{-1})}(h_V^{\frac{2}{p_1}},\dots,h_V^{\frac{2}{p_n}})-\phi_k(r_0r_1^{-1},\dots,r_{n-1}r_n^{-1})h_V^{\frac{2}{p}}\right\|_p=0.
$$

The main idea is to reduce the problem to the linear case using (3.3.1) and apply the following result for linear Fourier multipliers. For unimodular groups, this is just [CJKM23, Proposotion 3.9].

Proposition 3.3.3. *Let $\mathcal{V}$ be a symmetric neighbourhood basis of the identity of G. Let $2\leq q<p\leq\infty$ or $1\leq p<q\leq 2$. Assume $\psi\in C_b(G)$ is a p-Fourier symbol. Then we have*

$$
\lim_{V\in\mathcal{V}}\|T_\psi(h_V^{2/q})-\psi(1)h_V^{2/q}\|_q\to 0.
$$

The proof of Proposition 3.3.3 is a matter of combining results and remarks from [CJKM23, Proposition 3.9], [CPPR15, Claim B and Section 8] and applying Haagerup reduction to [CPR18, Lemma 3.1] to generalise that estimate to general von Neumann algebras. We give more details in Section 3.5.

Proof of Lemma 3.3.2. The idea is to use a dominated convergence argument in the last expression of (3.3.1). However, the functions ϕ_k need not be integrable. We work around this by multiplying with compactly supported functions that are close to 1 around e, so that as $V \in \mathcal{V}$ decreases to $\{e\}$ we are just 'multiplying by 1' in the limit. Define a function $\zeta \in C_c(G) \cap A(G)$ with $\zeta(e) = 1$ which is positive definite and (therefore) satisfies $\| T_\zeta : L_p(\mathcal{L}G) \to L_p(\mathcal{L}G)\| \leq 1$ for all $1 \leq p \leq \infty$ (cf. Remark 2.6.24). Next let

$$\zeta_j(s) = \zeta(r_{j-1}^{-1} s r_j), \qquad 1 \leq j \leq n, \; s \in G.$$

We define a product function as follows:

$$(\phi(\zeta_1, \dots, \zeta_n))(s_1, \dots, s_n) = \phi(s_1, \dots, s_n)\zeta_1(s_1) \dots \zeta_n(s_n).$$

Then

$$\begin{aligned}
&\| T_{\phi_k(r_0 \cdot r_1^{-1}, \dots, r_{n-1} \cdot r_n^{-1})}(h_V^{\frac{2}{p_1}}, \dots, h_V^{\frac{2}{p_n}}) - \phi_k(r_0 r_1^{-1}, \dots, r_{n-1} r_n^{-1}) h_V^{\frac{2}{p}} \|_p \\
&\leq \| (\phi(\zeta_1, \dots, \zeta_n))_k (r_0 r_1^{-1}, \dots, r_{n-1} r_n^{-1}) h_V^{\frac{2}{p}} - \phi_k(r_0 r_1^{-1}, \dots, r_{n-1} r_n^{-1}) h_V^{\frac{2}{p}} \|_p \\
&+ \| T_{(\phi(\zeta_1, \dots, \zeta_n))_k (r_0 \cdot r_1^{-1}, \dots, r_{n-1} \cdot r_n^{-1})}(h_V^{\frac{2}{p_1}}, \dots, h_V^{\frac{2}{p_n}}) - (\phi(\zeta_1, \dots, \zeta_n))_k (r_0 r_1^{-1}, \dots, r_{n-1} r_n^{-1}) h_V^{\frac{2}{p}} \|_p \\
&+ \| T_{\phi_k(r_0 \cdot r_1^{-1}, \dots, r_{n-1} \cdot r_n^{-1})}(h_V^{\frac{2}{p_1}}, \dots, h_V^{\frac{2}{p_n}}) - T_{(\phi(\zeta_1, \dots, \zeta_n))_k (r_0 \cdot r_1^{-1}, \dots, r_{n-1} \cdot r_n^{-1})}(h_V^{\frac{2}{p_1}}, \dots, h_V^{\frac{2}{p_n}}) \|_p \\
&=: A_{k,V} + B_{k,V} + C_{k,V}.
\end{aligned}$$

Here $\phi((\zeta_1, \dots, \zeta_n))_k$ is defined again by (3.3.1) for the same i. We will estimate these terms separately. We start by showing that $\lim_k \limsup_{V \in \mathcal{V}} A_{k,V}$ and $\lim_k \limsup_{V \in \mathcal{V}} C_{k,V}$ are 0, essentially reducing the problem to the integrable functions $\phi(\zeta_1, \dots, \zeta_n)$. We then apply the idea mentioned above to show that $\lim_{V \in \mathcal{V}} B_{k,V} = 0$ for any k.

Firstly, since $\psi_k \to \psi$ pointwise for any $\psi \in C_b(G)^{\times n}$ we have

$$\begin{aligned}
\limsup_{V \in \mathcal{V}} A_{k,V} &= |(\phi(\zeta_1, \dots, \zeta_n))_k (r_0 r_1^{-1}, \dots, r_{n-1} r_n^{-1}) - \phi_k(r_0 r_1^{-1}, \dots, r_{n-1} r_n^{-1})| \\
&\to |\phi(r_0 r_1^{-1}, \dots, r_{n-1} r_n^{-1})(1 - \zeta_1(r_0 r_1^{-1}) \dots \zeta_n(r_{n-1} r_n^{-1}))| = 0.
\end{aligned}$$

Next, we estimate the limit in k of $\limsup_{V \in \mathcal{V}} C_{k,V}$. Set $u_j = t_j^{-1} r_{j-1}$, $v_j = t_j r_j$ and

$$C_V(t_1, \dots, t_n) = \| T_\eta(h_V^{\frac{2}{p_1}}, \dots, h_V^{\frac{2}{p_n}}) \|_p,$$

where

$$\eta = (\phi - \phi(\zeta_1, \dots, \zeta_n))(u_1 \cdot u_2^{-1}, \dots, u_{i-1} \cdot u_i^{-1}, u_i \cdot r_i^{-1}, r_i \cdot v_{i+1}^{-1}, v_{i+1} \cdot v_{i+2}^{-1}, \dots, v_{n-1} \cdot v_n^{-1}).$$

Thanks to the strong continuity statement of Lemma 3.2.6, we can use Fubini to deduce:

$$C_{k,V} \leq \int_{G^{\times n}} C_V(t_1, \dots, t_n) \left(\prod_{i=1}^{n} |\varphi_k(t_j)| \right) dt_1 \dots dt_n. \tag{3.3.4}$$

Now set

$$y_{j,V} = \lambda_{u_j} h_V^{\frac{2}{p_j}} \lambda^*_{u_{j+1}} \text{ for } 1 \leq j \leq i-1, \qquad y_{i,V} = \lambda_{u_i} h_V^{\frac{2}{p_i}} \lambda^*_{r_i}, \qquad y_{i+1,V} = \lambda_{r_i} h_V^{\frac{2}{p_{i+1}}} \lambda^*_{v_{i+1}},$$
$$y_{j,V} = \lambda_{v_{j-1}} h_V^{\frac{2}{p_j}} \lambda^*_{v_j} \quad \text{for } i+2 \leq j \leq n.$$

Denote ι_q for the identity operator on $L_q(\mathcal{L}G)$. The symbol 1 is used both for the constant 1-function and the number 1. Then we get the following estimate, where we apply Lemma 3.2.6 and (2.6.6) in the first line and Lemma 3.2.7 and the assumption that T_ζ is a contraction in the third line:

$$\begin{aligned} C_V(t_1, \dots, t_n) &= \| T_{(\phi - \phi(\zeta_1, \dots, \zeta_n))}(y_{1,V}, \dots, y_{n,V}) \|_p \\ &\leq \sum_{j=1}^{n} \| T_{\phi(1, \dots, 1, (\zeta_j - 1), \zeta_{j+1}, \dots, \zeta_n)}(y_{1,V}, \dots, y_{n,V}) \|_p \\ &\leq \| T_\phi : L_{p_1} \times \dots \times L_{p_n} \to L_p \| \sum_{j=1}^{n} \left(\| (T_{\zeta_j} - \iota_{p_j})(y_{j,V}) \|_{p_j} \prod_{i \neq j} \| y_{i,V} \|_{p_i} \right). \end{aligned} \tag{3.3.5}$$

By (2.6.6), we have $\| y_{j,V} \|_{p_j} = 1$. Further, by applying again Lemma 3.2.6 and Proposition 3.3.3,

$$\| (T_{\zeta_j} - \iota_{p_j})(y_{j,V}) \|_{p_j} = \| T_{\zeta_j(u_j \cdot u_{j+1}^{-1}) - 1}(h_V^{\frac{2}{p_j}}) \|_{p_j} \to |\zeta_j(u_j u_{j+1}^{-1}) - 1|$$

for $1 \leq j \leq i-1$. Filling in the definition of ζ_j,

$$|\zeta_j(u_j u_{j+1}^{-1}) - 1| = |\zeta(r_{j-1}^{-1} t_j^{-1} r_{j-1} r_j^{-1} t_{j+1} r_j) - 1|$$

and this equals 0 when evaluated at $t_j, t_{j+1} = e$. Similarly, we find for $i \leq j \leq n$ that $\lim_{V \in \mathcal{V}} \| (T_{\zeta_j} - \iota)(y_{j,V}) \|_{L_{p_j}(\mathcal{L}G)}$ exists and equals 0 when evaluated at the identity in the corresponding $t_1, \dots, t_n$. Moreover, all these values are bounded by 2. Going back to (3.3.4), let us write $M := \| T_\phi : L_{p_1} \times \dots \times L_{p_n} \to L_p \|$. We find

$$C_{k,V} \leq \int_{G^{\times n}} M \left(\sum_{j=1}^{n} \| (T_{\zeta_j} - \iota)(y_{j,V}) \|_{p_j} \right) \left(\prod_{j=1}^{n} |\varphi_k(t_j)| \right) dt_1 \dots dt_n. \tag{3.3.6}$$

The integrand of (3.3.6) is bounded by the integrable function $2M \prod_{i=1}^{n} |\varphi_k(t_j)|$. Hence, by Lebesgue's dominated convergence theorem, the right hand side of (3.3.6) converges in V. We find that

$$\limsup_{V \in \mathcal{V}} C_{k,V} \leq M \int_{G^{\times n}} \left(\sum_{j=1}^{n} \lim_{V \in \mathcal{V}} \| (T_{\zeta_j} - \iota)(y_{j,V}) \|_{p_j} \right) \left(\prod_{i=1}^{n} |\varphi_k(t_j)| \right) dt_1 \dots dt_n.$$

This quantity goes to 0 in k. This concludes the proof for $C_{k,V}$.

Finally we prove that $\lim_{V\in\mathcal{V}} B_{k,V} = 0$ for any k. We fix a k for the remainder of the proof. Recall that since $\varphi_k \in A(G)$, T_{φ_k} is bounded on $L_q(\mathcal{L}G)$ for any $1 \le q \le \infty$. Moreover, since $\varphi_k \in C_c(G) \star C_c(G)$, we also have $\varphi_k \Delta \in C_c(G) \star C_c(G) \subseteq A(G)$ (cf. the calculation before (2.6.5)), hence $T_{\varphi_k \Delta}$ is also bounded on $L_q(\mathcal{L}G)$ for any q.

We may assume, by scaling φ_k if necessary, that $T_{\varphi_k} : L_q(\mathcal{L}G) \to L_q(\mathcal{L}G)$ and $T_{\varphi_k\Delta} : L_q(\mathcal{L}G) \to L_q(\mathcal{L}G)$ are contractions for any Hölder combination q of $p_1, \dots, p_n$. Of course, this means that $\|\varphi_k\|_1$ need no longer be 1 from now on. Set

$$\begin{aligned}\psi_k(s_1,\dots,s_n;t_1,\dots,t_n) :=& \left(\prod_{j=1}^{i} \varphi_k(r_{j-1}s_j\dots s_i r_i^{-1}t_j)\right)\\ &\times\left(\prod_{j=i+1}^{n} \varphi_k(t_j r_i s_{i+1}\dots s_j r_j^{-1})\Delta(r_i s_{i+1}\dots s_j r_j^{-1})\right)\\ =:& \,\psi_k^1(s_1,\dots,s_i;t_1,\dots,t_i)\psi_k^2(s_{i+1},\dots,s_n;t_{i+1},\dots,t_n).\end{aligned}$$

By using the last term of (3.3.1) and Fubini, we get

$$\begin{aligned}B_{k,V} \le \int_{G^{\times n}} &|(\phi(\zeta_1,\dots,\zeta_n))_{t_1,\dots,t_n}(e,\dots,e)|\\ &\times \| T_{\psi_k(\cdot;t_1,\dots,t_n)}(h_V^{\frac{2}{p_1}},\dots,h_V^{\frac{2}{p_n}}) - \psi_k(1,\dots,1;t_1,\dots t_n)h_V^{\frac{2}{p}}\|_p dt_1\dots dt_n.\end{aligned} \tag{3.3.7}$$

Note that $|\varphi_k| \le 1$ by the assumed contractivity of T_{φ_k}. Indeed, for $s \in G$, apply T_{φ_k} to λ_s to deduce that $|\varphi_k(s)| \le 1$. Hence, $|\psi_k(1,\dots,1;t_1,\dots,t_n)| \le \prod_{j=i+1}^n \Delta(r_i r_j^{-1})$. Moreover, from the expression (3.3.9) below we see that

$$\| T_{\psi_k(\cdot;t_1,\dots,t_n)}(h_V^{\frac{2}{p_1}},\dots,h_V^{\frac{2}{p_n}})\|_p \le \Delta((t_{i+1},\dots,t_n)^{-1}).$$

Since $\phi(\zeta_1,\dots,\zeta_n)$ is compactly supported, the integrand of (3.3.7) is dominated by an integrable function. Hence by the Lebesgue dominated convergence theorem, it suffices to show that the term

$$\| T_{\psi_k(\cdot;t_1,\dots,t_n)}(h_V^{\frac{2}{p_1}},\dots,h_V^{\frac{2}{p_n}}) - \psi_k(1,\dots,1;t_1,\dots t_n)h_V^{\frac{2}{p}}\|_p \tag{3.3.8}$$

goes to 0 in V for any choice of $t_1,\dots,t_n \in G$.

Fix $t_1,\dots,t_n \in G$. For $1 \le j \le i$, set $q_j^{-1} = \sum_{l=j}^{i} p_l^{-1}$ (so $q_1 = \bar{p}_1$) and $T_j = T_{\varphi_k(r_{j-1}\cdot r_i^{-1}t_j)}$. By Lemma 3.2.6, T_j is a contraction on $L_{q_j}(\mathcal{L}G)$. For $i+1 \le j \le n$, set $q_j^{-1} = \sum_{l=j}^{n} p_l^{-1}$ (so $q_{i+1} = \bar{p}_2$) and $T_j = T_{\varphi_k(t_j r_i\cdot r_j^{-1})\Delta(r_i\cdot r_j^{-1})}$. We can estimate the norm of $T_j : L_{q_j}(\mathcal{L}G) \to L_{q_j}(\mathcal{L}G)$ by using again Lemma 3.2.6:

$$\|T_j\| = \Delta(t_j^{-1})\|T_{\varphi_k(t_j r_i\cdot r_j^{-1})\Delta(t_j r_i\cdot r_j^{-1})}\| = \Delta(t_j^{-1})\|T_{\varphi_k\Delta}\| \le \Delta(t_j^{-1}).$$

Now, by Lemma 3.2.8, we have

$$T_{\psi_k^1(\cdot;t_1,\dots,t_i)}(h_V^{\frac{2}{p_1}},\dots,h_V^{\frac{2}{p_i}}) = T_1(h_V^{\frac{2}{p_1}}\, T_2(h_V^{\frac{2}{p_2}}\dots T_i(h_V^{\frac{2}{p_i}})\dots)),$$
$$T_{\psi_k^2(\cdot;t_{i+1},\dots,t_n)}(h_V^{\frac{2}{p_{i+1}}},\dots,h_V^{\frac{2}{p_n}}) = T_{i+1}(h_V^{\frac{2}{p_{i+1}}}\, T_{i+2}(h_V^{\frac{2}{p_{i+2}}}\dots T_n(h_V^{\frac{2}{p_n}})\dots)).$$

Clearly, $T_{\psi_k^1(\cdot;t_1,\dots,t_i)}$ is contractive as a map on $L_{p_1}(\mathscr{L}G)\times\dots\times L_{p_i}(\mathscr{L}G)$. Let $x_j\in L_{p_j}(\mathscr{L}G)$ with $\|x_j\|_{L_{p_j}(\mathscr{L}G)}\le 1$; then, from (3.2.4),

$$\begin{aligned}\|T_{\psi_k(\cdot;t_1,\dots,t_n)}(x_1,\dots,x_n)\|_p &\le \|T_{\psi_k^2(\cdot;t_1,\dots,t_n)}(x_{i+1},\dots,x_n)\|_{\tilde p_2}\\ &\le \Delta((t_{i+1}\dots t_n)^{-1}).\end{aligned} \tag{3.3.9}$$

This validates the use of the dominated convergence theorem above. Now we go back to estimating (3.3.8). Using subsequently the triangle inequality and Hölder's inequality (with again (3.2.4)), we find

$$\begin{aligned}
&\|T_{\psi_k(\cdot;t_1,\dots,t_n)}(h_V^{\frac{2}{p_1}},\dots,h_V^{\frac{2}{p_n}}) - \psi_k(1,\dots,1;t_1,\dots t_n)h_V^{\frac{2}{p}}\|_p\\
&\le \|T_{\psi_k^1(\cdot;t_1,\dots,t_i)}(h_V^{\frac{2}{p_1}},\dots,h_V^{\frac{2}{p_i}})\cdot\psi_k^2(1,\dots,1;t_{i+1},\dots,t_n)h_V^{\frac{2}{\tilde p_2}} - \psi_k(1,\dots,1;t_1,\dots t_n)h_V^{\frac{2}{p}}\|_p\\
&+\|T_{\psi_k(\cdot;t_1,\dots,t_n)}(h_V^{\frac{2}{p_1}},\dots,h_V^{\frac{2}{p_n}}) - T_{\psi_k^1(\cdot;t_1,\dots,t_i)}(h_V^{\frac{2}{p_1}},\dots,h_V^{\frac{2}{p_i}})\cdot\psi_k^2(1,\dots,1;t_{i+1},\dots,t_n)h_V^{\frac{2}{\tilde p_2}}\|_p\\
&\le\left(\prod_{j=i+1}^n \Delta(r_j r_j^{-1})\right)\|T_{\psi_k^1(\cdot;t_1,\dots,t_i)}(h_V^{\frac{2}{p_1}},\dots,h_V^{\frac{2}{p_i}}) - \psi_k^1(1,\dots,1;t_1,\dots,t_i)h_V^{\frac{2}{\tilde p_1}}\|_{\tilde p_1}\\
&+\|T_{\psi_k^2(\cdot;t_{i+1},\dots,t_n)}(h_V^{\frac{2}{p_{i+1}}},\dots,h_V^{\frac{2}{p_n}}) - \psi_k^2(1,\dots,1;t_{i+1},\dots,t_n)h_V^{\frac{2}{\tilde p_2}}\|_{\tilde p_2}\\
&=: B_{k,V}^1 + B_{k,V}^2.
\end{aligned}$$

We show only that $\lim_{V\in\mathcal{V}} B_{k,V}^2 = 0$; the equality $\lim_{V\in\mathcal{V}} B_{k,V}^1 = 0$ follows similarly and is in fact slightly easier since the T_j are contractions for $j\le i$. Now set, for $i\le j\le n$,

$$R_{j,V} := \left(\prod_{l=j+1}^n \varphi_k(t_l r_i r_l^{-1})\right) T_{i+1}(h_V^{\frac{2}{p_{i+1}}}\dots T_j(h_V^{\frac{2}{q_j}})\dots).$$

Here $R_{i,V} = \prod_{l=i+1}^{n-1}\varphi_k(t_l r_i r_l^{-1})h_V^{\frac{2}{q_1}}$. Then

$$B_{k,V}^2 \le \sum_{j=i+1}^n \|R_{j,V} - R_{j-1,V}\|_{\tilde p_2}.$$

Recall that $|\varphi_k|\le 1$. Hence

$$\begin{aligned}
&\|R_{j,V} - R_{j-1,V}\|_{L_{\tilde p_2}(\mathscr{L}G)}\\
&=\left(\prod_{l=j+1}^n |\varphi_k(t_l r_i r_l^{-1})|\right)\|T_{i+1}(h_V^{\frac{2}{p_{i+1}}}\dots T_{j-1}(h_V^{\frac{2}{p_{j-1}}}(T_j(h_V^{\frac{2}{q_j}}) - \varphi_k(t_j r_i r_j^{-1})h_V^{\frac{2}{q_j}}))\dots)\|_{\tilde p_2}\\
&\le \Delta((t_{i+1}\dots t_n)^{-1})\|T_j(h_V^{\frac{2}{q_j}}) - \varphi_k(t_j r_i r_j^{-1})h_V^{\frac{2}{q_j}}\|_{q_j}.
\end{aligned}$$

We know that $q_j > \bar{p}_2 > 1$ for any $i+1 \le j \le n$. Additionally, T_j is bounded on $\mathscr{L}G$ and $L_1(\mathscr{L}G)$. By Proposition 3.3.3, the above terms converge to 0 in V. Hence, $\lim_{V\in\mathcal{V}} B^2_{k,V} = 0$. This finishes the proof. □

Remark 3.3.4. As in the unimodular case (see [CKV23, Remark 3.3]) we do not know if Theorem 3.3.1 holds if $p = p_i = 1$ for some $1 \le i \le n$ (and $p_j = \infty$ for all $j \ne i$). The proof above fails in that case because we cannot apply Proposition 3.3.3.

3.4. SCHUR TO FOURIER TRANSFERENCE FOR AMENABLE GROUPS

In this section we prove the transference from Schur to Fourier multipliers for amenable groups. Here, we will indicate at one place how the proof simplifies in the unimodular case. We also fixed a small mistake in the version of [CKV23], as will be mentioned at the relevant spot.

Recall [Pat88, Theorem 4.10] that G is amenable iff it satisfies the following Følner condition: for any $\varepsilon > 0$ and any compact set $K \subseteq G$, there exists a compact set $F \subseteq G$ with non-zero measure such that $\frac{\mu((sF\setminus F)\cup(F\setminus sF))}{\mu(F)} < \varepsilon$ for all $s \in K$. This allows us to construct a net $F_{(\varepsilon,K)}$ of such Følner sets using the ordering $(\varepsilon_1, K_1) \le (\varepsilon_2, K_2)$ if $\varepsilon_1 \ge \varepsilon_2, K_1 \subseteq K_2$.

Theorem 3.4.1. *Let G be a locally compact, amenable group and let $1 \le p, p', p_1, \dots, p_n \le \infty$ be such that $\frac{1}{p} = \sum_{i=1}^n \frac{1}{p_i} = 1 - \frac{1}{p'}$. Let $\phi \in L_\infty(G^{\times n})$ and define $\widetilde{\phi} \in L_\infty(G^{\times n+1})$ by*

$$\widetilde{\phi}(s_0, \dots, s_n) = \phi(s_0 s_1^{-1}, s_1 s_2^{-1}, \dots, s_{n-1} s_n^{-1}), \qquad s_i \in G.$$

Assume that $\widetilde{\phi}$ is a $(p_1, \dots, p_n)$-Schur symbol. Then there is a net I and there are complete contractions $i_{q,\alpha} : L_q(\mathscr{L}G) \to S_q(L_2(G))$, $\alpha \in I$, such that for all $f_i, f \in C_c(G) \star C_c(G)$,

$$\left| \langle i_{p,\alpha}(T_\phi(x_1, \dots, x_n)), i_{p',\alpha}(y) \rangle - \langle M_{\widetilde{\phi}}(i_{p_1,\alpha}(x_1), \dots, i_{p_n,\alpha}(x_n)), i_{p',\alpha}(y) \rangle \right| \xrightarrow{\alpha} 0, \qquad (3.4.1)$$

where $x_i = \Delta^{a_i}\lambda(f_i)\Delta^{b_i} \in L^{p_i}(\mathscr{L}G)$, $y = \Delta^a \lambda(f) \Delta^b \in L^{p'}(\mathscr{L}G)$ (i.e. $a_i + b_i = \frac{1}{p_i}$). In a similar way, the matrix amplifications of $i_{q,\alpha}$ approximately intertwine the multiplicative amplifications of the Fourier and Schur multipliers.

Proof. Let $F_\alpha, \alpha \in I$ be a Følner net for G, where I is the index set consisting of pairs (ε, K) for $\varepsilon > 0$, $K \subseteq G$ compact and the ordering as described above.

Let $P_\alpha = P_{F_\alpha}$ be the projection of $L_2(G)$ onto $L_2(F_\alpha)$. Consider the maps

$$i_{p,\alpha} : L_p(\mathscr{L}G) \to S_p(L_2(G)), \qquad i_{p,\alpha}(x) = \mu(F_\alpha)^{-1/p} P_\alpha x P_\alpha$$

They are contractions by [CS15a, Theorem 5.1]. By replacing G by $G \times SU(2)$, one proves that they are in fact complete contractions (see also the last paragraph of [CS15a, Proof of Theorem 5.2]).

Now fix α. From (2.6.13), we deduce

$$\begin{aligned}
&M_{\tilde{\phi}}(i_{p_1,\alpha}(x_1),\dots,i_{p_n,\alpha}(x_n))(t_0,t_n)\\
&=\frac{1}{\mu(F_\alpha)^{1/p}}1_{F_\alpha}(t_0)1_{F_\alpha}(t_n)\int_{F_\alpha^{\times n-1}}\phi(t_0t_1^{-1},\dots,t_{n-1}t_n^{-1})f_1(t_0t_1^{-1})\dots f_n(t_{n-1}t_n^{-1})\times\\
&\qquad\Delta^{a_1}(t_0)\Delta^{b_1+a_2}(t_1)\dots\Delta^{b_n}(t_n)\Delta((t_1\dots t_n)^{-1})dt_1\dots dt_{n-1}.
\end{aligned}$$

From Lemma 3.2.9 we have a similar expression for the kernel of $i_{p,\alpha}(T_\phi(x_1,\dots,x_n))$:

$$\begin{aligned}
(t_0,t_n)\mapsto&\frac{1}{\mu(F_\alpha)^{1/p}}1_{F_\alpha}(t_0)1_{F_\alpha}(t_n)\int_{G^{\times n-1}}\phi(t_0t_1^{-1},\dots,t_{n-1}t_n^{-1})f_1(t_0t_1^{-1})\dots f_n(t_{n-1}t_n^{-1})\times\\
&\qquad\Delta^{a_1}(t_0)\Delta^{b_1+a_2}(t_1)\dots\Delta^{b_n}(t_n)\Delta((t_1\dots t_n)^{-1})dt_1\dots dt_{n-1}.
\end{aligned}$$

Now we need to take the pairing of these kernels with $i_{p',\alpha}(y)$ and calculate their difference. To that end, we define the following function Φ:

$$\begin{aligned}
\Phi(t_0,\dots,t_n)=&\phi(t_0t_1^{-1},\dots,t_{n-1}t_n^{-1})f_1(t_0t_1^{-1})\dots f_n(t_{n-1}t_n^{-1})f(t_nt_0^{-1})\times\\
&\Delta^{a_1+b}(t_0)\Delta^{b_1+a_2}(t_1)\dots\Delta^{b_n+a}(t_n)\Delta((t_0t_1\dots t_n)^{-1}),
\end{aligned}$$

and the function Ψ_α:

$$\Psi_\alpha(t_0,\dots,t_n)=1_{F_\alpha}(t_0)1_{F_\alpha}(t_n)-1_{F_\alpha^{\times n+1}}(t_0,\dots,t_n)=1_{F_\alpha\times(F_\alpha^{\times n-1})^c\times F_\alpha}(t_0,\dots,t_n).$$

Note that in [CKV23], the indicator function was mistakenly taken over $F_\alpha\times(F_\alpha^c)^{\times n-1}\times F_\alpha$ instead. This correction leads to an extra term n in the choice of the lower bound of α at the end. Also note that a priori, it is not clear that $T_\phi(x_1,\dots,x_n)$ lies in $L_p(\mathscr{L}G)$, and hence it is not clear that $i_{p,\alpha}(T_\phi(x_1,\dots,x_n))$ lies in $S_p(L_2(G))$. However, both $i_{p,\alpha}(T_\phi(x_1,\dots,x_n))$ and $i_{p',\alpha}(y)$ are given by integration against a kernel in $L_2(G\times G)$, so the pairing (2.6.8) is still well-defined as a pairing in $S_2(L_2(G))$ instead. Now we have:

$$\begin{aligned}
&|\langle i_{p,\alpha}(T_\phi(x_1,\dots,x_n)),i_{p',\alpha}(y)\rangle-\langle M_{\tilde{\phi}}(i_{p_1,\alpha}(x_1),\dots,i_{p_n,\alpha}(x_n)),i_{p',\alpha}(y)\rangle|\\
&=\left|\frac{1}{\mu(F_\alpha)}\int_{G^{\times n+1}}\Phi(t_0,\dots,t_n)\Psi_\alpha(t_0,\dots,t_n)dt_0\dots dt_n\right|
\end{aligned}\tag{3.4.2}$$

Let $K\subseteq G$ be some compact set such that $\mathrm{supp}(f_j),\mathrm{supp}(f)\subseteq K$ and $e\in K$. Let $t_0,\dots,t_n$ be such that both $\Phi(t_0,\dots,t_n)$ and $\Psi_\alpha(t_0,\dots,t_n)$ are nonzero. Since $\Psi_\alpha(t_0,\dots,t_n)$ is nonzero, we must have $t_0,t_n\in F_\alpha$ and $t_i\notin F_\alpha$ for some $i\in\{1,\dots,n-1\}$. Since $\Phi(t_0,\dots,t_n)$ is nonzero, there are $k_1,\dots,k_n\in K$ such that $t_{n-1}=k_nt_n$, $t_{n-2}=k_{n-1}k_nt_n$, $\dots$, $t_0=k_1\dots k_nt_n$. Hence we find

$$\begin{aligned}
t_n&\in F_\alpha\cap(k_1\dots k_n)^{-1}F_\alpha\setminus\big((k_2\dots k_n)^{-1}F_\alpha\cap\dots\cap k_n^{-1}F_\alpha\big)\\
&\subseteq F_\alpha\setminus\big((k_2\dots k_n)^{-1}F_\alpha\cap\dots\cap k_n^{-1}F_\alpha\big)\\
&=(F_\alpha\setminus(k_2\dots k_n)^{-1}F_\alpha)\cup\dots\cup(F_\alpha\setminus k_n^{-1}F_\alpha)
\end{aligned}\tag{3.4.3}$$

We want to apply change of variables in (3.4.2). We note that this is much simpler in the unimodular case; the reader interested only in that case can deduce (3.4.4) more directly and skip (3.4.5) completely. Let us first look at a simple case: assume $g \in L_1(G \times G)$ is such that $g(s,t) \neq 0$ only when $st^{-1} \in K$. Then

$$\begin{aligned}\int_{G^{\times 2}} g(s,t)dsdt &= \int_{G^{\times 2}} 1_K(st^{-1})g(s,t)dsdt = \int_{G^{\times 2}} 1_K(s)g(st,t)\Delta(t)dsdt \\ &= \int_G \int_K g(k_1t,t)\Delta(t)dk_1dt\end{aligned}$$

where we renamed the variable s in the last line. Applying the above formula twice for a function $g \in L_1(G^{\times 3})$ such that $g(r,s,t) \neq 0$ only when $rs^{-1} \in K$, $st^{-1} \in K$, we get

$$\begin{aligned}\int_{G^{\times 3}} g(r,s,t)drdsdt &= \int_{G^{\times 2}} \int_K g(k_1s,s,t)\Delta(s)dk_1dsdt \\ &= \int_G \int_{K^{\times 2}} g(k_1k_2t,k_2t,t)\Delta(k_2t)\Delta(t)dk_1dk_2dt.\end{aligned}$$

Carrying on like this, we obtain

$$\begin{aligned}&\left|\frac{1}{\mu(F_\alpha)} \int_{G^{\times n+1}} \Phi(t_0,\ldots,t_n)\Psi_\alpha(t_0,\ldots,t_n)dt_0\ldots dt_n\right| \\ &= \left|\frac{1}{\mu(F_\alpha)} \int_{K^{\times n}} \int_G \Phi(k_1\ldots k_nt_n,\ldots,k_nt_n,t_n)\Psi_\alpha(k_1\ldots k_nt_n,\ldots,k_nt_n,t_n)\times \right. \\ &\qquad \left. \Delta(k_2\ldots k_nt_n)\ldots\Delta(k_nt_n)\Delta(t_n)dt_ndk_1\ldots dk_n\right|.\end{aligned} \tag{3.4.4}$$

Note that $a+b+\sum_{i=1}^n a_i+b_i = 1$, hence

$$\begin{aligned}&|\Phi(k_1\ldots k_nt_n,\ldots,k_nt_n,t_n)|\Delta(k_2\ldots k_nt_n)\ldots\Delta(k_nt_n)\Delta(t_n) \\ &\leq \|\phi f_1\ldots f_nf\|_\infty \Delta^{a_1+b}(k_1\ldots k_nt_n)\Delta^{b_1+a_2+1}(k_2\ldots k_nt_n)\ldots\Delta^{b_{n-1}+a_n+1}(k_nt_n)\times \\ &\qquad \Delta^{b_n+a+1}(t_n)\Delta(k_1^{-1}k_2^{-2}\ldots k_n^{-n}t_n^{-n-1}) \\ &= \|\phi f_1\ldots f_nf\|_\infty \Delta^{a_1+b-1}(k_1)\Delta^{a_1+a_2+b_1+b-1}(k_2)\ldots\Delta^{1-b_n-a-1}(k_n) \\ &\leq \|\phi f_1\ldots f_nf\|_\infty C_{K,n} =: M.\end{aligned} \tag{3.4.5}$$

Here the constant $C_{K,n}$ can be chosen to be dependent only on K and n (and G).

Applying (3.4.2), (3.4.4), (3.4.5) and (3.4.3) consecutively we get

$$\begin{aligned}
&|\langle i_{p,\alpha}(T_\phi(x_1,\dots,x_n)), i_{p',\alpha}(y)\rangle_{p,p'} - \langle M_{\widetilde{\phi}}(i_{p_1,\alpha}(x_1),\dots,i_{p_n,\alpha}(x_n)), i_{p',\alpha}(y)\rangle| \\
&= \Big|\frac{1}{\mu(F_\alpha)}\int_{K^{\times n}}\int_G \Phi(k_1\dots k_n t_n,\dots,k_n t_n,t_n)\Psi_\alpha(k_1\dots k_n t_n,\dots,k_n t_n,t_n)\times \\
&\qquad \Delta(k_2\dots k_n t_n)\dots\Delta(k_n t_n)\Delta(t_n)\,dt_n\,dk_1\dots dk_n\Big| \\
&\le \frac{M}{\mu(F_\alpha)}\int_{K^{\times n}}\int_G \Psi_\alpha(k_1\dots k_n t_n,\dots,k_n t_n,t_n)\,dt_n\,dk_1\dots dk_n \\
&= \frac{M}{\mu(F_\alpha)}\int_{K^{\times n}} \mu\big(F_\alpha\cap(k_1\dots k_n)^{-1}F_\alpha\setminus\big((k_2\dots k_n)^{-1}F_\alpha\cap\dots\cap k_n^{-1}F_\alpha\big)\big)\,dk_1\dots dk_n \\
&\le \frac{M}{\mu(F_\alpha)}\int_{K^{\times n}}\sum_{i=2}^{n}\mu(F_\alpha\setminus(k_i\dots k_n)^{-1}F_\alpha)\,dk_1\dots dk_n \\
&\le M(n-1)\mu(K)^n \sup_{k\in K^{1-n}}\frac{\mu(F_\alpha\setminus kF_\alpha)}{\mu(F_\alpha)}.
\end{aligned} \tag{3.4.6}$$

Using the ordering described earlier, if the index $\alpha \ge (\varepsilon\times\big(Mn\mu_G(K)^n\big)^{-1}, K^{1-n})$, then the Følner condition implies that (3.4.6) is less than ε, and hence the limit (3.4.1) holds.

From (3.4.1), it follows from writing out the definitions that the matrix amplifications of $i_{p,\alpha}$ also approximately intertwine the multiplicative amplifications of the Fourier and Schur multipliers. i.e. for $\beta_i \in S^N_{p_i}, \beta\in S^N_{p'}$, we have

$$\begin{aligned}
&\Big|\langle \mathrm{id}\otimes i_{p,\alpha}(T^{(N)}_\phi(\beta_1\otimes x_1,\dots,\beta_n\otimes x_n)), \mathrm{id}\otimes i_{p',\alpha}(\beta\otimes y)\rangle_{p,p'} - \\
&\quad \langle M^{(N)}_{\widetilde{\phi}}(\mathrm{id}\otimes i_{p_1,\alpha}(\beta_1\otimes x_1),\dots,\mathrm{id}\otimes i_{p_n,\alpha}(\beta_n\otimes x_n)), \mathrm{id}\otimes i_{p',\alpha}(\beta\otimes y)\rangle\Big| \to 0
\end{aligned} \tag{3.4.7}$$

□

Corollary 3.4.2. *Let G be an amenable locally compact group and $1\le p,p_1,\dots,p_n\le\infty$ be such that $\frac{1}{p}=\sum_{i=1}^n\frac{1}{p_i}$. Let $\phi\in L_\infty(G^{\times n})$. If $\widetilde{\phi}$ is a $(p_1,\dots,p_n)$-Schur symbol (resp. $(p_1,\dots,p_n)$-mb Schur symbol) then ϕ is a $(p_1,\dots,p_n)$-Fourier symbol (resp. $(p_1,\dots,p_n)$-mb Fourier symbol). Moreover,*

$$\|T_\phi\|_{(p_1,\dots,p_n)} \le \|M_{\widetilde{\phi}}\|_{(p_1,\dots,p_n)}, \qquad \|T_\phi\|_{(p_1,\dots,p_n)-mb} \le \|M_{\widetilde{\phi}}\|_{(p_1,\dots,p_n)-mb}.$$

Proof. Let x_i be as in the hypotheses of Theorem 3.4.1 and let $i_{p,\alpha}$ be as in the proof of Theorem 3.4.1. Let p' be the Holder conjugate of p. In [CS15a, Theorem 5.2] it is proven that

$$\langle i_{p,\alpha}(x), i_{p',\alpha}(y)\rangle_{p,p'} \to \langle x,y\rangle_{p,p'}, \qquad x\in L_p(\mathscr{L}G),\ y\in L_{p'}(\mathscr{L}G). \tag{3.4.8}$$

Note that this inequality also holds, and in fact is explicitly proven for $p=\infty$; by symmetry it also holds for $p=1$. We remark that this result also uses the Følner condition.

Let $\varepsilon > 0$. Then we can find $y = \Delta^a \lambda(f)\Delta^b$, $f \in C_c(G) \star C_c(G)$ such that $\|y\|_{p'} \le 1$ and

$$\|T_\phi(x_1,\dots,x_n)\|_p \le |\langle T_\phi(x_1,\dots,x_n), y\rangle| + \varepsilon.$$

Next, by (3.4.8) and Theorem 3.4.1 we can find $\alpha \in I$ such that the following two inequalities hold:

$$|\langle T_\phi(x_1,\dots,x_n), y\rangle - \langle i_{p,\alpha}(T_\phi(x_1,\dots,x_n)), i_{p',\alpha}(y)\rangle| < \varepsilon$$

and

$$|\langle i_{p,\alpha}(T_\phi(x_1,\dots,x_n)), i_{p',\alpha}(y)\rangle_{p,p'} - \langle M_{\tilde\phi}(i_{p_1,\alpha}(x_1),\dots,i_{p_n,\alpha}(x_n)), i_{p',\alpha}(y)\rangle_{p,p'}| < \varepsilon.$$

By combining these inequalities we find

$$\begin{aligned}\|T_\phi(x_1,\dots,x_n)\|_p &\le |\langle M_{\tilde\phi}(i_{p_1,\alpha}(x_1),\dots,i_{p_n,\alpha}(x_n)), i_{p',\alpha}(y)\rangle_{p,p'}| + 3\varepsilon\\ &\le \|M_{\tilde\phi}\|_{(p_1,\dots,p_n)} \prod_{i=1}^n \|x_i\|_{p_i} + 3\varepsilon\end{aligned}$$

The elements x_i as chosen above are norm dense in $L_{p_i}(\mathcal{L}G)$ (resp. $C^*_\lambda(G)$ when $p_i = \infty$), hence we get the required bound. The multiplicative bound follows similarly. □

Remark 3.4.3. In [CS15a], [CKV23], the proof runs via an ultraproduct construction. The ultraproduct is not actually necessary as demonstrated above, as all limits are usual limits and not ultralimits.

As another corollary, we get the following multiplicatively bounded, non-unimodular version of [CJKM23, Theorem 4.5]. Moreover, we no longer need the SAIN condition and the subgroup need no longer be discrete.

Corollary 3.4.4. *Let G be a locally compact, first countable group and let $1 \le p \le \infty$ and $1 < p_1,\dots,p_n \le \infty$ with $p^{-1} = \sum_{i=1}^n p_i^{-1}$. Let $\phi \in C_b(G^{\times n})$ be a $(p_1,\dots,p_n)$-mb Fourier symbol and let $H \le G$ be an amenable subgroup. Then*

$$\|T_{\phi|_{H^{\times n}}}\|_{(p_1,\dots,p_n)-mb} \le \|T_\phi\|_{(p_1,\dots,p_n)-mb}$$

Proof. The associated inequality for Schur multipliers follows from Theorem 2.6.16. Now Corollary 3.4.2 (using amenability of H) and Theorem 3.3.1 yield the result. □

In the next corollary we prove a necessary condition for a 'Fourier multiplier' to satisfy (3.4.1) for the embeddings $i_{p,\alpha}$ defined above. This was used in the discussion in Section 3.2.

Corollary 3.4.5. *Fix $n > 1$, $1 \le p_1,\dots,p_n, p, p' \le \infty$ such that $\frac{1}{p} = \sum_{i=1}^n \frac{1}{p_i} = 1 - \frac{1}{p'}$ and let $\theta_1,\dots,\theta_n,\theta,\theta' \in [0,1]$. Let $i_{p,\alpha}$ be as in the proof of Theorem 3.4.1. Assume that for each $\phi \in L_\infty(G^{\times n})$, we have a map $S_\phi : \kappa_{p_i}^{\theta_1}(L) \times \dots \times \kappa_{p_n}^{\theta_n}(L) \to \kappa_p^\theta(L)$ satisfying*

$$\left|\langle i_{p,\alpha}(S_\phi(x_1,\dots,x_n)), i_{p',\alpha}(y)\rangle_{p,p'} - \langle M_{\tilde\phi}(i_{p_1,\alpha}(x_1),\dots,i_{p_n,\alpha}(x_n)), i_{p',\alpha}(y)\rangle_{p,p'}\right| \xrightarrow{\alpha} 0. \tag{3.4.9}$$

for $x_i \in \kappa_{p_i}^{\theta_i}(L)$, $y \in \kappa_{p'}^{\theta'}(L)$. *Now let* $\phi(s_1,\dots,s_n) = \phi_1(s_1)\dots\phi_n(s_n)$ *for some functions* $\phi_1,\dots,\phi_n \in L^\infty(G)$. *Then* S_ϕ *must satisfy*

$$S_\phi(x_1,\dots,x_n) = T_{\phi_1}(x_1)\dots T_{\phi_n}(x_n)$$

for $x_i \in \kappa_{p_i}^{\theta_i}(L)$, $i = 1,\dots,n$.

Proof. Fix some $y \in \kappa_{p'}^{\theta'}(L)$. By density, it suffices to show that

$$\langle S_\phi(x_1,\dots,x_n), y\rangle = \langle T_{\phi_1}(x_1)\dots T_{\phi_n}(x_n), y\rangle.$$

By (3.4.8), it suffices to show

$$\lim_{\alpha\in I} |\langle i_{p,\alpha}(S_\phi(x_1,\dots,x_n) - T_{\phi_1}(x_1)\dots T_{\phi_n}(x_n)), i_{p',\alpha}(y)\rangle| = 0.$$

By running the proof of Theorem 3.4.1 with the constant 1 function in place of ϕ and $T_{\phi_i}(x_i)$ in place of x_i, we find that

$$\lim_{\alpha\in I} |\langle i_{p,\alpha}(T_{\phi_1}(x_1)\dots T_{\phi_n}(x_n)) - i_{p_1,\alpha}(T_{\phi_1}(x_1))\dots i_{p_n,\alpha}(T_{\phi_n}(x_n)), i_{p',\alpha}(y)\rangle| = 0.$$

Since multiplication with ϕ_i only maps $C_c(G)\star C_c(G)$ to $C_c(G)$, we no longer need to have that $T_{\phi_i}(x_i) \in \kappa_{p_i}^{\theta_i}(L)$, so we cannot apply Theorem 3.4.1 directly. But since we have $\phi = 1$, this does not give any technical complications in the proof.

Using the kernel representations, it is straightforward to show that

$$\begin{aligned} i_{p_1,\alpha}(T_{\phi_1}(x_1))\dots i_{p_n,\alpha}(T_{\phi_n}(x_n)) &= M_{\widetilde{\phi_1}}(i_{p_1,\alpha}(x_1))\dots M_{\widetilde{\phi_n}}(i_{p_n,\alpha}(x_n)) \\ &= M_{\widetilde{\phi}}(i_{p_1,\alpha}(x_1),\dots,i_{p_n,\alpha}(x_n)). \end{aligned}$$

By combining the above observations with (3.4.9), we get the required result. □

3.5. LINEAR INTERTWINING RESULT

In this section, we sketch the proof of Proposition 3.3.3. The main ingredient to be added to already existing results is the extension of [CPR18, Lemma 3.1] to general von Neumann algebras via Haagerup reduction. The Haagerup reduction method is described by Theorem 3.5.1, proved for σ-finite von Neumann algebras in [HJX10] and extended to the weight case in [CPPR15, Section 8].

Recall from Section 2.3.1 that the centraliser $\mathcal{M}^\varphi$ of a nfs weight φ on a von Neumann algebra $\mathcal{M}$ is given by

$$\mathcal{M}^\varphi = \{x \in \mathcal{M} : \sigma_t^\varphi(x) = x \ \forall t \in \mathbb{R}\}.$$

Theorem 3.5.1. *Let* $(\mathcal{M},\varphi)$ *be any von Neumann algebra equipped with a nfs weight. There is another von Neumann algebra* $(\mathcal{R},\widehat{\varphi})$ *containing* $\mathcal{M}$ *and with nfs weight* $\widehat{\varphi}$ *extending* φ*, and elements* a_n *in the center of the centralizer of* $\widehat{\varphi}$ *such that the following properties hold:*

1. *There is a conditional expectation* $\mathscr{E} : \mathscr{R} \to \mathscr{M}$ *satisfying*

$$\varphi \circ \mathscr{E} = \widehat{\varphi}, \qquad \sigma_s^{\varphi} \circ \mathscr{E} = \mathscr{E} \circ \sigma_s^{\widehat{\varphi}}, \quad s \in \mathbb{R}.$$

2. *The centralisers* $\mathscr{R}_n := \mathscr{R}^{\varphi_n}$ *of the weights* $\varphi_n := \varphi(e^{-a_n}\cdot)$ *are semifinite for* $n \geq 1$.

3. *There are conditional expectations* $\mathscr{E}_n : \mathscr{R} \to \mathscr{R}_n$ *satisfying*

$$\widehat{\varphi} \circ \mathscr{E}_n = \widehat{\varphi}, \qquad \sigma_s^{\widehat{\varphi}} \circ \mathscr{E}_n = \mathscr{E}_n \circ \sigma_s^{\widehat{\varphi}}, \quad s \in \mathbb{R}$$

4. $\mathscr{E}_n(x) \to x$ σ*-strongly for* $x \in \mathfrak{n}_{\widehat{\varphi}}$, *and* $\bigcup_{n\geq 1} \mathscr{R}_n$ *is* σ*-strongly dense in* $\mathscr{R}$.

Now assume that $T : \mathscr{M} \to \mathscr{M}$ is unital completely positive (ucp) and satisfies $\varphi \circ T \leq \varphi$. Recall that by Proposition 2.4.17, T 'extends' to a map $T^{(p)}$ on $L_p(\mathscr{M})$, in the sense that $T^{(p)}(D_\varphi^{1/2p} x D_\varphi^{1/2p}) = D_\varphi^{1/2p} T(x) D_\varphi^{1/2p}$ for $x \in \mathfrak{m}_{\widehat{\varphi}}$. If T satisfies $\sigma_s^\varphi \circ T = T \circ \sigma_s^\varphi$, then we moreover have $T^{(p)}(D_\varphi^{\theta/p} x D_\varphi^{(1-\theta)/p}) = D_\varphi^{\theta/p} T(x) D_\varphi^{(1-\theta)/p}$ for any $0 \leq \theta \leq 1$ and $x \in \mathfrak{m}_{\widehat{\varphi}}$.

In particular, the conditional expectations $\mathscr{E}, \mathscr{E}_n$ 'extend' to maps $\mathscr{E}^{(p)}, \mathscr{E}_n^{(p)}$ from $L_p(\mathscr{R}, \widehat{\varphi})$ to $L_p(\mathscr{M}, \varphi)$ resp. $L_p(\mathscr{R}_n, \widehat{\varphi})$. The following statement is [CPPR15, Lemma 8.3]:

$$\lim_{n\to\infty} \|\mathscr{E}_n^{(p)}(x) - x\|_p = 0, \qquad 1 \leq p < \infty,\ x \in L_p(\mathscr{R}, \hat{\varphi}). \tag{3.5.1}$$

We need a few more facts; we refer to [CPPR15, Section 8.2] for the details. First, there is an isometric isomorphism $\kappa_p : L_p(\mathscr{R}_n, \widehat{\varphi}) \to L_p(\mathscr{R}_n, \varphi_n)$ given by $\kappa_p(D_{\widehat{\varphi}}^{1/2p} x D_{\widehat{\varphi}}^{1/2p}) = e^{a_n/2p} x e^{a_n/2p}$ for $x \in \mathfrak{m}_{\widehat{\varphi}}$. Next, assume that $T : \mathscr{M} \to \mathscr{M}$ is ucp and preserves φ and σ_s^φ. Then by [HJX10, Section 4] there exists an extension $\widehat{T} : \mathscr{R} \to \mathscr{R}$ which is also ucp and preserves $\widehat{\varphi}$ and $\sigma_s^{\widehat{\varphi}}$. Hence $\widehat{T}$ itself also 'extends' to the various noncommutative L_p-spaces. Moreover, the following diagram commutes:

$$\begin{array}{ccccc}
L_p(\mathscr{R}, \widehat{\varphi}) & \xrightarrow{\mathscr{E}_n^{(p)}} & L_p(\mathscr{R}_n, \widehat{\varphi}) & \xrightarrow{\kappa_p} & L_p(\mathscr{R}_n, \varphi_n) \\
{\scriptstyle \widehat{T}^{(p)}}\big\uparrow & & {\scriptstyle \widehat{T}^{(p)}}\big\uparrow & & {\scriptstyle \widehat{T}}\big\uparrow \\
L_p(\mathscr{R}, \widehat{\varphi}) & \xrightarrow{\mathscr{E}_n^{(p)}} & L_p(\mathscr{R}_n, \widehat{\varphi}) & \xrightarrow{\kappa_p} & L_p(\mathscr{R}_n, \varphi_n).
\end{array}$$

Note that since φ_n is tracial on $\mathscr{R}_n$, the $\widehat{T}$ in the rightmost upwards arrow is actually an extension of the operator $\widehat{T}$ on $\mathscr{R}_n$ so we do not need to use the notation $\widehat{T}^{(p)}$ here.

Finally, for $1 \leq p, q < \infty$ we define the Mazur maps $M_{p,q} : L_p(\mathscr{M}) \to L_q(\mathscr{M})$ by $x \mapsto u|x|^{p/q}$ where $x = u|x|$ is the polar decomposition of x. The Mazur maps satisfy $\kappa_q \circ M_{p,q} = M_{p,q} \circ \kappa_p$, see for instance [Ric15, end of Section 3]. We are now ready to state and prove the generalisation of [CPR18, Lemma 3.1] for general von Neumann algebras. This result was already shown for $2 < p < \infty$ in [CPPR15, Section 8], but we will prove the result for all $1 < p < \infty$ at once since this does not take any extra effort.

Lemma 3.5.2. *Let $(\mathcal{M},\varphi)$ be a von Neumann algebra equipped with nfs weight. Let $T : \mathcal{M} \to \mathcal{M}$ be a unital completely positive map satisfying $\varphi \circ T = \varphi$ and $T \circ \sigma_s^\varphi = \sigma_s^\varphi \circ T$ for all $s \in \mathbb{R}$. Then there exists a universal constant $C > 0$ such that for any $x \in L_2(\mathcal{M})$ and $1 < p < \infty$:*

$$\|T^{(p)}(M_{2,p}(x)) - M_{2,p}(x)\|_p \leq C\|T^{(2)}(x) - x\|_2^\theta \|x\|_2^{1-\theta},$$

where $\theta = \frac{1}{4}\min\{\frac{p}{2}, \frac{2}{p}\}$.

Proof. The proof runs via Haagerup reduction, using the estimates for the semifinite case from [CPPR15, Claim B] for $p > 2$ and [CPR18, Lemma 3.1] for $p < 2$. Note that the latter was stated only for finite von Neumann algebras, but the same proof works for the semifinite case as well.

Set $y = M_{2,p}(x)$. Since $T = \widehat{T}$ on $\mathcal{M}$ and $L_p(\mathcal{M},\varphi) \hookrightarrow L_p(\mathcal{R},\widehat{\varphi})$ canonically and isometrically, we have $T^{(p)}(y) = \widehat{T}^{(p)}(y)$ and

$$\|T^{(p)}(y) - y\|_{L_p(\mathcal{M},\varphi)} = \|\widehat{T}^{(p)}(y) - y\|_{L_p(\mathcal{R},\widehat{\varphi})}.$$

Now fix $n \geq 1$. Then

$$\begin{aligned}\|\mathcal{E}_n^{(p)}(\widehat{T}^{(p)}(y)) - \mathcal{E}_n^{(p)}(y)\|_{L_p(\mathcal{R}_n,\widehat{\varphi})} &= \|\kappa_p\left(\mathcal{E}_n^{(p)}(\widehat{T}^{(p)}(y)) - \mathcal{E}_n^{(p)}(y)\right)\|_{L_p(\mathcal{R}_n,\varphi_n)}\\ &= \|\widehat{T}(\kappa_p(\mathcal{E}_n^{(p)}(y))) - \kappa_p(\mathcal{E}_n^{(p)}(y))\|_{L_p(\mathcal{R}_n,\varphi_n)}.\end{aligned}$$

Now we can apply the result for the semifinite case on $\kappa_p(\mathcal{E}_n^{(p)}(y))$ to obtain

$$\begin{aligned}&\|\mathcal{E}_n^{(p)}(\widehat{T}^{(p)}(y)) - \mathcal{E}_n^{(p)}(y)\|_{L_p(\mathcal{R}_n,\widehat{\varphi})}\\ &\leq C\|\hat{T}(M_{p,2}(\kappa_p(\mathcal{E}_n^{(p)}(y)))) - M_{p,2}(\kappa_p(\mathcal{E}_n^{(p)}(y)))\|_{L_2(\mathcal{R}_n,\varphi_n)}^\theta \cdot \|M_{p,2}(\kappa_p(\mathcal{E}_n^{(p)}(y)))\|_{L_2(\mathcal{R}_n,\varphi_n)}^{1-\theta}\\ &= C\|\kappa_2(\hat{T}^{(2)}(M_{p,2}(\mathcal{E}_n^{(p)}(y)))) - \kappa_2(M_{p,2}(\mathcal{E}_n^{(p)}(y)))\|_{L_2(\mathcal{R}_n,\varphi_n)}^\theta \cdot \|\kappa_2(M_{p,2}(\mathcal{E}_n^{(p)}(y)))\|_{L_2(\mathcal{R}_n,\varphi_n)}^{1-\theta}\\ &= C\|\hat{T}^{(2)}(M_{p,2}(\mathcal{E}_n^{(p)}(y))) - M_{p,2}(\mathcal{E}_n^{(p)}(y))\|_{L_2(\mathcal{R}_n,\widehat{\varphi})}^\theta \cdot \|M_{p,2}(\mathcal{E}_n^{(p)}(y))\|_{L_2(\mathcal{R}_n,\widehat{\varphi})}^{1-\theta}\\ &=: CA_n^\theta B_n^{1-\theta}.\end{aligned}$$

By the triangle inequality, the main result from [Ric15] and (3.5.1), we find

$$\begin{aligned}B_n &\leq \|M_{p,2}(\mathcal{E}_n^{(p)}(y)) - M_{p,2}(y)\|_{L_2(\mathcal{R}_n,\widehat{\varphi})} + \|M_{p,2}(y)\|_{L_2(\mathcal{R}_n,\widehat{\varphi})}\\ &\leq C_{x,p}\|\mathcal{E}_n^{(p)}(y) - y\|_{L_p(\mathcal{R}_n,\widehat{\varphi})}^{\min\{\frac{p}{2},1\}} + \|x\|_{L_2(\mathcal{R}_n,\widehat{\varphi})} \to \|x\|_{L_2(\mathcal{M},\varphi)}\end{aligned}$$

for some constant $C_{x,p}$ independent of n. Similarly, we find

$$A_n \leq C_{x,p}\|\widehat{T}^{(2)} - 1_{\mathcal{R}}\|\|\mathcal{E}_n^{(p)}(y) - y\|_{L_p(\mathcal{R}_n,\widehat{\varphi})}^{\min\{\frac{p}{2},1\}} + \|\widehat{T}^{(2)}(x) - x\|_{L_2(\mathcal{R}_n,\widehat{\varphi})} \to \|T^{(2)}(x) - x\|_{L_2(\mathcal{M},\varphi)}.$$

Hence, taking limits and applying again (3.5.1), we conclude

$$\|T^{(p)}(y) - y\|_p = \lim_{n\to\infty}\|\mathcal{E}_n^{(p)}(\widehat{T}^{(p)}(y)) - \mathcal{E}_n^{(p)}(y)\|_{L_p(\mathcal{R}_n,\widehat{\varphi})} \leq C\|T^{(2)}(x) - x\|_2^\theta\|x\|_2^{1-\theta}.$$

□

Proof of Proposition 3.3.3. The proof is essentially a matter of adapting [CPPR15, Proof of Claim B] using results and remarks from [CJKM23, Proposition 3.9], [CPPR15, Section 8], and applying Lemma 3.5.2. We indicate only the changes to [CPPR15, Proof of Claim B]. The statement we have to prove is precisely [CPPR15, Equation (9)], but without the u_j (this is just a different choice based on convenience). The T_ζ constructed in [CPPR15, Proof of Claim B] is a φ-preserving ucp map that commutes with the modular automorphism group; one can see this from (2.6.3). Hence, we can apply Lemma 3.5.2 on T_ζ and h_V to show [CPPR15, Equation (10)] (but without the u_j). Then, setting $z_j = h_V^{2/q}$, the rest of the proof is the same. □

4

LOWER BOUNDS FOR CERTAIN BILINEAR MULTIPLIERS

This chapter is based on (part of) the following article:

1. **Martijn Caspers, Amudhan Krishnaswamy-Usha, Gerrit Vos**, *Multilinear transference of Fourier and Schur multipliers acting on non-commutative L_p-spaces*, Canadian Journal of Mathematics, **75**(6):1986-2006 (2023).

In this short chapter we prove a result about non-boundedness of the bilinear Hilbert transform and Calderon-Zygmund operators based on our multilinear transference techniques. This chapter is mostly based on the final section of [CKV23]. The results from Lemma 4.1.3 and Proposition 4.1.4 are new; they were obtained after the publication of [CKV23] but not published anywhere.

We consider the case of vector valued bilinear Fourier multipliers on $\mathbb{R}$. Lacey and Thiele have shown in [LT99] that the bilinear Hilbert transform is bounded from $L_{p_1}(\mathbb{R}) \times L_{p_2}(\mathbb{R}) \to L_p(\mathbb{R})$, when $\frac{2}{3} < p < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. The vector valued bilinear Hilbert transform is bounded as a map from

$$L_{p_1}(\mathbb{R}, S_{q_1}) \times L_{p_2}(\mathbb{R}, S_{q_2}) \to L_p(\mathbb{R}, S_q)$$

whenever $1 < \frac{1}{\max\{q,q'\}} + \frac{1}{\max\{q_1,q_1'\}} + \frac{1}{\max\{q_2,q_2'\}}$, as shown by Amenta and Uraltsev in [AU20] and Di Plinio, Li, Martikainen and Vourinen in [DLMV22]. In particular, this class does not include Hölder combinations of q_i. We show that this result does not extend to the case when $p_i = q_i, p = q = 1$, using a transference method similar to the ones used in earlier sections. To be precise, we prove the following result (Theorem 4.1.2).

Theorem C. *Let $1 < p_1, p_2 < \infty$ be such that $\frac{1}{p_1} + \frac{1}{p_2} = 1$ and set $h(s,t) = \chi_{\geq 0}(s-t)$. There exists an absolute constant $C > 0$ such that for every $m \in \mathbb{N}_{\geq 1}$ we have*

$$\| T_h^{(m)} : L_{p_1}(\mathbb{R}, S_{p_1}^m) \times L_{p_2}(\mathbb{R}, S_{p_2}^m) \to L_1(\mathbb{R}, S_1^m) \| > C \log(m).$$

Additionally, we show a similar result for Calderón-Zygmund operators. Here Grafakos and Torres [GT02] have shown that for a class of Calderón-Zygmund operators we have boundedness $L_1 \times L_1 \to L_{\frac{1}{2},\infty}$ in the Euclidean case. Later, a vector valued extension was obtained in [DLMV20]. Here for a class of Calderón-Zygmund operators the boundedness of the vector valued map was obtained for $L_{p_1} \times L_{p_2} \to L_p$ with $1 < p, p_1, p_2 < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Theorem 4.2.1 shows that the latter result cannot be extended to the case when $p = 1$.

The structure of the chapter is straightforward: in Section 4.1 we prove result on the bilinear Hilbert transform, and state some other (unpublished) results in this context. In Section 4.2, we prove the Calderón-Zygmund result.

4

4.1. LOWER BOUNDS FOR THE VECTOR VALUED BILINEAR HILBERT TRANSFORM

For $0 < p < \infty$ recall that $S_p^m = S_p(\mathbb{C}^m)$ is the Schatten L_p-space associated with linear operators on $\mathbb{C}^m$. For $0 < p < 1$ we have that S_p^m is a quasi-Banach space satisfying the quasi-triangle inequality:

$$\|x + y\|_p \leq 2^{\frac{1}{p}-1}(\|x\|_p + \|y\|_p), \qquad x, y \in S_p^m.$$

We set

$$h(\xi_1,\xi_2) = \chi_{\geq 0}(\xi_1 - \xi_2), \qquad \xi_1,\xi_2 \in \mathbb{R},$$

where we take the convention that the indicator function satisfies $\chi_{\geq 0}(0) = \frac{1}{2}$. The first statement of the following theorem is the main result of [LT99] and the latter statement of this theorem for $1 < p < \infty$ was proved in [AU20] and [DLMV22].

Theorem 4.1.1. *For every $1 < q_1, q_2, q, p_1, p_2 < \infty$, $\frac{2}{3} < p < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $m \in \mathbb{N}_{\geq 1}$ there exists a bounded linear map*

$$T_h^{(m)} : L_{p_1}(\mathbb{R}, S_{q_1}^m) \times L_{p_2}(\mathbb{R}, S_{q_2}^m) \to L_p(\mathbb{R}, S_q^m) \tag{4.1.1}$$

that is determined by

$$T_h^{(m)}(f_1, f_2)(s) = \int_{\mathbb{R}}\int_{\mathbb{R}} \widehat{f_1}(\xi_1)\widehat{f_2}(\xi_2)h(\xi_1,\xi_2)e^{is(\xi_1+\xi_2)}d\xi_1 d\xi_2,$$

where $s \in \mathbb{R}$ and f_i, $i = 1,2$ are functions in $L_{p_i}(\mathbb{R}, S_{q_i}^m)$ whose Fourier transforms $\widehat{f_i}$ are continuous compactly supported functions $\mathbb{R} \to S_{q_i}^m$. If $1 < p := (\frac{1}{p_1} + \frac{1}{p_2})^{-1} < \infty$ and $\frac{1}{\max\{q,q'\}} + \frac{1}{\max\{q_1,q_1'\}} + \frac{1}{\max\{q_2,q_2'\}} > 1$ we have that this operator is moreover uniformly bounded in m.

Note that the map $T_h^{(m)}$ as defined above coincides with the multiplicative amplification of the map $T_h := T_h^{(1)}$ as defined in Section 2.5.3, so this notation is consistent.

Our aim is to show that the results of [AU20] and [DLMV22] cannot be extended to the case $p_i = q_i$, $q = p = 1 = \frac{1}{p_1} + \frac{1}{p_2}$; i.e. the bound of (4.1.1) is not uniform in m. In particular we show that the bound can be estimated from below by $C\log(m)$ for some constant C independent of m.

For a function $\phi:\mathbb{R}^2\to\mathbb{C}$ we recall the definition

$$\widetilde{\phi}(\lambda_0,\lambda_1,\lambda_2)=\phi(\lambda_0-\lambda_1,\lambda_1-\lambda_2),\qquad \lambda_i\in\mathbb{R}.$$

Theorem 4.1.2. *Let* $1<p_1,p_2<\infty$ *be such that* $\frac{1}{p_1}+\frac{1}{p_2}=1$*. There exists an absolute constant* $C>0$ *such that for every* $m\in\mathbb{N}_{\geq 1}$ *we have*

$$A_{p_1,p_2,m}:=\|T_h^{(m)}:L_{p_1}(\mathbb{R},S_{p_1}^m)\times L_{p_2}(\mathbb{R},S_{p_2}^m)\to L_1(\mathbb{R},S_1^m)\|>C\log(m).$$

Proof. In the proof let $\mathbb{Z}_m=[-m,m]\cap\mathbb{Z}$. We may naturally identify $S_p(\ell_2(\mathbb{Z}_m))$ with S_p^{2m+1}. Let $\varphi\in C_c(\mathbb{R}),\varphi\geq 0$ be such that $\varphi(t)=\varphi(-t),t\in\mathbb{R}$, $\|\varphi\|_{L_1(G)}=1$ and its support is contained in $[-\frac{1}{2},\frac{1}{2}]$. Set for $s_1,s_2\in\mathbb{R}$,

$$H(s_1,s_2)=\int_{\mathbb{R}}h(s_1+t,-t+s_2)\varphi(t)dt.$$

Then H is continuous and H equals h on $\mathbb{Z}\times\mathbb{Z}$. As a consequence of Lemma 3.2.6 (or a multiplicatively bounded version of [CJKM23, Lemma 4.3]) we find

$$\|T_H^{(2m+1)}:L_{p_1}(\mathbb{R},S_{p_1}^{2m+1})\times L_{p_2}(\mathbb{R},S_{p_2}^{2m+1})\to L_1(\mathbb{R},S_1^{2m+1})\|\leq A_{p_1,p_2,2m+1}.$$

By the multilinear De Leeuw restriction theorem [CJKM23, Theorem C] applied to the subgroup $M_n(\mathbb{T})\subseteq M_n(\mathbb{R})$ we have

$$\|T_{H|_{\mathbb{Z}\times\mathbb{Z}}}^{(2m+1)}:L_{p_1}(\mathbb{T},S_{p_1}^{2m+1})\times L_{p_2}(\mathbb{T},S_{p_2}^{2m+1})\to L_1(\mathbb{T},S_1^{2m+1})\|\leq A_{p_1,p_2,2m+1}. \tag{4.1.2}$$

Let $\zeta_l(z)=z^l,z\in\mathbb{T},l\in\mathbb{Z}$. Set the unitary $U=\sum_{l=-m}^m p_l\otimes\zeta_l$ and for any $1<p<\infty$ the isometric map

$$\pi_p:S_p^{2m+1}\to S_p^{2m+1}\otimes L_p(\mathbb{T}):x\mapsto U(x\otimes 1)U^*.$$

Then, similarly to the proof of Proposition 2.6.21,

$$T_{H|_{\mathbb{Z}\times\mathbb{Z}}}^{(2m+1)}\circ(\pi_{p_1}\times\pi_{p_2})=\pi_p\circ M_{\widetilde{H}|_{\mathbb{Z}_m^3}}.$$

This together with (4.1.2) implies that

$$\|M_{\widetilde{H}|_{\mathbb{Z}_m^3}}:S_{p_1}^{2m+1}\times S_{p_2}^{2m+1}\to S_1^{2m+1}\|\leq A_{p_1,p_2,2m+1}. \tag{4.1.3}$$

Now set $H_j(s,t)=\widetilde{H}|_{\mathbb{Z}\times\mathbb{Z}}(s,j,t)$, $s,t\in\mathbb{Z}$. Note that

$$H_j(s,t)=\chi_{\geq 0}(s+t-2j).$$

By [PSST17, Theorem 2.3] we find that

$$\max_{-m\leq j\leq m}\|M_{H_j}:S_1^{2m+1}\to S_1^{2m+1}\|\leq\|M_{\widetilde{H}|_{\mathbb{Z}_m^3}}^{(m)}:S_{p_1}^{2m+1}\times S_{p_2}^{2m+1}\to S_1^{2m+1}\|. \tag{4.1.4}$$

For $j=0$ we have that M_{H_j} is the triangular truncation map and therefore by [Dav88, Proof of Lemma 10] (apply M_{H_0} to the matrix consisting of only 1's) there is a constant $C>0$ such that

$$C\log(2m+1) \le \|M_{H_0} : S_1^{2m+1} \to S_1^{2m+1}\|. \tag{4.1.5}$$

Combining (4.1.3), (4.1.4), (4.1.5) yields the result for $2m+1$. Since the norm of $T_h^{(m)}$ is increasing in m the result for even m also follows. □

The last part of the previous proof deals with finding a lower bound for a bilinear Schur multiplier. Let us state some other results in this context, where $p_1=\infty$ or $p_2=\infty$. These cannot be used directly to deduce lower bounds for multipliers as in Theorem 4.1.1, since the restriction theorem from [CJKM23, Theorem C] does not hold for $p_i=\infty$. We first prove the following lemma.

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Lemma 4.1.3. *Let $\phi\in\ell_\infty(\mathbb{Z}^3)$ and let $X\subseteq\mathbb{Z}$ finite with $|X|=m$. Set $\phi_X := \phi|_{X^3}$. Then for any $m\ge 1$,*

$$\|M_{\phi_X} : M_m\times M_m \to S_p^m\| \ge \sup_{k\in X}\|M_{\phi_X(\cdot,\cdot,k)} : S_1^m \to S_1^m\|.$$

Proof. Let $k\in X$ arbitrary and let $e_{k,k}$ be the corresponding matrix unit. The main trick is the observation

$$S_p^m e_{k,k} = S_2^m e_{k,k}, \qquad 1\le p\le\infty$$

Indeed, for $x=(x_i)_{i\in X}\in S_p^m e_{k,k}$,

$$x^*x = \left(\sum_{i\in K} x_i^2\right) e_{k,k}$$

and hence

$$\|x\|_p = \left(\sum_{i\in X} x_i^2\right)^{1/2} = \|x\|_2, \qquad 1\le p\le\infty.$$

We observe that M_{ϕ_X} restricts to a map $M_m\times M_m e_{k,k}\to S_p^m e_{k,k}$. Hence

$$\begin{aligned}
\|M_{\phi_X} : M_m\times M_m \to S_p^m\| &\ge \sup_{\substack{x\in M_m, y\in M_m e_{k,k},\\ \|x\|_\infty=\|y\|_\infty=1}} \|M_{\phi_X}(x,y)\|_p \\
&= \sup_{\substack{x\in M_m, y\in S_2^m e_{k,k},\\ \|x\|_\infty=\|y\|_2=1}} \|M_{\phi_X}(x,y)\|_2 \\
&= \sup_{\substack{x\in M_m, y\in S_2^m e_{k,k},\\ \|x\|_\infty=\|y\|_2=1}} \left\|\left(\sum_{j\in X}\phi(i,j,k)x_{ij}y_j\right)\right\|_2 \\
&= \sup_{x\in M_m, \|x\|_\infty=1} \|(\phi(i,j,k)x_{ij})_{i,j}\|_\infty \\
&= \|S_{\phi_X(\cdot,\cdot,k)} : M_n\to M_n\|.
\end{aligned}$$

By duality (see Lemma 2.6.13, although the matrix case is simpler), the last expression is equal to $\|S_{m(\cdot,\cdot,k)} : S_1^n\to S_1^n\|$. This proves the lemma. □

Proposition 4.1.4. *Let $\phi(i,j,k) = \chi_{\geq 0}(i+k-2j)$. For any $1 \leq p \leq \infty$, there exists an absolute constant $C > 0$ such that for every $m \in \mathbb{N}_{\geq 1}$ we have*

$$\| M_{\phi|_{\mathbb{Z}_m^3}} : M_{2m+1} \times S_p^{2m+1} \to S_p^{2m+1} \| > C \log(2m+1).$$

Proof. Set again $\phi_j(i,k) = \phi|_{\mathbb{Z}_m^3}(i,j,k)$. By Lemma 4.1.3 we have

$$\| M_{\phi|_{\mathbb{Z}_m^3}} : M_{2m+1} \times M_{2m+1} \to S_p^{2m+1} \| \geq \| M_{\phi_0} : S_1^{2m+1} \to S_1^{2m+1} \|.$$

Observe that M_{ϕ_0} is again the triangular truncation map. Hence, as we already saw in the proof of Theorem 4.1.2,

$$\| M_{\phi_0} : S_1^{2m+1} \to S_1^{2m+1} \| \geq C \log(2m+1).$$

Since $\| \cdot \|_{S_p^n} \leq \| \cdot \|_{S_q^n}$ for $1 \leq q \leq p \leq \infty$ (which is easily seen using the singular value formula for the norms), we get the result. □

4

4.2. LOWER BOUNDS FOR CALDERÓN-ZYGMUND OPERATORS

The aim of this section is to show a result similar to Theorem 4.1.2 for Calderón-Zygmund operators by considering an example. This shows that the results from [DLMV20] cannot be extended to the case where the range space is $p = 1$. This is in contrast with the commutative situation where Grafakos-Torres [GT02] have shown boundedness of a class of Calderón-Zygmund operators with natural size and smoothness conditions as maps $L_p \times \ldots \times L_p \to L_{p/n}$ for $p \in (1, \infty)$.

Consider any symbol ϕ that is smooth on $\mathbb{R}^2 \setminus \{0\}$, homogeneous and which is determined on one of the quadrants by

$$\phi(s,t) = \frac{s}{s-t}, \qquad s \in \mathbb{R}_{>0}, t \in \mathbb{R}_{<0}. \tag{4.2.1}$$

Here homogeneous means that $\phi(\lambda s, \lambda t) = \phi(s,t), s,t \in \mathbb{R}, \lambda > 0$. We assume moreover that ϕ is regulated at 0, by which we mean that

$$\phi(0) = \pi^{-1} r^{-2} \int_{\|(t_1,t_2)\|_2 \leq r} \phi(t_1,t_2) dt_1 dt_2, \qquad r > 0.$$

As ϕ is homogeneous this expression is independent of r. This type of symbol ϕ is important as it occurs naturally in the analysis of divided difference functions; for instance it plays a crucial role in [CSZ21].

Theorem 4.2.1. *Let $1 < p_1, p_2 < \infty$ be such that $\frac{1}{p_1} + \frac{1}{p_2} = 1$. There exists an absolute constant $C > 0$ such that*

$$B_{p_1,p_2,m} := \| T_\phi^{(m)} : L_{p_1}(\mathbb{R}, S_{p_1}^m) \times L_{p_2}(\mathbb{R}, S_{p_2}^m) \to L_1(\mathbb{R}, S_1^m) \| > C \log(m).$$

Proof. By [Dav88, Lemma 10] (and the proof of [Dav88, Corollary 11]) there exist constants $0 = \lambda_0 < \lambda_1 < \ldots < \lambda_m$ such that the function

$$\psi(i,j) = \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}, \qquad 1 \leq i,j \leq m,$$

is the symbol of a linear Schur multiplier $M_\psi : S_1^m \to S_1^m$ whose norm is at least $C\log(m)$ for some absolute constant $C > 0$. Without loss of generality we may assume that $\lambda_i \in K_m^{-1}\mathbb{Z}$ for some $K_m \in \mathbb{N}_{\geq 1}$ by an approximation argument. Then in this proof let $\Lambda_m = \{\lambda_0, \lambda_1, \ldots, \lambda_m\}$. We may naturally identify S_p^{m+1} with $S_p(\ell_2(\Lambda_m))$ by identifying $E_{i,j}$ with E_{λ_i,λ_j}. We proceed as in the proof of Theorem 4.1.2.

For $\lambda \in K_m^{-1}\mathbb{Z}$ let p_λ be the orthogonal projection of $\ell_2(K_m^{-1}\mathbb{Z})$ onto $\mathbb{C}\delta_\lambda$. Further for $\lambda \in K_m^{-1}\mathbb{Z}$ set $\zeta_\lambda : \mathbb{T} \to \mathbb{C}$ by $\zeta_\lambda(z) = z^{K_m\lambda}, \theta \in \mathbb{R}$. This way every $z \in \mathbb{T}$ determines a representation $\lambda \mapsto \zeta_\lambda(z)$ of $K_m^{-1}\mathbb{Z}$ and this identifies $\mathbb{T}$ with the Pontrjagin dual of $K_m^{-1}\mathbb{Z}$. Set the unitary $U = \sum_{\lambda\in\Lambda_m} p_\lambda \otimes \zeta_\lambda$ and for any $1 < p < \infty$ the isometric map

$$\pi_p : S_p^{m+1} \to S_p^{m+1} \otimes L_p(\mathbb{T}) : x \mapsto U(x\otimes 1)U^*.$$

4

For $r > 0$, consider the function

$$\phi_r(s_1, s_2) = \frac{1}{\pi r^2}\int_{\|(s_1-t_1,s_2-t_2)\|_2\leq r} \phi(t_1,t_2)\,dt_1\,dt_2.$$

This function is continuous and bounded, and hence we may apply the bilinear De Leeuw restriction theorem [CJKM23, Theorem C] to get

$$\begin{aligned} &\|T^{(m+1)}_{\phi_r|_{(K_m^{-1}\mathbb{Z})^2}} : L_{p_1}(\mathbb{T}, S_{p_1}^{m+1}) \times L_{p_2}(\mathbb{T}, S_{p_2}^{m+1}) \to L_1(\mathbb{T}, S_1^{m+1})\| \\ &\quad\leq \|T^{(m+1)}_{\phi_r} : L_{p_1}(\mathbb{R}, S_{p_1}^{m+1}) \times L_{p_2}(\mathbb{R}, S_{p_2}^{m+1}) \to L_1(\mathbb{R}, S_1^{m+1})\|. \end{aligned} \tag{4.2.2}$$

Since $\phi_r|_{(K_m^{-1}\mathbb{Z})^2}$ converges to $\phi|_{(K_m^{-1}\mathbb{Z})^2}$ pointwise, we obtain (by considering the action of the multiplier on functions with finite frequency support),

$$\begin{aligned} &\lim_{r\searrow 0}\|T^{(m+1)}_{\phi_r|_{(K_m^{-1}\mathbb{Z})^2}} : L_{p_1}(\mathbb{T}, S_{p_1}^{m+1}) \times L_{p_2}(\mathbb{T}, S_{p_2}^{m+1}) \to L_1(\mathbb{T}, S_1^{m+1})\| \\ &\quad= \|T^{(m+1)}_{\phi|_{(K_m^{-1}\mathbb{Z})^2}} : L_{p_1}(\mathbb{T}, S_{p_1}^{m+1}) \times L_{p_2}(\mathbb{T}, S_{p_2}^{m+1}) \to L_1(\mathbb{T}, S_1^{m+1})\|. \end{aligned} \tag{4.2.3}$$

Further, viewing ϕ_r as a convolution of ϕ with an $L_1(\mathbb{R}^2)$ function, from Lemma 3.2.6 (or [CJKM23, Lemma 4.3]),

$$\begin{aligned} &\|T^{(m+1)}_{\phi_r} : L_{p_1}(\mathbb{R}, S_{p_1}^{m+1}) \times L_{p_2}(\mathbb{R}, S_{p_2}^{m+1}) \to L_1(\mathbb{R}, S_1^{m+1})\| \\ &\quad\leq \|T^{(m+1)}_{\phi} : L_{p_1}(\mathbb{R}, S_{p_1}^{m+1}) \times L_{p_2}(\mathbb{R}, S_{p_2}^{m+1}) \to L_1(\mathbb{R}, S_1^{m+1})\| = B_{p_1,p_2,m}. \end{aligned} \tag{4.2.4}$$

Combining the estimates (4.2.3), (4.2.2), (4.2.4) we find that

$$\|T^{(m+1)}_{\phi|_{(K_m^{-1}\mathbb{Z})^2}} : L_{p_1}(\mathbb{T}, S_{p_1}^{m+1}) \times L_{p_2}(\mathbb{T}, S_{p_2}^{m+1}) \to L_1(\mathbb{T}, S_1^{m+1})\| \leq B_{p_1,p_2,m}. \tag{4.2.5}$$

We view $\widetilde{\phi}|_{\Lambda_m\times\Lambda_m\times\Lambda_m}$ as the symbol of a Schur multiplier $S_{p_1}^{m+1} \times S_{p_2}^{m+1} \to S_1^{m+1}$. Then,

$$T^{(m+1)}_{\phi|_{K_m^{-1}\mathbb{Z}}} \circ (\pi_{p_1} \times \pi_{p_2}) = \pi_p \circ M_{\widetilde{\phi}|_{\Lambda_m\times\Lambda_m\times\Lambda_m}}.$$

It follows with (4.2.5) that

$$\begin{aligned}&\| M_{\widetilde{\phi}|_{\Lambda_m\times\Lambda_m\times\Lambda_m}} : S_{p_1}^{m+1}\times S_{p_2}^{m+1}\to S_1^{m+1}\| \\ \leq &\| T^{(m+1)}_{\phi|_{K_m^{-1}\mathbb{Z}}} : L_{p_1}(\mathbb{T},S_{p_1}^{m+1})\times L_{p_2}(\mathbb{T},S_{p_2}^{m+1})\to L_1(\mathbb{T},S_p^{m+1})\| \leq B_{p_1,p_2,m+1}.\end{aligned} \tag{4.2.6}$$

By [PSST17, Theorem 2.3] we find that

$$\| M_{\widetilde{\phi}|_{\Lambda_m\times\Lambda_m\times\Lambda_m}(\cdot,0\cdot)} : S_1^{m+1}\to S_1^{m+1}\| \leq \| M_{\widetilde{\phi}|_{\Lambda_m\times\Lambda_m\times\Lambda_m}} : S_{p_1}^{m+1}\times S_{p_2}^{m+1}\to S_p^{m+1}\|. \tag{4.2.7}$$

Now for $s,t\in\mathbb{R}_{>0}$ we find

$$\widetilde{\phi}(s,0,t)=\phi(s-0,0-t)=\frac{s}{s+t}=\frac{1}{2}(1+\frac{s-t}{s+t})=\frac{1}{2}(1+\psi(s,t)).$$

It follows therefore by the first paragraph that for some constant $C>0$,

$$C\log(m)\leq\| M_{\widetilde{\phi}|_{\Lambda_m\times\Lambda_m\times\Lambda_m}(\cdot,0\cdot)} : S_1^{m+1}\to S_1^{m+1}\|.$$

The combination of the latter estimate with (4.2.6) yields the result. □

Remark 4.2.2. In [GT02] it is shown that for a natural class of Calderón-Zygmund operators, the associated convolution operator is bounded as a map $L_1\times L_1\to L_{\frac{1}{2},\infty}$ as well as $L_{p_1}\times L_{p_2}\to L_p$ with $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$ and $\frac{1}{2}<p<\infty, 1<p_1,p_2<\infty$. This applies in particular to the map T_ϕ with symbol ϕ as in (4.2.1), see [GT02, Proposition 6]. Our example shows that this result does not extend to the vector-valued setting in case $\frac{1}{2}<p\leq 1$. On the other hand, affirmative results in case $1<p<\infty$, and $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$ were obtained in [DLMV20]. The question remains open whether a weak L_1-bound $L_{p_1}\times L_{p_2}\to L_{1,\infty}$, $\frac{1}{p_1}+\frac{1}{p_2}=1$ holds, even in the case $p_1=p_2=2$.

5

NONCOMMUTATIVE BMO SPACES

This chapter is based on (part of) the following article:

1. **Martijn Caspers, Gerrit Vos**, *BMO spaces of σ-finite von Neumann algebras and Fourier-Schur multipliers on $SU_q(2)$*, Studia Mathematica **262**(1):45-91 (2022).

In this chapter, we define semigroup BMO spaces for σ-finite von Neumann algebras and prove duality and interpolation results. The material comes from the first half of [CV22]. We note that compared to the published version, there are some changes in Section 5.4. Namely, we no longer define a predual for BMO but only for the row and column spaces. These changes are outlined in the Erratum [CV].

As this was done in [CV22], we will for this chapter only use the convention that inner products are antilinear in the first component and linear in the second. In the entire chapter $\mathcal{M}$ is a σ-finite von Neumann algebra with faithful normal state φ. We also fix some nfs weight ψ on $\mathcal{M}'$ (the precise choice doesn't matter) and denote $D_\varphi := \frac{d\varphi}{d\psi}$ for the associated spatial derivative and $L_p(\mathcal{M}) := L_p(\mathcal{M}, \psi)$ for the Connes-Hilsum L_p-spaces.

We shall take the approach to BMO from [Mei08], [JM12] as a starting point. It assumes the existence of a Markov semi-group $\Phi = (\Phi_t)_{t\geq 0}$ on a finite (or semi-finite) von Neumann algebra $(\mathcal{M}, \tau)$, see Definition 5.3.1. [JM12] considers various BMO-norms associated with this and its subordinated Poisson semigroup. We only consider the norm $\|\cdot\|_{\mathrm{BMO}_\Phi}$ (or $\|\cdot\|_{\mathrm{BMO}(\Phi)}$ in the notation of [JM12]). For $x \in L_2(\mathcal{M})$ the column BMO-seminorm is then defined as

$$\|x\|^2_{\mathrm{BMO}^c_\Phi} = \sup_{t\geq 0} \|\Phi_t(|x - \Phi_t(x)|^2)\|_\infty, \tag{5.0.1}$$

where the Markov maps Φ_t extend naturally to $L_2(\mathcal{M})$ and $L_1(\mathcal{M})$. Then column space $\mathrm{BMO}^c(\mathcal{M}, \Phi)$ is defined as the space of elements from $L_2(\mathcal{M})$ (minus some degenerate part) where the norm (5.0.1) is finite. Finally, $\mathrm{BMO}(\mathcal{M}, \Phi)$ is defined as the intersection of $\mathrm{BMO}^c(\mathcal{M},\Phi)$ and its adjoint row space.

[JM12] establishes the natural interpolation results between BMO and L_p by making use of Markov dilations and interpolation results for martingale BMO spaces. [Cas19] obtains similar interpolation results in the more general context of σ-finite von Neumann algebras through the Haagerup reduction method [HJX10] and the finite case [JM12]. Both papers do this for several of the various BMO-norms defined in [JM12]. The main advantage of considering the BMO-norm (5.0.1) as opposed to the norm $\|\cdot\|_{\mathrm{bmo}_\Phi}$ is that the Markov dilation is not required to have a.u. continuous path in order to apply complex interpolation.

There is a very subtle but important point that makes a difference between this chapter and [Cas19]. In [Cas19] BMO is defined by only considering x in $\mathcal{M}$ and then taking an abstract completion with respect to the norm (5.0.1) (or one of the other BMO-norms). This 'smaller BMO space' has the benefit that basic properties like the triangle inequality and completeness follow rather easily. Here we stay closer to the 'larger BMO space' of L_2-elements with finite BMO-norm as defined above, and show that these basic properties still hold. We do this by proving a Fefferman-Stein duality result.

5

In this chapter, we study abstract BMO spaces of σ-finite von Neumann algebras. Instead of a direct H^1-BMO duality theorem, we will prove such a duality only for the column and row BMO spaces, which suffices for our purposes. This is Theorem 5.4.1.

Theorem D. *There exist Banach spaces $h_1^r(\mathcal{M},\Phi)$ and $h_1^c(\mathcal{M},\Phi)$ such that*

$$\mathrm{BMO}^c(\mathcal{M},\Phi) \cong h_1^r(\mathcal{M},\Phi)^*, \qquad \mathrm{BMO}^r(\mathcal{M},\Phi) \cong h_1^c(\mathcal{M},\Phi)^*.$$

The proof parallels the tracial proof in [JMP14]. The main difficulty lies in the fact that L_p spaces beyond tracial von Neumann algebras do not naturally intersect and we must deal with Tomita-Takesaki modular theory. It should be mentioned that the H^1 Hardy spaces we construct here are abstract in nature and the question of whether every (column) BMO space has a natural Hardy space as its predual remains open. We refer to [Mei08] and [JM12, Open problems, p. 741] for details about this question, where it was resolved under additional assumptions on the semi-group.

Within the construction of the preduals we need some L_p-module theory, see [Pas73] and [JS05]. In particular, we need to extend some results to the σ-finite case. We give an introduction to the theory and prove the necessary results in Sections 5.1 and 5.2. The existence of preduals for the column and row BMO space then settles important basic properties of the BMO space itself, namely the triangle inequality and completeness of the normed space.

Finally, we show that the interpolation result of [Cas19] still holds for our larger BMO space and extends [JM12] beyond the tracial case. We refer to Section 6.6 and [JM12], [Cas19] for the definition of a standard Markov dilation. The following is Theorem 5.5.6.

Theorem E. *If Φ is φ-modular and admits a φ-modular standard Markov dilation, then for all $1 \le p < \infty$, $1 < q < \infty$,*

$$[\mathrm{BMO}(\mathcal{M}, \Phi), L_p^\circ(\mathcal{M})]_{1/q} \approx_{pq} L_{pq}^\circ(\mathcal{M}).$$

Here $\approx_{pq}$ means that the Banach spaces are isomorphic and the norm of the isomorphism in both directions can be estimated by an absolute constant times pq.

We note that the modularity assumptions are only needed to carry out the Haagerup reduction method as in [Cas19]. Many natural Markov semi-groups are modular or can be averaged to a modular Markov semi-group in case φ is almost periodic, see [CS15b, Proposition 4.2], [OT15, Theorem 4.15].

Let us describe the structure of the chapter. We start by recalling some L_p-module theory as introduced in [JS05]. In the second section of this chapter, we extend some duality results to the σ-finite case; specifically, the duality relations of the L_p-module corresponding to the GNS modules. In Section 5.3, we introduce Markov semigroups and construct BMO spaces of σ-finite von Neumann algebras. In Section 5.4, we construct preduals for the associated column and row BMO spaces, using the theory of L_p-modules developed in the first two sections. In Section 5.5, we show that the interpolation results from [Cas19, Theorem 4.5] hold again for the current definition of BMO. Finally, in Section 5.6 we describe an operator space structure on a suitable subset of $\mathcal{M}$ with respect to the BMO-norm.

5

5.1. GENERAL THEORY OF L_p-MODULES

Definition 5.1.1. Let $1 \le p \le \infty$. A sesquilinear form $\langle \cdot, \cdot \rangle : X \times X \to L_{p/2}(\mathcal{M})$ on a right $\mathcal{M}$-module X is called an $L_{p/2}$*-valued inner product* if it satisfies for $x, y \in X$ and $a \in \mathcal{M}$:

(i) $\langle x, x \rangle \ge 0$,

(ii) $\langle x, x \rangle = 0 \iff x = 0$,

(iii) $\langle x, y \rangle = \langle y, x \rangle^*$,

(iv) $\langle x, ya \rangle = \langle x, y \rangle a$.

A $L_{p/2}$-valued inner product defines a norm on X given by

$$\|x\| := \|\langle x, x \rangle\|_{p/2}^{1/2}.$$

For $p < \infty$, X is called an L_p $\mathcal{M}$*-module* if it has a $L_{p/2}$-valued inner product and is complete with respect to the above norm. For $p = \infty$, we require that X has a L_∞-valued inner product and is complete in the topology generated by the seminorms

$$x \mapsto \omega(\langle x, x \rangle)^{1/2}, \quad \omega \in \mathcal{M}_*^+.$$

We call this the STOP topology (after [JM12]).

Lemma 5.1.2. *[JS05, Proposition 3.2] For $x, y \in X$ there exists some $T \in \mathcal{M}$ with $\|T\| \leq 1$ such that $\langle x, y\rangle = \langle x, x\rangle^{\frac{1}{2}} T \langle y, y\rangle^{\frac{1}{2}}$. This implies the '$L_p$-module Cauchy Schwarz inequality':*

$$\|\langle x, y\rangle\|_{p/2} \leq \|x\|\|y\|.$$

Remark 5.1.3. The norms defined here are a priori only quasinorms. However, Theorem 5.1.6 will show that they are in fact norms.

An important class of L_p $\mathcal{M}$-modules are the so-called *principal L_p-modules*. Recall the column space $L_p(\mathcal{M}; \ell_2^C(I))$ defined for $1 \leq p < \infty$ as the norm closure of finite sequences $x = (x_\alpha)_{\alpha \in I}$, $x_\alpha \in L_p(\mathcal{M})$, with respect to the norm

$$\|x\|_{L_p(\mathcal{M}; \ell_2^C)} := \|(\sum_{\alpha \in I} |x_\alpha|^2)^{1/2}\|_p.$$

These spaces are isometrically isomorphic to $L_p(\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2(I)))e_{1,1}$, the column subspace of $L_p(\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2(I)))$, via

$$(x_\alpha) \mapsto \begin{pmatrix} x_1 & 0 & \dots \\ x_2 & 0 & \dots \\ \vdots & \vdots & \end{pmatrix}.$$

For $p = \infty$, we take the space of all sequences in $L_\infty(\mathcal{M})$ such that its image under the above map is in $L_\infty(\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2(I)))$. See [PX97] for more details about the above construction.

Now let $1 \leq p \leq \infty$ be fixed, I be some index set and $(q_\alpha)_{\alpha \in I} \in \mathcal{M}$ be a set of projections. Consider the closed subspace

$$X_p = \{(x_\alpha)_{\alpha \in I} : x_\alpha \in q_\alpha L_p(\mathcal{M}), \sum_{\alpha \in I} x_\alpha^* x_\alpha \in L_{p/2}(\mathcal{M})\} \subseteq L_p(\mathcal{M}; \ell_2^C(I)).$$

We define an $L_{p/2}$-valued inner product on X_p by

$$\langle x, y\rangle = \sum_{\alpha \in I} (x_\alpha)^* y_\alpha.$$

We refer to [JS05] for the fact that this is indeed a well-defined $L_{p/2}$-valued inner product. This makes X_p into an L_p $\mathcal{M}$-module. We call X_p a *principal L_p-module* and denote it by $\bigoplus_I q_\alpha L_p(\mathcal{M})$.

Note that we have the isometric isomorphism

$$\bigoplus_I q_\alpha L_p(\mathcal{M}) \cong Q L_p(\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2(I)))e_{1,1}, \quad Q = \begin{pmatrix} q_1 & 0 & \dots \\ 0 & q_2 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}. \tag{5.1.1}$$

This equation combined with the following general lemma (which has nothing to do with L_p-modules) will show that the family of principal L_p-modules $\bigoplus_I q_\alpha L_p(\mathcal{M})$, $1 \leq p \leq \infty$, satisfies the expected duality relations (although the identifications become antilinear).

Lemma 5.1.4. *Let $\mathcal{N}$ be a σ-finite von Neumann algebra and let $P, Q \in \mathcal{N}$ projections. Then for $1 \le p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$ we have the following antilinear isometric isomorphism:*

$$(QL_p(\mathcal{N})P)^* \cong QL_{p'}(\mathcal{N})P.$$

Proof. Let $1 \le p < \infty$. Define $S_p := QL_p(\mathcal{N})P \subseteq L_p(\mathcal{N})$. It follows (see for instance [Con90, Theorem III.10.1]) that $S_p^* \cong L_{p'}(\mathcal{N})/S_p^\perp$, where $S_p^\perp = \{b \in L_{p'}(\mathcal{N}) : \mathrm{Tr}(S_p b) = 0\}$. Hence it suffices to prove $L_{p'}(\mathcal{N})/S_p^\perp \cong QL_{p'}(\mathcal{N})P$.

Let $a \in L_p(\mathcal{N})$, $b \in L_{p'}(\mathcal{N})$. Then $\mathrm{Tr}((QaP)b) = \mathrm{Tr}(a(PbQ))$, hence for $b \in L_{p'}(\mathcal{N})$:

$$b \in S_p^\perp \iff PbQ = 0 \iff Qb^*P = 0.$$

Therefore if we define the surjective map

$$\Psi : L_{p'}(\mathcal{N}) \to QL_{p'}(\mathcal{N})P, \qquad b \mapsto Qb^*P,$$

then $\ker\Psi = S_p^\perp$ and hence the induced map $\Phi : L_{p'}(\mathcal{N})/S_p^\perp \to QL_{p'}(\mathcal{N})P$ is an isomorphism. Ψ is contractive, hence Φ is also contractive. Conversely, for $b \in L_{p'}(\mathcal{N})$, we have

$$P(b - PbQ)Q = PbQ - PbQ = 0,$$

hence $b - PbQ \in S_p^\perp$, or in other words $PbQ \in b + S_p^\perp$. Thus

$$\|Qb^*P\| = \|PbQ\| \ge \|b + S_p^\perp\|.$$

This implies that Φ^{-1} is also contractive, so Φ is an isometric isomorphism. □

Corollary 5.1.5. *Let $(q_\alpha)_{\alpha\in I}$ be some family of projections. Then for $1 \le p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, we have an antilinear isometric identification*

$$\Big(\bigoplus_I q_\alpha L_p(\mathcal{M})\Big)^* \cong \bigoplus_I q_\alpha L_{p'}(\mathcal{M}).$$

The main theorem concerning L_p-modules states that every L_p-module is in fact isometrically isomorphic to a principal L_p-module.

Theorem 5.1.6 (Theorem 3.6 of [JS05])**.** *Let X be a right $L_p\,\mathcal{M}$-module. Then there exists some index set I and projections $(q_\alpha)_{\alpha\in I} \in \mathcal{M}$ such that*

$$X \cong \bigoplus_{\alpha\in I} q_\alpha L_p(\mathcal{M}).$$

The following lemma allows us to transfer the duality results of principal L_p-modules to general families of L_p-modules satisfying certain requirements. The lemma is essentially copied from [JP14, Corollary 1.13] with some adjustments to go from the finite to the σ-finite case. It is in fact slightly more general to circumvent difficulties with finding an embedding $X_\infty \hookrightarrow X_p$.

Lemma 5.1.7. *Let $(X_p)_{1\le p\le\infty}$ be a family of $L_p\,\mathcal{M}$-modules. Assume that there exist maps $I_{q,p}: X_q \to X_p$ ($q<\infty$) and $I_{\infty,p}: A \to X_p$ for some submodule $A\subseteq X_\infty$, that satisfy for $1\le p<r<q\le\infty$:*

i) $I_{q,p}(xa) = I_{q,p}(x)\sigma^{\varphi}_{i(\frac{1}{p}-\frac{1}{q})}(a)$ *for* $x\in X_q$ *(or* $x\in A$ *if* $q=\infty$*),* $a\in\mathcal{T}_\varphi$,

ii) $I_{r,p}\circ I_{q,r} = I_{q,p}$,

iii) $\kappa^{1/2}_{q/2,p/2}(\langle x,y\rangle_{X_q}) = \langle I_{q,p}(x), I_{q,p}(y)\rangle_{X_p}$ *for* $x,y\in X_q$ *(or* $x,y\in A$ *if* $q=\infty$*),*

iv) $I_{\infty,p}(A)$ *is dense in* X_p.

Then there exists a family of projections $(q_\alpha)_{\alpha\in I}\in\mathcal{M}$ such that $X_p\cong\bigoplus_{\alpha\in I} q_\alpha L_p(\mathcal{M})$, $1\le p\le\infty$.

Proof. We give details only for those parts that differ from [JP14, Corollary 1.13]. One shows that the maps $I_{q,p}$ are automatically contractive embeddings. By applying Theorem 5.1.6 (which holds for σ-finite von Neumann algebras) to the $p=\infty$ case we acquire projections (q_α) such that $X_\infty\cong\bigoplus_{\alpha\in I} q_\alpha L_\infty(\mathcal{M})$, say through an isometric isomorphism of L_∞-modules φ_∞. For $1\le p<\infty$, the embeddings $I_{\infty,p}$ allow us to 'transfer' this map to X_p:

$$\varphi_p : I_{\infty,p}(A)\to\bigoplus_{\alpha\in I} q_\alpha L_p(\mathcal{M}),\quad \varphi_p(I_{\infty,p}(x)) = \bigoplus_{\alpha\in I}\kappa^1_p(\varphi_\infty(x)_\alpha) = \bigoplus_{\alpha\in I}\varphi_\infty(x)_\alpha D_\varphi^{1/p}.$$

We show that φ_p preserves inner products; for $x,y\in A$:

$$\begin{aligned}\langle\varphi_p(I_{\infty,p}(x)),\varphi_p(I_{\infty,p}(y))\rangle_{\oplus q_\alpha L_p} &= \sum_\alpha D_\varphi^{1/p}(\varphi_\infty(x)_\alpha)^*\varphi_\infty(x)_\alpha D_\varphi^{1/p}\\ &= \kappa^{1/2}_{p/2}(\langle\varphi_\infty(x),\varphi_\infty(y)\rangle_{\oplus q_\alpha L_\infty}\\ &= \kappa^{1/2}_{p/2}(\langle x,y\rangle_{X_\infty}) = \langle I_{\infty,p}(x), I_{\infty,p}(y)\rangle_{X_p}.\end{aligned}$$

Since $I_{\infty,p}(A)$ is dense in X_p, φ_p extends to an isometric homomorphism on X_p. It turns out to be an isomorphism since we can use a similar argument to construct an inverse. Next we show that φ_p preserves the module structure (this was not an issue in the finite case); for $x\in A$, $a\in\mathcal{T}_\varphi$:

$$\begin{aligned}\varphi_p(I_{\infty,p}(x)a) = \varphi_p(I_{\infty,p}(x\sigma^\varphi_{-\frac{i}{p}}(a))) &= \bigoplus_{\alpha\in I}\varphi_\infty(x\sigma^\varphi_{-\frac{i}{p}}(a))_\alpha D_\varphi^{1/p}\\ = \bigoplus_{\alpha\in I}\varphi_\infty(x)_\alpha\sigma^\varphi_{-\frac{i}{p}}(a)D_\varphi^{1/p} &= \bigoplus_{\alpha\in I}\varphi_\infty(x)_\alpha D_\varphi^{1/p}a = \varphi_p(I_{\infty,p}(x))a.\end{aligned}\tag{5.1.2}$$

Now let $a\in\mathcal{M}$ be arbitrary. By Kaplansky and strong density of $\mathcal{T}_\varphi$ in $\mathcal{M}$, we may choose a bounded net $(a_\lambda)_\lambda$ in $\mathcal{T}_\varphi$ converging to a in the strong topology. Then by Proposition 2.4.30 we have

$$\|I_{\infty,p}(x)(a-a_\lambda)\|_{X_p} = \|(a-a_\lambda)^*\langle I_{\infty,p}(x), I_{\infty,p}(x)\rangle_{X_p}(a-a_\lambda)\|^{1/2}_{p/2}\to 0$$

and similarly $\|\varphi_p(I_{\infty,p}(x))(a-a_\lambda)\|_{\oplus q_\alpha L_p}\to 0$. Since φ_p is continuous it follows that (5.1.2) holds for any $a\in\mathcal{M}$. □

5.2. THE GNS-MODULE

We now describe the GNS-module as introduced by [Pas73], but in the context of von Neumann algebras. Let $\Phi : \mathcal{M} \to \mathcal{M}$ be a completely positive map of von Neumann algebras. We define the L_∞-valued inner product:

$$\langle \sum_i a_i \otimes b_i, \sum_j a'_j \otimes b'_j \rangle_\infty = \sum_{i,j} b_i^* \Phi(a_i^* a'_j) b'_j$$

and set $\mathcal{N}_0$ to be the quotient of $\mathcal{M} \otimes \mathcal{M}$ by the set $\{z \in \mathcal{M} \otimes \mathcal{M} : \langle z, z \rangle = 0\}$.

For $1 \le p < \infty$, we define the $L_{p/2}$-valued inner product by simply taking the inclusion of $\mathcal{M}$ into $L_{p/2}(\mathcal{M})$ (see Remark 2.4.11 for the case $1 \le p < 2$):

$$\langle z, z' \rangle_{p/2} = \kappa_{p/2}^{1/2}\left(\langle z, z' \rangle_\infty\right), \qquad z, z' \in \mathcal{M} \otimes \mathcal{M}. \tag{5.2.1}$$

This $L_{p/2}$-valued inner product gives rise to a norm $\|z\|_{p,\Phi} := \|\langle z, z \rangle_{p/2}\|_{p/2}^{1/2}$ on $\mathcal{N}_0$. We define $L_p(\mathcal{M} \otimes_\Phi \mathcal{M})$ to be the Banach space completion of $\mathcal{N}_0$ with respect to this norm.

Next we define a module structure on $L_p(\mathcal{M} \otimes_\Phi \mathcal{M})$. For $z \in \mathcal{M} \otimes \mathcal{M}$ and $a \in \mathcal{T}_\varphi$, it is given by

$$z \cdot a := z(1_{\mathcal{M}} \otimes \sigma_{-\frac{i}{p}}(a)). \tag{5.2.2}$$

Note that by (2.4.2), this module structure satisfies property (iv) of Definition 5.1.1. By Kaplansky and strong density of $\mathcal{T}_\varphi$ in $\mathcal{M}$, we can approach $a \in \mathcal{M}$ by a bounded net $(a_\lambda)_\lambda \in \mathcal{M}$ converging to a in the strong topology. Setting $b_{\lambda,\mu} = a_\lambda - a_\mu$ and using Proposition 2.4.30, we have

$$\|z \cdot b_{\lambda,\mu}\|_{p,\Phi} = \|\langle z \cdot b_{\lambda,\mu}, z \cdot b_{\lambda,\mu} \rangle_{p/2}\|_{p/2}^{1/2} = \|b_{\lambda,\mu}^* \langle z, z \rangle_{p/2} b_{\lambda,\mu}\|_{p/2}^{1/2} \to 0.$$

Hence we can extend (5.2.2) for elements $a \in \mathcal{M}$, where the right hand side takes values in $L_p(\mathcal{M} \otimes_\Phi \mathcal{M})$. This right action is then strong/$\| \, \|_{p,\Phi}$-continuous on the unit ball of $\mathcal{M}$.

By the L_p-module Cauchy Schwarz inequality, the $L_{p/2}$-valued inner product and the module structure extend to the space $L_p(\mathcal{M} \otimes_\Phi \mathcal{M})$. With this, $L_p(\mathcal{M} \otimes_\Phi \mathcal{M})$ turns into a well-defined L_p $\mathcal{M}$-module.

For $p = \infty$, we define $L_\infty(\mathcal{M} \otimes_\Phi \mathcal{M})$ to be the completion with respect to the STOP topology, i.e. the one generated by the seminorms $z \mapsto \omega(\langle z, z \rangle_\infty)^{1/2}$, $\omega \in \mathcal{M}_*$. $\langle \cdot, \cdot \rangle_\infty$ is continuous in both variables on $\mathcal{M} \otimes \mathcal{M}$ with respect to the STOP topology (and the weak-$*$ topology in the range); one can see this by writing $\langle z, z' \rangle_\infty = \langle z, z \rangle_\infty^{1/2} T \langle z', z' \rangle_\infty^{1/2}$ as in Lemma 5.1.2 and, for $\omega \in \mathcal{M}_*$, using the classical Cauchy Schwarz inequality on the bilinear form $(z, z') \mapsto \omega(\langle z, z' \rangle_\infty)$. Hence $\langle \cdot, \cdot \rangle_\infty$ extends to an $\mathcal{M}$-valued inner product on $L_\infty(\mathcal{M} \otimes_\Phi \mathcal{M})$. The module structure is simply given by $z \cdot a := z(1 \otimes a)$.

Proposition 5.2.1. *There exists a family of projections $(q_\alpha)_{\alpha \in I} \in \mathcal{M}$ such that $L_p(\mathcal{M} \otimes_\Phi \mathcal{M}) \cong \bigoplus_I q_\alpha L_p(\mathcal{M})$, $1 \le p \le \infty$.*

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Proof. To use Lemma 5.1.7, we must construct maps $I_{q,p}$ as in the assumptions of that lemma. The maps will be extensions of the identity map $\iota : \mathcal{M} \otimes \mathcal{M} \to \mathcal{M} \otimes \mathcal{M}$. For $q = \infty$, the space A from the lemma will be $\mathcal{M} \otimes \mathcal{M}$ and $I_{\infty,p}$ is simply the identity $\iota : A \to L_p(\mathcal{M} \otimes_\Phi \mathcal{M})$. For $p \le q < \infty$, the extensions exist because of the following estimate for $z \in \mathcal{M} \otimes \mathcal{M}$:

$$\|z\|_{q,\Phi} = \|\langle z,z\rangle_{q/2}\|_{q/2}^{1/2} = \|\kappa_{\infty,q/2}^{1/2}(\langle z,z\rangle_\infty)\|_{q/2}^{1/2} \ge \|\kappa_{q/2,p/2}^{1/2}(\kappa_{q/2}^{1/2}(\langle z,z\rangle_\infty))\|_{p/2}^{1/2}$$
$$= \|\kappa_{p/2}^{1/2}(\langle z,z\rangle_\infty)\|_{p/2}^{1/2} = \|z\|_{p,\Phi}.$$

It follows that ι extends to a contractive map $I_{q,p} : L_q(\mathcal{M} \otimes_\Phi \mathcal{M}) \to L_p(\mathcal{M} \otimes_\Phi \mathcal{M})$. The properties i)-iv) all follow from the previous constructions. Now we can apply Lemma 5.1.7 to deduce the result. □

Remark 5.2.2. We can deduce in hindsight the existence of the expected embedding

$$L_\infty(\mathcal{M} \otimes_\Phi \mathcal{M}) \hookrightarrow L_p(\mathcal{M} \otimes_\Phi \mathcal{M})$$

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through the identification with principal L_p-modules where the embedding is clear. We will need this observation later. In this case there is a common dense subset so there is no need to keep track of embeddings here; instead, we may 'redefine' the GNS-modules for $1 < p \le \infty$ to be closures within $L_1(\mathcal{M} \otimes_{\Phi_t} \mathcal{M})$ instead of abstract completions, so that $L_q(\mathcal{M} \otimes_{\Phi_t} \mathcal{M}) \subseteq L_p(\mathcal{M} \otimes_{\Phi_t} \mathcal{M})$ for $1 \le p \le q \le \infty$. Then through the identification with principal modules, we see that (5.2.1) also holds for $z, z' \in L_\infty(\mathcal{M} \otimes_{\Phi_t} \mathcal{M})$; this was not entirely trivial.

Our next goal is to define duality results on the GNS-modules. To define a dual relation, we need to show that the bracket can be extended to a map taking arguments from different spaces. This follows easily through the identification with principal modules where this extension is evident. In the GNS-picture, the bracket is given by

$$\langle x,y\rangle_{p,q} = D_\varphi^{1/p}\langle x,y\rangle_\infty D_\varphi^{1/q} = \kappa_r^{r/q}(\langle x,y\rangle_\infty) \tag{5.2.3}$$

for $x, y \in \mathcal{M} \otimes \mathcal{M}$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ with $1 \le p,q,r \le \infty$ but p and q not both ∞.

The (antilinear) duality pairing is then defined as follows:

$$(x,y) = \mathrm{Tr}(\langle x,y\rangle_{p,q}), \quad x \in L_p(\mathcal{M} \otimes_\Phi \mathcal{M}),\ y \in L_q(\mathcal{M} \otimes_\Phi \mathcal{M}),\ \frac{1}{p} + \frac{1}{q} = 1. \tag{5.2.4}$$

This duality identifies $L_p(\mathcal{M} \otimes_\Phi \mathcal{M})$ as a subspace of $L_q(\mathcal{M} \otimes_\Phi \mathcal{M})^*$. Using the identification with principal modules, we can show that this inclusion is an (isometric) isomorphism.

Corollary 5.2.3. *For* $1 \le p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, *we have an antilinear isomorphism*

$$(L_p(\mathcal{M} \otimes_\Phi \mathcal{M}))^* \cong L_q(\mathcal{M} \otimes_\Phi \mathcal{M}).$$

Proof. This follows from Proposition 5.2.1 and Corollary 5.1.5. □

Remark 5.2.4. The definition of $\langle\cdot,\cdot\rangle_{p,p}$ coincides with that of $\langle\cdot,\cdot\rangle_{p/2}$. Both notations make sense; the first refers to the inputs, the second to the output (and it corresponds to the term $L_{p/2}$-valued inner product). We will mostly be using the latter notation.

Remark 5.2.5. Due to the tracial property, the embedding we choose to define the duality bracket does not matter. In particular, if $x \in L_1(\mathcal{M} \otimes_\Phi \mathcal{M}) \cap L_2(\mathcal{M} \otimes_\Phi \mathcal{M})$ and $y \in L_\infty(\mathcal{M} \otimes_\Phi \mathcal{M}) \cap L_2(\mathcal{M} \otimes_\Phi \mathcal{M})$ then

$$\mathrm{Tr}(\langle x, y\rangle_1) = \mathrm{Tr}(\langle x, y\rangle_{1,\infty})$$

In the next lemma we check that the inner product behaves as expected when we use, informally speaking, elements from $L_p(\mathcal{M})$ in the first tensor leg as inputs. For this last lemma, we presume that Φ satisfies the conditions of Proposition 2.4.28 so that $\Phi^{(p/2)}$ exists.

Lemma 5.2.6. *Let $1 \le p < \infty$, and let Φ be a unital completely positive (ucp) φ-preserving map such that $\Phi \circ \sigma_t^\varphi = \sigma_t^\varphi \circ \Phi$ for all $t \in \mathbb{R}$. The map*

$$\Psi_p : \kappa_p^1(\mathcal{M}) \to L_p(\mathcal{M} \otimes_\Phi \mathcal{M}), \quad \kappa_p^1(x) \mapsto x \otimes 1$$

extends to a contractive mapping $\Psi_p : L_p(\mathcal{M}) \to L_p(\mathcal{M} \otimes_\Phi \mathcal{M})$. For $x, y \in L_p(\mathcal{M})$, $z = \sum_j a_j \otimes b_j \in \mathcal{M} \otimes \mathcal{M}$, it satisfies

$$\langle \Psi_p(x), \Psi_p(y)\rangle_{p/2} = \Phi^{(p/2)}(x^* y), \qquad 2 \le p < \infty,$$
$$\langle \Psi_p(x), z\rangle_{p/2} = \sum_j \Phi^{(p)}(x^* a_j) b_j D_\varphi^{1/p}, \quad 1 \le p < \infty.$$

Proof. We first note the following identity for $x, y \in \mathcal{M}$:

$$\langle x \otimes 1, y \otimes 1\rangle_{p/2} = \kappa_{p/2}^{1/2}(\Phi(x^* y)) = \Phi^{(p/2)}(\kappa_{p/2}^{1/2}(x^* y)) \overset{(2.4.11)}{=} \Phi^{(p/2)}(\kappa_p^1(x)^* \kappa_p^1(y)) \tag{5.2.5}$$

Hence, by the generalised Hölder inequality

$$\begin{aligned}\|x \otimes 1\|_{p,\Phi} &= \|\Phi^{(p/2)}(\kappa_p^1(x)^* \kappa_p^1(x))\|_{p/2}^{1/2} \le \|\kappa_p^1(x)^* \kappa_p^1(x)\|_{p/2}^{1/2} \\ &\le \|\kappa_p^1(x)^*\|_p^{1/2} \|\kappa_p^1(x)\|_p^{1/2} = \|\kappa_p^1(x)\|_p.\end{aligned}$$

This shows that Ψ_p is contractive on $\kappa_p^1(\mathcal{M})$ and hence extends to a contractive mapping on $L_p(\mathcal{M})$.

Now let $x, y \in L_p(\mathcal{M})$ and take $(x_n), (y_n) \in \mathcal{M}$ such that $\kappa_p^1(x_n) \to_p x$ and $\kappa_p^1(y_n) \to_p y$. From Minkowski's inequality and the generalised Hölder inequality it follows that

$$\kappa_p^1(x_n)^* \kappa_p^1(y_n) \to_{p/2} x^* y.$$

Hence by (5.2.5) and continuity of $\Phi^{(p/2)}$:

$$\langle \Psi_p(x), \Psi_p(y)\rangle_{p/2} = \lim_{n\to\infty} \langle x_n \otimes 1, y_n \otimes 1\rangle_{p/2} = \lim_{n\to\infty} \Phi^{(p/2)}(\kappa_p^1(x_n)^* \kappa_p^1(y_n)) = \Phi^{(p/2)}(x^* y).$$

The final equality is proved with a very similar method and is left to the reader. □

5.3. MARKOV SEMIGROUPS AND BMO SPACES

Definition 5.3.1. A (GNS-symmetric) Markov semigroup is a semigroup $(\Phi_t)_{t\geq 0}$ of linear maps $\mathcal{M} \to \mathcal{M}$ satisfying the following conditions:

i) Φ_t is normal ucp, $t \geq 0$,

ii) $\varphi(\Phi_t(x)y) = \varphi(x\Phi_t(y))$, $x, y \in \mathcal{M}$, $t \geq 0$ (GNS-symmetry)

iii) The mapping $t \mapsto \Phi_t(x)$ is strongly continuous, $x \in \mathcal{M}$.

The Markov semigroup is called φ-modular if $\Phi_t \circ \sigma_s^{\varphi} = \sigma_s^{\varphi} \circ \Phi_t$ for all $s \in \mathbb{R}$, $t \geq 0$.

Note that by condition ii), $\varphi(\Phi_t(x)) = \varphi(x)$; in particular, the Φ_t are faithful. If $\Phi := (\Phi_t)_{t\geq 0}$ is a φ-modular Markov semigroup, then by Proposition 2.4.28 there are extensions $\Phi_t^{(p)} : L_p(\mathcal{M}) \to L_p(\mathcal{M})$, where $\Phi_t^{(1)}$ is trace-preserving. Note that condition ii) implies, after appropriate approximations, that $\Phi_t^{(2)}$ is self-adjoint.

For the rest of this section we assume $\Phi = (\Phi_t)_{t\geq 0}$ to be a φ-modular Markov semigroup. We define closed subspaces of $\mathcal{M}$ and $L_p(\mathcal{M})$ as follows

$$\mathcal{M}^\circ = \{x \in \mathcal{M} \mid \Phi_t(x) \to 0 \ \sigma\text{-weakly as } t \to \infty\},$$
$$L_p^\circ(\mathcal{M}) = \{x \in L_p(\mathcal{M}) \mid \|\Phi_t^{(p)}(x)\|_p \to 0,\ t \to \infty\}.$$

Then [Cas19, Lemma 2.3] assures that the inclusions $\kappa_{q,p}^{\theta}$ restrict to contractive inclusions $L_q^\circ(\mathcal{M}) \to L_p^\circ(\mathcal{M})$ for $q \geq p$.

We record here two short lemmas for later use. We will need the generator A_2 of the semigroup $(\Phi_t^{(2)})_{t\geq 0}$, i.e. the positive self-adjoint unbounded operator such that $e^{-tA_2} = \Phi_t^{(2)}$; the existence is guaranteed by a very special case of the Hille-Yosida theorem and we refer to the papers [Cip97] and [GL95] for a more elaborate analysis of generators of Markovian semi-groups.

Lemma 5.3.2. *For each $x \in \mathcal{M}$, the net $\{\Phi_t(x)\}_{t\geq 0}$ converges σ-strongly as $t \to \infty$.*

Proof. Let $x \in \mathcal{M}$ and write $xD_\varphi^{1/2} = \xi_1 + \xi_2$ for $\xi_1 \in \ker(A_2), \xi_2 \in \ker(A_2)^\perp$. Then

$$\Phi_t(x)D_\varphi^{1/2} = \Phi_t^{(2)}(xD_\varphi^{1/2}) = e^{-tA_2}(\xi_1 + \xi_2) = \xi_1 + e^{-tA_2}\xi_2.$$

It follows by elementary spectral theory for unbounded operators that $e^{-tA_2}\xi_2 \to 0$ as $t \to \infty$. Therefore $\Phi_t(x)D_\varphi^{1/2}$ converges in the L_2-topology, i.e. $\Phi_t(x)$ is Cauchy within $\mathcal{M}$ in the $\|\cdot\|_2$-topology generated by the GNS inner product $\langle x, y\rangle = \varphi(x^*y), x, y \in \mathcal{M}$. Since the Φ_t are contractive, the net $\Phi_t(x)$ is bounded in $\mathcal{M}$. So as the $\|\cdot\|_2$-topology and the strong (and σ-strong) topology coincide on the unit ball, the net $\Phi_t(x)$ converges to an element in $\mathcal{M}$ in the strong (and σ-strong) topology. □

Lemma 5.3.3. *Assume that $x \in L_1^\circ(\mathcal{M})$ is such that* $\mathrm{Tr}(xz) = 0$ *for all $z \in \mathcal{M}^\circ$. Then $x = 0$.*

Proof. Let $y \in \mathcal{M}$ and set the σ-strong (hence σ-weak) limit $P(y) = \lim_{t\to\infty} \Phi_t(y)$, which exists by Lemma 5.3.2. Then $y - P(y) \in \mathcal{M}^\circ$, hence we have

$$\mathrm{Tr}(xy) = \mathrm{Tr}(x(y - P(y))) + \mathrm{Tr}(xP(y)) = \mathrm{Tr}(xP(y)).$$

Now using condition ii) of Definition 5.3.1 and appropriate approximation, we can show that $\mathrm{Tr}(w\Phi_t(z)) = \mathrm{Tr}(\Phi_t^{(1)}(w)z)$ for $w \in L_1(\mathcal{M})$, $z \in \mathcal{M}$. Hence

$$\mathrm{Tr}(xP(y)) = \lim_{t\to\infty} \mathrm{Tr}(x\Phi_t(y)) = \lim_{t\to\infty} \mathrm{Tr}(\Phi_t^{(1)}(x)y) = 0$$

since $x \in L_1^\circ(\mathcal{M})$. As $y \in \mathcal{M}$ was arbitrary, we must have $x = 0$. □

For $x \in \mathcal{M}$ we define the column and row BMO-norm:

$$\|x\|_{\mathrm{BMO}_\Phi^c} = \sup_{t\geq 0} \|\Phi_t(|x-\Phi_t(x)|^2)\|_\infty^{1/2}; \quad \|x\|_{\mathrm{BMO}_\Phi^r} = \|x^*\|_{\mathrm{BMO}_\Phi^c}.$$

The BMO-norm is defined as $\|x\|_{\mathrm{BMO}_\Phi} = \max\{\|x\|_{\mathrm{BMO}_\Phi^c}, \|x\|_{\mathrm{BMO}_\Phi^r}\}$. This defines a seminorm by [JM12, Proposition 2.1].

Since Φ is faithful, we see that for $x \in \mathcal{M}$, $\|x\|_{\mathrm{BMO}_\Phi} = 0$ implies that $x = \Phi_t(x)$ for all $t > 0$. This means that the above seminorms are actually norms on $\mathcal{M}^\circ$.

Next, we turn our attention to defining an analogous BMO-norm on the space $L_2(\mathcal{M})$ such as in [JM12]. This turns out to be more involved in the σ-finite case.

The embedding $\kappa_1^{1/2}$ allows us to define $\|\cdot\|_\infty$ on $L_1(\mathcal{M})$ (it takes values ∞ outside of $\kappa_1^{1/2}(\mathcal{M})$). We will also denote this by $\|\cdot\|_\infty$. Then we can define analogous column and row BMO-(semi)norms on $L_2(\mathcal{M})$ by

$$\|x\|_{\mathrm{BMO}_\Phi^c} = \sup_{t\geq 0} \|\Phi_t^{(1)}(|x-\Phi_t^{(2)}(x)|^2)\|_\infty^{1/2}; \quad \|x\|_{\mathrm{BMO}_\Phi^r} = \|x^*\|_{\mathrm{BMO}_\Phi^c} \tag{5.3.1}$$

We will only show later (at the end of this chapter) that these seminorms satisfy the triangle inequality. As with the corresponding norms on $\mathcal{M}$, these seminorms are norms on $L_2^\circ(\mathcal{M})$. Now we define the column BMO space as

$$\mathrm{BMO}^c(\mathcal{M},\Phi) = \{x \in L_2^\circ(\mathcal{M}) \mid \|x\|_{\mathrm{BMO}_\Phi^c} < \infty\}$$

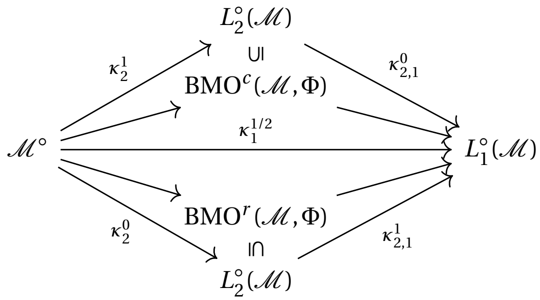
and we define the row BMO space as the adjoint of the column BMO space with norm as in (5.3.1). We emphasize that we have thus constructed a column (resp. row) BMO-norm both on $\mathcal{M}^\circ$ and $L_2^\circ(\mathcal{M})$ which by mild abuse of notation are denoted in the same way. They are identified by the right embedding for the column norm and the left embedding for the row norm:

$$\begin{aligned} \|\kappa_2^1(x)\|_{\mathrm{BMO}_\Phi^c} &= \|xD_\varphi^{1/2}\|_{\mathrm{BMO}_\Phi^c} = \|x\|_{\mathrm{BMO}_\Phi^c}, \\ \|\kappa_2^0(x)\|_{\mathrm{BMO}_\Phi^r} &= \|D_\varphi^{1/2}x\|_{\mathrm{BMO}_\Phi^r} = \|x\|_{\mathrm{BMO}_\Phi^r}, \end{aligned} \tag{5.3.2}$$

where $x \in \mathcal{M}^\circ$. These equalities are straightforward to check. Since clearly $\|x\|_{\mathrm{BMO}_\Phi^c} \leq 4\|x\|_\infty^2$ for $x \in \mathcal{M}^\circ$, it follows that κ_2^1 embeds $\mathcal{M}^\circ$ into $\mathrm{BMO}^c(\mathcal{M},\Phi)$, and similarly κ_2^0 embeds $\mathcal{M}^\circ$ into $\mathrm{BMO}^r(\mathcal{M},\Phi)$.

The first idea for a definition of the BMO-norm would be $\max\{\|x\|_{\mathrm{BMO}^c_\Phi}, \|x\|_{\mathrm{BMO}^r_\Phi}\}$, similarly to the definition on $\mathcal{M}$. However, this is not a suitable definition for the following reason. The equalities (5.3.2) show how the right and left embeddings of $\mathcal{M}$ in $L_2(\mathcal{M})$ preserve the column and row norms respectively. However, there is no embedding of $\mathcal{M}$ into $L_2(\mathcal{M})$ that would preserve the maximum of these norms.

Instead, we embed $\mathrm{BMO}^c(\mathcal{M},\Phi)$ and $\mathrm{BMO}^r(\mathcal{M},\Phi)$ in $L_1^\circ(\mathcal{M})$ through the embeddings $\kappa^0_{2,1}$ and $\kappa^1_{2,1}$ respectively. This turns $(\mathrm{BMO}^c(\mathcal{M},\Phi), \mathrm{BMO}^r(\mathcal{M},\Phi))$ into a compatible couple. The following diagram commutes:



We define

$$\mathrm{BMO}(\mathcal{M},\Phi) = \kappa^0_{2,1}(\mathrm{BMO}^c(\mathcal{M},\Phi)) \cap \kappa^1_{2,1}(\mathrm{BMO}^r(\mathcal{M},\Phi))$$

to be the intersection space, and for $x \in \mathrm{BMO}(\mathcal{M},\Phi)$ we denote by

$$x_c \in \mathrm{BMO}^c(\mathcal{M},\Phi), \qquad x_r \in \mathrm{BMO}^r(\mathcal{M},\Phi)$$

the elements such that $\kappa^0_{2,1}(x_c) = x = \kappa^1_{2,1}(x_r)$. The norm on $\mathrm{BMO}(\mathcal{M},\Phi)$ is defined as

$$\|x\|_{\mathrm{BMO}_\Phi} = \max\{\|x_c\|_{\mathrm{BMO}^c_\Phi}, \|x_r\|_{\mathrm{BMO}^r_\Phi}\}.$$

When no confusion can occur, we omit the reference to the semigroup in the notation of the various BMO-norms and just write, for instance, $\|\cdot\|_{\mathrm{BMO}}$. We check that $\kappa_1^{1/2}$ is indeed an embedding of $\mathcal{M}^\circ$ into $\mathrm{BMO}(\mathcal{M})$ that preserves $\|\cdot\|_{\mathrm{BMO}}$:

$$\begin{aligned}\|\kappa_1^{1/2}(z)\|_{\mathrm{BMO}} &= \max\{\|\kappa_2^1(z)\|_{\mathrm{BMO}^c}, \|\kappa_2^0(z)\|_{\mathrm{BMO}^r}\} \\ &= \max\{\|z\|_{\mathrm{BMO}^c}, \|z\|_{\mathrm{BMO}^r}\} = \|z\|_{\mathrm{BMO}}.\end{aligned}$$

The next estimate shows that $L_1^\circ(\mathcal{M})$ contains the closure of $\kappa_1^{1/2}(\mathcal{M}^\circ)$ with respect to $\|\cdot\|_{\mathrm{BMO}}$, as expected.

Lemma 5.3.4. *For $x \in L_2^\circ(\mathcal{M})$, we have $\|x\|_2 \le \|x\|_{\mathrm{BMO}^c}$ and $\|x\|_2 \le \|x\|_{\mathrm{BMO}^r}$. Hence for $x \in \mathrm{BMO}(\mathcal{M},\Phi)$, we have*

$$\|x\|_{\mathrm{BMO}} \ge \max\{\|x_c\|_2, \|x_r\|_2\} \ge \|x\|_1.$$

Proof. Let $x \in L_2^\circ(\mathcal{M})$. If $\|x\|_{\mathrm{BMO}^c} = \infty$ then the inequality trivially holds. Now assume that $\|x\|_{\mathrm{BMO}^c} < \infty$. Then for all $t \geq 0$ there exists a $y_t \in \mathcal{M}$ such that $\Phi_t^{(1)}|x - \Phi_t^{(2)}(x)|^2 = \kappa_1^{1/2}(y_t)$.

Let $\varepsilon > 0$. Then we can find $t > 0$ such that $\|\Phi_t^{(2)}(x)\|_2 < \varepsilon$. Then since $\Phi_t^{(1)}$ is trace-preserving:

$$\begin{aligned}\|x\|_2 &\leq \|x - \Phi_t^{(2)}(x)\|_2 + \varepsilon = \mathrm{Tr}(|x - \Phi_t^{(2)}(x)|^2)^{1/2} + \varepsilon = \mathrm{Tr}(\Phi_t^{(1)}|x - \Phi_t(x)|^2)^{1/2} + \varepsilon \\ &= \mathrm{Tr}(\kappa_1^{1/2}(y_t))^{1/2} + \varepsilon = \varphi(y_t)^{1/2} + \varepsilon \leq \|y_t\|_\infty^{1/2} + \varepsilon \leq \|x\|_{\mathrm{BMO}^c} + \varepsilon.\end{aligned}$$

Since $\|x\|_2 = \|x^*\|_2$, we also get $\|x\|_2 \leq \|x\|_{\mathrm{BMO}^r}$. The final statement follows from the definition of $\|\cdot\|_{\mathrm{BMO}}$ and contractivity of $\kappa_{2,1}^\theta$. This finishes the proof. □

It is not a priori clear whether $\mathrm{BMO}(\mathcal{M}, \Phi)$ is complete. However, this will follow as a corollary from the result of the next subsection, which provides an 'artificial' predual to $\mathrm{BMO}(\mathcal{M}, \Phi)$.

5.4. A PREDUAL OF BMO

We dedicate this section to proving the following theorem:

Theorem 5.4.1. *There exist Banach spaces $h_1^r(\mathcal{M}, \Phi)$ and $h_1^c(\mathcal{M}, \Phi)$ such that*

$$\mathrm{BMO}^c(\mathcal{M}, \Phi) \cong h_1^r(\mathcal{M}, \Phi)^*, \qquad \mathrm{BMO}^r(\mathcal{M}, \Phi) \cong h_1^c(\mathcal{M}, \Phi)^*.$$

In this part we will suppress the reference to $\mathcal{M}$ and Φ in the notation of $\mathrm{BMO}^c, \mathrm{BMO}^r$ and their preduals h_1^r, h_1^c.

In the finite case a predual for BMO was found in [JM12, Section 5.2.3], see also [JMP14, Appendix A]. Our proof mostly follows the lines of [JMP14], although we will not attempt to define a sum space h_1. Also, our predual of BMO^c will instead be h_1^r and vice versa, which makes the identification in Theorem 5.4.1 linear instead of antilinear.

Proof of Theorem 4.5. Since BMO^r lies within $L_2^\circ(\mathcal{M})$, we have at our disposal an inner product that can provide us with a duality bracket. We take the Hahn-Banach norm relation as the definition of the norm of h_1^c:

$$\|y\|_{h_1^c} = \sup_{\|x\|_{\mathrm{BMO}^r} \leq 1} |\mathrm{Tr}(xy)|, \qquad y \in L_2^\circ(\mathcal{M}).$$

which would be a well-defined norm even if $\|\cdot\|_{\mathrm{BMO}^r}$ wouldn't satisfy the triangle inequality. As we can always find $x \in \mathcal{M}^\circ$ such that $|\mathrm{Tr}(\kappa_2^0(x)y)| > 0$, we have $\|y\|_{h_1^c} > 0$ for $y \neq 0$ (for example take x such that $\kappa_2^0(x)$ is close to y^*).

Now by Lemma 5.3.4:

$$\|y\|_{h_1^c} \leq \sup_{\|x\|_2 \leq 1} |\mathrm{Tr}(xy)| = \|y\|_2.$$

Hence we define h_1^c to be the completion of $L_2^\circ(\mathcal{M})$ with respect to $\|\cdot\|_{h_1^c}$, and we obtain a contractive inclusion $L_2^\circ(\mathcal{M}) \subseteq h_1^c$. We define h_1^r analogously by taking the sup over x with $\|x\|_{\mathrm{BMO}^c} \leq 1$.

We will only show that $\mathrm{BMO}^r \cong (h_1^c)^*$ (the other case follows similarly). It is not hard to show that $\mathrm{BMO}^r \subseteq (h_1^c)^*$ contractively. Conversely, let $\psi \in (h_1^c)^*$. Then $\psi|_{L_2^\circ(\mathcal{M})} \in L_2^\circ(\mathcal{M})^*$ by Lemma 5.3.4. Hence by the Riesz representation theorem there exists an $x_0 \in L_2^\circ(\mathcal{M})$ such that

$$\psi(z) = \mathrm{Tr}(x_0^* z)$$

for all $z \in L_2^\circ(\mathcal{M})$. What remains to be shown is that $x_0^* \in \mathrm{BMO}^r$, with $\|x_0^*\|_{\mathrm{BMO}^r} \leq \|\psi\|_{(h_1^c)^*}$ (the other inequality follows from the definition of h_1^c). This is equivalent to requiring that $x_0 \in \mathrm{BMO}^c$ with $\|x_0\|_{\mathrm{BMO}^c} \leq \|\psi\|_{(h_1^c)^*}$

Fix $t > 0$. We will now use the L_p-modules $L_p(\mathcal{M} \otimes_{\Phi_t} \mathcal{M})$ corresponding to the ucp map Φ_t. Let Ψ_p be the embedding of Lemma 5.2.6. Then we can define the map

$$u_t : L_2^\circ(\mathcal{M}) \to L_2(\mathcal{M} \otimes_{\Phi_t} \mathcal{M}), \quad u_t(y) = \Psi_2(y - \Phi_t^{(2)}(y)).$$

Now it suffices to show that

$$u_t(x_0) \in L_\infty(\mathcal{M} \otimes_{\Phi_t} \mathcal{M}) \text{ and } \|u_t(x_0)\|_{\infty,\Phi_t} \leq \|\psi\|_{(h_1^c)^*}$$

since then

$$\begin{aligned} \|x_0\|_{\mathrm{BMO}^c} &= \sup_{t\geq 0} \|\Phi_t^{(1)}(|x_0 - \Phi_t^{(2)}(x_0)|^2)\|_\infty^{1/2} \overset{\text{Lem. 3.13}}{=} \sup_{t\geq 0} \|\langle u_t(x_0), u_t(x_0)\rangle_1\|_\infty^{1/2} \\ &\overset{\text{Rem. 3.9}}{=} \sup_{t\geq 0} \|\kappa_1^{1/2}(\langle u_t(x_0), u_t(x_0)\rangle_\infty)\|_\infty^{1/2} = \sup_{t\geq 0} \|\langle u_t(x_0), u_t(x_0)\rangle_\infty\|_\infty^{1/2} \\ &= \sup_{t\geq 0} \|u_t(x_0)\|_{\infty,\Phi_t} \leq \|\psi\|_{(h_1^c)^*}. \end{aligned}$$

where we have used respectively the first identity of Lemma 5.2.6, the last part of Remark 3.9 and the definition of $\|\cdot\|_\infty$ in $L_1(\mathcal{M})$.

Define $\varphi_{u_t(x_0)}$ to be the dual action of $u_t(x_0)$ on $L_2(\mathcal{M} \otimes_{\Phi_t} \mathcal{M})$ restricted to $\mathcal{M} \otimes \mathcal{M}$, i.e.

$$\varphi_{u_t(x_0)}(z) := \mathrm{Tr}(\langle u_t(x_0), z\rangle_1)$$

The goal is to prove that $u_t(x_0)$ also defines a dual action on $L_1(\mathcal{M} \otimes_{\Phi_t} \mathcal{M})$. The proof is rather technical, so we contain it in a separate lemma.

Lemma 5.4.2. *Let $z \in \mathcal{M} \otimes \mathcal{M}$. Then*

$$|\varphi_{u_t(x_0)}(z)| \leq \|\psi\|_{(h_1^c)^*} \|z\|_{1,\Phi_t}$$

In particular, $\varphi_{u_t(x_0)}$ extends to an element of $L_1(\mathcal{M} \otimes_\Phi \mathcal{M})^*$ with $\|\varphi_{u_t(x_0)}\| \leq \|\psi\|_{(h_1^c)^*}$

Proof. Let $z = \sum_j a_j \otimes b_j$. Using the second identity of Lemma 5.2.6 and the fact that $\Phi_t^{(2)}$ is self-adjoint we have

$$\begin{aligned}
\operatorname{Tr}(\langle u_t(x_0), z\rangle_1) &= \sum_j \operatorname{Tr}(\Phi_t^{(2)}((x_0 - \Phi_t^{(2)}(x_0))^* a_j) b_j D_\varphi^{1/2}) \\
&= \sum_j \operatorname{Tr}(\Phi_t^{(2)}(x_0^* a_j) b_j D_\varphi^{1/2}) - \operatorname{Tr}(\Phi_t^{(2)}(\Phi_t^{(2)}(x_0^*) a_j) b_j D_\varphi^{1/2}) \\
&= \sum_j \operatorname{Tr}(x_0^* a_j \Phi_t^{(2)}(b_j D_\varphi^{1/2})) - \operatorname{Tr}(\Phi_t^{(2)}(x_0^*) a_j \Phi_t^{(2)}(b_j D_\varphi^{1/2})) \\
&= \sum_j \operatorname{Tr}(x_0^* a_j \Phi_t^{(2)}(b_j D_\varphi^{1/2})) - \operatorname{Tr}(x_0^* \Phi_t^{(2)}(a_j \Phi_t^{(2)}(b_j D_\varphi^{1/2}))) \\
&= \sum_j \operatorname{Tr}(x_0^* [a_j \Phi_t^{(2)}(b_j D_\varphi^{1/2}) - \Phi_t^{(2)}(a_j \Phi_t^{(2)}(b_j D_\varphi^{1/2}))]) \\
&= \operatorname{Tr}(x_0^* u_t^*(z)).
\end{aligned}$$

Thus $u_t^*(z) := \sum_j a_j \Phi_t^{(2)}(b_j D_\varphi^{1/2}) - \Phi_t^{(2)}(a_j \Phi_t^{(2)}(b_j D_\varphi^{1/2})) \in L_2(\mathcal{M})$.

We are done if we can prove that $\|u_t^*(z)\|_{h_1^c} \leq \|z\|_{1,\Phi_t}$. However, we do not even have $u_t^*(z) \in L_2^\circ(\mathcal{M})$ in general, so this will not be possible. To circumvent this, let π be the projection $L_2(\mathcal{M}) \to L_2^\circ(\mathcal{M})$. Then π is self-adjoint and $\pi(x_0) = x_0$, hence

$$\operatorname{Tr}(x_0^* u_t^*(z)) = \operatorname{Tr}(x_0^* \pi(u_t^*(z))).$$

We claim that $\|\pi(u_t^*(z))\|_{h_1^c} \leq \|z\|_{1,\Phi_t}$. Indeed, by (5.2.3) and Remark 5.2.5:

$$\begin{aligned}
\|\pi(u_t^*(z))\|_{h_1^c} &= \sup_{\|y\|_{\mathrm{BMO}^r} \leq 1} |\operatorname{Tr}(y \pi(u_t^*(z)))| = \sup_{\|y\|_{\mathrm{BMO}^c} \leq 1} |\operatorname{Tr}(y^* \pi(u_t^*(z)))| \\
&= \sup_{\|y\|_{\mathrm{BMO}^c} \leq 1} |\operatorname{Tr}(\langle u_t(y), z\rangle_1)| = \sup_{\|y\|_{\mathrm{BMO}^c} \leq 1} |\operatorname{Tr}(\langle u_t(y), z\rangle_{\infty,1})| \\
&\leq \sup_{\|y\|_{\mathrm{BMO}^c} \leq 1} \|z\|_{1,\Phi_t} \|u_t(y)\|_{\infty,\Phi_t} = \|z\|_{1,\Phi_t}.
\end{aligned}$$

It follows that indeed

$$|\varphi_{u_t(x_0)}(z)| = |\operatorname{Tr}(x_0^* u_t^*(z))| \leq \sup_{\|h\|_{h_1^c} \leq 1} |\operatorname{Tr}(x_0^* h)| \|z\|_{1,\Phi_t} = \|\psi\|_{(h_1^c)^*} \|z\|_{1,\Phi_t}$$

□

Now through our duality result of Proposition 5.2.3, $u_t(x_0) \in L_\infty(\mathcal{M} \otimes_\Phi \mathcal{M})$ and

$$\|x_0\|_{\mathrm{BMO}^c} = \sup_{t \geq 0} \|u_t(x_0)\|_{\infty,\Phi_t} = \sup_{t \geq 0} \sup_{\|z\|_{1,\Phi_t} \leq 1} |\operatorname{Tr}(\langle u_t(x_0), z\rangle_{\infty,1})| \leq \|\psi\|_{(h_1^c)^*}.$$

This shows that indeed $\mathrm{BMO}^r \cong (h_1^c)^*$. □

Note that this also proves that $\|\cdot\|_{\mathrm{BMO}^c}$, $\|\cdot\|_{\mathrm{BMO}^r}$ satisfy the triangle inequality and that BMO^c and BMO^r are Banach spaces. Hence $(\mathrm{BMO}^c, \mathrm{BMO}^r)$ is a well-defined compatible couple and the intersection space BMO is also a well-defined Banach space:

Corollary 5.4.3. $\mathrm{BMO}(\mathcal{M},\Phi)$, $\mathrm{BMO}^c(\mathcal{M},\Phi)$ *and* $\mathrm{BMO}^r(\mathcal{M},\Phi)$ *are Banach spaces.*

Remark 5.4.4. In the absence of a predual for BMO, we will define a "weak-$*$ topology" in a different way, namely as the locally convex topology inherited from the topologies $\sigma(\mathrm{BMO}^c, h_1^r)$ and $\sigma(\mathrm{BMO}^r, h_1^c)$. By slight abuse of notation, we will call this the weak-$*$ topology. More precisely, recall that for $x \in \mathrm{BMO}$, we denoted by $x_c \in \mathrm{BMO}^c$ and $x_r \in \mathrm{BMO}^r$ those elements for which $x = \kappa^0_{2,1}(x_c) = \kappa^1_{2,1}(x_r)$. Then we say that a net $x^\lambda \in \mathrm{BMO}$ converges to $x \in \mathrm{BMO}$ in the weak-$*$ topology if $x_c^\lambda \to x_c$ in the weak-$*$ topology of BMO^c and $x_r^\lambda \to x_r$ in the weak-$*$ topology of BMO^r.

5.5. INTERPOLATION FOR BMO SPACE

In this section we show that [Cas19, Theorem 4.5] holds again for the current definition of BMO. Similar to how [Cas19, Theorem 4.5] is proved, the proof is a mutatis mutandis copy of the methods in [Cas19, Section 3] provided that conditional expectations extend to a contraction on BMO. In other words, we must show that [Cas19, Lemma 4.3] still holds in the current setup. This is done in Proposition 5.5.5 below. We start with some auxiliary lemmas that could be of independent interest.

Let us state some preliminary facts. By [Ter81, Theorem II.36], a standard form for $\mathcal{M}$ is $(\mathcal{M}, L_2(\mathcal{M}), J, L_2^+(\mathcal{M}))$, where J is the conjugation operator. Hence we will consider $\mathcal{M}$ as a von Neumann subalgebra of $\mathcal{B}(L_2(\mathcal{M}))$ by left multiplication. With an inclusion of von Neumann algebras $\mathcal{M}_1 \subseteq \mathcal{M}$ we mean a unital inclusion, meaning that the unit of $\mathcal{M}_1$ equals the unit of $\mathcal{M}$. It is a well known fact that $\mathcal{M}_1$ admits a φ-preserving conditional expectation if and only if $\sigma_t^\varphi(\mathcal{M}_1) = \mathcal{M}_1$ for all $t \in \mathbb{R}$, see [Tak03a, Theorem IX.4.2]. If $\mathcal{E}$ is a φ-preserving conditional expectation, then we can use Proposition 2.4.28 to extend it to a contraction $\mathcal{E}^{(p)} : L_p(\mathcal{M}) \to L_p(\mathcal{M})$, which can be checked to land in $L_p(\mathcal{M}_1)$.

Lemma 5.5.1. *Let $\mathcal{M}_1 \subseteq \mathcal{M}$ be a von Neumann subalgebra that admits a φ-preserving conditional expectation $\mathcal{E}$. Then for $x \in L_1(\mathcal{M})$ and $y \in \mathcal{M}$ we have*

$$\mathrm{Tr}(x\mathcal{E}(y)) = \mathrm{Tr}(\mathcal{E}^{(1)}(x)y).$$

Proof. If $x = D_\varphi x'$ with $x' \in \mathcal{M}$ we have since $\mathcal{E}^{(1)}$ is Tr-preserving,

$$\begin{aligned}\mathrm{Tr}(x\mathcal{E}(y)) &= \mathrm{Tr}(\mathcal{E}^{(1)}(x\mathcal{E}(y))) = \mathrm{Tr}(D_\varphi\mathcal{E}(x'\mathcal{E}(y))) = \mathrm{Tr}(D_\varphi\mathcal{E}(x')\mathcal{E}(y))\\ &=\mathrm{Tr}(D_\varphi\mathcal{E}(\mathcal{E}(x')y)) = \mathrm{Tr}(\mathcal{E}^{(1)}(D_\varphi\mathcal{E}(x')y)) = \mathrm{Tr}(\mathcal{E}^{(1)}(x)y).\end{aligned}$$

For general $x \in L_1(\mathcal{M})$ the statement follows by approximation. □

The following lemma is a variation of the Kadison-Schwarz inequality.

Lemma 5.5.2. *Let $\mathcal{M}_1 \subseteq \mathcal{M}$ be a von Neumann subalgebra that admits a φ-preserving conditional expectation $\mathcal{E}$. Then for $x \in L_2(\mathcal{M})$ we have the following inequality in $L_1(\mathcal{M})$,*

$$\mathcal{E}^{(2)}(x)\mathcal{E}^{(2)}(x)^* \le \mathcal{E}^{(1)}(xx^*).$$

Proof. Naturally $L_2(\mathcal{M}_1) \subseteq L_2(\mathcal{M})$ is a closed subspace and we have that $\mathcal{E}^{(2)}: L_2(\mathcal{M}) \to L_2(\mathcal{M}_1)$ is the orthogonal projection onto this subspace, see [Tak03a, Proof of Theorem IX.4.2]. $L_2(\mathcal{M}_1)$ is an invariant subspace for $\mathcal{M}_1$. Therefore $\mathcal{M}_1$ commutes with both $\mathcal{E}^{(2)}$ and $1-\mathcal{E}^{(2)}$. Hence, for $y \in \mathcal{M}_1$ and $x \in L_2(\mathcal{M})$ we have

$$\langle \mathcal{E}^{(2)}(x), y\mathcal{E}^{(2)}(x)\rangle + \langle (1-\mathcal{E}^{(2)})(x), y(1-\mathcal{E}^{(2)})(x)\rangle = \langle x, yx\rangle.$$

And so for $y \in \mathcal{M}^+$ we have

$$\mathrm{Tr}(\mathcal{E}(y)\mathcal{E}^{(2)}(x)\mathcal{E}^{(2)}(x)^*) = \langle \mathcal{E}^{(2)}(x), \mathcal{E}(y)\mathcal{E}^{(2)}(x)\rangle \leq \langle x, \mathcal{E}(y)x\rangle = \mathrm{Tr}(\mathcal{E}(y)xx^*). \tag{5.5.1}$$

We further have by Lemma 5.5.1,

$$\mathrm{Tr}(\mathcal{E}(y)xx^*) = \mathrm{Tr}(y\mathcal{E}^{(1)}(xx^*)),$$

and since $\mathcal{E}^{(1)}$ is a projection onto $L_1(\mathcal{M}_1)$

$$\mathrm{Tr}(\mathcal{E}(y)\mathcal{E}^{(2)}(x)\mathcal{E}^{(2)}(x)^*) = \mathrm{Tr}(y\mathcal{E}^{(1)}(\mathcal{E}^{(2)}(x)\mathcal{E}^{(2)}(x)^*)) = \mathrm{Tr}(y\mathcal{E}^{(2)}(x)\mathcal{E}^{(2)}(x)^*).$$

Therefore (5.5.1) shows that we have the following Kadison-Schwarz type inquality,

$$\mathcal{E}^{(2)}(x)\mathcal{E}^{(2)}(x)^* \leq \mathcal{E}^{(1)}(xx^*).$$

□

Lemma 5.5.3. *Let $\omega \in \mathcal{M}_*^+$. The following are equivalent:*

1. *We have $\omega \leq \varphi$.*
2. *There exists $x \in \mathcal{M}$ with $0 \leq x \leq 1$ such that $D_\varphi^{\frac{1}{2}} x D_\varphi^{\frac{1}{2}} = D_\omega$.*

Proof. For (1) ⇒ (2), consider the map

$$T: L_2(\mathcal{M}) \to L_2(\mathcal{M}): D_\varphi^{\frac{1}{2}} x \mapsto D_\omega^{\frac{1}{2}} x, \qquad x \in \mathcal{M}.$$

From the fact that $\omega \leq \varphi$ it follows that T is a well-defined contraction. Moreover, we claim that $T \in \mathcal{M}$. Indeed, the commutant of $\mathcal{M}$ acting on $L_2(\mathcal{M})$ is given by $J\mathcal{M}J$ where $J: \xi \mapsto \xi^*$ is the modular conjugation. Then it follows that for $x, y \in \mathcal{M}$ we have

$$TJyJD_\varphi^{\frac{1}{2}} x = TD_\varphi^{\frac{1}{2}} xy^* = D_\omega^{\frac{1}{2}} xy^* = JyJT(D_\varphi^{\frac{1}{2}} x).$$

Now set $x = T^*T \in \mathcal{M}$ so that $0 \leq x \leq 1$. We have $TD_\varphi^{\frac{1}{2}} = D_\omega^{\frac{1}{2}}$ so that $(D_\varphi^{\frac{1}{2}} T^*)(TD_\varphi^{\frac{1}{2}}) = D_\omega$.

The implication (2) ⇒ (1) follows as for $y \in \mathcal{M}$ we have

$$\begin{aligned}\omega(yy^*) &= \mathrm{Tr}(D_\omega yy^*) = \mathrm{Tr}(y^* D_\varphi^{\frac{1}{2}} x D_\varphi^{\frac{1}{2}} y) = \langle D_\varphi^{\frac{1}{2}} y, x D_\varphi^{\frac{1}{2}} y\rangle \\ &\leq \langle D_\varphi^{\frac{1}{2}} y, D_\varphi^{\frac{1}{2}} y\rangle = \mathrm{Tr}(y^* D_\varphi y) = \varphi(yy^*).\end{aligned}$$

□

Lemma 5.5.4. *Let $a, b \in L_1(\mathcal{M})^+$ and suppose that $a \leq b$ and $b = D_\varphi^{\frac{1}{2}} x_b D_\varphi^{\frac{1}{2}}$ with $x_b \in \mathcal{M}^+$. Then there exists $x_a \in \mathcal{M}^+$ such that $a = D_\varphi^{\frac{1}{2}} x_a D_\varphi^{\frac{1}{2}}$. Moreover $x_a \leq x_b$.*

Proof. Let φ_a and φ_b be in $\mathcal{M}_*^+$ such that $D_{\varphi_a} = a$ and $D_{\varphi_b} = b$. The assumptions and Lemma 5.5.3 imply that $\varphi_b \leq \|x_b\|\varphi$. We find that $\varphi_a \leq \varphi_b \leq \|x_b\|\varphi$. Therefore Lemma 5.5.3 implies that there exists $x_a \in \mathcal{M}$ with $0 \leq x_a \leq \|x_b\|$ such that $a = D_\varphi^{\frac{1}{2}} x_a D_\varphi^{\frac{1}{2}}$. We have moreover $x_a \leq x_b$ since $a \leq b$ implies that for $y \in \mathcal{M}$,

$$\begin{aligned} &\langle D_\varphi^{\frac{1}{2}} y, x_a D_\varphi^{\frac{1}{2}} y\rangle = \mathrm{Tr}(y^* D_\varphi^{\frac{1}{2}} x_a D_\varphi^{\frac{1}{2}} y) = \mathrm{Tr}(D_\varphi^{\frac{1}{2}} x_a D_\varphi^{\frac{1}{2}} y y^*) = \mathrm{Tr}(a y y^*) \\ \leq &\mathrm{Tr}(b y y^*) = \mathrm{Tr}(D_\varphi^{\frac{1}{2}} x_b D_\varphi^{\frac{1}{2}} y y^*) = \langle D_\varphi^{\frac{1}{2}} y, x_b D_\varphi^{\frac{1}{2}} y\rangle. \end{aligned}$$

□

Proposition 5.5.5. *Let $\mathcal{M}_1 \subseteq \mathcal{M}$ be a von Neumann subalgebra that admits a φ-preserving conditional expectation $\mathcal{E}$. Let $\Phi = (\Phi_t)_{t\geq 0}$ be a Markov semi-group on $\mathcal{M}$ that preserves $\mathcal{M}_1$. Then we have isometric 1-complemented inclusions*

$$\mathrm{BMO}(\mathcal{M}_1, \Phi) \subseteq \mathrm{BMO}(\mathcal{M}, \Phi).$$

Proof. That the isometric inclusion exists is clear from the definitions. We have to prove that the inclusion is 1-complemented. For $t \geq 0$ and $x \in \mathrm{BMO}_\Phi^c(\mathcal{M}) \subseteq L_2^\circ(\mathcal{M})$ we have the following (in)equalities in $L_1(\mathcal{M})$ by Lemma 5.5.2,

$$\begin{aligned} &|\mathcal{E}^{(2)}(x) - \Phi_t^{(2)}(\mathcal{E}^{(2)}(x))|^2 = \mathcal{E}^{(2)}(x - \Phi_t^{(2)}(x))^* \mathcal{E}^{(2)}(x - \Phi_t^{(2)}(x)) \\ \leq &\mathcal{E}^{(1)}((x - \Phi_t^{(2)}(x))^*(x - \Phi_t^{(2)}(x))). \end{aligned}$$

As $\Phi_t^{(1)}$ preserves positivity and commutes with $\mathcal{E}^{(1)}$,

$$\Phi_t^{(1)}(|\mathcal{E}^{(2)}(x) - \Phi_t^{(2)}(\mathcal{E}^{(2)}(x))|^2) \leq \mathcal{E}^{(1)}(\Phi_t^{(1)}((x - \Phi_t^{(2)}(x))^*(x - \Phi_t^{(2)}(x)))). \tag{5.5.2}$$

By assumption we may write

$$\Phi_t^{(1)}((x - \Phi_t^{(2)}(x))^*(x - \Phi_t^{(2)}(x)) = \kappa_1^{1/2}(x_t'),$$

for some $x_t' \in \mathcal{M}$. So the right hand side of (5.5.2) equals $\kappa_1^{1/2}(\mathcal{E}(x_t'))$. By Lemma 5.5.4 it follows that there exists $x_t'' \in \mathcal{M}$ with $0 \leq x_t'' \leq \mathcal{E}(x_t')$ such that

$$\Phi_t^{(1)}(|\mathcal{E}^{(2)}(x) - \Phi_t^{(2)}(\mathcal{E}^{(2)}(x))|^2) = \kappa_1^{1/2}(x_t'').$$

Taking norms we have

$$\|\mathcal{E}^{(2)}(x)\|_{\mathrm{BMO}^c} = \sup_{t\geq 0}\|x_t''\|_\infty \leq \sup_{t\geq 0}\|\mathcal{E}(x_t')\|_\infty \leq \sup_{t\geq 0}\|x_t'\|_\infty = \|x\|_{\mathrm{BMO}^c}.$$

The row BMO-estimate and the BMO-estimate follow similarly. □

We may now conclude the following theorem. The proof (based on the Haagerup reduction method) follows exactly as in [Cas19, Sections 3 and 4] where [Cas19, Lemma 4.3] needs to be replaced by Proposition 5.5.5. Note that in the statement of [Cas19, Theorem 4.5] the standard Markov dilation must be modular as well (this is a misprint in the text of [Cas19]).

Theorem 5.5.6. *Let Φ be a φ-modular Markov semigroup on a σ-finite von Neumann algebra $(\mathcal{M},\varphi)$ admitting a modular standard Markov dilation. Then for all $1 \le p < \infty, 1 < q < \infty$,*

$$[\mathrm{BMO}(\mathcal{M},\Phi), L_p^{\circ}(\mathcal{M})]_{1/q} \approx_{pq} L_{pq}^{\circ}(\mathcal{M}).$$

Here $\approx_{pq}$ means that the Banach spaces are isomorphic and the norm of the isomorphism in both directions can be estimated by an absolute constant times pq.

5.6. COMPLETELY BOUNDED MAPS WITH RESPECT TO THE BMO-NORM

As before, we let $\Phi = (\Phi_t)_{t\ge 0}$ be a Markov semigroup on $\mathcal{M}$. Fix some $n \ge 2$. Then the maps $\iota_{M_n} \otimes \Phi_t$ define a Markov semigroup on $M_n(\mathcal{M})$. Hence we can define the matrix BMO-norms $\|\cdot\|_{\mathrm{BMO}_n}$ on $M_n(\mathcal{M})^{\circ}$ with respect to the semigroup $S_n := (\iota_{M_n} \otimes \Phi_t)_{t\ge 0}$. Through a straightforward calculation, one also checks that $M_n(\mathcal{M})^{\circ} = M_n(\mathcal{M}^{\circ})$. Hence the above norms define matrix norms on $\mathcal{M}^{\circ}$. It is not hard to prove that these norms turn $\mathcal{M}^{\circ}$ into an operator space, which we denote by $(\mathcal{M}^{\circ}, \|\cdot\|_{\mathrm{BMO}})$. We leave the details to the reader.

Let $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$ be a σ-finite von Neumann algebra. Then $\mathcal{N}\bar{\otimes}\mathcal{M}$ is again a σ-finite von Neumann algebra. Similarly as in the matrix case, $S := (\iota_{\mathcal{N}} \otimes \Phi_t)_{t\ge 0}$ is a semigroup on $\mathcal{N}\bar{\otimes}\mathcal{M}$. In line with the main text, we denote $\|\cdot\|_{\mathrm{BMO}_S}$ for the corresponding BMO-norm on $(\mathcal{N}\bar{\otimes}\mathcal{M})^{\circ}$.

Using the fact that $\mathcal{N}_* \otimes \mathcal{M}_*$ is dense in $(\mathcal{N}\bar{\otimes}\mathcal{M})_*$ (see [Sak71, Chapter 1.22]) one can show that $\mathcal{N} \otimes \mathcal{M}^{\circ} \subseteq (\mathcal{N}\bar{\otimes}\mathcal{M})^{\circ}$.

Proposition 5.6.1. *Let $\mathcal{A} \subseteq \mathcal{M}$ be a linear subspace and $T : \mathcal{A} \to (\mathcal{M}^{\circ}, \|\cdot\|_{\mathrm{BMO}})$ be completely bounded. For $x \in \mathcal{N} \otimes \mathcal{A}$,*

$$\|(\iota_{\mathcal{N}} \otimes T)(x)\|_{\mathrm{BMO}_S} \le \|T\|_{cb} \|x\|_{\mathcal{B}(\mathcal{H}\otimes_2\mathcal{K})}.$$

Proof. Take $x \in \mathcal{N} \otimes \mathcal{A}$ and write $x = \sum_n x_n \otimes x_n'$. Let $z = (\iota_{\mathcal{N}} \otimes T)(x) \in \mathcal{N} \otimes \mathcal{M}^{\circ}$. Setting $w_n = T(x_n')$ we have $z = \sum_n x_n \otimes w_n$. For a finite dimensional subspace $F \subseteq \mathcal{H}$ let P_F be the projection onto F. Denote $x_n^F = P_F x_n P_F$ the truncation of x_n to F. Denote $z^F = \sum_n x_n^F \otimes w_n$ and $x^F = \sum_n x_n^F \otimes x_n'$.

Now we prove the column estimate. Let $\xi \in \mathcal{H} \otimes \mathcal{K}$ (algebraic tensor product) and write $\xi = \sum_k \xi_k \otimes \eta_k$. Define $F \subseteq \mathcal{H}$ to be

$$F = \mathrm{Span}\{\xi_k, x_m\xi_k, x_n^* x_m \xi_k \mid n, m, k\}.$$

Then we note that F is finite dimensional and $(x_n^F)^* x_m^F \xi_k = x_n^* x_m \xi_k$, Let $t \geq 0$ be arbitrary. Writing out the expression in the column BMO-norm gives

$$(\iota_{\mathcal{N}} \otimes \Phi_t)(|z - (\iota_{\mathcal{N}} \otimes \Phi_t)(z)|^2) = \sum_{n,m} x_n^* x_m \otimes \Phi_t((w_n - \Phi_t(w_n))^* (w_m - \Phi_t(w_m))).$$

Hence, denoting $S_F := (\iota_{\mathcal{B}(F)} \otimes \Phi_t)_{t \geq 0}$,

$$\begin{aligned} & \|(\iota_{\mathcal{N}} \otimes \Phi_t)(|z - (\iota_{\mathcal{N}} \otimes \Phi_t)(z)|^2)\xi\|_{\mathcal{H} \otimes_2 \mathcal{K}} \\ = \; & \|(\iota_{\mathcal{B}(\mathcal{F})} \otimes \Phi_t)(|z^F - (\iota_{\mathcal{B}(\mathcal{F})} \otimes \Phi_t)(z^F)|^2)\xi\|_{F \otimes \mathcal{K}} \\ \leq \; & \|(\iota_{\mathcal{B}(\mathcal{F})} \otimes \Phi_t)(|z^F - (\iota_{\mathcal{B}(\mathcal{F})} \otimes \Phi_t)(z^F)|^2)\|_{\mathcal{B}(F \otimes \mathcal{K})} \|\xi\| \\ \leq \; & \|z^F\|^2_{\mathrm{BMO}^c_{S_F}} \|\xi\| = \|(\iota_{\mathcal{B}(F)} \otimes T)(x^F)\|^2_{\mathrm{BMO}^c_{S_F}} \|\xi\| \leq \|T\|^2_{cb} \|x\|^2_{\mathcal{B}(\mathcal{H} \otimes_2 \mathcal{K})} \|\xi\|. \end{aligned}$$

In the last step, we used that T is also completely bounded when considering $\|\cdot\|_{\mathrm{BMO}^c}$ on the right. Taking the supremum over all $\xi \in \mathcal{H} \otimes \mathcal{K}$ with $\|\xi\| = 1$ and $t \geq 0$, we conclude

$$\|(\iota_{\mathcal{N}} \otimes T)(x)\|_{\mathrm{BMO}^c_S} \leq \|T\|_{cb} \|x\|_{\mathcal{B}(\mathcal{H} \otimes_2 \mathcal{K})}$$

The row BMO estimate follows similarly, from which the BMO estimate follows. □

Remark 5.6.2. In the case where $\mathcal{M}$ is a finite von Neumann algebra, we can extend the operator space structure to $\mathrm{BMO}(\mathcal{M}, \Phi)$. In the σ-finite case however, it seems to be more difficult than expected to prove that $M_n(\mathrm{BMO}(\mathcal{M}, \Phi)) \subseteq \mathrm{BMO}(M_n(\mathcal{M}), \iota_{M_n} \otimes \Phi)$.

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6

L_p-BOUNDEDNESS OF BMO-VALUED FOURIER-SCHUR MULTIPLIERS ON $SU_q(2)$

This chapter is based on (part of) the following article:

1. **Martijn Caspers, Gerrit Vos**, *BMO spaces of σ-finite von Neumann algebras and Fourier-Schur multipliers on $SU_q(2)$*, Studia Mathematica **262**(1):45-91 (2022).

In this chapter, we provide concrete examples of multipliers on the compact quantum group $SU_q(2)$ as an application of our BMO results of Chapter 5. This is based on the second half of [CV22]. There are some changes here to the original version in the proof of Theorem 6.5.1, which are described in [CV]. The proof had to be adapted because, as mentioned in the beginning of Chapter 5, we no longer define a predual for BMO but only for the row and column spaces.

The multipliers we consider are so-called Fourier-Schur multipliers. The notations from the following definition will be defined in Section 6.1.2.

Definition A. Let $\mathbb{G}$ be a compact quantum group and $T : \mathrm{Pol}(\mathbb{G}) \to \mathrm{Pol}(\mathbb{G})$ a linear map. We call T a Fourier-Schur multiplier if the following condition holds. Let u be any finite dimensional corepresentation on $\mathscr{H}$. Then there exists an orthogonal basis e_i such that if $u_{i,j}$ are the matrix coefficients with respect to this basis, then there exist numbers $c_{i,j} := c^u_{i,j} \in \mathbb{C}$ such that

$$Tu_{i,j} = c_{i,j} u_{i,j}.$$

In this case $(c^u_{i,j})_{i,j,u}$ is called the symbol of T.

Basically, Fourier-Schur multipliers are Schur multipliers acting on the 'Fourier domain'. This can be given meaning through the definition of the Fourier transform on quantum groups, see e.g. [Cas13, Section 4, 5]. We consider Fourier-Schur multipliers on $\mathbb{G}_q := SU_q(2), q \in (-1,1)\backslash\{0\}$ associated with completely bounded Fourier multipliers on the torus $\mathbb{T}$.

The semigroups we use to define BMO are the Heat semi-group on $\mathbb{T}$ and the Markov semigroup Φ on $\mathbb{G}_q$ constructed in Section 6.4. We use the shorthand notation $\mathrm{BMO}(\mathbb{T})$, $\mathrm{BMO}(\mathbb{G}_q)$ for the associated BMO spaces; see again Section 6.4. The core of this chapter is the following endpoint estimate, which is Theorem 6.5.1:

Theorem F. *Let $m \in \ell_\infty(\mathbb{Z})$ with $m(0) = 0$ be such that the Fourier multiplier T_m is completely bounded as a map $L_\infty(\mathbb{T}) \to \mathrm{BMO}(\mathbb{T})$. Let $\tilde{T}_m : \mathrm{Pol}(\mathbb{G}_\mathrm{q}) \to \mathrm{Pol}(\mathbb{G}_\mathrm{q})$ be the Fourier-Schur multiplier with symbol $(m(-i-j))_{i,j,l}$ with respect to the basis described in (6.3.1). Then $\tilde{T}_m$ extends to a bounded map*

$$\tilde{T}_m^{(\infty)} : L_\infty(\mathbb{G}_q) \to \mathrm{BMO}(\mathbb{G}_q).$$

Moreover $\|\tilde{T}_m^{(\infty)} : L_\infty(\mathbb{G}_q) \to \mathrm{BMO}(\mathbb{G}_q)\| \le \|T_m : L_\infty(\mathbb{T}) \to \mathrm{BMO}(\mathbb{T})\|_{cb}$.

We also prove a L_2-L_2 estimate in Section 6.3 by a careful analysis of the Peter-Weyl decomposition. Now, using the interpolation results of Section 5.5, also the corresponding $L_p \to L_p$ follow. This is proved in Theorem 6.7.1.

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In the proof of Theorem F we use our column and row H^1-BMO duality principle to show that Fourier-Schur multipliers extend from the weak-$*$ dense subalgebra of matrix coefficients of irreducible unitary corepresentations. The other important ingredient is a transference principle. Another fact we need is that the Markov semigroup Φ admits a Markov dilation, which we prove in a somewhat isolated Section 6.6.

Let us describe the structure of the chapter. We start by giving some preliminaries on BMO functions of the torus, quantum groups and $SU_q(2)$. In Section 6.2 we define Fourier-Schur multipliers and introduce the class of Fourier-Schur multipliers on $SU_q(2)$ that we will study. In Section 6.3 we prove the lower endpoint estimate for $p = 2$. In Section 6.4 we define the BMO spaces of $SU_q(2)$ using a transference principle. Then, in Section 6.5, we prove the upper endpoint estimate from L_∞ to BMO, which is the core of the chapter. Then we prove the final fact needed for interpolation, namely the existence of a markov dilation, in Section 6.6. Finally, in Section 6.7, we apply complex interpolation to prove that our Fourier-Schur multipliers are bounded on L_p.

6.1. PRELIMINARIES

6.1.1. BMO SPACES OF THE TORUS

Define trigonometric functions

$$\zeta_k : \mathbb{T} \to \mathbb{T} : z \mapsto z^k, \qquad k \in \mathbb{Z}.$$

Set the $*$-algebra $\mathrm{Pol}(\mathbb{T}) := \mathrm{Span}\{\zeta_k : k \in \mathbb{Z}\}$. For $m \in \ell_\infty(\mathbb{Z})$ let $T_m : L_2(\mathbb{T}) \to L_2(\mathbb{T})$ be the Fourier multiplier defined by $T_m(\zeta_k) = m(k)\zeta_k, k \in \mathbb{Z}$. For $t \ge 0$ let $h_t \in \ell_\infty(\mathbb{Z})$ be given by

$h_t(k) = e^{-tk^2}$. Then the maps T_{h_t} are well-known to define a Markov semigroup on the von Neumann algebra $L_\infty(\mathbb{T})$ (as they are restrictions of the Heat semi-group on $L_\infty(\mathbb{R})$). We use the shorthand notation

$$\mathrm{BMO}(\mathbb{T}) := \mathrm{BMO}(L_\infty(\mathbb{T}), (T_{h_t})_{t\geq 0}).$$

Let $m \in \ell_\infty(\mathbb{Z})$ be such that $m(0) = 0$. Then as $t \to \infty$,

$$\| T_{h_t}(T_m \zeta_k) \|_\infty = e^{-tk^2} |m(k)| \| \zeta_k \|_\infty \to 0.$$

So T_m maps $\mathrm{Pol}(\mathbb{T})$ to $L_\infty^\circ(\mathbb{T})$.

6.1.2. COMPACT QUANTUM GROUPS

For the theory of compact quantum groups we refer to [Wor98] or the notes [MV98] which follows the same lines.

Definition 6.1.1. A compact quantum group $\mathbb{G} = (C(\mathbb{G}), \Delta)$ consists of a unital C*-algebra $C(\mathbb{G})$ and a unital $*$-homomorphism $\Delta : C(\mathbb{G}) \to C(\mathbb{G}) \otimes_{\min} C(\mathbb{G})$ called the comultiplication such that $(\Delta \otimes \iota) \circ \Delta = (\iota \otimes \Delta) \circ \Delta$ (coassociativity) and such that both $\Delta(C(\mathbb{G}))(C(\mathbb{G}) \otimes 1)$ and $\Delta(C(\mathbb{G}))(1 \otimes C(\mathbb{G}))$ are dense in $C(\mathbb{G}) \otimes_{\min} C(\mathbb{G})$. Here $\iota : C(\mathbb{G}) \to C(\mathbb{G})$ is the identity map.

A finite dimensional (unitary) corepresentation is a unitary $u \in C(\mathbb{G}) \otimes M_n(\mathbb{C})$ such that $(\Delta \otimes \mathrm{id})(u) = u_{13} u_{23}$ where $u_{23} = 1 \otimes u$ and u_{13} is the flip applied to the first two tensor legs of u_{23}. All corepresentations are assumed to be unitary. The elements $(\mathrm{id} \otimes \omega)(u) \in C(\mathbb{G})$ with $\omega \in M_n(\mathbb{C})^*$ are called matrix coefficients. The span of all matrix coefficients is a $*$-algebra called $\mathrm{Pol}(\mathbb{G})$. Δ maps $\mathrm{Pol}(\mathbb{G})$ to $\mathrm{Pol}(\mathbb{G}) \otimes \mathrm{Pol}(\mathbb{G})$.

Here we shall mainly be concerned with the quantum group $SU_q(2)$ and we shall introduce further structure such as Haar states and von Neumann algebras for this case only.

6.1.3. INTRODUCTION $SU_q(2)$

Let $\mathbb{G}_q := SU_q(2)$ with $q \in (-1,1) \backslash \{0\}$. It was introduced by Woronowicz in [Wor87b]. Its C^*-algebra is the one generated by the operators α, γ on the Hilbert space $\mathcal{H} = \ell_2(\mathbb{N}) \otimes_2 \ell_2(\mathbb{Z})$ given by

$$\begin{aligned} \alpha(e_i \otimes f_j) &= \sqrt{1 - q^{2i}}\, e_{i-1} \otimes f_j, \\ \gamma(e_i \otimes f_j) &= q^i e_i \otimes f_{j+1}. \end{aligned}$$

where $e_i \otimes f_j, i \in \mathbb{N}, j \in \mathbb{Z}$ are the basis vectors of $\mathcal{H}$. The operators α, γ satisfy the following relations:

$$\begin{array}{ccc} \gamma^*\gamma = \gamma\gamma^*, & \alpha\gamma = q\gamma\alpha, & \alpha\gamma^* = q\gamma^*\alpha, \\ \alpha^*\alpha + \gamma^*\gamma = I, & \alpha\alpha^* + q^2\gamma^*\gamma = I. & \end{array}$$

The comultiplication is given by

$$\Delta(\alpha) = \alpha \otimes \alpha - q\gamma^* \otimes \gamma, \qquad \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma.$$

We define $L_\infty(\mathbb{G}_q) = \langle \alpha, \gamma \rangle'' \subseteq \mathcal{B}(\mathcal{H})$. The corresponding Connes-Hilsum L_p-spaces are written as $L_p(\mathbb{G}_q) := L_p(L_\infty(\mathbb{G}_q))$. We also define $\mathrm{Pol}(\mathbb{G}_q) \subseteq L_\infty(\mathbb{G}_q)$ to be the $*$-algebra generated by α, γ. This is equivalent to the definition given in Section 6.1.2. It is the linear span of elements $\alpha^k \gamma^l (\gamma^*)^m$, $k \in \mathbb{Z}, l, m \in \mathbb{N}$, where we set $\alpha^k = (\alpha^*)^{|k|}$ in case $k < 0$. Obviously, $\mathrm{Pol}(\mathbb{G}_q)$ is weakly (or weak-$*$) dense in $L_\infty(\mathbb{G}_q)$.

The Haar state on $L_\infty(\mathbb{G}_q)$ is given by the following formula:

$$\varphi(x) = (1-q^2) \sum_{k \in \mathbb{Z}_{\geq 0}} q^{2k} \langle e_k \otimes f_0, x(e_k \otimes f_0) \rangle. \tag{6.1.1}$$

See [Wor87a, Appendix A1] for the complete calculation. Note that $\varphi(\alpha^k \gamma^l (\gamma^*)^m)$ is non-zero if and only if $k = 0, l = m$. It is also faithful, as follows for instance from (6.1.1).

The modular automorphism group is given by

$$\sigma_t^\varphi(\alpha^k \gamma^l (\gamma^*)^m) = q^{-itk} \alpha^k \gamma^l (\gamma^*)^m. \tag{6.1.2}$$

This can be derived from [Tak03a, Theorem VIII.3.3], where the u_t from the theorem is equal to $(\gamma^* \gamma)^{it}$ and the ψ is a trace.

Remark 6.1.2. The above definition of $L_\infty(\mathbb{G}_q)$ is not the standard way to define the von Neumann algebra; usually this would be the double commutant within the GNS-representation corresponding to the Haar state ϕ. However, these von Neumann algebras are isomorphic, although they are not unitarily isomorphic.

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6.2. FOURIER-SCHUR MULTIPLIERS ON $SU_q(2)$

Definition 6.2.1. Let $\mathbb{G}$ be a compact quantum group and $T : \mathrm{Pol}(\mathbb{G}) \to \mathrm{Pol}(\mathbb{G})$ a linear map. We call T a Fourier-Schur multiplier if the following condition holds. Let u be any finite dimensional corepresentation on $\mathcal{H}$. Then there exists an orthogonal basis e_i such that if $u_{i,j}$ are the matrix coefficients with respect to this basis, then there exist numbers $c_{i,j} := c_{i,j}^u \in \mathbb{C}$ such that

$$T u_{i,j} = c_{i,j} u_{i,j}.$$

In this case $(c_{i,j}^u)_{i,j,u}$ is called the symbol of T.

Remark 6.2.2. If $\mathbb{G}$ comes from a classical abelian group G, i.e. if all irreducible corepresentations are one-dimensional, then the above definition coincides with the definition of a classical Fourier multiplier. In general, we see that $T = \mathcal{F} S \mathcal{F}^{-1}$, where S is a Schur multiplier. Hence the name 'Fourier-Schur multiplier'.

We will construct Fourier-Schur multipliers from Fourier multipliers on the torus $\mathbb{T} \subseteq \mathbb{C}$. We assume henceforth that $m \in \ell_\infty(\mathbb{Z})$ with $m(0) = 0$ such that $T_m : L_\infty(\mathbb{T}) \to \mathrm{BMO}(\mathbb{T})$ is completely bounded. In the remainder of this section, we will consider the map

$$\tilde{T}_m : \mathrm{Pol}(\mathbb{G}_q) \to \mathrm{Pol}(\mathbb{G}_q), \quad \alpha^k \gamma^l (\gamma^*)^m \mapsto m(k) \alpha^k \gamma^l (\gamma^*)^m \tag{6.2.1}$$

We will see after the next subsection that $\tilde{T}_m$ is indeed a Fourier-Schur multiplier. We remark that the symbol m is used both as an element of $\ell_\infty(\mathbb{Z})$ and a power of γ^*; the context will always make clear which is meant.

We introduce at this point the Markov semigroup that we will use to define the BMO space:

$$\Phi_t(\alpha^k\gamma^l(\gamma^*)^m) = e^{-tk^2}\alpha^k\gamma^l(\gamma^*)^m, \qquad k\in\mathbb{Z}, l,m\in\mathbb{N}, t\geq 0.$$

We will only prove in Section 6.4 that the maps Φ_t extend to form a Markov semigroup on $L_\infty(\mathbb{G}_q)$. However, for the sake of exposition it will be convenient to already define the corresponding spaces $L_p^\circ(\mathbb{G}_q)$ as in Section 5.3.

The final goal is to prove that this map extends boundedly to $L_p(\mathbb{G}_q)\to L_p^\circ(\mathbb{G}_q)$ for all $p\geq 2$. We do this through complex interpolation (Riesz-Torin). This requires 3 steps: (1) a lower endpoint estimate; (2) an upper endpoint estimate involving BMO spaces and (3) the construction of a Markov dilation in order to apply Theorem 5.5.6.

We treat the Markov dilation in Section 6.6. The remainder of this section is devoted to the endpoint estimates.

Similarly to the torus, we have

Lemma 6.2.3. *Let* $1\leq p\leq\infty$. *Then* $\kappa_p^1\circ\tilde{T}_m$ *maps* $\mathrm{Pol}(\mathbb{G}_\mathrm{q})$ *to* $L_p^\circ(\mathbb{G}_q)$.

Proof. Let $x=\alpha^k\gamma^l(\gamma^*)^m$. For $k=0$, we have $\tilde{T}_m(x)=0\in L_p^\circ(\mathbb{G}_q)$. Now assume $|k|>0$. Then for any $1\leq p\leq\infty$, we have as $t\to\infty$,

$$\|\Phi_t^{(p)}(\kappa_p^1(\tilde{T}_m x))\|_p = \|\kappa_p^1(\Phi_t(\tilde{T}_m(x)))\|_p = |m(k)e^{-tk^2}|\|\kappa_p^1(x)\|_p\to 0.$$

Since $\mathrm{Pol}(\mathbb{G}_\mathrm{q})$ is the span of elements $\alpha^k\gamma^l(\gamma^*)^m$, the result follows by linearity. (Note that for $p=\infty$, the σ-weak convergence follows from norm convergence.) □

6.3. L_2-ESTIMATE

In this subsection we prove that (6.2.1) extends to a bounded map $L_2(\mathbb{G}_q)\to L_2(\mathbb{G}_q)$. At the same time we prove (essentially) that it defines a Fourier-Schur multiplier. The main ingredient will be the Peter-Weyl decomposition of $\mathbb{G}_q$ (see [KS97, Theorem 4.17]) that we shall summarize now.

A complete set of mutually inequivalent irreducible corepresentations of $\mathbb{G}_q$ can be constructed as follows. They are labeled by half integers $l\in\frac{1}{2}\mathbb{N}$. Consider the vector space of linear combinations of the homogeneous polynomials in α,γ of degree $2l$. For some specific constants $C_{l,k,q}$, we define basis vectors as follows:

$$g_k^{(l)} = C_{l,k,q}\alpha^{l-k}\gamma^{l+k}, \quad k=-l,-l+1,\dots,l. \tag{6.3.1}$$

The precise value of the constant $C_{l,k,q}$ can be found in [KS97, Chapter 4.2.3]; it is of little importance to us. Next, we define the matrix $u^{(l)}\in\mathrm{Pol}(\mathbb{G}_q)\otimes M_{2l+1}(\mathbb{C})$ by

$$\Delta(g_k^{(l)}) = \sum_{i=-l}^{l} u_{k,i}^{(l)}\otimes g_i^{(l)}.$$

The Peter-Weyl theorem now takes the following form from which we derive the main result of this subsection in Proposition 6.3.2.

Lemma 6.3.1 (Proposition 4.16 and Theorem 4.17 of [KS97])**.** *The matrix coefficients of* $u^{(l)} \in M_{2l+1}(L_\infty(\mathbb{G}_q))$ *are a linear basis for* $\mathrm{Pol}(\mathbb{G}_q)$ *satisfying the orthogonality relations*

$$\varphi((u^{(l)}_{i,j})^* u^{(k)}_{r,s}) = C_i^{(l)} \delta_{l,k}\delta_{i,r}\delta_{j,s}.$$

for some constants $C_i^{(l)} \in \mathbb{C}$.

Proposition 6.3.2. *The* $u^{(l)}_{i,j}$ *form an orthogonal basis of eigenvectors for the map* $\tilde{T}_m$ *defined in* (6.2.1) *with eigenvalues* $m(-i-j)$.

Proof. To prove this, we will calculate an explicit expression for the matrix elements $u^{(l)}_{i,j}$. With our notation $\alpha\alpha^{-1} = \alpha\alpha^* = 1 - q^2\gamma^*\gamma$. Hence,

$$\begin{aligned}\alpha^k(\alpha^*)^k &= \alpha^{k-1}(1-q^2\gamma^*\gamma)(\alpha^*)^{k-1} = (1-q^{2k}\gamma^*\gamma)\alpha^{k-1}(\alpha^*)^{k-1}\\ &= \cdots = (1-q^{2k}\gamma^*\gamma)(1-q^{2k-2}\gamma^*\gamma)\dots(1-q^2\gamma^*\gamma) =: (q^2\gamma^*\gamma;q^2)_k.\end{aligned}$$

The notation $(a;b)_k$ is known as the Pochhammer symbol. We define $\left[{k \atop i}\right]_q$ to be the q-*binomial coefficients* from [KS97, Section 2.1.2]. They satisfy the formula

$$(v+w)^k = \sum_{i=0}^{k} \left[{k \atop i}\right]_{q^{-1}} v^i w^{k-i}.$$

for v, w satisfying $vw = qwv$. Below we will use this formula on both tensor legs simultaneously, which means that the subscript of the q-binomial coefficient becomes q^{-2}. Thus:

$$\begin{aligned}\Delta(g_i^{(l)}) &= C_{l,i,q}\Delta(\alpha^{l-i}\gamma^{l+i}) = C_{l,i,q}\Delta(\alpha)^{l-i}\Delta(\gamma)^{l+i}\\ &= C_{l,i,q}(\alpha\otimes\alpha - q\gamma^*\otimes\gamma)^{l-i}(\gamma\otimes\alpha+\alpha^*\otimes\gamma)^{l+i}\\ &= C_{l,i,q}\left(\sum_{a=0}^{l-i}(-q)^{l-i-a}\left[{l-i \atop a}\right]_{q^{-2}} \alpha^a(\gamma^*)^{l-i-a}\otimes\alpha^a\gamma^{l-i-a}\right)\\ &\quad\times \left(\sum_{s=0}^{l+i}\left[{l+i \atop s}\right]_{q^{-2}} \gamma^s(\alpha^*)^{l+i-s}\otimes\alpha^s\gamma^{l+i-s}\right)\\ &= C_{l,i,q}\sum_{a=0}^{l-i}\sum_{s=0}^{l+i} C'_{a,s}\alpha^{a+s-l-i}(\gamma^*)^{l-i-a}\gamma^s P_{a,s}(\gamma^*,\gamma)\otimes\alpha^{a+s}\gamma^{2l-a-s}\end{aligned}$$

where

$$C'_{a,s} := C'_{l,i,q,a,s} = (-q)^{l-i-a}q^{(l+i-s)(s+l-i-a)-s(l-i-a)}\left[{l-i \atop a}\right]_{q^{-2}}\left[{l+i \atop s}\right]_{q^{-2}}$$

and $P_{a,s}(\gamma^*,\gamma) := P_{l,i,q,a,s}(\gamma^*,\gamma)$ is some polynomial in the variables γ^*, γ depending on the minimum value of $\{a, l+i-s\}$. If the minimum value is $l+i-s$ then $P_{a,s}(\gamma^*,\gamma) = (q^2\gamma^*\gamma; q^2)_{\min(a,l+i-s)}$; if it is a then the Pochhammer symbol appears instead to the left of $\alpha^{a+s-l-i}$, so after interchanging we obtain extra powers of q in the terms of the polynomial.

Next, we substitute s by j where $j = l-a-s$ and set

$$P'_{a,j}(\gamma^*,\gamma) := (\gamma^*)^{l-i-a}\gamma^{l-j-a}P_{a,l-j-a}(\gamma^*,\gamma),\ C''_{a,j} := C'_{a,l-j-a}$$

with slight abuse of notation. This gives:

$$\begin{aligned}\Delta(g_i^{(l)}) &= C_{l,i,q}\sum_{a=0}^{l-i}\sum_{j=-a-i}^{l-a} C''_{a,j}\alpha^{-(i+j)}P'_{a,j}(\gamma^*,\gamma)\otimes\alpha^{l-j}\gamma^{l+j}\\ &= C_{l,i,q}\sum_{j=-l}^{l}\sum_{a=\max\{0,-i-j\}}^{\min\{l-i,l-j\}} C''_{a,j}\alpha^{-(i+j)}P'_{a,j}(\gamma^*,\gamma)\otimes C_{l,j,q}^{-1}g_j^{(l)}.\end{aligned}$$

Hence we find

$$u_{i,j}^{(l)} = \alpha^{-(i+j)}\cdot C_{l,i,q}C_{l,j,q}^{-1}\sum_a C''_{a,i,j,l,q}P'_{a,i,j,l,q}(\gamma^*,\gamma). \tag{6.3.2}$$

Now since the only power of α that occurs in (6.3.2) is $\alpha^{-(i+j)}$, the $u_{i,j}^{(l)}$ are eigenvectors for the maps $\tilde{T}_m$. □

Corollary 6.3.3. *The map (6.2.1) is a Fourier-Schur multiplier for $\mathbb{G}_q$ with symbol $(m(-i-j))_{i,j,l}$ where $l \in \frac{1}{2}\mathbb{N}$ indexes the corepresentation and $1 \le i,j \le 2l+1$.*

Corollary 6.3.4. *For every $m \in \ell_\infty(\mathbb{Z})$ there is a map $\tilde{T}_m^{(2)} : L_2(\mathbb{G}_q) \to L_2(\mathbb{G}_q)$ extending (6.2.1) by*

$$\tilde{T}_m^{(2)} \circ \kappa_{\frac{1}{2}} = \kappa_{\frac{1}{2}} \circ \tilde{T}_m$$

which is bounded with norm at most $\|m\|_\infty$. If $m(0) = 0$ then $\tilde{T}_m^{(2)} : L_2(\mathbb{G}_q) \to L_2^\circ(\mathbb{G}_q)$.

Proof. Define the φ-GNS inner product on $\mathrm{Pol}(\mathbb{G}_q)$ by $\langle x,y\rangle = \varphi(x^*y)$ and denote the associated GNS space by $\mathcal{H}_\varphi$. By Lemma 6.3.1 and Proposition 6.3.2 we see that $\tilde{T}_m : \mathrm{Pol}(\mathbb{G}_q) \to \mathrm{Pol}(\mathbb{G}_q)$ is bounded with respect to this inner product with bound at most $\|m\|_\infty$. Hence it extends to a map $\tilde{T}_m^\varphi : \mathcal{H}_\varphi \to \mathcal{H}_\varphi$. By Proposition 2.4.15 we have that

$$\mathrm{Pol}(\mathbb{G}_q) \to L_2(\mathbb{G}_q) : x \mapsto xD_\varphi^{1/2}$$

is an isometry with respect to this inner product on the left and hence extends to a unitary map $U : \mathcal{H}_\varphi \to L_2(\mathbb{G}_q)$. Then the map $\tilde{T}_m^{(2)} := U\tilde{T}_m^\varphi U^* : L_2(\mathbb{G}_q) \to L_2(\mathbb{G}_q)$ satisfies the conditions. The final statement is Lemma 6.2.3.

□

6

6.4. Transference principle and construction of the BMO space

In this subsection we construct the BMO spaces corresponding to $\mathbb{G}_q = SU_q(2)$ for $q \in (-1,1)\backslash\{0\}$ that we need for the upper endpoint estimate. The main tool behind both the construction of the BMO spaces and the proof of the actual upper endpoint estimate is the transference principle of Lemma 6.4.2. The idea is to obtain properties of Fourier-Schur multipliers on $L_\infty(\mathbb{G}_q)$ from properties of Fourier multipliers on $L_\infty(\mathbb{T})$.

Let $e_{i,j}$ be the matrix units in $\mathcal{B}(\ell_2(\mathbb{Z}_{\geq 0}))$ and recall that $\zeta_i : \mathbb{T} \to \mathbb{T}$ was defined by $z \mapsto z^i$. We define the unitary

$$U = \sum_{i=0}^{\infty} e_{i,i} \otimes 1_{\mathcal{B}(\ell_2(\mathbb{Z}))} \otimes \zeta_i \in \mathcal{B}(\mathcal{H}) \bar{\otimes} L_\infty(\mathbb{T}),$$

and the injective normal $*$-homomorphism

$$\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \bar{\otimes} L_\infty(\mathbb{T}) : x \mapsto U^*(x \otimes 1)U.$$

Lemma 6.4.1. *We have for $k \in \mathbb{Z}, l, m \in \mathbb{N}$ that*

$$\pi(\alpha^k \gamma^l (\gamma^*)^m) = \alpha^k \gamma^l (\gamma^*)^m \otimes \zeta_k. \tag{6.4.1}$$

Proof. For $\xi \in L_2(\mathbb{T})$, $i \in \mathbb{Z}_{\geq 0}$, $j \in \mathbb{Z}$,

$$\begin{aligned}
&\pi(\alpha^k \gamma^l (\gamma^*)^m)(e_i \otimes f_j \otimes \xi) \\
=& U^*(\alpha^k \gamma^l (\gamma^*)^m \otimes \mathrm{id})(e_i \otimes f_j \otimes \zeta_i \xi) \\
=& U^* \sqrt{(1-q^{2i})(1-q^{2i-2})\dots(1-q^{2i-2k+2})}\, q^{i(l+m)} e_{i-k} \otimes f_{j+l-m} \otimes \zeta_i \xi \\
=& \sqrt{(1-q^{2i})(1-q^{2i-2})\dots(1-q^{2i-2k+2})}\, q^{i(l+m)} e_{i-k} \otimes f_{j+l-m} \otimes \zeta_k \xi \\
=& (\alpha^k \gamma^l (\gamma^*)^m \otimes \zeta_k)(e_i \otimes f_j \otimes \xi).
\end{aligned}$$

□

This implies that π maps $\mathrm{Pol}(\mathbb{G}_q)$ into $\mathrm{Pol}(\mathbb{G}_q) \otimes \mathrm{L}_\infty(\mathbb{T})$. Hence by density, it maps $L_\infty(\mathbb{G}_q)$ into $L_\infty(\mathbb{G}_q) \bar{\otimes} L_\infty(\mathbb{T})$. We denote $\iota_{\mathcal{M}}$ for the identity operator $\mathcal{M} \to \mathcal{M}$ on a von Neumann algebra $\mathcal{M}$, reserving $1_{\mathcal{M}}$ for the unit of $\mathcal{M}$. The following identity is now immediate. We refer to this identity as the 'transference principle'.

Lemma 6.4.2. *Let $\tilde{m} \in \ell_\infty(\mathbb{Z})$. For $k \in \mathbb{Z}, l, m \in \mathbb{N}$ we have*

$$(\iota_{L_\infty(\mathbb{G}_q)} \otimes T_{\tilde{m}})\pi(\alpha^k \gamma^l (\gamma^*)^m) = \tilde{m}(k)\pi(\alpha^k \gamma^l (\gamma^*)^m).$$

Set again the Heat multipliers $h_t(k) = e^{-tk^2}, k \in \mathbb{Z}, t \geq 0$. Let us define a semigroup on $L_\infty(\mathbb{G}_q) \bar{\otimes} L_\infty(\mathbb{T})$ by $S = (S_t)_{t\geq 0}$ with $S_t := \iota_{L_\infty(\mathbb{G}_q)} \otimes T_{h_t}$. Recall that $(T_{h_t})_{t \geq 0}$ is a Markov semigroup (see Section 6.1.1). By approximation with elements from the algebraic tensor product and Proposition 2.5.5, one can prove that S is also a Markov semigroup. From this and the transference principle, we can now prove that the semigroup $(\Phi_t)_{t\geq 0}$ we defined in Section 6.2 is actually a well-defined Markov semigroup.

Proposition 6.4.3. *The family of maps given by the assignment*

$$\Phi_t(\alpha^k \gamma^l (\gamma^*)^m) = e^{-tk^2} \alpha^k \gamma^l (\gamma^*)^m, \qquad k \in \mathbb{Z}, l, m \in \mathbb{N}, t \geq 0,$$

extends to a Markov semigroup of Fourier-Schur multipliers $\Phi := (\Phi_t)_{t\geq 0}$ *on* $L_\infty(\mathbb{G}_q)$ *satisfying*

$$\pi \circ \Phi_t = S_t \circ \pi.$$

Moreover, the semi-group is modular.

Proof. By Lemma 6.4.2 we have the commutative diagram:

$$\begin{array}{ccc} L_\infty(\mathbb{G}_q)\bar{\otimes}L_\infty(\mathbb{T}) & \xrightarrow{S_t} & L_\infty(\mathbb{G}_q)\bar{\otimes}L_\infty(\mathbb{T}) \\ \pi\uparrow & & \pi\uparrow \\ \mathrm{Pol}(\mathbb{G}_\mathrm{q}) & \xrightarrow{\Phi_t} & L_\infty(\mathbb{G}_q) \end{array}$$

π is a normal injective $*$-homomorphism so that we may view $L_\infty(\mathbb{G}_q)$ as a (unital) von Neumann subalgebra of $L_\infty(\mathbb{G}_q)\bar{\otimes}L_\infty(\mathbb{T})$. We find that Φ_t, being the restriction of S_t to $\mathrm{Pol}(\mathbb{G}_q)$, is also a normal ucp map. This means that Φ_t extends to a normal ucp map on $L_\infty(\mathbb{G}_q)$. By the same argument, we deduce strong continuity of $t \mapsto \Phi_t(x)$. This shows properties (i) and (iii) of Definition 5.3.1.

To show property (ii), we recall (see (6.1.1)) that the Haar functional φ on $\mathbb{G}_q$ is non-zero on basis elements $\alpha^k\gamma^l(\gamma^*)^m$ only if $k = 0, l = m$. If $x = \alpha^k\gamma^l(\gamma^*)^m$, $y = \alpha^{k'}\gamma^{l'}(\gamma^*)^{m'}$, then $xy = C\alpha^{k+k'}\gamma^{l+l'}(\gamma^*)^{m+m'}$ for some constant C. Thus, $\varphi(x\Phi_t(y)) = \varphi(\Phi_t(x)y)$ on basis elements x, y, and hence everywhere. Finally, by the formula for the modular automorphism group (6.1.2), we find that Φ_t is φ-modular. □

We define corresponding BMO spaces for this semigroup. We use the shorthand notation $\mathrm{BMO}(\mathbb{G}_q)$ for $\mathrm{BMO}(L_\infty(\mathbb{G}_q), \Phi)$, and similarly for the column and row spaces. We can also define a BMO-norm $\|\cdot\|_{\mathrm{BMO}_S}$ on $(L_\infty(\mathbb{G}_q)\bar{\otimes}L_\infty(\mathbb{T}))^\circ$. We will do some of the estimates within the normed spaces $(L_\infty^\circ(\mathbb{G}_q), \|\cdot\|_{\mathrm{BMO}_\Phi})$ and $((L_\infty(\mathbb{G}_q)\bar{\otimes}L_\infty(\mathbb{T}))^\circ, \|\cdot\|_{\mathrm{BMO}_S})$ to avoid some technicalities.

Lemma 6.4.4. *The map* π *is isometric as a map between normed spaces*

$$\pi : (L_\infty^\circ(\mathbb{G}_q), \|\cdot\|_{\mathrm{BMO}_\Phi}) \to ((L_\infty(\mathbb{G}_q)\bar{\otimes}L_\infty(\mathbb{T}))^\circ, \|\cdot\|_{\mathrm{BMO}_S}).$$

Proof. This follows from the commutative diagram of Proposition 6.4.3 and the fact that π is an injective, hence isometric, $*$-homomorphism $L_\infty(\mathbb{G}_q) \to L_\infty(\mathbb{G}_q)\bar{\otimes}L_\infty(\mathbb{T})$. Indeed, for $x \in L_\infty(\mathbb{G}_q)^\circ$, we have that

$$\|S_t(\pi(x))\|_\infty = \|(\pi \circ \Phi_t)(x)\|_\infty \to 0,$$

which implies in particular σ-weak convergence. Hence $\pi(x) \in (L_\infty(\mathbb{G}_q)\bar{\otimes}L_\infty(\mathbb{T}))^\circ$. Also,

$$\begin{aligned}\|\pi(x)\|^2_{\mathrm{BMO}^c_{\mathrm{S}}} &= \sup_{t\geq 0}\|S_t(|\pi(x)-S_t(\pi(x))|^2)\| = \sup_{t\geq 0}\|S_t(|\pi(x)-\pi(\Phi_t(x))|^2)\| \\ &= \sup_{t\geq 0}\|S_t(\pi(|x-\Phi_t(x)|^2))\| = \sup_{t\geq 0}\|\pi(\Phi_t(|x-\Phi_t(x)|^2))\| \\ &= \sup_{t\geq 0}\|\Phi_t(|x-\Phi_t(x)|^2)\| = \|x\|^2_{\mathrm{BMO}^c_\Phi}.\end{aligned}$$

Replacing x by x^* yields isometry for the row BMO-norm from which it follows that π is isometric on BMO as well. □

6.5. L_∞-BMO ESTIMATE

We proceed to prove an upper end point estimate for $\tilde{T}_m$. Recall that we defined a “weak-$*$ topology" on $\mathrm{BMO}(\mathbb{G}_q)$ in Remark 5.4.4.

Theorem 6.5.1. *Let $m \in \ell_\infty(\mathbb{Z})$ with $m(0) = 0$ be such that $T_m : L_\infty(\mathbb{T}) \to \mathrm{BMO}(\mathbb{T})$ is completely bounded. Then there exists a bounded weak-$*$/weak-$*$ continuous map*

$$\tilde{T}_m^{(\infty)} : L_\infty(\mathbb{G}_q) \to \mathrm{BMO}(\mathbb{G}_\mathrm{q}),$$

satisfying $\tilde{T}_m^{(\infty)}(x) = \kappa_1^{1/2}(\tilde{T}_m(x))$ for $x \in \mathrm{Pol}(\mathbb{G}_\mathrm{q})$. Moreover,

$$\|\tilde{T}_m^{(\infty)} : L_\infty(\mathbb{G}_q) \to \mathrm{BMO}(\mathbb{G}_q)\| \leq \|T_m : L_\infty(\mathbb{T}) \to \mathrm{BMO}(\mathbb{T})\|_{cb}. \tag{6.5.1}$$

The proof consists of the following two lemmas. We first prove a BMO-norm estimate of $\tilde{T}_m$ for the polynomial algebra, using again the transference principle from Lemma 6.4.2.

Lemma 6.5.2. *Let $m \in \ell_\infty(\mathbb{Z})$ with $m(0) = 0$ be such that $T_m : L_\infty(\mathbb{T}) \to \mathrm{BMO}(\mathbb{T})$ is completely bounded. Then for $x \in \mathrm{Pol}(\mathbb{G}_\mathrm{q})$:*

$$\|\tilde{T}_m(x)\|_{\mathrm{BMO}_\Phi} \leq \|T_m : L_\infty(\mathbb{T}) \to \mathrm{BMO}(\mathbb{T})\|_{cb}\|x\|_\infty. \tag{6.5.2}$$

Proof. By Lemma 6.2.3, $\tilde{T}_m$ maps $\mathrm{Pol}(\mathbb{G}_\mathrm{q})$ to $L^\circ_\infty(\mathbb{G}_q)$. Note that π sends $\mathrm{Pol}(\mathbb{G}_\mathrm{q})$ to $L_\infty(\mathbb{G}_q)\otimes \mathrm{Pol}(\mathbb{T})$ and $\iota_{L_\infty(\mathbb{G}_q)}\otimes T_m$ sends $L_\infty(\mathbb{G}_q)\otimes\mathrm{Pol}(\mathbb{T})$ to $L_\infty(\mathbb{G}_q)\otimes L^\circ_\infty(\mathbb{T}) \subseteq (L_\infty(\mathbb{G}_q)\bar{\otimes}L_\infty(\mathbb{T}))^\circ$ (see also Appendix 5.6). Now Lemma 6.4.2 gives us a commutative diagram like in Proposition 6.4.3.

$$\begin{array}{ccc} L_\infty(\mathbb{G}_q)\otimes\mathrm{Pol}(\mathbb{T}) & \xrightarrow{\iota_{L_\infty(\mathbb{G}_q)}\otimes T_m} & (L_\infty(\mathbb{G}_q)\bar{\otimes}L_\infty(\mathbb{T}))^\circ \\ \pi\uparrow & & \pi\uparrow \\ \mathrm{Pol}(\mathbb{G}_\mathrm{q}) & \xrightarrow{\quad\tilde{T}_m\quad} & L^\circ_\infty(\mathbb{G}_q) \end{array}$$

Note that the restriction $T_m : \mathrm{Pol}(\mathbb{T}) \to (\mathrm{L}^\circ_\infty(\mathbb{T}), \|\cdot\|_{\mathrm{BMO}})$ is also completely bounded. Now Lemma 6.4.4 and Proposition 5.6.1 allows us to find a BMO-estimate on $\tilde{T}_m$ for $x \in \mathrm{Pol}(\mathbb{G}_\mathrm{q})$:

$$\|\tilde{T}_m(x)\|_{\mathrm{BMO}_\Phi} = \|\pi\circ\tilde{T}_m(x)\|_{\mathrm{BMO}_S} = \|(\iota_{L_\infty(\mathbb{G}_q)}\otimes T_m)\circ\pi(x)\|_{\mathrm{BMO}_S}$$
$$\leq \|T_m\|_{cb}\|\pi(x)\| = \|T_m\|_{cb}\|x\|_\infty.$$

where $\|T_m\|_{cb} = \|T_m : L_\infty(\mathbb{T}) \to \mathrm{BMO}(\mathbb{T})\|_{cb}$. □

Recall that $\kappa_1^{1/2}$ isometrically embeds the normed space $(L_\infty^\circ(\mathbb{G}_q), \|\cdot\|_{\mathrm{BMO}_\Phi})$ into $\mathrm{BMO}(\mathbb{G}_q)$. Now define $\tilde{T}_m^{(\infty)} = \kappa_1^{1/2}\circ\tilde{T}_m$, which we may consider as a bounded map from $\mathrm{Pol}(\mathbb{G}_q)$ to $\mathrm{BMO}(\mathbb{G}_q)$ by Lemma 6.5.2. It remains to prove that this map extends to $L_\infty(\mathbb{G}_q)$. The proof is essentially that of [JMP14, Lemma 1.6] together with a number of technicalities that we overcome here.

Lemma 6.5.3. *$\tilde{T}_m^{(\infty)}$ has a weak-$*$/weak-$*$ continuous extension to $L_\infty(\mathbb{G}_q) \to \mathrm{BMO}(\mathbb{G}_q)$.*

Proof. Let $h_1^c(\mathbb{G}_q) := h_1^c(L_\infty(\mathbb{G}_q),\Phi)$ and $h_1^r(\mathbb{G}_q) := h_1^r(L_\infty(\mathbb{G}_q),\Phi)$ be the preduals constructed in Section 5.4. We will construct maps $S_c : h_1^c(\mathbb{G}_q) \to L_1(\mathbb{G}_q)$ and $S_r : h_1^r(\mathbb{G}_q) \to L_1(\mathbb{G}_q)$ such that their adjoints are equal and extend $\tilde{T}_m^{(\infty)}$.

Construction of maps S_c and S_r. We construct the map $S_c : h_1^c(\mathbb{G}_q) \to L_1(\mathbb{G}_q)$ by proving that the map $\kappa_{2,1}^1\circ(\tilde{T}_m^{(2)})^*$ is bounded as a map $L_2^\circ(\mathbb{G}_q) \to L_1(\mathbb{G}_q)$ with respect to $\|\cdot\|_{h_1^c(\mathbb{G}_q)}$ on the left. For $y \in L_2^\circ(\mathbb{G}_q)$ and $z \in \mathrm{Pol}(\mathbb{G}_q)$ we find

$$\langle z, (\tilde{T}_m^{(2)})^*(y)D_\varphi^{1/2}\rangle = \langle D_\varphi^{1/2}z, (\tilde{T}_m^{(2)})^*(y)\rangle = \langle D_\varphi^{1/2}\tilde{T}_m(z), y\rangle. \tag{6.5.3}$$

By the Kaplansky density theorem and [Tak02, Theorem II.2.6] the unit ball of $\mathrm{Pol}(\mathbb{G}_q)$ is weak-$*$ dense in the unit ball of $L_\infty(\mathbb{G}_q)$. Hence for $y \in L_2^\circ(\mathbb{G}_q)$ we find:

$$\begin{aligned}\|\kappa_{2,1}^1((\tilde{T}_m^{(2)})^*y)\|_{L_1(\mathbb{G}_q)} &= \sup_{z\in\mathrm{Pol}(\mathbb{G}_q)_{\leq 1}} |\langle z, (\tilde{T}_m^{(2)})^*(y)D_\varphi^{1/2}\rangle| \\ &= \sup_{z\in\mathrm{Pol}(\mathbb{G}_q)_{\leq 1}} |\langle D_\varphi^{1/2}\tilde{T}_m(z), y\rangle| \leq \|T_m\|_{cb}\|y\|_{h_1^c(\mathbb{G}_q)}.\end{aligned}$$

In the last step we used that $\|\kappa_2^0(\tilde{T}_m(z))\|_{\mathrm{BMO}^r} = \|\tilde{T}_m(z)\|_{\mathrm{BMO}^r} \leq \|T_m\|_{cb}\|z\|_\infty$. We conclude that $\kappa_{2,1}^1\circ(\tilde{T}_m^{(2)})^*$ extends to a bounded map

$$S_c : h_1^c(\mathbb{G}_q) \to L_1(\mathbb{G}_q).$$

In a similar manner we can prove that the map $\kappa_{2,1}^0\circ(\tilde{T}_m^{(2)})^*$ extends to a bounded map

$$S_r : h_1^r(\mathbb{G}_q) \to L_1(\mathbb{G}_q).$$

By taking limits in (6.5.3), we can prove the following equalities for $z \in \mathrm{Pol}(\mathbb{G}_q)$, $y_a \in h_1^c(\mathbb{G}_q)$ and $y_b \in h_1^r(\mathbb{G}_q)$:

$$\langle z, S_c(y_a)\rangle = \langle D_\varphi^{1/2}\tilde{T}_m(z), y_a\rangle, \qquad \langle z, S_r(y_b)\rangle = \langle \tilde{T}_m(z)D_\varphi^{1/2}, y_b\rangle \tag{6.5.4}$$

Analysis of the adjoint maps. Now consider the adjoint maps $S_r^*: L_\infty(\mathbb{G}_q) \to \mathrm{BMO}^c(\mathbb{G}_q)$ and $S_c^*: L_\infty(\mathbb{G}_q) \to \mathrm{BMO}^r(\mathbb{G}_q)$. They are weak-$*$/weak-$*$ continuous by Proposition 2.1.1. By composition, we get maps

$$T_c := \kappa_{2,1}^0 \circ S_r^* : L_\infty(\mathbb{G}_q) \to L_1^\circ(\mathbb{G}_q), \qquad T_r := \kappa_{2,1}^1 \circ S_c^* : L_\infty(\mathbb{G}_q) \to L_1^\circ(\mathbb{G}_q).$$

Let $z \in \mathrm{Pol}(\mathbb{G}_q)$ and $y \in h_1^c(\mathbb{G}_q)$. Then (6.5.4) yields

$$\langle S_c^*(z), y\rangle = \langle z, S_c(y)\rangle = \langle D_\varphi^{1/2} \tilde{T}_m(z), y\rangle$$

so S_c^* extends $\kappa_2^0 \circ \tilde{T}_m$. Hence, T_r is a weak-$*$/weak-$*$ continuous extension of $\kappa_1^{1/2} \circ \tilde{T}_m = \tilde{T}_m^{(\infty)}$. In a similar way, we find that T_c is a weak-$*$/weak-$*$ continuous extension of $\tilde{T}_m^{(\infty)}$. In particular, T_c and T_r coincide on $\mathrm{Pol}(\mathbb{G}_q)$. It remains to prove that $T_c = T_r$; this implies that the image of this map is contained in $\mathrm{BMO}(\mathbb{G}_q)$, and hence it is the weak-$*$/weak-$*$ continuous extension of $\tilde{T}_m^{(\infty)}$ that we were looking for.

Proof of the equality $T_r = T_c$. Let $x \in L_\infty(\mathbb{G}_q)$ and take a net $x_\lambda \in \mathrm{Pol}(\mathbb{G}_q)$ such that $x_\lambda \to x$ in the weak-$*$ topology. By the weak-$*$/weak-$*$ continuity of S_r^*, we have $S_r^*(x_\lambda) \to S_r^*(x) =: y_c \in \mathrm{BMO}^c(\mathbb{G}_q)$ in the weak-$*$ topology. Similarly, we have $S_c^*(x_\lambda) \to S_c^*(x) =: y_r \in \mathrm{BMO}^r(\mathbb{G}_q)$ in the weak-$*$ topology. We need to prove that $D_\varphi^{1/2} y_c = y_r D_\varphi^{1/2}$ in $L_1^\circ(\mathbb{G}_q)$.

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First let $z \in L_\infty^\circ(\mathbb{G}_q)$. In that case, we have

$$\langle S_r^*(x_\lambda), z D_\varphi^{1/2}\rangle_{\mathrm{BMO}^c, h_1^r} = \langle T_c(x_\lambda), z\rangle_{1,\infty} = \langle T_r(x_\lambda), z\rangle_{1,\infty} = \langle S_c^*(x_\lambda), D_\varphi^{1/2} z\rangle_{\mathrm{BMO}^r, h_1^c}.$$

Hence,

$$\begin{aligned}\langle D_\varphi^{1/2} y_c, z\rangle_{1,\infty} &= \langle y_c, z D_\varphi^{1/2}\rangle_{\mathrm{BMO}^c, h_1^r} = \lim_\lambda \langle S_r^*(x_\lambda), z D_\varphi^{1/2}\rangle_{\mathrm{BMO}^c, h_1^r} \\ &= \lim_\lambda \langle S_c^*(x_\lambda), D_\varphi^{1/2} z\rangle_{\mathrm{BMO}^r, h_1^c} = \langle y_r, D_\varphi^{1/2} z\rangle_{\mathrm{BMO}^r, h_1^c} = \langle y_r D_\varphi^{1/2}, z\rangle_{1,\infty}.\end{aligned}$$

Now let $z \in L_\infty(\mathbb{G}_q)$. Let $\mathbb{E}$ be the projection of $L_\infty(\mathbb{G}_q)$ onto $L_\infty^\circ(\mathbb{G}_q)$. Then as in Proposition 2.4.28, we get a projection $\mathbb{E}^{(1)}: L_1(\mathbb{G}_q) \to L_1^\circ(\mathbb{G}_q)$ which is the adjoint of $\mathbb{E}$. Hence,

$$\langle D_\varphi^{1/2} y_c, z\rangle = \langle \mathbb{E}^{(1)}(D_\varphi^{1/2} y_c), z\rangle = \langle D_\varphi^{1/2} y_c, \mathbb{E}(z)\rangle = \langle y_r D_\varphi^{1/2}, \mathbb{E}(z)\rangle = \langle y_r D_\varphi^{1/2}, z\rangle.$$

We conclude that $D_\varphi^{1/2} y_c = y_r D_\varphi^{1/2}$; this finishes the proof. □

Proof of Theorem 6.5.1. The existence of $\tilde{T}_m^{(\infty)}$ follows from Lemma 6.5.2 and 6.5.3. The inequality in (6.5.1) follows from (6.5.2) and the Kaplansky density theorem. □

6.6. A Markov dilation of the Markov semigroup Φ

Definition 6.6.1. We say that a Markov semigroup Φ on a σ-finite von Neumann algebra $\mathcal{M}$ with faithful normal state φ admits a *standard Markov dilation* if there exist:

(i) a σ-finite von Neumann algebra $\mathcal{N}$ with normal faithful state $\varphi_{\mathcal{N}}$,

(ii) an increasing filtration $(\mathcal{N}_s)_{s\geq 0}$ with $\varphi_{\mathcal{N}}$-preserving conditional expectations $\mathcal{E}_s : \mathcal{N} \to \mathcal{N}_s$,

(iii) a $*$-homomorphisms $\pi_s : \mathcal{M} \to \mathcal{N}_s$ such that $\varphi_{\mathcal{N}} \circ \pi_s = \varphi_{\mathcal{N}}$ and

$$\mathcal{E}_s(\pi_t(x)) = \pi_s(\Phi_{t-s}(x)), \quad s < t, x \in \mathcal{M}.$$

A Markov dilation is called φ-modular if it additionally satisfies

$$\pi_s \circ \sigma_t^{\varphi} = \sigma_t^{\varphi_{\mathcal{N}}} \circ \pi_s, \quad s \geq 0, t \in \mathbb{R}.$$

One can analogously define the notion of a reversed Markov dilation; we refer to [CJSZ20, Definition 5.1] for the precise statement.

In this section, we construct a Markov dilation for Φ. To construct the Markov dilation, we use the fact that $L_\infty(\mathbb{G}_q)$ can be written as the tensor product of two relatively simple von Neumann algebras. This is a well-known fact; we give a sketch of the proof for the convenience of the reader. We let $\mathcal{L}(\mathbb{Z})$ be the group von Neumann algebra of $\mathbb{Z}$ generated by the left regular representation λ.

Proposition 6.6.2. $L_\infty(\mathbb{G}_q) = \mathcal{B}(\ell_2(\mathbb{Z}_{\geq 0}))\bar{\otimes}\mathcal{L}(\mathbb{Z})$.

Proof. Let $T_m, T_{\tilde{m}}$ be the multiplication maps on $\ell_2(\mathbb{Z}_{\geq 0})$ with symbols $m(k) = q^k$, $\tilde{m}(k) = \sqrt{1-q^{2k}}$. Then we can write

$$\gamma = T_m \otimes \lambda_{1,\mathbb{Z}}, \quad \alpha = (\lambda^*_{1,\mathbb{Z}_{\geq 0}} T_{\tilde{m}}) \otimes 1$$

where we denote $\lambda_{1,\mathbb{Z}}$ and $\lambda_{1,\mathbb{Z}_{\geq 0}}$ for the right shift on $\ell_2(\mathbb{Z})$ and $\ell_2(\mathbb{Z}_{\geq 0})$ respectively. From these expressions it is immediately clear that $L_\infty(\mathbb{G}_q) \subseteq \mathcal{B}(\ell_2(\mathbb{Z}_{\geq 0}))\bar{\otimes}\mathcal{L}(\mathbb{Z})$. For the other inclusion, note that the partial isometries in the polar decompositions of α, γ are $1 \otimes \lambda_{1,\mathbb{Z}}$ and $\lambda^*_{1,\mathbb{Z}_{\geq 0}} \otimes 1$ respectively. These elements generate $1 \otimes \mathcal{L}(\mathbb{Z})$ and $\mathcal{B}(\ell_2(\mathbb{Z}_{\geq 0})) \otimes 1$ respectively as von Neumann algebras. Hence the other inclusion follows from the definition of the von Neumann algebraic tensor product. □

Through this expression for $L_\infty(\mathbb{G}_q)$ we will show that Φ_t can be written as a Schur multiplier.

Proposition 6.6.3. *The semi-group Φ admits a (standard and reversed) φ-modular Markov dilation.*

Proof. We prove first that Φ_t can be written as a Schur multiplier on the left tensor leg of $L_\infty(\mathbb{G}_q)$. Let $x = \alpha^k \gamma^l (\gamma^*)^m$. x acts on basis vectors by

$$e_i \otimes f_r \overset{x}{\mapsto} c e_{i-k} \otimes f_{r+l-m}, \quad c := c_{q,k,l,m,i,r} = \sqrt{(1-q^{2i})(1-q^{2i-2})\dots(1-q^{2i-2k+2})}\, q^{i(l+m)}.$$

In other words, the matrix elements of x are given by

$$\langle x e_i \otimes f_r, e_j \otimes f_s \rangle = c\, \delta_{j,i-k} \delta_{s,r+l-m}.$$

Hence if we define $\Psi_t : \mathscr{B}(\ell_2(\mathbb{Z}_{\geq 0})) \to \mathscr{B}(\ell_2(\mathbb{Z}_{\geq 0}))$ as the Schur multiplier given by $\Psi_t(e_{i,j}) = e^{-t|i-j|^2} e_{i,j}$, then we have

$$\Phi_t(x) = e^{-tk^2} x = (\Psi_t \otimes \mathrm{id}_{\mathscr{L}(\mathbb{Z})})(x)$$

Hence Φ_t and $\Psi_t \otimes \mathrm{id}_{\mathscr{L}(\mathbb{Z})}$ coincide on $\mathrm{Pol}(\mathbb{G}_q)$. Since both are normal (Proposition 6.4.3 for Φ_t and Lemma 2.6.13 ii) for Ψ_t) they must coincide on $L_\infty(\mathbb{G}_q)$.

The proof from now on is essentially that of [Ric08] or [CJSZ20, Proposition 4.2] with the main difference that the unitary u below only sums over the indices of $\ell_2(\mathbb{Z}_{\geq 0})$. Let $\varepsilon > 0$ be arbitrary. We define a sesquilinear form on the real finite linear span $\mathscr{H}_0 = \mathrm{Span}_{\mathbb{R}}\{e_i, i \geq 0\} \subseteq \mathscr{H}$ by setting

$$\langle \xi, \eta \rangle = \sum_{i,j \geq 0} e^{-\varepsilon (j-i)^2} \xi_i \eta_j, \quad \xi, \eta \in \mathscr{H}_0$$

We define $\mathscr{H}_{\mathbb{R}}$ to be the completion of $\mathscr{H}_0$ with respect to $\langle \cdot, \cdot \rangle$ after quotienting out the degenerate part. Let $\Gamma = \Gamma(\mathscr{H}_{\mathbb{R}})$ be the associated exterior algebra (see [CJSZ20, Section 2.8]) with vacuum vector Ω and canonical vacuum state τ_Ω. The dilation von Neumann algebra $(\mathscr{B}, \varphi_{\mathscr{B}})$ will be given by

$$\mathscr{B} = L_\infty(\mathbb{G}_q) \bar{\otimes} \Gamma^{\bar{\otimes}\infty}, \qquad \varphi_{\mathscr{B}} = \varphi \otimes \tau_\Omega^{\otimes\infty}$$

where the infinite tensor product is taken with respect to τ_Ω. Next we describe the dilation homomorphisms π_s. We consider the unitary

$$u = \sum_{i \geq 0} e_{i,i} \otimes 1_{\mathscr{L}(\mathbb{Z})} \otimes s(e_i) \otimes 1_\Gamma^{\otimes\infty} \in L_\infty(\mathbb{G}_q) \bar{\otimes} \Gamma^{\bar{\otimes}\infty}$$

which is defined as a strong limit of sums. Let $S : v \mapsto 1 \otimes v$ be the tensor shift on $\Gamma^{\bar{\otimes}\infty}$, and let $\beta : \mathscr{B} \to \mathscr{B}$ be defined by $\beta(z) = u^*(\iota_{L_\infty(\mathbb{G}_q)} \otimes S)(z) u$. The $*$-homomorphisms $\pi_s : L_\infty(\mathbb{G}_q) \to \mathscr{B}$ are given by

$$\pi_0 : x \mapsto x \otimes 1 \otimes 1 \ldots, \qquad \pi_k : x \mapsto (\beta^k \circ \pi_0)(x), \quad k \geq 1.$$

One shows by induction that for $x \in L_\infty(\mathbb{G}_q)$

$$\pi_k(x) = \sum_{i,j \geq 0} e_{i,i} x e_{j,j} \otimes (s(e_i) s(e_j))^{\otimes k} \otimes 1_\Gamma^{\otimes\infty}.$$

By (6.1.1) it follows that π_k is state-preserving, and by [Tak03b, Proposition XIV.1.11], it is φ-modular.

Finally, the filtration is given by

$$\mathscr{B}_m = L_\infty(\mathbb{G}_q) \bar{\otimes} \Gamma^{\otimes m} \otimes 1_\Gamma^{\otimes\infty} \subseteq \mathscr{B}.$$

One checks that the associated conditional expectations satisfy

$$\begin{aligned} &\mathscr{E}_m(e_{i,i} x e_{j,j} \otimes (s(e_i) s(e_j))^{\otimes k} \otimes \mathrm{id}_\Gamma^{\otimes\infty}) \\ =&\tau_\Omega(s(e_i) s(e_j))^{k-m} e_{i,i} x e_{j,j} \otimes (s(e_i) s(e_j))^{\otimes m} \otimes 1_\Gamma^{\otimes\infty}. \end{aligned}$$

From this and the identity

$$\tau_\Omega(s(e_i)s(e_j)) = \langle s(e_j)\Omega, s(e_i)\Omega\rangle = e^{-\varepsilon(j-i)^2}$$

one deduces that indeed

$$(\mathscr{E}_m \circ \pi_k)(x) = \pi_m(\Phi_{\varepsilon(k-m)}(x)).$$

So the semigroup $(\Phi_{\varepsilon n})_{n\geq 0}$ admits a Markov dilation for any $\varepsilon > 0$. By [CJSZ20, Theorem 3.2], $(\Phi_t)_{t\geq 0}$ admits a standard Markov dilation. This theorem is stated only for finite von Neumann algebras, but it also holds in the σ-finite case with the same proof mutatis mutandis. A reversed Markov dilation can be obtained by essentially the same argument and a σ-finite analogue of [CJSZ20, Theorem 5.3]. □

6.7. CONSEQUENCES FOR L_p-FOURIER SCHUR MULTIPLIERS

Theorem 6.7.1. *Let $m \in \ell_\infty(\mathbb{Z})$ with $m(0) = 0$ be such that the Fourier multiplier $T_m : L_\infty(\mathbb{T}) \to \mathrm{BMO}(\mathbb{T})$ is completely bounded. Let $\tilde{T}_m : \mathrm{Pol}(\mathbb{G}_q) \to \mathrm{Pol}(\mathbb{G}_q)$ be the Fourier-Schur multiplier with symbol $(m(-i-j))_{i,j,l}$ with respect to the basis described in (6.3.1), where $l \in \frac{1}{2}\mathbb{N}$ indexes the corepresentation and $1 \leq i, j \leq 2l+1$. Then for $1 < p < \infty$, $\tilde{T}_m$ extends to a bounded map*

$$\tilde{T}_m^{(p)} : L_p(\mathbb{G}_q) \to L_p^\circ(\mathbb{G}_q),$$

where by 'extension' we mean that $\tilde{T}_m^{(p)}(\kappa_p^1(x)) = \kappa_p^1(\tilde{T}_m(x))$

Proof. Proposition 6.3.4 and Theorem 6.5.1 show that $\tilde{T}_m^{(\infty)}$ and $\tilde{T}_m^{(2)}$ together are compatible morphisms. Therefore, by Riesz-Torin (2.4.4), we get bounded maps on the interpolation spaces. Since Φ admits a Markov dilation (see Proposition 6.6.3), Theorem 5.5.6 tells us that

$$[\mathrm{BMO}(\mathbb{G}_q), L_2^\circ(\mathbb{G}_q)]_{2/p} \approx L_p^\circ(\mathbb{G}_q).$$

Also we have by (2.4.10) that

$$[L_\infty(\mathbb{G}_q), L_2(\mathbb{G}_q)]_{2/p} \approx L_p(\mathbb{G}_q).$$

This proves that for $2 \leq p < \infty$ we can construct bounded maps $\tilde{T}_m^{(p)} : L_p(\mathbb{G}_q) \to L_p^\circ(\mathbb{G}_q)$ that extend $\tilde{T}_m$ - or more precisely, they satisfy $\tilde{T}_m^{(p)}(\kappa_p^1(x)) = \kappa_p^1(\tilde{T}_m(x))$ for all $x \in \mathrm{Pol}(\mathbb{G}_q)$.

Now consider $1 < p < 2$ and let p' be such that $\frac{1}{p} + \frac{1}{p'} = 1$. Then the adjoint map $\tilde{T}_m^*$ is simply the Fourier multiplier with symbol $\bar{m}$, and hence by the above argument $\tilde{T}_m^*$ extends to a map on $L_{p'}(\mathbb{G}_q)$. Hence the map $\tilde{T}_m^{(p)} : L_p(\mathbb{G}_q) \to L_p(\mathbb{G}_q)$ given by the double adjoint is the extension we were looking for. □

Remark 6.7.2. The condition that $m(0) = 0$ is not very important: if we 'add a constant sequence to m', i.e. we switch to the map $T_{m+\lambda 1} = T_m + \lambda\iota_{L_\infty(\mathbb{T})}$, then this map still 'extends' (in the sense of the theorem) to a bounded map $L_p(\mathbb{G}_q) \to L_p(\mathbb{G}_q)$.

Remark 6.7.3. [JMP14, Lemma 3.3] constructs classes of completely bounded multipliers $L_\infty(\mathbb{T}) \to \mathrm{BMO}(\mathbb{T})$. Further, in [JMP14, Lemma 1.3] the connection between classical BMO-spaces and non-commutative semi-group BMO spaces is established giving further examples. This shows that indeed the class of symbols m to which Theorem 6.7.1 applies is non-empty and contains a reasonable class of examples.

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7

THE RELATIVE HAAGERUP PROPERTY

This chapter is based on the following article:

1. **Martijn Caspers, Mario Klisse, Adam Skalski, Gerrit Vos, Mateusz Wasilewski**, *Relative Haagerup property for arbitrary van Neumann algebras*, to appear in Advances of Mathematics.

In this chapter, we undertake a systematic study of the relative Haagerup property for unital expected inclusions of σ-finite von Neumann algebras. This chapter is somewhat isolated from the rest of the thesis, and does not use the theory of noncommutative L_p-spaces developed in Section 2.4.

We will study the relative Haagerup property in the following context: we have a unital inclusion $\mathcal{N} \subseteq \mathcal{M}$ of σ-finite von Neumann algebras equipped with a faithful normal conditional expectation $\mathbb{E}_{\mathcal{N}} : \mathcal{M} \to \mathcal{N}$. We first define it in terms of a fixed faithful normal state (preserved by $\mathbb{E}_{\mathcal{N}}$) but then quickly show that it depends only on the conditional expectation in question. Essentially, for a triple $(\mathcal{M}, \mathcal{N}, \mathbb{E}_{\mathcal{N}})$ to have the relative Haagerup property we require the existence of completely positive, normal, $\mathcal{N}$-bimodular maps on $\mathcal{M}$ which are $\mathbb{E}_{\mathcal{N}}$-decreasing, L_2-compact (in the sense determined by the conditional expectation $\mathbb{E}_{\mathcal{N}}$), uniformly bounded and converge point-strongly to the identity, see Definition 7.2.2. Much more can be said in the case where $\mathcal{N}$ is assumed to be finite; here we obtain the following theorem, which is a combination of Theorem 7.4.4 and Theorem 7.4.5. It is one of the main results of the chapter.

Theorem G. *Suppose that $\mathcal{N} \subseteq \mathcal{M}$ is a unital, expected inclusion of von Neumann algebras and assume that $\mathcal{N}$ is finite. Then the relative Haagerup property of the triple $(\mathcal{M}, \mathcal{N}, \mathbb{E}_{\mathcal{N}})$ does not depend on the choice of a faithful normal conditional expectation $\mathbb{E}_{\mathcal{N}} : \mathcal{M} \to \mathcal{N}$. Moreover if $(\mathcal{M}, \mathcal{N}, \mathbb{E}_{\mathcal{N}})$ has the relative Haagerup property and we are given a fixed state $\tau \in \mathcal{N}_*$ we can always assume that the approximating maps are unital and $\tau \circ \mathbb{E}_{\mathcal{N}}$-preserving.*

The key idea of the proof once again, as in [CS15c], uses crossed products by modular actions and the passage to the semifinite setting that (Takai-)Takesaki duality permits. However, the relative context makes the technical details much more demanding and makes adapting the earlier methods – including those developed in [BF11] – significantly more complicated. On the other hand, allowing non-trivial inclusions allows us to significantly broaden the class of examples fitting into our framework and yields certain facts which are new even in the context of the standard Haagerup property of finite von Neumann algebras. This is exemplified by the next key result of this work and its corollary (which also requires proving a general theorem on the relative Haagerup property of amalgamated free products). These are Theorem 7.6.6 and Corollary 7.7.2.

Theorem H. *Suppose that $\mathscr{N} \subseteq \mathscr{M}$ is a unital inclusion of von Neumann algebras equipped with a faithful normal conditional expectation $\mathbb{E}_{\mathscr{N}} : \mathscr{M} \to \mathscr{N}$ and assume that $\mathscr{N}$ is finite-dimensional. Then the relative Haagerup property of the triple $(\mathscr{M}, \mathscr{N}, \mathbb{E}_{\mathscr{N}})$ is equivalent to the usual Haagerup property of $\mathscr{M}$.*

Corollary I. *Suppose that $\mathscr{N} \subset \mathscr{M}_1$ and $\mathscr{N} \subset \mathscr{M}_2$ are unital inclusions of von Neumann algebras equipped with respective faithful normal conditional expectations. If $\mathscr{N}$ is finite-dimensional and both $\mathscr{M}_1, \mathscr{M}_2$ have the Haagerup property, then the amalgamated free product $\mathscr{M}_1 *_{\mathscr{N}} \mathscr{M}_2$ also has the Haagerup property.*

We illustrate our results with examples coming on one hand from the class of q-deformed Hecke-von Neumann algebras of (virtually free) Coxeter groups, and on the other hand from discrete quantum groups. Of particular interest is also the elementary case of $\mathscr{M} = \mathscr{B}(\mathscr{H})$ which provides us with both triples $(\mathscr{B}(\mathscr{H}), \mathscr{N}, \mathbb{E}_{\mathscr{N}})$ that do and do not have the relative Haagerup property. This is rather surprising as it gives examples of co-amenable inclusions of von Neumann algebras in the sense of [Pop86] (see also [BMO20]) without the relative Haagerup property.

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Let us describe the contents of the chapter. we recall some facts regarding von Neumann algebras, their modular theory and completely positive approximations in Section 7.1, and introduce (certain variants of) the definition of the relative Haagerup property in Section 7.2. The latter section also contains the initial discussion of the independence of our notion of various ingredients, mainly in the semifinite setting. Section 7.3, the most technical part of the chapter, introduces the crossed product arguments allowing us to reduce the general problem to the semifinite case. Section 7.4 contains the main general results of the paper, including Theorem G. In Section 7.5 we briefly describe the known examples of Haagerup inclusions related to Cartan subalgebras. Here we also study the case $\mathscr{M} = \mathscr{B}(\mathscr{H})$ which leads to very interesting counterexamples. In Section 7.6 we show that in the case of a finite-dimensional subalgebra the relative Haagerup property is equivalent to the Haagerup property of the larger algebra and prove Theorem H. In Section 7.7 we discuss the behaviour of the relative Haagerup property with respect to the amalgamated free product construction and show Corollary I. In Section 7.7 we also discuss briefly a consequence of these results for the Hecke-von Neumann algebras associated to virtually free Coxeter groups. Finally in a short Section 7.8 we present an example of a Haagerup inclusion coming from quantum groups, which in fact is even strongly of finite index.

7.1. PRELIMINARIES AND NOTATION

We stress that in this chapter, the symbol λ is used for the inclusion of $\mathbb{R}$ into the crossed product. It still behaves like the left regular representation, but it acts on the Hilbert space $L_2(\mathbb{R},\mathcal{H})$ instead of $L_2(\mathbb{R})$.

We will assume throughout this chapter that the von Neumann algebras we study are σ-finite, i.e. they admit faithful normal states. For a σ-finite von Neumann algebra $\mathcal{M}$ with normal faithful state φ, we denote by Ω_φ its cyclic vector. For this chapter, we also denote $L_2(\mathcal{M},\varphi) := \mathcal{H}_\varphi$ to be its GNS Hilbert space, so that we include the underlying von Neumann algebra in the notation. We denote the norm on $L_2(\mathcal{M},\varphi)$ by $\|\cdot\|_2$. We will usually identify $\mathcal{M}$ with its image under the GNS representation, so $\mathcal{M} \subseteq \mathcal{B}(L_2(\mathcal{M},\varphi))$. We further write $\|x\|_{2,\varphi} := \varphi(x^*x)^{1/2}$ for $x \in \mathcal{M}$ or, if φ is clear from the context, $\|x\|_2 := \|x\|_{2,\varphi}$. Recall that right multiplication of $\mathcal{M}$ on $L_2(\mathcal{M},\varphi)$ is given by $\xi x := J_\varphi x^* J_\varphi \xi$

We will always assume inclusions of von Neumann algebras $\mathcal{N} \subseteq \mathcal{M}$ to be unital in the sense that $1_\mathcal{M} \in \mathcal{N}$, and conditional expectations to be faithful and normal. We will usually repeat these conditions throughout the text. We recall that for a functional $\varphi \in \mathcal{M}_*$ and elements $a,b \in \mathcal{M}$ we denote by $a\varphi b \in \mathcal{M}_*$ the normal functional given by $(a\varphi b)(x) := \varphi(bax)$, $x \in \mathcal{M}$, and further write $a\varphi$ for $a\varphi 1$ and φb for $1\varphi b$.

Recall the definition of the centralizer $\mathcal{M}^\varphi$ from Section 2.3.1. We will often consider the situation where $\mathcal{N} \subseteq \mathcal{M}$ is a unital embedding, equipped with a faithful normal conditional expectation $\mathbb{E}_\mathcal{N} : \mathcal{M} \to \mathcal{N}$, $\tau \in \mathcal{N}_*$ is a faithful tracial state, and $\varphi = \tau \circ \mathbb{E}_\mathcal{N}$. Then an easy computation shows that $\mathcal{N} \subseteq \mathcal{M}^\varphi$.

Let us specialise Proposition 2.4.28 to the case $p = 2$, while remaining in the GNS picture. Let $(\mathcal{M},\varphi)$ and $(\mathcal{N},\psi)$ be σ-finite von Neumann algebras with normal faithful states φ and ψ. For a linear map $\Phi : \mathcal{M} \to \mathcal{N}$ we say that its *L^2-implementation* $\Phi^{(2)}$ (with respect to φ and ψ) exists if the map $x\Omega_\varphi \mapsto \Phi(x)\Omega_\psi$ extends to a bounded operator $\Phi^{(2)} : L_2(\mathcal{M},\varphi) \to L_2(\mathcal{N},\psi)$. This is the case if and only if there exists a constant $C > 0$ such that for all $x \in \mathcal{M}$,

$$\psi(\Phi(x)^*\Phi(x)) \le C\varphi(x^*x).$$

In particular, if Φ is 2-positive (see the beginning of Section 2.2) with $\psi \circ \Phi \le \varphi$, the Kadison-Schwarz inequality implies that $\Phi^{(2)}$ exists with $\|\Phi^{(2)}\| \le \|\Phi(1)\|^{1/2}$. Indeed, for $x \in \mathcal{M}$

$$\begin{aligned} \|\Phi(x)\Omega_\psi\|_2^2 &= \psi(\Phi(x)^*\Phi(x)) \le \|\Phi(1)\|\psi(\Phi(x^*x)) \\ &\le \|\Phi(1)\|\varphi(x^*x) = \|\Phi(1)\|\|x\Omega_\varphi\|_2^2. \end{aligned}$$

This implies that if Φ is contractive then so is $\Phi^{(2)}$.

The general principle of the following lemma was used as part of a proof in [CS15c] and [Jol02] a number of times. Here we present it separately. We will also need several straightforward variations of this lemma. Because they can be proved in a very similar way, we shall not state them here. The essence of the result is that, given two nets of maps with suitable properties that strongly converge to the identity, the composition of these maps gives rise to a net that also converges to the identity in the strong operator topology.

Lemma 7.1.1. *Let $(\mathcal{M},\varphi)$ and $(\mathcal{N},\varphi_j)$, $j\in\mathbb{N}$ be pairs of von Neumann algebras equipped with faithful normal states. Consider a normal completely positive map $\pi:\mathcal{M}\to\mathcal{N}$, a bounded sequence of normal completely positive maps $(\Psi_j:\mathcal{N}\to\mathcal{M})_{j\in\mathbb{N}}$ and for every $j\in\mathbb{N}$ a bounded net of completely positive maps $(\Phi_{j,k}:\mathcal{N}\to\mathcal{N})_{k\in K_j}$. Assume that for all $j\in\mathbb{N}$, $k\in K_j$ the inequalities $\varphi_j\circ\pi\le\varphi$, $\varphi\circ\Psi_j\le\varphi_j$ and $\varphi_j\circ\Phi_{j,k}\le\varphi_j$ hold, that $\Psi_j\circ\pi(x)\to x$ strongly in j for every $x\in\mathcal{M}$ and that for every $j\in\mathbb{N}$, $x\in\mathcal{N}$ we have $\Phi_{j,k}(x)\to x$ strongly in k. Then there exists a directed set $\mathcal{F}$ and a function $(\widetilde{j},\widetilde{k})\colon\mathcal{F}\to\{(j,k)\mid j\in\mathbb{N},k\in K_j\}$, $F\mapsto(\widetilde{j}(F),\widetilde{k}(F))$ such that $\Psi_{\widetilde{j}(F)}\circ\Phi_{\widetilde{j}(F),\widetilde{k}(F)}\circ\pi(x)\to x$ strongly in F for every $x\in\mathcal{M}$.*

Proof. For $j\in\mathbb{N}$ and $k\in K_j$ write

$$\begin{aligned}
\pi_j^{(2)} &: L^2(\mathcal{M},\varphi)\to L^2(\mathcal{N},\varphi_j), & x\Omega_\varphi &\mapsto \pi(x)\Omega_{\varphi_j},\\
\Psi_j^{(2)} &: L^2(\mathcal{N},\varphi_j)\to L^2(\mathcal{M},\varphi), & x\Omega_{\varphi_j} &\mapsto \Phi_{j,k}(x)\Omega_\varphi,\\
\Phi_{j,k}^{(2)} &: L^2(\mathcal{N},\varphi_j)\to L^2(\mathcal{N},\varphi_j), & x\Omega_{\varphi_j} &\mapsto \Phi_{j,k}(x)\Omega_{\varphi_j}
\end{aligned}$$

for the corresponding L^2-implementations with respect to φ and φ_j. Let $C\ge1$ be a bound for the norms of $(\Psi_j)_{j\in\mathbb{N}}$ and hence for the norms of $(\Psi_j^{(2)})_{j\in\mathbb{N}}$. We shall make use of the fact that on bounded sets the strong topology coincides with the L^2-topology determined by a state, see Proposition 2.4.29. Therefore we have strong limits $\Psi_j^{(2)}\pi_j^{(2)}\to 1$ in $\mathcal{B}(L^2(\mathcal{M},\varphi))$ and $\Phi_{j,k}^{(2)}\to 1$ in $\mathcal{B}(L^2(\mathcal{N},\varphi_j))$. Now let $F\subseteq L^2(\mathcal{M},\varphi)$ be a finite subset. We may find $j=\widetilde{j}(F)\in\mathbb{N}$ such that for all $\xi\in F$,

$$\|\Psi_j^{(2)}\pi_j^{(2)}\xi-\xi\|_2<|F|^{-1}.$$

In turn, we may find $k=\widetilde{k}(j,F)=\widetilde{k}(F)$ such that for all $\xi\in F$,

$$\|\Phi_{j,k}^{(2)}\pi_j^{(2)}\xi-\pi_j^{(2)}\xi\|_2<|F|^{-1}.$$

From the triangle inequality and by using that the operator norm of $\Psi_j^{(2)}$ is bounded by C,

$$\begin{aligned}
\|\Psi_j^{(2)}\Phi_{j,k}^{(2)}\pi_j^{(2)}\xi-\xi\|_2 &\le \|\Psi_j^{(2)}\Phi_{j,k}^{(2)}\pi_j^{(2)}\xi-\Psi_j^{(2)}\pi_j^{(2)}\xi\|_2+\|\Psi_j^{(2)}\pi_j^{(2)}\xi-\xi\|_2\\
&\le \|\Psi_j^{(2)}\|\,\|\Phi_{j,k}^{(2)}\pi_j^{(2)}\xi-\pi_j^{(2)}\xi\|_2+\|\Psi_j^{(2)}\pi_j^{(2)}\xi-\xi\|_2\\
&< (1+C)|F|^{-1}.
\end{aligned}$$

This implies that $\Psi_{\widetilde{j}(F)}^{(2)}\Phi_{\widetilde{j}(F),\widetilde{k}(F)}^{(2)}\pi_{\widetilde{j}(F)}^{(2)}\to 1$ strongly in $\mathcal{B}(L^2(\mathcal{M},\varphi))$ where the net is indexed by all finite subsets of $L^2(\mathcal{M},\varphi)$ partially ordered by inclusion. Using once more Proposition 2.4.29, one sees that for $x\in\mathcal{M}$ we have that $\Psi_{\widetilde{j}(F)}\circ\Phi_{\widetilde{j}(F),\widetilde{k}(F)}\circ\pi(x)\to x$ strongly. The claim follows. □

7.2. RELATIVE HAAGERUP PROPERTY

In this section we introduce the relative Haagerup property for inclusions of general σ-finite von Neumann algebras and consider natural variations of the definition. For this, fix a triple $(\mathcal{M},\mathcal{N},\varphi)$ where $\mathcal{N}\subseteq\mathcal{M}$ is a unital inclusion of von Neumann algebras and where φ is a faithful normal positive functional on $\mathcal{M}$ whose corresponding modular automorphism group $(\sigma_t^{\varphi})_{t\in\mathbb{R}}$ satisfies $\sigma_t^{\varphi}(\mathcal{N})\subseteq\mathcal{N}$ for all $t\in\mathbb{R}$. To keep the notation short, we will often just write $(\mathcal{M},\mathcal{N},\varphi)$ and will implicitly assume that the triple satisfies the mentioned conditions. By [Tak03a, Theorem IX.4.2] the assumption $\sigma_t^{\varphi}(\mathcal{N})\subseteq\mathcal{N}$, $t\in\mathbb{R}$ is equivalent to the existence of a (uniquely determined) φ-preserving (necessarily faithful) normal conditional expectation $\mathbb{E}_{\mathcal{N}}^{\varphi}:\mathcal{M}\to\mathcal{N}$. If the corresponding functional φ is clear, we will often just write $\mathbb{E}_{\mathcal{N}}$ instead of $\mathbb{E}_{\mathcal{N}}^{\varphi}$ (compare also with Subsection 7.2.2).

7.2.1. FIRST DEFINITION OF RELATIVE HAAGERUP PROPERTY

For a triple $(\mathcal{M},\mathcal{N},\varphi)$ as before define the *Jones projection*

$$e_{\mathcal{N}}^{\varphi}:=\mathbb{E}_{\mathcal{N}}^{(2)}:L^2(\mathcal{M},\varphi)\to L^2(\mathcal{M},\varphi),$$

to be the orthogonal projection onto $L^2(\mathcal{N},\varphi)\subseteq L^2(\mathcal{M},\varphi)$ and let $\langle\mathcal{M},\mathcal{N}\rangle\subseteq\mathcal{B}(L^2(\mathcal{M},\varphi))$ be the von Neumann subalgebra generated by $e_{\mathcal{N}}^{\varphi}$ and $\mathcal{M}$. This is the *Jones construction*. We will usually write $e_{\mathcal{N}}$ instead of $e_{\mathcal{N}}^{\varphi}$ if there is no ambiguity. Further set

$$\mathcal{K}_{00}(\mathcal{M},\mathcal{N},\varphi):=\mathrm{Span}\{xe_{\mathcal{N}}y\mid x,y\in\mathcal{M}\}\subseteq\mathcal{B}(L^2(\mathcal{M},\varphi))$$

and

$$\mathcal{K}(\mathcal{M},\mathcal{N},\varphi):=\overline{\mathcal{K}_{00}(\mathcal{M},\mathcal{N},\varphi)}.$$

Then $\mathcal{K}_{00}(\mathcal{M},\mathcal{N},\varphi)$ is a (not necessarily closed) two-sided ideal in $\langle\mathcal{M},\mathcal{N}\rangle$ whose elements are called the *finite rank operators* relative to $\mathcal{N}$. Similarly, $\mathcal{K}(\mathcal{M},\mathcal{N},\varphi)$ is a closed two-sided ideal in $\langle\mathcal{M},\mathcal{N}\rangle$ whose elements are called the *compact operators* relative to $\mathcal{N}$. Note that if $\mathcal{N}=\mathbb{C}1_{\mathcal{M}}$, then $e_{\mathcal{N}}$ is a rank one projection and the operators in $\mathcal{K}(\mathcal{M},\mathcal{N},\varphi)$ are precisely the compact operators on $L^2(\mathcal{M},\varphi)$.

Remark 7.2.1. In the following it is often convenient to identify a finite rank operator $ae_{\mathcal{N}}b\in\mathcal{K}_{00}(\mathcal{M},\mathcal{N},\varphi)$, $a,b\in\mathcal{M}$ with the map $a\mathbb{E}_{\mathcal{N}}(b\,\cdot):\mathcal{M}\to\mathcal{M}$. The latter does not depend on φ (but only on the conditional expectation $\mathbb{E}_{\mathcal{N}}$), and the notation is naturally compatible with the inclusion $\mathcal{M}\subset L^2(\mathcal{M},\varphi)$. We will often write $a\mathbb{E}_{\mathcal{N}}b:=a\mathbb{E}_{\mathcal{N}}(b\,\cdot)$.

Definition 7.2.2. Let $\mathcal{N}\subseteq\mathcal{M}$ be a unital inclusion of von Neumann algebras and let φ be a faithful normal positive functional on $\mathcal{M}$ with $\sigma_t^{\varphi}(\mathcal{N})\subseteq\mathcal{N}$ for all $t\in\mathbb{R}$. We say that the triple $(\mathcal{M},\mathcal{N},\varphi)$ has the *relative Haagerup property* (or just *property (rHAP)*) if there exists a net $(\Phi_i)_{i\in I}$ of normal maps $\Phi_i:\mathcal{M}\to\mathcal{M}$ such that

1. Φ_i is completely positive and $\sup_i\|\Phi_i\|<\infty$ for all $i\in I$;
2. Φ_i is an $\mathcal{N}$-$\mathcal{N}$-bimodule map for all $i\in I$;
3. $\Phi_i(x)\to x$ strongly for every $x\in\mathcal{M}$;
4. $\varphi\circ\Phi_i\leq\varphi$ for all $i\in I$;

5. For every $i \in I$ the L^2-implementation

$$\Phi_i^{(2)} : L^2(\mathcal{M},\varphi) \to L^2(\mathcal{M},\varphi), x\Omega_\varphi \mapsto \Phi_i(x)\Omega_\varphi,$$

is contained in $\mathcal{K}(\mathcal{M},\mathcal{N},\varphi)$.

Remark 7.2.3. (1) In many applications φ will be a faithful normal state, but for notational convenience we shall rather work in the more general setting. Note that we may always normalize φ to be a state and that the definition of the relative Haagerup property does not change under this normalization.
(2) Note that in [Pop06] (see also [Jol05]) a different notion of relative compactness is used to define the relative Haagerup property. It coincides with ours in case $\mathcal{N}' \cap \mathcal{M} \subseteq \mathcal{N}$. However, the alternative notion is not very suitable beyond the tracial situation since it requires the existence of finite projections; we will return to this issue in Section 7.5.1.
(3) In the case where $\mathcal{N} = \mathbb{C}1_{\mathcal{M}}$ Definition 7.2.2 recovers the usual definition of the (non-relative) Haagerup property, see [CS15c, Definition 3.1].

There is a number of immediate variations of Definition 7.2.2. For instance, one may replace the condition (1) by one of the following stronger conditions:

(1′) For every $i \in I$ the map Φ_i is contractive completely positive.

(1″) For every $i \in I$ the map Φ_i is unital completely positive.

We may also replace the condition (4) by the following condition:

(4′) $\varphi \circ \Phi_i = \varphi$.

One of the results that we shall prove is that if the subalgebra $\mathcal{N}$ is finite, then condition (4) is redundant. We will further prove that in this setting the approximating maps Φ_i, $i \in I$ can be chosen to be unital and state-preserving implying that all the variations of the relative Haagerup property from above coincide. To simplify the statements of the following sections, let us introduce the following auxiliary notion, which is a priori weaker (see Section 7.2.3).

Definition 7.2.4. Let $\mathcal{N} \subseteq \mathcal{M}$ be a unital inclusion of von Neumann algebras and let φ be a faithful normal positive functional on $\mathcal{M}$ with $\sigma_t^\varphi(\mathcal{N}) \subseteq \mathcal{N}$ for all $t \in \mathbb{R}$. We say that the triple $(\mathcal{M},\mathcal{N},\varphi)$ has *property (rHAP)$^-$* if there exists a net $(\Phi_i)_{i\in I}$ of normal maps $\Phi_i : \mathcal{M} \to \mathcal{M}$ such that

1. Φ_i is completely positive for all $i \in I$;
2. Φ_i is an $\mathcal{N}$-$\mathcal{N}$-bimodule map for all $i \in I$;
3. $\|\Phi_i(x) - x\|_{2,\varphi} \to 0$ for every $x \in \mathcal{M}$;
4. For every $i \in I$ the L^2-implementation

$$\Phi_i^{(2)}: L^2(\mathcal{M},\varphi) \to L^2(\mathcal{M},\varphi), x\Omega_\varphi \mapsto \Phi_i(x)\Omega_\varphi,$$

exists and is contained in $\mathcal{K}(\mathcal{M},\mathcal{N},\varphi)$.

7.2.2. DEPENDENCE ON THE POSITIVE FUNCTIONAL: REDUCTION TO THE DEPENDENCE ON THE CONDITIONAL EXPECTATION

Let $\mathcal{N} \subseteq \mathcal{M}$ be a unital inclusion of von Neumann algebras which admits a faithful normal conditional expectation $\mathbb{E}_{\mathcal{N}} : \mathcal{M} \to \mathcal{N}$. Recall ([Tak03a, Theorem IX.4.2]) that for every faithful normal positive functional φ on $\mathcal{M}$ with $\varphi \circ \mathbb{E}_{\mathcal{N}} = \varphi$ the corresponding modular automorphism group $(\sigma_t^{\varphi})_{t\in\mathbb{R}}$ satisfies $\sigma_t^{\varphi}(\mathcal{N}) = \mathcal{N}$ for all $t \in \mathbb{R}$. Note that such a functional always exists, as it suffices to pick a faithful normal state $\omega \in \mathcal{N}_*$ (which exists by our standing σ-finiteness assumption) and set $\varphi = \omega \circ \mathbb{E}_{\mathcal{N}}$. In this subsection we will examine the dependence of the relative Haagerup property of $(\mathcal{M}, \mathcal{N}, \varphi)$ on the functional φ. We shall prove that the property rather depends on the conditional expectation $\mathbb{E}_{\mathcal{N}}$ than on φ.

Lemma 7.2.5. *Let $\Phi : \mathcal{M} \to \mathcal{M}$ be $\mathcal{N}$-$\mathcal{N}$-bimodular. Then the following statements are equivalent:*

1. *$\mathbb{E}_{\mathcal{N}} \circ \Phi \leq \mathbb{E}_{\mathcal{N}}$ (resp. $\mathbb{E}_{\mathcal{N}} \circ \Phi = \mathbb{E}_{\mathcal{N}}$).*
2. *For all $\varphi \in \mathcal{M}_*^+$ with $\varphi \circ \mathbb{E}_{\mathcal{N}} = \varphi$ we have $\varphi \circ \Phi \leq \varphi$ (resp. $\varphi \circ \Phi = \varphi$).*
3. *There exists a faithful functional $\varphi \in \mathcal{M}_*^+$ with $\varphi \circ \mathbb{E}_{\mathcal{N}} = \varphi$ such that $\varphi \circ \Phi \leq \varphi$ (resp. $\varphi \circ \Phi = \varphi$).*

Further, the following statements are equivalent:

4. *There exists $C > 0$ such that $\mathbb{E}_{\mathcal{N}}(\Phi(x)^*\Phi(x)) \leq C\mathbb{E}_{\mathcal{N}}(x^*x)$ for all $x \in \mathcal{M}$.*
5. *There exists $C > 0$ such that for all $\varphi \in \mathcal{M}_*^+$ with $\varphi \circ \mathbb{E}_{\mathcal{N}} = \varphi$ and $x \in \mathcal{M}$ we have $\varphi(\Phi(x)^*\Phi(x)) \leq C\varphi(x^*x)$.*
6. *There exists $C > 0$ and a faithful functional $\varphi \in \mathcal{M}_*^+$ with $\varphi \circ \mathbb{E}_{\mathcal{N}} = \varphi$ such that for all $x \in \mathcal{M}$ we have $\varphi(\Phi(x)^*\Phi(x)) \leq C\varphi(x^*x)$.*

In particular, if the L^2-implementation of Φ with respect to φ exists, then it exists with respect to any other ψ with $\psi \circ \mathbb{E}_{\mathcal{N}} = \psi$.

Proof. We prove the statements for the inequalities; the respective cases with equalities follow similarly. The implications (1) ⇔ (2) ⇒ (3) of the first three statements are trivial. For the implication (3) ⇒ (1) take φ as in (3). For $x \in \mathcal{N}$ consider the positive functional $x^*\varphi x \in \mathcal{M}_*^+$ which again satisfies $(x^*\varphi x) \circ \mathbb{E}_{\mathcal{N}} = x^*\varphi x$. Then, for $y \in \mathcal{M}^+$

$$(x^*\varphi x) \circ \mathbb{E}_{\mathcal{N}} \circ \Phi(y) = \varphi \circ \mathbb{E}_{\mathcal{N}} \circ \Phi(xyx^*) \leq \varphi \circ \mathbb{E}_{\mathcal{N}}(xyx^*) = (x^*\varphi x) \circ \mathbb{E}_{\mathcal{N}}(y).$$

Since the restrictions of functionals $x^*\varphi x$, $x \in \mathcal{N}$ to $\mathcal{N}$ are dense in $\mathcal{N}_*^+$ we conclude that $\mathbb{E}_{\mathcal{N}} \circ \Phi \leq \mathbb{E}_{\mathcal{N}}$. The equivalence of the statements (4), (5) and (6) follows in a similar way. □

The following lemma shows that in good circumstances compactness of the L^2-implementations does not depend on the choice of the state.

Lemma 7.2.6. *Let $\varphi, \psi \in \mathcal{M}_*^+$ be faithful with $\varphi \circ \mathbb{E}_{\mathcal{N}} = \varphi$ and $\psi \circ \mathbb{E}_{\mathcal{N}} = \psi$. Let further $\Phi : \mathcal{M} \to \mathcal{M}$ be a completely positive $\mathcal{N}$-$\mathcal{N}$-bimodule map whose L^2-implementation $\Phi_\varphi^{(2)}$ with respect to φ exists (hence, by Lemma 7.2.5, the L^2-implementation $\Phi_\psi^{(2)}$ of Φ with respect to ψ exists as well). Then, $\Phi_\varphi^{(2)} \in \mathcal{K}(\mathcal{M}, \mathcal{N}, \varphi)$ if and only if $\Phi_\psi^{(2)} \in \mathcal{K}(\mathcal{M}, \mathcal{N}, \psi)$.*

Proof. Let U be the unique unitary mapping the standard form $(\mathcal{M}, L^2(\mathcal{M},\varphi), J_\varphi, P_\varphi)$ to the standard form $(\mathcal{M}, L^2(\mathcal{M},\psi), J_\psi, P_\psi)$, see Proposition 2.3.2. It restricts to the unique unitary map between the standard forms $(\mathcal{N}, L^2(\mathcal{N},\varphi), J_{\varphi|_\mathcal{N}}, P_{\varphi|_\mathcal{N}})$ and $(\mathcal{N}, L^2(\mathcal{N},\psi), J_{\psi|_\mathcal{N}}, P_{\psi|_\mathcal{N}})$. Indeed, for all $x \in \mathcal{M}$, $\varphi(x) = \langle xU\Omega_\varphi, U\Omega_\varphi\rangle$ and by [Haa75, Lemma 2.10], $U\Omega_\varphi$ is the unique element in $L^2(\mathcal{M},\psi)$ satisfying this equation. On the other hand, applying [Haa75, Lemma 2.10] to $\varphi|_\mathcal{N}$ implies the existence of a unique vector $\xi \in L^2(\mathcal{N},\psi)$ such that $\varphi(x) = \langle x\xi,\xi\rangle$ for all $x \in \mathcal{N}$. By approximating ξ by elements in $\mathcal{N}\Omega_\psi$ and by using the assumptions $\varphi\circ\mathbb{E}_\mathcal{N} = \varphi$ and $\psi\circ\mathbb{E}_\mathcal{N} = \psi$ one deduces that for all $x \in \mathcal{M}$,

$$\varphi(x) = \varphi\circ\mathbb{E}_\mathcal{N}(x) = \langle \mathbb{E}_\mathcal{N}(x)\xi,\xi\rangle = \langle x\xi,\xi\rangle$$

and hence $U\Omega_\varphi = \xi \in L^2(\mathcal{N},\psi)$. This implies $U(\mathcal{N}\Omega_\varphi) \subseteq L^2(\mathcal{N},\psi)$ and therefore (by density and symmetry) $U(L^2(\mathcal{N},\psi)) = L^2(\mathcal{N},\psi)$. Finally, using that $\sigma_t^\varphi(\mathcal{N}) = \mathcal{N}$ and hence $J_{\varphi|_\mathcal{N}} = (J_\varphi)|_{L^2(\mathcal{N},\varphi)}$ (and similarly $J_{\psi|_\mathcal{N}} = (J_\psi)|_{L^2(\mathcal{N},\psi)}$) it is straightforward to check that the restriction of U satisfies the other properties of the unique unitary mapping between the standard forms $(\mathcal{N}, L^2(\mathcal{N},\varphi), J_{\varphi|_\mathcal{N}}, P_{\varphi|_\mathcal{N}})$ and $(\mathcal{N}, L^2(\mathcal{N},\psi), J_{\psi|_\mathcal{N}}, P_{\psi|_\mathcal{N}})$.

Since $e_\mathcal{N}^\varphi = (\mathbb{E}_\mathcal{N}^\varphi)^{(2)}$ is the orthogonal projection of $L^2(\mathcal{M},\varphi)$ onto $L^2(\mathcal{N},\varphi)$ and $e_\mathcal{N}^\psi = (\mathbb{E}_\mathcal{N}^\psi)^{(2)}$ is the orthogonal projection of $L^2(\mathcal{M},\psi)$ onto $L^2(\mathcal{N},\psi)$, we see that $U^* e_\mathcal{N}^\psi U = e_\mathcal{N}^\varphi$. Hence, for every map Λ of the form $\Lambda = a\mathbb{E}_\mathcal{N} b$ with $a, b \in \mathcal{M}$ the L^2-implementation $\Lambda_\varphi^{(2)}$ with respect to φ and the L^2-implementation $\Lambda_\psi^{(2)}$ with respect to ψ exist with $\Lambda_\varphi^{(2)} = ae_\mathcal{N}^\varphi b = U^* ae_\mathcal{N}^\psi bU = U^*\Lambda_\psi^{(2)} U$.

Now assume that $\Phi_\varphi^{(2)} \in \mathcal{K}(\mathcal{M},\mathcal{N},\varphi)$. Then there exists a sequence $(\Phi_k)_{k\in\mathbb{N}}$ of maps $\mathcal{M} \to \mathcal{M}$ of the form $\Phi_k = \sum_{i=1}^{N_k} a_{i,k}\mathbb{E}_\mathcal{N} b_{i,k}$ with $N_k \in \mathbb{N}$ and $a_{1,k}, b_{1,k}, \ldots, a_{N_k,k}, b_{N_k,k} \in \mathcal{M}$ whose L^2-implementations $\Phi_{k,\varphi}^{(2)} \in \mathcal{K}_{00}(\mathcal{M},\mathcal{N},\varphi)$ (with respect to φ) norm-converge to $\Phi_\varphi^{(2)}$. By the above, $U\Phi_{k,\varphi}^{(2)} U^* \in \mathcal{K}_{00}(\mathcal{M},\mathcal{N},\psi)$ is given by $x\Omega_\psi \mapsto \Phi_k(x)\Omega_\psi$ for $x \in \mathcal{M}$. We claim that the sequence $(U\Phi_{k,\varphi}^{(2)} U^*)_{k\in\mathbb{N}} \subseteq \mathcal{K}_{00}(\mathcal{M},\mathcal{N},\psi)$ norm-converges to $\Phi_\psi^{(2)}$. Indeed, by the density of the set of all elements of the form $x(\varphi|_\mathcal{N})x^*$, $x \in \mathcal{N}$ in $\mathcal{N}_*^+$ we find a net $(x_i)_{i\in I} \subseteq \mathcal{N}$ such that $x_i(\varphi|_\mathcal{N})x_i^* \to \psi|_\mathcal{N}$. In combination with $\varphi\circ\mathbb{E}_\mathcal{N} = \varphi$ and $\psi\circ\mathbb{E}_\mathcal{N} = \psi$ this also implies $x_i\varphi x_i^* \to \psi$. For $y \in \mathcal{M}$ and $k \in \mathbb{N}$,

$$\begin{aligned}
\|(\Phi_\psi^{(2)} - U\Phi_{k,\varphi}^{(2)} U^*)y\Omega_\psi\|_{2,\psi}^2 &= \|(\Phi(y) - \Phi_k(y))\Omega_\psi\|_{2,\psi}^2 \\
&= \lim_i \|(\Phi(y) - \Phi_k(y))x_i\Omega_\varphi\|_{2,\varphi}^2 \\
&= \lim_i \|(\Phi_\varphi^{(2)} - \Phi_{k,\varphi}^{(2)})yx_i\Omega_\varphi\|_{2,\varphi}^2 \\
&\leq \lim_i \|\Phi_\varphi^{(2)} - \Phi_{k,\varphi}^{(2)}\|^2 \varphi(x_i^* y^* yx_i) \\
&= \|\Phi_\varphi^{(2)} - \Phi_{k,\varphi}^{(2)}\|^2 \psi(y^* y).
\end{aligned}$$

In the third step we used the $\mathcal{N}$-$\mathcal{N}$-bimodularity of Φ and the right $\mathcal{N}$-modularity of Φ_k. Now, $\Phi_{k,\varphi}^{(2)} \to \Phi_\varphi^{(2)}$ and $(U\Phi_{k,\varphi}^{(2)} U^*)_{k\in\mathbb{N}}$ is a Cauchy sequence, hence the above inequality leads to $U\Phi_{k,\varphi}^{(2)} U^* \to \Phi_\psi^{(2)}$ as claimed. In particular, $\Phi_\psi^{(2)} \in \mathcal{K}(\mathcal{M},\mathcal{N},\psi)$ which finishes the proof. □

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Theorem 7.2.7. *Let $\mathcal{N} \subseteq \mathcal{M}$ be a unital inclusion of von Neumann algebras which admits a faithful normal conditional expectation $\mathbb{E}_{\mathcal{N}} : \mathcal{M} \to \mathcal{N}$. Further, let $\varphi, \psi \in \mathcal{M}_*^+$ be faithful normal positive functionals with $\varphi \circ \mathbb{E}_{\mathcal{N}} = \varphi$ and $\psi \circ \mathbb{E}_{\mathcal{N}} = \psi$. Then the triple $(\mathcal{M}, \mathcal{N}, \varphi)$ has property (rHAP) (resp. property (rHAP)$^-$) if and only if the triple $(\mathcal{M}, \mathcal{N}, \psi)$ has property (rHAP) (resp. property (rHAP)$^-$). In particular, property (rHAP) (resp. property (rHAP)$^-$) only depends on the triple $(\mathcal{M}, \mathcal{N}, \mathbb{E}_{\mathcal{N}})$.*

Proof. It follows from Lemma 7.2.5 and Lemma 7.2.6 that if $(\Phi_j)_{j \in J}$ is a net of approximating maps witnessing the relative Haagerup property of $(\mathcal{M}, \mathcal{N}, \varphi)$ (resp. property (rHAP)$^-$ of $(\mathcal{M}, \mathcal{N}, \varphi)$), then it also witnesses the Haagerup property of $(\mathcal{M}, \mathcal{N}, \psi)$ (resp. property (rHAP)$^-$ of $(\mathcal{M}, \mathcal{N}, \psi)$) and vice versa. □

We will later see that in the case where the von Neumann subalgebra $\mathcal{N}$ is finite the statement in Theorem 7.2.7 can be strengthened: in this case property (rHAP) (and equivalently property (rHAP)$^-$) does not even depend on the choice of the conditional expectation $\mathbb{E}_{\mathcal{N}}$.

Motivated by Theorem 7.2.7 we introduce the following natural definition.

Definition 7.2.8. Let $\mathcal{N} \subseteq \mathcal{M}$ be a unital inclusion of von Neumann algebras which admits a faithful normal conditional expectation $\mathbb{E}_{\mathcal{N}} : \mathcal{M} \to \mathcal{N}$. We say that the triple $(\mathcal{M}, \mathcal{N}, \mathbb{E}_{\mathcal{N}})$ has the *relative Haagerup property* (or just *property (rHAP)*) if $(\mathcal{M}, \mathcal{N}, \varphi)$ has the relative Haagerup property for some (equivalently any) faithful normal positive functional $\varphi \in \mathcal{M}_*^+$ with $\varphi \circ \mathbb{E}_{\mathcal{N}} = \varphi$. The same terminology shall be adopted for property (rHAP)$^-$.

7.2.3. STATE PRESERVATION, CONTRACTIVITY AND UNITALITY OF THE APPROXIMATING MAPS IN A SPECIAL CASE

In this subsection we will prove that the relative Haagerup property of certain triples $(\mathcal{M}, \mathcal{N}, \mathbb{E}_{\mathcal{N}})$ may be witnessed by approximating maps that satisfy extra conditions, such as state-preservation, contractivity and unitality. This will play a crucial role in Section 7.4. The approach is inspired by [BF11, Section 2], where ideas from [Jol02] were used.

Lemma 7.2.9. *Let $\mathcal{M}$ be a von Neumann algebra, $\varphi \in \mathcal{M}_*$ a faithful normal state and $y \in \mathcal{M}$. If $y\varphi = \varphi y$ (i.e. $y \in \mathcal{M}^\varphi$), then $y\Omega_\varphi = \Omega_\varphi y$.*

Proof. As mentioned in Section 2.3.1, we have that $\sigma_t^\varphi(y) = y$ for all $t \in \mathbb{R}$. But then y is analytic and moreover $\sigma_{-i/2}^\varphi(y) = y$. Hence

$$\Omega_\varphi y = J_\varphi y^* J_\varphi \Omega_\varphi = J_\varphi \sigma_{-i/2}^\varphi(y^*) J_\varphi \Omega_\varphi = J_\varphi \Delta_\varphi^{1/2} y^* \Delta_\varphi^{-1/2} J_\varphi \Omega_\varphi = S_\varphi y^* S_\varphi \Omega_\varphi = y\Omega_\varphi.$$

The claim follows. □

Proposition 7.2.10. *Let $\mathcal{N} \subseteq \mathcal{M}$ be a unital inclusion of von Neumann algebras that admits a faithful normal conditional expectation $\mathbb{E}_{\mathcal{N}}$. Assume that $\mathcal{N}$ is finite and let $\tau \in \mathcal{N}_*$ be a faithful normal tracial state that we extend to a state $\varphi := \tau \circ \mathbb{E}_{\mathcal{N}}$ on $\mathcal{M}$. Let further $\Phi : \mathcal{M} \to \mathcal{M}$ be a normal, completely positive, $\mathcal{N}$-$\mathcal{N}$-bimodular map for which there exists $\delta > 0$ such that $c := \Phi(1) \leq 1 - \delta$ and $\varphi \circ \Phi \leq (1-\delta)\varphi$. Then one can find $a, b \in \mathcal{N}' \cap \mathcal{M}$ such that $a \geq 0$, $\mathbb{E}_{\mathcal{N}}(a) = 1$, $a\mathbb{E}_{\mathcal{N}}(b^* b) = \mathbb{E}_{\mathcal{N}}(b^* b) a = 1 - c$ and $b\varphi b^* = \varphi - \varphi \circ \Phi$.*

Proof. The complete positivity of Φ implies that $0 \leq \|\Phi\| = \|\Phi(1)\| = \|c\| \leq 1-\delta$, hence the map Φ must be contractive. It is clear that $c = \Phi(1) \geq 0$. Further, since $\mathbb{E}_{\mathcal{N}}(1-c) > \mathbb{E}_{\mathcal{N}}(\delta) = \delta$, the element $\mathbb{E}_{\mathcal{N}}(1-c) \in \mathcal{N}$ is boundedly invertible. Additionally, the $\mathcal{N}$-$\mathcal{N}$-bimodularity of Φ implies that for every $n \in \mathcal{N}$,

$$nc = n\Phi(1) = \Phi(n) = \Phi(1)n = cn,$$

so $c \in \mathcal{N}' \cap \mathcal{M}$. The latter two observations imply that for

$$a := (1-c)(\mathbb{E}_{\mathcal{N}}(1-c))^{-1}$$

we have $a \in \mathcal{N}' \cap \mathcal{M}$, $a \geq 0$ and $\mathbb{E}_{\mathcal{N}}(a) = 1$.

Consider the positive normal functional $\varphi - \varphi \circ \Phi \in \mathcal{M}_*$. By [Haa75, Lemma 2.10] there exists a unique vector $\xi \in L^2(\mathcal{M},\varphi)^+$ such that $(\varphi - \varphi\circ\Phi)(x) = \langle x\xi, \xi\rangle$ for all $x \in \mathcal{M}$. Note that $\{J_\varphi x \Omega_\varphi \mid x \in \mathcal{M}\}$ is dense in $L^2(\mathcal{M},\varphi)$ and define the linear map

$$b : L^2(\mathcal{M},\varphi) \to L^2(\mathcal{M},\varphi),\ J_\varphi x\Omega_\varphi \mapsto J_\varphi x\xi.$$

It is contractive since

$$\|b(J_\varphi x\Omega_\varphi)\|_2^2 = \|J_\varphi x\xi\|_2^2 = \|x\xi\|_2^2 = (\varphi - \varphi\circ\Phi)(x^*x) \leq \varphi(x^*x) = \|x\Omega_\varphi\|_2^2 = \|J_\varphi x\Omega_\varphi\|_2^2$$

for all $x \in \mathcal{M}$. Further, for $x, y \in \mathcal{M}$,

$$bJ_\varphi xJ_\varphi(J_\varphi y\Omega_\varphi) = bJ_\varphi xy\Omega_\varphi = J_\varphi xy\xi = J_\varphi xJ_\varphi J_\varphi y\xi = J_\varphi xJ_\varphi b(J_\varphi y\Omega_\varphi).$$

It hence follows that b and $J_\varphi xJ_\varphi$ commute and therefore that $b \in (J_\varphi \mathcal{M} J_\varphi)' = \mathcal{M}'' = \mathcal{M}$.

We claim that a and b from above satisfy the required conditions. It remains to show that $b \in \mathcal{N}' \cap \mathcal{M}$, $b\varphi b^* = \varphi - \varphi\circ\Phi$ and $a\mathbb{E}_{\mathcal{N}}(b^*b) = \mathbb{E}_{\mathcal{N}}(b^*b)a = 1-c$.

- $b \in \mathcal{N}' \cap \mathcal{M}$*:* By the assumption we have $\varphi - \varphi\circ\Phi \geq \delta\varphi$ and therefore $\varphi - \varphi\circ\Phi$ is a faithful normal functional. For $x \in \mathcal{M}$, $n \in \mathcal{N}$ the $\mathcal{N}$-$\mathcal{N}$-bimodularity of Φ and the traciality of φ on $\mathcal{N}$ (implying that n is contained in the centralizer of φ) imply that $\varphi\circ\Phi(xn) = \varphi(\Phi(x)n) = \varphi(n\Phi(x)) = \varphi\circ\Phi(nx)$, hence $n(\varphi-\varphi\circ\Phi) = (\varphi-\varphi\circ\phi)n$. The unique isomorphism between the standard forms induced by φ and $\varphi - \varphi\circ\Phi$ maps ξ to the canonical cyclic vector in $L^2(\mathcal{M},\varphi-\varphi\circ\Phi)$. Hence, from Lemma 7.2.9 applied to $\varphi - \varphi\circ\Phi$ we get $n\xi = \xi n$ for all $n \in \mathcal{N}$, which, together with the fact that $J_\varphi n\Omega_\varphi = n^*\Omega_\varphi$, implies that for $x \in \mathcal{M}$

$$\begin{aligned} bn(J_\varphi x\Omega_\varphi) &= bJ_\varphi xJ_\varphi n\Omega_\varphi = bJ_\varphi xn^*\Omega_\varphi = J_\varphi xn^*\xi \\ &= J_\varphi x\xi n^* = J_\varphi xJ_\varphi nJ_\varphi\xi = nJ_\varphi x\xi = nb(J_\varphi x\Omega_\varphi), \end{aligned}$$

  so $b \in \mathcal{N}' \cap \mathcal{M}$ by the density of $\{J_\varphi x\Omega_\varphi \mid x \in \mathcal{M}\}$ in $L^2(\mathcal{M},\varphi)$.

- $b\varphi b^* = \varphi - \varphi\circ\Phi$*:* For every $x \in \mathcal{M}$ the equality

$$(b\varphi b^*)(x) = \left\langle xb\Omega_\varphi, b\Omega_\varphi\right\rangle = \langle x\xi,\xi\rangle = (\varphi - \varphi\circ\Phi)(x)$$

  holds, i.e. $b\varphi b^* = \varphi - \varphi\circ\Phi$.

- $a\mathbb{E}_{\mathcal{N}}(b^*b) = \mathbb{E}_{\mathcal{N}}(b^*b)a = 1-c$: For $x \in \mathcal{M}$ we find by $b \in \mathcal{N}' \cap \mathcal{M}$ and $b\varphi b^* = \varphi - \varphi \circ \Phi$ that

$$\begin{aligned} & \varphi(x\mathbb{E}_{\mathcal{N}}(b^*b)) = \varphi(\mathbb{E}_{\mathcal{N}}(x)b^*b) = \varphi(b^*\mathbb{E}_{\mathcal{N}}(x)b) = (\varphi - \varphi\circ\Phi)(\mathbb{E}_{\mathcal{N}}(x)) \\ = \; & \varphi(\mathbb{E}_{\mathcal{N}}(x)) - \varphi(\mathbb{E}_{\mathcal{N}}(x)\Phi(1)) = \varphi(x) - \varphi(x\mathbb{E}_{\mathcal{N}}(\Phi(1))) = \varphi(x\mathbb{E}_{\mathcal{N}}(1-c)) \end{aligned}$$

  and hence $\mathbb{E}_{\mathcal{N}}(1-c) = \mathbb{E}_{\mathcal{N}}(b^*b)$. It follows by the definition of a that $a\mathbb{E}_{\mathcal{N}}(b^*b) = a\mathbb{E}_{\mathcal{N}}(1-c) = 1-c$ and similarly, as $a \in \mathcal{N}' \cap \mathcal{M}$, we have $\mathbb{E}_{\mathcal{N}}(b^*b)a = 1-c$.

□

Lemma 7.2.11. *Let $\mathcal{N} \subseteq \mathcal{M}$ be a unital inclusion of von Neumann algebras which admits a faithful normal conditional expectation $\mathbb{E}_{\mathcal{N}}$. Assume that $\mathcal{N}$ is finite and let $\tau \in \mathcal{N}_*$ be a faithful normal tracial state that we extend to a state $\varphi := \tau \circ \mathbb{E}_{\mathcal{N}}$ on $\mathcal{M}$. Let further $x \in \mathcal{N}' \cap \mathcal{M}$ be an element which is analytic for σ^{φ}. Then $\mathbb{E}_{\mathcal{N}}(yx) = \mathbb{E}_{\mathcal{N}}(\sigma_i^{\varphi}(x)y)$ for all $y \in \mathcal{M}$.*

Proof. For $n \in \mathcal{N}$ we have by the traciality of φ on $\mathcal{N}$ (implying that n is contained in the centralizer of φ) that

$$n\sigma_z^{\varphi}(x) = \sigma_z^{\varphi}(n)\sigma_z^{\varphi}(x) = \sigma_z^{\varphi}(nx) = \sigma_z^{\varphi}(xn) = \sigma_z^{\varphi}(x)\sigma_z^{\varphi}(n) = \sigma_z^{\varphi}(x)n.$$

for all $z \in \mathbb{C}$. Therefore, $\sigma_z^{\varphi}(x) \in \mathcal{N}' \cap \mathcal{M}$ and in particular $\sigma_i^{\varphi}(x) \in \mathcal{N}' \cap \mathcal{M}$. One further calculates that for $y \in \mathcal{M}$,

$$\begin{aligned} (\varphi n)(\mathbb{E}_{\mathcal{N}}(yx)) &= \varphi(\mathbb{E}_{\mathcal{N}}(nyx)) = \varphi(nyx) = \varphi(\sigma_i^{\varphi}(x)ny) \\ &= \varphi(n\sigma_i^{\varphi}(x)y) = (\varphi n)(\mathbb{E}_{\mathcal{N}}(\sigma_i^{\varphi}(x)y)). \end{aligned}$$

Since the set of functionals of the form φn, $n \in \mathcal{N}$ is dense in $\mathcal{N}_*$ we find that $\mathbb{E}_{\mathcal{N}}(yx) = \mathbb{E}_{\mathcal{N}}(\sigma_i^{\varphi}(x)y)$, as claimed. □

Lemma 7.2.12. *Let $\mathcal{N} \subseteq \mathcal{M}$ be a unital inclusion of von Neumann algebras that admits a faithful normal conditional expectation $\mathbb{E}_{\mathcal{N}}$. Assume that $\mathcal{N}$ is finite and let $\tau \in \mathcal{N}_*$ be a faithful normal tracial state that we extend to a state $\varphi := \tau \circ \mathbb{E}_{\mathcal{N}}$ on $\mathcal{M}$. Let $h_1, h_2 \in \mathcal{M}$ and let $h_3, h_4 \in \mathcal{N}' \cap \mathcal{M}$ be analytic for σ^{φ}. Suppose that $\Phi : \mathcal{M} \to \mathcal{M}$ is a normal map such that $\Phi_{\varphi}^{(2)} \in \mathcal{K}(\mathcal{M}, \mathcal{N}, \varphi)$ and define the map $\tilde{\Phi} := h_1\Phi(h_2 \cdot h_3)h_4$. Then we also have that $\tilde{\Phi}_{\varphi}^{(2)} \in \mathcal{K}(\mathcal{M}, \mathcal{N}, \varphi)$.*

Proof. First, by [Tak03a, VIII.3.18(i)] (and its proof), $\tilde{\Phi}$ has a bounded L^2-implementation with $\|\tilde{\Phi}_{\varphi}^{(2)}\| \leq C\|\Phi_{\varphi}^{(2)}\|$, where the constant $C > 0$ depends on h_1, h_2, h_3, h_4. It thus suffices to show that the passage $\Phi \to \tilde{\Phi}$ preserves the property of having a finite-rank implementation. Let then $a, b \in \mathcal{M}$ so that ae_Nb is in $\mathcal{K}_{00}(\mathcal{M}, \mathcal{N}, \varphi)$. So for $x \in \mathcal{M}$ we have by Lemma 7.2.11,

$$h_1(ae_Nb)(h_2xh_3)h_4\Omega_{\varphi} = h_1ah_4e_N(\sigma_i^{\varphi}(h_3)bh_2x)\Omega_{\varphi},$$

and so this map is in $\mathcal{K}_{00}(\mathcal{M}, \mathcal{N}, \varphi)$. □

We are now ready to formulate the main result of this subsection. In combination with Lemma 7.2.14 it will later allow us to deduce that the relative Haagerup property of a triple $(\mathcal{M},\mathcal{N},\mathbb{E}_{\mathcal{N}})$ with finite $\mathcal{N}$ may be witnessed by unital and state-preserving maps. Its proof is inspired by [BF11, Section 2].

Theorem 7.2.13. *Let $\mathcal{N}\subseteq\mathcal{M}$ be a unital inclusion of von Neumann algebras which admits a faithful normal conditional expectation $\mathbb{E}_{\mathcal{N}}$. Assume that $\mathcal{N}$ is finite, let $\tau\in\mathcal{N}_*$ be a faithful normal tracial state that we extend to a state $\varphi:=\tau\circ\mathbb{E}_{\mathcal{N}}$ on $\mathcal{M}$ and suppose that the triple $(\mathcal{M},\mathcal{N},\varphi)$ has property (rHAP) witnessed by contractive approximating maps. Then, if all the elements of $\mathcal{M}$ are analytic with respect to the modular automorphism group of φ – for example if there exists a boundedly invertible element $h\in\mathcal{M}^+$ with $\sigma_t^{\varphi}(x)=h^{it}xh^{-it}$ for all $t\in\mathbb{R}$, $x\in\mathcal{M}$, property (rHAP) of $(\mathcal{M},\mathcal{N},\varphi)$ may be witnessed by unital and state-preserving approximating maps, i.e. we may assume that (1″) and (4′) hold.*

Proof. Let $(\Phi_j)_{j\in J_1}$ be a net of contractive approximating maps witnessing property (rHAP) of the triple $(\mathcal{M},\mathcal{N},\varphi)$ and choose a net $(\delta_j)_{j\in J_2}$ with $\delta_j\to 0$. We now set $J=J_1\times J_2$ with the product partial order and for $j=(j_1,j_2)\in J$ we set $\Phi_j=\Phi_{j_1}$ and $\delta_j=\delta_{j_2}$. Then for all $j\in J$,

$$c_j:=(1-\delta_j)\Phi_j(1)\leq 1-\delta_j \qquad \text{and} \qquad (1-\delta_j)(\varphi\circ\Phi_j)\leq(1-\delta_j)\varphi.$$

In particular, we may apply Proposition 7.2.10 to $(1-\delta_j)\Phi_j$ to find elements $a_j,b_j\in\mathcal{N}'\cap\mathcal{M}$ with $a_j\geq 0$, $\mathbb{E}_{\mathcal{N}}(a_j)=1$, $a_j\mathbb{E}_{\mathcal{N}}(b_j^*b_j)=\mathbb{E}_{\mathcal{N}}(b_j^*b_j)a_j=1-c_j$ and $b_j\varphi b_j^*=\varphi-(1-\delta_j)(\varphi\circ\Phi_j)$. For $j\in J$ define

$$\Psi_j:\mathcal{M}\to\mathcal{M},\ \Psi_j(x):=(1-\delta_j)\Phi_j(x)+a_j\mathbb{E}_{\mathcal{N}}(b_j^*xb_j).$$

It is clear that Ψ_j is normal completely positive and $\mathcal{N}$-$\mathcal{N}$-bimodular. Further,

$$\Psi_j(1)=(1-\delta_j)\Phi_j(1)+a_j\mathbb{E}_{\mathcal{N}}(b_j^*b_j)=c_j+(1-c_j)=1$$

and for any $x\in\mathcal{M}$

$$\begin{aligned}\varphi\circ\Psi_j(x) &= (1-\delta_j)\varphi(\Phi_j(x))+\varphi(a_j\mathbb{E}_{\mathcal{N}}(b_j^*xb_j))\\ &= (1-\delta_j)\varphi(\Phi_j(x))+\varphi(\mathbb{E}_{\mathcal{N}}(a_j)b_j^*xb_j)\\ &= (1-\delta_j)\varphi(\Phi_j(x))+(b_j\varphi b_j^*)(x)\\ &= (1-\delta_j)\varphi(\Phi_j(x))+\varphi(x)-(1-\delta_j)\varphi(\Phi_j(x))\\ &= \varphi(x),\end{aligned}$$

so the Ψ_j are unital and φ-preserving.

For the relative compactness note that by the assumption that every element in $\mathcal{M}$ is analytic for σ^{φ}, Lemma 7.2.11 implies that for all $x\in\mathcal{M}$

$$\Psi_j(x)=(1-\delta_j)\Phi_j(x)+a_j\mathbb{E}_{\mathcal{N}}(\sigma_i^{\varphi}(b_j)b_j^*x),$$

hence,

$$\Psi_j^{(2)} = (1-\delta_j)\Phi_j^{(2)} + a_j e_{\mathcal{N}} \sigma_i^{\varphi}(b_j) b_j^* \in \mathcal{K}(\mathcal{M},\mathcal{N},\varphi).$$

It remains to show that for every $x \in \mathcal{M}$, $\Psi_j(x) \to x$ strongly. For this, estimate for $x \geq 0$,

$$\begin{aligned}
(\Psi_j - (1-\delta_j)\Phi_j)(x) &= a_j^{1/2}\mathbb{E}_{\mathcal{N}}(b_j^* x b_j) a_j^{1/2} \\
&\leq \|x\| a_j^{1/2}\mathbb{E}_{\mathcal{N}}(b_j^* b_j) a_j^{1/2} \\
&= \|x\|(1-c_j).
\end{aligned}$$

Since $c_j = (1-\delta_j)\Phi_j(1) \to 1$ and $(1-\delta_j)\Phi_j(x) \to x$ strongly it then follows that

$$\Psi_j(x) = (\Psi_j(x) - (1-\delta_j)\Phi_j(x)) + (1-\delta_j)\Phi_j(x) \to x$$

strongly for every $x \in \mathcal{M}$. This completes the proof. □

Another important statement that was proved in [CS15b] in case of the usual (non-relative) Haagerup property is the following lemma. It will later ensure the contractivity of certain approximating maps and allow us to apply Theorem 7.2.13 in a suitable setting.

Lemma 7.2.14. *Let $\mathcal{M}$ be a finite von Neumann algebra equipped with a faithful normal tracial state $\tau \in \mathcal{M}_*$ and let $\mathcal{N} \subseteq \mathcal{M}$ be a unital inclusion. Assume that $h \in \mathcal{N}' \cap \mathcal{M}$ is a boundedly invertible self-adjoint element and define $\varphi \in \mathcal{M}_*$ by $\varphi(x) := \tau(hxh)$ for $x \in \mathcal{M}$. Then, if $(\mathcal{M},\mathcal{N},\varphi)$ has property (rHAP), the approximating maps $(\Phi_i)_{i\in I}$ witnessing property (rHAP) may be chosen contractively, i.e. we may assume that $(1')$ holds.*

Proof. The proof is given in [CS15b, Lemma 4.3]. One only needs to check that the condition $h \in \mathcal{N}' \cap \mathcal{M}^+$ ensures that the maps Φ_k', Φ_k^l and Ψ_j defined there are $\mathcal{N}$-$\mathcal{N}$-bimodule maps that are compact relative to $\mathcal{N}$. Let us comment on this. Firstly, in Step 1 of the proof of [CS15b, Lemma 4.3] it is shown that the approximating maps Φ_k witnessing the Haagerup property maybe chosen such that $\sup_k \|\Phi_k\| < \infty$. In the current setup of (rHAP) this is automatic (see Definition 7.2.2) and so we may skip this step.

We now turn to Step 2 in the proof of [CS15b, Lemma 4.3]. Let Φ_k be the approximating maps witnessing the (rHAP) for $(\mathcal{M},\mathcal{N},\varphi)$. In particular Φ_k is $\mathcal{N}$-$\mathcal{N}$-bimodular and $\Phi_k^{(2)} \in \mathcal{K}(\mathcal{M},\mathcal{N},\varphi)$. By [Tak03a, Theorem VIII.2.211] we have $\sigma_t^{\varphi}(x) = h^{it}xh^{-it}$, $t \in \mathbb{R}, x \in \mathcal{M}$. Now recall the map defined in [CS15b, Lemma 4.3] given by

$$\begin{aligned}
\Phi_k^l(x) &= \sqrt{\frac{1}{l\pi}} \int_{-\infty}^{\infty} e^{-t^2/l} \sigma_t^{\varphi}(\Phi_k(\sigma_{-t}^{\varphi}(x))) dt \\
&= \sqrt{\frac{1}{l\pi}} \int_{-\infty}^{\infty} e^{-t^2/l} h^{it}\Phi_k(h^{-it}xh^{it})h^{-it} dt.
\end{aligned} \tag{7.2.1}$$

Since $h \in \mathcal{N}' \cap \mathcal{M}$ this map is $\mathcal{N}$-$\mathcal{N}$-bimodular. Since $\sigma_t^{\varphi}(h^{is}) = h^{is}$, $s,t \in \mathbb{R}$ it follows from Lemma 7.2.12 that the L^2-implementation of

$$x \mapsto \sigma_t^{\varphi}(\Phi_k(\sigma_{-t}^{\varphi}(x))) = h^{it}\Phi_k(h^{-it}xh^{it})h^{-it}, \qquad t \in \mathbb{R}, \tag{7.2.2}$$

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exists and is compact, i.e. contained in $\mathcal{K}(\mathcal{M},\mathcal{N},\varphi)$. By assumption h is boundedly invertible and so $t \mapsto h^{it}$ depends continuously (in norm) on t. Hence the map (7.2.2) depends continuously on t and it follows that (7.2.1) is compact.

Next, in the proof of [CS15b, Lemma 4.3] the following operators were defined:

$$g_k^l = \Phi_k^l(1), \qquad f_k^{n,l} = F_n(g_k^l),$$

where $F_n(z) = e^{-n(z-1)^2}, z \in \mathbb{C}, n \in \mathbb{N}$. Since Φ_k^l is $\mathcal{N}$-$\mathcal{N}$ bimodular it follows that $g_k^l \in \mathcal{N}' \cap \mathcal{M}$. Therefore also $f_k^{n,l} \in \mathcal{N}' \cap \mathcal{M}$. Then the proof of [CS15b, Lemma 4.3] defines for suitable $n(j), k(j), l(j) \in \mathbb{N}, \varepsilon_j > 0$ depending on some j in a directed set the map $\Psi_j : \mathcal{M} \to \mathcal{M}$ via the formula:

$$\Psi_j(\cdot) = \frac{1}{(1+\varepsilon_j)^2} f_{k(j)}^{n(j),l(j)} \Phi_{k(j)}^{l(j)}(\cdot) f_{k(j)}^{n(j),l(j)}.$$

Since $f_{k(j)}^{n(j),l(j)} \in \mathcal{N}' \cap \mathcal{M}$ it follows that Ψ_j is both $\mathcal{N}$-$\mathcal{N}$-bimodular and compact, i.e. $\Psi_j^{(2)} \in \mathcal{K}(\mathcal{M},\mathcal{N},\varphi)$. The last part of the proof of [CS15b, Lemma 4.3] shows that $\Psi_j^{(2)} \to 1$ strongly and this holds true here as well with the same proof. By Proposition 2.4.29 this shows that for every $x \in \mathcal{M}$ we have $\Psi_j(x) \to x$ strongly.

□

7.3. For finite $\mathcal{N}$: translation into the finite setting

Let again $\mathcal{N} \subseteq \mathcal{M}$ be a unital inclusion of von Neumann algebras which admits a faithful normal conditional expectation $\mathbb{E}_{\mathcal{N}}$. Assume moreover that $\mathcal{N}$ is a general σ-finite von Neumann algebra, though in many of the statements below we shall add the assumption that $\mathcal{N}$ is finite. The aim of this section is to characterise the relative Haagerup property (resp. property (rHAP)$^-$) of the triple $(\mathcal{M},\mathcal{N},\mathbb{E}_{\mathcal{N}})$ in terms of the structure of certain corners of crossed product von Neumann algebras associated with the modular automorphism group of some faithful $\varphi \in \mathcal{M}_*^+$ with $\varphi \circ \mathbb{E}_{\mathcal{N}} = \varphi$. These statements will play a crucial role in Section 7.4.

7.3.1. Crossed products

Let us first recall some of the theory of crossed product von Neumann algebras and their duality for which we refer to [Tak03a, Section X.2]. For this, fix an action $\mathbb{R} \curvearrowright^{\alpha} \mathcal{M}$ on $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$, define the corresponding *fixed point algebra*

$$\mathcal{M}^{\alpha} := \{x \in \mathcal{M} \mid \alpha_t(x) = x \text{ for all } t \in \mathbb{R}\}$$

and let $\mathcal{M} \rtimes_{\alpha} \mathbb{R} \subseteq \mathcal{B}(\mathcal{H} \otimes L^2(\mathbb{R})) \cong \mathcal{B}(L^2(\mathbb{R},\mathcal{H}))$ be the corresponding *crossed product von Neumann algebra*. It is generated by the operators $\pi_{\alpha}(x)$, $x \in \mathcal{M}$ and $\lambda_t := \lambda_t^{\alpha}$, $t \in \mathbb{R}$ where

$$(\pi_{\alpha}(x)\xi)(t) = \alpha_{-t}(x)\xi(t) \qquad \text{and} \qquad (\lambda_t\xi)(s) = \xi(s-t)$$

for $s,t \in \mathbb{R}$, $x \in \mathcal{M}$, $\xi \in \mathcal{H} \otimes L^2(\mathbb{R})$; we will also occasionally use λ to denote the left regular representation on $L^2(\mathbb{R})$, which should not cause any confusion. Recall that this construction does not depend on the choice of the embedding $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ and that $\mathcal{M} \cong \pi_\alpha(\mathcal{M})$. For notational convenience we will therefore omit the faithful normal representation π_α in our notation and identify $\mathcal{M}$ with $\pi_\alpha(\mathcal{M})$ and $\mathcal{N}$ with $\pi_\alpha(\mathcal{N})$. Note that $\pi_\alpha(x) = x \otimes 1$ for all $x \in \mathcal{M}^\alpha$. Set further $\lambda(f) := \int_{\mathbb{R}} f(t)\lambda_t dt$ for $f \in L^1(\mathbb{R})$ and

$$\mathcal{L}(\mathbb{R}) := \{\lambda(f) \mid f \in L^1(\mathbb{R})\}'' = \{\lambda_s \mid s \in \mathbb{R}\}'' \subseteq \mathcal{B}(\mathcal{H} \otimes L^2(\mathbb{R})).$$

Remark 7.3.1. For $f \in L^1(\mathbb{R})$ we denote by

$$\widehat{f}(s) = \int_{\mathbb{R}} f(t) e^{ist} dt \in L^\infty(\mathbb{R}),$$

its Fourier transform. Let $\mathcal{F}_2 : L^2(\mathbb{R}) \to L^2(\mathbb{R}) : f \mapsto (2\pi)^{-\frac{1}{2}} \widehat{f}$ be the unitary Fourier transform operator on $L^2(\mathbb{R})$. Then $\mathcal{F}_2 \lambda(f) \mathcal{F}_2^*$ is the multiplication operator with $\widehat{f}$. We shall occasionally extend our notation in the following way. Let $f \in L^2(\mathbb{R})$ be such that its Fourier transform $\widehat{f}$ is in $L^\infty(\mathbb{R})$. We shall write $\lambda(f)$ for $\mathcal{F}_2^* \widehat{f} \mathcal{F}_2$ where we view $\widehat{f}$ as a multiplication operator. This is naturally compatible with the earlier notation for $f \in L^1(\mathbb{R})$

Let $\mathbb{R} \overset{\widehat{\alpha}}{\curvearrowright} \mathcal{M} \rtimes_\alpha \mathbb{R}$ be the *dual action* determined by

$$\widehat{\alpha}_t(x) = x, \qquad \text{and} \qquad \widehat{\alpha}_t(\lambda_s) = \exp(-ist)\lambda_s, \tag{7.3.1}$$

for $x \in \mathcal{M}$, $s, t \in \mathbb{R}$ and recall that its fixed point algebra is given by

$$\mathcal{M} = (\mathcal{M} \rtimes_\alpha \mathbb{R})^{\widehat{\alpha}}. \tag{7.3.2}$$

The expression

$$T_{\widehat{\alpha}}(x) := \int_{\mathbb{R}} \widehat{\alpha}_s(x) ds, \qquad x \in (\mathcal{M} \rtimes_\alpha \mathbb{R})^+,$$

defines a faithful normal semi-finite operator valued weight on $\mathcal{M} \rtimes_\alpha \mathbb{R}$ which takes values in the extended positive part of $\mathcal{M}$. Choose $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ with $\|f\|_2 = 1$ such that the support of the Fourier transform $\widehat{f}$ equals $\mathbb{R}$. We keep f fixed throughout the whole section. One has $T_{\widehat{\alpha}}(\lambda(f)^* \lambda(f)) = \|f\|_2^2 = 1$, hence we may define the unital normal completely positive map

$$T_f := T_{f,\widehat{\alpha}} : \mathcal{M} \rtimes_\alpha \mathbb{R} \to \mathcal{M},\ x \mapsto T_{\widehat{\alpha}}(\lambda(f)^* x \lambda(f)).$$

By Proposition 2.4.29 T_f is strongly continuous on the unit ball. For a given map $\Phi : \mathcal{M} \rtimes_\alpha \mathbb{R} \to \mathcal{M} \rtimes_\alpha \mathbb{R}$ and a positive normal functional $\varphi \in \mathcal{M}_*$ we further define

$$\widetilde{\Phi}_f : \mathcal{M} \to \mathcal{M}, \quad \widetilde{\Phi}_f(x) = T_f(\Phi(x)).$$

and

$$\widehat{\varphi}_f : \mathcal{M} \rtimes_\alpha \mathbb{R} \to \mathbb{C}, \quad \widehat{\varphi}_f(x) = \varphi(T_f(x)).$$

The functional $\widehat{\varphi}_f$ is normal and positive. It is moreover a state if φ is a state. Since we assumed the support of $\widehat{f}$ to be equal to $\mathbb{R}$, by Remark 7.3.1 the support projection of $\lambda(f)$ equals 1. It follows that $\widehat{\varphi}_f$ is faithful if and only if φ is faithful.

Lemma 7.3.2. *Assume that* $\mathcal{N} \subseteq \mathcal{M}^\alpha$. *Then* T_f *is* $\mathcal{N}$*-*$\mathcal{N}$*-bimodular, meaning that for* $x, y \in \mathcal{N}$, $a \in \mathcal{M} \rtimes_\alpha \mathbb{R}$ *we have* $T_f(xay) = xT_f(a)y$.

Proof. As $\mathcal{N} \subseteq \mathcal{M}^\alpha$ we have that $\mathcal{N}$ and $\lambda(f)$ commute. From the definition of $T_{\widehat{\alpha}}$ and (7.3.2) we get that for $x, y \in \mathcal{N}$ and $a \in \mathcal{M} \rtimes_\alpha \mathbb{R}$,

$$T_{\widehat{\alpha}}(\lambda(f)^* xay\lambda(f)) = T_{\widehat{\alpha}}(x\lambda(f)^* a\lambda(f)y) = xT_{\widehat{\alpha}}(\lambda(f)^* a\lambda(f))y.$$

This concludes the proof. □

We recall the following formula which was proved in [CS15c, Lemma 5.2] (which extends [Haa78, Theorem 3.1 (c)]) in case $k = g$; the general case then follows from the polarization identity. For $k, g \in L^2(\mathbb{R})$ such that $\widehat{k}, \widehat{g} \in L^\infty(\mathbb{R})$ we have:

$$T_{\widehat{\alpha}}(\lambda(k)^* x\lambda(g)) = \int_{\mathbb{R}} \overline{k(t)} g(t)\alpha_{-t}(x)dt, \qquad x \in \mathcal{M}. \tag{7.3.3}$$

We shall need the following consequence of it. For $g \in L^1(\mathbb{R})$ define $g^*(t) := \overline{g(-t)}$, which is the involution for the convolution algebra $L^1(\mathbb{R})$.

Lemma 7.3.3. *Let* $h \in C_c(\mathbb{R})$ *and let* $x \in \mathcal{M}$. *Then, for* $k, g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ *and* $a := \lambda(h)x$,

$$T_{\widehat{\alpha}}(\lambda(k)^* a\lambda(g)) = \int_{\mathbb{R}}\int_{\mathbb{R}} k^*(s)g(t)h(-s-t)\alpha_{-t}(x)dsdt,$$

and

$$T_{\widehat{\alpha}}(\lambda(k)^*\lambda(g)a) = \int_{\mathbb{R}}\int_{\mathbb{R}k^*(s)g}(t)h(-s-t)xdsdt.$$

Proof. We have $\lambda(k)^* a = \lambda(h^* * k)^* x$. The equality (7.3.3) then implies

$$\begin{aligned} T_{\widehat{\alpha}}(\lambda(k)^* a\lambda(g)) &= T_{\widehat{\alpha}}(\lambda(h^* * k)^* x\lambda(g)) \\ &= \int_{\mathbb{R}} \overline{(h^* * k)}(t)g(t)\alpha_{-t}(x)dt \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} k^*(s)g(t)h(-s-t)\alpha_{-t}(x)dsdt. \end{aligned}$$

This concludes the proof of the first formula. The second formula follows from the first after observing that $T_{\widehat{\alpha}}(\lambda(k)^*\lambda(g)a) = T_{\widehat{\alpha}}(\lambda(k)^*\lambda(g)\lambda(h))x$. □

7.3.2. PASSAGE TO CROSSED PRODUCTS

Let us now study the stability of the relative Haagerup property with respect to certain crossed products. The setting is the same as in Subsection 7.3.1.

Proposition 7.3.4. *Let* $\Phi : \mathcal{M} \rtimes_\alpha \mathbb{R} \to \mathcal{M} \rtimes_\alpha \mathbb{R}$ *be a linear map and fix* $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ *as before. Then the following statements hold:*

1. *If* Φ *is completely positive then so is* $\widetilde{\Phi}_f$.
2. *Assume that* $\mathcal{N} \subseteq \mathcal{M}^\alpha$. *If* Φ *is an* $\mathcal{N}$*-*$\mathcal{N}$*-bimodule map then* $\widetilde{\Phi}_f$ *is an* $\mathcal{N}$*-*$\mathcal{N}$*-bimodule map.*

In the remaining statements let $\varphi \in \mathcal{M}_*^+$ be a faithful normal positive functional with $\varphi \circ \mathbb{E}_{\mathcal{N}} = \varphi$ and $\varphi \circ \alpha_t = \varphi$ for all $t \in \mathbb{R}$. Then:

3. *If $\widehat{\varphi}_f \circ \Phi \leq \widehat{\varphi}_f$ (resp. $\widehat{\varphi}_f \circ \Phi = \widehat{\varphi}_f$) then $\varphi \circ \widetilde{\Phi}_f \leq \varphi$ (resp. $\varphi \circ \widetilde{\Phi}_f = \varphi$).*

4. *If the L^2-implementation of Φ with respect to $\widehat{\varphi}_f$ exists, then the L^2-implementation of $\widetilde{\Phi}_f$ with respect to φ exists as well.*

Now, if $\mathcal{N} \subseteq \mathcal{M}^\alpha$, $\mathbb{E}_{\mathcal{N}} \circ \alpha_t = \mathbb{E}_{\mathcal{N}}$ for all $t \in \mathbb{R}$ and f is continuous, then:

5. *If $\Phi \in \mathcal{K}_{00}(\mathcal{M} \rtimes_\alpha \mathbb{R}, \mathcal{N}, \widehat{\varphi}_f)$, then $\widetilde{\Phi}_f \in \mathcal{K}_{00}(\mathcal{M}, \mathcal{N}, \varphi)$.*

6. *If $\Phi \in \mathcal{K}(\mathcal{M} \rtimes_\alpha \mathbb{R}, \mathcal{N}, \widehat{\varphi}_f)$, then $\widetilde{\Phi}_f \in \mathcal{K}(\mathcal{M}, \mathcal{N}, \varphi)$.*

Proof. (1) is straightforward from the constructions and (2) follows from Lemma 7.3.2.

(3): If $\widehat{\varphi}_f \circ \Phi \leq \widehat{\varphi}_f$ we have for $x \in \mathcal{M}^+$, using (7.3.3) and the α-invariance of φ,

$$\begin{aligned} \varphi \circ \widetilde{\Phi}_f(x) &= \varphi(T_f(\Phi(x))) = \widehat{\varphi}_f(\Phi(x)) \leq \widehat{\varphi}_f(x) \\ &= \varphi(T_{\widehat{\alpha}}(\lambda(f)^* x \lambda(f))) = \int_{\mathbb{R}} |f(t)|^2 \varphi(\alpha_{-t}(x))\,dt = \varphi(x). \end{aligned}$$

Moreover, if $\widehat{\varphi}_f \circ \Phi = \widehat{\varphi}_f$ then the inequality above is actually an equality.

(4): Assume that there exists a constant $C > 0$ such that $\widehat{\varphi}_f(\Phi(x)^*\Phi(x)) \leq C\widehat{\varphi}_f(x^*x)$ for all $x \in \mathcal{M}$. Then, by the Kadison-Schwarz inequality and (7.3.3),

$$\varphi(\widetilde{\Phi}_f(x)^*\widetilde{\Phi}_f(x)) = \varphi(T_f(\Phi(x))^* T_f(\Phi(x))) \leq \widehat{\varphi}_f(\Phi(x)^*\Phi(x)) \leq C\widehat{\varphi}_f(x^*x) = C\varphi(x^*x)$$

for all $x \in \mathcal{M}$, where we use the fact (proved above) that φ and $\hat{\varphi}_f$ coincide on $\mathcal{M}_+$. This implies that the L^2-implementation of $\widetilde{\Phi}_f$ with respect to φ exists.

(5): By Lemma 7.3.2 and the discussion before, $\mathbb{F}_{\mathcal{N}} = \mathbb{E}_{\mathcal{N}} \circ T_f$ is the unique faithful normal $\widehat{\varphi}_f$-preserving conditional expectation of $\mathcal{M} \rtimes_\alpha \mathbb{R}$ onto $\mathcal{N}$. Let $a, b \in \mathcal{M} \rtimes_\alpha \mathbb{R}$. By $\mathcal{N} \subseteq \mathcal{M}^\alpha$ we have for $x \in \mathcal{M}$,

$$\widetilde{(a\mathbb{F}_{\mathcal{N}} b)}_f(x) := T_f(a\mathbb{F}_{\mathcal{N}}(bx)) = T_f(a)\mathbb{F}_{\mathcal{N}}(bx) \tag{7.3.4}$$

We shall show that $\mathbb{F}_{\mathcal{N}}(bx) = \mathbb{E}_{\mathcal{N}}(\widetilde{b}x)$ for all $x \in \mathcal{M}$, where $\widetilde{b} := T_{\widehat{\alpha}}(\lambda(f)^*\lambda(f)b)$. For this it suffices to consider the case where $b = \lambda(h)y$ for some compactly supported function $h \in C_c(\mathbb{R})$ and $y \in \mathcal{M}$, since such elements span a σ-weakly dense subset of $\mathcal{M} \rtimes_\alpha \mathbb{R}$ and the map $b \mapsto \widetilde{b}$ is σ-weakly continuous. Using Lemma 7.3.3 twice and the fact that

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$\mathbb{E}_{\mathcal{N}} \circ \alpha_t = \mathbb{E}_{\mathcal{N}}$ for all $t \in \mathbb{R}$ one has

$$\begin{aligned}
\mathbb{F}_{\mathcal{N}}(bx) &= \mathbb{E}_{\mathcal{N}} \circ T_f(bx) \\
&= \mathbb{E}_{\mathcal{N}}\left(\int_{\mathbb{R}}\int_{\mathbb{R}} f^*(s)f(t)h(-s-t)\alpha_{-t}(yx)dsdt\right) \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} f^*(s)f(t)h(-s-t)\mathbb{E}_{\mathcal{N}}(yx)dsdt \\
&= \mathbb{E}_{\mathcal{N}}\left(\int_{\mathbb{R}}\int_{\mathbb{R}} f^*(s)f(t)h(-s-t)ydsdt\; x\right) \\
&= \mathbb{E}_{\mathcal{N}}(\widetilde{b}x),
\end{aligned}$$

as claimed. Combining this equality and (7.3.4) we get that $(\widetilde{a\mathbb{E}_{\mathcal{N}} b})_f = T_f(a)\mathbb{F}_{\mathcal{N}}\widetilde{b}$. By considering linear combinations of such expressions one gets that if $\Phi \in \mathcal{K}_{00}(\mathcal{M} \rtimes_\alpha \mathbb{R}, \mathcal{N}, \widehat{\varphi}_f)$ then also $\widetilde{\Phi}_f \in \mathcal{K}_{00}(\mathcal{M}, \mathcal{N}, \varphi)$. This proves (5).

(6): This follows directly from (5) by approximation and the fact that $\|\widetilde{\Phi}_f\| \leq \|\Phi\|$. □

In the following we will direct our attention to certain choices of functions $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ with $\|f\|_2 = 1$ whose support of the Fourier transform $\widehat{f}$ equals $\mathbb{R}$. For this, define for $j \in \mathbb{N}$ the L^2-normalized *Gaussian*

$$f_j : \mathbb{R} \to \mathbb{R},\ f_j(s) := \left(\frac{j}{\pi}\right)^{1/4} \exp(-js^2/2).$$

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Further set for a given map $\Phi : \mathcal{M} \rtimes_\alpha \mathbb{R} \to \mathcal{M} \rtimes_\alpha \mathbb{R}$ and a positive normal functional $\varphi \in \mathcal{M}_*$

$$\widehat{\varphi}_j := \widehat{\varphi}_{f_j} \qquad \text{and} \qquad \widetilde{\Phi}_j := \widetilde{\Phi}_{f_j}.$$

Theorem 7.3.5. *Let $\mathcal{N} \subseteq \mathcal{M}$ be a unital inclusion of von Neumann algebras with a faithful normal conditional expectation $\mathbb{E}_{\mathcal{N}} : \mathcal{M} \to \mathcal{N}$. Let further $\varphi \in \mathcal{M}_*^+$ be a faithful normal positive functional with $\varphi \circ \mathbb{E}_{\mathcal{N}} = \varphi$ and $\mathbb{R} \curvearrowright^\alpha \mathcal{M}$ be an action such that $\mathcal{N} \subseteq \mathcal{M}^\alpha$. Finally assume that $\mathbb{E}_{\mathcal{N}} \circ \alpha_t = \mathbb{E}_{\mathcal{N}}$ (or, equivalently under the earlier assumptions, that $\varphi = \varphi \circ \alpha_t$) for all $t \in \mathbb{R}$. Then the following statements hold:*

1. *If the triple $(\mathcal{M} \rtimes_\alpha \mathbb{R}, \mathcal{N}, \widehat{\varphi}_j)$ has property (rHAP) (resp. property (rHAP)$^-$) for all $j \in \mathbb{N}$, then $(\mathcal{M}, \mathcal{N}, \varphi)$ has property (rHAP) (resp. property (rHAP)$^-$).*

2. *If for all $j \in \mathbb{N}$ property (rHAP) of $(\mathcal{M} \rtimes_\alpha \mathbb{R}, \mathcal{N}, \widehat{\varphi}_j)$ is witnessed by unital (resp. $\widehat{\varphi}_j$-preserving) approximating maps (see (1$''$) and (4$'$) in Section 7.2), then also property (rHAP) of $(\mathcal{M}, \mathcal{N}, \varphi)$ may be witnessed by unital (resp. φ-preserving) approximating maps.*

Proof. (1): For fixed $j \in \mathbb{N}$ let $(\Phi_{j,k})_{k \in K_j}$ be a bounded net of normal completely positive maps witnessing the relative Haagerup property of $(\mathcal{M} \rtimes_\alpha \mathbb{R}, \mathcal{N}, \widehat{\varphi}_j)$. In particular, $\Phi_{j,k} \to 1$ in the point-strong topology in k. Set $\widetilde{\Phi}_{j,k} := T_{f_j} \circ \Phi_{j,k}$. As $s \mapsto \alpha_s(x)$ is strongly continuous for $x \in \mathcal{M}$ and f_j is L^2-normalized with mass concentrated around 0, Lemma 7.3.3 shows that for $x \in \mathcal{M}$,

$$T_{f_j}(x) = \int_{\mathbb{R}} |f_j(s)|^2 \alpha_s(x) ds \to x$$

as $j \to \infty$ in the strong topology. Lemma 7.1.1 then shows that we may find a directed set $\mathcal{F}$ and a function $(\widetilde{j}, \widetilde{k}) : \mathcal{F} \to \{(j,k) \mid j \in \mathbb{N}, k \in K_j\}$, $F \mapsto (\widetilde{j}(F), \widetilde{k}(F))$ such that the net $(\widetilde{\Phi}_{\widetilde{j}(F), \widetilde{k}(F)})_{F \in \mathcal{F}}$ converges to the identity in the point-strong topology. By Proposition 7.3.4 these maps then witness the relative Haagerup property for $(\mathcal{M}, \mathcal{N}, \varphi)$. In the same way, using a variant of Lemma 7.1.1, we can deduce that if $(\mathcal{M} \rtimes_\alpha \mathbb{R}, \mathcal{N}, \widehat{\varphi}_j)$ has property (rHAP)$^-$, then $(\mathcal{M}, \mathcal{N}, \varphi)$ has property (rHAP)$^-$ as well.

(2): Note that if $\Phi_{j,k}$ is unital for all $k \in \mathbb{N}$, then $\widetilde{\Phi}_{j,k}$ is unital as well and if $\Phi_{j,k}$ is $\widehat{\varphi}_j$-preserving for all $k \in \mathbb{N}$, then $\widetilde{\Phi}_{j,k}$ is φ-preserving, c.f. Proposition 7.3.4. □

We will now apply this theorem to the modular automorphism group σ^φ of φ as well as its dual action.

Theorem 7.3.6. *Let $\mathcal{N} \subseteq \mathcal{M}$ be a unital inclusion of von Neumann algebras with a faithful normal conditional expectation $\mathbb{E}_\mathcal{N} : \mathcal{M} \to \mathcal{N}$. Assume that $\mathcal{N}$ is finite and let $\tau \in \mathcal{N}_*$ be a faithful normal tracial state. Further define the faithful normal (possibly non-tracial) state $\varphi := \tau \circ \mathbb{E}_\mathcal{N}$ on $\mathcal{M}$. Then the following statements hold:*

1. *If for all $j \in \mathbb{N}$ the triple $(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}, \mathcal{N}, \widehat{\varphi}_j)$ has property (rHAP) (resp. property (rHAP)$^-$), then $(\mathcal{M}, \mathcal{N}, \varphi)$ has property (rHAP) (resp. property (rHAP)$^-$).*
2. *If for all $j \in \mathbb{N}$ property (rHAP) of $(\mathcal{M} \rtimes_\alpha \mathbb{R}, \mathcal{N}, \widehat{\varphi}_j)$ is witnessed by unital (resp. $\widehat{\varphi}_j$-preserving) approximating maps, then also property (rHAP) of $(\mathcal{M}, \mathcal{N}, \varphi)$ may be witnessed by unital (resp. φ-preserving) approximating maps.*

Proof. This is Theorem 7.3.5 for $\alpha = \sigma^\varphi$; the assumptions are satisfied, as follows from the fact that $\mathcal{N} \subset \mathcal{M}^\varphi$. □

We will also prove the converse of Theorem 7.3.6 by using crossed product duality. We first recall the following lemma which is well-known. We will use the fact that every function $g \in L^\infty(\mathbb{R})$ may be viewed as a multiplication operator on $L^2(\mathbb{R})$.

Lemma 7.3.7. *For $g, h \in C_b(\mathbb{R}) \cap L^2(\mathbb{R})$ we have that $g\lambda(h) \in \mathcal{B}(L^2(\mathbb{R}))$ is Hilbert-Schmidt with*

$$\mathrm{Tr}((g\lambda(h))^* g\lambda(h)) = \|g\|_2^2 \|h\|_2^2.$$

Proof. Let $\mathcal{S}_2(\mathcal{H})$ denote the Hilbert-Schmidt operators on a Hilbert space $\mathcal{H}$. We have linear identifications $\mathcal{H} \otimes \overline{\mathcal{H}} \simeq \mathcal{S}_2(\mathcal{H})$ where $\xi \otimes \overline{\eta}$ corresponds to the rank 1 operator $v \mapsto \xi\eta^*(v)$. We identify $L^2(\mathbb{R})$ with $\overline{L^2(\mathbb{R})}$ linearly and isometrically through the pairing $\langle \xi, \eta \rangle = \int_{\mathbb{R}} \xi(s)\eta(s) ds$. Therefore we have isometric linear identifications

$$\mathcal{S}_2(L^2(\mathbb{R})) \simeq L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) \simeq L^2(\mathbb{R}^2), \tag{7.3.5}$$

where the rank 1 operator $\xi\eta^*$ corresponds to the function $(s,t)\mapsto \xi(s)\eta(t)$.

Now, $g\lambda(h)$ is an integral operator on $L^2(\mathbb{R})$ with a square-integrable kernel $K(x,y) := g(x)h(x-y)$. Then $g\lambda(h)$ is Hilbert-Schmidt and corresponds to $K\in L^2(\mathbb{R}^2)$ in (7.3.5), so that $\|g\lambda(h)\|_{\mathcal{S}_2} = \|K\|_2 = \|g\|_2\|h\|_2$. □

Further recall that for $j\in\mathbb{N}$ the Gaussian f_j was defined by $f_j(s) := j^{1/4}\pi^{-1/4}e^{-js^2/2}$, $s\in\mathbb{R}$ and $\widehat{f_j}$ denotes its Fourier transform. Both these functions are L^2-normalized by definition and the Plancherel identity. Define for $i,j\in\mathbb{N}$ a positive linear functional $\psi_{i,j}$ on $\mathcal{B}(L^2(\mathbb{R}))$ by

$$\psi_{i,j}(x) := \mathrm{Tr}((\widehat{f}_i\lambda(f_j))^* x\widehat{f}_i\lambda(f_j)).$$

It is a state by Lemma 7.3.7. We will need the following elementary lemma for which we give a short non-explicit proof following from the results in [CS15c].

Lemma 7.3.8. *For all $i,j\in\mathbb{N}$ the pair $(\mathcal{B}(L^2(\mathbb{R})),\psi_{i,j})$ has the Haagerup property in the sense that the triple $(\mathcal{B}(L^2(\mathbb{R})),\mathbb{C},\psi_{i,j})$ has the relative Haagerup property, see [CS15c, Definition 3.1]. Moreover, the approximating maps may be chosen to be unital and $\psi_{i,j}$-preserving.*

Proof. According to [CS15c, Proposition 3.4], $(\mathcal{B}(L^2(\mathbb{R}),\mathrm{Tr})$ has the Haagerup property. By [CS15c, Theorem 1.3] the Haagerup property does not depend on the choice of the faithful normal semi-finite weight and hence $(\mathcal{B}(L^2(\mathbb{R})),\psi_{i,j})$ has the Haagerup property for all $i,j\in\mathbb{N}$. In [CS15b, Theorem 5.1] it was proved that the approximating maps may be taken unital and state preserving. This finishes the proof. □

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As before, let $\mathcal{N}\subseteq\mathcal{M}$ be a unital inclusion of von Neumann algebras which admits a faithful normal conditional expectation $\mathbb{E}_{\mathcal{N}}$ and fix a faithful normal state φ on $\mathcal{M}$ with $\varphi=\varphi\circ\mathbb{E}_{\mathcal{N}}$. Let σ^φ be the corresponding modular automorphism group, $\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R}$ the crossed product von Neumann algebra and let

$$\theta := \widehat{\sigma^\varphi} : \mathbb{R}\curvearrowright \mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R}$$

be the dual action as defined in (7.3.1). Define for $j\in\mathbb{N}$ the state $\widehat{\varphi}_j := \varphi\circ T_{f_j,\theta}$ on $\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R}$ as before and recall that $\mathcal{M}$ (hence also $\mathcal{N}$) is invariant under θ. We may in turn consider the double crossed product which admits an isomorphism of von Neumann algebras (i.e. a bijective $*$-homomorphism, which is automatically normal by Sakai [Sak71, Theorem 1.13.2]),

$$(\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R})\rtimes_\theta\mathbb{R}\cong\mathcal{M}\otimes\mathcal{B}(L^2(\mathbb{R})). \tag{7.3.6}$$

Let us describe what this isomorphism looks like. For $g\in L^\infty(\mathbb{R})$ write $\mu(g) := 1_{\mathcal{M}}\otimes g\in\mathcal{M}\otimes\mathcal{B}(L^2(\mathbb{R}))$ for the multiplication operator acting in the second tensor leg. The double crossed product above is generated by $\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R}$ and the left regular representation of the second copy of $\mathbb{R}$, denoted here by λ_t^θ, $t\in\mathbb{R}$. Under the isomorphism, $\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R}$ is identified as a subalgebra of $\mathcal{M}\otimes\mathcal{B}(L^2(\mathbb{R}))$ via inclusion. Further, λ_t^θ is identified for every $t\in\mathbb{R}$ with $\mu(e_t)=1_{\mathcal{M}}\otimes e_t$ where $e_t(s) := \exp(-ist)$ for $s\in\mathbb{R}$. Under this correspondence, $\lambda^\theta(f_j)=\mu(\widehat{f}_j)$. We find that for $x\in\mathcal{M}\otimes\mathcal{B}(L^2(\mathbb{R}))$,

$$\begin{aligned}(\varphi\circ T_{f_j,\theta}\circ T_{f_i,\widehat{\theta}})(x) &= \varphi\left(T_\theta\left(\lambda(f_j)^* T_{\widehat{\theta}}\left(\mu(\widehat{f}_i)^* x\mu(\widehat{f}_i)\right)\lambda(f_j)\right)\right)\\ &= \varphi\left(T_\theta\left(T_{\widehat{\theta}}\left(\lambda(f_j)^*\mu(\widehat{f}_i)^* x\mu(\widehat{f}_i)\lambda(f_j)\right)\right)\right).\end{aligned}$$

By [Tak03a, Theorem X.2.3] and the fact that $\varphi \circ \sigma_t^{\varphi} = \varphi$ we have that (formally, being imprecise about domains) the normal semi-finite faithful weight $\varphi \circ T_\theta \circ T_{\widehat{\theta}}$ coincides with $\varphi \otimes \mathrm{Tr}$. Hence, for $i, j \in \mathbb{N}$ we have equality of states

$$\varphi \circ T_{f_j,\theta} \circ T_{f_i,\widehat{\theta}} = \varphi \otimes \psi_{i,j}.$$

The following theorem now provides a passage to study the relative Haagerup property on the continuous core of a von Neumann algebra, which is semi-finite.

Theorem 7.3.9. *Let $\mathcal{N} \subseteq \mathcal{M}$ be a unital inclusion of von Neumann algebras which admits a faithful normal conditional expectation $\mathbb{E}_{\mathcal{N}}$ and assume that $\mathcal{N}$ is finite with a faithful normal tracial state $\tau \in \mathcal{N}_*$. Set $\varphi = \tau \circ \mathbb{E}_{\mathcal{N}} \in \mathcal{M}_*$. Then the following two statements hold:*

1. *The triple $(\mathcal{M}, \mathcal{N}, \varphi)$ has property (rHAP) (resp. property (rHAP)$^-$) if and only if $(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}, \mathcal{N}, \widehat{\varphi}_j)$ has property (rHAP) (resp. property (rHAP)$^-$) for all $j \in \mathbb{N}$.*
2. *If property (rHAP) of $(\mathcal{M}, \mathcal{N}, \varphi)$ is witnessed by unital (resp. φ-preserving) maps, then for all $j \in \mathbb{N}$ property (rHAP) of $(\mathcal{M} \rtimes_\alpha \mathbb{R}, \mathcal{N}, \widehat{\varphi}_j)$ is witnessed by unital (resp. $\widehat{\varphi}_j$-preserving) maps, and vice versa.*

Proof. The if statements were proven in Theorem 7.3.6. For the converse of (1) assume that $(\mathcal{M}, \mathcal{N}, \varphi)$ has the relative Haagerup property. $(\mathcal{B}(L^2(\mathbb{R})), \mathbb{C}, \psi_{i,j})$ has the relative Haagerup property for all $i, j \in \mathbb{N}$, see Lemma 7.3.8. Therefore by a suitable modification of [CS15c, Lemma 3.5], we see that $(\mathcal{M} \otimes \mathcal{B}(L^2(\mathbb{R})), \mathcal{N} \otimes \mathbb{C}, \varphi \otimes \psi_{i,j})$ has the relative Haagerup property for all $i, j \in \mathbb{N}$. It follows from Theorem 7.3.5 and the discussion above that $(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}, \mathcal{N}, \widehat{\varphi}_j)$ has the relative Haagerup property [1]. The statements in (2) and the statement about property (rHAP)$^-$ follow in the same way. □

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7.3.3. PASSAGE TO CORNERS OF CROSSED PRODUCTS

In Section 7.3.2 we characterised the relative Haagerup property of the triple $(\mathcal{M}, \mathcal{N}, \varphi)$ for finite $\mathcal{N}$ with a faithful normal tracial state $\tau \in \mathcal{N}_*$ and $\varphi := \tau \circ \mathbb{E}_{\mathcal{N}} \in \mathcal{M}_*$ in terms of the crossed product triples $(\mathcal{M} \rtimes_\alpha \mathbb{R}, \mathcal{N}, \widehat{\varphi}_j)$, $j \in \mathbb{N}$. In the following we will pass over to suitable corners of these crossed products which allows to translate our investigations into the setting of finite von Neumann algebras. In this setting the following lemma will be useful.

Lemma 7.3.10. *Let $\mathcal{N} \subseteq \mathcal{M}$ be a unital inclusion of finite von Neumann algebras, let $\tau \in \mathcal{M}_*$ be a faithful normal tracial state and let $\mathbb{E}_{\mathcal{N}} : \mathcal{M} \to \mathcal{N}$ be the unique τ-preserving faithful normal conditional expectation onto $\mathcal{N}$. Further, let $h \in \mathcal{N}' \cap \mathcal{M}$ be self-adjoint and boundedly invertible. For a linear completely positive map $\Phi : \mathcal{M} \to \mathcal{M}$ set*

$$\Phi^h(x) = h^{-1} \Phi(hxh) h^{-1}.$$

Then, the L^2-implementation $\Phi^{(2)}$ of Φ with respect to τ exists if and only if the L^2- implementation $(\Phi^h)^{(2)}$ of Φ^h with respect to $h\tau h$ exists. Further, $\Phi^{(2)} \in \mathcal{K}(\mathcal{M}, \mathcal{N}, \tau)$ if and only if $(\Phi^h)^{(2)} \in \mathcal{K}(\mathcal{M}, \mathcal{N}, h\tau h)$.

[1] Note that in the picture above $\pi(x) = x \otimes 1$ and hence $\pi(\mathcal{N}) = \mathcal{N} \otimes \mathbb{C}$ since φ is tracial on $\mathcal{N}$. This is used implicitly in the identifications of $\mathcal{N}$ in the double crossed product isomorphism (7.3.6).

Proof. Note first that the assumptions on h imply that $\mathbb{E}_{\mathcal{N}}(h^2)$ is a positive boundedly invertible element of the center $Z(\mathcal{N})$ of $\mathcal{N}$. Indeed, we have for all $n \in \mathcal{N}$ the equality $n\mathbb{E}_{\mathcal{N}}(h^2) = \mathbb{E}_{\mathcal{N}}(nh^2) = \mathbb{E}_{\mathcal{N}}(h^2 n) = \mathbb{E}_{\mathcal{N}}(h^2)n$, and if h is boundedly invertible, then $h^2 \geq c1_{\mathcal{M}}$ for some $c > 0$, hence $\mathbb{E}_{\mathcal{N}}(h^2) \geq c1_{\mathcal{M}}$.

Now the map $\mathbb{E}^h_{\mathcal{N}} : x \mapsto \mathbb{E}_{\mathcal{N}}(h^2)^{-1/2}\mathbb{E}_{\mathcal{N}}(hxh)\mathbb{E}_{\mathcal{N}}(h^2)^{-1/2}$ is the unique normal $h\tau h$-preserving conditional expectation onto $\mathcal{N}$. Indeed, we can verify it is an idempotent, normal, ucp map with image equal to $\mathcal{N}$ and for any $x \in \mathcal{M}$ we have

$$\begin{aligned}
(h\tau h)(\mathbb{E}^h_{\mathcal{N}}(x)) &= \tau(h\mathbb{E}_{\mathcal{N}}(h^2)^{-1/2}\mathbb{E}_{\mathcal{N}}(hxh)\mathbb{E}_{\mathcal{N}}(h^2)^{-1/2}h)\\
&= \tau(h^2\mathbb{E}_{\mathcal{N}}(h^2)^{-1}\mathbb{E}_{\mathcal{N}}(hxh))\\
&= \tau(\mathbb{E}_{\mathcal{N}}(h^2\mathbb{E}_{\mathcal{N}}(h^2)^{-1}\mathbb{E}_{\mathcal{N}}(hxh)))\\
&= \tau(\mathbb{E}_{\mathcal{N}}(h^2)\mathbb{E}_{\mathcal{N}}(h^2)^{-1}\mathbb{E}_{\mathcal{N}}(hxh))\\
&= \tau(\mathbb{E}_{\mathcal{N}}(hxh))\\
&= \tau(hxh)\\
&= (h\tau h)(x).
\end{aligned}$$

Now assume that the L^2-implementation $\Phi^{(2)}$ of Φ with respect to τ exists, i.e. that there exists a constant $C > 0$ such that $\tau(\Phi(x)^*\Phi(x)) \leq C\tau(x^*x)$ for all $x \in \mathcal{M}$. Then

$$\begin{aligned}
&(h\tau h)(\Phi^h(x)^*\Phi^h(x)) = \tau(\Phi(hx^*h)h^{-2}\Phi(hxh)) \leq \left\|h^{-2}\right\|\tau(\Phi(hx^*h)\Phi(hxh))\\
\leq\ & C\left\|h^{-2}\right\|\tau(hx^*hhxh) \leq C\left\|h^{-2}\right\|\left\|h^2\right\|\tau(hx^*xh) = C\left\|h^{-2}\right\|\left\|h^2\right\|(h\tau h)(x^*x)
\end{aligned}$$

for all $x \in \mathcal{M}$, so the L^2-implementation $(\Phi^h)^{(2)}$ exists as well. The converse implication also follows, as $\Phi = (\Phi^h)^{h^{-1}}$.

For elements $a, b, x \in \mathcal{M}$ the equality

$$\begin{aligned}
(a\mathbb{E}_{\mathcal{N}}b)^h(x) &= h^{-1}a\mathbb{E}_{\mathcal{N}}(bhxh)h^{-1}\\
&= h^{-1}a\mathbb{E}_{\mathcal{N}}(h^2)^{1/2}\mathbb{E}^h_{\mathcal{N}}(h^{-1}bhx)\mathbb{E}_{\mathcal{N}}(h^2)^{1/2}h^{-1}\\
&= (h^{-1}a\mathbb{E}_{\mathcal{N}}(h^2)h^{-1})\mathbb{E}^h_{\mathcal{N}}(h^{-1}bhx)\\
&= (h^{-1}a\mathbb{E}_{\mathcal{N}}(h^2)h^{-1}\mathbb{E}^h_{\mathcal{N}}h^{-1}bh)(x)
\end{aligned}$$

implies by taking linear combinations and approximation that if $\Phi^{(2)} \in \mathcal{K}(\mathcal{M},\mathcal{N},\tau)$, then $(\Phi^h)^{(2)} \in \mathcal{K}(\mathcal{M},\mathcal{N},h\tau h)$. The converse statement follows as before, which finishes the proof. □

Now, for a triple $(\mathcal{M},\mathcal{N},\varphi)$ let h be the unique (possibly unbounded) positive self-adjoint operator affiliated with $\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$ such that $h^{it} = \lambda_t$ for all $t \in \mathbb{R}$. If we further assume that $\mathcal{N} \subseteq \mathcal{M}^{\sigma^\varphi}$ (which implies that $\mathcal{N}$ is finite with a tracial state $\varphi|_N$) we have for $x \in \mathcal{N}$ that $\lambda_t x \lambda_t^* = \sigma^\varphi_t(x) = x$ and hence $\lambda_t \in \mathcal{N}' \cap (\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})$. This implies that h is affiliated with $\mathcal{N}' \cap (\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})$ and so its finite spectral projections are elements in $\mathcal{N}' \cap (\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})$. Set for $k \in \mathbb{N}$

$$p_k = \chi_{[k^{-1},k]}(h) \qquad \text{and} \qquad h_k = hp_k.$$

Here $\chi_{[k^{-1},k]}$ denotes the indicator function of $[k^{-1},k]\subseteq\mathbb{R}$ and p_k is the corresponding spectral projection. Then, for every $k\in\mathbb{N}$, h_k is boundedly invertible in the corner algebra $p_k(\mathscr{M}\rtimes_{\sigma^\varphi}\mathbb{R})p_k$ and we write h_k^{-1} for its inverse which we view as an operator in $\mathscr{M}\rtimes_{\sigma^\varphi}\mathbb{R}$.

Denote by $\widehat{\varphi}:=\varphi\circ T_\theta$ the dual weight of φ and let $\tau_\rtimes$ be the unique faithful normal semi-finite weight on $\mathscr{M}\rtimes_{\sigma^\varphi}\mathbb{R}$ whose Connes cocycle derivative satisfies $(D\widehat{\varphi}/D\tau_\rtimes)_t=h^{it}$ for all $t\in\mathbb{R}$ (we refer to [Haa79b, Lemma 5.2]; the proofs below stay within the realm of bounded functionals). It is a trace on $\mathscr{M}\rtimes_{\sigma^\varphi}\mathbb{R}$ which is formally given by

$$\tau_\rtimes(x)=\varphi\circ T_\theta(h^{-\frac{1}{2}}xh^{-\frac{1}{2}}),\qquad x\in(\mathscr{M}\rtimes_{\sigma^\varphi}\mathbb{R})^+.$$

By construction we have

$$\widehat{\varphi}_j(p_kxp_k)=\tau_\rtimes(h_k^{\frac{1}{2}}\lambda(f_j)^*x\lambda(f_j)h_k^{\frac{1}{2}}),\qquad x\in\mathscr{M}\rtimes_{\sigma^\varphi}\mathbb{R}. \tag{7.3.7}$$

for all $j\in\mathbb{N}$, where $\widehat{\varphi}_j$ and f_j are defined as in Subsection 7.3.2. Further note that the operators $\lambda(f_j)$ and h_k commute.

Remark 7.3.11. Following Remark 7.3.1, for $k\in\mathbb{N}$ the operators p_k and h_k can be described in terms of multiplication operators conjugated with the Fourier unitary $\mathscr{F}_2$. Indeed, $\mathscr{F}_2\lambda_t\mathscr{F}_2^*$ is the multiplication operator on $L^2(\mathbb{R},\mathscr{H})$ with the function $(s\mapsto e^{its})$ and therefore (under proper identification of the domains) $\mathscr{F}_2h\mathscr{F}_2^*$ coincides with the multiplication operator with $(s\mapsto e^s)$. It follows that for all $k\in\mathbb{N}$, $\mathscr{F}_2p_k\mathscr{F}_2^*$ is the multiplication with $(I_k:s\mapsto\chi_{[-\log(k),\log(k)]}(s))$ and $\mathscr{F}_2h_k\mathscr{F}_2^*$ is the multiplication with $(J_k:s\mapsto\chi_{[-\log(k),\log(k)]}(s)e^s)$. Therefore, by Remark 7.3.1,

$$p_k=\lambda(\widehat{I_k}),\qquad h_k=\lambda(\widehat{J_k}),\qquad\text{and}\qquad h_k^{-1}=\lambda(\widehat{J_k^{-1}}),$$

where J_k^{-1} is the function $(s\mapsto\chi_{[-\log(k),\log(k)]}e^{-s})$. We also have that

$$\lambda(f_j)h_k=\lambda(f_j)\lambda(\widehat{J_k})=\lambda(f_j*\widehat{J_k})=\mathscr{F}_2^*\widehat{f_j}J_k\mathscr{F}_2, \tag{7.3.8}$$

where we view the product $\widehat{f_j}J_k$ as a multiplication operator. Since the Fourier transform of f_j is Gaussian we see that $\mathscr{F}_2^*\widehat{f_j}J_k\mathscr{F}_2$ is positive and boundedly invertible in the corner algebra $p_k(\mathscr{M}\rtimes_{\sigma^\varphi}\mathbb{R})p_k$. Further, by (7.3.3) and the Plancherel identity,

$$T_\theta(h_k^{-1})=T_\theta(\lambda(\widehat{J_k^{-1/2}})\lambda(\widehat{J_k^{-1/2}}))=\|\widehat{J_k^{-1/2}}\|_2^2=\|J_k^{-1/2}\|_2^2=k-k^{-1}.$$

It follows that

$$\tau_\rtimes(p_k)=\varphi(T_\theta(h^{-1/2}p_kh^{-1/2}))=\varphi(T_\theta(h_k^{-1}))=k-k^{-1}.$$

In particular, $\tau_\rtimes(p_k)<\infty$. Since $\tau_\rtimes$ is tracial we also have for $x\in\mathscr{M}\rtimes_{\sigma^\varphi}\mathbb{R}$,

$$\tau_\rtimes(p_kxp_k)=\varphi\circ T_\theta(h_k^{-1}p_kxp_k). \tag{7.3.9}$$

7

In the next statements we will work with property (rHAP) (resp. property (rHAP)$^-$) for general faithful normal positive functionals instead of just states, see Remark 7.2.3. This is notationally more convenient. Note that $p_k \widehat{\varphi}_j p_k$, $j \in \mathbb{N}$ is not a state, but a positive scalar multiple of a state.

We shall use the fact that the unique faithful normal $\widehat{\varphi}_j$-preserving conditional expectation $\mathbb{E}_{\mathcal{N}}^{\widehat{\varphi}_j}$ of $\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$ onto $\mathcal{N}$ is given by $\mathbb{E}_{\mathcal{N}}^{\widehat{\varphi}_j} = \mathbb{E}_{\mathcal{N}} \circ T_{f_j}$. This fact was used in the proof of Proposition 7.3.4 already.

Lemma 7.3.12. *For every $k \in \mathbb{N}$, $j \in \mathbb{N}$ there is a faithful normal $p_k \widehat{\varphi}_j p_k$-preserving conditional expectation of $p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k$ onto $p_k \mathcal{N} p_k$ given by*

$$x \mapsto \mu_k^{-1} p_k \mathbb{E}_{\mathcal{N}}(T_{f_j}(x)) p_k = \mu_k^{-1} p_k \mathbb{E}_{\mathcal{N}}^{\widehat{\varphi}_j}(x) p_k \tag{7.3.10}$$

where $\mu_k := T_{f_j}(p_k) = \| \widehat{f}_j \chi_{[-\log(k), \log(k)]} \|_2^2$. In particular, $T_{f_j}(p_k)$ is a scalar multiple of the identity.

Proof. First note that by Remark 7.3.1 and Remark 7.3.11 the operator $p_k \lambda(f_j)$ coincides with $\lambda(g_{j,k})$ where $g_{j,k}$ is the inverse Fourier transform of the function $\widehat{f}_j \chi_{[-\log(k), \log(k)]}$. The equality (7.3.3) then implies that

$$T_{f_j}(p_k) = T_\theta(\lambda(f_j)^* p_k \lambda(f_j)) = T_\theta(\lambda(g_{j,k})^* \lambda(g_{j,k})) = \|g_{j,k}\|_2^2 = \mu_k \tag{7.3.11}$$

is a multiple of the identity. For $x \in p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k$ expand

$$\begin{aligned} (p_k \widehat{\varphi}_j p_k)(p_k \mathbb{E}_{\mathcal{N}}^{\widehat{\varphi}_j}(x) p_k) &= \widehat{\varphi}_j(p_k \mathbb{E}_{\mathcal{N}}^{\widehat{\varphi}_j}(x) p_k) \\ &= (\varphi \circ T_{f_j})(p_k \mathbb{E}_{\mathcal{N}}(T_{f_j}(x)) p_k) \\ &= (\varphi \circ T_\theta)(\lambda(f_j)^* p_k \mathbb{E}_{\mathcal{N}}(T_{f_j}(x)) p_k \lambda(f_j)). \end{aligned}$$

Since $\mathcal{N} \subseteq \mathcal{M}^{\sigma^\varphi}$ we see that

$$\begin{aligned} (p_k \widehat{\varphi}_j p_k)(p_k \mathbb{E}_{\mathcal{N}}^{\widehat{\varphi}_j}(x) p_k) &= (\varphi \circ T_\theta)(\mathbb{E}_{\mathcal{N}}(T_{f_j}(x)) \lambda(f_j)^* p_k \lambda(f_j)) \\ &= \varphi(\mathbb{E}_{\mathcal{N}}(T_{f_j}(x)) T_{f_j}(p_k)). \end{aligned}$$

With (7.3.11) we can continue as follows:

$$(p_k \widehat{\varphi}_j p_k)(p_k \mathbb{E}_{\mathcal{N}}^{\widehat{\varphi}_j}(x) p_k) = \mu_k \varphi(\mathbb{E}_{\mathcal{N}}(T_{f_j}(x))) = \mu_k \varphi(T_{f_j}(x)) = \mu_k \widehat{\varphi}_j(x) = \mu_k \widehat{\varphi}_j(p_k x p_k).$$

This proves that (7.3.10) is $p_k \widehat{\varphi}_j p_k$-preserving, as claimed. For $x \in \mathcal{N} \subseteq \mathcal{M}^{\sigma^\varphi}$ we have that x and p_k commute. Therefore, using the $\mathcal{N}$-module property of the maps involved,

$$p_k \mathbb{E}_{\mathcal{N}}(T_{f_j}(p_k x p_k)) p_k = p_k x p_k \mathbb{E}_{\mathcal{N}}(T_{f_j}(p_k)) = \mu_k p_k x p_k.$$

This shows that the map $x \mapsto \mu_k^{-1} p_k \mathbb{E}_{\mathcal{N}}(T_{f_j}(x)) p_k$ is a unital (the unit being p_k) normal completely positive projection onto $p_k \mathcal{N} p_k$ (see [BO08b, Theorem 1.5.10]). □

Lemma 7.3.13. *Let $\mathcal{N} \subseteq \mathcal{M}$ be a unital inclusion of von Neumann algebras which admits a faithful normal conditional expectation $\mathbb{E}_{\mathcal{N}}$. Assume that $\mathcal{N}$ is finite and let $\tau \in \mathcal{N}_*$ be a faithful normal tracial state that we extend to a state $\varphi := \tau \circ \mathbb{E}_{\mathcal{N}}$ on $\mathcal{M}$. Then we have $\mathbb{E}_{\mathcal{N}}(T_{f_j}(xa)) = \mathbb{E}_{\mathcal{N}}(T_{f_j}(ax))$ and $\mathbb{E}_{\mathcal{N}}(T_\theta(xa)) = \mathbb{E}_{\mathcal{N}}(T_\theta(ax))$ for every $j \in \mathbb{N}$, $a \in \mathcal{L}(\mathbb{R})$ and $x \in \mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$.*

Proof. We first prove that $\mathbb{E}_{\mathcal{N}}(T_{f_j}(xa)) = \mathbb{E}_{\mathcal{N}}(T_{f_j}(ax))$. Suppose $a = \lambda(k)$ and $x = y\lambda(g)$ for $y \in \mathcal{M}$, $k \in L^1(\mathbb{R})$ and $g \in C_c(\mathbb{R})$. Let us first compute $T_{f_j}(xa)$ and $T_{f_j}(ax)$. By the formula (7.3.3) we have

$$T_{f_j}(xa) = \int_{\mathbb{R}} f_j^*(-t)(g * k * f_j)(t)\sigma^\varphi_{-t}(y)dt.$$

By a similar computation we get

$$T_{f_j}(ax) = \int_{\mathbb{R}} (f_j^* * k)(-t)(g * f_j)(t)\sigma^\varphi_{-t}(y)dt.$$

We now apply $\mathbb{E}_{\mathcal{N}}$ to these expressions and use the fact that $\mathcal{N}$ is contained in the centralizer of φ, so $\mathbb{E}_{\mathcal{N}}(\sigma^\varphi_{-t}(y)) = \mathbb{E}_{\mathcal{N}}(y)$. It therefore suffices to prove the equality of the integrals $\int_{\mathbb{R}} f_j^*(-t)(g * k * f_j)(t)dt$ and $\int_{\mathbb{R}} (f_j^* * k)(-t)(g * f_j)(t)dt$; Using the commutativity of the convolution on $\mathbb{R}$, we can rewrite the first one as

$$\int_{\mathbb{R}}\int_{\mathbb{R}} f_j^*(-t)(g * f_j)(t-s)k(s)dsdt$$

and the second one is equal to

$$\int_{\mathbb{R}}\int_{\mathbb{R}} f_j^*(-t-s)k(s)(g * f_j(t))dsdt.$$

In the second integral we can introduce a new variable $t' := t + s$ and it transforms into

$$\int_{\mathbb{R}}\int_{\mathbb{R}} f_j^*(-t')(g * f_j)(t'-s)k(s)dsdt',$$

which is equal to the first one. For arbitrary $a \in \mathcal{L}(\mathbb{R})$ and $x \in \mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$ we can find bounded nets $(a_i)_{i\in I}$ and $(x_i)_{i\in I}$ formed by linear combinations of elements of the form discussed above that converge strongly to a and x, respectively, as a consequence of Kaplansky's density theorem. As multiplication is strongly continuous on bounded subsets, we have strong limits $\lim_{i\in I} a_i x_i = ax$ and $\lim_{i\in I} x_i a_i$. As both $\mathbb{E}_{\mathcal{N}}$ and T_{f_j} are strongly continuous on bounded subsets, we may conclude. The equality $\mathbb{E}_{\mathcal{N}}(T_\theta(xa)) = \mathbb{E}_{\mathcal{N}}(T_\theta(ax))$ follows by a similar computation. □

The ideas appearing in the proof of the next statements are of a similar type.

Proposition 7.3.14. *Let $\mathcal{N} \subseteq \mathcal{M}$ be a unital inclusion of von Neumann algebras which admits a faithful normal conditional expectation $\mathbb{E}_{\mathcal{N}}$. Assume that $\mathcal{N}$ is finite and let $\tau \in \mathcal{N}_*$ be a faithful normal tracial state that we extend to a state $\varphi := \tau \circ \mathbb{E}_{\mathcal{N}}$ on $\mathcal{M}$. Then, for every $j \in \mathbb{N}$, the following statements hold:*

1. *The triple $(\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R},\mathcal{N},\widehat{\varphi}_j)$ satisfies property (rHAP) if and only if, for every $k\in\mathbb{N}$, the triple $(p_k(\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R})p_k, p_k\mathcal{N}p_k, p_k\widehat{\varphi}_j p_k)$ satisfies property (rHAP). Moreover, the (rHAP) may be witnessed by contractive maps, i.e. we may assume that (1′) holds.*

2. *If for every $k\in\mathbb{N}$ the property (rHAP) of the triple $(p_k(\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R})p_k, p_k\mathcal{N}p_k, p_k\widehat{\varphi}_j p_k)$ is witnessed by unital $p_k\widehat{\varphi}_j p_k$-preserving approximating maps, then the relative Haagerup property of $(\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R},\mathcal{N},\widehat{\varphi}_j)$ is witnessed by unital $\widehat{\varphi}_j$-preserving maps.*

3. *If the triple $(\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R},\mathcal{N},\widehat{\varphi}_j)$ satisfies property (rHAP)$^-$ then the triple $(p_k(\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R})p_k, p_k\mathcal{N}p_k, p_k\widehat{\varphi}_j p_k)$ satisfies property (rHAP)$^-$ as well for every $k\in\mathbb{N}$.*

Proof. First part of (1): For the "⇒" direction assume that $(\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R},\mathcal{N},\widehat{\varphi}_j)$ satisfies property (rHAP) and that it is witnessed by a net of maps $(\Phi_i)_{i\in I}$. Fix $k\in\mathbb{N}$. We will show that $(p_k\Phi_i(\,\cdot\,)p_k)_{i\in I}$ is a net of approximating maps witnessing the relative Haagerup property of $(p_k(\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R})p_k, p_k\mathcal{N}p_k, p_k\widehat{\varphi}_j p_k)$.

It is clear that for every $i\in I$ the map $p_k\Phi_i(\,\cdot\,)p_k$ is completely positive, that the net $(p_k\Phi_i(\,\cdot\,)p_k)_{i\in I}$ admits a uniform bound on its norms and that $p_k\Phi_i(\,\cdot\,)p_k\to\mathrm{id}$ in the point-strong topology in i as maps on $p_k(\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R})p_k$. By our assumptions, $\mathcal{N}\subseteq\mathcal{M}^{\sigma^\varphi}$ and hence p_k and $\mathcal{N}$ commute. Hence for $a,b\in\mathcal{N}$, $x\in p_k(\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R})p_k$ we have

$$\begin{aligned} p_k\Phi_i(p_k a p_k x p_k b p_k)p_k &= p_k\Phi_i(a p_k x p_k b)p_k \\ &= p_k a\Phi_i(p_k x p_k)b p_k = p_k a p_k\Phi_i(p_k x p_k)p_k b p_k, \end{aligned}$$

which shows that $p_k\Phi_i(\,\cdot\,)p_k$ is a $p_k\mathcal{N}p_k$-$p_k\mathcal{N}p_k$-bimodule map for every $i\in I$.

We have by [Tak03a, Theorem VIII.3.19.(vi)], [Tak03a, Theorem X.1.17.(ii)] and the fact that p_k and $\lambda(f_j)$ commute that

$$\sigma_t^{\widehat{\varphi}_j}(p_k)=\lambda(f_j)^{it}\sigma_t^{\widehat{\varphi}}(p_k)\lambda(f_j)^{-it}=\lambda(f_j)^{it}p_k\lambda(f_j)^{-it}=p_k.$$

Therefore by [CS15c, Lemma 2.3], for $x\in p_k(\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R})p_k$ positive,

$$(p_k\widehat{\varphi}_j p_k)(p_k\Phi_i(x)p_k)=\widehat{\varphi}_j(p_k\Phi_i(x)p_k)\le\widehat{\varphi}_j(\Phi_i(x))\le\widehat{\varphi}_j(x)=(p_k\widehat{\varphi}_j p_k)(x),$$

i.e. $(p_k\widehat{\varphi}_j p_k)\circ(p_k\Phi_i(\,\cdot\,)p_k)\le p_k\widehat{\varphi}_j p_k$.

Now, for every map Φ on $\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R}$ of the form $\Phi=a\mathbb{E}_{\mathcal{N}}^{\widehat{\varphi}_j}(\cdot)b$ with $a,b\in\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R}$ and $x\in p_k(\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R})p_k$ we have, using Lemma 7.3.13 (recalling that $\mathbb{E}_{\mathcal{N}}^{\widehat{\varphi}_j}=\mathbb{E}_{\mathcal{N}}\circ T_{f_j}$) and the fact that p_k commutes with $\mathcal{N}$, that

$$p_k\Phi(x)p_k=p_k a\mathbb{E}_{\mathcal{N}}^{\widehat{\varphi}_j}(b p_k x p_k)p_k=p_k a\mathbb{E}_{\mathcal{N}}^{\widehat{\varphi}_j}(p_k b p_k x)p_k=(p_k a p_k)\mathbb{E}_{\mathcal{N}}^{\widehat{\varphi}_j}(p_k b p_k x).$$

Lemma 7.3.12 then implies that $(p_k\Phi(\,\cdot\,)p_k)^{(2)}\in\mathcal{K}_{00}(p_k(\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R})p_k, p_k\mathcal{N}p_k, p_k\widehat{\varphi}_j p_k)$. By taking linear combinations and approximating we see that for maps Φ on $\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R}$

with $\Phi^{(2)} \in \mathcal{K}(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}, \mathcal{N}, \widehat{\varphi}_j)$ we must have $(p_k\Phi(\cdot)p_k)^{(2)} \in \mathcal{K}(p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_k\mathcal{N}p_k, p_k\widehat{\varphi}_j p_k)$. Therefore for the approximating maps Φ_i, $i \in I$ we conclude that

$$(p_k\Phi_i(\cdot)p_k)^{(2)} \in \mathcal{K}(p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_k\mathcal{N}p_k, p_k\widehat{\varphi}_j p_k).$$

This shows that $(p_k\Phi_i(\cdot)p_k)_{i\in I}$ indeed witnesses the relative Haagerup property of the triple $(p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_k\mathcal{N}p_k, p_k\widehat{\varphi}_j p_k)$.

(3): Note that if $(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}, \mathcal{N}, \widehat{\varphi}_j)$ has property (rHAP)$^-$ witnessed by the net $(\Phi_i)_{i\in I}$, then property (rHAP)$^-$ of $(p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_k\mathcal{N}p_k, p_k\widehat{\varphi}_j p_k)$ follows in a very similar way as above. The only condition that remains to be checked is that the L^2-implementation $(p_k\Phi_i(\cdot)p_k)^{(2)}$ exists. For this, assume that there exists $C > 0$ with $\widehat{\varphi}_j(\Phi_i(x)^*\Phi_i(x)) \leq C\widehat{\varphi}_j(x^*x)$ for all $x \in \mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$. Then, using again [CS15c, Lemma2.3] for the second inequality,

$$\begin{aligned}
(p_k\widehat{\varphi}_j p_k)((p_k\Phi_i(x)p_k)^*(p_k\Phi_i(x)p_k)) &= \widehat{\varphi}_j(p_k\Phi_i(x^*)p_k\Phi_i(x)p_k) \\
&\leq \widehat{\varphi}_j(p_k\Phi_i(x)^*\Phi_i(x)p_k) \\
&\leq \widehat{\varphi}_j(\Phi_i(x)^*\Phi_i(x)) \\
&\leq C\widehat{\varphi}_j(x^*x) \\
&= C(p_k\widehat{\varphi}_j p_k)(x^*x)
\end{aligned}$$

for all $x \in p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k$. The claim follows.

Second part of (1): For the "$\Leftarrow$" direction assume that for every $k \in \mathbb{N}$ the triple $(p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_k\mathcal{N}p_k, p_k\widehat{\varphi}_j p_k)$ satisfies property (rHAP) witnessed by approximating maps $(\Phi_{k,i})_{i\in I_k}$. We wish to apply Lemma 7.2.14 for which we check the conditions. By $\mathcal{N} \subseteq \mathcal{M}^{\sigma^\varphi}$ we have that $\mathcal{N}$ and λ_t commute for every $t \in \mathbb{R}$ and hence so do $\mathcal{N}$ and h_k. In particular, $h_k \in (p_k\mathcal{N}p_k)' \cap p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k$. By (7.3.8) and the remarks after it, it follows that $\lambda(f_j)h_k$ is positive and boundedly invertible. Now, from (7.3.7) we see that the conditions of Lemma 7.2.14 are fulfilled and this lemma shows that the maps of the net $(\Phi_{k,i})_{i\in I_k}$ can be chosen contractively, i.e. we may assume that (1$'$) holds. We shall prove that $(\Phi_{k,i}(p_k \cdot p_k))_{k\in\mathbb{N}, i\in I_k}$ induces a net witnessing property (rHAP) of $(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}, \mathcal{N}, \widehat{\varphi}_j)$. This in particular shows that we may assume (1$'$).

By the contractivity of the $\Phi_{k,i}$ it is clear that the maps $\Phi_{k,i}(p_k \cdot p_k)$ are completely positive with a uniform bound on their norms. Since $\mathcal{N}$ and p_k commute we see that for $a, b \in \mathcal{N}$ and $x \in \mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$

$$\begin{aligned}
\Phi_{k,i}(p_k axb p_k) &= \Phi_{k,i}(p_k a p_k x p_k b p_k) \\
&= p_k a p_k \Phi_{k,i}(p_k x p_k) p_k b p_k \\
&= a p_k \Phi_{k,i}(p_k x p_k) p_k b \\
&= a\Phi_{k,i}(p_k x p_k) b.
\end{aligned}$$

Therefore $\Phi_{k,i}(p_k \cdot p_k)$ is an $\mathcal{N}$-$\mathcal{N}$ bimodule map for every $k \in \mathbb{N}$, $i \in I_k$. We have, using again [CS15c, Lemma 2.3], that for $x \in (\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})^+$

$$\begin{aligned}\widehat{\varphi}_j(\Phi_{k,i}(p_k x p_k)) &= \widehat{\varphi}_j(p_k \Phi_{k,i}(p_k x p_k) p_k) \\ &= (p_k \widehat{\varphi}_j p_k)(\Phi_{k,i}(p_k x p_k)) \\ &\leq (p_k \widehat{\varphi}_j p_k)(p_k x p_k) \\ &= \widehat{\varphi}_j(p_k x p_k) \leq \widehat{\varphi}_j(x).\end{aligned}$$

i.e. $\widehat{\varphi}_j \circ \Phi_{k,i}(p_k \cdot p_k) \leq \widehat{\varphi}_j$.

We claim that $(\Phi_{k,i}(p_k \cdot p_k))^{(2)} \in \mathcal{K}(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}, \mathcal{N}, \widehat{\varphi}_j)$ for all $k \in \mathbb{N}$, $i \in I_k$. Indeed, take an arbitrary map Φ of the form $\Phi(x) = p_k a p_k \mathbb{E}_{\mathcal{N}}(T_{f_j}(p_k b p_k x))$ for $x \in \mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$ where $a, b \in \mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$. The L^2-implementations of such operators span $\mathcal{K}_{00}(p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_k\mathcal{N}p_k, p_k\widehat{\varphi}_j p_k)$ by Lemma 7.3.12. Lemma 7.3.13 and the fact that p_k and $\mathcal{N}$ commute show that for $x \in \mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$,

$$\Phi(p_k x p_k) = p_k a p_k \mathbb{E}_{\mathcal{N}}(T_{f_j}(p_k b p_k x p_k)) = p_k a p_k \mathbb{E}_{\mathcal{N}}(T_{f_j}(p_k b p_k x)).$$

Then, since $\mathbb{E}_{\mathcal{N}} \circ T_{f_j}$ is the faithful normal $\widehat{\varphi}_j$-preserving conditional expectation of $\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$ onto $\mathcal{N}$, this implies that $(\Phi(p_k \cdot p_k))^{(2)} \in \mathcal{K}_{00}(\mathcal{M} \rtimes^{\sigma^\varphi} \mathbb{R}, \mathcal{N}, \widehat{\varphi}_j)$. By taking linear combinations and approximation we see that if $\Phi^{(2)} \in \mathcal{K}(p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_k\mathcal{N}p_k, p_k\widehat{\varphi}_j p_k)$, then $(\Phi(p_k \cdot p_k))^{(2)} \in \mathcal{K}(\mathcal{M} \rtimes^{\sigma^\varphi} \mathbb{R}, \mathcal{N}, \widehat{\varphi}_j)$. We conclude that $(\Phi_{k,i}(p_k \cdot p_k))^{(2)} \in \mathcal{K}(\mathcal{M} \rtimes^{\sigma^\varphi} \mathbb{R}, \mathcal{N}, \widehat{\varphi}_j)$.

Now, for $x \in \mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$ we see that

$$\lim_{k \to \infty} \lim_{i \in I_k} \Phi_{k,i}(p_k x p_k) = x$$

in the strong topology. Then a variant of Lemma 7.1.1 shows that there is a directed set $\mathcal{F}$ and a function $(\widetilde{k}, \widetilde{i}) : \mathcal{F} \to \{(k,i) \mid k \in \mathbb{N}, i \in I_k\}$, $F \mapsto (\widetilde{k}(F), \widetilde{i}(F))$ such $(\Phi_{\widetilde{k}(F), \widetilde{i}(F)})_{F \in \mathcal{F}}$ witnesses the relative Haagerup property of $(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}, \mathcal{N}, \widehat{\varphi}_j)$.

(2): It only remains to show that if for every $k \in \mathbb{N}$ the property (rHAP) of the triple $(p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_k\mathcal{N}p_k, p_k\widehat{\varphi}_j p_k)$ is witnessed by unital $p_k\widehat{\varphi}_j p_k$-preserving approximating maps, then the relative Haagerup property of $(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}, \mathcal{N}, \widehat{\varphi}_j)$ is witnessed by unital $\widehat{\varphi}_j$-preserving maps. For this, assume that the maps $(\Phi_{k,i})_{i \in I}$ from before are unital and $p_k\widehat{\varphi}_j p_k$-preserving and choose a sequence $(\varepsilon_k)_{k \in \mathbb{N}} \subseteq (0,1)$ with $\varepsilon_k \to 0$. Recall that $p_k \in \mathcal{N}' \cap (\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})$ and note that $\mathbb{E}_{\mathcal{N}}^{\widehat{\varphi}_j}(1 - (1-\varepsilon_k)p_k) \geq \varepsilon_k$. We then have $\mathbb{E}_{\mathcal{N}}^{\widehat{\varphi}_j}(1 - (1-\varepsilon_k)p_k) \in \mathcal{N} \cap \mathcal{N}'$, the inverse $(\mathbb{E}_{\mathcal{N}}^{\widehat{\varphi}_j}(1 - (1-\varepsilon_k)p_k))^{-1} \in \mathcal{N} \cap \mathcal{N}'$ exists and $a_k := (1 - (1-\varepsilon_k)p_k)(\mathbb{E}_{\mathcal{N}}^{\widehat{\varphi}_j}(1 - (1-\varepsilon_k)p_k))^{-1} \in \mathcal{N}' \cap (\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})$ is positive. Set $b_k := 1 - (1-\varepsilon_k)p_k geq 0$. Define the maps

$$\widetilde{\Phi}_{k,i}(\cdot) := (1-\varepsilon_k)\Phi_{k,i}(p_k \cdot p_k) + a_k \mathbb{E}_{\mathcal{N}}^{\widehat{\varphi}_j}(b_k^{1/2} \cdot b_k^{1/2}).$$

Obviously $\widetilde{\Phi}_{k,i}$ is normal, completely positive and $\mathcal{N}$-$\mathcal{N}$-bimodular. We may finish the proof as in Theorem 7.2.13 now; since the statement of that theorem is not directly applicable here we will give the complete proof for the convenience of the reader. We have

$$\widetilde{\Phi}_{k,i}(1) = (1-\varepsilon_k)\Phi_{k,i}(p_k) + a_k\mathbb{E}_{\mathcal{N}}^{\widehat{\varphi}_j}(b_k) = (1-\varepsilon_k)p_k + (1-(1-\varepsilon_k)p_k) = 1.$$

Now, since $\Phi_{k,i}$ is $p_k\widehat{\varphi}_j p_k$-preserving we have that $\widehat{\varphi}_j \circ \Phi_{k,i}(p_k x p_k) = \widehat{\varphi_j}(p_k x p_k)$ for all $x \in \mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$, and hence with Lemma 7.3.13 we deduce that

$$\begin{aligned}
\widehat{\varphi}_j \circ \widetilde{\Phi}_{k,i}(x) &= (1-\varepsilon_k)\widehat{\varphi}_j(\Phi_{k,i}(p_k x p_k)) + \widehat{\varphi}_j(a_k \mathbb{E}_{\mathcal{N}}^{\widehat{\varphi}_j}(b_k^{1/2} x b_k^{1/2})) \\
&= (1-\varepsilon_k)\widehat{\varphi}_j(p_k x p_k) + \widehat{\varphi}_j(\mathbb{E}_{\mathcal{N}}^{\widehat{\varphi}_j}(a_k) b_k^{1/2} x b_k^{1/2}) \\
&= (1-\varepsilon_k)\widehat{\varphi}_j \circ \mathbb{E}_{\mathcal{N}}^{\widehat{\varphi}_j}(p_k x p_k) + \widehat{\varphi}_j \circ \mathbb{E}_{\mathcal{N}}^{\widehat{\varphi}_j}(b_k^{1/2} x b_k^{1/2}) \\
&= (1-\varepsilon_k)\widehat{\varphi}_j \circ \mathbb{E}_{\mathcal{N}}^{\widehat{\varphi}_j}(p_k x) + \widehat{\varphi}_j \circ \mathbb{E}_{\mathcal{N}}^{\widehat{\varphi}_j}(b_k x) \\
&= (1-\varepsilon_k)\widehat{\varphi}_j(p_k x) + \widehat{\varphi}_j(b_k x) \\
&= \widehat{\varphi}_j(x).
\end{aligned}$$

By the fact that $(\Phi_{k,i}(p_k \cdot p_k))^{(2)} \in \mathcal{K}(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}, \mathcal{N}, \widehat{\varphi}_j)$ and by Lemma 7.3.13, we have

$$\widetilde{\Phi}_{k,i}^{(2)} = (1-\varepsilon_k)(\Phi_{k,i}(p_k \cdot p_k))^{(2)} + a_k e_{\mathcal{N}}^{\widehat{\varphi}_j} b_k \in \mathcal{K}(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}, \mathcal{N}, \widehat{\varphi}_j).$$

Further, for every $x \in (\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})_+$,

$$\begin{aligned}
\widetilde{\Phi}_{k,i}(x) - (1-\varepsilon_k)\Phi_{k,i}(p_k x p_k) &= a_k \mathbb{E}_{\mathcal{N}}^{\widehat{\varphi}_j}(b_k^{1/2} x b_k^{1/2}) \\
&\le \|x\|\, a_k \mathbb{E}_{\mathcal{N}}^{\widehat{\varphi}_j}(b_k) \\
&= \|x\|\,(1-(1-\varepsilon_k)p_k),
\end{aligned}$$

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from which we deduce that $\lim_{F\in\mathcal{F}} \widetilde{\Phi}_{\widetilde{k}(F),\widetilde{i}(F)} = \mathrm{id}_{\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R}}$. Thus, the net $(\widetilde{\Phi}_{\widetilde{k}(F),\widetilde{i}(F)})_{F\in\mathcal{F}}$ of unital $\widehat{\varphi}_j$-preserving maps witnesses the relative Haagerup property of $(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}, \mathcal{N}, \widehat{\varphi}_j)$. □

We are now ready to formulate the key statement of this section. Note that for every $k \in \mathbb{N}$ the von Neumann algebra $p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k$ is finite with a faithful normal tracial state $p_k \tau_{\rtimes} p_k$.

Theorem 7.3.15. *Let $\mathcal{N} \subseteq \mathcal{M}$ be a unital inclusion of von Neumann algebras which admits a faithful normal conditional expectation $\mathbb{E}_{\mathcal{N}}$. Assume that $\mathcal{N}$ is finite and let $\tau \in \mathcal{N}_*$ be a faithful normal tracial state that we extend to a state $\varphi := \tau \circ \mathbb{E}_{\mathcal{N}}$ on $\mathcal{M}$. Then the following are equivalent:*

1. *The triple $(\mathcal{M}, \mathcal{N}, \varphi)$ has property (rHAP);*
2. *$(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}, \mathcal{N}, \widehat{\varphi}_j)$ has property (rHAP) for every $j \in \mathbb{N}$;*
3. *$(p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_k\mathcal{N}p_k, p_k\tau_{\rtimes}p_k)$ has property (rHAP) for every $k \in \mathbb{N}$.*

Further, the following statement holds:

4. *If the triple* $(\mathcal{M},\mathcal{N},\varphi)$ *has property (rHAP)*$^-$*, then for every* $k\in\mathbb{N}$*,* $(p_k(\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R})p_k, p_k\mathcal{N}p_k, p_k\tau_\rtimes p_k)$ *has property (rHAP)*$^-$*.*

Proof. The equivalence "(1) ⇔ (2)" was proved in Theorem 7.3.9.

"(2) ⇒ (3)": Assume that for $j\in\mathbb{N}$ the triple $(\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R},\mathcal{N},\widehat{\varphi}_j)$ has property (rHAP) and fix $k\in\mathbb{N}$. Then by Proposition 7.3.14, the triple $(p_k(\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R})p_k, p_k\mathcal{N}p_k, p_k\widehat{\varphi}_j p_k)$ also has the (rHAP). Let $(\Phi_i)_{i\in I}$ be a net of suitable approximating maps and define the self-adjoint boundedly invertible operator $A_{j,k}:=\lambda(f_j)h_k^{1/2}\in(p_k\mathcal{N}p_k)'\cap(p_k(\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R})p_k)$. By (7.3.7) for every $x\in p_k(\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R})p_k$ the equality

$$(p_k\widehat{\varphi}_j p_k)(x)=\tau_\rtimes(A_{j,k}^*xA_{j,k})=(A_{j,k}p_k\tau_\rtimes p_kA_{j,k})(x)$$

holds and hence Lemma 7.3.10 implies that the L^2-implementation of the map $\Phi_i'(\cdot):=A_{j,k}\Phi_i(A_{j,k}^{-1}\cdot A_{j,k}^{-1})A_{j,k}$ exists and is contained in $\mathcal{K}(p_k(\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R})p_k, p_k\mathcal{N}p_k, p_k\tau_\rtimes p_k)$. Similarly to the proof of Proposition 7.3.14 one checks that the net $(\Phi_i')_{i\in I}$ witnesses property (rHAP) of $(p_k(\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R})p_k, p_k\mathcal{N}p_k, p_k\tau_\rtimes p_k)$. We omit the details.

"(2) ⇐ (3)" Now assume that the triple $(p_k(\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R})p_k, p_k\mathcal{N}p_k, p_k\tau_\rtimes p_k)$ has property (rHAP) for every $k\in\mathbb{N}$. It suffices to show that the triple $(p_k(\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R})p_k, p_k\mathcal{N}p_k, p_k\widehat{\varphi}_j p_k)$ has property (rHAP) as it implies the desired statement by Proposition 7.3.14. So let $(\Phi_i)_{i\in I}$ be a net that witnesses property (rHAP) of the triple $(p_k(\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R})p_k, p_k\mathcal{N}p_k, p_k\tau_\rtimes p_k)$ and set $\Phi_i':=A_{j,k}^{-1}\Phi_i(A_{j,k}\cdot A_{j,k})A_{j,k}^{-1}$. Lemma 7.3.10 and (7.3.7) imply that for every $i\in I$ the L^2-implementation $(\Phi_i')^{(2)}$ of Φ_i' with respect to the positive functional $p_k\widehat{\varphi_j}p_k$ is contained in $\mathcal{K}(p_k(\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R})p_k, p_k\mathcal{N}p_k, p_k\widehat{\varphi}_j p_k)$. Again, similarly to the proof of Proposition 7.3.14 one checks that the net $(\Phi_i')_{i\in I}$ witnesses property (rHAP).

It remains to show (4). The statement easily follows from Proposition 7.3.9, Proposition 7.3.14 and the arguments used in the proof of the implication "(2) ⇒ (3)". □

7.4. MAIN RESULTS

After the main work has been done in Section 7.3 we can now put the pieces together. This allows us to show that in the case of a finite von Neumann subalgebra the notion of relative Haagerup property is independent of the choice of the corresponding faithful normal conditional expectation, that the approximating maps may be chosen to be unital and state-preserving and that property (rHAP) and property (rHAP)$^-$ are equivalent. The general notation will be the same as in Section 7.3.

7.4.1. INDEPENDENCE OF THE CONDITIONAL EXPECTATION

Let $\mathcal{N}\subseteq\mathcal{M}$ be a unital inclusion of von Neumann algebras for which $\mathcal{N}$ is finite with a faithful normal tracial state $\tau\in\mathcal{N}_*$. Let further $\mathbb{E}_\mathcal{N},\mathbb{F}_\mathcal{N}:\mathcal{M}\to\mathcal{N}$ be two faithful normal conditional expectations and extend τ to states $\varphi:=\tau\circ\mathbb{E}_\mathcal{N}$ and $\psi:=\tau\circ\mathbb{F}_\mathcal{N}$ on $\mathcal{M}$. In this subsection we will prove that the triple $(\mathcal{M},\mathcal{N},\mathbb{E}_\mathcal{N})$ has property (rHAP) if and only if the triple $(\mathcal{M},\mathcal{N},\mathbb{F}_\mathcal{N})$ does, i.e. the relative Haagerup property is an intrinsic invariant of the inclusion $\mathcal{N} \subseteq \mathcal{M}$. Let us first introduce some notation.

As in Section 7.3 consider the crossed product von Neumann algebra $\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$ which contains the projections $p_k \in \mathcal{N}' \cap (\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})$, $k \in \mathbb{N}$ and carries the canonical normal semi-finite tracial weight $\tau_\rtimes$ which we will from now on denote by $\tau_{\rtimes,1}$. For $t \in \mathbb{R}$ write λ_t^φ for the left regular representation operators in $\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$. Similarly, we write $\tau_{\rtimes,2}$ for the canonical normal semi-finite tracial weight on $\mathcal{M} \rtimes_{\sigma^\psi} \mathbb{R}$ and denote the corresponding left regular representation operators by λ_t^ψ, $t \in \mathbb{R}$.

For $t \in \mathbb{R}$ let $u_t := (D\varphi / D\psi)_t \in \mathcal{M}$ be the Connes cocycle Radon-Nikodym derivative, so in particular $u_t \sigma_t^\varphi(u_s) = u_{t+s}$ and $\sigma_t^\psi(x) = u_t^* \sigma_t^\varphi(x) u_t$ hold for all $s, t \in \mathbb{R}$. Then (see [Tak03a, Proof of Theorem X.1.7]) there exists an isomorphism $\rho : \mathcal{M} \rtimes_{\sigma^\psi} \mathbb{R} \to \mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$ of von Neumann algebras which restricts to the identity on $\mathcal{M}$ and for which $\rho(\lambda_t^\psi) = u_t \lambda_t^\varphi$ for all $t \in \mathbb{R}$. This implies that the dual actions θ^φ and θ^ψ of σ^φ and σ^ψ respectively are related by the equality $\theta_t^\varphi \circ \rho = \rho \circ \theta_t^\psi$, $t \in \mathbb{R}$. Further, $\tau_{\rtimes,1} \circ \rho = \tau_{\rtimes,2}$ (see the footnote [2]). Denote by h_ψ the unique unbounded self-adjoint positive operator affiliated with $\mathcal{M} \rtimes_{\sigma^\psi} \mathbb{R}$ such that $h_\psi^{it} = \lambda_t^\psi$ for all $t \in \mathbb{R}$ and set

$$p_{\psi,k} := \chi_{[k^{-1},k]}(h_\psi) \qquad \text{and} \qquad q_k := \rho(p_{\psi,k}).$$

for $k \in \mathbb{N}$. Further, define

$$h_{\psi,k} := \rho(\chi_{[k^{-1},k]}(h_\psi) h_\psi) = \rho(p_{\psi,k} h_\psi).$$

Recall that for all $k \in \mathbb{N}$ and $t \in \mathbb{R}$ we write $h^{it} = \lambda_t^\varphi$, $p_k := \chi_{[k^{-1},k]}(h)$, and $h_k := p_k h$. The following statement compares to Lemma 7.3.12.

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Lemma 7.4.1. *For every $k \in \mathbb{N}$ there is a (unique) faithful normal $p_k \tau_{\rtimes,1} p_k$-preserving conditional expectation $\mathbb{E}_{1,k} : p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}) p_k \to p_k \mathcal{N} p_k$ given by*

$$x \mapsto \nu_k^{-1} p_k \mathbb{E}_\mathcal{N}(T_{\theta^\varphi}(h_k^{-1} x)) p_k,$$

where $\nu_k := T_{\theta^\varphi}(h_k^{-1}) = k - k^{-1}$. In particular, $T_{\theta^\varphi}(h_k^{-1})$ is a scalar multiple of the identity.

Proof. The proof is essentially the same as that of Lemma 7.3.12. First note that by Remark 7.3.11 the operator h_k coincides with $\lambda(\widehat{J_k})$ where $J_k(s) = \chi_{[-\log(k),\log(k)]} e^s$ and that $\nu_k = T_{\theta^\varphi}(h_k^{-1}) = k - k^{-1}$ is a multiple of the identity. For $x \in p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}) p_k$ one checks

[2] This is well-known to specialists, but it seems that the statement does not appear explicitly in [Tak03a]. The argument goes as follows. Firstly, as ρ intertwines the dual actions on $\mathcal{M} \rtimes_{\sigma^\psi} \mathbb{R}$ and $\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$ we find that $\widehat{\varphi} \circ \rho$ is the dual weight of φ in the crossed product $\mathcal{M} \rtimes_{\sigma^\psi} \mathbb{R}$. Let $t \in \mathbb{R}$. By [Tak03a, Theorem X.1.17] we have Connes cocycle derivative $\left(\frac{D\widehat{\psi}}{D\widehat{\varphi}\circ\rho}\right)_t = u_t = \rho(u_t)$. Then by the chain rule [Tak03a, Theorem VIII.3.7],

$$\left(\frac{D\tau_{\rtimes,2}}{D\tau_{\rtimes,1}\circ\rho}\right)_t = \left(\frac{D\tau_{\rtimes,2}}{D\widehat{\psi}}\right)_t \left(\frac{D\widehat{\psi}}{D\widehat{\varphi}\circ\rho}\right)_t \left(\frac{D\widehat{\varphi}\circ\rho}{D\tau_{\rtimes,1}\circ\rho}\right)_t = \lambda_{-t}^\psi \rho^{-1}(u_t \lambda_t^\varphi) = 1.$$

Hence $\tau_{\rtimes,1} \circ \rho = \tau_{\rtimes,2}$.

using (7.3.9) for the second and last equality, that

$$\begin{aligned}
(p_k\tau_{\rtimes,1}p_k)(p_k\mathbb{E}_{\mathcal{N}}(T_{\theta^\varphi}(h_k^{-1}x))p_k) &= \tau_{\rtimes,1}(p_k\mathbb{E}_{\mathcal{N}}(T_{\theta^\varphi}(h_k^{-1}x))p_k) \\
&= \varphi\circ T_{\theta^\varphi}(p_k h_k^{-1}\mathbb{E}_{\mathcal{N}}(T_{\theta^\varphi}(h_k^{-1}x))p_k) \\
&= \varphi\circ T_{\theta^\varphi}(h_k^{-1}\mathbb{E}_{\mathcal{N}}(T_{\theta^\varphi}(h_k^{-1}x))) \\
&= \varphi\left(T_{\theta^\varphi}(h_k^{-1})\mathbb{E}_{\mathcal{N}}(T_{\theta^\varphi}(h_k^{-1}x))\right) \\
&= \nu_k\varphi\left(\mathbb{E}_{\mathcal{N}}(T_{\theta^\varphi}(h_k^{-1}x))\right) \\
&= \nu_k\varphi\circ T_{\theta^\varphi}(h_k^{-1}x)) \\
&= \nu_k\tau_{\rtimes,1}(p_k x p_k),
\end{aligned}$$

hence $\mathbb{E}_{1,k}$ is indeed $p_k\tau_{\rtimes,1}p_k$-preserving. Here we used in the fourth line that $\mathcal{N}$ is invariant under the dual action θ^φ and in the fifth line that $T_{\theta^\varphi}(h_k^{-1})$ is a multiple of the identity.

From Lemma 7.3.13 we see that

$$\nu_k^{-1}p_k\mathbb{E}_{\mathcal{N}}(T_{\theta^\varphi}(h_k^{-1}\,\cdot))p_k = \nu_k^{-1}p_k\mathbb{E}_{\mathcal{N}}(T_{\theta^\varphi}(h_k^{-1/2}\cdot h_k^{-1/2}))p_k,$$

and from the right hand side of this expression it is clear that (7.4.1) is completely positive. The remaining statements (i.e. that $\mathbb{E}_{1,k}$ is a unital faithful normal $p_k\mathcal{N}p_k$-$p_k\mathcal{N}p_k$-bimodule map) are then easy to check.

□

The following lemma provides the analogous statement for the functional $q_k\tau_{\rtimes,1}q_k$ and the inclusion $q_k\mathcal{N}q_k \subseteq q_k(\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R})q_k$. We omit the proof.

Lemma 7.4.2. *For every $k\in\mathbb{N}$ there is a (unique) faithful normal $q_k\tau_{\rtimes,1}q_k$-preserving conditional expectation $\mathbb{E}_{2,k}: q_k(\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R})q_k \to q_k\mathcal{N}q_k$ given by*

$$x\mapsto \nu_k^{-1}q_k\mathbb{E}_{\mathcal{N}}(T_{\theta^\varphi}(h_{\psi,k}^{-1}x))q_k,$$

where $\nu_k := k-k^{-1}$ as before.

Proposition 7.4.3. *Let $\mathcal{N}\subseteq\mathcal{M}$ be a unital inclusion of von Neumann algebras for which $\mathcal{N}$ is finite with a faithful normal tracial state $\tau\in\mathcal{N}_*$. Let further $\mathbb{E}_{\mathcal{N}},\mathbb{F}_{\mathcal{N}}:\mathcal{M}\to\mathcal{N}$ be two faithful normal conditional expectations and extend τ to states $\varphi := \tau\circ\mathbb{E}_{\mathcal{N}}$ and $\psi := \tau\circ\mathbb{F}_{\mathcal{N}}$ on $\mathcal{M}$. Then the following statements are equivalent:*

1. *For every $k\in\mathbb{N}$ the triple $(p_k(\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R})p_k, p_k\mathcal{N}p_k, p_k\tau_{\rtimes,1}p_k)$ has property (rHAP).*

2. *For every $k\in\mathbb{N}$ the triple $(q_k(\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R})q_k, q_k\mathcal{N}q_k, q_k\tau_{\rtimes,1}q_k)$ has property (rHAP).*

Proof. By symmetry it suffices to consider the direction "(2) ⇒ (1)". For this, fix $k,l\in\mathbb{N}$ and let $(\Phi_{l,i})_{i\in I_l}$ be a net of maps witnessing the relative Haagerup property of the triple $(q_l(\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R})q_l, q_l\mathcal{N}q_l, q_l\tau_{\rtimes,1}q_l)$, which we can assume to be contractive by Lemma 7.2.14. Define for $i\in I_l$ the normal completely positive contractive map

$$\Phi'_{k,l,i}: p_k(\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R})p_k \to p_k(\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R})p_k : x\mapsto p_k\Phi_{l,i}(q_l x q_l)p_k.$$

As $\mathcal{N} \subseteq \mathcal{M}^{\sigma^\varphi}$ and $\mathcal{N} \subseteq \mathcal{M}^{\sigma^\psi}$, $\mathcal{N}$ commutes with both q_l and p_k. Thus we have that for $x \in p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k$ and $a, b \in \mathcal{N}$

$$\begin{aligned}\Phi'_{k,l,i}(p_k a p_k x p_k b p_k) &= p_k \Phi_{l,i}(q_l p_k a p_k x p_k b p_k q_l) p_k = p_k \Phi_{l,i}(q_l a x b q_l) p_k \\ &= p_k \Phi_{l,i}(q_l a q_l x q_l b q_l) p_k = p_k q_l a q_l \Phi_{l,i}(q_l x q_l) q_l b q_l p_k \\ &= p_k a \Phi_{l,i}(q_l x q_l) b p_k = p_k a p_k \Phi_{l,i}(q_l x q_l) p_k b p_k \\ &= p_k a p_k \Phi'_{k,l,i}(x) p_k b p_k,\end{aligned}$$

i.e. $\Phi'_{k,l,i}$ is $p_k\mathcal{N}p_k$-$p_k\mathcal{N}p_k$-bimodular. Further, $(p_k \tau_{\rtimes,1} p_k) \circ \Phi'_{k,l,i} \leq p_k \tau_{\rtimes,1} p_k$ since for all positive $x \in p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k$ we have

$$\begin{aligned}(p_k \tau_{\rtimes,1} p_k) \circ \Phi'_{k,l,i}(x) &= \tau_{\rtimes 1}(p_k \Phi_{l,i}(q_l p_k x p_k q_l) p_k) \leq \tau_{\rtimes,1}(\Phi_{l,i}(q_l p_k x p_k q_l)) \\ &= \tau_{\rtimes,1}(q_l \Phi_{l,i}(q_l p_k x p_k q_l) q_l) \leq \tau_{\rtimes,1}(q_l p_k x p_k q_l) \\ &\leq (p_k \tau_{\rtimes,1} p_k)(x).\end{aligned}$$

For every map Φ on $q_l(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})q_l$ of the form $\Phi = a\mathbb{E}_{2,l} b$ with $a, b \in q_l(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})q_l$ and $x \in p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k$ we have by Lemma 7.4.2 that

$$\begin{aligned}p_k \Phi(q_l x q_l) p_k &= p_k a \mathbb{E}_{2,l}(b q_l x q_l) p_k \\ &= \nu_l^{-1} p_k a q_l \mathbb{E}_{\mathcal{N}}(T_{\theta^\varphi}(h_{\psi,l}^{-1} b q_l x q_l)) q_l p_k \\ &= \nu_l^{-1} p_k a p_k \mathbb{E}_{\mathcal{N}}(T_{\theta^\varphi}(h_{\psi,l}^{-1} b q_l x q_l)).\end{aligned}$$

Now we may use the isomorphism ρ and apply Lemma 7.3.13 to $\mathcal{M} \rtimes_{\sigma^\psi} \mathbb{R}$ to get

$$\begin{aligned}p_k \Phi(q_l x q_l) p_k &= \nu_l^{-1} p_k a p_k \mathbb{E}_{\mathcal{N}}(T_{\theta^\psi}(p_{\psi,l} h_\psi^{-1} \rho^{-1}(b) p_{\psi,l} \rho^{-1}(x) p_{\psi,l})) \\ &= \nu_l^{-1} p_k a p_k \mathbb{E}_{\mathcal{N}}(T_{\theta^\psi}(p_{\psi,l} h_\psi^{-1} \rho^{-1}(b) p_{\psi,l} \rho^{-1}(x)) \\ &= \nu_l^{-1} p_k a p_k \mathbb{E}_{\mathcal{N}}(T_{\theta^\varphi}(q_l h_{\psi,l}^{-1} b q_l x)).\end{aligned}$$

Then by Lemma 7.3.13 applied to $\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$ for the second equality and Lemma 7.4.1 for the last equality, we find

$$\begin{aligned}p_k \Phi(q_l x q_l) p_k &= \nu_l^{-1} p_k a p_k \mathbb{E}_{\mathcal{N}}(T_{\theta^\varphi}(q_l h_{\psi,l}^{-1} b q_l x p_k)) \\ &= \nu_l^{-1} p_k a p_k \mathbb{E}_{\mathcal{N}}(T_{\theta^\varphi}(h_k^{-1}(h_k q_l h_{\psi,l}^{-1} b q_l x))) p_k \\ &= \nu_k \nu_l^{-1} p_k a \mathbb{E}_{1,k}((h_k q_l h_{\psi,l}^{-1} b q_l) x).\end{aligned}$$

Thus $(p_k \Phi(q_l \cdot q_l) p_k)^{(2)} \in \mathcal{K}_{00}(p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_k\mathcal{N}p_k, p_k \tau_{\rtimes,1} p_k)$. Taking linear combinations and approximation, we see that if $\Phi^{(2)} \in \mathcal{K}(q_l(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})q_l, q_l\mathcal{N}q_l, q_l \tau_{\rtimes,1} q_l)$, then also $p_k \Phi(q_l \cdot q_l)p_k)^{(2)} \in \mathcal{K}(p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_k\mathcal{N}p_k, p_k \tau_{\rtimes,1} p_k)$. In particular, we find $(\Phi'_{k,l,i})^{(2)} \in \mathcal{K}(p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_k\mathcal{N}p_k, p_k \tau_{\rtimes,1} p_k)$ for $k, l \in \mathbb{N}$ and $i \in I_l$.

For every $x \in p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k$ we have that

$$\lim_{l \to \infty} \lim_{i \in I_l} \Phi'_{k,l,i}(x) = x$$

in the strong topology. A variant of Lemma 7.1.1 then shows that there is a directed set $\mathscr{F}$ and an increasing function $(\tilde{l},\tilde{i}) : \mathscr{F} \to \{(l,i) \mid k \in \mathbb{N}, i \in I_l\}$, $F \mapsto (\tilde{l}(F),\tilde{i}(F))$ such that $(\Phi'_{k,\tilde{l}(F),\tilde{i}(F)})_{F\in\mathscr{F}}$ witnesses the relative Haagerup property of $(p_k(\mathscr{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_k\mathscr{N}p_k, p_k\tau_{\rtimes,1}p_k)$. □

Theorem 7.4.4. *Let $\mathscr{N} \subseteq \mathscr{M}$ be a unital inclusion of von Neumann algebras with $\mathscr{N}$ finite. Let $\mathbb{E}_{\mathscr{N}}, \mathbb{F}_{\mathscr{N}} : \mathscr{M} \to \mathscr{N}$ be two faithful normal conditional expectations. Then the triple $(\mathscr{M},\mathscr{N},\mathbb{E}_{\mathscr{N}})$ has (rHAP) if and only if the triple $(\mathscr{M},\mathscr{N},\mathbb{F}_{\mathscr{N}})$ has (rHAP).*

Proof. Assume that the triple $(\mathscr{M},\mathscr{N},\mathbb{E}_{\mathscr{N}})$ has the relative Haagerup property. Let τ be a faithful normal tracial state on $\mathscr{N}$ that we extend to a state $\varphi := \tau\circ\mathbb{E}_{\mathscr{N}}$ on $\mathscr{M}$. Theorem 7.3.15 implies that for every $k \in \mathbb{N}$ the triple $(p_k(\mathscr{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_k\mathscr{N}p_k, p_k\tau_{\rtimes,1}p_k)$ has the (rHAP). With Proposition 7.4.3 we get that for every $k \in \mathbb{N}$ the triple $(q_k(\mathscr{M} \rtimes_{\sigma^\varphi} \mathbb{R})q_k, q_k\mathscr{N}q_k, q_k\tau_{\rtimes,1}q_k)$ has the (rHAP). The isomorphism ρ restricts to an isomorphism $q_k(\mathscr{M} \rtimes_{\sigma^\varphi} \mathbb{R})q_k \cong p_{\psi,k}(\mathscr{M} \rtimes_{\sigma^\psi} \mathbb{R})p_{\psi,k}$ which maps $q_k\mathscr{N}q_k$ onto $p_{\psi,k}\mathscr{N}p_{\psi,k}$ and for which $(q_k\tau_{\rtimes,1}q_k)\circ\rho = p_{\psi,k}\tau_{\rtimes,2}p_{\psi,k}$. Combining this with Theorem 7.3.15 implies that $(\mathscr{M},\mathscr{N},\mathbb{F}_{\mathscr{N}})$ has the (rHAP). □

7.4.2. UNITALITY AND STATE-PRESERVATION OF THE APPROXIMATING MAPS

The following theorem states that for triples $(\mathscr{M},\mathscr{N},\varphi)$ with $\mathscr{N}$ finite the approximating maps may be assumed to be unital and state-preserving. The proof combines the passage to suitable crossed products and corners of crossed products from Section 7.3 with the case considered in Subsection 7.2.3.

Theorem 7.4.5. *Let $\mathscr{N} \subseteq \mathscr{M}$ be a unital inclusion of von Neumann algebras which admits a faithful normal conditional expectation $\mathbb{E}_{\mathscr{N}}$. Assume that $\mathscr{N}$ is finite. Let $\tau \in \mathscr{N}_*$ be a faithful normal (possibly non-tracial) state that we extend to a state $\varphi := \tau\circ\mathbb{E}_{\mathscr{N}}$ on $\mathscr{M}$ and assume that the triple $(\mathscr{M},\mathscr{N},\varphi)$ has property (rHAP). Then property (rHAP) may be witnessed by a net of unital and φ-preserving approximating maps, i.e. we may assume (1″) and (4′).*

Proof. First assume that τ is tracial. Since the triple $(\mathscr{M},\mathscr{N},\varphi)$ has property (rHAP) we get with Theorem 7.3.9 and Proposition 7.3.14 that for all $j \in \mathbb{N}$, $k \in \mathbb{N}$ the triple $(p_k(\mathscr{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_k\mathscr{N}p_k, p_k\widehat{\varphi}_j p_k)$ has property (rHAP) as well and that it may be witnessed by a net of contractive approximating maps. As we have seen before, for every $k \in \mathbb{N}$ the element $h_k^{1/2}\lambda(f_j) \in p_k(\mathscr{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k$ is positive and boundedly invertible in $p_k(\mathscr{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k$. Further, by (7.3.7) and [Tak03a, Theorem VIII.2.11] the equality

$$\sigma_t^{p_k\widehat{\varphi}_j p_k}(x) = (h_k^{1/2}\lambda(f_j))^{it}x(h_k^{1/2}\lambda(f_j))^{-it}$$

holds for all $x \in p_k(\mathscr{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k$, $t \in \mathbb{R}$. Theorem 7.2.13 then implies that property (rHAP) of $(p_k(\mathscr{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_k\mathscr{N}p_k, p_k\widehat{\varphi}_j p_k)$ may for every $j,k \in \mathbb{N}$ be witnessed by a net of unital $(p_k\widehat{\varphi}_j p_k)$-preserving maps. By applying the converse directions of Proposition 7.3.14 and Theorem 7.3.9 we deduce the claimed statement.

Now we show that we may replace τ by any non-tracial faithful state in N_*. Let still $\tau \in N_*$ be a faithful tracial state. Let $(\Phi_i)_{i\in I}$ be approximating maps witnessing the (rHAP) for $(\mathscr{M},\mathscr{N},\tau\circ\mathbb{E}_{\mathscr{N}})$ which by the previous paragraph may be taken unital and $\tau\circ\mathbb{E}_{\mathscr{N}}$-preserving. The proof of Theorem 7.2.7, exploiting Lemmas 7.2.5 and 7.2.6 shows that $(\Phi_i)_{i\in I}$ also witness the (rHAP) for $(\mathscr{M},\mathscr{N},\varphi\circ\mathbb{E}_{\mathscr{N}})$ for any faithful state $\varphi\in N_*$. Further Lemma 7.2.5 shows that Φ_i is $\varphi\circ\mathbb{E}_{\mathscr{N}}$-preserving and we are done. □

7.4.3. EQUIVALENCE OF (RHAP) AND (RHAP)$^-$

In [BF11] among other things Bannon and Fang prove that for triples $(\mathscr{M},\mathscr{N},\tau)$ of finite von Neumann algebras with a tracial state $\tau\in\mathscr{M}_*$ the subtraciality condition in Popa's notion of the relative Haagerup property is redundant. It is easy to check that their proof translates into our setting, which leads to the following variation of [BF11, Theorem 2.2].

Theorem 7.4.6 (Bannon-Fang)**.** *Let $\mathscr{M}$ be a finite von Neumann algebra equipped with a faithful normal tracial state $\tau\in\mathscr{M}_*$ and let $\mathscr{N}\subseteq\mathscr{M}$ be a unital inclusion of von Neumann algebras. If the triple $(\mathscr{M},\mathscr{N},\tau)$ has property (rHAP)$^-$, then it has property (rHAP). Further, property (rHAP) may be witnessed by unital and trace-preserving approximating maps.*

In combination with Theorem 7.4.5 the following theorem provides a generalisation of Theorem 7.4.6.

Theorem 7.4.7. *Let $\mathscr{N}\subseteq\mathscr{M}$ be a unital inclusion of von Neumann algebras which admits a faithful normal conditional expectation $\mathbb{E}_{\mathscr{N}}$. Assume that $\mathscr{N}$ is finite. Let $\tau\in\mathscr{N}_*$ be a faithful normal state that we extend to a state $\varphi:=\tau\circ\mathbb{E}_{\mathscr{N}}$ on $\mathscr{M}$. Then the triple $(\mathscr{M},\mathscr{N},\varphi)$ has property (rHAP) if and only if it has property (rHAP)$^-$.*

Proof. By Theorem 7.2.7 we may without loss of generality assume that τ is tracial on $\mathscr{N}$. It is clear that property (rHAP) implies property (rHAP)$^-$. Conversely, if the triple $(\mathscr{M},\mathscr{N},\varphi)$ has property (rHAP)$^-$, then we deduce from Theorem 7.3.15 that for every $k\in\mathbb{N}$ the triple $(p_k(\mathscr{M}\rtimes_{\sigma^\varphi}\mathbb{R})p_k, p_k\mathscr{N}p_k, p_k\tau_{\rtimes,1}p_k)$ has property (rHAP)$^-$ as well. Recall that $p_k(\mathscr{M}\rtimes_{\sigma^\varphi}\mathbb{R})p_k$ is finite since $p_k\tau_{\rtimes}p_k$ is a faithful normal tracial state. We can hence apply Theorem 7.4.6 to deduce that $(p_k(\mathscr{M}\rtimes_{\sigma^\varphi}\mathbb{R})p_k, p_k\mathscr{N}p_k, p_k\tau_{\rtimes,1}p_k)$ has (rHAP) for every $k\in\mathbb{N}$. In combination with Theorem 7.3.15 this implies that the triple $(\mathscr{M},\mathscr{N},\varphi)$ has property (rHAP). □

We finish this subsection with an easy lemma which will be needed later on. It could be formulated in a greater generality, but this is the form we will use in Section 7.7.

Lemma 7.4.8. *Let $\mathscr{N}\subseteq\mathscr{M}_1\subset\mathscr{M}$ be a unital inclusion of von Neumann algebras with $\mathscr{N}$ finite. Assume that we have faithful normal conditional expectations $\mathbb{E}_1:\mathscr{M}_1\to\mathscr{N}$ and $\mathbb{F}_1:\mathscr{M}\to\mathscr{M}_1$ and a faithful tracial state $\tau\in N_*$. Set $\varphi=\tau\circ\mathbb{E}_1\circ\mathbb{F}_1$. Then if the triple $(\mathscr{M},\mathscr{N},\varphi)$ has property (rHAP) then the triple $(\mathscr{M}_1,\mathscr{N},\varphi|_{\mathscr{M}_1})$ also has property (rHAP).*

Proof. Suppose that $(\Phi_i)_{i\in I}$ is a net of approximations (unital, φ-preserving maps on $\mathscr{M}$) satisfying the conditions in the property (rHAP) for the triple $(\mathscr{M},\mathscr{N},\varphi)$. For each $i\in I$ define $\Psi_i:=\mathbb{F}_1\circ\Phi_i|_{\mathscr{M}_1}$. Our conditions guarantee that $\mathbb{F}_1$ is φ-preserving, so Ψ_i is a normal, ucp, $\mathscr{N}$-bimodular, $\varphi|_{\mathscr{M}_1}$ preserving map on $\mathscr{M}_1$. Due to the last theorem, we need only to check that $(\Psi_i)_{i\in I}$ satisfy the conditions in the property (rHAP)$^-$ (for the triple $(\mathscr{M}_1,\mathscr{N},\varphi|_{\mathscr{M}_1})$). Condition (iii) holds as for $x\in\mathscr{M}_1$ we have $\Psi_i(x)-x=\mathbb{F}_1(\Phi_i(x)-x)$ and

$\mathbb{F}_1^{(2)}$ is the orthogonal projection from $L^2(\mathscr{M},\varphi)$ onto $L^2(\mathscr{M}_1,\varphi|_{\mathscr{M}_1})$.

To verify the last condition we assume first that Φ_i is of the form $a(\mathbb{E}_1\circ\mathbb{F}_1)(b\cdot)$ for some $a,b\in\mathscr{M}$. But then for $x\in\mathscr{M}_1$ we have

$$\Psi_i(x)=\mathbb{F}_1(a(\mathbb{E}_1\circ\mathbb{F}_1)(bx))=\mathbb{F}_1(a)(\mathbb{E}_1\circ\mathbb{F}_1)(bx)))=\mathbb{F}_1(a)\mathbb{E}_1(\mathbb{F}_1(b)x),$$

so we get that $\Psi_i^{(2)}\in\mathscr{K}_{00}(\mathscr{M}_1,\mathscr{N},\varphi|_{\mathscr{M}_1})$. Taking linear combinations and approximation ends the proof. □

7.5. FIRST EXAMPLES

In this section we first put our definitions and main results in concrete context, discussing examples of the Haagerup (and non-Haagerup) inclusions arising in the framework of Cartan subalgebras, as studied in [Jol02], [Ued06] and [Ana13], and then present the case of the big algebra being just $\mathscr{B}(\mathscr{H})$. The examples related to the latter situation show that the relative Haagerup property is not implied by coamenability as defined in [Pop86].

7.5.1. EXAMPLES FROM EQUIVALENCE RELATIONS AND GROUPOIDS

In this subsection we will discuss examples of inclusions of von Neumann algebras which satisfy the relative Haagerup property and have already appeared in the literature. As mentioned in the introduction, the notion of the Haagerup property regarding the von Neumann inclusions beyond the finite context first appeared in the study of von Neumann algebras associated with groupoids/equivalence relations.

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The first result here is due to [Jol02], still in the finite context. Note that Jolissaint uses the definition of the Haagerup inclusion $\mathscr{N}\subset\mathscr{M}$ due to Popa in [Pop06], namely the one using the larger ideal of 'generalised compacts' than the one employed in this paper, but also note that due to [Pop06, Proposition 2.2] both notions coincide if $\mathscr{N}'\cap\mathscr{M}\subset\mathscr{N}$, so for example if $\mathscr{N}$ is a maximal abelian subalgebra in $\mathscr{M}$, which is the case of interest for the result below.

Theorem 7.5.1. *[Jol02, Theorem 2.1] Let $\mathscr{R}$ be a measure preserving standard equivalence relation on a set X (with the measure ν on $\mathscr{R}$ induced by the invariant probability measure μ on X). Then the following are equivalent:*

(i) $\mathscr{R}$ has the Haagerup property, i.e. it admits a sequence of positive-definite functions $(\varphi_n:\mathscr{R}\to\mathbb{C})_{n\in\mathbb{N}}$ which are bounded by 1 on the diagonal, converge to 1 ν-almost everywhere and satisfy the 'vanishing property': for every $n\in\mathbb{N}$ and $\varepsilon>0$ there is

$$\nu(\{(x,y)\in\mathscr{R}:|\varphi_n(x,y)|>\varepsilon\})<\infty;$$

(ii) the von Neumann inclusion (of finite von Neumann algebras)

$$L^\infty(X,\mu)\subset\mathscr{L}(\mathscr{R})$$

has the relative Haagerup property.

The definition beyond the finite case has first been considered in [Ued06]; a more detailed study has been conducted by Anantharaman-Delaroche in [Ana13]. Note that both these papers use the notion of the relative Haagerup property for arbitrary (expected) von Neumann inclusions identical to the one studied here. We will now describe the setup.

Let $\mathscr{G}$ be a measured groupoid with countable fibers, equipped with a quasi-invariant probability measure μ on the unit space $\mathscr{G}^{(0)}$ (note that a measure preserving standard equivalence relation as considered above is one source of such examples). Again μ induces a measure ν on $\mathscr{G}$; we further obtain a (not necessarily finite) von Neumann algebra $\mathscr{L}(\mathscr{G}) \subset \mathscr{B}(L^2(\mathscr{G},\nu))$. The following result holds.

Theorem 7.5.2. *[Ana13, Theorem 1] Let $\mathscr{G}$ be a measured groupoid with countable fibers, as above. Then the following conditions are equivalent:*

(i) $\mathscr{G}$ has the Haagerup property, i.e. it admits a sequence of positive-definite functions $(F_n : \mathscr{G} \to \mathbb{C})_{n\in\mathbb{N}}$ which are equal to 1 on $\mathscr{G}^{(0)}$, converge to 1 ν-almost everywhere and satisfy the 'vanishing property': for every $n \in \mathbb{N}$ and $\varepsilon > 0$ there is

$$\nu(\{g \in \mathscr{G} : |\varphi_n(g)| > \varepsilon\}) < \infty;$$

(ii) the von Neumann inclusion

$$L^\infty(\mathscr{G}^{(0)},\mu) \subset \mathscr{L}(\mathscr{G})$$

has the relative Haagerup property.

Ueda shows in [Ued06, Lemma 5] (and then Anantharaman-Delaroche reproves it in [Ana13, Theorem 3]) that a property of a groupoid as above called *treeability* implies the Haagerup property. [Ana13, Theorem 5] also shows that for ergodic measured groupoid with countable fibers the Haagerup property is incompatible with Property (T); we are however not aware of explicit examples of such Property (T) groupoids leading to von Neumann algebras which are not finite, and a general intuition regarding Property (T) objects says that these should naturally lead to finite von Neumann algebras (for example discrete property (T) quantum groups are necessarily unimodular, see [Fim10]).

7.5.2. EXAMPLES AND COUNTEREXAMPLES WITH $\mathscr{M} = \mathscr{B}(\mathscr{H})$

We end this section by studying which triples $(\mathscr{B}(\mathscr{H}),\mathscr{N},\mathbb{E}_\mathscr{N})$ have (rHAP). Since the conditional expectation $\mathbb{E}_\mathscr{N}$ is assumed to be normal it follows by a result of Tomiyama from [Tom59] that $\mathscr{N}$ must be a direct sum of type I factors, so $\mathscr{N} \simeq \oplus_{i\in I}\mathscr{B}(\mathscr{K}_i)$ for some index set I. Note that each $\mathscr{B}(\mathscr{K}_i)$ may occur in $\mathscr{B}(\mathscr{H})$ with a certain multiplicity $m_i \in \mathbb{N}\cup\{\infty\}$. In general, we have that $\mathscr{N}$ is spatially isomorphic to $\oplus_{i\in I}\mathscr{B}(\mathscr{K}_i) \otimes \mathbb{C}1_{m_i}$ where 1_{m_i} is the identity acting on a Hilbert space of dimension m_i. For simplicity in the examples below we assume that all multiplicities m_i equal 1 and ignore the spatial isomorphism. In that case the normal conditional expectation of $\mathscr{B}(\mathscr{H})$ onto $\oplus_{i\in I}\mathscr{B}(\mathscr{K}_i)$ is unique and determined by $\mathbb{E}_\mathscr{N}(x) = \sum_{i\in I} p_i x p_i$ where p_i is the projection onto $\mathscr{K}_i$. Therefore, in this case we can speak not only of the Haagerup property of the inclusion $\mathscr{N} \subseteq \mathscr{B}(\mathscr{H})$, but also about maps being compact and of finite index relative to this inclusion.

Theorem 7.5.3. *Assume that $\mathcal{H}$ is a separable Hilbert space, that $\mathcal{H} = \bigoplus_{i\in I} \mathcal{K}_i$, where I is an index set and that the dimension of $\mathcal{K}_i$ does not depend on $i \in I$. Put $\mathcal{N} = \bigoplus_{i\in I} \mathcal{B}(\mathcal{K}_i) \subset \mathcal{B}(\mathcal{H})$. Then the triple $(\mathcal{B}(\mathcal{H}), \mathcal{N}, \mathbb{E}_{\mathcal{N}})$ has the property (rHAP).*

Proof. We may assume that $\mathcal{K}_i = \mathcal{K}$ for a single (separable) Hilbert space $\mathcal{K}$. The inclusion $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$ is then isomorphic to the inclusion $\ell^\infty(I) \otimes \mathcal{B}(\mathcal{K}) \subseteq \mathcal{B}(\ell^2(I)) \otimes \mathcal{B}(\mathcal{K})$. In the case where I is finite $\ell^\infty(I) \subseteq \mathcal{B}(\ell^2(I))$ is a finite dimensional inclusion which clearly has (rHAP). In the case where I is infinite we may assume that $I = \mathbb{Z}$ and the inclusion $\ell^\infty(\mathbb{Z}) \subseteq \mathcal{B}(\ell^2(\mathbb{Z}))$ has the (rHAP) with approximating maps given by the (Fejér-)Herz-Schur multipliers T_n with

$$T_n((x_{i,j})_{i,j\in\mathbb{Z}}) = (W(i-j)x_{i,j})_{i,j\in\mathbb{Z}}, \qquad W(k) := \max(1 - \frac{|k|}{n}, 0).$$

Since $W = \frac{1}{n}(\chi_{[0,n]})^* * \chi_{[0,n]}$ is positive definite and converges to the identity pointwise it follows that T_n is completely positive and $T_n^{(2)}$ converges to the identity strongly. Further $T_n^{(2)}$ is finite rank relative to $\ell^\infty(\mathbb{Z})$, so certainly compact. In both cases (I being finite or infinite), we tensor the approximating maps with $\mathrm{Id}_{\mathcal{B}(\mathcal{K})}$ and find that $\ell^\infty(I) \otimes \mathcal{B}(\mathcal{K}) \subseteq \mathcal{B}(\ell^2(I)) \otimes \mathcal{B}(\mathcal{K})$ has (rHAP). □

With a bit more work Theorem 7.5.3 could be proved in larger generality by relaxing the assumption that the multiplicities are trivial and that the dimension is constant (as opposed to say for example uniformly bounded). However, we cannot admit just any subalgebra $\mathcal{N}$ as the following counterexample shows.

Theorem 7.5.4. *Let $\mathcal{H} = \mathcal{K}_1 \oplus \mathcal{K}_2$, where $\mathcal{K}_1, \mathcal{K}_2$ are Hilbert spaces such that* $\dim(\mathcal{K}_1) < \infty$ *and* $\dim(\mathcal{K}_2) = \infty$. *Set $\mathcal{N} = \mathcal{B}(\mathcal{K}_1) \oplus \mathcal{B}(\mathcal{K}_2)$. Then the triple $(\mathcal{B}(\mathcal{H}), \mathcal{N}, \mathbb{E}_{\mathcal{N}})$ does not have the property (rHAP).*

Proof. Let p be the projection of $\mathcal{H}$ onto $\mathcal{K}_1$. Let $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a normal linear map. The proof is based on two claims.

Claim 1: If Φ is an $\mathcal{N}$-$\mathcal{N}$ bimodule map then $\mathcal{B}(\mathcal{H})p$ is an invariant subspace. Moreover, the restriction of Φ to $\mathcal{B}(\mathcal{H})p$ lies in the linear span of the two maps $xp \mapsto pxp$ and $xp \mapsto (1-p)xp$.

Proof of Claim 1. Note that p is contained in $\mathcal{N}$ from which the first statement follows. For the second part let $E^i_{k,l}$ be matrix units with respect to some basis of $\mathcal{K}_i$. Then for $x \in \mathcal{B}(\mathcal{H})$ we have $\Phi(E^i_{k,k} x E^i_{l,l}) = E^i_{k,k}\Phi(x)E^i_{k,k}$ so that $E^i_{k,k}\mathcal{B}(\mathcal{H})E^i_{k,k}$ is an eigenspace of Φ (i.e. Φ is a Herz-Schur multiplier). Moreover $\Phi(E^i_{k',k'} x E^i_{l',l'}) = E^i_{k',k}\Phi(E^i_{k,k'} x E^i_{l',l})E^i_{l,l'}$ so that the eigenvalues of these spaces only depend on i. This in particular implies the claim.

Claim 2: If Φ is compact relative to the inclusion $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$ then $\mathcal{B}(\mathcal{H})p$ is an invariant subspace. Moreover, the restriction of Φ to $\mathcal{B}(\mathcal{H})p$ is compact (in the non-relative sense).

Proof of Claim 2. By approximation it suffices to prove Claim 2 with 'compact' replaced by 'finite rank'. So assume that $\Phi = a\mathbb{E}_{\mathcal{N}} b$ with $a, b \in \mathcal{B}(\mathcal{H})$. Note that $p \in \mathcal{N} \cap \mathcal{N}'$ and therefore $a\mathbb{E}_{\mathcal{N}}(bxp) = ap\mathbb{E}_{\mathcal{N}}(bx)p = a\mathbb{E}_{\mathcal{N}}(pbxp)$. The first of these equalities shows that $\mathcal{B}(\mathcal{H})p$ is invariant. Further $x \mapsto (pxp)$ is finite rank as p projects onto a finite dimensional space. This proves the claim.

Remainder of the proof. Suppose that Φ is both $\mathcal{N}$-$\mathcal{N}$ bimodular and compact relative to $\mathcal{N}$. By Claim 1 we know that there are scalars $\lambda_1, \lambda_2 \in \mathbb{C}$ such that $\Phi(xp) = \lambda_1 pxp + \lambda_2(1-p)xp$. If $\lambda_2 \neq 0$ then the associated L^2-map is not compact (in the non-relative sense) since $(1-p)$ projects onto an infinite dimensional Hilbert space. This contradicts Claim 2 because the restriction of Φ to $\mathcal{B}(\mathcal{H})p$ is compact. We conclude that $\lambda_2 = 0$ for any normal map $\Phi : \mathcal{M} \to \mathcal{M}$ that is $\mathcal{N}$-$\mathcal{N}$-bimodular and compact relative to $\mathcal{N}$. But then we can never find a net of such maps that approximates the identity map on $\mathcal{B}(\mathcal{H})$ in the point-strong topology. Hence the inclusion $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$ fails to have (rHAP). □

Remark 7.5.5. Recall that a unital inclusion of von Neumann algebras $\mathcal{N} \subseteq \mathcal{M}$ is said to be *co-amenable* if there exists a (not necessarily normal) conditional expectation from $\mathcal{N}'$ onto $\mathcal{M}'$, where the commutants are taken with respect to any Hilbert space realization of $\mathcal{M}$. Theorem 7.5.4 shows – surprisingly – that a co-amenable inclusion in general need not have (rHAP).

Note that this also means that a naive extension of the definition of relative Haagerup property in terms of correspondences, modelled on the notion of *strictly mixing bimodules* [OOT17, Theorem 9] valid for the non-relative Haagerup property, cannot be equivalent to the definition studied in our paper. Indeed, the last fact, together with the examples above, would contradict [BMO20, Theorem 2.4].

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7.6. PROPERTY (RHAP) FOR FINITE-DIMENSIONAL SUBALGEBRAS

In this section we consider the case of finite-dimensional subalgebras and show equivalence of the relative Haagerup property and the non-relative Haagerup property. For this, we fix a unital inclusion $\mathcal{N} \subseteq \mathcal{M}$ of von Neumann algebras and assume that it admits a faithful normal conditional expectation $\mathbb{E}_{\mathcal{N}}$. Assume that $\mathcal{N}$ is finite-dimensional and let $\tau \in \mathcal{N}_*$ be a faithful normal tracial state on $\mathcal{N}$ that we extend to a state $\varphi := \tau \circ \mathbb{E}_{\mathcal{N}}$ on $\mathcal{M}$. We will prove that the triple $(\mathcal{M}, \mathcal{N}, \varphi)$ has property (rHAP) if and only if $(\mathcal{M}, \mathbb{C}, \varphi)$ does. Recall that by Theorem 7.2.7 the Haagerup property of $(\mathcal{M}, \mathcal{N}, \varphi)$ does not depend on the choice of the state τ.

Denote by $z_1, ..., z_n \in \mathcal{Z}(\mathcal{N})$ the minimal central projections of $\mathcal{N}$. There exist natural numbers $n_1, ..., n_k \in \mathbb{N}$ such that $z_k\mathcal{N} \cong M_{n_k}(\mathbb{C})$ for $k = 1, ..., n$. Let $(f_i^k)_{1 \le i \le n_k}$ be an orthonormal basis of $\mathbb{C}^{n_k}$, write $E_{i,j}^k$, $1 \le i, j \le n_k$ for the matrix units with respect to this basis and set $E_i^k := E_{i,i}^k$ for the diagonal projections. We have that $E_{i,j}^k f_l^k = \delta_{j,l} f_i^k$ for all $k \in \mathbb{N}$, $1 \le i, j, l \le n_k$ and $\sum_{k=1}^n \sum_{i=1}^{n_k} E_i^k = 1$. Set $d := \sum_{k=1}^n n_k$, choose an orthonormal basis $(f_{k,i})_{1 \le k \le n,\, 1 \le i \le n_k}$ of $\mathbb{C}^d$ with corresponding matrix units $e_{(k,i),(l,j)} \in M_d(\mathbb{C})$ where

$1 \leq k, l \leq n$, $1 \leq i \leq n_k$, $1 \leq j \leq n_l$ and define

$$p := \sum_{k=1}^{n} E_1^k. \tag{7.6.1}$$

For a general linear map $\Phi \colon p\mathcal{M}p \to p\mathcal{M}p$ we may define a linear map $\widetilde{\Phi} \colon \mathcal{M} \to \mathcal{M}$ by

$$\widetilde{\Phi}(E_i^k x E_j^l) := E_{i,1}^k \Phi(E_{1,i}^k x E_{j,1}^l) E_{1,j}^l \tag{7.6.2}$$

for all $1 \leq k, l \leq n$, $1 \leq i \leq n_k$, $1 \leq j \leq n_l$ and $x \in \mathcal{M}$. Let us study the properties of $\widetilde{\Phi}$.

Lemma 7.6.1. *Let $\Phi : p\mathcal{M}p \to p\mathcal{M}p$ be a linear map. Define*

$$U := \sum_{k=1}^{n} \sum_{i=1}^{n_k} e_{(1,1),(k,i)} \otimes E_{i,1}^k \in M_d(\mathbb{C}) \otimes \mathcal{M}, \qquad V := \sum_{k=1}^{n} \sum_{i=1}^{n_k} f_{k,i} \otimes E_{1,i}^k \in \mathbb{C}^d \otimes \mathcal{M}.$$

Then,

$$\widetilde{\Phi}(x) = V^* (id_{\mathcal{B}(L^2(\mathcal{N},\tau))} \otimes \Phi) \big(U^* (1 \otimes x) U\big) V.$$

Proof. We have for $x \in z_k M z_l$ with $1 \leq k, l \leq n$ that

$$U^* (1 \otimes x) U = \sum_{i=1}^{n_k} \sum_{j=1}^{n_l} e_{(k,i),(l,j)} \otimes E_{1,i}^k x E_{j,1}^l$$

so that

$$V^* (\mathrm{id}_{\mathcal{B}(L^2(\mathcal{N},\tau))} \otimes \Phi) \big(U^* (1 \otimes x) U\big) V = \sum_{i=1}^{n_k} \sum_{j=1}^{n_l} E_{i,1}^k \Phi(E_{1,i}^k x E_{j,1}^l) E_{1,j}^l.$$

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By definition this expression coincides with $\widetilde{\Phi}(x)$. The claim follows. □

Lemma 7.6.2. *If $\Phi \colon p\mathcal{M}p \to p\mathcal{M}p$ is a unital normal completely positive map, then $\widetilde{\Phi}$ is contractive, normal and completely positive.*

Proof. The normality and the complete positivity follow from Lemma 7.6.1. We further have

$$\begin{aligned} \|\widetilde{\Phi}\| &= \widetilde{\Phi}(1) = \widetilde{\Phi}\left(\sum_{k=1}^{n} \sum_{i=1}^{n_k} E_i^k\right) = \sum_{k=1}^{n} \sum_{i=1}^{n_k} E_{i,1}^k \Phi(E_1^k) E_{1,i}^k \\ &\leq \sum_{k=1}^{n} \sum_{i=1}^{n_k} E_{i,1}^k E_{1,i}^k = \sum_{k=1}^{n} \sum_{i=1}^{n_k} E_i^k = 1, \end{aligned}$$

i.e. $\widetilde{\Phi}$ is contractive. □

Lemma 7.6.3. *Let $\Phi \colon p\mathcal{M}p \to p\mathcal{M}p$ be a linear map. Then $\widetilde{\Phi}$ is an $\mathcal{N}$-$\mathcal{N}$-bimodule map.*

Proof. Let $x \in \mathcal{M}$. For $1 \leq l, k, m \leq n$ and $1 \leq r, s \leq n_l$, $1 \leq i \leq n_k$, $1 \leq j \leq n_m$ we have

$$\begin{aligned} E_{r,s}^l \widetilde{\Phi}(E_i^k x E_j^m) &= E_{r,s}^l E_{i,1}^k \Phi(E_{1,i}^k x E_{j,1}^m) E_{1,j}^m \\ &= \delta_{s,i} \delta_{l,k} E_{r,1}^k \Phi(E_{1,i}^k x E_{j,1}^m) E_{1,j}^m \\ &= E_{r,1}^k \Phi(E_{1,r}^k E_{r,s}^l E_i^k x E_j^m E_{j,1}^m) E_{1,j}^m. \end{aligned}$$

We hence find that for $y \in E_i^k x E_j^m$, $E_{r,s}^l \widetilde{\Phi}(y) = \widetilde{\Phi}(E_{r,s}^l y)$. The linearity of $\widetilde{\Phi}$ then implies that it is a left $\mathcal{N}$-module map. A similar argument applies to the right-handed case. □

Proposition 7.6.4. *Define the map*

$$\mathrm{Diag}\colon p\mathcal{M}p \to p\mathcal{M}p,\ x \mapsto \sum_{k=1}^{n} \frac{\varphi(E_1^k x E_1^k)}{\varphi(E_1^k)} E_1^k.$$

Then $\widetilde{\mathrm{Diag}} = \mathbb{E}_{\mathcal{N}}$.

Proof. It is clear that the map Diag is linear, unital, normal and completely positive. Hence, by Lemma 7.6.2 and Lemma 7.6.3, $\widetilde{\mathrm{Diag}}$ is contractive normal completely positive and $\mathcal{N}$-$\mathcal{N}$-bimodular. It is easy to check that $\widetilde{\mathrm{Diag}}$ is even unital. In particular, $\widetilde{\mathrm{Diag}}$ restricts to the identity on $\mathcal{N}$. It is further clear that $\widetilde{\mathrm{Diag}}$ is faithful and that it maps $\mathcal{M}$ onto $\mathcal{N}$, so $\widetilde{\mathrm{Diag}}$ is a faithful normal conditional expectation. For $x \in \mathcal{M}$ and $1 \le k, l \le n$, $1 \le i \le n_k$, $1 \le j \le n_l$ we have

$$\begin{aligned}
\varphi \circ \widetilde{\mathrm{Diag}}(E_i^k x E_j^l) &= \varphi\left(E_{i,1}^k \mathrm{Diag}(E_{1,i}^k x E_{j,1}^l) E_{1,j}^l\right) \\
&= \sum_{m=1}^{n} \frac{\varphi(E_1^m E_{1,i}^k x E_{j,1}^l E_1^m)}{\varphi(E_1^m)} \varphi\left(E_{i,1}^k E_1^m E_{1,j}^l\right) \\
&= \frac{\varphi(E_1^l E_{1,i}^k x E_{j,1}^l E_1^l)}{\varphi(E_1^l)} \varphi(E_{i,1}^k E_1^l E_{1,j}^l) \\
&= \delta_{k,l} \frac{\varphi(E_{1,i}^l x E_{j,1}^l)}{\varphi(E_1^l)} \varphi(E_{i,1}^l E_{1,j}^l) \\
&= \delta_{k,l} \frac{\varphi(E_{1,i}^l x E_{j,1}^l)}{\tau(E_1^l)} \tau(E_{i,j}^l).
\end{aligned}$$

But then, since τ is tracial,

$$\begin{aligned}
\varphi \circ \widetilde{\mathrm{Diag}}(E_i^k x E_j^l) &= \delta_{i,j}\delta_{k,l}\varphi(E_{1,i}^l x E_{i,1}^l) = \delta_{i,j}\delta_{k,l}\tau(\mathbb{E}_{\mathcal{N}}(E_{1,i}^l x E_{i,1}^l)) \\
&= \delta_{i,j}\delta_{k,l}\tau(E_{1,i}^l \mathbb{E}_{\mathcal{N}}(x) E_{i,1}^l) = \tau(E_i^k \mathbb{E}_{\mathcal{N}}(x) E_j^l) \\
&= \varphi(E_i^k x E_j^l),
\end{aligned}$$

i.e. $\widetilde{\mathrm{Diag}}$ is φ-preserving. Since $\mathbb{E}_{\mathcal{N}}$ is the unique faithful normal φ-preserving conditional expectation onto $\mathcal{N}$, we get that $\widetilde{\mathrm{Diag}} = \mathbb{E}_{\mathcal{N}}$. □

Lemma 7.6.5. *Let $\Phi\colon p\mathcal{M}p \to p\mathcal{M}p$ be a normal completely positive map with $\varphi \circ \Phi \le \varphi$ and assume that the L^2-implementation $\Phi^{(2)}$ of Φ with respect to $\varphi|_{p\mathcal{M}p}$ is a compact operator. Then $\widetilde{\Phi}$ satisfies $\varphi \circ \widetilde{\Phi} \le \varphi$ and $(\widetilde{\Phi})^{(2)} \in \mathcal{K}(\mathcal{M}, \mathcal{N}, \varphi)$.*

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Proof. For $1 \le k,l \le n$, $1 \le i \le n_k$, $1 \le j \le n_l$ and $x \in \mathscr{M}$ positive we have by the traciality of τ,

$$\begin{aligned}
\varphi \circ \widetilde{\Phi}(E_i^k x E_j^l) &= \varphi\left(E_{i,1}^k \Phi(E_{1,i}^k x E_{j,1}^l) E_{1,j}^l\right) \\
&= \tau\left(E_{i,1}^k \mathbb{E}_{\mathscr{N}}\left(\Phi(E_{1,i}^k x E_{j,1}^l)\right) E_{1,j}^l\right) \\
&= \delta_{i,j}\delta_{k,l}\tau\left(E_1^k \mathbb{E}_{\mathscr{N}}(\Phi(E_{1,i}^k x E_{i,1}^k))\right),
\end{aligned}$$

so in particular $\varphi \circ \widetilde{\Phi}(E_i^k x E_j^l) \ge 0$. We get (as $\mathscr{N}$ is contained in the centralizer $\mathscr{M}^\varphi$)

$$\begin{aligned}
\varphi \circ \widetilde{\Phi}(E_i^k x E_j^l) &= \delta_{i,j}\delta_{k,l}\tau\left(E_1^k \mathbb{E}_{\mathscr{N}}(\Phi(E_{1,i}^k x E_{i,1}^k))\right) \\
&\le \delta_{i,j}\delta_{k,l}\tau\left(\mathbb{E}_{\mathscr{N}}(\Phi(E_{1,i}^k x E_{i,1}^k))\right) \\
&\le \delta_{i,j}\delta_{k,l}\varphi(E_{1,i}^k x E_{i,1}^k) \\
&= \varphi(E_i^k x E_j^l).
\end{aligned}$$

This implies that $\widetilde{\Phi}$ indeed satisfies $\varphi \circ \widetilde{\Phi} \le \varphi$. In particular, the L^2-implementation of $\widetilde{\Phi}$ with respect to φ exists.

It remains to show that $(\widetilde{\Phi})^{(2)} \in \mathscr{K}(\mathscr{M},\mathscr{N},\varphi)$. For this, let $\Psi\colon p\mathscr{M}p \to p\mathscr{M}p$ be a map with $\Psi^{(2)} = ae_{\mathbb{C}}b$ where $a,b \in p\mathscr{M}p$ and $e_{\mathbb{C}}$ denotes the rank one projection $(\varphi|_{p\mathscr{M}p}(\cdot)p)^{(2)} \in \mathscr{B}(L^2(p\mathscr{M}p, \varphi|_{p\mathscr{M}p}))$. For $1 \le k,l \le n$, $1 \le i \le n_k$, $1 \le j \le n_l$ and $x \in \mathscr{M}$ we then have

$$\widetilde{\Psi}(E_i^k x E_j^l) = E_{i,1}^k a\varphi(bE_{1,i}^k x E_{j,1}^l)E_{1,j}^l = E_{i,1}^k a\varphi(bE_{1,i}^k x E_{j,1}^l)E_1^l E_{1,j}^l.$$

Note that by Proposition 7.6.4,

$$\begin{aligned}
\mathbb{E}_{\mathscr{N}}(bE_{1,i}^k x E_{j,1}^l) &= \sum_{r=1}^n \mathbb{E}_{\mathscr{N}}(E_1^r bE_{1,i}^k x E_{j,1}^l) = \sum_{r=1}^n \widetilde{\mathrm{Diag}}(E_1^r bE_{1,i}^k x E_{j,1}^l) \\
&= \sum_{r=1}^n E_1^r \mathrm{Diag}(E_1^r bE_{1,i}^k x E_{j,1}^l)E_1^l \\
&= \sum_{r=1}^n \sum_{m=1}^n \frac{\varphi(E_1^m E_1^r bE_{1,i}^k x E_{j,1}^l E_1^m)}{\varphi(E_1^m)} E_1^r E_1^m E_1^l \\
&= \frac{\varphi(E_1^l bE_{1,i}^k x E_{j,1}^l)}{\varphi(E_1^l)} E_1^l = \frac{\varphi(bE_{1,i}^k x E_{j,1}^l)}{\varphi(E_1^l)} E_1^l,
\end{aligned}$$

where in the last equality we again used that τ is tracial. Hence

$$\widetilde{\Psi}(E_i^k x E_j^l) = \varphi(E_1^l)E_{i,1}^k a\mathbb{E}_{\mathscr{N}}(bE_{1,i}^k x E_{j,1}^l)E_{1,j}^l = \varphi(E_1^l)E_{i,1}^k a\mathbb{E}_{\mathscr{N}}(bE_{1,i}^k E_i^k x E_j^l). \tag{7.6.3}$$

Fix now suitable t_0, j_0, k_0, l_0 and $x \in \mathcal{M}$ and compute the following expression:

$$\begin{aligned}\left(\sum_{k=1}^{n}\sum_{l=1}^{n}\sum_{r=1}^{n}\sum_{t=1}^{n_k} \varphi(E_1^r)E_{t,1}^k a E_1^l e_{\mathcal{N}} E_1^r b E_{1,t}^k\right)(E_{t_0}^{k_0} x E_{j_0}^{l_0}\Omega_\varphi) \\ = \sum_{l=1}^{n}\sum_{r=1}^{n} \varphi(E_1^r)E_{t_0,1}^{k_0} a E_1^l \mathbb{E}_{\mathcal{N}}(E_1^r b E_{1,t_0}^{k_0} x E_{j_0}^{l_0})\Omega_\varphi \\ = \sum_{l=1}^{n} \varphi(E_1^l)E_{t_0,1}^{k_0} a E_1^l \mathbb{E}_{\mathcal{N}}(b E_{1,t_0}^{k_0} x E_{j_0,1}^{l_0}) E_{1,j_0}^{l_0}\Omega_\varphi\end{aligned}$$

Now the equality (7.6.3) implies that the value of the conditional expectation appearing in the last formula is a scalar multiple of $E_1^{l_0}$, so the whole expression equals

$$\varphi(E_1^{l_0})E_{t_0,1}^{k_0} a \mathbb{E}_{\mathcal{N}}(b E_{1,t_0}^{k_0} x E_{j_0}^{l_0})\Omega_\varphi = \widetilde{\Psi}(E_{t_0}^{k_0} x E_{j_0}^{l_0})\Omega_\varphi.$$

Hence we arrive at

$$(\widetilde{\Psi})^{(2)} = \sum_{k=1}^{n}\sum_{l=1}^{n}\sum_{r=1}^{n}\sum_{t=1}^{n_k} \varphi(E_1^r)E_{t,1}^k a E_1^l e_{\mathcal{N}} E_1^r b E_{1,t}^k \in \mathcal{K}_{00}(\mathcal{M},\mathcal{N},\varphi).$$

Thus, by taking linear combinations, for every map Ψ with $\Psi^{(2)} \in \mathcal{K}_{00}(p\mathcal{M}p,\mathbb{C},\varphi)$ the L^2-implementation of $\widetilde{\Psi}$ is contained in $\mathcal{K}_{00}(\mathcal{M},\mathcal{N},\varphi)$. Via approximation we then see that $(\widetilde{\Phi})^{(2)} \in \mathcal{K}(\mathcal{M},\mathcal{N},\varphi)$. □

We are now ready to prove the main theorem of this section.

Theorem 7.6.6. *Let $\mathcal{N} \subseteq \mathcal{M}$ be a unital inclusion of von Neumann algebras and assume that it admits a faithful normal conditional expectation $\mathbb{E}_{\mathcal{N}} : \mathcal{M} \to \mathcal{N}$. Assume that $\mathcal{N}$ is finite-dimensional and let $\tau \in \mathcal{N}_*$ be a faithful state on $\mathcal{N}$ that we extend to a state $\varphi := \tau \circ \mathbb{E}_{\mathcal{N}}$ on $\mathcal{M}$. Then $\mathcal{M}$ has the Haagerup property (in the sense that the triple $(\mathcal{M},\mathbb{C},\varphi)$ has the relative Haagerup property) if and only if the triple $(\mathcal{M},\mathcal{N},\varphi)$ has the relative Haagerup property.*

Proof. By Theorem 7.2.7 we may assume without loss of generality that τ is tracial.

"⇐" Assume that the triple $(\mathcal{M},\mathcal{N},\varphi)$ has the relative Haagerup property and let $(\Phi_i)_{i\in I}$ be a net of normal completely positive maps witnessing it. Since $\mathcal{N}$ is finite dimensional, $e_{\mathcal{N}}$ is a finite rank projection. In particular, $\mathcal{K}_{00}(\mathcal{M},\mathcal{N},\varphi)$ consists of finite rank operators and hence $\mathcal{K}(\mathcal{M},\mathcal{N},\varphi) \subseteq \mathcal{K}(\mathcal{M},\mathbb{C},\varphi)$. In particular, $\Phi_i^{(2)} \in \mathcal{K}(\mathcal{M},\mathbb{C},\varphi)$ for every $i \in I$. Further, $\Phi_i(x) \to x$ strongly for every $x \in \mathcal{M}$. This implies that the net $(\Phi_i)_{i\in I}$ also witnesses the relative Haagerup property of the triple $(\mathcal{M},\mathbb{C},\varphi)$.

"⇒" Assume that $\mathcal{M}$ has the Haagerup property. Recall that the projection p was defined in (7.6.1). By [CS15c, Lemma 4.1] the triple $(p\mathcal{M}p,\mathbb{C},\varphi|_{p\mathcal{M}p})$ also has the relative Haagerup property and by Theorem 7.4.5 we find a net $(\Phi_i)_{i\in I}$ of unital normal completely positive φ-preserving maps witnessing it. By Lemma 7.6.2, Lemma 7.6.3 and Lemma 7.6.5 we find that $\widetilde{\Phi}_i$ is a contractive normal completely positive $\mathcal{N}$-$\mathcal{N}$-bimodule map with $\varphi \circ \widetilde{\Phi} \leq \varphi$ and $(\widetilde{\Phi})^{(2)} \in \mathcal{K}(\mathcal{M},\mathcal{N},\varphi)$ for every $i \in I$. It follows directly from the prescription (7.6.2) that $\widetilde{\Phi}_i(x) \to x$ strongly for every $x \in \mathcal{M}$. It follows that the net $(\widetilde{\Phi}_i)_{i\in I}$ witnesses the relative Haagerup property of the triple $(\mathcal{M}, \mathcal{N}, \varphi)$. □

7.7. THE RELATIVE HAAGERUP PROPERTY FOR AMALGAMATED FREE PRODUCTS

In this section we study the notion of the relative Haagerup property in the context of amalgamated free products of von Neumann algebras. We will further apply our results to the class of virtually free Hecke-von Neumann algebras.

7.7.1. PRESERVATION UNDER AMALGAMATED FREE PRODUCTS

The following theorem demonstrates that in the setting of Section 7.4, property (rHAP) is preserved under taking amalgamated free products (for details on operator algebraic amalgamated free products see [Voi85] or also [BO08b]). For finite inclusions of von Neumann algebras this has been proved in [Boc93, Proposition 3.9].

Theorem 7.7.1. *Let $\mathcal{N} \subseteq \mathcal{M}_1$ and $\mathcal{N} \subseteq \mathcal{M}_2$ be unital embeddings of von Neumann algebras which admit faithful normal conditional expectations $\mathbb{E}_1 : \mathcal{M}_1 \to \mathcal{N}$, $\mathbb{E}_2 : \mathcal{M}_2 \to \mathcal{N}$ and for which $\mathcal{N}$ is finite. Denote by $\mathcal{M} := (\mathcal{M}_1, \mathbb{E}_1) *_{\mathcal{N}} (\mathcal{M}_2, \mathbb{E}_2)$ the amalgamated free product von Neumann algebra of $\mathcal{M}_1$ and $\mathcal{M}_2$ with respect to the expectations $\mathbb{E}_1$, $\mathbb{E}_2$ and let $\mathbb{E}_{\mathcal{N}}$ be the corresponding conditional expectation of $\mathcal{M}$ onto $\mathcal{N}$. Then $(\mathcal{M}_1, \mathcal{N}, \mathbb{E}_1)$ and $(\mathcal{M}_2, \mathcal{N}, \mathbb{E}_2)$ have the relative Haagerup property if and only if the triple $(\mathcal{M}, \mathcal{N}, \mathbb{E}_{\mathcal{N}})$ has the relative Haagerup property.*

Proof. "⇒" Assume that both $(\mathcal{M}_1, \mathcal{N}, \mathbb{E}_1)$ and $(\mathcal{M}_2, \mathcal{N}, \mathbb{E}_2)$ have the relative Haagerup property, let $\tau \in \mathcal{N}_*$ be a faithful normal tracial state and set $\varphi_1 := \tau \circ \mathbb{E}_1$, $\varphi_2 := \tau \circ \mathbb{E}_2$. Then the triples $(\mathcal{M}_1, \mathcal{N}, \varphi_1)$ and $(\mathcal{M}_2, \mathcal{N}, \varphi_2)$ have the relative Haagerup property. Without loss of generality we can assume that the corresponding nets $(\Phi_{i,1})_{i\in I}$ and $(\Phi_{i,2})_{i\in I}$ witnessing the relative Haagerup property are indexed by the same set I. By Theorem 7.4.5 we can also assume that the maps are unital with $\varphi_1 \circ \Phi_{i,1} = \varphi_1$, $\varphi_2 \circ \Phi_{i,2} = \varphi_2$ for all $i \in I$, which then implies that $\Phi_{i,1}|_{\mathcal{N}} = \Phi_{i,2}|_{\mathcal{N}} = \mathrm{id}_{\mathcal{N}}$ and that $\mathbb{E}_1 \circ \Phi_{i,1} = \mathbb{E}_1$, $\mathbb{E}_2 \circ \Phi_{i,2} = \mathbb{E}_2$. Choose a net $(\varepsilon_i)_{i\in I}$ (we can use the same indexing set, modifying it if necessary) with $\varepsilon_i \to 0$ and define unital normal completely positive $\mathcal{N}$-$\mathcal{N}$-bimodular maps $\Phi'_{i,1} := \frac{1}{1+\varepsilon_i}(\Phi_{i,1} + \varepsilon_i \mathbb{E}_1)$, $\Phi'_{i,2} := \frac{1}{1+\varepsilon_i}(\Phi_{i,2} + \varepsilon_i \mathbb{E}_2)$.

In the following we will need to work with certain sets of multi-indices: for each $n \in \mathbb{N}$ set $\mathcal{J}_n = \{\mathbf{j} = (j_1, \ldots, j_n) : j_k \in \{1,2\} \text{ and } j_k \neq j_{k+1} \text{ for } k = 1, \ldots, n-1\}$; put also $\mathcal{J} = \bigcup_{n\in\mathbb{N}} \mathcal{J}_n$.

Set $\varphi := \tau \circ \mathbb{E}_{\mathcal{N}}$, let $\Psi_i := \Phi_{i,1} * \Phi_{i,2} \colon \mathcal{M} \to \mathcal{M}$ be the unital normal completely positive map with $\Psi_i|_{\mathcal{N}} = \mathrm{id}_{\mathcal{N}}$ and $\Psi_i(x_1 \ldots x_n) = \Phi'_{i,j_1}(x_1) \ldots \Phi'_{i,j_n}(x_n)$ for $\mathbf{j} \in \mathcal{J}_n$ and $x_k \in M_{j_k} \cap \ker(\mathbb{E}_{j_k})$ for $k = 1, \ldots, n$ (see [BD01, Theorem 3.8]) and define $\Psi'_i := \Phi'_{i,1} * \Phi'_{i,2}$ analogously. We claim that the net $(\Psi'_i)_{i\in I}$ witnesses the relative Haagerup property of the triple $(\mathcal{M}, \mathcal{N}, \varphi)$. Indeed, it is clear that the maps satisfy the conditions (1), (2) and (4) of Definition 7.2.2. It remains to show that $\Psi'_i(x) \to x$ strongly for every $x \in \mathcal{M}$ and that the

L^2-implementations $(\Psi_i')^{(2)}$ are contained in $\mathcal{K}(\mathcal{M},\mathcal{N},\varphi)$.

Define for $n \in \mathbb{N}$ and $\mathbf{j} \in \mathcal{J}_n$ the Hilbert subspace

$$\mathcal{H}_{\mathbf{j}} := \overline{\text{Span}}\left\{x_1 \ldots x_n \Omega_\varphi \mid x_1 \in \ker(\mathbb{E}_{j_1}), \ldots, x_n \in \ker(\mathbb{E}_{j_n})\right\} \subseteq L^2(\mathcal{M},\varphi)$$

and let $P_{\mathbf{j}} \in \mathcal{B}(L^2(\mathcal{M},\varphi))$ be the orthogonal projection onto $\mathcal{H}_{\mathbf{j}}$. Note that these Hilbert subspaces are pairwise orthogonal for different multi-indices $\mathbf{j}_1, \mathbf{j}_2 \in \mathcal{J}$, orthogonal to $\mathcal{N}\Omega_\varphi \subseteq L^2(\mathcal{M},\varphi)$, one has inclusions $\Psi_i^{(2)}\mathcal{H}_{\mathbf{j}} \subseteq \mathcal{H}_{\mathbf{j}}$, $(\Psi_i')^{(2)}\mathcal{H}_{\mathbf{j}} \subseteq \mathcal{H}_{\mathbf{j}}$ and the span of the union of all $\mathcal{H}_{\mathbf{j}}$ ($\mathbf{j} \in \mathcal{J}$) with $\mathcal{N}\Omega_\varphi$ is dense in $L^2(\mathcal{M},\varphi)$.

For the strong convergence it suffices to show that $\|\Psi_i^{(2)}\xi - \xi\|_2 \to 0$ for all $\xi \in \mathcal{H}_{\mathbf{j}}$, $\mathbf{j} \in \mathcal{J}$. So let $n \in \mathbb{N}$, $\mathbf{j} \in \mathcal{J}_n$, $x_1 \in \ker(\mathbb{E}_{j_1})$, ..., $x_n \in \ker(\mathbb{E}_{j_n})$. Then,

$$\begin{aligned}
& \|(\Psi_i')^{(2)}(x_1 \ldots x_n \Omega_\varphi) - x_1 \ldots x_n \Omega_\varphi\|_2 = \|\Phi'_{i,j_1}(x_1) \ldots \Phi'_{i,j_n}(x_n)\Omega_\varphi - x_1 \ldots x_n \Omega_\varphi\|_2 \\
\leq\ & \|(\Phi'_{i,j_1}(x_1) - x_1)\Phi'_{i,j_2}(x_2) \ldots \Phi'_{i,j_n}(x_n)\Omega_\varphi\|_2 \\
& \qquad + \|x_1\| \|\Phi'_{i,j_2}(x_2) \ldots \Phi'_{i,j_n}(x_n)\Omega_\varphi - x_2 \ldots x_n \Omega_\varphi\|_2 \\
\leq\ & \ldots \leq \|(\Phi'_{i,j_1}(x_1) - x_1)\Phi'_{i,j_2}(x_2) \ldots \Phi'_{i,j_n}(x_n)\Omega_\varphi\|_2 \\
& \qquad + \|x_1\| \|(\Phi'_{i,j_2}(x_2) - x_2)\Phi'_{i,j_3}(x_3) \ldots \Phi'_{i,j_n}(x_n)\Omega_\varphi\|_2 \\
& \qquad + \ldots + \|x_1\| \ldots \|x_{n-1}\| \|\Phi'_{i,j_n}(x_n)\Omega_\varphi - x_n \Omega_\varphi\|_2 \to 0.
\end{aligned}$$

This implies that indeed $\Psi_i(x) \to x$ strongly for every $x \in \mathcal{M}$.

To treat the relative compactness, express the operators $(\Phi'_{i,1})^{(2)} \in \mathcal{K}(\mathcal{M}_1,\mathcal{N},\varphi_1)$, $(\Phi'_{i,2})^{(2)} \in \mathcal{K}(\mathcal{M}_2,\mathcal{N},\varphi_2)$ as norm-limits

$$(\Phi'_{i,1})^{(2)} = \lim_{l\to\infty} \sum_{k=1}^{N_l^{(i,1)}} a_{k,l}^{(i,1)} e_{\mathcal{N}}^{\varphi_1} b_{k,j}^{(i,1)} \quad \text{and} \quad (\Phi'_{i,2})^{(2)} = \lim_{l\to\infty} \sum_{k=1}^{N_j^{(i,2)}} a_{k,l}^{(i,2)} e_{\mathcal{N}}^{\varphi_2} b_{k,l}^{(i,2)}$$

for suitable $N_l^{(i,1)}, N_j^{(i,2)} \in \mathbb{N}$, $a_{k,l}^{(i,1)}, b_{k,l}^{(i,1)} \in \mathcal{M}_1$ and $a_{k,l}^{(i,2)}, b_{k,l}^{(i,2)} \in \mathcal{M}_2$.

Claim. For $n \in \mathbb{N}$, $\mathbf{j} \in \mathcal{J}_n$, we have

$$\left\|(\Psi_i')^{(2)} P_{\mathbf{j}}\right\| \leq \left(\frac{1}{1+\varepsilon_i}\right)^n \tag{7.7.1}$$

and

$$(\Psi_i')^{(2)} P_{\mathbf{j}} = \lim_{l_1,\ldots,l_n\to\infty} \sum_{k_1,\ldots,k_n} a_{k_1,l_1}^{(i,j_1)} \ldots a_{k_n,l_n}^{(i,j_n)} e_{\mathcal{N}} b_{k_n,l_n}^{(i,j_n)} \ldots b_{k_1,l_1}^{(i,j_1)}, \tag{7.7.2}$$

where the convergence is in norm.

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Proof of the claim. For $x_1 \in \ker(\mathbb{E}_{j_1})$, ..., $x_n \in \ker(\mathbb{E}_{j_n})$ one calculates

$$\begin{aligned}(\Psi_i')^{(2)} P_{\mathbf{j}}(x_1 ... x_n \Omega_\varphi) &= \Phi'_{i,j_1}(x_1) ... \Phi'_{i,j_n}(x_n)\Omega_\varphi \\ &= \left(\frac{1}{1+\varepsilon_i}\right)^n \Phi_{i,j_1}(x_1)...\Phi_{i,j_n}(x_n)\Omega_\varphi \\ &= \left(\frac{1}{1+\varepsilon_i}\right)^n \Psi_i^{(2)}(x_1...x_n\Omega_\varphi)\end{aligned}$$

and hence $(\Psi_i')^{(2)} P_{\mathbf{j}} = (1+\varepsilon_i)^{-n}\Psi_i^{(2)} P_{\mathbf{j}}$. By the unitality of $\Phi_{i,1}$ and $\Phi_{i,2}$ the inequality (7.7.1) then follows from

$$\left\|(\Psi_i')^{(2)} P_{\mathbf{j}}\right\| = \left(\frac{1}{1+\varepsilon_i}\right)^n \left\|\Psi_i^{(2)} P_{\mathbf{j}}\right\| \leq \left(\frac{1}{1+\varepsilon_i}\right)^n \left\|\Psi_i^{(2)}\right\| \leq \left(\frac{1}{1+\varepsilon_i}\right)^n \|\Psi_i\| = \left(\frac{1}{1+\varepsilon_i}\right)^n.$$

We proceed by induction over n. For $n = 1$ the equality (7.7.2) is clear. Assume that the equality (7.7.2) holds for $\mathbf{j} \in \mathscr{J}_{n-1}$ and let $j_n \in \{1,2\}$ with $j_n \neq i_{n-1}$, $\mathbf{j}' = (\mathbf{j}, j_n)$. One easily checks that the left- and right-hand side of (7.7.2) both vanish on the orthogonal complement of $\mathscr{H}_{\mathbf{j}'}$. Further, for $x_1 \in \ker(\mathbb{E}_{j_1})$, ..., $x_n \in \ker(\mathbb{E}_{j_n})$, we get by the assumption

$$\begin{aligned}&(\Psi_i')^{(2)}(x_1...x_n\Omega_\varphi) = \Psi_i'(x_1...x_{n-1})\Phi'_{i,j_n}(x_n)\Omega_\varphi = \Psi_i'(x_1...x_{n-1})(\Phi'_{i,j_n})^{(2)}(x_n\Omega_\varphi) \\ &= \lim_{l_1,...,l_n} \sum_{k_1,...,k_{n-1}} a_{k_1,l_1}^{(i,j_1)} ... a_{k_{n-1},l_{n-1}}^{(i,j_{n-1})} \mathbb{E}_{\mathscr{N}}\left(b_{k_{n-1},l_{n-1}}^{(i,j_{n-1})} ... b_{k_1,l_1}^{(i,j_1)} x_1 ... x_{n-1}\right) \times \\ &\qquad \left(\sum_{k_n} a_{k_n,l_n}^{(i,j_n)} e_{\mathscr{N}} b_{k_n,l_n}^{(i,j_n)}\right) x_n\Omega_\varphi.\end{aligned}$$

Since the $\Phi'_{i,1}$ and $\Phi'_{i,2}$ are $\mathscr{N}$-$\mathscr{N}$-bimodular, we have $(\Phi'_{i,j_n})^{(2)} \in \mathscr{N}' \cap \langle \mathscr{N}, \mathscr{M}\rangle$ and hence

$$(\Psi_i')^{(2)}(x_1...x_n\Omega_\varphi) = \lim_{l_1,...,l_n} \sum_{k_1,...,k_n} a_{k_1,l_1}^{(i,j_1)} ... a_{k_n,l_n}^{(i,j_n)} \mathbb{E}_{\mathscr{N}}\left(b_{k_n,l_n}^{(i,j_n)} ... b_{k_1,l_1}^{(i,j_1)} x_1 ... x_n\right)\Omega_\varphi,$$

i.e.

$$\sum_{k_1,...,k_n} a_{k_1,l_1}^{(i,j_1)} ... a_{k_n,l_n}^{(i,j_n)} e_{\mathscr{N}} b_{k_n,l_n}^{(i,j_n)} ... b_{k_1,l_1}^{(i,j_1)} \to (\Psi_i')^{(2)}$$

strongly in $l_1, ..., l_n$. The second part of the claim, i.e. (7.7.2), then follows from noticing that

$$\left(\sum_{k_1,...,k_n} a_{k_1,l_1}^{(i,j_1)} ... a_{k_n,l_n}^{(i,j_n)} e_{\mathscr{N}} b_{k_n,l_n}^{(i,j_n)} ... b_{k_1,l_1}^{(i,j_1)}\right)_{l_1,...,l_n}$$

is a Cauchy sequence (compare with [Boc93, Section 3]).

The (in)equalities (7.7.1) and (7.7.2) in particular imply that $(\Psi_i')^{(2)}$ can be expressed as a norm limit

$$(\Psi_i')^{(2)} = e_{\mathscr{N}} + \lim_{n\to\infty} \sum_{\mathbf{j}\in\mathscr{J}_n} \Psi_i^{(2)} P_{\mathbf{j}}$$

and hence $(\Psi_i')^{(2)} \in \mathscr{K}(\mathscr{M},\mathscr{N},\varphi)$ for all $i \in I$. This finishes the direction "⇒".

"⇐" It suffices to prove the result for $\mathscr{M}_1$. Note first that [BD01, Lemma 3.5] shows that we have a normal conditional expectation $\mathbb{F}_1 : \mathscr{M} \to \mathscr{M}_1$ such that $\mathbb{E}_1 \circ \mathbb{F}_1 = \mathbb{E}_{\mathscr{N}}$. Hence Lemma 7.4.8 ends the proof. □

In combination with Theorem 7.6.6, Theorem 7.7.1 leads to the following corollary. This generalizes a result by Freslon [Fre13, Theorem 2.3.19] who showed this corollary in the realm of von Neumann algebras of discrete quantum groups, and the analogous property for classical groups was first shown in [Jol00] (see also [CCJJV01, Section 6]). To the authors' best knowledge even for inclusions of finite von Neumann algebras the statement of the following corollary is new.

Corollary 7.7.2. *Let $\mathscr{N} \subseteq \mathscr{M}_1$ and $\mathscr{N} \subseteq \mathscr{M}_2$ be unital embeddings of von Neumann algebras which admit faithful normal conditional expectations $\mathbb{E}_1 : \mathscr{M}_1 \to \mathscr{N}$, $\mathbb{E}_2 : \mathscr{M}_2 \to \mathscr{N}$ and assume that $\mathscr{N}$ is finite-dimensional. Assume moreover that $\mathscr{M}_1$ and $\mathscr{M}_2$ have the Haagerup property. Then $(\mathscr{M}_1,\mathbb{E}_1) *_{\mathscr{N}} (\mathscr{M}_2,\mathbb{E}_2)$, the amalgamated free product von Neumann algebra, has the Haagerup property as well.*

7.7.2. HAAGERUP PROPERTY FOR HECKE-VON NEUMANN ALGEBRAS OF VIRTUALLY FREE COXETER GROUPS

Let us now demonstrate the application of Corollary 7.7.2 in the context of virtually free Hecke-von Neumann algebras.

A *Coxeter system* (W,S) consists of a set S and a group W freely generated by S with respect to relations of the form $(st)^{m_{st}} = e$ where $m_{st} \in \{1,2,...,\infty\}$ with $m_{ss} = 1$, $m_{st} \geq 2$ for all $s \neq t$ and $m_{st} = m_{ts}$. By $m_{st} = \infty$ we mean that no relation of the form $(st)^m = e$ with $m \in \mathbb{N}$ is imposed, i.e. s and t are free with respect to each other; and the system is said to be *right-angled* if $m_{st} \in \{2,\infty\}$ for all $s,t \in S$, $s \neq t$. The system is of *finite rank* if the generating set S is finite. A subgroup of (W,S) is called *special* if it is generated by a subset of S.

With every Coxeter system one can associate a family of von Neumann algebras, its *Hecke-von Neumann algebras*, which can be viewed as q-deformations of the group von Neumann algebra $\mathscr{L}(W)$ of the Coxeter group W. For this, fix a multi-parameter $q := (q_s)_{s\in S} \in \mathbb{R}^S_{>0}$ with $q_s = q_t$ for all $s,t \in S$ which are conjugate in W. Further, write $p_s(q) := (q_s - 1)/\sqrt{q_s}$ for $s \in S$. Then the corresponding Hecke-von Neumann algebra $\mathscr{N}_q(W)$ is the von Neumann subalgebra of $\mathscr{B}(\ell^2(W))$ generated by the operators $T_s^{(q)}$, $s \in S$ where $T_s^{(q)} : \mathscr{B}(\ell^2(W)) \to \mathscr{B}(\ell^2(W))$ is defined by

$$T_s^{(q)}\delta_{\mathbf{w}} = \begin{cases} \delta_{s\mathbf{w}} & ,\text{ if } |s\mathbf{w}| > |\mathbf{w}| \\ \delta_{s\mathbf{w}} + p_s(q)\delta_{\mathbf{w}} & ,\text{ if } |s\mathbf{w}| < |\mathbf{w}| \end{cases}.$$

Here $|\cdot|$ denotes the word length function with respect to S and $(\delta_{\mathbf{w}})_{\mathbf{w}\in W} \subseteq \ell^2(W)$ is the canonical orthonormal basis of $\ell^2(W)$. For a group element $\mathbf{w} \in W$ which can be expressed by a reduced expression of the form $\mathbf{w} = s_1 ... s_n$ with $s_1,...,s_n \in S$ we set $T_{\mathbf{w}}^{(q)} :=$

$T_{s_1}^{(q)} \dots T_{s_n}^{(q)} \in \mathcal{N}_q(W)$. This operator does not depend on the choice of the expression $s_1 \dots s_n$ and the span of such operators is dense in $\mathcal{N}_q(W)$. Further, the von Neumann algebra $\mathcal{N}_q(W)$ carries a canonical faithful normal tracial state τ_q defined by $\tau_q(x) := \langle x\delta_e, \delta_e \rangle$ for $x \in \mathcal{N}_q(W)$. For more details on Hecke-von Neumann algebras see [Dav08, Chapter 20].

The aim of this subsection is to study the Haagerup property of Hecke-von Neumann algebras of virtually free Coxeter groups. We will approach this by decomposing these Hecke-von Neumann algebras as suitable amalgamated free products over finite-dimensional subalgebras. In the case of right-angled Hecke-von Neumann algebras the Haagerup property has been obtained in [Cas20, Theorem 3.9].

Fix a finite rank Coxeter system (W, S). A subset $T \subseteq S$ is called *spherical* if the *special subgroup* $W_T \subseteq W$ generated by T is finite. (W, S) is called spherical if S is a spherical subset.

If W is an arbitrary group which decomposes as an amalgamated free product $W = W_1 *_{W_0} W_2$ where (W_1, S_1), (W_2, S_2) are Coxeter systems with $W_0 = W_1 \cap W_2$ and $S_0 := S_1 \cap S_2$ generates W_0, then $(W, S_1 \cup S_2)$ is a Coxeter system as well. We may now define the class of *virtually free* Coxeter systems as the smallest class of Coxeter groups containing all spherical Coxeter groups and which is stable under taking amalgamated free products over special spherical subgroups. Note that the original definition of virtually free Coxeter systems is different, but by [Dav08, Proposition 8.8.5] equivalent to the one used here.

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Now, for a multi-parameter $q := (q_s)_{s\in S} \in \mathbb{R}_{>0}^S$ as above we have a natural unital embedding $\mathcal{N}_q(W_0) \subseteq \mathcal{N}_q(W)$ (see [Dav08, p. 19.2.2]). Let $\mathbb{E}_{\mathcal{N}_q(W_0)} : \mathcal{N}_q(W) \to \mathcal{N}_q(W_0)$ be the unique faithful normal trace-preserving conditional expectation onto $\mathcal{N}_q(W_0)$. Then, for $\mathbf{w} \in W$ the following equality holds:

$$\mathbb{E}_{\mathcal{N}_q(W_0)}(T_{\mathbf{w}}^{(q)}) = \begin{cases} T_{\mathbf{w}}^{(q)}, & \text{if } \mathbf{w} \in W_0 \\ 0, & \text{if } \mathbf{w} \notin W_0. \end{cases}$$

Let us show that the amalgamated free product decomposition of a Coxeter group translates into the Hecke-von Neumann algebra setting. Note that the arguments using the (iterated) amalgamated free product description of Hecke-deformed Coxeter group C^*-algebras appear for example in [RS22], exploiting the earlier work on operator algebraic graph products in [CF17].

Proposition 7.7.3. *Let (W, S) be a finite rank Coxeter system that decomposes as $W = W_1 *_{W_0} W_2$ where (W_1, S_1), (W_2, S_2) are Coxeter systems with $S = S_1 \cup S_2$, $W_0 = W_1 \cap W_2$ such that $S_0 := S_1 \cap S_2$ generates W_0. For a multi-parameter $q = (q_s)_{s\in S}$ with $q_s = q_t$ for all $s, t \in S$ which are conjugate in W the corresponding Hecke-von Neumann algebra $\mathcal{N}_q(W)$ decomposes as an amalgamated free product of the form*

$$\mathcal{N}_q(W) = \mathcal{N}_{q_1}(W_1) *_{\mathcal{N}_{q_0}(W_0)} \mathcal{N}_{q_2}(W_2),$$

where $q_0 := (q_s)_{s\in S_0}$, $q_1 := (q_s)_{s\in S_1}$ *and* $q_2 := (q_s)_{s\in S_2}$. *The decomposition is taken with respect to the restricted conditional expectations* $(\mathbb{E}_{\mathcal{N}_q(W_0)})|_{\mathcal{N}_{q_1}(W_1)}$ *and* $(\mathbb{E}_{\mathcal{N}_q(W_0)})|_{\mathcal{N}_{q_2}(W_2)}$.

Proof. We will use the multi-index notation from the proof of Theorem 7.7.1. By the uniqueness of the amalgamated free product construction in combination with our previous discussion, it suffices to show that $\mathbb{E}_{\mathcal{N}_q(W_0)}(a_1\dots a_n) = 0$ for all $n \in \mathbb{N}$, $\mathbf{j} \in \mathcal{J}_n$ and $a_k \in \mathcal{N}_{q_{j_k}}(W_{j_k})) \cap \ker(\mathbb{E}_{\mathcal{N}_q(W_0)})$. Let $\mathcal{N}_{q_i}(W_i)_1$ denote the unit ball of $\mathcal{N}_{q_i}(W_i)$. Let $\overline{\text{Span}}$ denote the strong closure of the linear span. By Kaplansky's density theorem,

$$\mathcal{N}_{q_1}(W_1)_1 \cap \ker(\mathbb{E}_{\mathcal{N}_q(W_0)}) = \mathcal{N}_{q_1}(W_1)_1 \cap \overline{\text{Span}}\{T^{(q)}_{\mathbf{w}} \mid \mathbf{w} \in W_1 \setminus W_0\},$$

and

$$\mathcal{N}_{q_2}(W_2)_1 \cap \ker(\mathbb{E}_{\mathcal{N}_q(W_0)}) = \mathcal{N}_{q_2}(W_2)_1 \cap \overline{\text{Span}}\{T^{(q)}_{\mathbf{w}} \mid \mathbf{w} \in W_2 \setminus W_0\}.$$

By [Mur90, Remark 4.3.1] the element $a_1\dots a_n$ can hence be approximated strongly by a bounded net of linear combinations of reduced expressions of the form $T^{(q)}_{\mathbf{w}_1}\dots T^{(q)}_{\mathbf{w}_n}$ with $\mathbf{w}_k \in W_{j_k} \setminus W_0$. But this expression coincides with $T^{(q)}_{\mathbf{w}_1\dots\mathbf{w}_n}$ where $\mathbf{w}_1\dots\mathbf{w}_n \in W \setminus W_0$ is non-trivial, so $\mathbb{E}_{\mathcal{N}_q(W_0)}(a_1\dots a_n) = 0$ since $\mathbb{E}_{\mathcal{N}_q}$ is normal and hence weakly continuous on bounded sets. □

The following corollary is an example of an application of Corollary 7.7.2 in a setting which is not covered by the results in [Fre13].

Corollary 7.7.4. *Let* (W,S) *be a finite rank Coxeter system, let* $q = (q_s)_{s\in S} \in \mathbb{R}^S_{>0}$ *be a multi-parameter with* $q_s = q_t$ *if* $s,t \in S$ *are conjugate in* W *and assume that* W *is virtually free. Then the corresponding Hecke-von Neumann algebra* $\mathcal{N}_q(W)$ *has the Haagerup property.*

Proof. This follows from a combination of [Cas20, Theorem 3.9], Proposition 7.7.3 and Corollary 7.7.2. □

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7.8. INCLUSIONS OF FINITE INDEX

In this section we will discuss finite index inclusions for not necessarily tracial von Neumann algebras defined in [BDH88]. We will pick one of the (possibly nonequivalent) definitions, which is most suitable in our context, and then we will illustrate this notion using certain compact quantum groups, namely free orthogonal quantum groups.

Definition 7.8.1. Let $\mathcal{N} \subseteq \mathcal{M}$ be an inclusion of von Neumann algebras with a faithful normal conditional expectation $\mathbb{E}_{\mathcal{N}} : \mathcal{M} \to \mathcal{N}$. We say that a family of elements $(m_i)_{i\in I}$ is an *orthonormal basis* of the right $\mathcal{N}$-module $L^2(\mathcal{M})_{\mathcal{N}}$ if

1. for each $i,j \in I$ we have $\mathbb{E}_{\mathcal{N}}(m_i^* m_j) = \delta_{ij} p_j$, where p_j is a projection in $\mathcal{N}$;
2. $\overline{\sum_{i\in I} m_i \mathcal{N}} = L^2(\mathcal{M})$.

We say that the inclusion $\mathcal{N} \subseteq \mathcal{M}$ is strongly of finite index if it admits a finite orthonormal basis.

Lemma 7.8.2. *If an inclusion* $\mathcal{N} \subseteq \mathcal{M}$ *is strongly of finite index then it has the Haagerup property.*

Proof. Let $m_1, \dots, m_n$ be a finite orthonormal basis for our inclusion. It suffices to show that $x = \sum_{i=1}^{n} m_i \mathbb{E}_{\mathcal{N}}(m_i^* x)$ for each $x \in \mathcal{M}$. Indeed, this would show that the identity map on $L^2(\mathcal{M})$ is relatively compact with respect to $\mathcal{N}$, so clearly the triple $(\mathcal{M}, \mathcal{N}, \mathbb{E}_{\mathcal{N}})$ satisfies the relative Haagerup property. The equality $x = \sum_{i=1}^{n} m_i \mathbb{E}_{\mathcal{N}}(m_i^* x)$ has been already observed by Popa (see [Pop95, Section 1]) in a more general context. □

7.8.1. FREE ORTHOGONAL QUANTUM GROUPS

We will now present a certain inclusion arising in the theory of compact quantum groups that has the relative Haagerup property. For the theory of compact quantum groups we refer the reader to the excellent book [NT13].

Definition 7.8.3 ([VW96])**.** Let $n \geqslant 2$ be an integer and let $F \in M_n(\mathbb{C})$ be a matrix such that $F\overline{F} = c\mathbb{1}$ for some $c \in \mathbb{R} \setminus \{0\}$. Let $\mathrm{Pol}(O_F^+)$ be the universal $*$-algebra generated by the entries of a unitary matrix $U \in M_n(\mathrm{Pol}(O_F^+))$, denoted u_{ij}, subject to the condition $U = F\overline{U}F^{-1}$, where $(\overline{U})_{ij} := (u_{ij})^*$ for all $i, j = 1, \dots, n$. Then the unique $*$-homomorphic extension of the map $\Delta(u_{ij}) := \sum_{k=1}^{n} u_{ik} \otimes u_{kj}$ makes $\mathrm{Pol}(O_F^+)$ into a Hopf $*$-algebra, whose universal C^*-algebra completion yields a compact quantum group.

Remark 7.8.4. As every compact quantum group admits a Haar state, we can use the GNS construction to construct a von Neumann algebra $L^\infty(O_F^+)$.

In [Ban96] Banica classified irreducible representations of the compact quantum group O_F^+. He showed that they are indexed by natural numbers, U^k, where U^0 is the trivial representation and $U^1 = U$ is the fundamental representation U. Moreover, the fusion rules satisfied by these representations are the following:

$$U^k \otimes U^l \simeq U^{k+l} \oplus U^{k+l-2} \oplus \cdots \oplus U^{|k-l|}, \quad k, l \in \mathbb{N},$$

just like for the classical compact group $SU(2)$. From the fusion rules one can infer that the coefficients of representations indexed by even numbers form a subalgebra. Further, one can use the defining relation $U = F\overline{U}F^{-1}$ to show that they form a $*$-subalgebra.

Definition 7.8.5. Let $\mathcal{M} := L^\infty(O_F^+)$. We define the *even subalgebra* $\mathcal{N}$ to be the von Neumann subalgebra of $\mathcal{M}$ generated by the elements $(u_{ij}u_{kl})_{1 \leqslant i,j,k,l \leqslant n}$. It is equal to the von Neumann algebra generated by the coefficients of the even representations; in fact it is related to the *projective version* of O_F^+, usually denoted PO_F^+.

Remark 7.8.6. It has been shown by Brannan in [Bra12b] that $\mathcal{N} \subseteq \mathcal{M}$ is a subfactor of index 2 in case that $F = \mathbb{1}$ (it is then an inclusion of finite von Neumann algebras).

We now roughly outline Brannan's argument and then mention why it cannot immediately be translated into our setting. There is an automorphism Φ of $\mathcal{M}$ such that $\Phi(u_{ij}) = -u_{ij}$; Φ can be first defined on $\mathrm{Pol}(O_F^+)$ by the universal property but it also preserves the Haar state, so can be extended to an automorphism of $L^\infty(O_F^+)$. The fixed point subalgebra of Φ is equal to the even subalgebra $\mathcal{N}$ and therefore $\mathbb{E}_{\mathcal{N}} := \frac{1}{2}(\mathrm{Id} + \Phi)$ is a conditional expectation onto $\mathcal{N}$ that preserves the Haar state. As a consequence $\mathbb{E}_{\mathcal{N}} - \frac{1}{2}\mathrm{Id}$ is a completely positive map, so one can use the Pimsner-Popa inequality, which works for II_1-factors, to conclude that the index of $\mathcal{N} \subseteq \mathcal{M}$ is at most 2. On the other hand, any proper inclusion has index at least 2, so the result follows. Unfortunately in the non-tracial case it is not clear if the condition that $\mathbb{E}_{\mathcal{N}} - \frac{1}{2}\,\mathrm{Id}$ is completely positive implies that the inclusion $\mathcal{N} \subseteq \mathcal{M}$ is strongly of finite index; so far it is only known that it implies being of finite index in a weaker sense (see [BDH88, Théorème 3.5]). Fortunately in our case it is possible to explicitly define a finite orthonormal basis.

Proposition 7.8.7. *Let $n \geqslant 2$ be an integer and let $F \in M_n(\mathbb{C})$ be a matrix such that $F\overline{F} = c\mathbb{1}$ for some $c \in \mathbb{R} \setminus \{0\}$. Let $\mathcal{M} := L^\infty(O_F^+)$ and let $\mathcal{N}$ be the even von Neumann subalgebra of $\mathcal{M}$. Then the inclusion $\mathcal{N} \subseteq \mathcal{M}$ is strongly of finite index. Moreover, one can find an orthonormal basis consisting of at most $n^2 + 1$ elements.*

Proof. One can verify by an explicit computation that $\mathcal{N}$ is left globally invariant by the modular automorphism group of the Haar state h of $L^\infty(O_F^+)$, so we do have a faithful normal h-preserving conditional expectation $\mathbb{E}_{\mathcal{N}} : \mathcal{M} \to \mathcal{N}$. We start with $n^2 + 1$ elements of $\mathcal{M}$, namely $\mathbb{1}$ and all the u_{ij}'s. Since we have all the coefficients of the fundamental representation, it follows from the fusion rules of O_F^+ that $\mathcal{N} \oplus \sum_{i,j=1}^n u_{ij}\mathcal{N}$ is a dense submodule of $L^2(\mathcal{M})_{\mathcal{N}}$.

Note that all the elements u_{ij} are odd, i.e. $\Phi(u_{ij}) = -u_{ij}$ for $i, j = 1, \ldots, n$. Suppose that we have a family $x_1, \ldots, x_k$ of odd elements. Then we can perform a Gram-Schmidt process to make this set orthonormal. To do it, first notice that $x_i^* x_i$ is an even element, hence so is $|x_i|$ – we conclude that the partial isometry in the polar decomposition $x_i = v_i|x_i|$ is odd as well. Our process works as follows: we first replace x_1 by the corresponding partial isometry v_1. Then we define $\tilde{x}_2 := x_2 - v_1 v_1^* x_2$. Because v_1 is a partial isometry, we get $v_1^* \tilde{x}_2 = v_1^* x_2 - v_1^* v_1 v_1^* x_2 = 0$. We then define v_2 to be the partial isometry appearing in the polar decomposition of $\tilde{x}_2$; it still holds that v_2 is odd and $v_1^* v_2 = 0$. We can continue this process just like the usual Gram-Schmidt process and obtain an orthonormal set of odd partial isometries v_i such that $\sum_{i=1}^k x_i\mathcal{N} \subset \sum_{i=1}^k v_i\mathcal{N}$; note that the projections $v_i^* v_i$ belong to $\mathcal{N}$. If we apply this procedure to the family $(u_{ij})_{1\leqslant i,j\leqslant n}$, we obtain a finite orthonormal basis for the inclusion $\mathcal{N} \subseteq \mathcal{M}$. □

Corollary 7.8.8. *The inclusion $\mathcal{N} \subseteq \mathcal{M} := L^\infty(O_F^+)$ has the relative Haagerup property.*

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Curriculum Vitæ

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List of Publications

This thesis is based on the following articles:

Accepted articles

3. **Martijn Caspers, Amudhan Krishnaswamy-Usha, Gerrit Vos**, *Multilinear transference of Fourier and Schur multipliers acting on non-commutative L_p-spaces*, Canadian Journal of Mathematics, **75**(6):1986-2006 (2023).

2. **Martijn Caspers, Mario Klisse, Adam Skalski, Gerrit Vos, Mateusz Wasilewski**, *Relative Haagerup property for arbitrary van Neumann algebras*, to appear in Advances of Mathematics. Preprint: 2021.arXiv:2110.15078.

1. **Martijn Caspers, Gerrit Vos**, *BMO spaces of σ-finite von Neumann algebras and Fourier-Schur multipliers on $SU_q(2)$*, Studia Mathematica **262**,(1):45-91 (2022).

Preprints

- **Gerrit Vos**, *Transference of multilinear Fourier and Schur multipliers acting on non-commutative L_p-spaces for non-unimodular groups*. Preprint: 2023.arXiv:2308.16595

During my PhD I was advised by dr. M.P.T. Caspers (daily supervisor and copromotor) and prof. dr. J.M.A.M. van Neerven (promotor). This PhD position was supported by the NWO Vidi grant 'Non-commutative harmonic analysis and rigidity of operator algebras', VI.Vidi.192.018.


$\|x\|_{BMO^c} = \sup_{t\geq 0} \|\Phi_t(|x - \Phi_t(x)|^2)\|^{1/2}$
$\|f\|_2 = \|\hat{f}\|_2$
$T_\phi(\lambda(f)) = \int_G \phi(s) f(s) \lambda_s ds$
$\kappa_p^{(\theta)}(x) = D_\varphi^{\frac{1-\theta}{p}} x D_\varphi^{\frac{\theta}{p}}$