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An Improved Source Term for Finite-Element Modelling with the Stress-Velocity Formulation of the Wave Equation

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Summary

For seismic modelling, imaging and inversion, finite-difference methods are still the workhorse of the industry despite their inability to meet the increasing demand for improved accuracy in subsurface imaging. Finite-element methods offer better accuracy but at a higher computational cost. A stress-velocity formulation with linear elements and an iterative method, defect correction, for inverting the mass matrix offers fourth-order super-convergence but is susceptible to numerical noise if waves in the wrong part of the dispersion curve are excited. We propose an improved source term that reduces that noise and investigate the accuracy of the method on structured triangular meshes as well as on unstructured rotated meshes. With an optimised source function, it is seen that the dispersive wavelengths can be avoided, giving the defect-correction approach a better performance than the mass-lumped formulation with only a marginal increase in compute effort.

Introduction

The finite-element method (FEM) offers generality, accuracy and adaptivity not available in the finite-difference method (FDM). The latter is popular for production runs in the hydrocarbon industry because of its relative ease of coding and reasonable computational efficiency. With the increasing demand for more accuracy in subsurface imaging as well as more abundant compute cores, the FEM may become more attractive. They have a higher computational cost than FDMs but high-order versions have superior accuracy in the presence of hard contrasts and topography and better exploit modern computer hardware. Alternatively, we may look for less compute-intensive formulations. The second-order formulation of the wave equation is often used for modelling seismic wave propagation with spectral methods, both for box-like elements on quadrilaterals and hexahedra (Komatitsch et al., 1999, e.g.) as well as for simplex-based elements on triangles (Mulder, 1996, 2013) or tetrahedra (Zhebel et al., 2014; Mulder and Shamasundar, 2016). For some applications, a first-order formulation may be desirable. Ainsworth (2014) claims better accuracy than with the second-order formulation in terms of numerical dispersion for odd-degree polynomial basis functions if the consistent mass matrix is used. Shamasundar and Mulder (2016a), however, showed that for elements of degree higher than one, projection errors on the spurious modes interfere and make the first-order formulation inferior to the second-order one. That only leaves the lowest-degree linear element as a serious candidate. Mass lumping, required to avoid the cost of inverting the large sparse mass matrix, will unfortunately decrease its accuracy from fourth to second order. A single iteration with an iterative method like defect correction will solve this problem, as shown in (Shamasundar and Mulder, 2016a) for the 1-D case. Shamasundar and Mulder (2016b) presented a first step towards extending the method to 2D, but initial results were extremely noisy unless a gaussian source was used that suppressed waves in the high-wavenumber part of the numerical dispersion curve, where the group velocity has the wrong sign. Here, we propose the use a tapered-sinc source term in the finite-element discretization that band-limits the wavenumber spectrum, similar to the one of (Hicks, 2002) for the FDM, and examine its performance on a homogeneous problem with an exact solution, using structured or unstructured triangular meshes. An application of the method to a less trivial model is included.

Method

The first-order formulation of the constant-density acoustic wave equation is

$$\rho^{-1}c^{-2}\partial_t p = \partial_x v_x + \partial_z v_z + f, \quad \rho\partial_t v_x = \partial_x p, \quad \rho\partial_t v_z = \partial_z p,$$

in contrast to the second-order formulation,

$$\rho^{-1}c^{-2}\partial_{tt} p = \partial_x(\rho^{-1}\partial_x p) + \partial_z(\rho^{-1}\partial_z p) + f'.$$

These are equivalent but their finite-element discretizations are usually different. In the 2-D case, the first-order or velocity-stress formulation involves three global mass matrices, which may or may not be lumped, whereas the second-order formulation has only one. The wavefields p,v_x,v_z are represented by polynomial interpolants $\psi_k(\xi,\eta)$ on the reference triangle. The triangles in the mesh are mapped from Cartesian to natural coordinates, $(\xi,\eta,1-\xi-\eta)$, on the reference triangle with vertices (0,0), (0,1), (1,0). In each element, there is a local mass matrix A and first-derivative matrices D^1 and D^2, each with entries

$$A_{k,l} = \int_0^1 \mathrm{d}\xi \int_0^{1-\xi} \mathrm{d}\eta\, \psi_k(\xi,\eta)\psi_l(\xi,\eta),$$

$$D^1_{k,l} = \int_0^1 \mathrm{d}\xi \int_0^{1-\xi} \mathrm{d}\eta\, \psi_k(\xi,\eta)\frac{\mathrm{d}}{\mathrm{d}\xi}\psi_l(\xi,\eta), \quad D^2_{k,l} = \int_0^1 \mathrm{d}\xi \int_0^{1-\xi} \mathrm{d}\eta\, \psi_k(\xi,\eta)\frac{\mathrm{d}}{\mathrm{d}\eta}\psi_l(\xi,\eta).$$

Together with an additional iteration, this leads to a higher cost per time step, which is offset by a less restrictive time step limit for stability and a much improved accuracy. Whether or not this gain in accuracy can be achieved on an unstructured mesh is the question addressed here.

To discretize the source term, we can integrate the FEMs linear basis functions against a delta function. With the current first-order formulation, this produces very noisy results, as shown earlier (Shamasundar

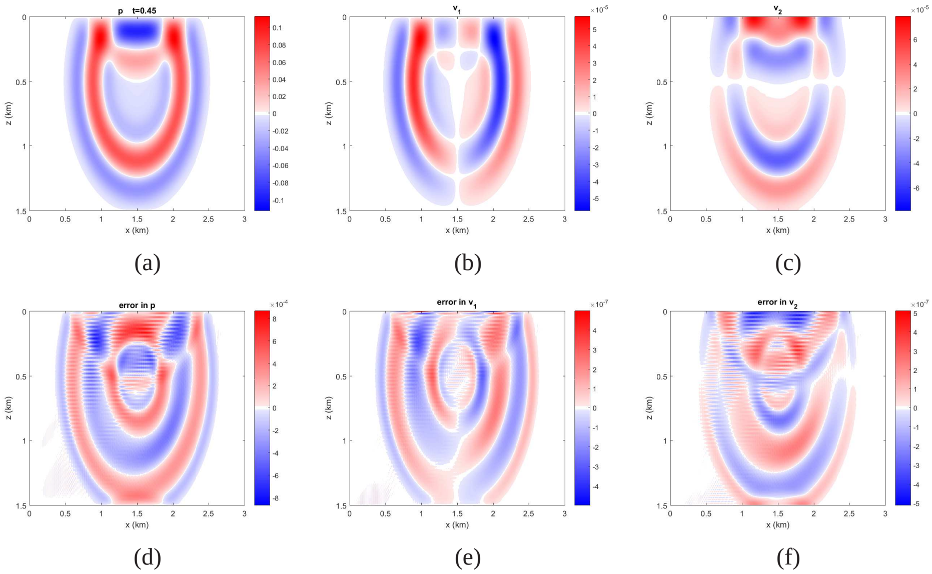


Figure 1 *Wavefields at the end of the simulation (0.45 s) on a structured mesh with 20301 vertices. The top row shows the numerical solution for* p *(a),* v_x *(b) and* v_z *(c), the bottom row their errors.*

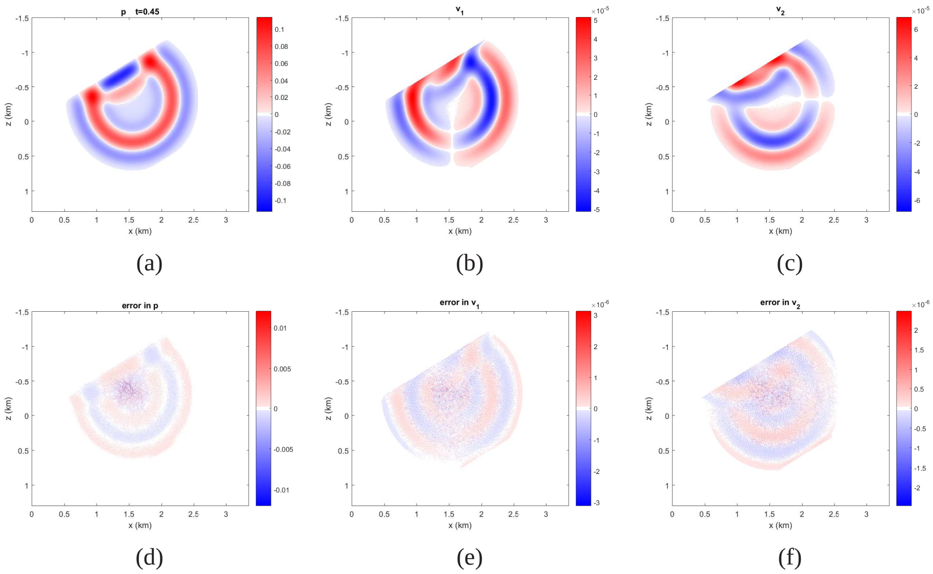


Figure 2 *As Figure 1 but on an unstructured mesh with 26542 vertices, rotated by 30°.*

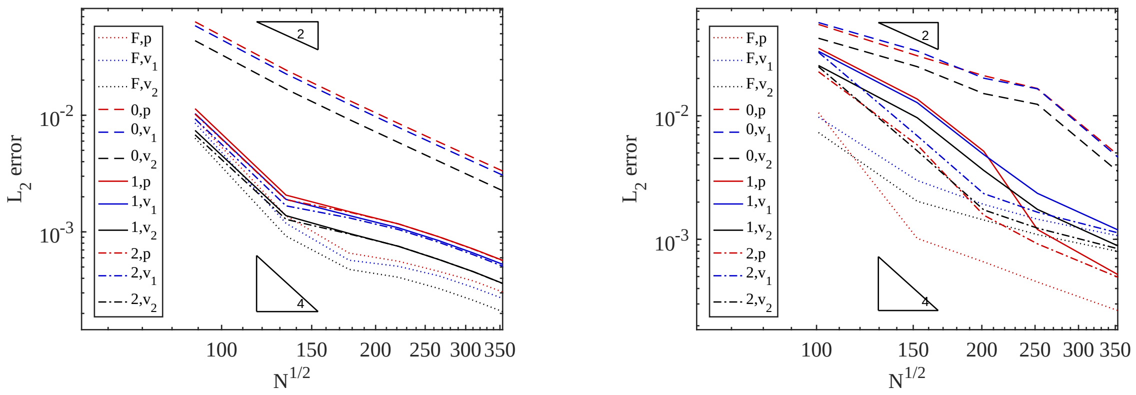
and Mulder, 2016b). An alternative is to integrate against a gaussian. This effectively suppressed the noise produced by that part of the dispersion curve where the group velocity has the wrong sign but also increases the numerical error for wavelengths in the correct part of the dispersion curve. A third option is integration against a tapered-sinc function of the form

$$\frac{1}{2}\left[1+\cos\left(\frac{\pi\zeta}{n_w+1}\right)\right]\frac{\sin\pi\zeta}{\pi\zeta}, \quad \zeta=\frac{\sqrt{(x-x_s)^2+(z-z_s)^2}}{r_s}\le(1+n_w),$$

and zero otherwise. Here, the source is located at (x_s,z_s) and r_s and n_w are parameters that need to be optimized.

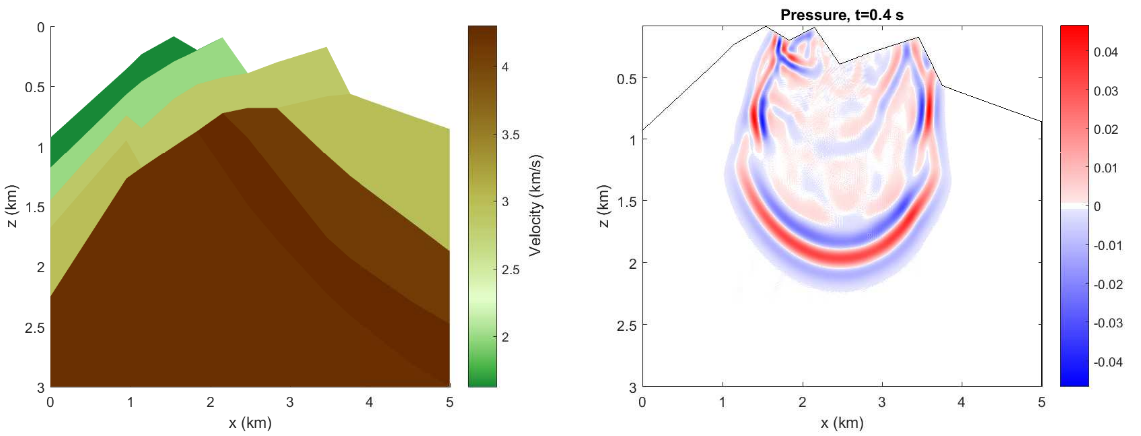
Results

We tested our method first on a homogeneous velocity model of 1.5 km/s in a domain of size $[0,3.0]\times[0,1.5]\,\mathrm{km}^2$ with zero Dirichlet pressure boundary conditions on all sides. The numerical solution was compared against the exact solution. The source was placed at $x_s=1.5$ km and $z_s=0.5$ km. The wavelet had compact support and was $w(t)=-(T_w/8)^2\frac{\mathrm{d}}{\mathrm{d}t}[1-(2t/T_w)^2]^8$. for $|t|<\frac{1}{2}T_w$ and zero otherwise. We



(a) (b)

Figure 3 *Convergence on a structured (a) and unstructured (b) mesh with the consistent mass matrix (dotted line), lumped mass matrix (dashes), 1 iteration (drawn) or 2 (dash-dotted line). Although the formal fourth-order accuracy is not obtained, the accuracy improvement with just one iteration compared to the second-order mass-lumped scheme is substantial.*



(a) (b)

Figure 4 *Velocity model (a) and a snapshot of the pressure wavefield (b).*

set $T_w = 0.31$ s, which in the second-order formulation would correspond to a peak frequency of 3 Hz, and let the computation run from time $-\frac{1}{2}T_w$ to 0.45 s. The convergence behaviour on both a structured and an unstructured mesh were examined. The latter was rotated by 30°. Figure 1 displays modelling results for one particular run on a structured mesh and Figure 2 for the rotated unstructured mesh. Figure 3 shows the convergence in terms of the root-mean-square error, divided by the maximum amplitude of the corresponding wavefield, as a function of mesh size. The latter has been replace by $N^{1/2}$, where N is the number of degrees of freedom for just the pressure or one of the velocity components. Fourier analysis predicts fourth-order convergence on structured triangular meshes (Shamasundar and Mulder, 2016a) but that rate is not obtained, partly because of the second-order order time-stepping error showing up on finer meshes, despite the fact that the time step was chosen deliberately smaller than its maximum allowable value. Although the fourth-order convergence rate is not obtained, the improvement over just mass-lumping with one or two inexpensive iterations is significant. The use of the consistent mass matrix is computationally unattractive and its results are only included for reference. We have repeated runs like these for a range of parameters n_w and r_s and found that $n_w = 3$ and $r_s = 2$ are good choices for both the structured and unstructured test problem.

Figure 4 displays a more complicated model, along with the wavefield at a time of 0.4s. The mesh was generated such that the triangle sizes scale with the local velocity. The Ricker wavelet had a peak frequency of 12 Hz and the source was located at $x_s = 2468.4$ m and $z_s = 410.35$ m.

Conclusions

We have demonstrated that the first-order formulation of the wave equation for constant-density acoustics with mass lumping and defect correction and linear elements can provide a substantial gain in accuracy compared to just mass lumping. A judiciously chosen source function, which avoids the excitation of waves in the 'wrong' part of the dispersion curve, enables this gain, although the formal fourth-order super-convergence predicted by local-mode analysis on structured triangular meshes was not obtained in our test problems. It remains to be seen if the method can outperform the less accurate but also less compute intensive second-order formulation, in particular in 3D.

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