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Zegeling, André; Kooij, Robert E.

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Several Bifurcation Mechanisms for Limit Cycles in a Predator–Prey System

André Zegeling[1] · **Robert E. Kooij**[2]

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Abstract

The research presented in this paper compares the occurrence of limit cycles under different bifurcation mechanisms in a simple system of two-dimensional autonomous predator–prey ODEs. Surprisingly two unconventional approaches, for a singular system and for a system with a center, turn out to produce more limit cycles than the traditional Andronov–Hopf bifurcation. The system has a functional response function which is a monotonically increasing cubic function of x for $0 \leq x \leq 1$ where x represents the prey density, and which is constant for $x > 1$. It acts as a proxy for investigating more general systems. The following results are obtained. For the *Andronov–Hopf bifurcation* the highest order of the weak focus is 2 and at most 2 small-amplitude limit cycles can be created. In the *center bifurcation* cases are shown to exist with at least 3 limit cycles. In the *singular perturbation* cases are shown to exist with at least 4 limit cycles and in some cases an exact upper bound of 2 limit cycles is obtained. Finally we indicate how the conclusions can be extended to more general systems. We show how an arbitrary number of limit cycles can be created by choosing an appropriate functional response function and growth function for the prey. One special situation is the system with group defense: the three bifurcation mechanisms typically produce less limit cycles if a group defense element is included.

Keywords Generalized Gause model · Center · Singular perturbation · Holling · Functional response · Limit cycle

Mathematics Subject Classification 34C15 · 92D25

✉ André Zegeling
zegela1@yahoo.com

1 Guilin University of Aerospace Technology, Jinji Road 2, Guilin, China

2 Faculty of Electrical Engineering, Mathematics & Computer Science, Delft University of Technology, Mekelweg 4, 2628 CD Delft, The Netherlands

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1 Introduction

We consider the following predator–prey system describing the interaction between preys (denoted by their density $x(t)$) and predators (denoted by their density $y(t)$). It is referred to as the generalized Gause system [10]:

$$\begin{aligned}\frac{dx(t)}{dt} &= h(x) - p(x)y,\\ \frac{dy(t)}{dt} &= -\delta y + \gamma p(x)y.\end{aligned} \tag{1.1}$$

This type of modelling ignores the effect of spatial diffusion and the interaction with other species. The other restriction is that we model the death rate of the predator in absence of any predators with an exponential decay.

The function $p(x)$ is the functional response function describing the effectiveness of the interaction between predator and prey, i.e. the net average amount of prey killed during the interaction with one predator. The functional response function $p(x)$ is a positive function for all $x > 0$ and $p(0) = 0$. Since it is assumed that there is no group defense by the prey population the response function is taken to be monotonically increasing: $p'(x) \geq 0$. On a finite interval, which we take to be $0 \leq x \leq 1$, $p(x)$ is parameterized as a cubic function. On this interval we will impose that $p(x)$ is absolute monotonically increasing, i.e. $p'(x) > 0$. The justification of choosing the boundary to be at $x = 1$ is that the variable x can be rescaled without changing the qualitative properties of the functions in the system. For $x > 1$ the function $p(x)$ is set equal to 1. This cut-off is a generalization of the Holling I type functional response.

The motivation from a biological point of view is that in reality it is difficult to determine the correct behaviour of the functional response function for large prey densities (see for example the experimental data in the original Holling paper [13]). Therefore a natural starting point is to take it to be a constant instead of choosing some specific asymptotic behaviour like in the cases of the Holling II and Holling III types. Moreover, there are some biological arguments to justify this cut-off, see [16]. Another reason for this choice is that it allows for some general extensions by making a small perturbation of the functional response function.

The motivation from a mathematical point of view is that this system is rich and covers many aspects of the other functional responses used in the literature. It is relatively easy to study the bifurcations in this form and to extend them to more general cases as will be done in the “Discussion” section.

Definition 1.1 The following functional response function is used:

- $p(x) = x(1 + (x - 1)(a_0 + a_1 x))$ for $0 \leq x \leq 1,\ a_0, a_1 \in \mathbb{R}$,
- $p(x) = 1$ for $x > 1$.

The functional form on the interval $0 \leq x \leq 1$ is a parametrized extension of the Holling type I function which corresponds to the special case $a_0 = a_1 = 0$. The two parameters a_0 and a_1 do not have an intrinsic biological meaning. As was discussed in [16], there are two kinds of modelling, one is phenomenological where the functional response is chosen in such a way to fit experimental data, the other are the so-called mechanistic models where the function is derived from fundamental biological properties. Our choice is of the phenomenological type similar to what Holling did in his paper [13]. The extension allows for different types of convexity on the interval $0 \le x \le 1$, whereas for the original Holling I function ($p(x) = x$), the second derivative is identically equal to zero.

On the interval $0 \le x \le 1$ we need to impose additional conditions on the parameters a_0 and a_1 to ensure that the function $p(x)$ satisfies the basic requirements of the functional response. The requirement of having no group defense translates into the condition that the functional response function $p(x)$ is monotonically increasing on the interval $0 \le x \le 1$.

A simple calculation shows that this holds true under the condition that the parameters a_0 and a_1 lie in a bounded region in the (a_0, a_1) plane determined by boundaries $\{a_0^2 + a_0a_1 + a_1^2 - 3a_1 = 0 \wedge a_1 > 1\}$, $a_0 = 1$ and $a_0 + a_1 + 1 = 0$. For convenience we will refer to the boundary $a_0^2 + a_0a_1 + a_1^2 - 3a_1 = 0 \wedge a_1 > 1$ as $D = 0$. It corresponds to the case where $p(x)$ has an inflection point on the interval $0 < x < 1$. This boundary case itself is excluded from the analysis while in principle we will include the other two boundary cases because they correspond to $p'(x) = 0$ at the end points of the interval $0 \le x \le 1$, i.e. at $x = 0$ and $x = 1$.

The three conditions on the parameters can be summarized as:

Definition 1.2 The parameter range W_1 for which the functional response $p(x)$ in (1.1) is increasing on the interval $0 \le x \le 1$ is given by the following conditions on the parameters:

$$\{a_1 > 1 \wedge a_0^2 + a_0a_1 + a_1^2 - 3a_1 < 0\} \vee \{a_1 \le 1 \ \wedge a_0 \le 1 \wedge a_0 + a_1 + 1 \ge 0\}.$$

Here the border cases correspond to the following mathematical interpretation:

- $D = 0$: $a_0^2 + a_0a_1 + a_1^2 - 3a_1 = 0 \wedge a_1 > 1$ iff $p'(\bar{x}) = p''(\bar{x}) = 0$, $p'''(\bar{x}) \ne 0$ for some $\bar{x} \in (0, 1)$.
- $a_0 = 1$ iff $p'(0) = 0$.
- $a_0 + a_1 + 1 = 0$ iff $p'(1) = 0$.

The region of parameters W_1 is depicted in Fig. 1 (for fixed values of k, δ, ϕ). The boundary of the bounded region W_1 has a reversed raindrop form in the a_0, a_1 plane given by a conic and two lines as given in Definition 1.2. The boundary curve defined through $a_0^2 + a_0a_1 + a_1^2 - 3a_1 = 0$ was excluded because in that case $p'(x)$ can become zero for some $0 < x = x_c < 1$. Even though the analysis for this case is very similar to the rest of this paper, it allows for the possibility of the functional response to be not strictly monotonic on the open interval $0 < x < 1$ which we prefer to avoid.

This choice of $p(x)$ ensures that $p(0) = 0$ and $p(1) = 1$. A rescaling in x, y, t was used to scale the parameter γ to become equal to 1, to position the cut-off at $x = 1$ and to ensure that $p(1) = 1$. It is similar to what was done in the predator–prey system of type Holling I, see [14].

The function $h(x)$ represents the growth of the prey population in the absence of predators, which we will take to be the traditional logistic growth function.

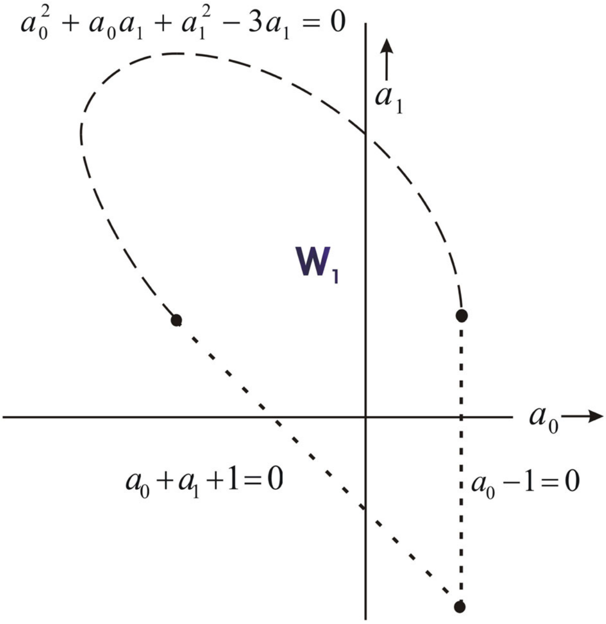


Fig. 1 The region of parameters W_1 for which the functional response $p(x)$ of (1.1) is non-decreasing on the interval $0 \le x \le 1$

Definition 1.3 The following growth rate function is used: $h(x) = \phi x(1 - \frac{x}{k})$, where $\phi > 0$ and $k > 2$.

The parameter k is the carrying capacity of the prey. In this paper we will discuss the case where $k > 2$, i.e. $h(x) > 0$ on the interval $0 < x < k$ and $h'(x)$ has a unique zero x_h for $x > 1$: $1 < x_h = \frac{k}{2} < k$.

The parameter δ is the death rate of the predator when there are no preys, i.e. when there is no food to be consumed.

We summarize and define the system:

$$\begin{aligned} \frac{dx(t)}{dt} &= h(x) - p(x)y, \\ \frac{dy(t)}{dt} &= -\delta y + p(x)y. \end{aligned} \tag{1.2}$$

where the functions $p(x)$ and $h(x)$ are defined through the Definitions 1.1 and 1.3 respectively. The parameters in the functional response are restricted according to Definition 1.2.

The aim of this paper is twofold.

First we discuss three bifurcation mechanisms for the creation of limit cycles in system (1.2). The first is the standard Andronov–Hopf bifurcation which can be understood through a straightforward calculation. Then we discuss two other unusual types of bifurcation corresponding to special values of the parameters in the system: the bifurcation from an irregular center case and the bifurcation from a singular system. Both surprisingly produce more limit cycles than the Andronov–Hopf bifurcation.

Then we study the generalization of these concepts to systems of Gause type (1.1) where the functions $h(x)$ and $p(x)$ are left in a general form and determine under which conditions the conclusions about the creation of limit cycles from the Andronov–Hopf bifurcation, center bifurcation and singular bifurcation are still true. Another question is: how do these bifurcations change when a group defense component is added? With this analysis we hope to shed some light on how the number of limit cycles depends on the prey growth rate function and the functional response function in general predator–prey systems.

In [2] it was observed that the occurrence of limit cycles is rare in typical families of autonomous planar ordinary differential equations. In the case of so-called polynomial systems a numerical experiment revealed that very few systems had limit cycles. The system we study in this paper on the contrary is rich with limit cycles and paradoxically it seems to be even more difficult to find situations in which no limit cycles occur! Therefore we consider this predator–prey system as a proxy for studying limit cycle problems in general. It seems to be a good starting point to explore new techniques for detecting limit cycles. We hope that the techniques and results in this paper will inspire other researchers to extend the results to other, more general, cases. The relevance of the study of limit cycles stems from the still unsolved 16th Hilbert problem asking for an upper bound on the number of limit cycles in polynomial systems.

2 Singularities

The existence of a singularity in the first quadrant of the phase plane $x > 0$, $y > 0$ is determined by the value of the parameter δ, i.e. the death rate of the predator density:

Lemma 2.1 *If $0 < \delta < 1$, there exists a singularity A in the first quadrant of the phase plane with coordinates: $x = x_g$, $y = y_g \equiv \frac{\phi(k-x_g)x_g}{k\delta}$ where x_g satisfies:*

$$p_g \equiv p(x_g) = x_g(1 + (x_g - 1)(a_0 + a_1 x_g)) = \delta. \tag{2.1}$$

The singularity A only exists for $0 < x_g < 1$. With the restriction $p'(x_g) > 0$, A is an elementary singularity of anti-saddle type, i.e. node, focus or center.

Proof The product of the eigenvalues of the linearized system at the singularity A is given by:

$$p_g p'(x_g) y_g,$$

which is positive under the assumptions of (1.2). □

It is natural in the following to introduce a new parameter x_g corresponding to the singularity of anti-saddle type related to δ through Eq. (2.1) and require that $0 < x_g < 1$.

There are two singularities on the coordinate axes: a saddle O at the origin $x = 0$, $y = 0$ and a saddle K on the axis $x = k$, $y = 0$. The saddle K has the important property that an unstable separatrix is leaving the saddle and is entering the first quadrant. The vertical line $x = k$ is a line without contact and limit cycles in the system cannot cross it, i.e. they lie in the strip $0 < x < k$.

2.1 Stability of the Singularity

To study the stability of the anti-saddle A at (x_g, y_g) of Lemma 2.1 we transform (1.2) to a Liénard system, because there it is more convenient to establish the stability and order of a weak focus. Moreover, it allows for the application of uniqueness and non-existence theorems for limit cycles in special situations. Since the singularity lies in the strip $0 < x < 1$, for establishing its local stability we can restrict our attention to the part of the system where $p(x)$ is a cubic polynomial and $h(x)$ exhibits logistic growth. We define:

Definition 2.2 Generalized Liénard system:

$$\begin{aligned} \frac{dx(t)}{dt} &= F(x) - \psi(y), \\ \frac{dy(t)}{dt} &= g(x), \end{aligned} \tag{2.2}$$

defined in a region $x \in (x_-, x_+)$, $y \in (y_-, y_+)$.

After applying the transformations $t \to \frac{t}{p(x)}$ and $y = e^v$ to (1.2) we arrive at the form of a generalized Liénard system (2.2) (relabelling $v \to y$) for the interval $0 < x < 1$ with:

$$\begin{aligned} F(x) &= \frac{\phi x(1 - \frac{x}{k})}{p(x)} - e^{y_g} = \frac{\phi(k - x)}{k(1 + (x - 1)(a_0 + a_1 x))} - e^{y_g}, \\ g(x) &= 1 - \frac{\delta}{p(x)} = 1 - \frac{p_g}{x(1 + (x - 1)(a_0 + a_1 x))}, \\ \psi(y) &= e^y - e^{y_g}, \end{aligned} \tag{2.3}$$

with $x \in (x_-, x_+) = (0, 1)$, $y \in (y_-, y_+) = (-\infty, \infty)$. For the interval $x > 1$ the same transformation yields:

$$\begin{aligned} F(x) &= \frac{\phi x(1 - \frac{x}{k})}{p(x)} - e^{y_g} = \frac{\phi x(k - x)}{k} - e^{y_g}, \\ g(x) &= 1 - \frac{\delta}{p(x)} = 1 - p_g, \\ \psi(y) &= e^y - e^{y_g}, \end{aligned} \tag{2.4}$$

with $x \in (x_-, x_+) = (1, \infty)$, $y \in (y_-, y_+) = (-\infty, \infty)$. Here we used that $p(x) = 1$ for $x > 1$.

The parameter y_g is the y-coordinate of the singularity in the strip $0 \leq x \leq 1$ and is given by $y_g = ln(\frac{\phi x_g(1-\frac{x_g}{k})}{p_g})$. With this definition of the functions we have ensured that $F(x_g) = 0$ and $\psi(y_g) = 0$ at the singularity corresponding to the anti-saddle of the system. This is necessary in the following when applying theorems for uniqueness of limit cycles in system (2.3) and (2.4).

Of importance for the study of limit cycles is the divergence of the vector field (2.2) with functions (2.3) and (2.4), denoted by the following notation:

$$\begin{aligned} f(x) &\equiv F'(x) = \frac{\phi(a_1x^2 - 2a_1kx + ka_1 + (1-k)a_0 - 1)}{k(1+(x-1)(a_0+a_1x))^2} \\ &\equiv \frac{\phi f_2(x)}{k(1+(x-1)(a_0+a_1x))^2} \end{aligned} \tag{2.5}$$

for $0 \leq x \leq 1$ and

$$f(x) \equiv F'(x) = \frac{\phi(k-2x)}{k} \tag{2.6}$$

for $x > 1$.

2.2 Strong Singularity

If $f(x_g) \neq 0$ in (2.5), then the singularity A is strong and its stability is determined by the sign of $f(x_g)$:

Lemma 2.3 *System (1.2) with $0 < x_g < 1$ has a unique singularity A in the first quadrant located at $x = x_g$, $y = y_g$. It is a strong anti-saddle which is stable (unstable) if $f_2(x_g) = a_1x_g^2 - 2a_1kx_g + ka_1 + (1-k)a_0 - 1 < 0$ ($f_2(x_g) > 0$). This condition can be reformulated as: A is stable (unstable) if $a_0 > a_0^{wf}(a_1)$ ($a_0 < a_0^{wf}(a_1)$), where $a_0^{wf}(a_1) \equiv \tau^{wf}a_1 + \frac{1}{1-k}$, with $\tau^{wf} \equiv \frac{x_g^2-2kx_g+k}{k-1}$.*

The condition $a_0 = a_0^{wf}(a_1)$ corresponds to a line l_{wf} with slope $\frac{1}{\tau^{wf}}$ in the (a_0, a_1) plane for fixed k, ϕ, x_g, i.e. the plane is divided into two regions with different stability of the strong singularity.

We will use the following properties of the weak-focus line l_{wf}:

Lemma 2.4 *The weak-focus line l_{wf} defined through $a_0 = a_0^{wf}(a_1)$ passes through the point C: (a_0^c, a_1^c) where $a_0^c = \frac{1}{1-k} < 0$, $a_1^c = 0$ for all $0 < x_g < 1$. The two limiting cases of the family of lines l_{wf} as a function of the parameter x_g are $x_g = 0$ and $x_g = 1$. For $x_g = 0$ the line passes through ($a_0 = 1$, $a_1 = 1$). For $x_g = 1$ the line is parallel to one of the boundaries of W_1 ($a_0 + a_1 + 1 = 0$).*

Lemma 2.5 *The point C lies inside region W_1. The line l_{wf} intersects the boundary of W_1 in exactly two points for fixed $0 < x_g < 1$. Of the two intersection points, one is with the conic defined by $D = 0$, while the other is with either the boundary line $a_0 + a_1 + 1 = 0$ or the vertical boundary line $a_0 = 1$.*

2.3 Center

The point C mentioned in Lemma 2.4 in the (a_0, a_1) plane has a special meaning for system (1.2):

Lemma 2.6 *For the parameter choice C: $(a_0 = \frac{1}{1-k}, a_1 = 0)$, the singularity A is a center in system* (1.2).

Proof Under the conditions of the lemma the system becomes integrable, because we can write $p(x) = x(1 + (x-1)(\frac{1}{1-k})) = \frac{kx}{(k-1)}(1 - \frac{x}{k})$:

$$\begin{aligned} \frac{dx(t)}{dt} &= \phi x\left(1 - \frac{x}{k}\right) - p(x)y = x\left(1 - \frac{x}{k}\right)\left(\phi - \frac{k}{(k-1)}y\right), \\ \frac{dy(t)}{dt} &= y(-\delta + p(x)). \end{aligned} \tag{2.7}$$

Through separation of variables this system can be integrated. Formally the expression for the integral of the system can be written as:

$$Z_1(y) + Z_2(x) = h, \tag{2.8}$$

with

$$\begin{aligned} Z_1(y) &= \int_{y_g}^{y} \frac{\phi(k-1) - k\tilde{y}}{(k-1)\tilde{y}} d\tilde{y}, \\ Z_2(x) &= \int_{x_g}^{x} \frac{\tilde{x}(\tilde{x}-k)}{k(-\delta + p(\tilde{x}))} d\tilde{x}. \end{aligned}$$

This establishes that the anti-saddle A in the first quadrant is indeed a center for the nonlinear system (1.2). Another simpler way to establish that A is a center is by considering the Liénard form (2.3) and by observing that for the center case the function $F(x)$ becomes a constant, i.e. the divergence of the vector field is identically equal to zero and the system becomes a Hamiltonian system.

For each $h \in (0, h^*)$ the integral (2.8) represents a cycle Γ_h in the period annulus surrounding the center A. The value $h = 0$ corresponds with the inner boundary of the period annulus: the singularity A itself. The value $h = h^*$ corresponds with the cycle Γ_{h^*} tangent to the vertical line $x = 1$.

Since the system is cut off at $x = 1$, the period annulus does not extend beyond $x = 1$ making it an irregular center case. □

2.4 Order and Stability of the Weak Focus

To study the stability of the weak focus of system (1.2) we impose the condition that $f(x_g) = 0$ in (2.5). This leads according to Lemma 2.3 to:

$$a_0^{wf}(a_1) = \tau^{wf} a_1 + \frac{1}{1-k}, \tag{2.9}$$

with $\tau^{wf} \equiv \frac{x_g^2 - 2kx_g + k}{k-1}$. The stability of the weak focus then follows from the sign of the first focal value V_1 which was calculated using the computer software program Maple.

The usual way to calculate focal values is to transform the system to a normal form. In the case of a Liénard system this can be achieved quite easily through some rescalings. However, we chose a more direct way to get the focal values as suggested in [11]. In that paper the stability of a weak focus for a Liénard system was shown to be determined through an elegant iterative mechanism avoiding the use of normal form transformations. They suggested that this algorithm could also determine the order of the weak focus and stated that in the thesis [19] a proof was given for the formulas of the first two focal values. Since we have not been able to access this thesis, we present a full proof of the statement here. The general case will be dealt with in a forthcoming paper.

Proposition 2.7 *Consider system* (2.2) *on the open strip* $x \in (r_1, r_2)$*, with a continuous function* $\Psi(y)$ *satisfying* $\Psi(0) = 0$*,* $y\Psi(y) > 0$ *for* $y \neq 0$*. The functions* $g(x)$ *and* $F(x)$ *are arbitrarily many times differentiable. Suppose there exists a unique* $x_g \in (r_1, r_2)$ *such that* $(x - x_g)g(x) > 0$ *for* $x \neq x_g$*,* $g'(x_g) > 0$*. Furthermore* $F'(0) = f(0) = 0$*. Then the first two focal values of the weak focus at* $x = x_g$*,* $y = 0$ *are proportional in sign to:*

$$V_1 \sim f''(x_g)g'(x_g) - f'(x_g)g''(x_g), \tag{2.10}$$

$$\begin{aligned} V_2 \sim\; & 10f'(x_g)g''(x_g)g'''(x_g) - 10f'''(x_g)g'(x_g)g''(x_g) \\ & +3f^{iv}(x_g)g'(x_g)^2 - 3f'(x_g)g'(x_g)g^{iv}(x_g). \end{aligned} \tag{2.11}$$

In (2.11) *the condition* $V_1 = 0$ *from* (2.10) *was used to simplify the expression.*

Proof For the convenience of the argument we assume that the singularity resides at $x = 0$ after a shift in the x-variable.

Consider $G(x_-) = G(x_+)$ with $x_- < 0 < x_+$, where $G(x) \equiv \int_0^x g(\bar{x})d\bar{x}$. In a small enough neighborhood of $x = 0$ the solutions to this equation can be written in the form $x_- = \alpha(x_+)$ where $\alpha(x_+) = -x_+ + \mathcal{O}(x_+^2)$. The expansion of $F(x_+) - F(\alpha(x_+)) = \sum_{i\geq 1} B_i(x_+)^i$ has coefficients B_i which will determine the order of the weak focus according to Theorem 2.5.1 in [12]. There it was shown that if there exists a $k \geq 1$ such that $B_j = 0$, $j = 0, \ldots, 2k$ and $B_{2k+1} < 0$ (> 0), then the origin is a stable (unstable) weak focus of order k.

Note that in [12] the definition of the Liénard system differs by a minus sign from system (2.2). Therefore the statement of the result from Theorem 2.5.1 [12] was changed by a minus sign to reflect this difference.

To obtain expressions for the signs of the focal value the leading term B_i in the expansion of $F(x_+) - F(\alpha(x_+))$ needs to be found. The easiest way to establish this seems to be by expanding the auxiliary function $L(h) = F(x_+(h)) - F(x_-(h))$, with $G(x_-(h)) = h$ and $G(x_+(h)) = h$, with $x_- < 0$, $x_+ > 0$, $h > 0$. Here h will also be small by construction when x_+ and x_- are small. By using the Lagrange inverse theorem the expansions of $x_-(h)$ and $x_+(h)$ can easily be obtained since we can write $G(x) = x^2 G_1(x)$ where $G_1(x) > 0$ for small enough x. The latter condition is satisfied because by assumption $g'(0) > 0$. The equation $G(x) = h$ can be written in the form $x\sqrt{G_1(x)} = \sqrt{h}$ for $x > 0$ and $x\sqrt{G_1(x)} = -\sqrt{h}$ for $x < 0$. The Lagrange inverse theorem then gives a formal expansion in the form:

$$x_+(h) = \sum_{n=1}^{\infty} \lambda_n \frac{\left(\sqrt{h}\right)^n}{n!} \tag{2.12}$$

and

$$x_-(h) = \sum_{n=1}^{\infty} (-1)^n \lambda_n \frac{\left(\sqrt{h}\right)^n}{n!}, \tag{2.13}$$

where λ_n is given by:

$$\lim_{x\to 0} \frac{d^{n-1}}{dx^{n-1}} \left[\frac{1}{\mu^n(x)} \right],$$

with $\mu(x) \equiv \sqrt{G_1(x)}$.

First we formally expand the function $F(x)$ and $G(x)$:

$$\begin{aligned} F(x) &= \frac{1}{2}F''(0)x^2 + \frac{1}{6}F'''(0)x^3 + \frac{1}{24}F^{iv}(0)x^4 + \frac{1}{120}F^{v}(0)x^5 + o(x^5), \\ G(x) &= \frac{1}{2}G''(0)x^2 + \frac{1}{6}G'''(0)x^3 + \frac{1}{24}G^{iv}(0)x^4 + \frac{1}{120}G^{v}(0)x^5 + o(x^5). \end{aligned}$$

The function $\mu(x) = \sqrt{G_1(x)}$ becomes:

$$\mu(x) = \sqrt{\frac{1}{2}G''(0) + \frac{1}{6}G'''(0)x + \frac{1}{24}G^{iv}(0)x^2 + \frac{1}{120}G^{v}(0)x^3 + o(x^3)}.$$

According to the Lagrange inverse theorem the expansion of $x_+(h)$ becomes:

$$x_+(h) = \lambda_1 h^{\frac{1}{2}} + \frac{1}{2}\lambda_2 h + \frac{1}{6}\lambda_3 h^{\frac{3}{2}} + \frac{1}{24}\lambda_4 h^2 + o(h^2),$$

with

$$\begin{aligned} \lambda_1 &= \frac{1}{30}\frac{\sqrt{1800}}{\sqrt{G''(0)}}, \\ \lambda_2 &= -\frac{2}{3}\frac{G'''(0)}{G''(0)^2}, \end{aligned}$$

$$\lambda_3 = -\frac{1}{180}\sqrt{1800}\frac{-5G'''(0)^2 + 3G^{iv}(0)G''(0)}{G''(0)^{\frac{7}{2}}},$$
$$\lambda_4 = -\frac{4}{45}\frac{40G'''(0)^3 - 45G'''(0)G^{iv}(0)G''(0) + 9G^{v}(0)G''(0)^2}{G''(0)^5}.$$

The expansion for $x_-(h)$ is obtained by replacing $h^{\frac{1}{2}}$ in the expansion for $x_+(h)$ by $-h^{\frac{1}{2}}$.

Next these two expansions are used in the expression $F(x_+(h)) - F(x_-(h))$. The lowest order term in h becomes:

$$\frac{2\sqrt{2}}{3}\frac{F'''(0)G''(0) - F''(0)G'''(0)}{G''(0)^{\frac{5}{2}}}h^{\frac{3}{2}}.$$

Since $h = G(x_+) = \frac{1}{2}G''(0)x_+^2 + o(x_+^2)$, in terms of the variable x_+ used in the expansion of Theorem 2.5.1 of [12] the above expansion term becomes:

$$\frac{1}{3}\frac{F'''(0)G''(0) - F''(0)G'''(0)}{G''(0)}x_+^3.$$

According to [12] this implies that if $F'''(0)G''(0) - F''(0)G'''(0) > 0$ (< 0), the weak focus is unstable (stable) of order one. Since $F''(0) = f'(0)$ etcetera, the first focal value is proportional in sign to:

$$f''(0)g'(0) - f'(0)g''(0) > 0.$$

If we impose the condition that $-F'''(0)G''(0) + F''(0)G'''(0) = 0$, i.e. write e.g. $f''(0) = \frac{f'(0)g''(0)}{g'(0)}$, then we need to look at the next non-vanishing term in the expansion. In $F(x_+(h)) - F(x_-(h))$ this becomes:

$$\frac{\sqrt{2}}{45}\frac{3F^{v}(0)G''(0)^2 - 3G''(0)F''(0)G^{v}(0) - 10G''(0)F^{iv}(0)G'''(0) + 10F''(0)G'''(0)G^{iv}(0)}{G''(0)^{\frac{9}{2}}}h^{\frac{5}{2}}.$$

In terms of the variable x_+ used in the expansion of Theorem 2.5.1 of [11] the above expansion term becomes:

$$\frac{1}{180}\frac{3F^{v}(0)G''(0)^2 - 3G''(0)F''(0)G^{v}(0) - 10G''(0)F^{iv}(0)G'''(0) + 10F''(0)G'''(0)G^{iv}(0)}{G''(0)^2}x_+^5.$$

The factor in the numerator determines the sign of the leading coefficient. From this the expression in the proposition for V_2 follows. □

2.5 Weak Focus of Order 1

Using the expressions of the previous section and the functions in (2.3) we find that the first focal value for system (1.2) at the singularity at $x = x_g$, $y = y_g$ under condition

(2.9) has the same sign as:

$$V_1 \sim \frac{-\phi(k-1)^2 a_1 S(x_g,k,a_1)}{k(a_1x_g^2-2a_1x_g+a_1-1)^3(k-x_g)(k-2x_g)^2}, \tag{2.14}$$

where $S(x_g,k,a_1) = 4a_1k^2x_g^2 - 5a_1kx_g^3 - 4a_1k^2x_g + 4a_1kx_g^2 + 2a_1x_g^3 + 2a_1k^2 - 3a_1kx_g - 2k^2 + 3kx_g$.

The sign in this expression is determined by the numerator factors $-a_1S(x_g,k,a_1)$ and denominator factors $(a_1x_g^2-2a_1x_g+a_1-1)^3(k-x_g)$. All other terms are obviously positive. The factor $(k-x_g)$ is positive because $x_g < 1 < k$.

The factor a_1 in the numerator corresponds to the center case C discussed in Lemma 2.6. The term $a_1x_g^2-2a_1x_g+a_1-1$ corresponds to the intersection of l_{wf} with the boundary of region W_1. It follows that it has fixed sign in W_1 and its sign is negative, because for $a_1=0$ it is equal to -1.

The expression $S(x_g,k,a_1)$ is a linear function in the parameter a_1. Setting V_1 equal to zero and solving for a_1 leads to:

$$a_1^{wf_2} = \frac{k(2k-3x_g)}{4k^2x_g^2-5kx_g^3-4k^2x_g+4kx_g^2+2x_g^3+2k^2-3kx_g} \equiv \frac{k(2k-3x_g)}{R(x_g,k)}, \tag{2.15}$$

and a_0^{wf} becomes after substituting (2.15) into (2.9)

$$a_0^{wf_2} = \frac{-4k^2x_g+4kx_g^2+2x_g^3+2k^2-3kx_g}{4k^2x_g^2-5kx_g^3-4k^2x_g+4kx_g^2+2x_g^3+2k^2-3kx_g} \equiv \frac{-4k^2x_g+4kx_g^2+2x_g^3+2k^2-3kx_g}{R(x_g,k)}. \tag{2.16}$$

Lemma 2.8 *The function $R(x_g,k)$ in the denominators of (2.15) and (2.16) is positive for the parameter values for which a weak focus occurs as described in Lemma 2.3.*

Proof $R(x_g,k)$ viewed as a cubic function of x_g for fixed k has a discriminant equal to $-4(64k^3-112k^2+99k-54)(k-1)^2k^3$. By writing $k=u+2$, the critical factor in this expression $64k^3-112k^2+99k-54$ becomes $64u^3+272u^2+419u+208$ which is positive for positive u, i.e. for $k>2$. It implies that for $k>2$ the function $R(x_g,k)$ has one real zero. Since we have $R(0,k)=2k^2>0$ and $R(1,k)=2(k-1)^2>0$, for each $x_g\in(0,1)$ the expression $R(x_g,k)>0$. □

Lemma 2.9 *In (2.15) $a_1^{wf_2} > 1$ for the parameter values for which a weak focus occurs as described in Lemma 2.3.*

Proof Substitution of (2.15) into $a_1^{wf_2}-1$ gives:

$$a_1^{wf_2} - 1 = \frac{x_g\big(-4k^2x_g+5kx_g^2+4k^2-4kx_g-2x_g^2\big)}{R(x_g,k)} \equiv \frac{x_gT(x_g,k)}{R(x_g,k)}.$$

The numerator contains a quadratic factor $T(x_g, k)$ which is quadratic in x_g. At $x_g = 0$, $T(0, k) = 4k^2 > 0$ and $\frac{\partial T(x_g,k)|}{\partial x_g}|_{x_g=0} = -4k(k+1) < 0$. At the end point $x_g = 1$, $T(1, k) = k - 2 > 0$ and $\frac{\partial T(x_g,k)|}{\partial x_g}|_{x_g=1} = -4k^2 + 6k - 4 < 0$, implying that $T(x_g, k) > 0$ for $k > 2$ and $0 < x_g < 1$. Concluding we find that $a_1^{wf2} - 1 > 0$ for the parameter values satisfying (2.15). □

For the original expression (2.14) we find that the denominator is negative and the sign of the numerator is determined by the expression $-a_1(R(x_g, k)a_1 + R_2(x_g, k))$. Since we have just shown that $R(x_g, k) > 0$ and that the zero of $R(x_g, k)a_1 + R_2(x_g, k)$ occurs for positive a_1, it follows that:

Lemma 2.10 *In system* (1.2) *the singularity at* $x = x_g$, $y = y_g$ *is a weak focus under condition* (2.9) *with additional conditions* $k \geq 2$, $0 < x_g < 1$. *It is a first order weak focus which is unstable for* $a_1 < 0$ *and* $a_1 > a_1^{wf_2}$ *and which is stable for* $0 < a_1 < a_1^{wf_2}$, *where* $a_1^{wf_2}$ *satisfies* (2.15).

The lemma does not state if a_0^{wf}, $a_1^{wf_2}$ corresponds to a point inside W_1. To resolve this we observe that the parametrized curve defined by (2.15) and (2.16) lies inside part of the ellipse formed by $D = 0$, the upper part of the boundary of W_1 according to Lemma 2.9. It is easy to check that the parametrized curve intersects $D = 0$, exactly for $x_g = \frac{k}{2} > 1$, i.e. not for a value of x_g acceptable in our model.

2.6 Weak Focus of Order 2

Finally we need to establish the stability of the weak focus when the first focal value V_1 vanishes. Using Maple to calculate the next focal value V_2 under the condition that V_1 vanishes, we get that the sign of V_2 is determined by the following expression after substitution of (2.15) and (2.16) into (2.11):

$$V_2 \sim -\frac{3\phi k(2k - 3x_g)R(x_g, k)\left(4k^3 - 10k^2 x_g + 6kx_g^2 + x_g^3\right)}{8x_g^5(k - x_g)^4(k - 2x_g)^3}.$$

The factor $R(x_g, k)$, as we have shown before, is positive. The factors $(k - 2x_g)^{-3}$ and $(2k - 3x_g)$ are positive for $k > 2$, $0 < x_g < 1$. Finally, in a similar way as was done for other factors before, we can prove that $4k^3 - 10k^2 x_g + 6kx_g^2 + x_g^3 > 0$. It follows that $V_2 < 0$, i.e. the weak focus is stable and of second order. Summarizing:

Lemma 2.11 *In system* (1.2) *the singularity A at* $x = x_g$, $y = y_g$ *is a center for* ($a_0 = \frac{1}{1-k}$, $a_1 = 0$) *and a stable weak focus of second order for the parameter values* $a_0^{wf_2}$ *and* $a_1^{wf_2}$ *defined by* (2.15) *and* (2.16) *respectively. The curve corresponding to a weak focus of order 2 in the* (a_0, a_1) *plane, parametrized by* x_g, *starts for* $x_g = 0$ *at* $(a_0 = 1, a_1 = 1)$ *and remains inside* W_1 *with* $a_1^{wf_2} > 1$ *for* $0 < x_g < 1$. *The weak focus cannot be of an order higher than 2, except when the center of Lemma 2.6 appears.*

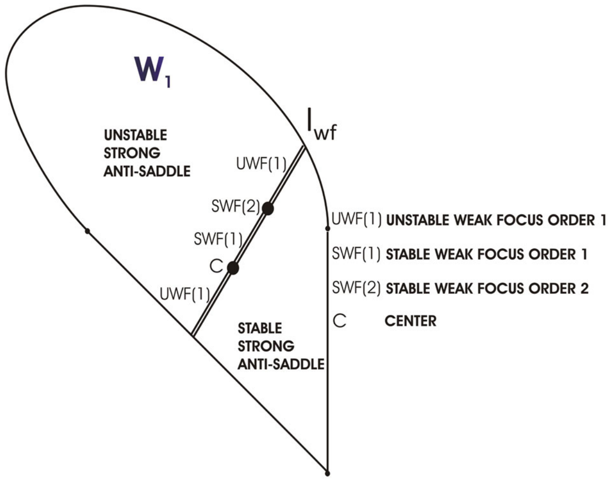


Fig. 2 Stability of the singularity in the first quadrant of system (1.2) in the (a_0, a_1) parameter plane for fixed ϕ, k, x_g

By combining the results of Lemmas 2.10 and 2.11 the stability of the singularity in the first quadrant is determined in the following way:

Proposition 2.12 *In system* (1.2) *the singularity A at* $x = x_g$, $y = y_g$ *is an anti-saddle in region* W_1 *(given in Definition* 1.2*) of the parameter space. For* $a_0 > a_0^{wf_1}(a_1)$ $(a_0 < a_0^{wf_1}(a_1))$ *the anti-saddle is strong and stable (unstable). For* $a_0 = \tau^{wf} a_1 + \frac{1}{1-k}$ *the singularity is a weak singularity:*

- *a stable weak focus of order 1 for* $0 < a_1 < a_1^{wf_2}$ *where* $a_1^{wf_2}$ *is given by expression* (2.15)*, an unstable weak focus of order 1 for* $a_1 < 0$ *or* $a_1 > a_1^{wf_2}$.
- *a center for* $a_1 = 0$.
- *a stable weak focus of order 2 for* $a_0 = a_0^{wf_2} = a_0^{wf}(a_1^{wf_2})$, $a_1 = a_1^{wf_2}$. *The parameter pair* $(a_0^{wf_2}, a_1^{wf_2})$ *lies inside* W_1 *with* $a_1^{wf_2} > 1$.

The different possibilities are shown in Fig. 2.

3 Global Properties

To facilitate the analysis we introduce some useful lemmas for showing the existence of limit cycles in (1.2).

As was shown in [16] system (1.1) is bounded under the conditions which we imposed on $h(x)$, $p(x)$ and δ in Definitions 1.1, 1.2 and 1.3, i.e. any solution starting in the first quadrant of the phase plane will enter (and stay in) a bounded region.

Lemma 3.1 *System* (1.2) *is a bounded system. The line tangent* l_{sep} *to the critical direction of the saddle singularity at* $(x = k, y = 0)$*, entering the first quadrant, forms together with the lines* $x = 0$ *and* $y = 0$ *a triangle T with the property that*

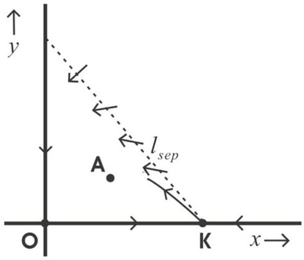


Fig. 3 Region of inflow for the bounded system (1.2)

all solutions starting in the first quadrant on l_{sep} will enter T and remain inside. See Fig. 3.

A consequence of this lemma is that limit cycles of the system will lie inside the triangle T. Since the most right point of T is the singularity at $(x = k, y = 0)$, limit cycles cannot cross the vertical line $x = k$. Moreover it follows immediately that if the system has exactly one singularity A—an anti-saddle—in the first quadrant, then according to the Poincaré–Bendixson theorem the singularity A is surrounded by (at least) one stable limit cycle if the singularity is unstable. Since the anti-saddle A is unstable for parameter values in a significant part of region W_1, we can easily indicate a region in the parameter space where limit cycles must exist.

Lemma 3.2 *In system (1.2) an odd (even) number of limit cycles surround the singularity A if it is unstable (stable).*

In the lemma the situation with 0 limit cycles is categorized as "even".

Since we have established in Proposition 2.12 for which parameter values the singularity is A unstable, we can immediately show the existence of (at least) one limit cycle for the following parameter values in region W_1:

Proposition 3.3 *In system (1.2) an odd number (counting multiplicities) of limit cycles surround the singularity A (as defined in Lemma 2.1) if and only if one of the following conditions is satisfied:*

- $a_0 < \tau^{wf} a_1 + \frac{1}{1-k}$.
- $a_0 = \tau^{wf} a_1 + \frac{1}{1-k}$ *with* $a_1 < 0 \vee a_1 > a_1^{wf_2}$.

with τ^{wf} defined in Lemma 2.3.

The conclusions of the proposition are indicated in Fig. 4.

For convenience of discussion in the following we will distinguish three types of limit cycles:

Definition 3.4 A "small" limit cycle in the strip $0 < x < 1$ is referred to as a limit cycle of type I. For the unique zero of $f(x)$ at $x = \frac{k}{2}$ for $x > 1$ we write x_f. A "medium" limit cycle intersecting $x = 1$ but not $x = x_f$ is referred to as a limit cycle of type II. A "big" limit cycle intersecting the line $x = x_f$ is referred to as a cycle of type III. See Fig. 5.

For system (1.2) it means that the three types correspond to the different ways a limit cycle can cross the vertical lines where $f(x)$, i.e. the divergence of the Liénard system,

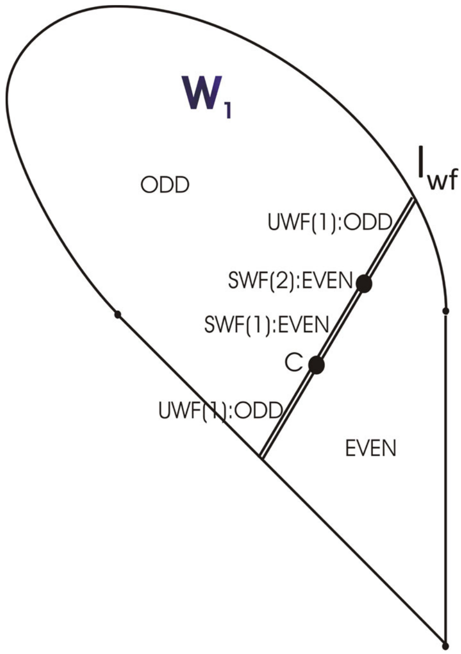


Fig. 4 Existence of an odd or even number of limit cycles due to the Poincaré–Bendixson theorem in (1.2)

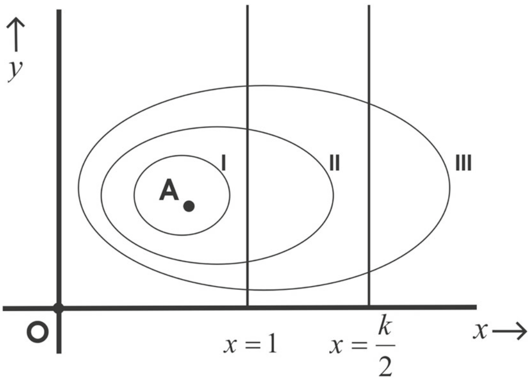


Fig. 5 Definition of cycle types according to Definition 3.4

changes sign (i.e. the function $f(x)$). However, this is not a strict definition because not for all parameter values $f(x)$ changes sign in the interval $0 < x < 1$, and not always at $x = 1$. There is always a change of sign at $x = x_f$ though.

4 Andronov–Hopf Bifurcation

With the results of the stability analysis of the previous sections we can indicate regions in the parameter space where small-amplitude limit cycles bifurcate according to the Andronov–Hopf bifurcation near the weak-focus cases. This gives a natural lower bound of 2 for the number of limit cycles in (1.2).

Using the results of Proposition 2.12 about the stability of the anti-saddle we can easily conclude the following standard bifurcation mechanisms of small-amplitude limit cycles. The results are deduced by taking into account that any change in stability of the focus will lead to the creation of a small-amplitude limit cycle.

The first result is the perturbation from the weak focus case into the strong focus case.

Lemma 4.1 *Consider a small perturbation of a weak focus at a point* (a_0^0, a_1^0) *in* W_1 *for fixed* k, ϕ, x_g, *with* $a_0^0 = \tau^{wf} a_1^0 + \frac{1}{1-k}$, *(with* τ^{wf} *defined in Lemma 2.3) in system (1.2) of the form:*

$$a_0 = a_0^0 + \lambda\epsilon, a_1 = a_1^0 + \mu\epsilon, 0 < \epsilon \ll 1.$$

Here λ *and* μ *define the direction of bifurcation in the* a_0, a_1 *parameter space for fixed* k, ϕ, x_g. *Define* $Z \equiv \lambda - \tau^{wf}\mu$.

- *A stable small-amplitude limit cycle is created for* $Z < 0$, *if* $0 < a_1 \leq a_1^{wf_2}$.
- *An unstable small-amplitude limit cycle is created for* $Z > 0$, *if* $a_1 < 0$ *or* $a_1 > a_1^{wf_2}$.
- *No small amplitude limit cycles are created for the other cases with* $Z \neq 0$.

Note that in this lemma the stable second order weak focus case $a_1 = a_1^{wf_2}$ is included. The center case C does not generate a small-amplitude limit cycle according to these bifurcation directions in parameter space.

The case not discussed in Lemma 4.1 is $Z = 0$, which is a bifurcation in the direction of the weak focus line l_{wf}. In the case of a weak focus of order 1 no limit cycles are created in this direction, because the stability of the focus does not change.

The center case bifurcation does not generate a small-amplitude limit cycle in that direction either except perhaps for some higher order bifurcations. However, the results of the next sections using a uniqueness theorem for limit cycles in Liénard systems will show that nothing special happens in that case either.

Lemma 4.2 *Consider a small perturbation from a second order weak focus at a point* (a_0^0, a_1^0) *in* W_1 *for fixed* k, ϕ, x_g, *with* $a_0 = a_0^{wf_2} + \tau^{wf}\epsilon$, $a_1 = a_1^{wf_2} + \epsilon + \sigma\epsilon^2$, $0 < \epsilon \ll 1$ *in system (1.2):*

- *For* $\sigma = 0$ *a stable small-amplitude limit cycle is created surrounding a first order weak focus.*
- *For* $0 < \sigma < \sigma^*$ *two small-amplitude limit cycles are created surrounding a strong stable focus. The inner (outer) limit cycle is unstable (stable).*
- *For* $\sigma = \sigma^*$ *a semi-stable small-amplitude limit cycle is created surrounding a strong focus.*
- *For other values of* σ *no small-amplitude limit cycles are created.*

with τ^{wf} *defined in Lemma 2.3.*

Proof For the weak focus of order 2, the perturbation along $Z = 0$ takes the form $a_0 = a_0^{wf_2} + \tau^{wf}\epsilon$, $a_1 = a_1^{wf_2} + \epsilon + \sigma\epsilon^2$, $0 < \epsilon \ll 1$ because we need to consider

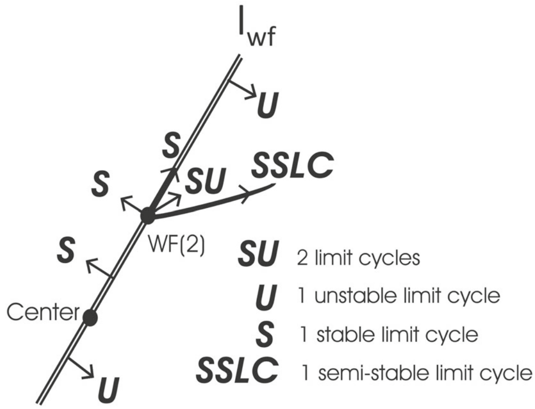


Fig. 6 Andronov–Hopf bifurcation of small-amplitude limit cycles in (1.2)

higher order perturbations. For $\sigma = 0$ a stable small-amplitude limit cycle is created surrounding an unstable weak focus of order 1, because we take $\epsilon > 0$: in that situation the singularity changes from a stable second order weak focus into an unstable first order weak focus. According to Lemma 4.1 an additional unstable limit cycle can be created by perturbing the unstable first order weak focus if $Z > 0$: two limit cycles appear by perturbing the second order weak focus in an appropriate direction, i.e. taking $\sigma > 0$, ensuring that $Z > 0$. It is well-known that in such a bifurcation mechanism a semi-stable limit cycle will occur if σ is taken sufficiently large. □

The results are shown in Fig. 6.

Summarizing the Andronov–Hopf bifurcation results:

Proposition 4.3 *At most two small-amplitude limit cycles can be created from a weak focus in system* (1.2) *surrounding singularity A in the first quadrant. From the point* $a_0 = a_0^{wf_2}$, $a_1 = a_1^{wf_2}$ *where the system has a second order weak focus a bifurcation curve emerges in parameter space corresponding to a semi-stable limit cycle. Moreover, there exist directions such that two small-amplitude limit cycles occur. All these small-amplitude limit cycles are of type I as in Definition* 3.4, *because the cycles do not cross* $x = 1$.

5 Center Bifurcation

For the bifurcation from the center case we consider different possibilities. First we check the existence of limit cycles for the center case itself, then we check the perturbation of limit cycles from the period annulus surrounding the center singularity.

5.1 Existence of Limit Cycle Surrounding the Center

For the parameter values $a_0 = \frac{1}{1-k}$, $a_1 = 0$ system (1.2) has a center at $x = x_g$, $y = y_g$ in the phase plane according to Lemma 2.6. This means that in the strip $0 \le x \le 1$ the singularity is surrounded by a continuous annulus of periodic orbits, the so-called period annulus. The period annulus is bounded on the outside by the periodic orbit which is tangent to the line $x = 1$. We denote this periodic orbit by Γ_{h^*}. The period annulus will not continue beyond $x = 1$. Solutions near the outer boundary of the annulus Γ_{h^*}, i.e. crossing the line $x = 1$, will spiral outwards or inwards depending on the properties of the function $h(x)$ and $p(x)$ for $x = 1 + \epsilon$. We consider a more general situation (but not necessarily the most general center case of the Gause system) when this can occur, i.e. when system (1.1) has the restriction:

$$h(x) = cp(x),\ 0 \le x \le 1. \tag{5.1}$$

Since we impose continuity of the functions at $x = 1$, necessarily $c = \frac{h(1)}{p(1)}$. No further restrictions on $p(x)$ are imposed for $x > 1$. System (1.2) is a special case with $p(x) = 1$ for $x > 1$. The outer stability of the period annulus is determined in the following way.

Lemma 5.1 *For system* (1.1) *with* (5.1) *and* $\gamma = 1$*, the period annulus defined for* $0 \le x \le 1$ *surrounding the center singularity* C *at* $x = x_g$, $y = y_g$*, is stable (unstable) on the outside if* $h(x)p(1) < h(1)p(x)$ $(h(x)p(1) > h(1)p(x))$ *for* $x \in (1, 1 + \epsilon)$ *where* $0 < \epsilon \ll 1$.

Proof For the proof we compare two systems. The first system S_1 is the system for which we want to determine the outer stability of the period annulus, i.e. system (1.1) with conditions (5.1), $\gamma = 1$ and $p'(x) > 0$ on the interval $0 \le x \le 1$. For $x > 1$ we leave $p(x)$ in a general form and impose only that $p(x)$ is continuous, i.e. we impose that $\lim_{x\downarrow 1} p(x) = 1$. The second system S_2 is (1.1) with $h(x) = cp(x)$, $p'(x) > 0$ for all $x > 0$, i.e. the same system as S_1 but with the extra condition on $p(x)$ imposed on $x > 1$ as well.

Denote by $B_h(x, y) \equiv K_1(y) + K_2(x) = h$, where $K_1(y) \equiv \int_{y_g}^{y} \frac{\tilde{y}-c}{\tilde{y}} d\tilde{y}$, $K_2(x) \equiv \int_{x_g}^{x} \frac{p(\tilde{x})-\delta}{p(\tilde{x})} d\tilde{x}$, the solution of system S_2. For $0 < h < h^*$ the closed curves $B_h(x, y)$ lie in the strip $0 \le x \le 1$, for $h = h^*$ the curve is tangent to $x = 1$ and for $h > h^*$ the closed curves intersect $x = 1$.

In the strip $0 \le x \le 1$ the vector field of S_1 is tangent to $B_h(x, y)$ by construction since they are solutions to the same system.

The vector field of system S_1 is not necessarily tangent to the vector field of system S_2 for $x > 1$. A simple calculation shows that for the vector field of system S_1 we have $\frac{dB_h(x,y)}{dt} = (1 - \frac{p(x_g)}{p(x)})(h(x) - cp(x))$ on the strip $x \in (1, 1 + \epsilon)$. Here the factor $(1 - \frac{p(x_g)}{p(x)})$ is positive. It follows that the stability of the outer cycle tangent to $x = 1$ is determined by the sign of $h(x) - cp(x)$ for $x \in (1, 1 + \epsilon)$. Since $c = \frac{h(1)}{p(1)}$ the conclusion of the lemma follows. □

Since for system (1.2) $h'(x) > 0$ and $p'(x) = 0$ for $x = 1 + \epsilon$, it follows that $h(1 + \epsilon)p(1) > h(1)p(1 + \epsilon)$ and the previous lemma shows:

Lemma 5.2 *Consider the center case for system* (1.2) *with* $a_0 = \frac{1}{1-k}$, $a_1 = 0$*. The period annulus bounded by the periodic orbit* Γ_{h^*}*, which is tangent to the vertical line* $x = 1$*, is unstable on the outside.*

In the case of an unstable annulus we can apply the Poincaré–Bendixson theorem to show the existence of a limit cycle in an annular region without singularities because (1.2) is a bounded system according to Lemma 3.1.

Lemma 5.3 *Consider the center case for system* (1.2) *with* $a_0 = \frac{1}{1-k}$, $a_1 = 0$. *The period annulus is surrounded by (at least) one stable limit cycle. According to Definition* 3.4 *this is a limit cycle of type III, i.e. a big cycle intersecting the line* $x = x_f$.

Proof It follows from Lemma 5.2 that the annulus bounded by the cycle tangent to $x = 1$ is unstable on the outside. The Poincaré–Bendixson theorem then states that there is an odd number of limit cycles outside this cycle not crossing the line $x = k$. These limit cycles are of type II or type III. However, the type II cycles can be excluded in the following way. The solutions of the system S_2 introduced in the proof of Lemma 5.1 act as a Lyapunov-function for an open region of system S_1. The quantity $\frac{dB_h(x,y)}{dt} = (1 - \frac{p(x_g)}{p(x)})(h(x) - cp(x))$ in the proof of Lemma 5.1 explicitly becomes (for system (1.2) with $a_0 = \frac{1}{1-k}, a_1 = 0$): $\frac{dB_h(x,y)}{dt} = (1 - p(x_g))\frac{\phi}{k}(x-1)(k-1-x)$. This expression has fixed sign for $x < k - 1$. Since we imposed that $k > 2$ it follows that $k - 1 > \frac{k}{2}$. Therefore limit cycles in the system have to cross the line $x = \frac{k}{2}$ showing that limit cycles can only be of type III. Another way to see this would be to consider the Liénard form of the system. The value $x = k - 1$ would correspond to a zero of $F(x)$. Limit cycles in such a system would have to cross the line $x = x_F$ where $F(x_F) = 0$. □

In [15] a condition was derived for the uniqueness of the limit cycle in the situation of Lemma 5.3:

Lemma 5.4 [15] *Consider the generalized Liénard system* (2.2) *and let* $F(x)$, $g(x)$ *be continuous, piecewise differentiable functions on the open interval* (r_1, r_2), *and let* $\Psi(y)$ *be a continuously differentiable function on* $\mathbb{R}$ *such that*

(i) *there exists* $x_g \in (r_1, r_2)$ *such that* $(x - x_g)g(x) > 0$ *for* $x \neq x_g$,
(ii) $\Psi(y)$ *is monotonically increasing,* $\Psi(0) = 0$,
(iii) $F(x) \equiv 0$ *for* $r_1 \leq x \leq x_0$, *with* $x_g < x_0$,
(iv) *there exists an* $x_F > x_0$ *such that* $F(x_F) = 0$,
(v) $F(x) > 0$ *for* $x_0 < x < x_F$,
(vi) $f(x) < 0$ *for* $x_F \leq x \leq r_2$ *where the function* $f(x)$ *is defined by* $\frac{dF(x)}{dx}$, *then in the strip* $r_1 < x < r_2$ *system* (2.2) *has at most one limit cycle, which is stable and hyperbolic if it exists.*

This lemma can be applied to the center case of system (1.2):

Proposition 5.5 *Consider the center case for system* (1.2) *with* $a_0 = \frac{1}{1-k}$, $a_1 = 0$. *The period annulus is surrounded by exactly one hyperbolic stable limit cycle. This limit cycle is of type III.*

Proof In the case of system (1.2) we take the functions as defined in (2.3) and (2.4). The functions in Lemma 5.4 at the center case C satisfy: $r_1 = 0$, $r_2 = k$, $\Psi(y) = e^y - 1$, $x_0 = 1$, which verifies conditions (i), (ii), (iii). For $x > 1$: $F(x) = \frac{\phi x(k-x)}{k} - \frac{\phi(k-1)}{k}$, $g(x) = 1 - p(x_g)$, where we used that $p(x) = 1$ for $x > 1$. It follows that $x_F = k - 1$

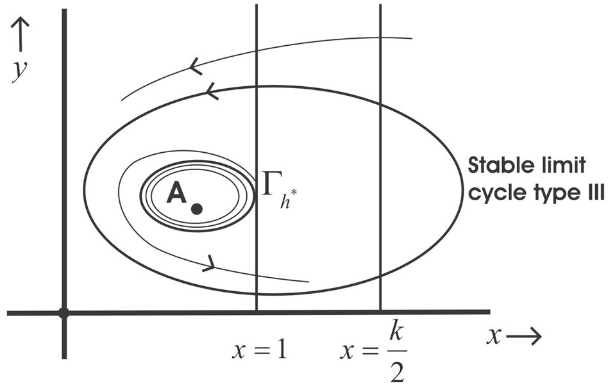


Fig. 7 Center case in (1.2) with period annulus surrounded by stable limit cycle according to Proposition 5.5

and $F(x) > 0$ for $1 < x < k - 1$ verifying conditions (iv) and (v). Moreover $f(x) = \frac{\phi(k-2x)}{k} < 0$ for $x > x_F$ verifying condition (vi). Together with Lemma 5.3 showing the existence of the limit cyle, the proposition is proved. The limit cycle has to cross the line $x = x_F > \frac{k}{2}$ and therefore it is a type III cycle as defined in Definition 3.4. □

The results are shown in Fig. 7.

5.2 Perturbation of the Period Annulus

Next we consider a small neighborhood of the center case C in the parameter space by varying the two parameters a_0, a_1 for fixed ϕ, k, x_g. The perturbation takes the form $a_0 = \frac{1}{1-k} + \tau\epsilon$, $a_1 = \epsilon$, with $0 < |\epsilon| \ll 1$. The period annulus at the center case is surrounded by a hyperbolic limit cycle according to Lemma 5.3. By changing the parameters the hyperbolic cycle will persist. It implies that after perturbation a unique stable limit cycle of type III will exist surrounding the singularity.

Proposition 5.6 *Consider the perturbation* $a_0 = \frac{1}{1-k} + \tau\epsilon$, $a_1 = \epsilon$ *from the period annulus surrounding the center singularity of system* (1.2). *For* $0 < |\epsilon| \ll 1$ *a unique stable hyperbolic limit cycle exists of type III according to Definition* 3.4.

Any other perturbed limit cycle will be of type I or II. Limit cycles can be perturbed from the period annulus parameterized through the parameter h as indicated in equation (2.8).

A limit cycle perturbed from a cycle Γ_h with $0 < h < h^*$ will not cross the line $x = 1$ and is therefore a type I cycle.

A limit cycle perturbed from the center singularity itself is a small-amplitude cycle and is also of type I. This bifurcation we will not study because we are looking for lower bounds on the number of perturbed limit cycles and the center-Hopf bifurcation does not give a higher number than we already would expect.

The bifurcation from the other end point of the interval $h = h^*$ is complicated because on the inside of the cycle Γ_{h^*} the unperturbed system has a continuum of periodic orbits (i.e. the period annulus) while on the outside nearby solutions are not closed but spiral outwards. The Poincaré-map near Γ_{h^*} is non-analytical for the unperturbed system. Therefore it is difficult to study its bifurcation. Nevertheless by using some simple stability arguments we can draw some conclusions about this bifurcation. The perturbed cycle(s) in this case could be of type I or II. It is not clear in advance which possibilities can be realized.

First we study the perturbation from the period annulus for $0 < h < h^*$. The occurrence of limit cycles after perturbation is governed by the zeroes of the so-called Pontyragin-integrals (see [1]). Consider the integral of the perturbed divergence of the vector field over the interior of an unperturbed orbit Γ_{h^0}, denoted by $I(h)$. If $I(h_0) = 0$, $\frac{dI(h)}{dh}|_{h=h_0} \neq 0$, then a unique hyperbolic limit cycle is created from Γ_{h^0}. In general the analysis of such integrals is quite complicated but in our case we can combine a uniqueness theorem for Liénard systems with a continuity argument to arrive at the conclusions we need.

To set up the Pontryagin-integrals for system (1.2) the Liénard form (2.3) is most suitable. By writing the perturbation in terms of one small parameter ϵ in the form $a_0 = \frac{1}{1-k} + \tau\epsilon$, $a_1 = \epsilon$, the divergence of the vector field $f(x)$ in (2.5) can be expanded in terms of ϵ:

$$f(x) = \epsilon(\tau U_0(x) + U_1(x)) + \mathcal{O}(\epsilon^2) \equiv -\epsilon \left(\tau \frac{\phi(k-1)^3}{k(k-x)^2} + \frac{\phi(-x^2+2kx-k)(k-1)^2}{k(k-x)^2} \right) + \mathcal{O}(\epsilon^2). \quad (5.2)$$

The corresponding Pontryagin-integrals take the form:

$$I(h) = \tau \iint_{Int(\Gamma_h)} U_0(x)dxdy + \iint_{Int(\Gamma_h)} U_1(x)dxdy \equiv \tau I_0(h) + I_1(h). \quad (5.3)$$

A necessary condition for the creation of a limit cycle from an orbit Γ_h is that $I(h) = 0$. Since the integrand $U_0(x)$ of $I_0(h)$ has fixed sign, i.e. negative, the integral $I_0(h)$ itself cannot be zero and the condition $I(h) = 0$ can be rewritten as:

$$\tau(h) = -\frac{I_1(h)}{I_0(h)}, \quad (5.4)$$

which is a continuous well-defined function of h for every $h \in (0, h^*)$. The quotient $\tau(h)$ corresponds to a bifurcation direction in the (a_0, a_1) plane. The study of this function is not easy in general, but we can apply a uniqueness theorem for limit cycles to prove that it is monotonic in h. The direction of the weak focus l_{wf} corresponds to the value τ_{wf} which was discussed in the previous section, i.e. $\lim_{h\downarrow 0} \tau(h) = \tau_{wf}$.

The essence of our analysis is that the Pontryagin-integrals are complicated and difficult to be analyzed directly. Therefore we apply an alternative technique to avoid the direct study of the integrals by applying a uniqueness theorem for Liénard systems. This is similar in spirit to what was done in [17]. For other perturbations for more general systems than (1.2), more limit cycles could be expected and currently no alternative techniques exist avoiding the direct study of the zeroes of the Pontryagin integrals.

Proposition 5.7 *Consider the perturbation* $a_0 = \frac{1}{1-k} + \tau\epsilon$, $a_1 = \epsilon$ *from the period annulus surrounding the center singularity of system* (1.2)*. No limit cycles are created for* $\tau \geq \tau_{wf}$ *or* $\tau \leq -1$*. For* $-1 < \tau < \tau_{wf}$ *at most one limit cycle is created which is hyperbolic if it exists. The cycle if it exists is of type I according to Definition* 3.4.

Proof For small ϵ the divergence of the perturbed vector field $f(x)$ to first order according to (5.2) is $\epsilon(\tau U_0(x) + U_1(x))$. It is not difficult to see that this is a quadratic expression in x which has fixed sign on the interval $0 < x < 1$ if $\tau \geq \frac{k}{k-1} > 0$ or $\tau \leq -1$. In that case no limit cycles will occur.

It leaves to prove that for $\tau_{wf} \leq \tau < \frac{k}{k-1}$ no limit cycles can occur and that for $-1 < \tau < \tau_{wf}$ at most one limit cycle can occur. A critical element in the following is the function $\frac{f(x)}{g(x)}$. Using the same perturbation form of the parameters $a_0 = \frac{1}{1-k} + \tau\epsilon$, $a_1 = \epsilon$ to first order in ϵ this becomes:

$$\frac{f(x)}{g(x)} = \frac{1}{(x - x_g)} \left[\epsilon \left(\frac{\phi(k-1)^2 x(\tau(1-k) + (x^2 - 2kx + k))}{k(k-x)(k - x_g - x)} \right) + \mathcal{O}(\epsilon^2) \right]. \tag{5.5}$$

For small ϵ the first order term in (5.5) determines the behaviour of the quotient $\frac{f(x)}{g(x)}$ on the interval $0 \leq x \leq 1$. This is justified because the structure for this quotient for all admissible parameters in system (1.2) is given by (with the use of (2.2) and (2.5)):

$$\frac{f(x)}{g(x)} \sim \frac{1}{(x - x_g)} \left[\frac{\phi x f_2(x)}{(1 + (x-1)(a_0 + a_1 x)) g_2(x)} \right] = \frac{1}{(x - x_g)} Y(x), \tag{5.6}$$

where $g_2(x)$ is a quadratic function which has fixed sign in $0 \leq x \leq 1$. The factor $(1 + (x-1)(a_0 + a_1 x)$ does not have zeroes on that interval either. Since $Y(x)$ will remain bounded in $Y(x)$ for $0 < x < 1$, the higher order contribution will not change the behaviour of the first order contribution.

In the following we consider the first order term in ϵ of (5.5) only and define the essential factor by:

$$T_1(x) \equiv \frac{x(x^2 - 2kx + k + \tau(1-k))}{(x-k)(x - x_g)(x - k + x_g)}. \tag{5.7}$$

In expression (5.7) we left out the positive constant factors $\frac{\phi(k-1)^2}{k}$ which are of no importance in the following arguments. To prove existence and uniqueness or non-existence of limit cycles after perturbation for cycles of type I on the interval $0 \leq x \leq 1$, we employ two standard theorems. The first is a standard non-existence theorem, which basically asks for a “gap” in the function $T_1(x)$ near $x = x_g$.

Lemma 5.8 *If the functions in* (2.2) *satisfy the following conditions:*

(i) $(x - x_g)g(x) > 0$ *for* $x \neq x_g$,
(ii) $(x - x_f)f(x) > 0$ *or* $(x - x_f)f(x) < 0$, $x \neq x_f$, $x_f > x_g$,
(iii) $\frac{d\psi(y)}{dy} > 0$, $\psi(0) = 0$,
(iv) $\exists c \in R$ *such that* $\frac{f(x)}{g(x)} = c$ *does not have a pair of solutions* (x_-, x_+) *with* $x_{min} < x_- < x_g < x_+ < x_{max}$, *then the system has no limit cycles surrounding the singularity at* $x = x_g$ *in the strip* $x_{min} < x < x_{max}$.

In the case where limit cycles can exist we will apply the well-known generalization of the so-called Zhang Zhifen theorem, see [18]:

Lemma 5.9 *If the functions in* (2.2) *satisfy the following conditions for* $x_{min} < x < x_{max}$:

(i) $(x - x_g)g(x) > 0$ *for* $x \neq x_g$,
(ii) $(x - x_f)f(x) > 0$ *or* $(x - x_f)f(x) < 0$, $x \neq x_f$, $x_f > x_g$,
(iii) $\frac{d\psi(y)}{dy} > 0$, $\psi(0) = 0$,
(iv) $\frac{f(x)}{g(x)}$ *is nondecreasing or nonincreasing in* $x < x_g$ *and* $x > x_f$, *then the system has at most one limit cycle surrounding the singularity at* $x = x_g$ *in the strip* $x_{min} < x < x_{max}$. *It is hyperbolic if it exists.*

These two lemmas in general can be combined to prove uniqueness of limit cycles for Liénard systems in many cases (see e.g. [17]). In Lemma 5.9 the stability of the limit cycle was left open. It will follow from the stability of the singularity, i.e. the sign of $f(x_g)$.

Conditions (i) and (iii) in both lemmas are clearly satisfied for system (2.2) and (2.3). As was shown before for $\tau \leq -1$ of $\tau \geq \frac{k}{k-1} > 0$ the function $f(x)$ has fixed sign and no limit cycles can occur due to the fact that the divergence of the vector field has fixed sign.

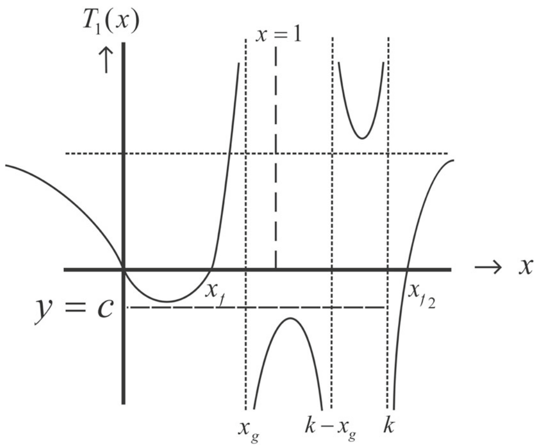
For $-1 < \tau < \tau_{wf}$ ($\tau \geq \tau_{wf}$) the function $f(x)$ has a zero x_f on the interval $0 < x < 1$ with $x_f < x_g$ ($x_f > x_g$) and condition (ii) is satisfied as well. The critical condition is therefore condition (iv) in both lemmas for which we will study $T_1(x)$ in (5.7).

The function $T_1(x)$ has easily verifiable properties. It has three vertical asymptotes at x_1, x_2, x_3: $0 < x_1 = x_g < 1$, $1 < x_2 = k - x_g < k$ and $x_3 = k > 1$. The horizontal asymptote for $x \to \pm\infty$ is $y = 1$. Each horizontal line $y = c$ can have at most three intersections with the graph of the function. If for some $y = c$ the graph has exactly three intersections then necessarily at all three points $T_1'(x) \neq 0$. With this information we will show now that always condition (iv) of Lemma 5.8 or condition (iv) of Lemma 5.9 is satisfied.

Case 1 $\tau_{wf} \leq \tau < \frac{k}{k-1}$. In this case $f(x)$ has two zeros. One satisfies $0 < x_f < x_g < 1$, the other $x_{f_2} > k$, i.e. to the right of the 3 vertical asymptotes, the largest of which is positioned at $x = k$. Since $T_1(x)$ has zeros at $x = 0$ and $x = x_f < x_g$, there must exist a local minimum at $0 < x^* < x_f$ with value $T_1(x^*) < 0$.

Consider the horizontal line $y = c = T_1(x^*)$. By construction it is tangent to the graph of $y = T_1(x)$ where it has a minimum. The equation $T_1(x) = c$ is a cubic

Fig. 8 Existence of a gap in the function $T_1(x)$ in (5.7) under the condition $\tau_{wf} \leq \tau < \frac{k}{k-1}$



equation in x with a double zero at $x = x^*$. Furthermore it has another solution for $x > k$ because $T_1(x)$ has a vertical asymptote at $x = k$, a zero at x_{f_2} and satisfies $T_1(x) < 0$ for $k < x < x_{f_2}$. It follows that the equation $T_1(x) = c - \epsilon_1$, $0 < \epsilon_1 \ll 1$, has only one zero, i.e. for $k < x < x_{f_2}$, because $T_1(x^*)$ has a local minimum for $x = x^*$. In particular it follows that $T_1(x) = c - \epsilon_1$ does not have solutions in the interval $0 < x < 1$.

In Fig. 8 the situation is illustrated. For the limiting case $\tau = \tau_{wf}$ a continuity argument gives the same result, because it is a limiting position of $\tau_{wf} < \tau$. Therefore condition (iv) in Lemma 5.8 is satisfied if $\tau_{wf} \leq \tau < \frac{k}{k-1}$ and no limit cycles occur in the strip $0 \leq x \leq 1$.

Case 2 $-1 < \tau < \tau_{wf}$. We will show that the function $T_1'(x)$ has fixed sign on the interval $0 < x < 1$ Since $T_1(x) < 0$ for $0 < x < x_g$ and $x_f < x < 1$ the conclusion of Lemma 5.9 will hold. Since in this case $0 < x_g < x_f < 1 < k - x_g < k + 2 - x_g < x_{f_2}$ it is easy to see that $T_1(x) < 0$ for $0 < x < x_g, x_f < x < k - x_g$ and $k + 2 < x < x_{f_2}$. In each of these strips one boundary of the interval represents a zero of $T_1(x)$ ($x = 0$, $x = x_f$ and $x = x_{f_2}$ respectively) and the other boundary represents a vertical asymptote where $T_1(x) \downarrow -\infty$ ($x = x_g$, $x = k - x_g$, $x = k + 2 - x_g$ respectively). Since in each strip no other zeroes or asymptotes occur, it follows from the continuity of the function $T_1(x)$ on these intervals that for each $c < 0$ the equation $T_1(x) = c$ has a solution in each interval, i.e. three solutions for each $c < 0$ in total. Since the intervals $0 < x < x_g$ and $x_f < x < 1$ are contained in the three intervals we can immediately conclude that $T_1'(x)$ cannot become 0, showing that condition (iv) of Lemma 5.9 is satisfied. This proves that under the condition $-1 < \tau < \tau_{wf}$ at most one limit cycle occurs on the interval $0 \leq x \leq 1$ and if it appears it is hyperbolic. In Fig. 9 this situation is illustrated.

This completes the proof of the proposition. □

Proposition 5.7 does not make a statement about the existence of a limit cycle. However, the Pontryagin-integrals show the existence of limit cycles through expression (5.4). For each h a bifurcation direction τ is defined for which a perturbed limit

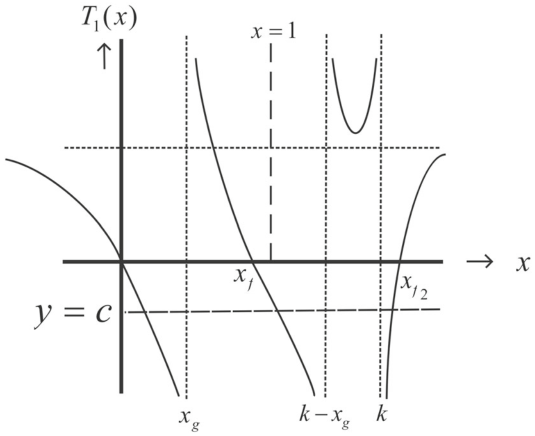


Fig. 9 Monotonicity of the function $T_1(x)$ in (5.7) under the condition $-1 < \tau < \tau_{wf}$

cycle will occur from Γ_h. Since the previous proposition showed that at most one limit cycle can occur, the function $\tau(h)$ has to be monotonic. If it were not monotonic, then there would exist a τ^* such that $\tau^* = \tau(h)$ has (at least) two different solutions h_1 and h_2, i.e. more than one limit cycle would be created in the τ^* direction in parameter space.

The annulus has the boundary cycle Γ_{h^*} which is tangent to $x = 1$. Since for every $0 < h < h^*$ a unique limit cycle is perturbed in the direction $\tau(h)$, by continuity also for $h = h^*$ a limit cycle is created. However, since the unperturbed cycle is tangent to $x = 1$ the previous argument about uniqueness does not apply anymore, because it only showed that at most one limit cycle is perturbed in the strip $0 < x < 1$. For $\tau < \tau^*$ no limit cycles are perturbed because they correspond to cycles crossing the line $x = 1$. These cycles do not occur in (1.2) due to the cut-off of the function $p(x)$ at $x = 1$.

It remains to be determined what the stability is of this unique limit cycle created in the direction $\tau_1 < \tau < \tau_{wf}$. For $\epsilon > 0$ (< 0) the singularity A is a strong unstable (stable) focus. Therefore the unique hyperbolic limit cycle in the strip $0 < x < 1$ is stable (unstable).

Summarizing the argument:

Proposition 5.10 *Consider the perturbation* $a_0 = \frac{1}{1-k} + \tau\epsilon$, $a_1 = \epsilon$ *from the period annulus surrounding the center singularity of system* (1.2). *No limit cycles are created for* $\tau \geq \tau_{wf}$ *or* $\tau < \tau^*$. *For* $\tau^* < \tau < \tau_{wf}$ *exactly one limit cycle is created which is hyperbolic if it exists. The limit cycle is of type I according to Definition* 3.4. *For* $\epsilon > 0$ (< 0) *the limit cycle is stable (unstable).*

5.3 Additional Limit Cycles Perturbed from the Center Case

The results of the previous two sections showed that after perturbation of the center case a unique (big) stable limit cycle of type III exists (Proposition 5.6) and a unique (small) limit cycle of type I exists for the bifurcation directions $\tau^* < \tau < \tau_{wf}$ with $a_0 = \frac{1}{1-k} + \tau\epsilon$, $a_1 = \epsilon$ (Proposition 5.10). These results give a complete picture of the limit cycles of type I and III after perturbation of the center case. It remains to be determined what happens with medium sized limit cycles of type II. These are the cycles crossing $x = 1$ but not the line $x = \frac{k}{2}$, the zero of $f(x)$ for $x > 1$. Such limit cycles can only be perturbed from the cycle γ_{h^*}, the periodic orbit in the period annulus of the center case tangent to the line $x = 1$. This bifurcation mechanism is difficult to analyze due to the non-analytical behaviour of the Poincaré-map near the cycle. However, using a continuity argument we can get some lower bounds on the number of created type II cycles. In order to do so we will use the simple but powerful Proposition 3.3. In terms of the bifurcation parameters near the center case, according to Lemmas 2.3 and 2.10, the stability of the singularity A is :

Lemma 5.11 *Consider the perturbation* $a_0 = \frac{1}{1-k} + \tau\epsilon$, $a_1 = \epsilon$ *from the period annulus surrounding the center singularity of system* (1.2). *For* $0 < |\epsilon| \ll 1$ *the singularity* A *is:*

- *a stable (unstable) strong focus for* $\epsilon(\tau - \tau_{wf}) > 0$ *(* < 0*).*
- *a stable (unstable) weak focus of order 1 for* $\tau = \tau_{wf}$ *and* $\epsilon > 0$ *(*< 0*).*

According to Lemma 3.2 an even number of limit cycles surround a stable singularity. It was established in Proposition 5.6 that a unique limit cycle of type III exists for all bifurcation parameters. Together with the stability criteria of Lemma 5.11 it follows immediately:

Lemma 5.12 *Consider the perturbation* $a_0 = \frac{1}{1-k} + \tau\epsilon$, $a_1 = \epsilon$ *from the period annulus surrounding the center singularity of system* (1.2). *For* $0 < |\epsilon| \ll 1$*, at least two limit cycles surround the singularity* A *if* $\epsilon(\tau - \tau_{wf}) > 0$ *or* $\tau = \tau_{wf}$, $\epsilon > 0$.

In this lemma the largest limit cycle is the limit cycle of type III of which the existence was proved before. The inner limit cycle could be of type I or type II depending on the parameter choices. In the case of $\epsilon(\tau - \tau_{wf}) > 0$, Proposition 5.10 indicates a subregion in parameter space where the inner cycle is a type I cycle. In the (unlikely) case that type II cycles exist they would come in pairs bringing the total number above 2. In the remainder of the region $\epsilon(\tau - \tau_{wf}) > 0$ no small limit cycles exist and the inner limit cycle needs to be a medium limit cycle of type II. For this to be true there has to be a curve in parameter space tangent to the direction $\tau = \tau^*$ where a limit cycle is tangent to $x = 1$, i.e. it is a transition curve where a cycle of type I becomes a type II upon crossing $\tau = \tau^*$.

Lemma 5.13 *Consider the perturbation* $a_0 = \frac{1}{1-k} + \tau\epsilon$, $a_1 = \epsilon$ *from the period annulus surrounding the center singularity of system* (1.2). *For* $0 < |\epsilon| \ll 1$*, there is a curve* $a_0 = P(a_1)$ *in parameter space starting at the center case* $a_0 = \frac{1}{1-k}$, $a_1 = 0$ *for which singularity* A *has a limit cycle tangent to the line* $x = 1$. *On one side of the curve the limit cycle is of type I, on the other of type II according to Definition 3.4. The curve has an asymptotic direction* $P'(a_1) = \tau^*$ *at the center case.*

The previous results covered the case where the perturbed singularity was stable. Next we consider the perturbation into the region where the singularity is unstable.

In the case $\tau = \tau_{wf}$, $\epsilon > 0$, a stable weak focus of order 1 exists after perturbation. No type I cycle will occur according to Proposition 5.10 and the inner limit cycle has to be a medium unstable limit cycle of type II crossing $x = 1$. By perturbation of the weak focus in such a way that it changes into an unstable strong focus, an additional type I cycle is created in an Andronov–Hopf bifurcation. It follows that situations occur with at least three limit cycles. To see how big the region in parameter space is with at least three limit cycles, the following simple argument gives an rough estimate.

The limit cycle of type I (if it exists) is stable if $\epsilon > 0$ according to Proposition 5.10. Since the big limit cycle is stable as well, it immediately implies that in that case (at least one) another unstable limit cycle must exist in between the two stable limit cycles because two stable limit cycles cannot be adjacent.

A different way to arrive at this conclusion is to observe that the anti-saddle A is an unstable strong focus for $\tau^* < \tau < \tau_{wf}$ and $\epsilon > 0$. According to Proposition 3.3 an odd number of limit cycles must surround A. Since we already found two limit cycles, a third one has to exist as well. The type I and type III limit cycles are unique, so the third limit cycle has to be an unstable (medium) limit cycle of type II.

Lemma 5.14 *Consider the perturbation* $a_0 = \frac{1}{1-k} + \tau\epsilon$, $a_1 = \epsilon$ *from the period annulus surrounding the center singularity of system* (1.2). *For* $0 < \epsilon < 1$, *an unstable limit cycle of type II exists surrounding singularity A for* $\tau^* < \tau < \tau_{wf}$.

It remains to be determined what will happen with the type I and type II limit cycles when the bifurcation direction becomes $\tau < \tau^*$. No type I cycles occur according to Proposition 5.10, meaning that just like in the situation of Lemma 5.13 a transition curve exists where the type I cycle changes into a type II cycle. This curve is tangent to the line in the $\tau = \tau^*$ direction. Therefore there will be a region where two type II cycles exist simultaneously together with the type III cycle.

To understand what happens to the two medium type II cycles, consider the bifurcation directions $\tau < -1 < \tau^*$. In that direction the divergence of the vector field (5.2) has fixed sign on the interval $0 < x < 1$ and only changes sign for the full (1.2) system at $x = \frac{k}{2}$. It follows that:

Lemma 5.15 *Consider the perturbation* $a_0 = \frac{1}{1-k} + \tau\epsilon$, $a_1 = \epsilon$ *from the period annulus surrounding the center singularity of system* (1.2). *For* $0 < \epsilon < 1$, *no type I or type II limit cycle (according to the Definition* 3.4) *occurs for* $\tau < -1$. *The system has a unique stable limit cycle of type III.*

The lemma indicates that the type I and type II limit cyles for $\tau^* < \tau < \tau_{wf}$ and $\epsilon > 0$ must disappear if τ is decreased from τ^* before reaching $\tau = -1$. This can only be achieved through the occurrence of a semi-stable limit cycle of type II, since the type III cycle is unique and hyperbolic in this perturbation.

Lemma 5.16 *Consider the perturbation* $a_0 = \frac{1}{1-k} + \tau\epsilon$, $a_1 = \epsilon$ *from the period annulus surrounding the center singularity of system* (1.2). *For* $0 < \epsilon < 1$, *there is a curve* $a_0 = P_2(a_1)$ *in parameter space starting at the center case* $a_0 = \frac{1}{1-k}$, $a_1 = 0$ *for which singularity A has a semi-stable limit cycle of type II according to the Definition* 3.4. *The curve has an asymptotic direction* $P_1'(a_1) = \tilde{\tau}$ *at the center case, with* $-1 < \tilde{\tau} \leq \tau^*$. *With* $0 < \epsilon_2 < 1$, *for* $\tau = \tilde{\tau} - \epsilon_2$, *the system has no medium limit cycles in the neighborhood of the semi-stable limit cycle. For* $\tau = \tilde{\tau} + \epsilon_2$, *the system has two medium limit cycles in the neighborhood of the semi-stable limit cycle.*

Remark: the bifurcation direction $\tilde{\tau}$ is unknown to us, although it seems reasonable (confirmed by a numerical experiment) that it is actually equal to τ^*.

Combining all the lemmas of this section, the following overall conclusion can be reached about the bifurcation of limit cycles from the center case in system (1.2):

Theorem 5.17 *Consider the perturbation* $a_0 = \frac{1}{1-k} + \tau\epsilon$, $a_1 = \epsilon$ *from the period annulus surrounding the center singularity of system* (1.2) *for* $0 < |\epsilon| \ll 1$.

For all bifurcation directions a unique limit cycle of type III exists.

- *For* $\epsilon < 0$ $\tau < \tilde{\tau}$ *one limit cycle of type II exists.*
  $\tau = \tilde{\tau}$ *one limit cycle tangent to* $x = 1$ *exists.*
  $\tilde{\tau} < \tau < \tau_{wf}$ *one limit cycle of type I exists.*
  $\tau \geq \tau_{wf}$ *no limit cycles of type I or II exist.*
- *For* $\epsilon > 0$
  $\tau < \tilde{\tau}$ *no limit cycles of type I or II exist.*
  $\tau = \tilde{\tau}$ *one semi-stable limit cycle of type II exists.*
  $\tilde{\tau} < \tau < \tau^*$ *two limit cycles of type II exist.*
  $\tau = \tau^*$ *one limit cycle tangent to* $x = 1$ *exists inside a limit cycle of type II.*
  $\tau^* < \tau < \tau_{wf}$ *one limit cycle of type I exists inside a limit cycle of type II.*
  $\tau \geq \tau_{wf}$ *one limit cycle of type II exists.*

A direct consequence of this theorem is that there exist bifurcations from the center case with (at least) three limit cycles surrounding A. To prove that the number is exactly three the bifurcation mechanism of limit cycles from the boundary of the period annulus Γ_{h^*} needs to be understood which is out of the scope of this paper.

Proposition 5.18 *Consider the center case for system* (1.2) *with* $a_0 = \frac{1}{1-k}$, $a_1 = 0$. *There exists a perturbation of* a_0, a_1 *with fixed* δ, k, ϕ *such that (at least) three limit cycles appear surrounding the strong focus.*

The cases are shown in Fig. 10.

Some numerical results for these bifurcations are shown in Figs. 11, 12, 13 and 14.

6 Singular Bifurcation Near $\delta = 1$

The previous bifurcation mechanisms showed the existence of two limit cycles created from a Andronov–Hopf bifurcation and three limit cycles from a center case bifurcation. In [16] the limit of $\delta \uparrow 1$ was discussed for the situation of $a_0 = a_1 = 0$ in (1.2), a system traditionally referred to as the Holling I system. It was proved that when the singularity is a focus, then exactly two limit cycles are created after perturbation of the singular system $\delta = 1$. In our case (1.2) also degenerates into a singular system when $x_g = 1$ which is equivalent to $\delta = 1$. Using the same techniques we can study the bifurcation of limit cycles when $x_g = 1 - \epsilon$ in (1.2). We obtain results for special cases, but we note that the general singular perturbation problem in system (1.2) with $\delta \uparrow 1$ remains open.

The singularity A after perturbation is an anti-saddle, which means it can be either a focus or a node. In this section we only consider the focus-case because the node case is much more complicated and leads to more difficult singular problems.

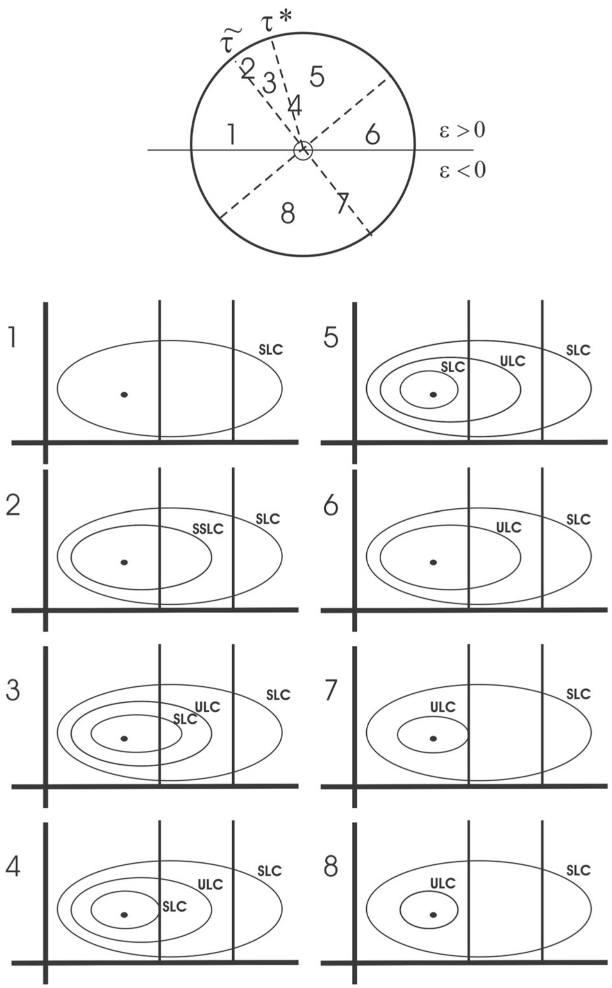
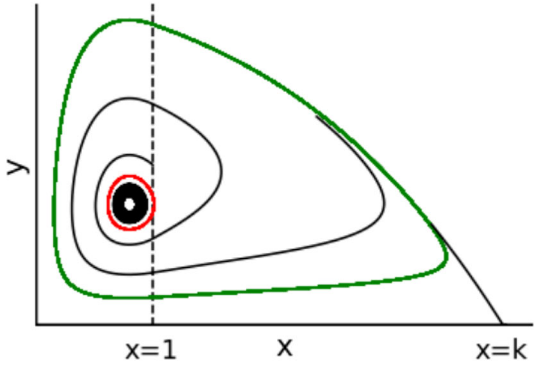


Fig. 10 Bifurcation of limit cycles from the center case in system (1.2) according to Theorem 5.17

Fig. 11 Perturbation of the center case C for $a_0 = \frac{1}{1-k}$, $a_1 = 0$, in system (1.2). The case $\epsilon > 0$, $\tau = \tau_{wf}$ of Theorem 5.17: one type II and one type III limit cycle surrounding a weak focus of order one. The parameters in system (1.2) are: $\phi = 1$, $k = 4$, $\delta = \frac{7936}{9375}$, $a_0 = -\frac{169}{375}$, $a_1 = \frac{1}{5}$. The stable limit cycle is displayed in green, the unstable limit cycle in red (color figure online)



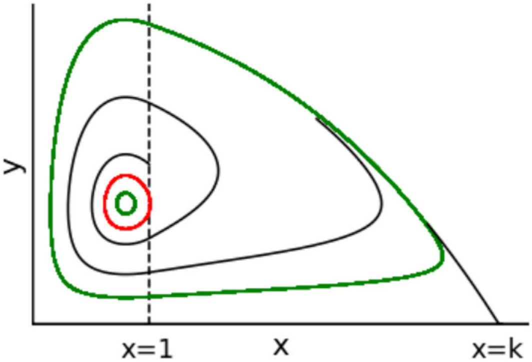


Fig. 12 Perturbation of the center case C for $a_0 = \frac{1}{1-k}$, $a_1 = 0$, in system (1.2). The case $\epsilon > 0$, $\tau^* < \tau < \tau_{wf}$ of Theorem 5.17: one type I, one type II and one type III limit cycle surrounding a strong focus. The parameters in system (1.2) are: $\phi = 1$, $k = 4$, $\delta = \frac{7936}{9375}$, $a_0 = -\frac{341}{750}$, $a_1 = \frac{41}{200}$. The stable limit cycles are displayed in green, the unstable limit cycle in red (color figure online)

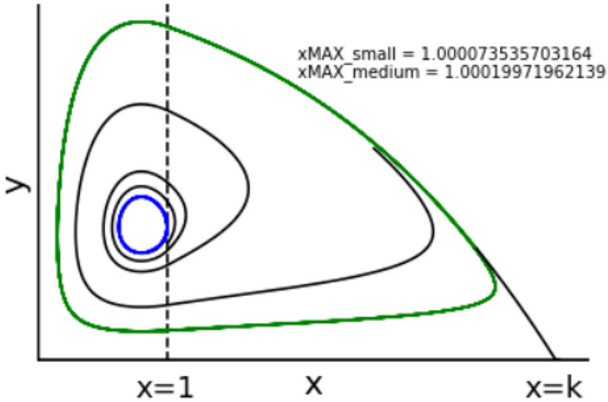


Fig. 13 Perturbation of the center case C for $a_0 = \frac{1}{1-k}$, $a_1 = 0$, in system (1.2). The case $\epsilon > 0$, $\tilde{\tau} < \tau < \tau^*$ of Theorem 5.17: two type II limit cycles and one type III limit cycle surrounding a strong focus. The parameters in system (1.2) are: $\phi = 1$, $k = 4$, $\delta = \frac{7936}{9375}$, $a_0 = -\frac{22,334}{46,875}$, $a_1 = \frac{2903}{12,500}$. The stable limit cycle of type III is displayed in green, the two medium limit cycles of type II in blue. Since both medium limit cycles are perturbed from the boundary cycle Γ_{h^*}, it is difficult to distinguish them in the figure. The two values $xMAX\_small$ and $xMAX\_medium$ represent the largest x-values these two cycles reach, showing how close the cycles are positioned (color figure online)

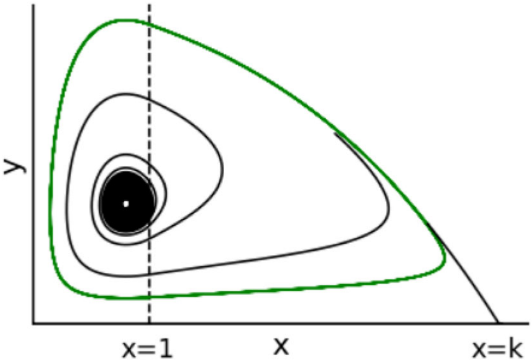
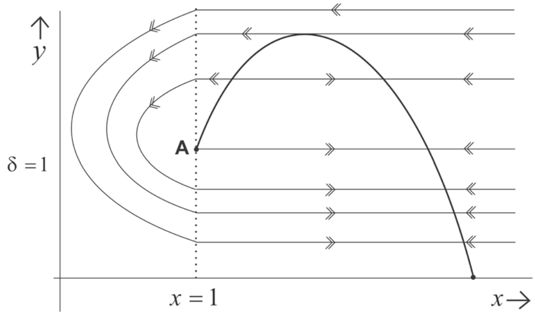


Fig. 14 Perturbation of the center case C for $a_0 = \frac{1}{1-k}$, $a_1 = 0$, in system (1.2). The case $\epsilon > 0$, $\tau < \tilde{\tau}$ of Theorem 5.17: just after a semi-stable limit cycle of type II has disappeared and one type III limit cycle surrounding a strong focus is left. The parameters in system (1.2) are: $\phi = 1$, $k = 4$, $\delta = \frac{7936}{9375}$, $a_0 = -\frac{89,339}{187,500}$, $a_1 = \frac{11,613}{50,000}$. The stable limit cycle is displayed in green (color figure online)

Fig. 15 Singular system (1.2) with $\delta = 1$



The condition for A to be a focus is:

$$\phi < \frac{4p'(x_g)\left(1 - \frac{x_g}{k}\right)}{\left(\left(1 - \frac{2x_g}{k}\right)p(x_g) - p'(x_g)\left(1 - \frac{x_g}{k}\right)\right)^2}.$$

For x_g close to 1, this condition reduces to:

$$\phi < \frac{4p'(1)\left(1 - \frac{1}{k}\right)}{\left(\left(1 - \frac{2}{k}\right) - p'(1)\left(1 - \frac{1}{k}\right)\right)^2},$$

where we used that $p(1) = 1$.

This condition means: for fixed a_0, a_1, k, x_g, the singularity A is a focus if ϕ is chosen to be sufficiently small. In the following it is assumed that this choice has been made.

6.1 The Degenerate System for $x_g = 1$

For $x_g = 1$ system (1.2) degenerates in the following way. For $0 \le x < 1$ the system remains regular. The singularity A at $x_g = 1$ moved to the line $x = 1$. For $x \ge 1$ the system becomes singular because $\frac{dy}{dt} \equiv 0$. The system has a continuum of singularities lying on the parabola $y = \phi x(1 - \frac{x}{k})$. Outside the parabola the solution set consists of horizontal lines $y = c$. After perturbation, i.e. $x_g = 1 - \epsilon$, the solutions moving close to the parabola are referred to as the slow solutions and the solutions near the horizontal lines as fast solutions, because of the different time scales these solutions refer to.

The unperturbed situation is shown in Fig. 15.

6.2 Perturbation of the Degenerate System

A conclusion from [16] is that the perturbed singular system in the case of a focus can only contain limit cycles coming from the degenerate singularity at $x = x_g = 1$ and from a big singular cycle as indicated in Fig. 16. The big singular cycle consists of 3

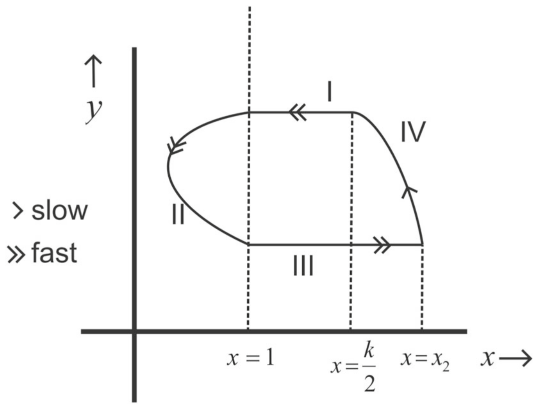


Fig. 16 Singular cycle consisting of 3 fast parts (I, II, III) and 1 slow part (IV) for system (1.2) with $\delta = 1$

fast solutions -one fast solution (II) of the regular system on the interval $0 < x < 1$ and 2 fast solutions (I, III) formed by horizontal lines for $x > 1$- and a slow solution (IV) being part of the continuum of singularities for $x > 1$. This situation is shown in Fig. 16.

The important observation is that the slow part of the big cycle is formed by the continuum of solutions defined for $\frac{k}{2} < x < k$, i.e. the part where the divergence of the perturbed vector field has fixed sign (because the unique zero of $f(x)$ for $x > 1$ occurs for $x = \frac{k}{2}$). According to the singular bifurcation theory (see [3–9]) the stability of perturbed cycles is determined by the sign of the slow divergence integral, which is essentially the integral of the perturbed divergence over the slow part. In this case that integral has to be negative because $f(x) < 0$ for the slow part. It follows immediately that in our case this integral has to be negative and therefore that only one limit cycle can be created from perturbing the big cycle in the unperturbed system for the focus case. It is stable and it is hyperbolic when it exists.

The analysis of the singular system in [16] was based on a blow-up of the singularity A through a transformation of the type $x_1 = \frac{x-x_g}{1-x_g}$, $y_1 = \frac{y-y_g}{1-x_g}$ which maps the singularity to the origin while keeping the boundary $x = 1$ in tact. After a rescaling in time $t \to kt$ the blown-up system takes the form (after restoring the notation $x_1 \to x$, $y_1 \to y$) for the region $x > 1$:

$x > 1$

$$\begin{aligned}
\frac{dx(t)}{dt} &= P_0(x, y) + \mathcal{O}(\epsilon),\\
\frac{dy(t)}{dt} &= Q_0(x, y) + \mathcal{O}(\epsilon),\\
P_0(x, y) &= -a_0k\phi - a_1k\phi + a_0\phi + a_1\phi - k\phi + \phi + (k\phi - 2\phi)x - ky,\\
Q_0(x, y) &= \phi(k-1)(1 + a_0 + a_1).
\end{aligned} \tag{6.1}$$

The choice of the blow-up transformation was motivated by the fact that it ensured that the line $x = 1$ remained in the finite part of the plane, i.e. we kept it at $x_1 = 1$. The scaling in y was adjusted accordingly in such a way that the singular nature of the system disappeared after transformation. The original interval $0 \le x \le 1$ is mapped onto the interval $x \le 1$ in the new coordinates.

The zeroth order terms in ϵ are sufficient for the analysis in the region $x > 1$ because for no parameter values the terms will vanish identically, i.e. it is a regular linear system, and the phase portrait is structurally stable under small perturbations. Here we exclude the boundary case $a_0 + a_1 + 1 = 0$ of W_1. The system for $x > 1$ is exactly the same as obtained in [16] which is not surprising because for $x > 1$ we imposed the same Holling I structure.

For $x \le 1$ the analysis requires higher order terms under certain conditions. It turns out that two cases need to be considered related to the limiting position of the weak focus line in the a_0, a_1 parameter plane.

6.2.1 Singular Perturbation for $a_0 \ne \frac{a_1 k - a_1 + 1}{1-k}$

For fixed k and ϕ the singular system with $\epsilon = 0$ depends on 2 parameters a_0 and a_1. The condition $a_0 = \frac{a_1 k - a_1 + 1}{1-k}$ corresponds to the limiting position of the weak focus line l_{wf} of the previous sections. It implies: if $a_0 = \frac{a_1 k - a_1 + 1}{1-k}$, then a weak focus can be perturbed from the singular system, but if $a_0 \ne \frac{a_1 k - a_1 + 1}{1-k}$ for $\epsilon = 0$ then only a strong anti-saddle can be perturbed. In the latter case the perturbation is similar to that studied in [16]. The equilibrium point at the origin of the blown-up system is a strong singularity. For the study of the perturbed system in that case it is sufficient to consider the zeroth order term only:

$x \le 1, a_0 \ne \frac{a_1 k - a_1 + 1}{1-k}$

$$
\begin{aligned}
\frac{dx(t)}{dt} &= P_0(x, y) + \mathcal{O}(\epsilon), \\
\frac{dy(t)}{dt} &= Q_0(x, y) + \mathcal{O}(\epsilon), \\
P_0(x, y) &= \phi((1-k)(a_0 + a_1) - 1)x - ky, \\
Q_0(x, y) &= (k-1)\phi(1 + a_0 + a_1)x.
\end{aligned}
\tag{6.2}
$$

The system for $x > 1$ has one important property: it has an invariant line representing the stable separatrix of a saddle singularity at infinity. This solution exits the blown-up system at this infinite saddle.

Lemma 6.1 *System (6.1) has one invariant line l given by $y = y_l(x) \equiv \frac{\phi(k-2)}{k}x - \frac{(k-1)(k+(k-2)\phi)}{k(k-2)}$. System (6.1) has two singularities at infinity: a saddle S with a stable separatrix, formed by the invariant line l, approaching it from the finite part of the phase plane and a stable node N at the end of the x-axis. See Fig. 17 where the phase portrait is shown on the Poincaré sphere.*

The lemma implies immediately that there should exist a stable big limit cycle outside the blown-up system (6.1) and (6.2). Since the only possible other limiting position of a limit cycle outside the blown-up singularity A is the big cycle of Fig. 16, we conclude:

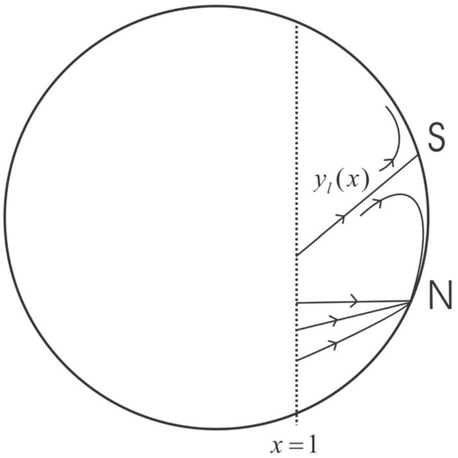


Fig. 17 Invariant line and phase portrait of the system (6.1) for $x > 1$ with $\epsilon = 0$ for the focus case, depicted on the Poincaré sphere

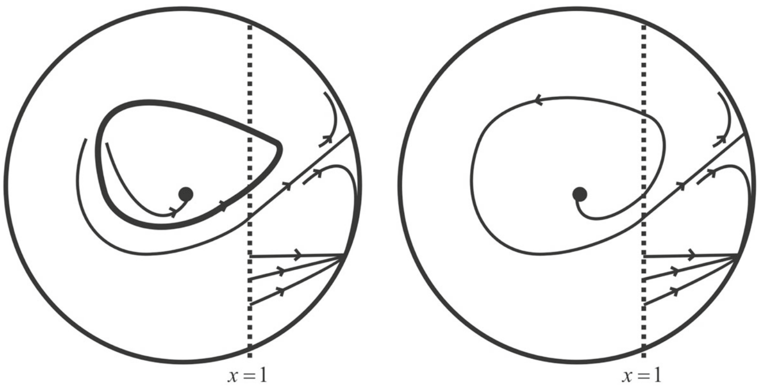


Fig. 18 The two cases of the blown-up system (6.1) and (6.2) for $\epsilon = 0$. In the case of an unstable focus no limit cycles occur, while for the stable focus a hyperbolic unstable limit cycle occurs

Lemma 6.2 *For system* (1.2) *with* $x_g = 1 - \epsilon$, $0 < \epsilon \ll 1$, *a unique hyperbolic stable limit cycle exists of type III, if* $a_0 \neq \frac{a_1 k - a_1 + 1}{1-k}$.

According to Proposition 3.3 we can conclude that for the case of a stable singularity A after perturbation a second limit cycle must exist. This follows also from singular perturbation theory. System (6.1) and (6.2) is essentially the same system studied in [16] where it was proved that the blown-up system contains a unique unstable limit cycle crossing the line $x = 1$ if the singularity A is a stable focus. If it is an unstable focus it is quite easy to prove that no cycles are contained in the blown-up system. In Fig. 18 the two situations in the blown-up system are depicted.

The conclusion is that for this case a sharp limit on the existence and upper bound of perturbed limit cycles can be given:

Proposition 6.3 *For system* (1.2) *with* $x_g = 1 - \epsilon$, $0 < \epsilon \ll 1$, $a_0 \neq \frac{a_1 k - a_1 + 1}{1-k}$, *a unique hyperbolic stable limit cycle exists if A is a focus. This stable limit cycle is of type III. If the focus is stable then an additional limit cycle exists which is hyperbolic* *and unstable. This stable limit cycle is of type II. For an unstable focus no other limit cycles exist.*

6.2.2 Singular Perturbation for $a_0 = \frac{a_1 k - a_1 + 1}{1-k}$

If $a_0 = \frac{a_1 k - a_1 + 1}{1-k}$ for $\epsilon = 0$, then the perturbation of the singular system for fixed ϕ and k is much more complicated, because the blown-up system will be structurally unstable in contrast to the previous section where the blown-up system was structurally stable.

The first observation is that the existence of a unique stable limit cycle of type III follows in the same way as was shown in the previous section. The reason being that the system for $0 \leq x \leq 1$ in the original coordinates has no influence on the behaviour of the solutions, because Lemma 6.1 still holds in the region $x > 1$.

However, in this particular case there is an even simpler proof possible because the system will resemble the center case of (1.2), where we proved the global existence and uniqueness of the type III limit cycle.

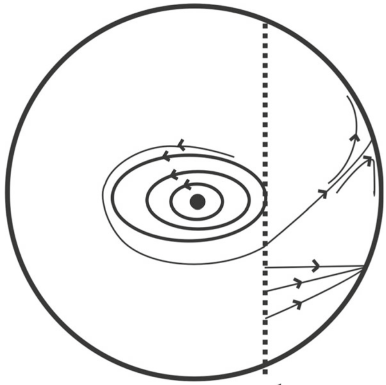
The first step is to blow up the system around the singularity A in the same way as for the other cases. For $x > 1$ nothing essentially changed and system (6.1) can be used. For $x \leq 1$ higher order terms in ϵ will play a role, for $a_0 = \frac{a_1 k - a_1 + 1}{1-k}$ we get:

$$x \leq 1,\ a_0 = \frac{a_1 k - a_1 + 1}{1-k}$$

$$\begin{aligned}
\frac{dx(t)}{dt} &= P_0(x, y) + \epsilon P_1(x, y) + \mathcal{O}(\epsilon^2), \\
\frac{dy(t)}{dt} &= Q_0(x, y) + \epsilon Q_1(x, y) + \mathcal{O}(\epsilon^2), \\
P_0(x, y) &= -ky, \\
P_1(x, y) &= \frac{1}{1-k}\big(\big(-2a_1 k^2 \phi - 2a_1 \phi + 4a_1 \phi k\big)x + (2k - k^2)y \\
&\quad + (k^2 - 2k)xy + \big(a_1 k^2 \phi - 2a_1 k \phi + a_1 \phi\big)x^2\big), \\
Q_0(x, y) &= \phi(k-2)x, \\
Q_1(x, y) &= \frac{-x}{1-k}\big(-2\phi k^2 a_1 - 2\phi - 2\phi a_1 + 2\phi k + 4\phi a_1 k + (k^2 - 2k)y \\
&\quad + \big(a_1 k^2 \phi - k\phi - 2a_1 k \phi + \phi + a_1 \phi\big)x\big).
\end{aligned} \tag{6.3}$$

Here for $\epsilon = 0$ the blown-up system for $x \leq 1$ contains a period annulus formed by the ellipses which are the solutions of the corresponding linear system. It is therefore structurally unstable, implying that it is not clear what will happen for $\epsilon \neq 0$. The period annulus is bounded on the outside by the ellipse tangent to the line $x = 1$. This creates a similar situation in the blown-up system as for the center case we studied in the previous sections. The limit cycles created from the period annulus (i.e. the cycles lying in the region $x < 1$) after perturbation correspond to the cycles labelled type I. They basically correspond to a regular Andronov–Hopf-bifurcation. The upper bound on the number of small cycles should be equal to the maximum order of the weak

Fig. 19 The blown-up system (6.1), (6.3) for $\epsilon = 0$, $a_0 = \frac{a_1 k - a_1 + 1}{1-k}$. A period annulus formed by ellipses exists for $x \leq 1$, which is unstable on the outside



focus, which is two in the case of system (1.2), but the full proof is not given here because it involves calculations involving higher order expansions in ϵ.

Essentially we have established that for $a_0 = \frac{a_1 k - a_1 + 1}{1-k}$ the unperturbed system has a similar structure as for the center case of (1.2) and the techniques of the previous sections can be applied.

The stability of the period annulus in the blown-up system is determined through Lemma 5.1. A simple check using the system for $x > 1$ of the blown-up system (6.1) shows that:

Lemma 6.4 *For system* (1.2) *with* $x_g = 1 - \epsilon$, $0 < \epsilon \ll 1$, $a_0 = \frac{a_1 k - a_1 + 1}{1-k}$, *the system* (6.1), (6.3) *contains a period annulus in the strip* $0 \leq x \leq 1$ *consisting of ellipses and is unstable on the outside for* $x > 1$ *in the limit* $\epsilon \downarrow 0$.

The situation is depicted in Fig. 19.

The conclusion from Lemma 6.4 is that the blown-up system has an unstable period annulus. Combined with the fact the system (1.2) is a bounded system the conclusion from the Poincaré–Bendixson theorem then shows the existence of the big stable limit cycle after perturbation. After all, limit cycles can only be perturbed from the singularity A and the big singular cycle from Fig. 16:

Lemma 6.5 *If system* (1.2) *contains a focus or center, then a unique stable hyperbolic limit cycle of type III is created from the singular system* $x_g = 1 - \epsilon$, $0 < \epsilon \ll 1$. *This big cycle exists independent of the stability of the perturbed singularity.*

Similar to the observations in the previous sections, it follows that if the perturbed system has a stable singularity, then there exist two limit cycles, i.e. an additional limit cycle exists which is unstable lying inside the stable cycle of type III. The type of this second cycle is not clear, because in principle it could have been created from the outer cycle in the period annulus touching the line $x = 1$ or it could directly have come from the period annulus itself.

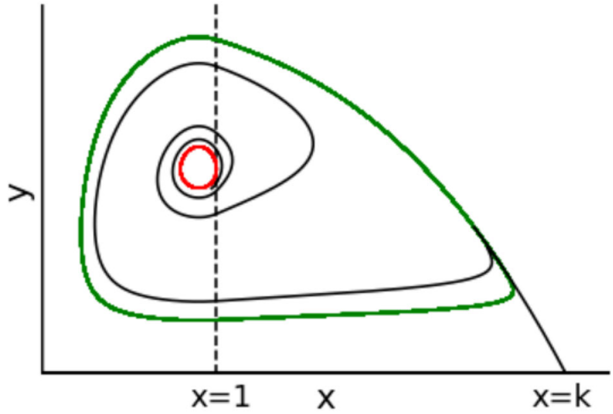


Fig. 20 Two limit cycles created surrounding a second order weak focus in (1.2) for $x_g = 1 - \epsilon$, $a_0 = \frac{a_1 k - a_1 + 1}{1-k}$. The parameters in system (1.2) are: $\phi = 1$, $k = 3$, $\delta = \frac{35{,}721}{38{,}350}$, $a_0 = -\frac{1258}{767}$, $a_1 = \frac{1100}{767}$. The stable limit cycle is displayed in green, the unstable limit cycle in red (color figure online)

6.2.3 Singular Perturbation of a Weak Focus of Order 2 for $a_0 = \frac{a_1 k - a_1 + 1}{1-k}$

The full analysis of the singular perturbation is left for further research. For our purposes we observe that system (1.2) contains parameter values for which a stable second order weak focus occurs according to Proposition 2.12. Therefore the previous discussion shows that:

Proposition 6.6 *If a second order weak focus is perturbed from the singular system* (1.2) *with* $x_g = 1$, *then the second order weak focus is surrounded by (at least) two limit cycles. One limit cycle is of type III, stable and hyperbolic. The other limit cycle is of type I or II and unstable.*

In Fig. 20 a numerical solution is shown for this proposition. The two limit cycles can be observed surrounding a second order weak focus in (1.2).

Next we perturb exactly two small-amplitude limit cycles from the second order weak focus through the standard Andronov–Hopf bifurcation as described in Proposition 4.2 leading to:

Proposition 6.7 *The singular system* (1.2) *with* $x_g = 1$ *can be perturbed in such a way that exactly 4 limit cycles are created surrounding the singularity. The outer limit cycle is stable and of type III, one is an unstable limit cycle of type I or II, and two are small-amplitude limit cycles of type I.*

An example of these 4 perturbed limit cycles is shown in Fig. 21.

7 Bifurcation Curves

An important but difficult question is how to explain the existence of 3 limit cycles in a center bifurcation and the existence of 4 limit cycles in a singular bifurcation

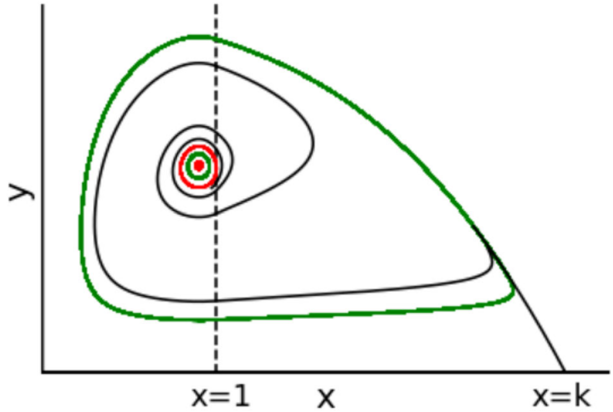


Fig. 21 A numerical example of 4 perturbed limit cycles from a singular system (1.2) with $\delta = 1$. The parameters in system (1.2) are: $\phi = 1, k = 3, \delta = \frac{35{,}719{,}184{,}511}{38{,}350{,}000{,}000}, a_0 = -\frac{630{,}524{,}029}{383{,}500{,}000}, a_1 = \frac{220{,}767}{153{,}400}$. The stable limit cycles are displayed in green, the unstable limit cycles in red (color figure online)

in the parameter space by continuously varying the parameters. The information of the previous sections can be summarized in terms of bifurcations curves for semi-stable limit cycles. These bifurcation curves essentially determine the number of limit cycles when parameter values are varied. In this section the approach will be to find an explanation in terms of bifurcation curves for semi-stable limit cycles. The terminology "curve" is used to indicate that the parameter set is typically displayed in the (a_0, a_1) plane for fixed k, ϕ, x_g. In this plane the semi-stable limit cycles typically occur for parameter sets represented by curves, although a formal proof for this is difficult because the standard technique of so-called rotated vector field parameters does not seem to apply to system (1.2).

In region W_1 of the parameter space near the center case C a bifurcation curve for semi-stable limit cycles S_C emerges on one side of the weak focus line l_{wf}. In the same fashion a bifurcation curve S_{wf} emerges from the second order weak focus point. The situation is displayed in Fig. 22.

These two bifurcation curves alone cannot explain the results we obtained for x_g close to 1. In that singular perturbation 4 limit cycles were found near the second order weak focus point. Moreover, the two semi-stable limit cycle curves S_C and S_{wf} correspond to essentially different types of semi-stable limit cycles. Near the center case C the semi-stable limit cycle is created from two limit cycles nearest to the unstable singularity A, i.e. the semi-stable limit cycle is stable on the inside and unstable on the outside. On the other hand in the Andronov–Hopf bifurcation near the second order weak focus, the singularity inside the semi-stable limit cycle is stable, i.e. the semi-stable limit cycle is unstable on the inside and stable on the outside. Therefore the bifurcation curves cannot be connected in a straightforward way.

To explain the co-existence of two essentially different semi-stable limit cycle curves and the occurrence of 4 limit cycles in a singular perturbation, we arrive at the picture of Fig. 23. A third semi-stable limit cycle bifurcation curve S_3 is shown. It is connected together with the curve S_C at a point T where the system has a triple

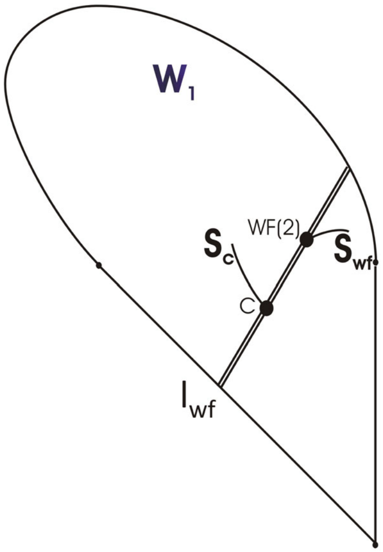


Fig. 22 Two bifurcation curves for semi-stable limit cycles emerging from the center case C and the second order weak focus point for system (1.2) in the parameter region for (a_0, a_1) in W_1 for fixed k, ϕ, x_g

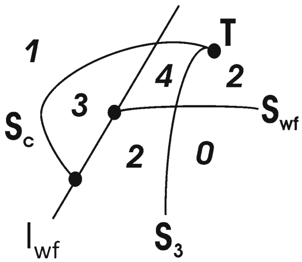


Fig. 23 Three bifurcation curves for semi-stable limit cycles for system (1.2) in the parameter region for (a_0, a_1) in W_1 for fixed k, ϕ, x_g. There is a point T where a limit cycle of multiplicity 3 occurs. Note that there is an intersection of two bifurcation curves for semi-stable limit cycles where the system has a singularity surrounded by two semi-stable limit cycles

limit cycle. The curve S_3 represents semi-stable limit cycles of the same type as for curve S_{wf}, i.e. unstable on the inside and stable on the outside. The figure indicates the number of limit cycles in the regions of the parameter space. Note that W_1 is not indicated here because it is not clear how the relative position of T and S_3 are relative to the boundaries of W_1. There is numerical evidence that the point T does enter W_1 for some x_g.

An interesting special case is the origin in the (a_0, a_1) plane: can it lie in the region where 4 limit cycles occur? Numerically it seems to be unlikely but a formal proof is difficult. The origin, where $a_0 = a_1 = 0$, corresponds to the case of a Holling I type functional response where singular pertubation from the case $x_g = 1$ showed that at most two limit cycles occur (see e.g. [16]). It is not a priori clear if perhaps the Holling I case could lie in the parameter region where 4 limit cycles appear when x_g is varied.

Another reason why T should enter W_1 is that for $k \downarrow 2$ the bifurcation curves become restricted in their behaviour. This is due to the fact that for $k = 2$ the period

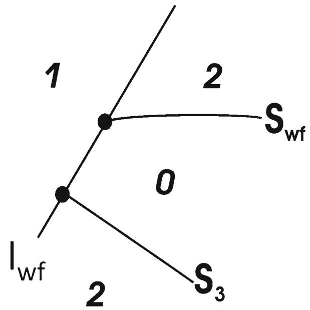


Fig. 24 Two bifurcation curves for semi-stable limit cycles emerging from the center case C and the second order weak focus point for system (1.2) with $k = 2$ in the parameter region for (a_0, a_1) in W_1 for fixed ϕ, x_g

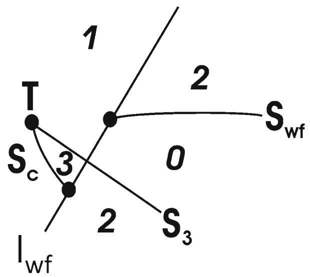


Fig. 25 The emerging of the semi-stable limit cycle bifurcation curve S_C and triple limit cycle point T for system (1.2) with $k = 2 + \epsilon$ in the parameter region for (a_0, a_1) in W_1 for fixed ϕ, x_g

annulus becomes stable according to the criterion in Lemma 5.1. It is not difficult to prove that the period annulus for $k = 2$ is not surrounded by limit cycles. It implies that for $k = 2 + \epsilon$ in the center case a stable limit cycle is created from the boundary cycle of the annulus. The argument leading to three limit cycles does not apply anymore for $k = 2$ and near the center case only two limit cycles can be created, not three. A simple check using the stabilities of the weak focus shows that for $k = 2$ a semi-stable limit cycle bifurcation curve of the type S_3 will emerge in the region $a_0 > \frac{1}{1-k}$. The result is shown in Fig. 24.

The problem then becomes to explain how the curve S_C and the point T, which exist for $k > 2$, disappear when k approaches 2. The explanation is that for $k = 2 + \epsilon$ the curve S_C and triple limit cycle point T need to emerge from the center point C. This is displayed in Fig. 25.

Then finally by continuation of parameters, varying x_g to arrive at the situation of the singularly perturbed system $x_g = 1 - \epsilon$ as shown in Fig. 23 , it is clear that the point T will move from close to C (Fig. 25) with $a_0 < \frac{1}{1-k}$ to a point $a_0 > a_0^{wf_2}$ (Fig. 23). Doing so it has to cross the weak focus line l_{wf} for some x_g, i.e. there are parameter values in system (1.2) for which an unstable first order weak focus is surrounded by a stable third order multiple limit cycle. The bifurcation diagram is displayed in Fig. 26.

8 Discussion

In this paper we introduced three bifurcation mechanisms for a special type of predator–prey system. The natural question arises how these results can be extended to general

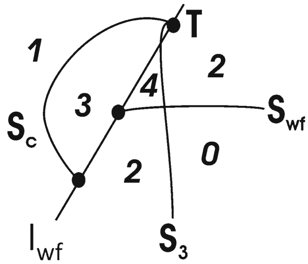


Fig. 26 Existence of a triple limit cycle surrounding a stable first order weak foucs for system (1.2) in the parameter region for (a_0, a_1) in W_1 for fixed k, ϕ, x_g

systems of the form (1.1). Each of the three bifurcation mechanisms used some special information of the function form of (1.2) but not all. In particular:

- The Andronov–Hopf bifurcation used the functions on the interval $0 \le x \le 1$ only because the calculation of the focal values is based on the behaviour of the functions near the singularity at $x = x_g$. The choices of $p(x) = 1$ and logistic growth function $h(x)$ (according to definition (1.3)) for $x > 1$ have no impact: the results of Proposition 2.12 hold true for any $p(x)$ and $h(x)$ defined for $x > 1$.
- The center bifurcation occurs for a specific choice of $p(x)$ and $h(x)$ on the interval $0 \le x \le 1$, i.e. they are proportional. The bifurcation mechanism can be easily extended to more general $p(x)$ and $h(x)$ defined on $x > 1$.
- The singular perturbation is not influenced by the choice of $p(x)$ and $h(x)$ on the interval $0 \le x \le 1$, as long as the system has a singularity of focus type collapsing onto the line $x = 1$ for $x_g \uparrow 1$. The restriction is basically that $p(x) = 1$ for $x > 1$. Even the functional form of $h(x)$ does not have to be fixed for $x > 1$.

In short it means that the Andronov–Hopf bifurcation mechanism allows for general $p(x)$, $h(x)$ on $x > 1$, the center bifurcation mechanism allows for general $p(x)$, $h(x)$ on $x > 1$ when $p(x)$ and $h(x)$ are proportional on $0 \le x \le 1$, and the singular perturbation mechanism essentially needs $p(x) = 1$ for $x > 1$. The three bifurcation mechanisms cover different aspects of modelling small, medium and large prey densities.

Moreover the results can give information about smooth perturbations of these choices. For example: in the singular case perturbation results can be applied to a situation where $p(x) = 1 + \epsilon(x)$ with $0 < |\epsilon(x)| \ll 1$. This is a situation where one assumes that the functional response function does not change much for larger x.

In the next subsections we briefly indicate some interesting observations that can be made for these three bifurcation mechanisms. It is virtually impossible to draw conclusions about the general system (1.1) but we would like to point out some distinct trends in the bifurcations.

8.1 Andronov–Hopf Bifurcation

For the general Gause system (1.1) with $\gamma = 1$ the conditions to have a weak focus are as follows:

To have an anti-saddle at (x_g, y_g) we must have $p'(x_g) > 0$.

Additionally for this anti-saddle to be a weak focus

$$h'(x_g)p(x_g) - h(x_g)p'(x_g) = 0$$

must hold true.

The stability of the weak focus is determined by the sign of the first focal value. The expression determining this sign is (using Maple):

$$-h'(x_g)^2h''(x_g)p(x_g)^2 + h(x_g)h'(x_g)^2p(x_g)p''(x_g) + h(x_g)h'(x_g)h'''(x_g)p(x_g)^2 + \\ -h(x_g)^2h'(x_g)p(x_g)p'''(x_g) - h(x_g)^2h''(x_g)p(x_g)p''(x_g) + h(x_g)^3p''(x_g)^2,$$

where we used that it is a weak focus by writing $p'(x_g) = \frac{h'(x_g)p(x_g)}{h(x_g)}$.

If this expression is equal to zero, then we get a second (or higher) order weak focus. Summarizing we get:

Lemma 8.1 *The singularity A of system (1.1) with $\gamma = 1$ at (x_g, y_g), where $y_g = \frac{h(x_g)}{\delta}$ and $p(x_g) = \delta$, is an elementary anti-saddle iff $p'(x_g) > 0$. If $p'(x_g) \neq \frac{h'(x_g)p(x_g)}{h(x_g)}$, it is a strong anti-saddle. If $p'(x_g) = \frac{h'(x_g)p(x_g)}{h(x_g)}$, then it is a first order weak focus if $p'''(x_g) \neq \frac{W_1(x_g)}{h(x_g)^2h'(x_g)p(x_g)}$, where*

$$\begin{aligned} W_1(x_g) = &-h'(x_g)^2h''(x_g)p(x_g)^2 + h(x_g)h'(x_g)^2p(x_g)p''(x_g) \\ &+ h(x_g)h'(x_g)h'''(x_g)p(x_g)^2 + \\ &-h(x_g)^2h''(x_g)p(x_g)p''(x_g) + h(x_g)^3p''(x_g)^2. \end{aligned}$$

It is a second (or higher) order weak focus iff $p'(x_g) > 0$, $p'(x_g) = \frac{h'(x_g)p(x_g)}{h(x_g)}$, $p'''(x_g) = \frac{W_1(x_g)}{h(x_g)^2h'(x_g)p(x_g)}$.

It is difficult to draw general conclusions from this lemma. However, a simple observation is that for any weak focus necessarily $p'(x_g) > 0$. It means that adding a group defense component does not have an impact on the order of the weak focus. Group defense would imply the existence of values of x where $p'(x) < 0$. At those points though a weak focus cannot appear. The Andronov–Hopf bifurcation for the general Gause-system is therefore determined by the functions $p(x)$ and $h(x)$ for those values of the prey densitities where group defense does not play a role.

For the choice of a cubic functional response $p(x)$, it turned out that a second order weak focus can occur for parameter values in W_1. It was stable. Interestingly enough, if a second order weak focus in system (1.1) would be unstable, Lemma 3.2 shows that it would be surrounded by a stable limit cycle, so generically, independent of the behaviour of $p(x)$, $h(x)$ for $x > 1$ the system could have at least three limit cycles! It is not clear if there is a deeper reason why in system (1.2) the second order weak focus turned out to be stable.

For system (1.2) it is easy to check that at the weak focus of order 2 the convexity of $p(x)$ is fixed. Substitution of the values for a_0 and a_1 according to (2.16) and (2.15) respectively into $p''(x_g)$ gives:

$$p''(x_g) \equiv x_g(2k - x_g)(k - 2x_g) > 0.$$

The second order weak focus can only appear on the part of $p(x)$ where the function is concave up. It is a remarkable feature which does not necessarily seem to be true for the general case. At the center case C it can easily be checked that $p''(x_g) < 0$.

Another interesting observation is that if we allow group defense into system (1.2) then no weak focus of order two can appear in system (1.2). It is the situation when the region of interest W_1 is extended with parameter values such that $p(x)$ has exactly one local maximum on the interval $0 \leq x \leq 1$. It is easily verified that this corresponds to a region W_2 defined through $a_0 + a_1 + 1 < 0 \wedge a_1 < 1$. By following similar calculations as were done for the region W_1 the conclusion is that no second order weak focus occurs for $k > 1$. Again it is not clear if this has a deeper reason or if it is a manifestation of the specific choices of the functions $p(x)$ and $h(x)$. In general, systems with group defense have been found with a second order weak focus. Nevertheless in our restricted family it does not occur.

8.2 Center Bifurcation

The bifurcation of 3 limit cycles from the center case C in (1.2) was essentially based on two mechanisms. The center case itself contained a stable limit cycle surrounding the period annulus and it was possible to bifurcate a stable weak focus of order one from the center which created a second unstable limit cycle from the outer boundary of the period annulus. A third limit cycle was then created from the weak focus in an Andronov–Hopf bifurcation.

How does this extend to general cases? Suppose system (1.1) has a period annulus on the interval $0 \leq x \leq 1$, i.e. $h(x) = cp(x)$. According to Lemma 5.1 the local behaviour of the two functions for $x = 1 + \epsilon$ will determine the outer stability of the annulus. In particular the graphical interpretation is that if the graph H of $y = h(x)$ lies above (below) the graph P of $y = cp(x)$, then the annulus is unstable (stable). Under rather general conditions system (1.1) will be bounded and the Poincaré–Bendixson shows that at least one stable limit cycle surrounds the annulus if H lies above P. Since we assume here that $p(x)$ is increasing and that in the general case $h(x)$ will have a zero for $x > 1$, it follows that the graphs of H and P will intersect at least once. See Fig. 27.

This case was discussed in more detail in [15] where the case of uniqueness of the surrounding limit cycle was discussed as well.

Since the relative position of the graphs H and P determines the stability of the period annulus it is easy to create an example of a center case surrounded by M limit cycles with $M \in \mathbb{N}$. This is established by starting with a situation where H lies above P for $x = 1 + \epsilon$. This implies the existence of a stable limit cycle surrounding the annulus. For fixed $h(x)$ we will perturb $p(x)$ in such a way that a second limit cycle

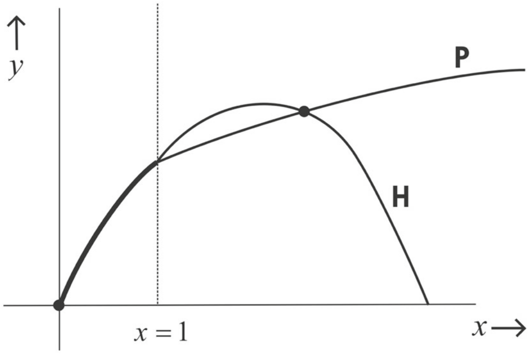


Fig. 27 General position of the graphs of H: $y = h(x)$ and P: $y = cp(x)$ for which a stable limit cycle surrounds the center case in system (1.1) if for $0 \leq x \leq 1$, $h(x) = cp(x)$

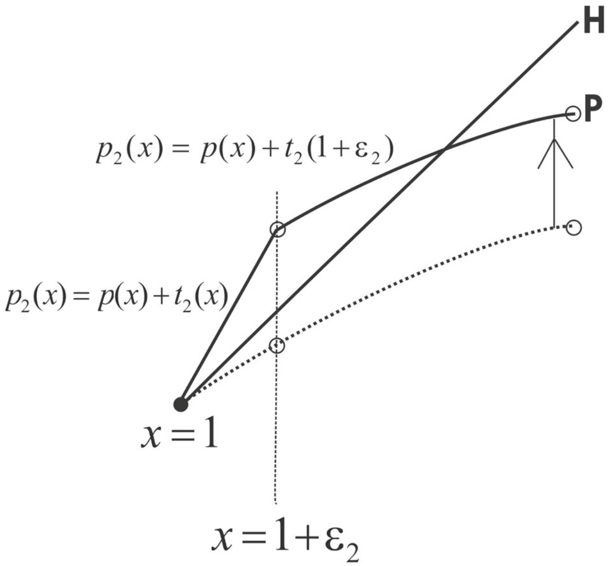


Fig. 28 The graphs of H: $y = h(x)$ and P: $y = cp(x)$ such that an unstable limit cycle is created from the outer boundary surrounding the center case in system (1.1) if for $0 \leq x \leq 1$, $h(x) = cp(x)$

is created from the cycle tangent to $x = 1$. Change $p(x)$ into $p_2(x) = p(1) + t_2(x)$ on the interval $1 < x < 1 + \epsilon_2$ with $t_2(1) = 0$ and $c(p'(1) + t_2'(1)) > h'(1)$, with $0 < \epsilon_2 \ll 1$. The function $t_2(x)$ is assumed to be differentiable at $x = 1$ and its derivative is chosen large enough to make P lie above H near $x = 1$. On the interval $x \geq 1 + \epsilon_2$ continue with the original $p(x)$ lifted to make $p_2(x)$ is continuous, i.e. $p_2(x) = p(x) + t_2(1 + \epsilon_2)$. Since $t_2(x)$ was assumed to be differentiable the constant $t_2(1 + \epsilon_2)$ can be made small enough to ensure that the original intersection between H and P is not removed. The situation is illustrated in Fig. 28.

By choosing ϵ_2 sufficiently small the original stable limit cycle is kept. However, the period annulus has become stable on the outside because the relative positions of H and P have changed. The Poincaré–Bendixson theorem then states that inside the original stable limit cycle and the outer boundary of the annulus (i.e. the cycle tangent

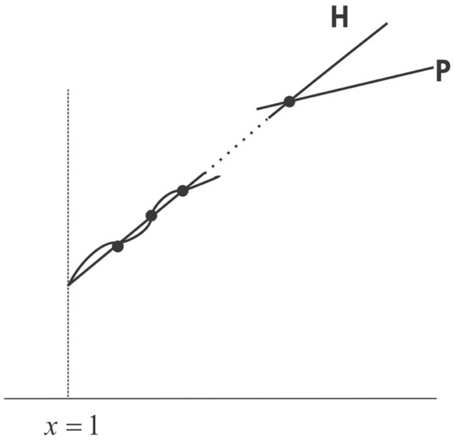


Fig. 29 The graphs of $y = h(x)$ and $y = cp(x)$ such that $M - 1$ limit cycles are created from the outer boundary surrounding the center case in system (1.1) if for $0 \le x \le 1$, $h(x) = cp(x)$

to $x = 1$) an unstable limit cycle must exist. It was created from the outer boundary of the annulus. It is not difficult to see that by continuing this procedure of switching the positions of H and P near $x = 1$ an arbitrary number of limit cycles can be created, see Fig. 29. Essentially it means that locally $M - 1$ extra intersections of the graphs H and P need to be created. These $M - 1$ new limit cycles are of the type II as defined in 3.4. The first limit cycle will be of type III because it has to cross the value of x where the first intersection of H and P occurs. This intersection wil lie to the right of the zero of the function $f(x)$ in the Liénard form of the system.

This shows that with little effort systems of the type (1.1) can be constructed where $M - 1$ type II limit cycles occur and 1 limit cycle of type III. Next we can perturb the center and try to create a stable weak focus of order one. A simple analysis of the first focal value as given in Lemma 8.1 shows that it is possible to do this generically. This is due to the fact that from a center typically a weak focus can be created and that it can be unstable or stable depending on the bifurcation direction.

According to this heuristic argument generically if a center situation occurs of this type, $M + 2$ limit cycles can be created from a center case where 2 are of type I, $M - 1$ are of type II and 1 is of type III.

We note that for systems with group defense a similar argument could hold true for the creation of the $M - 1$ limit cycles of type II and the one limit cyle of type III. Additional conditions need to be imposed though, because systems with group defense can have a saddle singularity in the first quadrant which could prevent the use of the Poincaré–Bendixson theorem. It shows again that systems without group defense are more flexible for creating a multiple number of limit cycles.

Finally we note that these mechanisms give a minimum number of limit cycles after bifurcation but not an upper bound which is obviously a much more difficult task. For some special cases this can be done though. For example: in the limit $\phi \downarrow 0$ the system becomes singular and perturbation techniques can be applied to find the maximum number of the perturbed limit cycles. This will be done in a forthcoming paper.

8.3 Singular Bifurcation

Essentially the analysis of the system (1.2) for $\delta \uparrow 1$ extends literally to the general case if we impose that $p(x) = 1$ for $x > 1$. Following the argument of the previous sections we get that the perturbed system will always have exactly two limit cycles, one of type II and one of type III if the perturbed singularity is a strong stable focus and the function $h(x)$ satisfies $h'(1) > 0$. This latter condition is necessary to ensure that the blown-up system contains an unstable manifold entering the region $x > 1$. These two limit cycles will also exist for the perturbation into the weak focus cases if a stable weak focus is created. If a stable weak focus of order 1 can be perturbed then in total 3 limit cyles can occur. If a stable focus of second order can be created then 4 limit cycles can occur after perturbation, etcetera.

Here also the choice of a group defense element in $p(x)$ will complicate the analysis and it is not easy to see how many limit cycles can be created, mainly because of the possibility of a saddle singularity in the phase plane.

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