Delft University of Technology
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Symmetry groups of regular polytopes in three and four dimensions.

The Platonic Solids, Binary Groups and Regular Polytopes in four-dimensional space.

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Abstract

A pentagon is an example of a highly symmetric polygon in two-dimensional space. The three-and four-dimensional analogue of these polygons are the regular polyhedra and the regular polytopes. There exist five regular polyhedra in three-dimensional space and these are called the Platonic solids. These five Platonic solids are the tetrahedron, cube, octahedron, dodecahedron and the icosahedron. In four-dimensional space, the regular polytopes are the 5-cell, the 8-cell, also called the tesseract, the 16-cell, the 24-cell, the 120-cell and the 600-cell.

The aim of this thesis is to give an introduction to some symmetry groups of the regular polytopes in three and four dimensional space at undergraduate mathematical level. The main focus of this thesis is to describe the symmetry group of the icosahedron, to introduce the Icosians, which are related to the rotation group of the icosahedron, and to study the action of the symmetries of the 600-cell on the twenty-five 24-cells it circumscribes.

First, the symmetry groups of the Platonic Solids, the regular polytopes in three dimensional space, will be established. Then it will be shown that there exists a two-to-one map from the the group $\mathbb{H}_1$ of unit quaternions to the group SO(3), the group of 3×3 orientation-preserving matrices. This map will be used to describe the binary groups, which are double covers of the rotation groups of the Platonic solids. After that, the symmetry group of the tesseract will be studied both via an isomorphism between $G := \{\pm 1\}^4 \times S_4$ and the symmetry group of the tesseract as well as geometrically via rotation planes. Then, the 24-cell and the 600-cell will be defined as the four-dimensional regular polytopes whose vertices are the quaternions from the binary tetrahedral group and the binary icosahedral group, the Icosians. It will be shown that twenty-five 24-cells inscribe a 600-cell and that there are 10 ways to decompose the vertices of a 600-cell into the vertices of 5 disjoint 24-cells. Next, it will be shown that these 10 decompositions are chiral, 5 being 'left-handed' and 5 being 'right-handed'. Finally, it is shown that the symmetry group of the 600-cell acts on these $5+5$ decompositions by permutation, each permutation being described by an element from $A_5 \times A_5 \rtimes \{\pm 1\}$, where -1 acts on $A_5 \times A_5$ by interchanging the factors of $A_5 \times A_5$.

Acknowledgements

I am extremely grateful to my supervisors dr. Jeroen Spandaw and dr. Paul Visser, foremost for their extensive explanations, patient guidance and useful ideas during our weekly Zoom meetings. Additionally, their critiques and recommendations together with the extensive collaboration in the last few weeks of this research has been very much appreciated. Lastly, a special thanks goes to my brother, for his unwavering patience to give me valuable advice and clear explanations throughout my bachelor.

Contents

List of symbols

Symbol	Description
I	the icosahedral rotation group
I_h	the full icosahedral group, also denoted H_3
H_3	I_h
T	the tetrahedral rotation group
T_d	the full tetrahedral symmetry group
O	octahedral rotation group
O_h	the full octahedral group
$\mathbb{H}$	the group of quaternions
$\mathbb{H}_1$	the group of unit quaternions
$2I$	the binary icosahedral group, the extension of I
$2O$	the binary octahedral group, the extension of O
$2T$	the binary tetrahedral group, the extension of T
C_8	the tesseract, 8-cell, hypercube of dimension 4
O_4	the rotation group of the tesseract
$O_{4,h}$	the full symmetry group of the tesseract
Q_8	the group of 8 quaternions given by the permutations of $(\pm 1, 0, 0, 0)$
$[3,3,5]$	The Coxeter notation for the symmetry group of the 600-cell
S_n	group of permutations of n elements
A_n	the group of even permutations of n elements
V_4	the Klein Vierergruppe
C_n	the cyclic group of n elements
C_2	the cyclic group of 2 elements, which we view as the multiplicative group $\{\pm 1\}$
$O(3)$	the group of orthogonal 3×3 matrices, (determinant ± 1)
$O(4)$	the group of orthogonal 4×4 matrices, (determinant ± 1)
$SO(3)$	the group of orthogonal and orientation preserving 3×3 matrices, (determinant 1)
$SO(4)$	the group of orthogonal and orientation preserving 4×4 matrices, (determinant 1)

1 Introduction

1.1 Symmetries in the branches of science

The world is full of symmetries. Symmetries are often considered to be beautiful and fascinating as they represent a kind of evenness, sameness, order or regularity. The Greeks, for example, were already fascinated by symmetries. They hoped the symmetries present in objects would be present in the structure of nature as well. Plato described some symmetries in mathematics, although he did not have a formal definition for symmetries nor was he interested in these symmetries in themselves. For him, the search for Beauty was the motivation to consider symmetric objects. [25]

However, throughout the years, symmetries have become highly useful in scientific research as well. Gross even claims that it would be *hard to imagine that much progress could have been made in deducing the laws of nature without the existence of certain symmetries*. [20] Penrose even points out the existence of symmetries in quite some important laws in nature. Het writes:

> All the successful equations of physics are symmetrical in time. They can be used equally well in one direction in time as in the other. The future and the past seem physically to be on a completely equal footing. Newton's laws, Hamilton's equations, Maxwell's equations, Einstein's general relativity, Dirac's equation, the Schrödinger's equation - all remain effectively unaltered if we reverse the direction of time. [30]

In chemistry, for example, symmetry are found as well. The borohydride-anion, dodecahedrane and the pure carbon atom, for instance, denoted by $B_{12}H_{12}^{2-}$, $C_{20}H_{20}$ and C_{60} respectively are said to have icosahedral symmetry. [23] Icosahedral symmetries can be described by the rotation group of the icosahedron, a three-dimensional regular polytope. As a matter of fact, the investigation of this rotation group is one of the subjects of this thesis. Another symmetry appearing in chemistry is found in the organization of protein cells enclosing the DNA of a virus. Those protein cells form the capsid and are often found to be highly symmetric. In fact, there are many viruses whose capsids have icosahedral symmetry. [9]

Furthermore, the icosahedral symmetries can be described by a group of quaternions, called the the Icosians. Those Icosians are closely related to the three- and four-dimensional regular polytopes. This group seems to be rich and filled with beautiful symmetries that are not yet fully explored. A possible research area concering those Icosians lies in the quantum mechanics. Namely, there seems to be a *tantalizing similarity to the structure of the known particles, the Standard*

Model from the particle physics. The understanding of the subgroups and conjugacy classes and the root systems of the Icosians, is a step on the way to examine whether there might exists *a meaningful mapping between the (basis states of) elementary particles and the group of Icosians.* [35]. Such a mapping might be useful to calculate the interactions between particles from the Standard Model more effectively. At the moment, scattering processes between fundamental particles *must be calculated by summing over all Feynman diagrams that describe the allowed interactions.*

This thesis has without doubt been a great and fascinating learning experience. However, I hope that the symmetries of the 600-cell that are not yet explored will contribute to determine the analogies between the Icosians and the Standard Model in particle physics.

1.2 Thesis outline

The aim of this thesis is to give an account of the regular polytopes in three- and four-dimensional space by studying their symmetry groups. The study of all the symmetry groups of these regular polytopes would be beyond the scope of a bachelor thesis. Therefore, this thesis will mainly be focused on the symmetry group of the icosahedron and the binary icosahedral group, called the Icosians, extending the rotation group of the icosahedron. These Icosians form the vertices of a four-dimensional regular polytope, called the 600-cell. This four-dimensional regular polytope will be studied extensively in this thesis as well. Although not all symmetry groups of the regular polytopes in three and four dimensional space are the main focus of this thesis, an account of the symmetry groups of the Platonic solids and the tesseract, together with an investigation of the 24-cell and the binary tetrahedral group and binary octahedral group will be given.

This thesis is written in such a way that it is understandable for any undergraduate mathematics student familiar with some Algebra and Linear Algebra. All required knowledge in these fields are included in the Appendix A.1.

This thesis is organised as follows. First of all, the Platonic Solids are introduced in Section 2. Those Platonic Solids are the three-dimensional regular polytopes: the tetrahedron, cube, octahedron, icosahedron and dodecahedron. The symmetry groups of these Platonic solids solids will be studied together with their duality. Also, the inscription of tetrahedra in a cube and cubes in a dodecahedron will be studied.

In Section 3, the quaternions will be formalized and the Euler's rotation theorem will be given. It will be shown that the rotation in three-dimensional space can be represented by quaternions. Representing the rotations from the rotations groups of the Platonic Solids by quaternions, the binary groups of these quaternions will be studied: the icosahedral group, tetraheral group and the octahedral group.

Next, in Section 4, symmetry groups in four dimensional space will be introduced by investigating the symmetry group of the tesseract. We will show that the symmetry group of the tesseract is isomorphic with the set $G := \{\pm 1\}^4 \times S_4$. This set is used to describe the symmetries of the tesseract. Afterwards, the symmetries of the tesseract will also be described geometrically using rotations planes. After that, the 24-cell will be investigated and we will show that there are three tesseract inscribed in a 24-cell as well as three 16-cells. As crowning achievement of the investigation of symmetry groups, the 600-cell will be studied. In particular, we will show that the 600-cell can be decomposed into 5 disjoint 24-cells in 10 different ways. It will be shown that those 10 decompositions are chiral, that is, 5 of them are 'left-handed' and 5 are 'right-handed'. It will be shown that the 600-cell acts on those $5+5$ decompositions by permutation. In fact, it will be shown that a symmetry of the 600-cell maps the left-handed decompositions to left-handed decompositions and right-handed decompositions to right-handed decompositions if and only if it preserves the orientation of the 600-cell.

The symmetries of the icosahedron and dodecahedron will run like a thread through this entire thesis, starting in Section 2, coming back in the binary icosahedral group in Section 3.2 and appearing again in the study of the 600-cell in Section 4.3. The symmetries of the 600-cell and the appearance of the icosahedral rotation group over and over again, must make mathematicians agree with Plato that there is a fascinating amount of beauty in mathematics.

1.3 Preliminary remarks

There are two remarks that need to be made in advance. The first remark concerns the figures used in thesis. Any figure without reference to a source is produced by myself using either the TikZ-package in LaTeX or using Mathematica. Secondly, some theorems in thesis have been found and been proved computationally. The Mathematica code used to find those result have been included in the Appendix B. However, together with my supervisors Dr. J.G. Spandaw and P.M. Visser, these theorems have been proved using geometry, group theory and root systems as well. Some of these proofs contain more details. However, these details are not included in this bachelor's thesis. The theorems concerned are:

- Theorem 4.17
- Theorem 4.18
- Theorem 4.20
- Theorem 4.21
- Theorem 4.22
- Remark after Definition 4.23
- Theorem 4.26
- Conjecture 4.27.

2 Three dimensions

2.1 Platonic Solids

Definition 2.1. In three-dimensional space, a Platonic solid is a regular, convex polyhedron. It is constructed by congruent, regular polygonal faces with the same number of faces meeting at each vertex. [13]

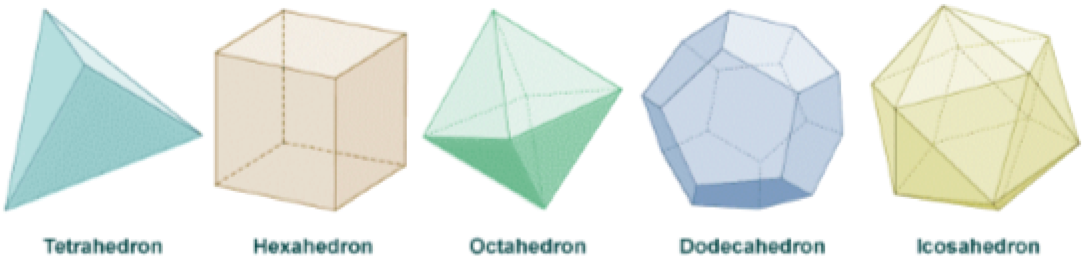


Figure 1: The five Platonic Solids. *[Image retrieved from [5]]*

There are exactly five regular, convex polyhedra: the tetrahedron, the cube or hexahedron, the octahedron, dodecahedron and the icosahedron. The existence of those Platonic Solids can be shown directly from the definition of the coordinates of its vertices and the regularity of the polyhedral faces. The converse, the fact that there are only five Platonic Solids can be obtained from the observation that for such a solid the sum of the angles between the edges of the faces meeting in a vertex, has to be strictly smaller than 360°. Indeed, if this sum add up to exactly 360°, the figure obtained is two-dimensional. If the angles add up to a number smaller than 360 there is space left to fold the faces meeting in a vertex in such a way that a three-dimensional polyhedron is constructed.

We will prove that there are at most five Platonic Solids. We will not prove the existence of the Platonic solids. That is, we will not prove that there are at least five Platonic Solids. However, an elaborate construction of the Platonic Solids can be found in Propostion 13 to 17 of the thirteenth book of *Euclid's elements* [12].

Theorem 2.2. *There are at most five Platonic solids: the tetrahedron, the cube, the octahedron, icosahedron and dodecahedron.*

We establish a formula for the edges, vertices and faces that such a solid has to satisfy. Together with Euler's polyedral formula in Theorem 2.3, we show that there are at most five Platonic solids.

Theorem 2.3 (Euler's polyhedral formula). *The number of vertices V, faces F, and edges E in a convex three-dimensional polyhedron satisfy*

$$V - E + F = 2.$$

Any Platonic solid is by definition a convex 3-dimensional polyhedron and thus satisfies Euler's polyhedral formula.
Define V to be the number of vertices of a polyhedron, E to be the number of its edges, F the number of its faces, E_F the number of edges of each face and F_V the number of faces that meet at each vertex. Then for any Platonic solid, the following equations have to be satisfied:

$$\begin{cases} E_F \cdot F = 2E = F_V \cdot V & (1) \\ V - E + F = 2 & (2) \end{cases}$$

The first equality in Equation (1) comes from the fact that each edge is incident to two vertices. The number of faces meeting in each vertex is the same as the number of edges meeting in a vertex. Thus, the product of the number of faces incident to a vertex and the total number of vertices is exactly twice the number of edges. The second equality in Equation (1) is derived from the fact that the each edge is incident to two faces. Thus, counting the total number of faces and multiplying this number with the number of edges of each face equals twice the number of edges as well.

Substitution of Equation (1) into Equation (2), dividing the obtained equation by 2 and taking E outside the brackets, gives the following equality:

$$E\left(\frac{1}{F_V} - \frac{1}{2} + \frac{1}{E_F}\right) = 1.$$

Since $E > 0$, it follows that :

$$\frac{1}{F_V} + \frac{1}{E_F} > \frac{1}{2}.$$

An explicit formula for V, E and F can also be found from substitutions of Equation (1) into Equation (2). These formulas for V, E and F are used to make Table 1.

$$\begin{cases} V = \dfrac{4E_F}{4-(F_V-2)(E_F-2)} & (3) \\ E = \dfrac{2E_F \cdot F_V}{4-(F_V-2)(E_F-2)} & (4) \\ F = \dfrac{4F_V}{4-(F_V-2)(E_F-2)} & (5) \end{cases}$$

Obviously, $E_F, F_V \geq 3$ to construct a three-dimensional object. If $F_V, E_F \geq 5$, Equation (4) implies that $E < 0$, which is impossible. Thus $3 \leq F_V, E_F \leq 5$. Furthermore, if $E_F = F_V = 4$, the system of equations in Equation (1) and Equation (2) becomes unsolvable. Geometrically, this can be explained as follows. If $E_F = F_V = 4$ there are 4 squares meeting at each vertex. Since the total angle meeting at the vertex equals $4 \cdot 90 = 360°$, the constructed object is again two-dimensional. Hence, $E_F = V_F = 4$ flattens the object and is not a Platonic Solid.

After substituting all possible values for E_F and F_V into the Equations (3), (4) and (5), we obtain the five Platonic solids: the tetrahedron, cube, octahedron, dodecahedron and the icosahedron. These Platonic solids, together with the number V, E, F, F_V and E_F, are visualized in Table 1 below. [24]

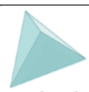
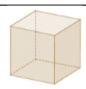
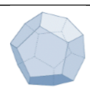
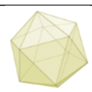
$F_V = 3$ $E_F = 3$	$V = 4$ $V = 6$ $F = 4$	 Tetrahedron
$F_V = 3$ $E_F = 4$	$V = 8$ $E = 12$ $F = 6$	 Cube
$F_V = 3$ $E_F = 5$	$V = 20$ $E = 30$ $F = 12$	 Dodecahedron
$F_V = 4$ $E_F = 3$	$V = 6$ $E = 12$ $F = 8$	 Octahedron
$F_V = 5$ $E_F = 5$	$V = 12$ $E = 30$ $F = 20$	 Icosahedron

Table 1: The five regular, convex 3-dimensional polyhedra, known as the Platonic Solids, with their number of vertices, edges and faces. *[Image retrieved from [5]]*

2.2 Icosahedron

This chapter is about the icosahedron, one of the five Platonic solids. The number of vertices, edges, faces as well as the symmetry group is examined. The rotations and reflections of the icosahedron will be made explicit and we will set up an isomorphism between the rotation group of the icosahedron and the alternating group A_5.

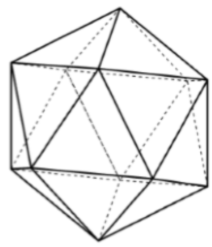


Figure 2: An icosahedron.

2.2.1 Definition of the icosahedron

Definition 2.4 (Icosahedron). The icosahedron is a regular polyhedron whose faces are 20 equiliteral triangles of which 5 meet in each vertex.

Since the icosahedron is made up of 20 triangles and each triangle is circumscribed by 3 edges, the icosahedron has $\frac{20\cdot 3}{2} = 30$ edges. In each vertex, 5 edges meet and each edge is incident to 2 vertices. Hence, the total number of vertices of the icosahedron is $\frac{30\cdot 2}{5} = 12$ vertices. In summary, the icosahedron meets the following criteria:

Icosahedron	
Triangular faces	20
Edges	30
Vertices	12

Another useful representation of the icosahedron is the coordinate representation of the vertices. Namely, the vertices of the icosahedron lie exactly at the corner points of three golden frames inscribing the icosahedron (Figure 3). These golden frames are characterized by the property that the ratio of its sides is given by the golden ratio, that is $1 : \phi$, where $\phi = \frac{1+\sqrt{5}}{2}$.

The coordinates of an icosahedron can be given using these golden rectangles. If we take the the length of the golden rectangles to be 2ϕ, the width of the rectangle is set to 2 and the distance between two adjacent vertices is 2. The coordinates of the vertices of an icosahedron defined by these golden frames are given by all possible sign combinations of the coordinates:

$$(0, \pm 1, \pm\phi), \quad (\pm\phi, \pm 1, 0), \quad (\pm 1, \pm\phi, 0). \tag{6}$$

2.2.2 Rotation group of the icosahedron

Before we describe the rotations of the icosahedron by their rotation axis, we shortly introduce the definition of a rotation in $\mathbb{R}^3$.

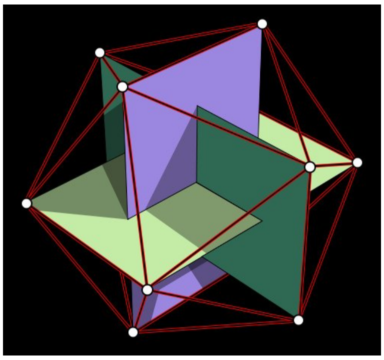


Figure 3: Golden frames inscribed in an icosahedron. *[Image retrieved from [8]]*

Definition 2.5 (Rotation in $\mathbb{R}^3$). A rotation in $\mathbb{R}^3$ is an orthogonal transformation that preserves orientation and can be described by an element from the matrix group O(3).

Intuitively, a rotation rotates about a certain rotation axis with a specified rotation angle. The rotation axis is left invariant under the rotation. This intuitive definition appears to be equivalent to the definition given. The equivalence can be found using the matrix defined in Theorem A.40.

The total number of rotations of the icosahedron can be found using a standard orbit-stabilizer argument. The icosahedron has 20 triangular faces and each triangle can be mapped to one of the other 20 triangles of the icosahedron. Each triangular face is fixed by only 3 rotations of the icosahedron. It thus follows that rotation group of the icosahedron contains $20 \cdot 3 = 60$ rotations.

These 60 rotations can be split up into four types. One type is the identity rotation and the other three types distinguish themselves from the other rotations by the geometric description of the points that get fixed under a rotation. To be more specific, one of these 59 rotations either fixes two opposite vertices, two opposite edges or two opposite triangles.

Let us introduce some notation for the rotation group and symmetry group of the icosahedron.

Definition 2.6 (Icosahedral group). The icosahedral group, denoted I, is the group consisting of all rotations of the icosahedron and has order 60.

Definition 2.7 (Full icosahedral group). The full icosahedral group, denoted I_h or H_3, is the group consisting of all symmetries of the icosahedron and has order 120.

In the last part of this section we determine the number of rotations for each of the three non-trivial rotations described earlier. In Section 2.2.3 we investigate the conjugacy classes of these four types of rotations.

Consider the rotation of the icosahedron fixing two opposite vertices. It follows that there are $\frac{12}{2}$ possible rotation axes. Those rotations have a rotation angle of $\frac{360°}{5} = 72°$, since 5 triangles meet in each vertex. This type of rotation can be seen as rotation of the 5 adjacent triangles over 72°. Notice that after performing 5 rotations about 72° about the same rotation axis and in the same direction of rotation, places the icosahedron in its original position. Thus, in total there are $6 \cdot 4 = 24$ non-trivial rotations about an axis through two vertices.

Next, we consider the rotations that fix two triangles of the icosahedron. Since the icosahedron has 20 triangles, it follows that there are $\frac{20}{2}$ possible ways to pick two opposite triangles from the icosahedron. Since a triangle has three edges, or equivalently, three adjacent triangles, it follows that these rotations rotate the adjacent edges or triangles respectively. To place an adjacent triangle in the positions of an adjacent triangle, the icosahedron needs to be rotated over 120°. Thus, only a rotation about 120° about a rotation axis through two midpoints of two opposite triangles rotates the icosahedron non-trivially. It follows that there are $10 \cdot 2 = 20$ non-trivial rotations that fix two opposite triangles.

Lastly, the rotation group of the icosahedron also contains rotations fixing two opposite edges. Since each edge is an edge of two triangles, a rotation that fixes two edges permutes the two triangles sharing such an edge. Since there are $\frac{30}{2}$ ways to construct a rotation axis through two opposite edges, it follows that there are 15 rotations over 180°.

In total we found the $24 + 20 + 15 + 1 = 60$ rotations of the icosahedron. A visualization of the rotations, the number of those rotations, their order, angle and rotation axis are summarized in Figure 2.

2.2.3 Conjugacy classes of the rotations of the icosahedron

To classify the conjugacy classes of I, we will use that I is a subgroup of SO(3), the group of all orthogonal and orientation preserving 3×3 matrices. In SO(3), it is a well-known fact that the rotations of the icosahedron are conjugate iff they have the same rotational angle. In any subgroup of SO(3) this result has to hold as well. However, it might be the case that the conjugacy classes from SO(3) fall apart in multiple conjugacy classes in I. In Section 2.3.3 we will show that the conjugacy classes of I have sizes $1, 12, 12, 15, 20$ and thus that rotations are conjugate iff they have the same rotational angle. First, we prove that rotations in SO(3) are conjugate iff they have the same rotational angle. After that, we show that the conjugacy classes in I do not split up any further. Lastly, we show in Section 2.4.4 that the rotation group of the icosahedron is isomorphic to the alternating group A_5, using the inscription of five cubes in the dodecahedron formalized

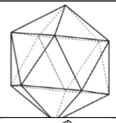
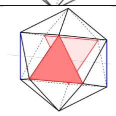
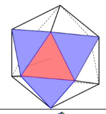
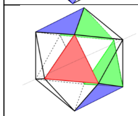
	Rotation axis	Order	Angle	Number of rotations
		1	360°	1
	Edges	2	180°	15
	Triangles	3	120°	20
	Vertices	5	72°	24
				$1+15+20+24=60$

Table 2: Orders, angles and number of rotations of the icosahedron.

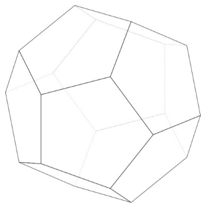


Figure 4: A dodecahedron.

in Section 2.4.2. As the fact that $I \cong A_5$ is an important result, we already state this result in the theorem below, although its proof needs to wait till Section 2.4.4.

Theorem 2.8. *The rotation group I of the icosahedron is isomorphic to A_5.*

2.3 Dodecahedron

2.3.1 Definition of the dodecahedron

Definition 2.9 (Dodecahedron). The dodecahedron is a regular convex polyhedron, whose faces are regular pentagons with 3 pentagons meeting in each vertex. The dodecahedron is the dual of the icosahedron.

The dodecahedron is a Platonic solid and is the dual of the icosahedron. The relation between dual polyhedra is *usually taught as a property of the Platonic solids, by pointing out that that the number of vertices of the cube and the number of faces of the octahedron are equal and vice versa*. [14] In a same way, the number of vertices and faces of the dodecahedron equals the number of faces and vertices of the icosahedron in that order. Thus, the dodecahedron can be described by:

Dodecahedron	
Pentagonal faces	12
Edges	30
Vertices	20

It is also possible to describe the dodecahedron by its coordinates. There are multiple ways to calculate these coordinates. One way is by taking the midpoints of the equiliteral triangles of the icosahedron. Alternatively, we can build up a dodecahedron from a single pentagon. For example, we can take a pentagon with distance 2 between two non-adjacent vertices. If we take for instance the coordinates $(1,1,1)$ and $(1,1,-1)$, the coordinates of three other vertices of a pentagon are for example given by:

$$\left(0,\phi,\frac{1}{\phi}\right), \qquad \left(0,\phi,-\frac{1}{\phi}\right), \qquad \left(\phi,\frac{1}{\phi},0\right).$$

We can find more vertices of the dodecahedron by calculating the vertices of any of the adjacent pentagons to this first pentagon. However, a dodecahedron obtained in this way is not unique. This is not a problem, since any pair of dodecahedra whose edge have the same length are isomorphic. As mentioned earlier, an elaborate construction of the dodecahedron has been done by Euclid in Proposition 17 of Euclid's Elements [12].

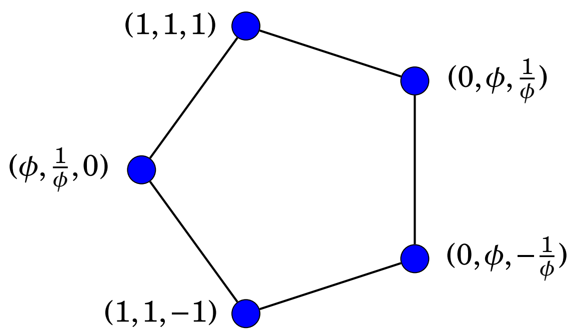


Figure 5: A regular pentagon.

Both computations lead to 20 vertices of a dodecahedron. The vertices of a dodecahedron can be described by:

$$(\pm 1,\pm 1,\pm 1), \qquad \left(0,\pm\phi,\pm\frac{1}{\phi}\right), \qquad \left(\pm\phi,\pm\frac{1}{\phi},0\right), \qquad \left(\pm\frac{1}{\phi},0,\pm\phi\right). \tag{7}$$

2.3.2 Symmetry group of the dodecahedron

Because of the duality between the icosahedron and the dodecahedron, they share their symmetry group. However, the geometrical interpretation of the rotation axes differs a bit between the icosahedron and dodecahedron. For example, the rotations fixing two vertices of the icosahedron fix the midpoints of two pentagons in the dodecahedron.

To describe the full symmetry group of the dodecahedron, I_h, we will first describe the reflection planes of the dodecahedron and relate it to a reflection in the center of the dodecahedron plus a consecutive rotation. After that, we consider the commutativity of composition of rotation and the point reflection. In the end, we give an overview of all reflections.

To distinguish between reflections in a plane and reflections that arise from a reflection plus a consecutive rotation, we introduce some terminology.

Definition 2.10 (Reflection). A reflection on the Euclidean space V is a linear operator s_α that sends some nonzero vector $v \in V$ to $-v$ while each vector lying in the hyperplane H_α orthogonal to α is fixed. The reflection is given by the formula:

$$s_\alpha v = v - \frac{2 < v, \alpha >}{< \alpha, \alpha >} \alpha,$$

where $<.,.>$ is the bilinear form of V and $v \in V$.

In this section, we work with $V = \mathbb{R}^3$. The hyperplane H_α fixed by a reflection is called the plane of reflection for $V = \mathbb{R}^3$. Now, we introduce the notion of a rotation-reflection.

Definition 2.11 (Rotation-reflection $\mathbb{R}^3$). A rotoreflection in $\mathbb{R}^3$, also called an improper rotation or rotation reflection, is a combination of a rotation around an axis and a reflection through the plane perpendicular to the rotation axis. [6]

With these definitions we can give an account of the reflections and rotoreflections of the dodecahedron. First of all, fifteen reflection planes are inscribed in the dodecahedron. Two sides of such reflection plane are described by two opposite edges of the dodecahedron. Two examples are given in Figure 6. Another reflection in the dodecahedron is a reflection in the center point $(0,0,0)$. Interestingly, the reflections through a reflection plane are the same as first reflecting in $(0,0,0)$ followed by a rotation of order 2. [16] An example is given in Figure 8.

We denote the rotations of the dodecahedron by the set $\{\mathrm{id}, \rho, r, R\}$ where ρ, r, R have order $2,3,5$ respectively. We denote the point reflection, with order 2, by σ. With this notation, all reflections and rotoreflections of the dodecahedron can be represented as in Table 3.

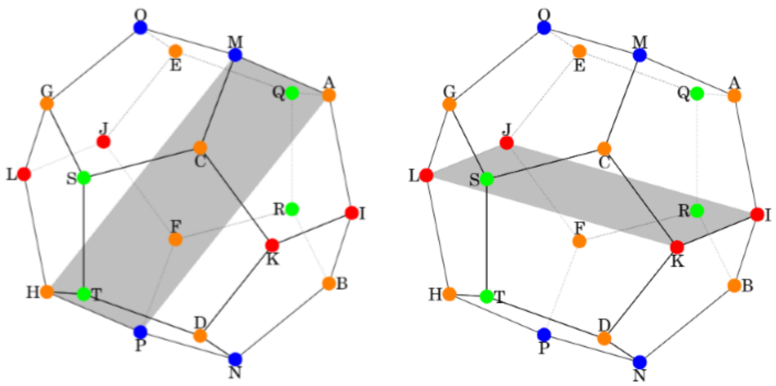


Figure 6: Two reflection planes of the dodecahedron.

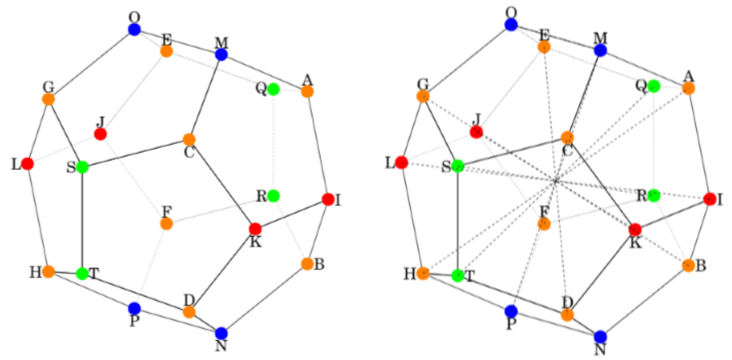
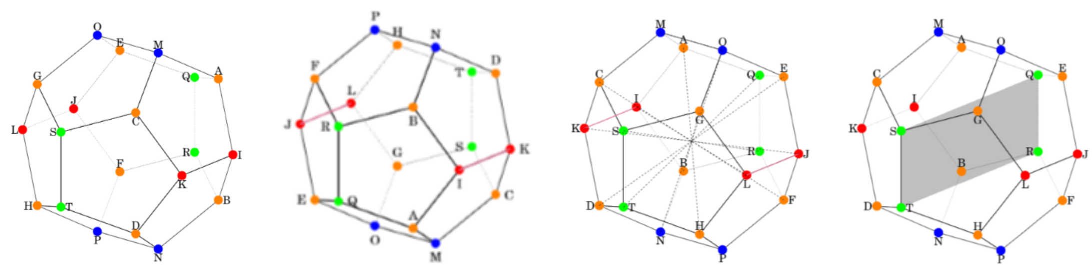


Figure 7: The dodecahedron and the reflection in the center point of the dodecahedron.



Dodecahedron **Rotation** **Point reflection + rotation** **Plane reflection**

Figure 8: A plane reflection in the dodecahedron is the same as a rotation of order 2 of the dodecahedron plus the reflection in the center point.

Reflection type	Notation	Number of reflections	Order
Point symmetry	σ	1	2
Plane reflection	$\rho\sigma$	15	2
Rotoreflection	$r\sigma$	10	6
Rotoreflection	$r^2\sigma$	10	6
Rotoreflection	$R\sigma$	6	10
Rotoreflection	$R^2\sigma$	6	10
Rotoreflection	$R^3\sigma$	6	10
Rotoreflection	$R^4\sigma$	6	10

Table 3: The reflections and rotoreflections of the dodecahedron.

An example of the composition of a rotation r of order 3 about 120° with the reflection in the origin is given in Figure 9. An example of a of a rotation R of order 5 about 72° plus the reflection in the origin is given in Figure 10.

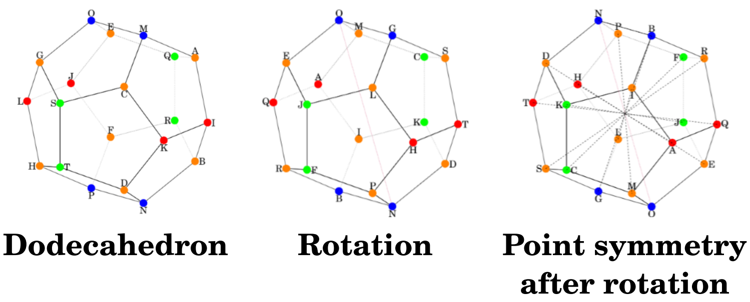


Figure 9: Point symmetry plus rotation r of order 2 about 120°.

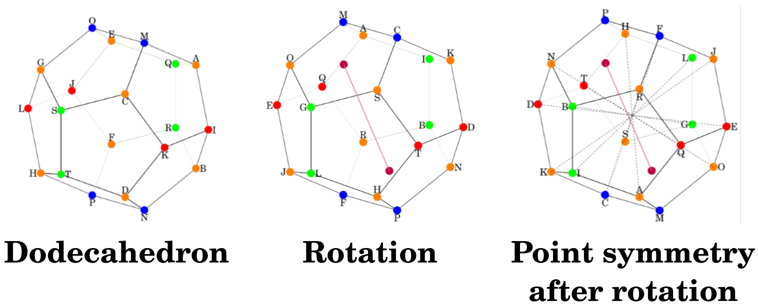
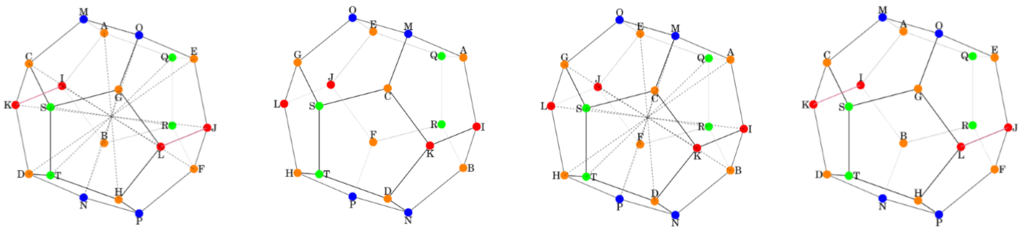


Figure 10: Point symmetry plus rotation R of order 5 about 72°

Another interesting property of the point reflection is that it commutes with any order rotation and with each reflection of the dodecahedron. Indeed, for any $R \in O(3)$ and I the identity matrix it holds that $-RI = -IR$ where $-I \in O(3)$ acts is the reflection in the center point. For a rotation of order 2 the commutativity is illustrated in Figure 11.



Point symmetry after rotation **Dodecahedron** **Point symmetry** **Rotation after point symmetry**

Figure 11: Commuting point symmetry plus a rotation ρ of order 2.

We conclude this chapter with stating that the the full symmetry group of the icosahedron, I_h, is isomorphic to $A_5 \times C_2$. Since we have not yet shown that $I \cong A_5$, we postpone the proof of this result to Section 2.4.4.

Theorem 2.12. *I_h is isomorphic to $A_5 \times C_2$.*

2.3.3 Conjugacy classes of the dodecahedron

In this subsection, we show that the conjugacy classes of the rotation group of the dodecahedron are given by the rotation angle. This result enables us to prove that $I \cong A_5$ and $I_h \cong A_5 \times C_2$.

We start with showing that two elements of SO(3) are conjugate if and only if their rotation angle is the same.

Theorem 2.13. *Let $R \in \mathrm{SO}(3)$ be a rotation with rotation angle α, where $0 \le \alpha \le \pi$. Define the rotation*

$$R_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{pmatrix}.$$

Then R and R_0 are conjugate in SO(3). That is, there exist a $P \in \mathrm{SO}(3)$ such that $R = PR_0P^{-1}$.

Proof. Take a unit vector $b_1 = (x, y, z)$ on the rotation axis of R. As a matter of fact, this vector is a unit eigenvector corresponding to the eigenvalue 1 of R. Construct an orthonormal basis $\mathscr{B} = \{b_1, b_2, b_3\}$ of $\mathbb{R}^3$. Denote the standard basis in $\mathbb{R}^3$ by $\mathscr{E} = \{e_1, e_2, e_3\}$ where e_i is the vector in $\mathbb{R}^3$ whose i-th coordinate takes value 1 and all the others zero. There exist a basis transformation from $\mathscr{B}$ to the orthonormal basis $\mathscr{E}$. If we denote the matrix representing this transformation by A, we have that $A \in \mathrm{O}(3)$. Indeed, the columns of A consists of the vectors b_i expressed in the basis $\mathscr{E}$. For example, the first column of A is given by $(x, y, z)^T$. It follows that $A \in \mathrm{O}(3)$. If $\mathrm{Det}(A) = 1$ with $A \in \mathrm{SO}(3)$, we are done. If $\mathrm{Det}(A) = -1$, we swap the second and third column of A and obtain a basis transformation that maps the b_1 to the rotation axis of R_0 but maps b_2 to e_3 and b_3 to e_2. Denote the matrix corresponding to this basis transformation by A' and note that $\mathrm{Det}(A') = 1$. Hence, we now have $R = A'R_0(A')^{-1}$ which proves the theorem. □

Corollary 2.13.1. *Suppose $R, R' \in \mathrm{SO}(3)$ are both rotations about an angle α, where $0 \le \alpha \le \pi$. Then R and R' are conjugate.*

Proof. Since there exist A_1 and A_2 such that $R = A_1 R_0 A_1^{-1}$ and $R' = A_2 R_0 A_2^{-1}$ it follows that $R = A_2 A_1^{1} R_0 A_1 A_2^{-1}$. □

The show that the conjugacy classes I are given by the rotation angle as well, we only need to show that the proof given in Theorem 2.13 and Corollary 2.13.1 works for the rotations in I as well.

Theorem 2.14. *The rotations in the icosahedron with same rotation angle $\pm\alpha$ are conjugate.*

Proof. To argue as in Theorem 2.13, we need to position the icosahedron in $\mathbb{R}^3$ such that one of the rotation axes through two vertices of the icosahedron lies on the x-axis. Obviously, there exists an orthogonal transformation in SO(3) that does so.

Furthermore, the entire proofs from Theorem 2.13 and Corollary 2.13.1 still holds for the rotaitons in I. The only tricky point in these proofs is where we need to swap the columns of the basis transformation from $\mathscr{B}$ to $\mathscr{E}$ to get a matrix representing the basis transformation with determinant 1. In this step we use and really need that I_h contains the reflection that maps the coordinates (x, y, z) on the icosahedron to (x, z, y) which again lie on the icosahedron. □

We state the consequence of the conjugation of rotations in I whenever the rotation angle is the same up to their sign in the following theorem.

Theorem 2.15. *The rotations in I conjugate if and only if they have the same rotation angle. Thus, the sizes of the conjugacy classes in I are* $1, 12, 12, 15, 20$.

2.4 Cube

2.4.1 Symmetry group of the cube

In this section, we study the symmetry group of the cube. Also, we formalize the inscription of five cubes in the dodecahedron. This enables us to prove that I is isomorphic to A_5. We will denote the full symmetry of the cube, called the full octahedral group, by O_h, and the octahedral rotation group by O.

We start with the rotations of the cube. The order of O is 24, using the orbit stabilizer theorem again. Each face of a cube can be mapped to 6 other faces and each face is mapped to itself by 4 rotations, giving the the $6 \cdot 4 = 24$ rotations of the cube.

The rotations of the cube either fix two vertices or two midpoints of the edges or two midpoints of the faces. Each rotation of the cube fixing two midpoints of faces permutes the adjacent 4 faces. It follows that the number of rotations of the cube with order 4 equals $\frac{6}{2} \cdot 3$ 9. Next, the rotations of a cube with rotation axis through the midpoints of two opposite pair of edges, permutes the adjacent faces of the edge. This means that there are $\frac{12}{2} = 6$ rotations of order 2. Lastly, a rotation that fixed two vertices permutes three adjacent faces. Hence, there are $\frac{8}{2} \cdot 2 = 8$ rotations of order 3. Thus, the rotation group of the cube consists of $9+6+8+1 = 24$ rotations, since the identity rotation is the last rotation that needs to be added.

We show that $O_h \cong S_4 \times C_2$ in Theorem 4.14. Before we are able to do so, we describe the action of the rotations by permutations of the four diagonals of the cube. Furthermore, we use that the reflections of the cube can be described by a reflection in the center of the cube followed by a rotation, as we saw for the dodecahedron as well.

Theorem 2.16. *The rotation group of the cube can be described by its action on its four diagonals.*

Proof. We show that each rotation permutes the diagonals and that each rotations corresponds to a different permutation. If we label the diagonals by colored letters b, g, y and r the rotations through two midpoints of two opposite faces of the cube give a permutation (b g yr). The rotation with rotation axes perpendicular to this one permute the diagonals described by the permutation (b yrg) or (b rg y). In a similar way, one can show that the rotations with rotation axis through the vertices permutes three of the four diagonals, each time fixing a different diagonal. The rotations with the rotation axis through the midpoints of two opposite edges permutes two diagonals while leaving the other two invariant. The diagonals that are swapped are the two diagonals going through the vertices incident to the edge that intersects the rotation axis. [15] □

An example of a rotation of the cube with order 2 that swaps 2 diagonals of the cube and leaving 2 diagonals invarian is illustrated in Figure 12.

Finally, to prove Theorem 2.17, we write the rotations of the cube as the permutation of the four diagonals as in Table 4. The diagonals are named by their

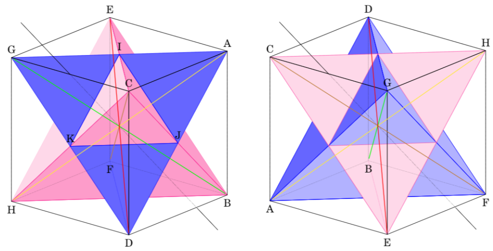


Figure 12: Rotation of order 2 of the cube, which swaps 2 diagonals of the cube, while it leaves the other 2 diagonals invariant.

color in the figure in the last column of the table.

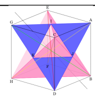
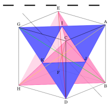
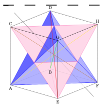
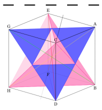
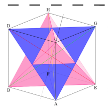
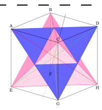
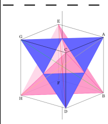
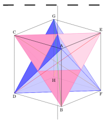
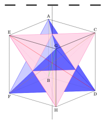
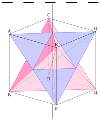
Order	Angle	Number	Representative	Figure
1		1	id	
2	180°	6	(bg)(y)(r)	 
3	120,240°	8	(ryg)(b)	  
4	90,240°	6	(rybg)	  
4	180°	3	(rb)(yg)	

Table 4: Rotations of the cube.

The reflections and rotoreflections of the cube can be obtained using similar arguments as we used for the dodecahedron. If we denote the rotations of the cube by the set {id, ρ, r, R } where ρ, r, R are of order 1,2,3,4 respectively, these reflections and rotoreflections of the cube can be summarized as in Table 5.

Using that the rotations of the cube can be described by its action on the four diagonals of the cube, this result is not hard to prove. However, in Section 2.4.4, we will formally introduce the notion of a direct product and how one can show that a group G is the direct product of two subgroups $H_1, H_2 \subset G$. The proof of

Reflection type	Notation	Number of reflections	Order
Point symmetry	σ	1	2
Plane reflection	$r\sigma$	6	2
	$R^2\sigma$	3	2
Rotoreflection	$\sigma\rho$	4	6
Rotoreflection	$\sigma\rho^2$	4	6
Rotoreflection	$R\sigma$	3	4
Rotoreflection	$R^3\sigma$	3	4
		$1+6+3+4+4+3+3=24$	

Table 5: Reflections and rotoreflections of the cube.

Theorem 2.17 will therefore be given in Section 2.4.4.

Theorem 2.17. *The full symmetry group of the cube is isomorphic to* $S_4 \times C_2$.

2.4.2 Inscription of five cubes in the dodecahedron

If we look at the vertices of the dodecahedron given in Equation (7), one can see that the vertices $(\pm 1, \pm 1, \pm 1)$ are precisely the vertices of a cube inscribed in this dodecahedron. In fact, there exactly 5 cubes are inscribed in any dodecahedron.

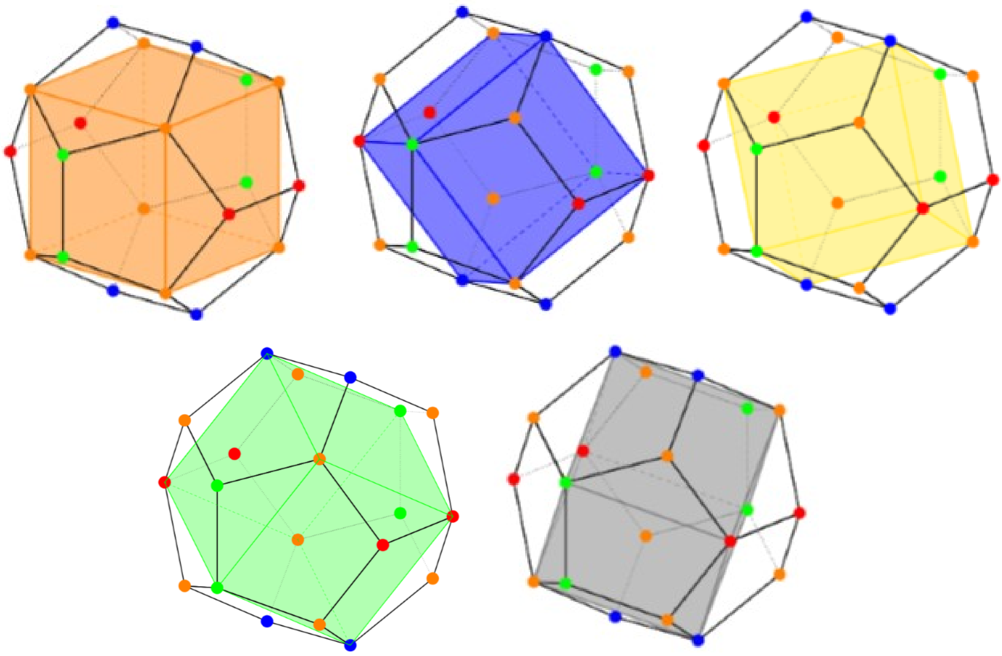


Figure 13: The five inscribed cubes of a dodecahedron.

In Figure 14, the edges of these five cubes are drawn in the dodecahedron. To show that these cubes are the only five cubes inscribed in a dodecahedron, we start with a pentagonal face of the dodecahedron. In this pentagon, we draw five diagonals connecting two different vertices of the face. The position of the cube inside a fixed dodecahedron is entirely fixed by one such an edge in a pentagon. Indeed, if we take one such edge in a pentagon of a dodecahedron, there are only 6 vertices of the dodecahedron with distance 2 to a vertex of the edge we began with. However, only 2 of these vertices can be connected with the vertex on our original edge, such that the angle between these edges is 90°. Since there are $\binom{5}{2} - 5 = 5$ possible ways to choose two non-adjacent vertices in a regular pentagon and connect them by an edge, this implies that there at most 5 cubes inscribe the dodecahedron.

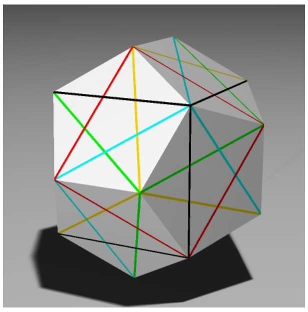


Figure 14: Inscribed cubes of the dodecahedron by their edges. *[Image retrieved from [37]]*

2.4.3 Rotation group of the icosahedron and its action on the five cubes in the dodecahedron

Having shown that $I \cong A_5$, we can describe the rotation group of the icosahedron by its action on the five cubes inscribed in a dodecahedron. We already know that each rotation will permute these five cubes according to a permutation of A_5.

Firstly, we consider the rotations of order 2. Given a rotation about two edges with two vertices of one edge called A and B, this rotation is illustrated in Table 6. Interestingly, the blue and black edges are swapped and so do the yellow and the green ones. However, the red edges remain unchanged under the rotation. Since one edge of a cube fixes the orientation of the cube inside the dodecahedron entirely, it follows that a rotation of order 2 swaps two sets of cubes and leaves the fifth cube invariant. This permutation can be represented by (bk)(gy)(r).

For a fifth order rotation, it is enough to consider action on a single pentagon that has its midpoint through the rotation axis.

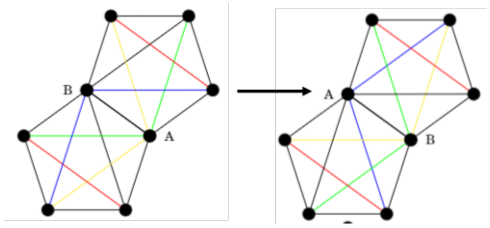


Table 6: Permutation of the inscribed cubes of the dodecahedron for a rotation of order 2.

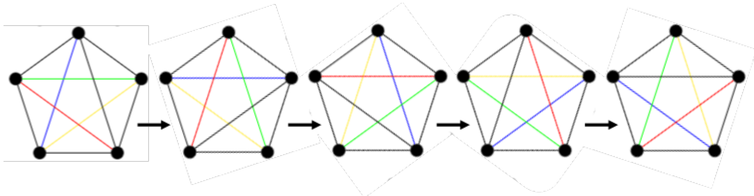
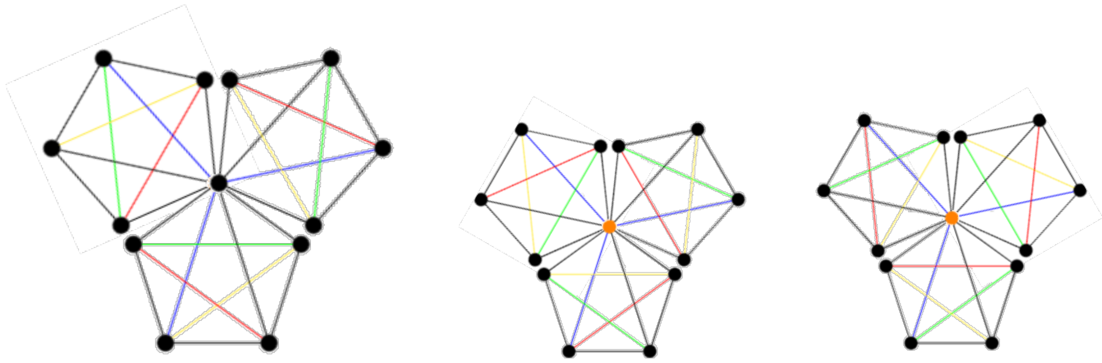


Figure 15: Permutation of the inscribed cubes of the dodecahedron for a rotation of order 5.

The action of a fifth order rotation on the cubes inscribed in the dodecahedron, can be described by the action on the diagonal edges of one of the pentagon that intersects the rotation axis. Since the vertices are rotated, it follows that the the rotation maps the diagonal edges of the pentagon to another edge in the same pentagon. This means that the cubes are permuted in a cycle of length 5. This permutation can be represented by (rbgky) and is visualized in Figure 15.

Lastly, a rotation of order 3 can be described by the permutation of three adjacent pentagons sharing the same vertex on the rotation axis. In Figure 7, the rotations are visualized by a projection on the plane of those three adjacent pentagons. Comparing the vertices in Figure 7**b** with the ones in Figure 7**a** and rotating about the vertex the pentagons share, the permutation of the cubes can be read off. For example, in each pentagon, the blue and the black edge are incident to the rotation vertex and their orientation does not change under the rotation. However, the green edge takes the place of the red edge after a clockwise rotation over 120°. The red edge is mapped to the yellow one, which goes to green one. In other words, a third order rotation permutes three cubes non-trivially and maps two cubes to itself. The cycle representation of this permutation is given by (gry).



(a) Projection on the plane **(b) Rotation 120°** **(c) Rotation 240°**

Table 7: Action of a rotation of order 3 of the dodecahedron on 15 edges of the 5 inscribed cubes of the dodecahedron.

2.4.4 Isomorphism between the rotation group of the icosahedron and the alternating group of elements

After introducing the notion of simpleness of a group, we prove that I is simple and with that result we prove that I is isomorphic to A_5 and I_h to $A_5 \times C_2$.

Definition 2.18. (Simple group) A nontrivial group is called simple if it has no other normal subgroups than the trivial subgroup and the entire group itself.

Theorem 2.19. *The rotation group of the icosahedron is simple.*

Proof. The Lagrange theorem that states that the order of a subgroup divides the order of the group. As a consequence, the order of the normal subgroups of I must divide $|I| = 60$. Additionally, any subgroup contains the id of I. Since normal subgroups consists of the union of conjugacy classes, going over all possible unions of the found conjugacy classes of size $1, 12, 12, 15, 20$, it follows that there are no combinations of conjugation classes that with id-rotation added to it divides 60. Thus, the only normal subgroups of I are id and I itself. Hence, I is a simple group. □

Theorem 2.20. *The rotation group I of the icosahedron is isomorphic to A_5.*

Proof. By describing the rotations of the icosahedron as permutations of the five inscribed cubes in the dodecahedron, one finds a homomorphism between the rotations of the icosahedron and its action on the five cubes. We denote this homomorphism by $f : I \to S_5$. Since the kernel of a homomorphism from a group G_1 to a group G_2 is a normal subgroup of G_1, it follows that ker(f) is a normal subgroup

of I (Theorem A.5). Since I is simple by Theorem 2.19, the kernel is either the id or I itself. However, the cubes are permuted non-trivially, so $\ker(f) = \text{id}$. The image of I in S_5 is a group of order 60. Since A_5 is the only subgroup of S_5 with index 2 and order 60 (Theorem A.28), it follows that $f(I) = A_5$.
Another way to conclude that the image of I in S_5 is A_5 is by using that the permutations of the cubes are all even permutations and are necessarily mapped to the even permutation in S_5. □

To show that $I_h \cong A_5 \times C_2$, let us first define what the direct product between two groups is together with some properties that imply that a group can be written as the direct product of two sets.

Definition 2.21 (Direct product). If G_1 and G_2 are groups, then the direct product of G_1 and G_2 is the set

$$G_1 \times G_2 := \{(g_1, g_2) : g_1 \in G_1, g_2 \in G_2\}$$

with the operation

$$(g_1, g_2) \cdot (h_1, h_2) = (g_1 h_1, g_2, h_2)$$

for all $g_1, h_2 \in G_1$ and $g_2, h_2 \in G_2$. In other words, the operation is component-wise multiplication. [18]

Theorem 2.22. *Let G be a group and let $H_1, H_2 \subset G$ be subgroups and $e \in G$ the identify element. Suppose the following properties hold:*

1. *$h_1 h_2 = h_2 h_1$ for all $h_1 \in H_1$ and $h_2 \in H_2$;*
2. *$H_1 \cap H_2 = \{e\}$;*
3. *Every $g \in G$ can be written as $g = h_1 h_2$, with $h_1 \in H_1$ and $h_2 \in H_2$.*

Then $G \cong H_1 \times H_2$. [18]

We will now show, using Theorem 2.22, that $I_h \cong A_5 \times C_2$

Theorem 2.23. *I_h is isomorphic to $A_5 \times C_2$.*

Proof. First of all, we identify the symmetry group of the icosahedron with a subgroup of O(3). From Theorem 2.20 we know that I is a subgroup of SO(3) isomorphic to A_5. The group $\{I_3, -I_3\} \subset \text{O}(3)$ is isomorphic to C_2. So all we need to show is that the properties from Theorem 2.22 hold. First of all, for any $R \in \text{SO}(3)$, $I_3 R = R I_3$ and $-R I_3 = -I_3 R$. Then second requirement is clearly also satisfied, as $-I_3$ has determinant -1 and $I_3 \in \text{SO}(3)$. Lastly, any $g \in I_h$ is either a rotation, rotoreflection or reflection in the center point of the icosahedron. A rotation can can be written as $R \times I_3$, a rotoreflection as $R \times -I_3$ and the reflection in the center point as $I_3 \times -I_3$. It follows that $I_h \cong A_5 \times C_2$. □

In a similar way as we proved that $I_h \cong A_5 \times C_2$, we can show that $O_h \cong S_4 \times C_2$.

Theorem 2.24. *The full symmetry group of the cube is isomorphic to* $S_4 \times C_2$.

Proof. We have a homomorphism f from the action of the rotations of the cube on the diagonals to S_4. From Table 5 in Section 2.4.1 the explicit action of f on O is given, which shows that the rotation group is isomorphic to S_4.
The fact that the full symmetry group is $S_4 \times C_2$ follows in a similar way as we will do for the icosahedron in Theorem 2.23. □

2.5 Tetrahedron

2.5.1 Symmetry group of the tetrahedron

In this section, we very briefly study the symmetry group of the tetrahedron T_d and the rotation group of the tetrahedron T, as this symmetry group was already studied in the Algebra 1 course in [36].

The order of the rotation group T is $4 \cdot 3 = 12$, since each face can be mapped to another face in 3 ways. Those rotations can be described by their rotation axis. There are 4 rotations that fix a vertex and a midpoint of a face. Those rotations have order 3. Thus, there are $4 \cdot 2 = 8$ rotations of this kind. Next, there are 3 rotations that fixing two edges of a tetrahedron. Those rotations have order 2. Hence, there are 3 rotations of this kind. Together with the identity rotation, we found all $8+3+1 = 12$ rotations of the tetrahedron.

The conjugacy classes of these rotations are not given by the rotation angle. Indeed, as a consequence of Lagrange's theorem, the order of the conjugation classes should divide the groups order. That is, the rotations of order 3 in T cannot lie in the same conjugacy class as 8 does not divide 12. What happens is that the rotations of order 3 are divided over two conjugacy classes, one containing 4 rotations about 120° and the other conjugacy classes their inverses about 240°.

Another way to describe the rotation group of the tetrahedron is by the action of the rotations of the tetrahedron on its vertices. We show that each rotation permutes the vertices of the tetrahedron in a different way. It then follows that T acts on its vertices by permutation. First, label the vertices of the tetrahedron with numbers $1,2,3,4$ (Figure 16). Consider the rotations of order 3. Take the rotation with rotation axis through vertex with label 1 and the midpoint of the face with the vertices with labels $2,3,4$ as its corner points. This rotation permutes the vertices $2,3,4$ while fixing vertex 1. Rotating clockwise gives the permutation (234) while rotating clockwise permutes the vertices according to the cycle (243). Since each rotation of order 3 fixes a different vertex, it follows that these rotations describe different permutations of the vertices. Then, the rotations of order 2 swaps

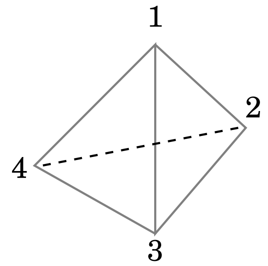


Figure 16: A tetrahedron.

two pairs of vertices. That is, these rotations correspond to the permutation cycles (12)(34), (13)(24) and (14)(23) of the vertices. Thus, the group T acts on the vertices by permutation. In fact, it follows that $T \cong A_4$. In [36] it was already shown that $T_d \cong S_4$.

2.5.2 Inscription of two tetrahedra in the cube

In Section 2.4.1 the rotations of the cubes were described by the action on the 4 diagonals. We can also consider the action of the cube on the inscribed tetrahedra in the cube, which were already drawn in Figures 12 and 4. These tetrahedra can be obtained by alternating labeling of the vertices of the cube.

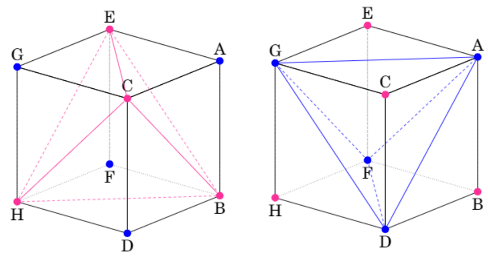


Figure 17: Alternated labeling giving two tetrahedra inscribed in a cube.

The action of the rotation group of the cube on these two tetrahedra is interesting. Namely, the action of a rotation of the cube on the inscribed tetrahedra either preserves both inscribed tetrahedra or it permutes these inscribed tetrahedra non-trivially. Any rotation of the cube of order 3 preserves the inscribed tetrahedra, but permutes the vertices of these tetrahedra. The rotation of order 4 over permutes the diagonals in a cycle of length 4. However, applying twice the same rotation of order 4 preserves the inscribed tetrahedra. The rotations of order 2 interchange the inscribed tetrahedra. It follows that the even permutations of the diagonals of the cube are precisely the permutations that leave the inscribed tetrahedra invariant. An example of each of these rotations is given in Figures 18, 19 and 20. In these figures, both the swapping of tetrahedra as the change in orientation of the tetrahedra is very well visible by the change in color and its intensity.

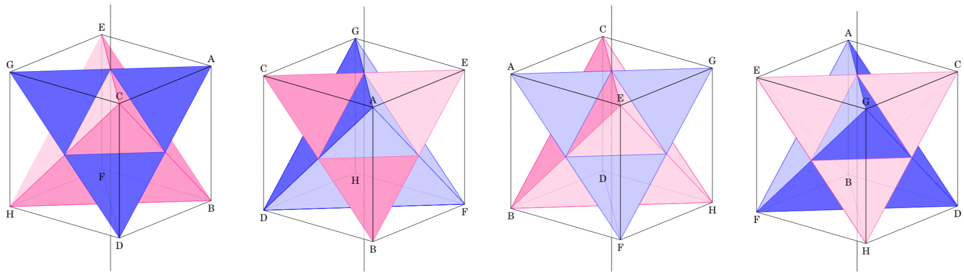


Figure 18: Rotation of order 4 of the cube acting on two inscribed tetrahedra.

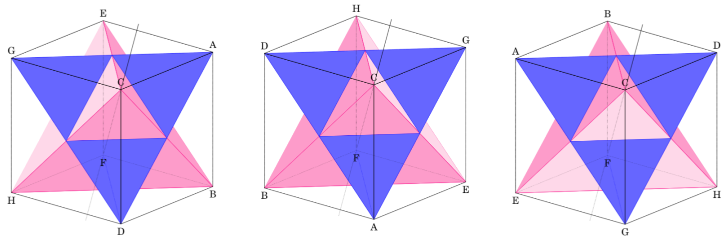


Figure 19: Rotation of order 3 of the cube acting on two inscribed tetrahedra.

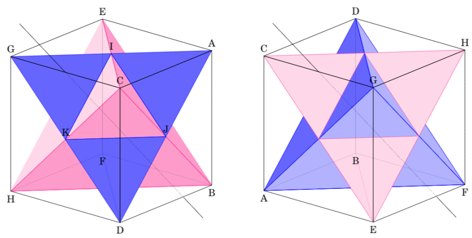
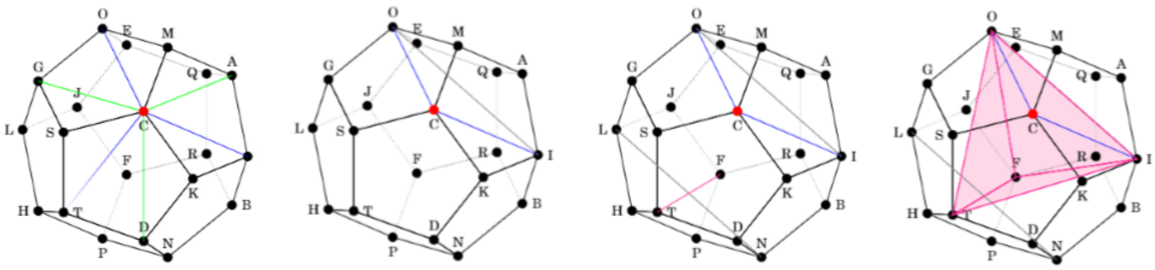


Figure 20: Rotation of order 2 of the cube acting on two inscribed tetrahedra.

2.5.3 Inscription of ten tetrahedra in the dodecahedron

Since 5 cubes inscribe a dodecahedron, it seems that 10 tetrahedra also inscribe the dodecahedron. Definitely, at least 10 tetrahedra inscribe the dodecahedron. We now show that at most 10 tetrahedra inscribe the dodecahedron. So, pick a vertex of the dodecahedron. Take all possible 6 edges to a non-adjacent vertex in a adjacent pentagon as shown in Figure 21**a**. From those diagonals, there are

$2 \cdot \binom{3}{2} = 6$ possible ways to pick two perpendicular edges. Connecting those edges by a diagonal as in Figure 21**b**, we constructed a face-diagonal of the tetrahedron. From this face-diagonal, the opposite face-diagonal can be constructed as well as done in Figure 21**c**. The tetrahedron is now uniquely determined by taking the face-diagonal perpendicular to the opposite face-diagonal. The vertices of those 2 face-diagonals are the vertices of the tetrahedron. Hence, the tetrahedron is constructed as in Figure 21**c**. However, 3 of the 6 possible choices for 2 perpendicular edges at the start of the construction, give rise to the same tetrahedron. Additionally, two vertices are incident to the same face-diagonal, we have a total of $\frac{20\cdot 2}{2} = 20$ ways to inscribe a tetrahedron in the dodecahedron. However, two distinct face-diagonals describe the same tetrahedron. That means that we counted each tetrahedron twice again. Thus, there are only 10 tetrahedra inscribed in the dodecahedron.



(a) **(b)** **(c)** **(d)**

Figure 21: Construction of a tetrahedron inscribed in dodecahedron.

In Section 2.2.2 we established the rotation group of the icosahedron and the dodecahedron. The length of the orbit of a cube inscribed in the dodecahedron is 5. Thus, the length of the orbit of a tetrahedron inscribed in the dodecahedron must be of length 5 as well. This implies by Theorem A.19 that the index of the stabilizer group of a tetrahedron, $\text{Stab}_G(x)$ has order 12 as its index is 5. We know that $\text{Stab}_G(x)$ is a subgroup of the rotation group of the cube it inscribes. Indeed, the rotations of the cube permute the diagonals, which determine the action under rotation of the tetrahedra completely. So, the stabilizer group of a tetrahedron is a group of order 12 and a subgroup of the group of rotations S_4 of the cube. The only subgroup of order 12 is the group A_4 of all even permutation. As we saw in Section 2.5.2, the even permutations preserve the tetrahedra. Hence, the rotations of the dodecahedra also preserve the tetrahedra. Differently phrased, the two inscribed tetrahedra of one cube, lie in different orbits in the dodecahedron, both of length 5.

3 Quaternions and quaternion groups

The rotation groups of the Platonic solids can be represented by sets of quaternions. These sets behave pretty beautifully. The quaternion representation of the rotation group I, for example, describes the vertices of the four-dimensional regular polytope called the 600-cell. In this section we consider the the sets of quaternions whose elements are quaternions representing either the rotations of the tetrahedron, the cube or the icosahedron.

3.1 Quaternions

We start with the definition of quaternions and the relation between the rotations in three-dimensional space and the quaternions. This definition of the quaternions is based on [18] and [21], while the rotation theorem (Theorem 3.11) comes from [29].

Definition 3.1 (Hamilton quaternion). Hamilton quaternions are expressions of the form

$$q = r + xi + yj + zk, \qquad \text{with } r, x, y, z \in \mathbb{R}.$$

These Hamilton quaternions can also be denoted by a vector $(r, x, y, z) \in \mathbb{R}^4$.
The **real part** of quaternion q is r, while the **imaginary part**, sometimes called the vector part of a quaternion, is $xi + yj + zk$.

We make the convention to refer to the Hamilton quaternions simply by quaternions, implicitly assuming that we work with the quaternions over the field $\mathbb{R}$ defined by Hamilton. We define quaternion addition and quaternion multiplication, the norm of a quaternions and its complex conjugate and show what the inverse of a quaternions is.

Definition 3.2 (Quaternion addition). Quaternion addition is defined compenentwise. That is,

$$(r + xi + yj + zk) + (r' + x'i + y'j + z'k) = (r + r') + (x + x')i + (y + y')j + (z + z')k,$$

with $r, x, y, z, r', x', y', z' \in \mathbb{R}$.

Quaternion multiplication is based on the following calculation rules:

	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	i	-1

Table 8: Calculation rules for the basis quaternions $1, i, j, k$ of $\mathbb{H}$.

Using these rules, we can define quaternion multiplication.

Definition 3.3. (Quaternion multiplication) Quaternion multiplication is given by:

$$\begin{aligned}(r + xi + yj + zk)\cdot(r' + x'i + y'j + z'k) = &(rr' - xx' - yy' - zz')\\ &+(rx' + xr' + yz' - zy')i\\ &+(ry' - xz' + yr' + zx')j\\ &+(rz' + xy' - yx' + zr')k.\end{aligned}$$

Definition 3.4. (Complex conjugate) The complex conjugate of a quaternion q, denoted $\overline{q}$, is given by:

$$\overline{q} = r - xi - yj - zk.$$

Definition 3.5. (Norm) The norm or length of a quaternion is defined to be

$$||q||^2 = q\cdot\overline{q} = r^2 + x^2 + y^2 + z^2.$$

Lemma 3.6. ***(Inverse)*** *The inverse of a quaternion $q \neq 0$ is given by:*

$$q^{-1} = \frac{\overline{q}}{||q||^2}.$$

This is a consequence of the fact that $||q||^2 = q\cdot\overline{q}$.

Since the addition and multiplication of quaternions in distributive and the inverse quaternion is well-defined for non-zero quaternions, the Hamilton quaternions form a division algebra. This division algebra is denoted by $\mathbb{H}^*$. The set of all Hamilton quaternions is denoted $\mathbb{H}$. In fact, $\mathbb{H}^* := \mathbb{H}\setminus\{0\}$.

Definition 3.7 ($\mathbb{H}^*$)**.** The set of Hamilton quaternions $\mathbb{H}^*$ forms a division algebra. That is, $\mathbb{H}^*$ is closed under the above defined quaternion addition and quaternion multiplication. In addition, each non-zero quaternion has a multiplicative inverse.

The quaternions with unit length are the quaternions that we are mainly intersected in. Let us thus define a subgroup of $\mathbb{H}^*$ that consists of these quaternions.

Definition 3.8 ($\mathbb{H}_1$)**.** The subgroup of quaternions with unit length is denoted by $\mathbb{H}_1$ and defined as:

$$\mathbb{H}_1 = \{q \in \mathbb{H}^* \mid ||q|| = 1\}.$$

Definition 3.9 (Quaternion group Q_8). Q_8, called the quaternion group, is a group consisting of 8 quaternions. The group is defined as:

$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}.$$

Theorem 3.10. *Given two quaternions $q, v \in \mathbb{H}_1$, then the quaternion conjugation qvq^{-1} preserves $Re(v)$.*

Proof. Write $q = (r, x, y, z)$ and $v = (r', a, b, c)$ for two quaternions in $\mathbb{H}_1$. The conjugation qvq^{-1} can be written as $q\text{Re}(v)q^{-1} + q\text{Im}(v)q^{-1}$. It then follows that

$$\text{Re}(qvq^{-1}) = r'qq^{-1} + \text{Re}(q\text{Im}(v)q^{-1}) = r',$$

where we use that scalar multiplication is commutative in $\mathbb{H}^*$, hence in $\mathbb{H}_1$, and the fact that quaternion conjugation with a pure imaginary quaternion gives another pure imaginary quaternion. Namely, if we denote $v = (v_0, v_1, v_2, v_3)$, then applying the calculation rules the real part is given by:

$$\begin{aligned}\text{Re}(q\text{Im}(v)q^{-1}) =& \text{Re}(q(0, v_1, v_2, v_3)q^{-1}) = (yz - yz + xr - xr)v_1 + \\ & (xz - xz + ry - ry)v_2 + (rz - rz + xy - xy)v_3 = 0\end{aligned}$$

□

Having defined the quaternions and the division algebra $\mathbb{H}_1$, we continue with Euler's rotation theorem which will be used extensively to relate the rotation groups I, T and O to the binary groups $2I, 2T$ and $2O$ which we introduce in Section 3.2, Section 3.3 and Section 3.4.

Theorem 3.11 (Euler's rotation theorem)**.** *For $R \in \text{SO}(3)$ there is a non-zero vector $\mathbf{v} \in \mathbb{R}^3$ such that $R\mathbf{v} = \mathbf{v}$.* [29]

Differently phrased, Euler's rotation theorem states that any rotation in 3-dimensional space is a rotation about a certain rotation axis, that may be represented by a vector that lies on the axis. Since this is a very important and useful consequence, we will state it in the following Corollary.

Corollary 3.11.1. *Any rotation in three-dimensional space can be described by a unit vector $v = (x, y, z)$ on the rotation axis and a rotation angle θ.*

From now on, we make the convention that any reference to a rotation by only a vector and an angle is a short way to describe a three-dimensional rotation by a vector on the rotation axis about the given angle.

A very useful way to represent three-dimensional rotations is by quaternions. The purely imaginary part of the quaternion decribed the rotation axis while the real part describe the rotation angle.

Definition 3.12 (Representation rotation in $\mathbb{R}^3$ by quaternion). Represent a rotation in three-dimensional space with rotation angle θ and a unit vector $v = (x, y, z) \in \mathbb{R}^3$ by the unit quaternion:

$$e^{\frac{1}{2}\theta(xi+yj+zk)} = \cos\left(\tfrac{1}{2}\theta\right) + (xi + yj + zk)\sin\left(\tfrac{1}{2}\theta\right).$$

The equality follows from the Taylor expansion of the exponential. Furthermore, it follows that conjugation of a purely imaginary quaternion v with a quaternion $q = (r, x, y, z) \in \mathbb{H}_1$ is the three-dimensional rotation of v about (x, y, z) with angle $2\arccos(r)$. As a consequence of the fact that we can represent a rotation by a unit quaternion, the rotation angle can be extracted from such a quaternion $q = r + xi + yj + zk \in \mathbb{H}_1$ using that $\theta = 2\arccos(r)$.

Lemma 3.13. *For a $v \in \mathbb{H}_1$ with $r = 0$ and $q \in \mathbb{H}_1$, the rotation described by $v \mapsto qvq^{-1}$ is the same rotation in $\mathbb{R}^3$ as the rotation described by $v \mapsto (-q)v(-q)^{-1}$.*

Proof. It directly follows from the commutativity of scalars and quaternions that $(-q)v(-q)^{-1} = --qvq^{-1} = qvq^{-1}$. □

An important consequence of Lemma 3.13 is that $\mathbb{H}_1$ is a double cover of SO(3). That is, each rotation in SO(3) is described by exactly two quaternions in $\mathbb{H}_1$. We state this consequence in Theorem 3.14. Since the proof is a bit tedious, we included it in Appendix A.4.2 in Theorem A.40.

Theorem 3.14. *The map $f : \mathbb{H}_1 \to \mathrm{SO}(3)$ given by $q \mapsto qvq^{-1}$ is surjective and $\mathbb{H}_1$ is a double cover of $SO(3)$. That is, the map $f : \mathbb{H}_1 \to SO(3)$ is two-to-one.*

3.2 Binary icosahedral group

3.2.1 Definition of $2I$

The binary icosahedral group is the the group of quaternions obtained from the double cover of the the rotation group of the icosahedron I under the map from Theorem 3.14. It is denoted by $2I$, has order 120 and is the extension of I. This group is called $2I$. The quaternions in $2I$ form, when interpreted as four-dimensional coordinates, the vertices of a 600-cell. This will be shown in Section 3.2.2. The 600-cell itself will be discussed in Section 4.3.

3.2.2 Pre-images of the quaternions in $2I$ in the 3-sphere

In this section, we show that the vertices of a 600-cell indeed describe the 60 rotations in I. To do so, we take the vertices of the 600-cell described by the quaternions in $2I$ as described in [27]. We show that each of these quaternions describes a rotation in I and that each rotation in I is described twice.

First of all, 16 vertices of the 600-cell are obtained from the coordinates:

$$\tfrac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1) \tag{8}$$

Next, eight quaternions are described by the quaternions from Q_8:

$$(0,0,0,\pm 1), \quad (0,0,\pm 1,0), \quad (0,\pm 1,0,0), \quad (\pm 1,0,0,0). \tag{9}$$

Lastly, there are 96 quaternions obtained from the coordinates:

$$\begin{array}{llll}
\frac{1}{2}\left(\pm 1,\pm\phi,\pm\frac{1}{\phi},0\right), & \frac{1}{2}\left(\pm\phi,\pm\frac{1}{\phi},\pm 1,0\right), & \frac{1}{2}\left(\pm\phi,\pm 1,0,\pm\frac{1}{\phi}\right), & \frac{1}{2}\left(\pm\frac{1}{\phi},\pm 1,\pm\phi,0\right) \\
\frac{1}{2}\left(0,\pm 1,\pm\frac{1}{\phi},\pm\phi\right), & \frac{1}{2}\left(\pm\phi,0,\pm\frac{1}{\phi},\pm 1\right), & \frac{1}{2}\left(0,\pm\phi,\pm 1,\pm\frac{1}{\phi}\right), & \frac{1}{2}\left(\pm 1,\pm\frac{1}{\phi},0,\pm\phi\right) \\
\frac{1}{2}\left(0,\pm\frac{1}{\phi},\pm\phi,\pm 1\right), & \frac{1}{2}\left(\pm\frac{1}{\phi},\pm\phi,0,\pm 1\right), & \frac{1}{2}\left(\pm\frac{1}{\phi},0,\pm 1,\pm\phi\right), & \frac{1}{2}\left(\pm 1,\pm\phi,\pm\phi,\pm\frac{1}{\phi}\right),
\end{array}$$

where ϕ denotes the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$.Those quaternions are the odd permutations of $\left(\pm\phi,\pm 1,\pm\frac{1}{2},0\right)$. One could choose to work with the even permutations as well, since it is shown in [27] that both sets of vertices form the vertex set of a 600-cell. Since the Icosians generated in the Mathematica code in Appendix B are the odd permutations, we work with the odd permutations for consistency.

To show that these 120 quaternions are the double cover of the rotations in I, we consider the sets of quaternions with same real part. We start with the 30 quaternions with $r = 0$. These are given by the quaternions:

$$\begin{gathered}
(0,0,0,\pm 1), \quad (0,0,\pm 1,0), \quad (0,\pm 1,0,0) \\
\frac{1}{2}\left(0,\pm\frac{1}{\phi},\pm\phi,0\right), \quad \frac{1}{2}\left(0,\pm\phi,\pm 1,\pm\frac{1}{\phi}\right), \quad \frac{1}{2}\left(0,\pm 1,\pm\frac{1}{\phi},\pm\phi\right)
\end{gathered}$$

Half of these lie exactly at the midpoints of the edges joining two vertices of the icosahedron. The other half represent their antipodals. The rotations those quaternions described are twice the rotations of order 2 that fix those midpoints of the edges. The vertices in three-dimensional space, give rise to another polyhedron with 20 triangular faces and 12 pentagonal faces. At each vertex, exactly two triangles and two pentagons meet. This polyhedron is called a icosidodecahedron (Figure 22).

Next we consider the 20 quaternions with $r = \frac{1}{2}$:

$$\frac{1}{2}(1,\pm 1,\pm 1,\pm 1), \quad \frac{1}{2}\left(1,\pm\frac{1}{\phi},0,\pm\phi\right), \quad \frac{1}{2}\left(1,0,\pm\phi,\pm\frac{1}{\phi}\right), \quad \frac{1}{2}\left(1,\pm\phi,\pm\frac{1}{\phi},0\right).$$

The imaginary parts of these quaternions describe the 20 vertices of the dodecahedron, or the rotations about 120° in the icosahedron.

For the quaternions with real part equal to $-\frac{1}{2}$, again the pre-image is given by the 20 vertices of the dodecahedron. As we saw in Lemma 3.13, the same rotation about 120° of the icosahedron is then described by a quaternion q with $r = \frac{1}{2}$ and the quaternion $-q$ with $r = -\frac{1}{2}$.

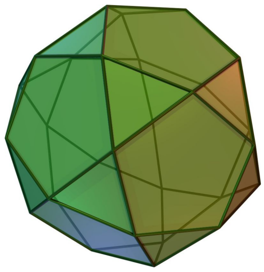


Figure 22: The icosidodecahedron. *[Image retrieved from [4]]*

For the quaternions with $r = \frac{1}{2}\phi$, we get the following quaternions:

$$\tfrac{1}{2}\left(\phi, \pm 1, 0, \pm \tfrac{1}{\phi}\right), \tfrac{1}{2}\left(\phi, 0, \pm \tfrac{1}{\phi}, \pm 1\right), \tfrac{1}{2}\left(\phi, \pm \tfrac{1}{\phi}, \pm 1, 0\right).$$

The imaginary parts of these 12 quaternions represent the vertices of a unit icosahedron. Thus, these quaternions describe the rotations of order 5 about 72°.

For the pre-image of the quaternions with $r = -\frac{1}{2}\phi$, the imaginary part again describes the vertices of an icosahedron. However, the rotation angle is the solution of $\cos\left(\frac{1}{2}r\right) = -\frac{1}{2}\phi$, which is $\frac{2}{5}\pi$.

Taking the quaternions with $r = \frac{1}{2\phi}$ we get the quaternions:

$$\tfrac{1}{2}\left(\pm \tfrac{1}{\phi}, 0, \pm 1, \pm\phi\right), \tfrac{1}{2}\left(\pm \tfrac{1}{\phi}, \pm\phi, 0, \pm 1\right), \tfrac{1}{2}\left(\pm \tfrac{1}{\phi}, \pm 1, \pm\phi, 0\right).$$

Those 24 quaternions describe the rotations of order 5 about an angle of 144°.

Lastly, we have the quaternions with $r = \pm 1$:

$$(\pm 1, 0, 0, 0).$$

These quaternions correspond to the identity rotation of the icosahedron.

All in all, we found that the vertices of the 600-cell we chose, describe the rotations in I twice and that each rotation is described.

3.2.3 The conjugacy classes of $2I$

We already know that the real part of a quaternion gives the rotation angle and that each quaternions in $2I$ represent a rotation of the icosahedron. With our knowledge from I, we would expect that the quaternions in $2I$ conjugate iff they have the same real part and order. It appears that the quaternions with same real part are conjugate in $2I$ as well. However, it is not true to conclude that whatever conjugates in I is conjugate in $2I$ as $2I$ is a double cover of I. Actually, there are conjugacy classes in $2I$ with a higher order than the conjugacy class of the corresponding rotation of the icosahedron. A Mathematica script that determines

Representative in $2I$	Order	Number
$(1,0,0,0)$	1	1
$(-1,0,0,0)$	2	1
$(0,\frac{1}{2},\frac{1}{2\phi},\frac{\phi}{2})$	4	30
$(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	6	20
$(-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	3	20
$(\frac{\phi}{2},\frac{1}{2\phi},\frac{1}{2},0)$	10	12
$(-\frac{\phi}{2},\frac{1}{2\phi},\frac{1}{2},0)$	5	12
$(\frac{1}{2\phi},\frac{\phi}{2},0,1)$	5	12
$(-\frac{1}{2\phi},\frac{\phi}{2},0,1)$	10	12

Table 9: Orders of the conjugacy classes in $2I$

the conjugacy classes and orders of elements in $2I$ can be found in Appendix B.1.1. From these computations it follows that the conjugacy classes in $2I$ have sizes $1,1,30,20,20,12,12,12,12$ with corresponding orders $1,2,4,6,3,10,5,5,10$. The conjugacy classes of $2I$ together with the order of the group and the number of members is summarized in Figure 9.

3.2.4 Normal subgroups of $2I$

A normal subgroup of $2I$ divides the group order 120 and is the union of conjugacy classes (Theorem A.7). Considering Figure 9, it follows that the only non-trivial normal subgroup is $\{\pm 1\}$ with order 2.

However, another way to show that the only normal subgroups of $2I$ are $2I$ itself, the trivial group and the group of order 2, is by using that homomorphisms map normal subgroups map to normal subgroups.

Theorem 3.15. *Let G_1 and G_2 be groups and $N \subseteq G_1$ a normal subgroup. Let $f : G_1 \to G_2$ be a surjective homomorphism. Then $f(N)$ is a normal subgroup of G_2.*

Proof. For any $g \in G_1$ and $n \in N$ it holds that there exist a $n' \in N$ such that $n' = gng^{-1}$. Then $f(gng^{-1}) = f(g)f(n)f(g)^{-1} = f(n')$. Also, each $h \in G_2$ and $n \in N$ satisfies $hf(n)h^{-1} \in f(N)$ as f is surjective. □

Now suppose $N \subset 2I$ is a normal subgroup. Suppose there is a one-to-one homomorphism, that is an isomorphism, $g : 2I \to I$. Then $g(N)$ is a normal subgroup of I and we also know that the g preserves the order of the elements in N from Theorem A.6. The only normal subgroups of I are isomorphic to A_5 or to the trivial group. That means that $g(N)$ has order 1 or 60. Suppose $|g(N)| = 60$ and $|N| = 60$. Since A_5 contains 15 elements of order 2, while $2I$ has only one such element, it follows that such an isomorphism g does not exist. In case $|f(N)| = 1$ and $|N| = 1$, we can find an isomorphism g, the trivial map. We find that $\{1\}$ is a normal subgroup of $2I$.

Now suppose that we have the two-to-one homomorphism f from Theorem 3.14 and restrict it to $h : 2I \to I$. We then know that h maps a normal subgroup $N \subset 2I$ to a normal subgroup of I. Thus, N is mapped two-to-one to I or the trivial group. We already found that $2I$ maps to I two-to-one. We do find here another subgroup of $2I$. Namely, $N = \{\pm 1\}$ as $\{\pm 1\}$ maps two-to-one to the identity rotation in I.

Thus, all the normal subgroups of $2I$ are $2I$ itself, the group $\{\pm 1\}$ and the trivial group.

From these observations, we can conclude that $\ker(h) = \{\pm 1\}$. Indeed, the kernel is a normal subgroup of $2I$, so there are only three cases to consider. Firstly, the $\ker(h)$ is not trivial, as h is not injective. Next, the kernel is not $2I$ itself, as there are elements in $2I$ that are not mapped to I trivially. For example, h does not map the quaternion $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ to the identity rotation of the icosahedron. Hence, $\ker(h) = \{\pm 1\}$.

3.3 Binary tetrahedral group $2T$

3.3.1 Definition of the binary tetrahedral group $2T$

The rotations of the tetrahedron can be represented by quaternions as well. This set, the binary tetrahedral group, is denoted $2T$ and is subgroup of $2I$. The coordinate representation of the 24 quaternions are given by the permutation of coordinates given by:

$$(\pm 1, 0, 0, 0) \qquad (\pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}).$$

These coordinates also form the vertices of a 24-cell in our coordinate system. Indeed, the quaternions with coordinates equal to $\pm\frac{1}{2}$ form a tesseract, while the quaternions with one coordinate equal to ± 1 form a 16-cell. The quaternions that form the vertices of a tesseract are also called unit Hurwitz quaternions. The other quaternions are called unit Lipschitz quaternions.

Theorem 3.16. *$2T$ is a subgroup of $\mathbb{H}_{\mathbb{1}}$.*

Proof. Since $2T = 2I \cap \mathbb{Q}^4$ and both $2T$ and Q^4 are groups, it follows that $2T$ is a group. [35] □

In Appendix B.1.3, a Mathematica code to compute that $2T$ is a group is included as well.

3.3.2 Pre-images of the quaternions in $2T$ in the 3-sphere

Consider a tetrahedron with vertices

$$(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}), \qquad (\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2}), \qquad (-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2}), \qquad (-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2}).$$

We show that the image of the quaternions in the binary tetrahedral group $2T$ as subgroup of $2I$ defined in Section 3.2, describe the rotations of the tetrahedron. The 24 quaternions in $2T$ are given all permutations of:

$$(\pm 1,0,0,0), \quad (\pm\tfrac{1}{2},\pm\tfrac{1}{2},\pm\tfrac{1}{2},\pm\tfrac{1}{2}).$$

The rotations with $r = \pm 1$ describe the identity rotation of the tetrahedron under the map from Theorem 3.14. The quaternions with $r = 0$ describe the rotations fixing two edges of the tetrahedron twice. Lastly, the rotations fixing a vertex and a midpoint of a face of the tetrahedron, are twice described by the quaternions with $r = \frac{1}{2}$.

3.3.3 Conjugacy classes and normal subgroups of $2T$

The order of the elements in $2T$ need not be the same order as the order of the corresponding rotation in the tetrahedron. For example, the quaternions with $r = \pm\frac{1}{2}$ are the rotation of about 120° or −120°. In the rotation group of the tetrahedron, those have order 3. However, one can compute that

$$(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2})^3 = (-1,0,0,0) \quad \text{and} \quad (-\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2})^3 = (1,0,0,0)$$

This means that in $2T$, the conjugacy class that contains $(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$ has order 3, but the conjugacy class containing $(-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$ has order 6.

Before we compute the orders of the elements in the conjugacy classes in $2T$, we need to determine the sizes of those conjugacy classes. It appears that the quaternions with $r = \pm\frac{1}{2}$ both split into two conjugacy classes. This result is not too surprising, as the rotation group of the tetrahedron is isomorphic to the alternation group A_4. In A_4, not all 3-cycles are conjugate as the transpositions in S_4, the elements that conjugate 3-cycles in S_4, does not lie in A_4.

<table>
<tr><th></th><th>Conjugacy class</th><th>Order in $2T$</th><th>Number in $2T$</th><th>Order in $2I$</th><th>Number in $2I$</th></tr>
<tr><td>A</td><td>$(1,0,0,0)$</td><td>1</td><td>1</td><td>1</td><td>1</td></tr>
<tr><td>B</td><td>$(-1,0,0,0)$</td><td>2</td><td>1</td><td>2</td><td>1</td></tr>
<tr><td>C</td><td>$(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}),(\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2})$
$(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}),(\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$</td><td>6</td><td>4</td><td rowspan="2">6</td><td rowspan="2">20</td></tr>
<tr><td>D</td><td>$(\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}),(\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2})$
$(\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}),(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$</td><td>6</td><td>4</td></tr>
<tr><td>E</td><td>$(-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}),(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2})$
$(-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}),(-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$</td><td>3</td><td>4</td><td rowspan="2">3</td><td rowspan="2">20</td></tr>
<tr><td>F</td><td>$(-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}),(-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2})$
$(-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}),(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$</td><td>3</td><td>4</td></tr>
<tr><td>G</td><td>$(0,1,0,0),(0,-1,0,0),(0,0,1,0)$
$(0,0,-1,0),(0,0,0,1),(0,0,0,-1)$</td><td>4</td><td>6</td><td>4</td><td>30</td></tr>
</table>

Table 10: Conjugacy classes of $2T$.

It follows from Table 10 that the normal subgroups of $2T$ must be the trivial group, $\{\pm 1\}, Q_8$ and $2T$ itself. The subgroup $\{\pm 1\}$ is normal as a consequence of the fact that quaternion multiplication by purely real quaternions is commutative. The subgroup Q_8 is also normal, since multiplication with a basis quaternion gives another basis quaternion as defined in Table 8. A computation is also included in Section B.1.3.

The algebraic way to prove that T has four normal subgroups, is by considering the normal subgroups of the T. Since $T \cong A_4$, the image of a normal subgroup under the homomorphism from $2T$ to T has to be A_4 itself, the Klein Vierergruppe V_4 or the trivial group. If we have a homomorphism from $2T$ to T, we must have that any normal subgroup $N \subset 2T$ maps either two-to-one or one-to-one to T. First, any subgroup of $2T$ with index 2 must be normal. Its normal image has either order 6 or order 12. Since the rotation group of the tetrahedron has no normal subgroup of order 6, the image should be A_4 itself. However, A_4 contains 9 elements of order 2, while $2T$ only contains one. It follows that $2T$ cannot contain a normal subgroup of order 12.

Then, any subgroup of order 8 in $2O$ maps to a normal subgroup of order 8 or 4. The Klein Vierergruppe is a normal subgroup of order 4. Hence, the double cover of the rotations described by the Klein Vierergruppe, that is Q_8, is a normal subgroup of $2T$.

Certainly, the identity rotation of the tetrahedron is described by two quaternions of $2T$. Those quaternions form a normal subgroup of order 2. Thus, the normal subgroups of $2T$ are the trivial group, $\{\pm 1\}, Q_8$ and $2T$ itself.

3.3.4 $2T$ as subgroup of $2I$

$2T$ is a subgroup of $2I$, but it is not a normal subgroup. However, we will show that $2I$ contains five conjugate subgroups of $2T$. This is a very useful and important result, as we will see in Section 4.3.2 when we consider the inscription of 24-cells in a 600-cell.

Although the reader should be familiar with the following theorems and definitions, we state them here again, as we will really need them in the proof of Theorem 3.20. The theorems and definitions come from [18], unless otherwise specified.

Definition 3.17 (Stabilizer of an element x). Let G be a group acting on a set X. The stabilizer of an $x \in X$, denoted $\text{Stab}_G(x)$, is a group and defined by:

$$\text{Stab}_G(x) := \{g \in G | gx = x\}.$$

Definition 3.18 (Normalizer of a subgroup). Let G be a group and let H be a subset, not necessarily subgroup, of G. The normalizer of H is then

$$N_G(H) := \{g \in G | ghg^{-1} \in H \text{ for all } h \in H\}.$$

Theorem 3.19. *If a group G acts on a subset X of G by conjugation, then the stabilizer of $x \in X$ is equal to the normaliser of $x \in X$. [36]*

In particular, the number of conjugate subgroups to H in G equals the index of the normalizer $N_G(H)$. [36]

Theorem 3.20. *$2I$ contains five conjugate subgroups of $2T$.*

Proof. The idea of the proof is to show that the normalizer $N_{2I}(2T)$ of $2T$ in $2I$ is $2T$ itself. It then follows that the are five conjugate subgroups of $2T$ in $2I$.

Let $2I$ act on $2T$ by conjugation, where $2T$ is seen as an element in the set of subgroups of $2I$. That is,

$$\text{Stab}_{2I}(2T) := \{g \in 2I | g2T = 2T\} = \{g \in 2I | gx = x \text{ for all } x \in 2T\}.$$

It follows from Theorem 3.19 that the normalizer $N_{2I}(2T)$ equals the stabilizer of $2T$ in $2I$. Hence, the $N_{2I}(2T) \subset 2I$, since the stabilizer is a subgroup of $2I$. Therefore, $N_{2I}(2T)$ is subgroup whose index lies between 5, the index of $2T$ in

$2I$, and 1, the index of $2I$ itself. Since $2T$ is a subgroup of its normalizer by definition, it follows that the index of $N_{2I}(2T)$ must divide 5. This implies that $N_{2I}(2T)$ is either $2T$ or $2I$. However, there are quaternions in $2I$ whose conjugation with quaternions in $2T$ lie outside $2T$. For example, conjugation of the quaternion $(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$ with the quaternion $(\frac{1}{2},\frac{\phi}{2},\frac{1}{2\phi},0)$ gives $(\frac{1}{2},\frac{\phi}{2},-\frac{1}{2\phi},0)$. It follows that $N_{2I}(2T) = 2T$. Since the number of conjugate subgroups of $2T$ equals the index of its normalizer, we have shown that there are five conjugate subgroups of $2T$ in $2I$. □

3.4 Binary octahedral group $2O$

3.4.1 Definition of the binary octahedral group $2O$

The binary octahedral group $2O$ is a set of 48 quaternions. These quaternions describe the rotations of the octahedron, which we show in Section 3.4.2. Additionally, those 48 quaternions form a group, which is proved in Theorem 3.21. Furthermore, these quaternions form the vertices of the compound of two 24-cells, which we show in Section 4.2.5. The coordinate representation of the 48 quaternions are given by the permutation of the coordinates:

$$(\pm 1,0,0,0), \qquad (\pm\tfrac{1}{2},\pm\tfrac{1}{2},\pm\tfrac{1}{2},\pm\tfrac{1}{2}), \qquad (\pm\tfrac{1}{\sqrt{2}},\pm\tfrac{1}{\sqrt{2}},0,0).$$

Theorem 3.21. *$2O$ is a subgroup of $\mathbb{H}_1$.*

Proof. The inverse of all quaternions in $2O$ lies in $2O$ as well. Also, the identity of $\mathbb{H}$ lies in $2O$. The computation in Section (B.1.2) shows that all products of quaternions lie in $2O$ as well. □

3.4.2 Pre-images of the quaternions in $2O$ in the 3-sphere

A convenient way to show that the quaternions in $2O$ describe the rotation group of the octahedron is to use the fact that the dual of the octahedron is the cube. This duality implies that the rotation group of the cube and the octahedron is the same. In Figure 23, a cube is inscribed in the octahedron. The rotation axes of the cube of three non-trivial rotations of different order are represented by three gray axes. The rotations through the midpoints of the faces of the cube, are rotations through the vertices of the octahedron. Furthermore, the rotations through the vertices of the cube are described by the rotation through the midpoints of triangular faces of the octahedron. Lastly, the rotation axes through the edges of the cube are also rotations through two midpoints of edges of the octahedron.

Consider an octahedron with vertices

$$(\pm\tfrac{1}{\sqrt{2}},0,0), \qquad (0,\pm\tfrac{1}{\sqrt{2}},0), \qquad (0,0,\pm\tfrac{1}{\sqrt{2}}).$$

We show that the image of the group $2O$ under the map from Theorem 3.14 is the rotation group of the octahedron. We describe the rotation the quaternions in $2O$ are mapped to and we also consider the conjugacy classes of $2O$.

First of all, the quaternions ± 1 act as the identity rotation of the octahedron under the map from Theorem 3.14. The quaternions with $r = \frac{1}{2}$ describe the rotations over 120° through the midpoints of the triangular faces of the octahedron. These midpoints of the triangles of the octahedron with edges of unit length have coordinates $(\pm\frac{1}{3\sqrt{2}}, \pm\frac{1}{3\sqrt{2}}, \pm\frac{1}{3\sqrt{2}})$. The quaternions with $r = -\frac{1}{2}$ describe the same rotation as the quaternions with real part $\frac{1}{2}$.

Then, the rotations about 90° and 270° with their rotation axis through two opposite vertices of the octahedron are described by the quaternions with $r = \pm\frac{1}{\sqrt{2}}$. The rotations with $r = 0$ and a single coordinate non-zero, describe the rotations through the vertices of the octahedron with rotation angle 180°. Lastly, the quaternions with $r = 0$ and described by the permutations of $(0, \pm 1, 0, 0)$ describe the rotations through the midpoints of the edge of the octahedron with unit edge length.

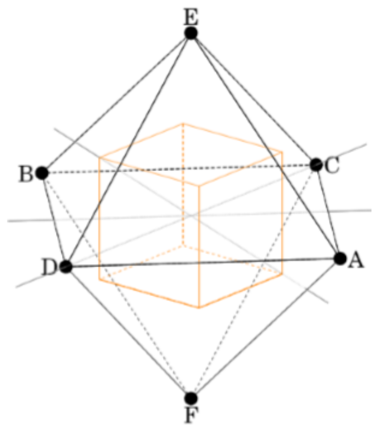


Figure 23: Inscription of a cube in the octahedron.

3.4.3 Conjugacy classes and normal subgroups of $2O$

The conjugacy classes of the quaternions in $2O$ are given in Table 11. The quaternions in $2O$ are conjugate iff they have the same real part. A computation of these conjugacy classes can be found in Appendix B.1.2.

Next, we consider the normal subgroups of $2O$. First of all, the trivial group, $2O$ itself and $\{\pm 1\}$ are normal subgroups of $2T$. Furthermore, Q_8 and $2T$ are normal in $2O$ as well. The fact that Q_8 is normal in $2O$ follows directly from

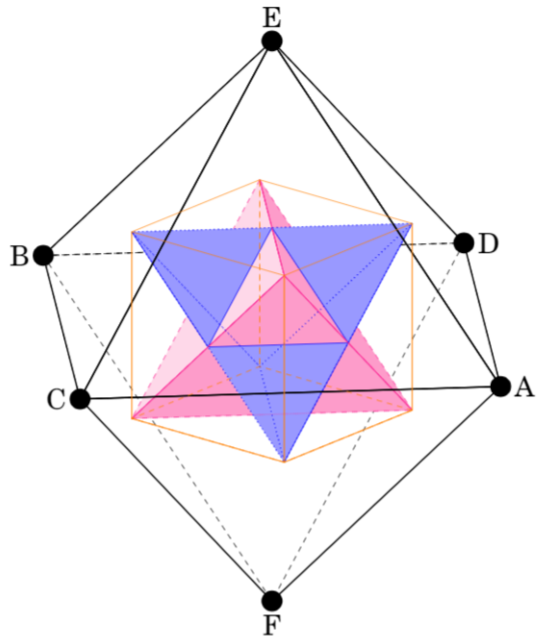


Figure 24: Inscription of two tetrahedra in a cube in an octahedron.

Conjugacy class	Order	Number
$(1,0,0,0)$	1	1
$(-1,0,0,0)$	2	1
$(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	6	8
$(-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	3	8
$(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0,0)$	8	6
$(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0,0)$	8	6
$(0,1,0,0)$	4	6
$(0,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0)$	4	12

Table 11: Conjugacy classes of $2O$.

the multiplication table in Table 8 and the normality of $2T$ follows from the fact that it has index 2 in $2O$. A computation of the normality can also be found in Appendix B.1.2.

However, we can also show that the normal subgroups are given by the trivial group, $\{\pm 1\}, Q_8, 2T$ and $2O$ itself, using the normal subgroups of O. We know $O \cong S_4$ and that the normal subgroups of S_4 are S_4, A_4, the Klein Vierergruppe V_4 and the trivial group.

Suppose there were a normal subgroup of $2O$ isomorphic to a normal subgroup of O. The number of elements of order 2 in this normal subgroup and in S_4 would be equal. However, S_4 contains 9 elements of order 2, while $2O$ only contains one such element. Any subgroup of S_4 other than the trivial group also contains mroe than one element with order 2. It follows that $2O$ cannot have a normal subgroup other than the trival group, that is isomorphic to normal subgroup of S_4, as any non-trivial isomorphism would not preserve the orders of elements.

However, if we map $2O$ two-to-one to $O \subset \mathrm{SO}(3)$ via the map from Theorem 3.14, we can map $2O$ to the normal subgroup O and $2T$ to the normal subgroup A_4. Furthermore, Q_8 can be mapped to the normal subgroup V_4 and $\{\pm 1\}$ to the the trivial normal subgroup.

It thus follows that $2O$ contains five normal subgroups which have orders $1, 2, 8, 24, 48$ respectively.

4 Four dimensions

4.1 Tesseract

4.1.1 Definition tesseract

Definition 4.1. The tesseract is the four-dimensional regular polytope and it is the analogue of the cube. The tesseract is also called the hypercube or the 8-cell, denoted C_8.

There are various other ways to describe what a tesseract is, how it can be constructed or how one could think of this four-dimensional cube. Each description gives an interesting insight in the properties or geometry of the tesseract. We will treat three more descriptions of the tesseract.

A first description of the tesseract is given by explicitly stating 16 vertices that form a tesseract. A convenient choice for these vertices is all possible combinations of $(\pm1, \pm1, \pm1, \pm1)$. Any tesseract with different vertices is similar to this one. The faces of the tesseract are squares and the cells are three-dimensional cubes.

Another insightful way of describing the relation between the tesseract and the three-dimensional cube, found amongst others in *The Panenmentalist Philosophy of Science* [19] is given as follows:

The cube is to the tesseract what the square is to the cube.

Lastly, the construction of the tesseract by starting with a hypercube of dimension 0 up tot a hypercube of dimension 4, the tesseract, is very insightful as well. A hypercube of dimension 0 is nothing more than a single vertex. The hypercube of dimension 1 is obtained by adding an extra dimension, an edge, starting at each vertex of the 0-cube. Since an edge must be incident to two vertices, we need to add an additional vertex as well. Thus, the hypercube of dimension 1 is a line segment incident to two vertices.

Adding another dimension gives the hypercube of dimensions 2. Again, we add to each vertex of the hypercube of dimension 1 an edge and add vertices in such a way that each edge is incident to two vertices. The obtained hypercube of dimension 2 is, as expected, a square.

For the third dimension, we again add from each vertex of the 2-cube the edges and necessary vertices to obtain the cube.

Hence, the hypercube of dimension 4, is nothing more than adding to each vertex of the cube an edge and at the end of such an edge a vertex. In this way, 7 more cubes are formed. Namely, each face of the cube gives rise to a cube of the tesseract. Together with the original cube and the cube formed by the 8 new vertices at the end of the edges added to lift the cube to the fourth dimension, it follows that the tesseract contains 8 cubes. In Figure 25, this construction is illustrated.

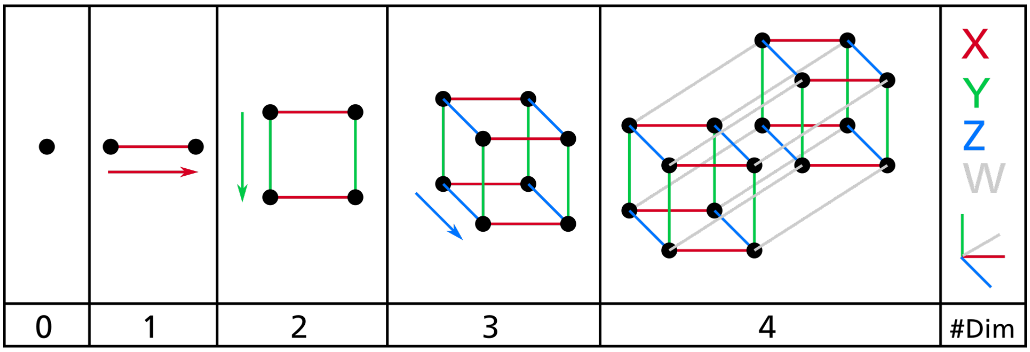


Figure 25: Construction tesseract from dimension 0 to dimension 4. *[Image retrieved from [3]]*

The number of vertices, edges, faces and cells of the tesseract are given as in Figure 26. The number of vertices in a tesseract are obtained by adding to each vertex of a cube in three-dimensional space an edge with each edge incident to a new vertex. Thus, the tesseract has twice as many vertices as the cube.

The 16 vertices are incident to 4 edges each, for each dimension exactly one. That means that the total number of edges is given by $\frac{16\cdot 4}{2} = 32$.

To determine the number of faces of the tesseract, one should notice that each set of 2 edges at a vertex describes a face of the tesseract. That means that $\binom{4}{2}\cdot 16 = 96$ faces are described by the vertices. However, one face is described by 4 edge-pairs and each face is thus counted 4 times. Hence, the total number of faces of the tesseract is 24.

Lastly, there are 8 cells in the tesseract, as we already reasoned in the construction of the tesseract from a hypercube of dimension 0.

n-cube	Name	Vertices	Edges	Faces	Cubes	Tesseracts
0-cube	-	1	-	-	-	-
1-cube	-	2	1	-	-	-
2-cube	Square	4	4	1	-	-
3-cube	Cube	8	12	6	1	-
4-cube	Tesseract	16	32	24	8	1

Figure 26: Number of n-cubes in a n-cube for $n = 0,1,2,3,4$.

The 8 cubes lie in pairs in the tesseract. Namely, each cube give rise to an opposite cube of which the vertices lie at the negative coordinates of the original cube. These 8 cubes are visualized in Figure 27.

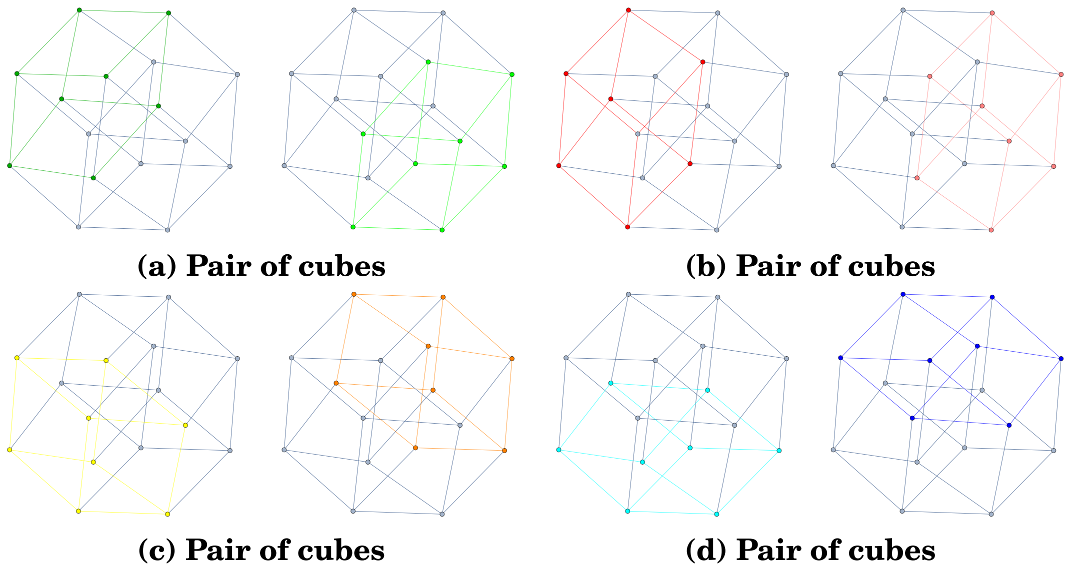


Figure 27: 4 pairs of cubes in the tesseract.

There exists a general formula to obtain the number k-cubes inside a n-cube for $k \leq n$ (Theorem 4.1.1). Furthermore, the Euler's polyhedron formula for n-cubes (Theorem 4.1.1) describes the relation between the number of k-cubes in a n-cube.

Theorem 4.2. *The number of k-cubes in a n-cube is* $2^{n-k} \cdot \dfrac{n!}{k!(n-k)!}$.

Proof. Each k-cube has 2^k vertices. Choosing 2^k vertices from the total 2^n vertices of the n, gives thus rise to an inscribed k-cube of the n-cube. Choosing 2^k vertices can be done by fixing the value of $n-k$ coordinates of a vertex to be either $\{-1,1\}$ and varying the other k of the coordinates. The k free coordinates indeed determine 2^k vertices with equal entries for the $n-k$ fixed coordinates. There are $\binom{n}{k}$ ways to choose k coordinates of the n to vary. Since there are 2^{n-k} ways to choose fixed values for the fixed $n-k$ coordinates, the total number of k-cubes in a n-cube is the product of these two numbers, $2^{n-k} \cdot \dfrac{n!}{k!(n-k)!}$. □

For example, the number of cubes in the tesseract has to equal $2^{4-3} \cdot \frac{4!}{3! \cdot 1!} = 8$, precisely as we already argued before. Also, the alternating sum over all these number of k-cubes in a n-cube equals 0 as shown in Theorem 4.3.

Theorem 4.3 (Euler's polyhedron formula for n-cubes). *The alternating sum over all k-cubes ($k < n$) of a n-cube is* 0 *if* n *is odd and* 2 *if* n *is even.*

Proof. The number of k-cubes in a n-cube is $2^{n-k}\binom{n}{k}$. So the sum over all k cubes is given by:

$$\binom{n}{0}2^n(-1)^0 + \binom{n}{1}2^{n-1}(-1)^1 + \cdots + \binom{n}{n-1}2^1(-1)^{n-1} =$$
$$\binom{n}{0}2^n(-1)^0 + \binom{n}{1}2^{n-1}(-1)^1 + \cdots + \binom{n}{n-1}2^1(-1)^{n-1} + \binom{n}{n}2^0(-1)^n - \binom{n}{n}2^0(-1)^n =$$
$$(2-1)^n - 1\cdot(-1)^n = 1 + (-1)^{n+1} = \begin{cases} 0 & \text{n odd} \\ 2 & \text{n even.} \end{cases}$$

□

4.1.2 Rotation group of the tesseract

We let $O_{4,h}$ denote the symmetry group of the tesseract. Furthermore, we denote the rotation group of the tesseract by O_4.

Theorem 4.4. *The order of the symmetry group $O_{4,h}$ of the tesseract is* 384.

Proof. This can be proved using the orbit-stabilizer theorem (Theorem A.19). If we let $O_{4,h}$ act on the set of 8 cubes in the tesseract, it follows that $|O_{4,h}| = 8\cdot 48 = 384$, since the stabilizer of a cube has order 48. □

Recalling the definition of a semi-direct product (Definition A.33), we may define the following set.

Definition 4.5. Define $G = \{\pm1\}^4 \rtimes_\rho S_4$ where $\rho : S_4 \to \mathrm{Aut}(\{\pm1\}^4)$ where ρ acts on the components $\{\pm1\}$ by permutation and $\{\pm1\}^4 = \{\pm1\}\times\{\pm1\}\times\{\pm1\}\times\{\pm1\}$.

Definition 4.6. Let G act on $\mathbb{R}^4$ by

$$(\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4;\sigma)(x_1,x_2,x_3,x_4) = (\epsilon_1 x_{\sigma^{-1}(1)}, \epsilon_2 x_{\sigma^{-1}(2)}, \epsilon_3 x_{\sigma^{-1}(3)}, \epsilon_4 x_{\sigma^{-1}(4)}),$$

where $(x_1,x_2,x_3,x_4) \in \mathbb{R}^4$.

Theorem 4.7. *G is isomorphic to $O_{4,h}$.*

Proof. First show that the order of G and the order of $O_{4,h}$ are the same. Next we show that the map $f : G \to O_{4,h}$ where f is as defined in Definition 4.6 maps injectively to $O_{4,h}$. It then follows that $G \cong O_{4,h}$.
The order of G is $2^4\cdot 24 = 384$ since $|\{\pm1\}| = 2^4$ and $|S_4| = 24$. Furthermore, $(\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4;\sigma) \in \ker(f)$ if and only if $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = 1$ and $\sigma = \mathrm{id}$. Hence the kernel is trivial implying that f injective. □

Definition 4.8. Define $\chi : C_2^4 \rtimes S_4 \to \{\pm1\}$ given by

$$\chi(\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4;\sigma) = \epsilon_1\epsilon_2\epsilon_3\epsilon_4 \operatorname{sgn}(\sigma),$$

where we identify C_2 with the set $\{\pm1\}$ and where $\sigma \in S_4$.

Theorem 4.9. *$\chi : C_2^4 \rtimes S_4 \to \{\pm1\}$ is a homomorphism.*

Proof. We need to show that

$$(\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4;\sigma)(\epsilon_1',\epsilon_2',\epsilon_3',\epsilon_4';\sigma') = (\epsilon_1\epsilon'_{\sigma(1)},\epsilon_2\epsilon'_{\sigma(2)},\epsilon_3\epsilon'_{\sigma(3)},\epsilon_4\epsilon'_{\sigma(4)};\sigma\sigma'). \tag{10}$$

is mapped by χ to $\epsilon_1\epsilon_2\epsilon_3\epsilon_4\operatorname{sgn}(\sigma)\epsilon_1'\epsilon_2'\epsilon_3'\epsilon_4'\operatorname{sgn}(\sigma')$. Of course it is:

$$\chi(\epsilon_1\epsilon'_{\sigma(1)},\epsilon_2\epsilon'_{\sigma(2)},\epsilon_3\epsilon'_{\sigma(3)},\epsilon_4\epsilon'_{\sigma(4)};\sigma\sigma') = \epsilon_1\epsilon'_{\sigma(1)}\epsilon_2\epsilon'_{\sigma(2)}\epsilon_3\epsilon'_{\sigma(3)}\epsilon_4\epsilon'_{\sigma(4)} \operatorname{sgn}(\sigma\sigma')$$

which is the same product with its terms in different order and using $\operatorname{sgn}(\sigma\sigma') = \operatorname{sgn}(\sigma)\operatorname{sgn}(\sigma')$. □

From Theorem 4.7 we now know that $O_{4,h} \cong G$. We now show that the rotation subgroup O_4 of the tesseract is described by $\ker(\chi)$.

Theorem 4.10. *The rotation subgroup O_4 is given by $O_4 = \ker(\chi)$.*

Proof. The $\ker(\chi)$ consist of all $(\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4;\sigma)$ with $\epsilon_1\epsilon_2\epsilon_3\epsilon_4\sigma = 1$.

It is obvious that the elements $(1,1,1,1;\sigma) = \mathrm{id} \rtimes \sigma$ and $(-1,1,1,1;\mathrm{id}) = -1 \times 1 \times 1 \times 1 \rtimes \mathrm{id}$, with σ an odd permutation, do not lie in $\ker(f)$. We show that these elements of O_4 describe all reflections. Next, we show that the index of these reflections in $\{\pm1\}$ is 2. Then, it follows that the elements in O_4 with $\epsilon_1\epsilon_2\epsilon_3\epsilon_4\sigma = 1$ are elements of O_4.

The element $(-1,1,1,1;\mathrm{id})$ maps a vector $(x_1,x_2,x_3,x_4) \in \mathbb{R}^4$ to $(-x_1,x_2,x_3,x_4)$ which is a reflection. This can be seen from the matrix corresponding to the transformation $(x_1,x_2,x_3,x_4) \mapsto (-x_1,x_2,x_3,x_4)$ which is matrix with ones on the diagonals, zeroes in all other entries and the first diagonal place -1. The determinant of this matrix is -1.

The element $(1,1,1,1;\sigma)$ with σ an odd permutation, permutes an odd number of coordinates of a vector $(x_1,x_2,x_3,x_4) \in \mathbb{R}^4$. This is again a reflection. The matrix belonging to this transformation is obtained from the identity matrix by interchanging an odd number of columns which changes the determinant 1 of the identity matrix to -1.

Since $(-1,1,1,1;\mathrm{id})^2 = (-1\cdot -1, 1\cdot 1, 1\cdot 1, 1\cdot 1, \mathrm{id})$, the index of $(-1,1,1,1;\mathrm{id})$ in $\{\pm1\}$ is 2. Furthermore, $(1,1,1,1;\sigma)^2 = (1,1,1,1;\sigma^2)$ and σ^2 is an even permutation. It follows that the index of $(1,1,1,1;\sigma)$ in $\{\pm1\}$ is 2 as well. □

In Section 4.1.4 it is convenient to work with a different notation for the symmetries of the tesseract. Although it is very similar to the definition of the action of G on $\mathbb{R}^4$ from Definition 4.6, we will introduce the notion of signed permutation cycles already here.

Definition 4.11 (Signed (permutation) cycle). A signed permutation cycle of a signed symmetric group of n elements represent a permutation of the set $\{-n,-n+1,\ldots,-1,1,\ldots,n-1,n\}$.

In Theorem 4.6, we described the action of G on $\mathbb{R}^4$ by the coordinates that a $x \in \mathbb{R}^4$ is mapped to. Another way of representing the action of G on $\mathbb{R}^4$ can be done by signed permutations cycles. These cycles represent which entries of a vector $v=(r,x,y,z) \in \mathbb{R}^4$ are permuted and in which entries a minus sign is added. We choose to work with the letters r,x,y,z rather than x_1,x_2,x_3,x_4 for readability. Let us give an example of a signed cycle representation of the map $(r,x,y,z) \mapsto (y,x,-r,z)$. The signed cycle representation of this map is $(r-y)$ where the minus sign belongs to the entry in front of it. Thus, we should read the signed cycle $(r-y)$ as r is send to $-y$ and y to r, while x and z are mapped back to themselves. As second example we take the map $(r,x,y,z) \mapsto (z,-r,-x,y)$. This map can be represented by the signed cycle $(r\text{-}x\text{-}yz)$.

We now describe the 192 rotations of the tesseract. The rotations are given in Table 12, where we use the result from Theorem 4.10 and denoted the signed cycle representation of each rotation as well. The rotations in four-dimensional space can be separated into two different kinds: the simple and non-simple rotations. Simple rotations are rotations that fix one plane pointwise. Non-simple rotations, on the other hand, first rotate with respect to a first plane of rotation and afterwards with respect to a second. Those two planes can be chosen to be orthogonal planes. That is, the normal vectors of these planes are orthogonal. In four-dimensional space, each plane has two normal vectors. Thus, planes in four-dimensional space are orthogonal if both normal vectors of both planes are orthogonal to each other.

The tesseract has 86 simple rotations and 106 non-simple rotations. The simple rotations can be represented by the following rotations from Table 12:

- 24 rotations $(\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4;(12))$ of order 2
- 32 rotations $(\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4;(123))$ of order 3
- 12 rotations $(\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4;(12)(34))$of order 2
- 12 rotations $(\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4;(12)(34))$ of order 4
- 6 rotations $(\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4;\mathrm{id})$ of order 2

The cycles used to denoted the $\sigma \in S_4$ are representatives of the cycles in S_4 of that cycle type. In Section 4.1.5, the rotations of the tesseract are considered in a geometrical way and work with the notions of simple and non-simple rotations.

$\sigma \in S_4$	Order of (ϵ,σ)	$(\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4)$	Number	Signed cycle	Signed cycle type
(12)	2	(1,1,-1,1) (-1,-1,-1,1)	$6\cdot 4 = 24$	$(r\ x\ )(y-)$ $(r-\ x-)(y-)$	$2-1$
(12)	4	(-1,1,1,1) (1,-1,-1,-1)	$6\cdot 2 = 24$ $6\cdot 2 = 24$	$(r\ x-)$ $(r-\ x)(y-)(z-)$	$-2+1+1$ $-2-1-1$
(123)	3	(-1,1,1,-1)	$4\cdot 8 = 32$	$(r\ y\ x-)(z-)$	$-3-1$
(123)	6	(-1,-1,1,1)	$4\cdot 8 = 32$	$(r\ y-\ x-)$	$3+1$
(1234)	8	(-1,1,1,1) (-1,-1,-1,1)	$8\cdot 6 = 48$	$(r\ z\ y\ x-)$ $(r\ z-\ y-\ x-)$	-4
(12)(34)	2	(-1,-1,1,1)	$4\cdot 3 = 12$	$(r-\ x-)$	$2+2$
(12)(34)	4	(-1,1,-1,1)	$4\cdot 3 = 12$	$(r\ x-)(y\ z-)$	$-2-2$
id	2	(-1,-1,1,1)	6	$(r-)(x-)$	$-1-1+1+1$
id	2	(-1,-1,-1,-1)	1	$(r-)(x-)(y-)(z-)$	$-1-1-1-1$
id	1	(1,1,1,1)	1	id	$1+1+1+1$
			192		

Table 12: Rotations tesseract represented by signed cycles and coordinate maps.

4.1.3 Symmetry group of the tesseract

Using the map from Theorem 4.9, we can construct the full symmetry group of the tesseract, with order 384. The orientation reversing isometries of the tesseract are given in Table 13. Table 12 and Table 13 together describe the entire symmetry group $O_{4,h}$.

All symmetries of the tesseract, ordered by the order of the symmetries, are given in Figure 14.

4.1.4 The conjugacy classes and normal subgroups of the symmetry group of the tesseract.

In the Section 4.1.3, we established the symmetry group O and connected them to the signed cycle representation. The conjugacy classes of $O_{4,h}$ are given exactly by those different signed cycle types, which is a consequence of the following theorem.

Theorem 4.12. *Two compositions of signed cycles in disjoint cycle decomposition are conjugate and geometrically similar if and only if their cycles types are the same. [26]*

Since the symmetry group $O_{4,h}$ contains 20 signed cycle types it follows that $O_{4,h}$ has 20 conjugacy classes. Those conjugacy classes are presented in Figure 15. The dashed line in the table separates the rotations of the tesseract above the line from the non-rotational symmetries below it.

S_4	Order	$(\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4)$	Number	Signed cycle	Signed cycle type
(12)	2	(-1,-1,1,1) (1,1,1,1)	$6+6=12$	$(r-\ x-)$ $(r\ x)$	2 + 1 + 1
(12)	2	(1,1,-1,-1) (-1,-1,-1,-1)	$6+6=12$	$(r\ x)(y-)(z-)$ $(r-\ x-)(y-)(z-)$	2 - 1 - 1
(12)	4	(1,-1,-1,1)	$4\cdot 6=24$	$(r-\ x)(y-)$	-2 - 1 + 1
(123)	6	(-1,1,1,1) (-1,-1,-1,1)	$8\cdot 3+8=32$	$(r\ y\ x-)$ $(r-\ y-\ x-)$	-3 + 1
(123)	6	(1,1,1,-1) (-1,-1,1,-1)	$8+8\cdot 3=32$	$(r\ y\ x)(z-)$ $(r\ x-\ y-)(z-)$	3 - 1
(1234)	4	(-1,-1,1,1) (1,1,1,1) (-1,-1,-1,-1)	$6\cdot 6+6+6=48$	$(r\ z\ y-\ x-)$ $(r\ z\ y\ x)$ $(r-\ z-\ y-\ x-)$	4
(12)(34)	4	(-1,1,1,1) (-1,-1 -1,1)	$3\cdot 4+3\cdot 4=24$	$(r\ x-)(y\ z)$ $(r-\ x-)(y\ z-)$	-2 + 2
(id)	2	(-1,1,1,1)	4	$(r-)$	-1 + 1 + 1 + 1
(id)	2	(-1,-1 -1,1)	4	$(r-)(x-)(y-)$	-1 - 1 - 1 + 1
			192		

Table 13: Orientation reversing isometries of the tesseract.

Order symmetry	Number	Rotations
1	1	1
2	69	43
3	32	32
4	138	36
6	96	32
8	48	48
	384	192

Table 14: All symmetries of the tesseract sorted by the order of the symmetry.

Cycle type	Order	Number
1 + 1 + 1 + 1	1	1
-1 - 1 - 1 - 1	2	1
-1 - 1 + 1 + 1	2	6
2 - 1 + 1	2	24
-2 - 1 - 1	4	12
-2 + 1 + 1	4	12
-3 - 1	6	32
3 + 1	3	32
-4	8	48
2 + 2	2	12
-2 - 2	4	12
−1 + 1 + 1 + 1	2	4
-1 - 1 - 1 + 1	2	4
2 + 1 + 1	2	12
2 - 1 - 1	2	12
-2 - 1 + 1	4	24
-3 + 1	6	32
3 - 1	6	32
4	4	48
-2 + 2	4	24

Table 15: All symmetries of the tesseract represented by their conjugacy classes.

To investigate whether the group O contains normal subgroups, we first consider the normal subgroups of the O_4 of order 192. Its divisors are given by: $2, 3, 4, 6, 8, 12, 16, 24, 32, 48, 64, 96, 192$. From Figure 15 it follows that there is a normal subgroup of order 2, containing the signed cycles $1+1+1+1$ and $-1-1-1-1$. However, due to the sizes of conjugacy classes, it also follows that there cannot be normal subgroups of order $3, 4, 6$ or 48. To see whether the rotation group has a subgroup of order 32, 64 or 96, we use the following approach. We consider the cycle types of products of elements in each conjugacy class. As each normal subgroup is a group, the product of arbitrary elements in the normal subgroup must lie in the normal subgroup again. That means that if a normal subgroup contains a particular conjugacy class, it must necessarily also contain all conjugacy classes that contain a product of from that particular conjugacy class.

In Figure 28, the conjugacy class of the rotation group are drawn. Each class is represented by its signed cycle type. If the product of two elements of a conjugacy class lie in a different nontrivial conjugacy class, a black arrow is drawn pointing to the conjugacy class the product lies in. The arrow pointing towards the conjugacy class of the identity element is represented by a dashed gray arrow, since each conjugacy class has products of elements that lie in this conjugacy class.

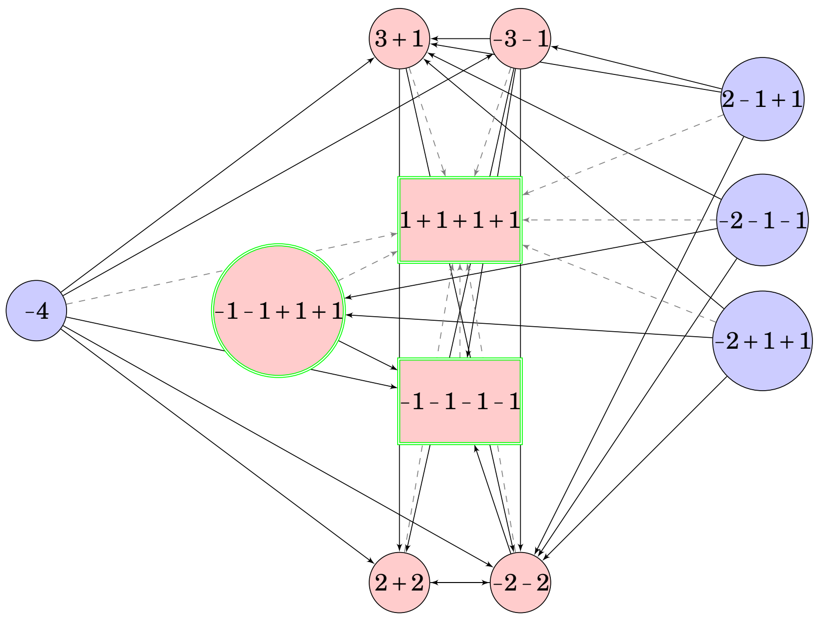


Figure 28: Product of elements in conjugacy classes of the rotations in O_4.

Any subgroup of order 32 consists of a conjugacy class of order 24 together with the normal subgroup of order 8 or of two conjugacy classes of order 12 together with the normal subgroup of order 8. The only way to choose conjugation classes that will form a group is to take the conjugacy classes represented by the

signed cycles $-2-2$ and $2+2$ together with the normal subgroup of order 8. This group is indeed normal, since any product of cycles must has an even number of minus signs and consist of two cycles of length 2 or of four cycles of length 1. Hence, there is a normal subgroup of order 32.
However, a normal subgroup of order 64 does not exist, although it seems possible to take the conjugacy classes of cycles with cycle type $3+1$ and the normal subgroup of order 32. However, the product

$$(x\ y\ z)(x\ y-)(z\ w-) = (x\ z\ w-)(y-)$$

does not lie in this union of conjugacy classes and the union is therefore not a group,
Next, a normal subgroup of order 96 exists. Take the conjugacy classes of the cycles of types

$$3+1, \quad -3-1, \quad 2+2, \quad -2-2.$$

and the normal subgroup of order 8. This group consists of even cycles whose cycle types have an even sign. Since both the product of even cycles is even and the product of cycles with an even number of minus signs is even, it follows that the product of any cycles in this set of 96 cycles lies in the same set. Furthermore, the inverses of each cycle lie in the set and the identity lies in it as well. Thus, the rotation group of the tesseract contains a normal subgroup of order 96.

The normal subgroup of order 96 can directly be related to the sign of the cycle types of the symmetry in $O_{4,h}$ and the sign of the product of the element in C_2^4 that describes part of the rotation. Define the map χ_1 and χ_2 that map from $C_2^4 \rtimes S_4 \to \{\pm 1\}$ by

$$\chi_1(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4; \sigma)) = \epsilon_1\epsilon_2\epsilon_3\epsilon_4 \qquad \chi_2(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4; \sigma)) = \text{sgn}(\sigma).$$

Then $\ker(\chi_1) \cap \ker(\chi_2)$ is the normal subgroup of order 96.

To investigate the normal subgroups of the full symmetry group of the tesseract, we need to investigate all 20 conjugacy classes. The conjugacy classes consisting of only products of 1-cycles, but with an arbitrary number of minus signs, gives rise to a normal subgroup of order 16.

For the other divisors of 384, we cannot form any new normal subgroups. Any subgroup of order 32 containing some conjugacy class form the non-rotations of the tesseract, must amongst other conjugacy classes, contain the conjugacy class of cycle type $2+2$. Any normal subgroup of order 64 must contain a conjugacy class of order 32. However, those conjugacy classes have products lying in a rotation conjugacy class of order 32. Thus, a normal subgroup of order 64 does not exist.

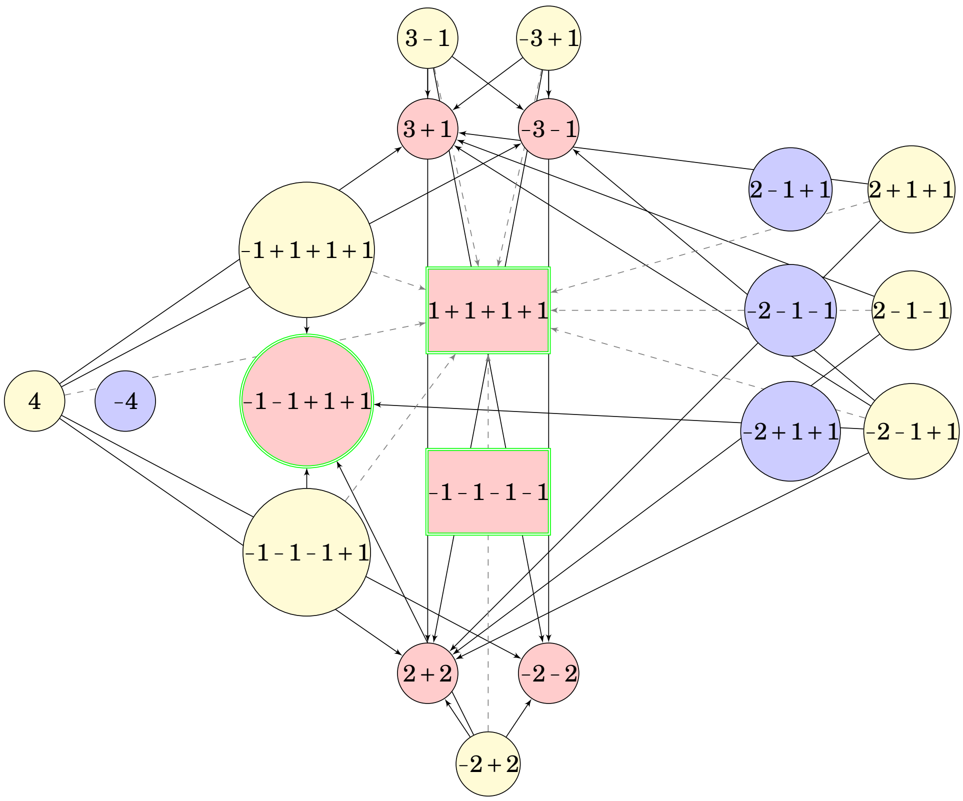


Figure 29: Product of elements in conjugacy classes of the orientation-reversing isometries in $O_{4,h}$.

Furthermore, a subgroup of order 96, contains 2 conjugacy classes of order 32. However, taking the two conjugacy classes of the cycles -3+1 and 3-1, the normal subgroup must also contain the conjugacy classes of cycles 3+1 and -3-1. Lastly, a subgroup of order 128 seems to be normal from Figure 29. However, taking the conjugacy classes with cycle types

$$3\,\text{-}\,1, \qquad 3+1, \qquad \text{-}1\,\text{-}\,1, \qquad 2+2, \qquad \text{-}2\,\text{-}\,2$$

and the subset of order 8, there are multiple products of elements in the subset that do not lie in the set. For example,

$$(x\ y\ z-)(x-\ y-\ z-)(w-) = (x-\ z-\ y)(w-),$$

lies in the conjugacy classes of the signed cycle $3-1$.

In conclusion, we found the 8 normal subgroups of $O_{4,h}$. We formalize the result in the following theorem.

Theorem 4.13. *The normal subgroups of the symmetry group $O_{4,h}$ of the tesseract have orders $2, 8, 16, 32, 96$ and 192. In Table 16, each of these normal subgroups is represented by the conjugation classes it contains.*

Normal subgroup	Conjugation classes	Order
Trivial	1+1+1+1	1
A	1+1+1+1, -1,-1,-1,-1	2
B	A, -1-1+1+1	8
C	B, 2+2, -2-2	32
D	C, 3+1, -3-1	96
E	B, -1+1+1+1, -1-1-1+1	16
Rotation group	D, 2-1+1, -2-1-1 -2+1+1, −4	192
Symmetry group	All conjugacy classes	384

Table 16: Normal subgroups of the symmetry group $O_{4,h}$ of the tesseract.

4.1.5 Geometrical interpretation of the rotations of the tesseract

To describe the rotation group of the tesseract in a geometrical way, we need to make an important remark that we will use throughout this section. That is, each rotation of the tesseract acts on a pair of cubes: a cube and its opposite defined by the negatives of the vertices of the first cube (Figure 27).

In three-dimensional space, rotations can be described by the rotation axis. An alternative way would be to describe the plane that is left invariant under the rotation. For three-dimensional space, the orthogonal complement is the plane that is left invariant under a rotation. An important remark here is that although the plane is left invariant, the vectors that lie in the plane are in general not invariant under the rotation. However, the points lying on the rotation axis itself are pointwise invariant under the rotation corresponding to that particular rotation axis. That means that any point on the rotation axis is mapped to itself, although any vector in the orthogonal complement of the rotation axis is mapped to another vector lying in the same plane.

In four-dimensional space, the rotation axis makes place for a rotation plane. This rotation plane is a fixed plane for the corresponding rotation, while each plane perpendicular to the rotation plane is only invariant under the rotation.

The geometrical description of the rotations of the tesseract will be described by the orthogonal complement of the rotation plane. That means that a rotation of the tesseract maps each vector in the orthogonal plane we describe has to be mapped to another point in that plane. We consider the action of a rotation on the vertices, edges, faces and cubes of the tesseract in the orthogonal plane. This geometrical account of the rotation group of the tesseract is based on [11]. However, before we describe O_4 by the geometrical description of these vertices, edges, faces and cubes in an invariant plane, we need to convince ourselves that such an approach will describe all rotations of the tesseract. This is the case and we state in in Theorem 4.14 which is proved in [11].

Theorem 4.14. *The action on the* $(n-1)$*-cubes uniquely determine the symmetry of the* n*-cube.* [11]

It is convenient to explicitly use the coordinates of the centres of the cubes, faces, edges and vertices in the standard tesseract with vertices $(\pm 1, \pm 1, \pm 1, \pm 1)$ to describe the rotations of the tesseract. These coordinates can be described as in Theorem 4.15 and is proved in [11].

Theorem 4.15. *The centre of a* k*-dimensional cube inside an* n*-dimensional cube* ($k \leq n$) *has* k *of its coordinates equal to* 0 *and* $n-k$ *of its coordinates equal to* ± 1. [11]

It follows that for the standrad tesseract, the centers of the k-cubes are given by the permutations of the coordinates given in Table 30.

We encounter all rotations of the tesseract by systematically considering those planes left invariant under a rotation of the tesseract by the k-cubes centers lying

k-cube	Coordinates center k-cube
Vertex	$(\pm1,\pm1,\pm1,\pm1)$
Edge	$(\pm1,\pm1,\pm1,0)$
Face	$(\pm1,\pm1,0,0$
Cube	$(\pm1,0,0,0)$
Tesseract	$(0,0,0,0)$

Figure 30: Coordinates of the k-cubes of the tesseract.

in the plane. First, we describe the simple rotations of the tesseract. To encounter all possible rotation planes, we start describing those planes that contain 3-cube centers. After that, we continue with planes that do not contain any 3-cube centers, but do contains 2-cube centers. We continue this process until we have described the last rotation by a plane that only contains vertices of the tesseract. Furthermore, in our approach we use the remark that k-cube centers lie in the invariant planes in pairs. That means that if a k-cube center lies in a plane, its opposite k-cube center must lie in the same plane. This result is very useful to show that certain planes do not describe a rotation.

An overview of the k-cubes centers in the hyperplane that describe the 86 simple rotations of the tesseract are given in Table 17. The gray rotations describe planes in the tesseract, but those planes do not describe a rotation. The full account of rotations of the tesseract is given in Figure 18. The greenblue-colored rotation in this table is a transformation of the tesseract that does not describe a simple rotation. However, together with a rotation about 180°, it does describe a non-simple rotation of the tesseract.

The first rotation is described by 2 cube-center pairs or four cube-centers. It follows that the plane spanned by those vectors must contain four face-centers as well. In the given example, those face-centers are given by $(\pm1,\pm1,0,0)$. Since there are $\binom{4}{2}$ ways to pick two cube-face center pairs and each center can be mapped to three other centers of the same type, these planes gives 18 rotations. Six of the rotations are about an angle of 90°, six about 180° and six about 270 °.

The next plane to consider is a plane that is describe by one cube-center pair and one face-centre pair. These planes need to contain two edge-center pairs as well. Indeed, in the example the coordinates of those edge center pairs are given by $(1,\pm1,\pm1,0),(-1,\pm1,\pm1)$. There are four possible ways to choose one cube center pair and $\binom{3}{2}\cdot 2$ ways to choose two coordinates that do not overlap with the non-zero coordinate of the cube-center. However, there are two ways to choose the signs of the face-centre coordinates, so in total there are 24 rotations described by those planes.

k	Highest k	0	1	2	3	Overlap	Example	Number	Degree
Number of k-cube centers	3			4	4		$\lambda(1,0,0,0)+\mu(0,1,0,0)$	$\binom{4}{2}\cdot 3=18$	$90,180,270°$
	3		4	2	2		$\lambda(1,0,0,0)+\mu(0,1,1,0)$	$4\cdot\binom{3}{2}\cdot 2=24$	$180°$
	3	4	2		2		$\lambda(1,0,0,0)+\mu(0,1,1,1)$	$4\cdot 4=16$	$180°$
	3	2	2		2		$\lambda(1,0,0,0)+\mu(1,1,1,1)$		$180°$
	2			4	2	2	$\lambda(1,1,0,0)+\mu(1,-1,0,0)$		$180°$
	2			6	-	1	$\lambda(1,1,0,0)+\mu(0,1,1,0)$	$4\cdot 2\cdot 3\cdot\frac{2}{3}\cdot 2=32$	$120,240°$
	2	4		4	-	0	$\lambda(1,1,0,0)+\mu(0,0,1,1)$	$3\cdot 2\cdot 2=12$	$120,180,270°$
	2	0	2	2	-	2	$\lambda(1,1,0,0)+\mu(1,-1,1,0)$	-	$180°$
	2		4	2	-	1	$\lambda(1,1,0,0)+\mu(0,1,1,1)$	-	$180°$
	2	2		2	-		$\lambda(1,1,0,0)+(1,-1,1,1)$	-	$180°$
	1		4	1	-	3	$\lambda(1,1,1,0)+\mu(1,-1,1,0)$	-	$180°$
	1		4	-	-	2	$\lambda(1,1,1,0)+\mu(0,1,-1,1)$	-	$90,180,270°$
	1	2	2	-	-		$\lambda(1,1,1,1)+\mu(0,-1,1,1)$	-	$180°$
	0	4		>0			$\lambda(1,1,1,1)+\mu(1,-1,1,-1)$		
								86	

Table 17: Geometrical description of the invariant planes of the simple rotations of the tesseract.

The third plane does not describe a simple rotation of the tesseract. The invariant plane is describe by one cube-centre pair and one edge-centre pair. Such a plane necessarily contains two more vertices-centres. In the examples, those four vertices are described by the coordinates $(1,\pm1,\pm1,\pm1),(-1,\pm1,\pm1,\pm1)$. However, any affine space, parallel to the invariant plane, is also invariant under the rotation. That means that any centre that lies in such an affine space, needs to be mapped to another centre in the plane or it must lie in the rotation plane that fixes its points under the rotation pointwise. The 3-cube centers $(0,1,0,0)$ do not lie in the invariant plane spanned by the vectors $(1,0,0,0)$ and $(0,1,1,1)$. We call this plane V. The affine plane parallel to V, call it U, is given by

$$(0,1,0,0)+u(1,0,0,0)+v(0,1,1,1)=(u,1+v,v,v) \qquad \text{with } u,v\in\mathbb{R}.$$

Since any 3-cube center has a single non-zero coordinates we must have $v=0$. As a consequence, we find that $u=0$. The only 3-cube center that lies in the affine plane described by $u=0=v$ is $(0,1,0,0)$. Thus, $(0,1,0,0)$ gets mapped to itself under the rotation. That means that the rotation does either not exist or $(0,1,0,0)$ lies in the rotation plane. The rotation plane is orthogonal to V, but $(0,1,0,0)$ is not. Thus, $(0,1,0,0)$ does not lie on the rotation plane and thus this rotation cannot exist.

The fourth invariant plane is described by a cube-centre and a vertex-centre. However, this plane also contains an edge center. For instance, in the example the edge-center $(0,1,1,1)$ lies in the plane as well. As in the previous case, we already conclude that such a plane cannot give rise to a simple rotation.

The next planes we are going to describe may not include any cube-centers, since those planes have already been investigated. An invariant plane described by two face-centers can either have zero or one or two overlapping coordinates. If the two vectors that describe the invariant plane have two overlapping coordinates, the plane must contain a cube-centre, which was not allowed. So we continue to look at an invariant plane spanned by two edge-centres that overlap in a single coordinate. Those planes contain a third face-centre pair. There are four choices for the overlapping coordinate, two choices for the relative sign of this overlapping coordinate, three choices for the second non-zero coordinate of one of the face-centers and two for a non-overlapping coordinate of the second face-centre-pair. However, in this way, each plane is counted by three different face-centre pairs. Together with the fact that each face-centre pair can be mapped to two other face-centre-pairs in the plane under such rotation, the total number of these rotations is given by $4 \cdot 2 \cdot 3 \frac{2}{3} \cdot 2 = 32$.

The last invariant plane that describes a rotation is the plane described by two face-centre pairs. This plane necessarily also includes two vertex-centre pairs. In the example, their coordinates are given by $(1,1,\pm 1,\pm 1),(-1,-1,\pm 1,\pm 1)$.

For the consecutive two planes, it is enough to consider the affine plane translated by the vector $(0,0,1,0)$, parallel to the defined invariant plane. This plane contains no other cube-centers and is not orthogonal to the rotation plane, so those rotations are ruled out.

Describing those planes which have as highest k-cube an edge-center, the plane described by two edge-center pairs directly fails to describe a rotation of this kind. Any plane containing two edge-center-pairs, necessarily contains a face-centre. Thus, this rotation does not add anything new to the rotation group of the tesseract.

For the last three invariant planes, the first two are ruled out considering the affine plane translated by the vector $(1,0,0,0)$. The last invariant plane spanned by two vertex-centre pairs contains an edge-centre and is thus ruled out as well.

The rotation planes of the tesseract that are orthogonal to each other add another 105 non-simple rotations to O_4. We consider six combinations of invariant planes of the tesseract which are given by the first six rows in Figure 18.

The planes described by two cube-centre pairs lie three mutually orthogonal pairs. Both planes can rotate about four different angles. However the composition of two rotations about 0° gives the identity and the composition of two rotations about 180° gives the inversion. Hence, there are 14 rotations left per mutually orthogonal pair of planes. In total, there are $3 \cdot 14 = 42$ rotations of this type, of which 18 rotations are simple.

Rotations described by one cube-centre pair and one face-centre pair, lie in orthogonal pairs, but their composition describes the inversion. So, those rotations only describe the simple rotations we already found.

The invariant plane described in greenblue in Figure 17, do not describe a simple rotation. However, the composition with planes described by two face-centre pairs with a single overlapping coordinate. For each teal plane, the composition with the rotation about 120° or 240 gives a non-simple rotation of the tesseract. In total, there are $16 \cdot 2 = 32$ non-simple rotations of this kind.

The planes spanned by two edge-centre-cubes and described by a face-centre and a vertex-centre describe 32 and 12 simple rotations respectively. Both types of planes do not lie in orthogonal pairs.

Lastly there is one type of rotation whose invariant planes do not contain any k-cubes. This rotation is the cyclic permutation of rotation axes. Since there are six ways to make distinct 4-cycles and there are four ways to give them a single minus sign and four ways to give them three minus signs. It follows that there are $6 \cdot 8 = 48$ rotations of this type.

Number of k-cubes				Number of rotations		
0	1	2	3	Simple	Non-simple	Total
0	0	4	4	18	24	42
0	4	2	2	24	0	24
4 0	0 6	2 0	2 0	0	32	32
0	6	0	0	32	0	32
4	0	4	0	12	0	12
0	0	0	0	0	48	48
Reflection				0	1	1
Identity				1	0	1
				$86+1$	$104+1$	$190+1+1$

Table 18: The simple and non-simple rotations of the tesseract.

4.2 The 24-cell

4.2.1 Definition of the 24-cell

Definition 4.16. The 24-cell is the four-dimensional convex regular polytope constructed of 24 octahedral cells. It is denoted C_{24} and also called the octaplex or octacube.

At each vertex of the 24-cell, six octahedral cells meet. Furthermore, at each edge, three octahedral cells meet. Since each octahedron has twelve edges, the total number of edges is $\frac{24 \cdot 12}{3} = 96$. Furthermore, each edge lies in three faces, but each face is also counted by three edges. Thus, the total number of faces in the 24-cell is 96 as well. Lastly, since a octahedron has six vertices, but the two vertices incident to an edge are shared by three octahedral cells, the total number of vertices in a 24-cell is given by $\frac{6 \cdot 24}{2 \cdot 3} = 24$. That means that in each vertex $\frac{96 \cdot 2}{24} = 8$ edges meet. These facts about the 24-cell are summarized in Figure 31.

24-cell	
Vertices	24
Edges	96
Faces	96
Octahedra	24

Figure 31: Number of vertices, edges, faces and octahedra in a 24-cell.

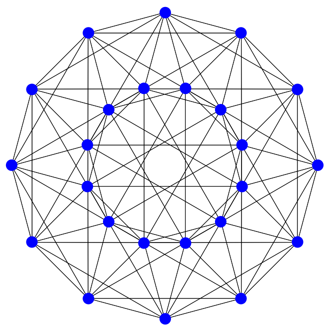
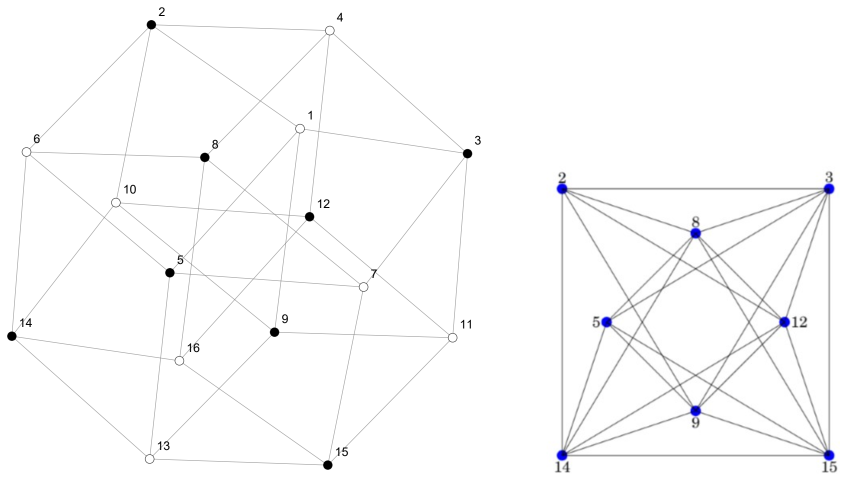


Figure 32: Orthographic projection of the 24-cell.

4.2.2 Demi-hypercubes in a tesseract

As two tetrahedra are inscribed in a cube, two 16-cells inscribe a tesseract. Those two 16-cells are obtained from the alternation of the vertices of such a tesseract. The obtained polytope with half of the vertices of a tesseract is called a demi-tesseract or demi-hypercube. The alternation of the vertices of a unit tesseract described by the vertices $\frac{1}{2}(\pm1,\pm1,\pm1,\pm1)$, gives two sets of vertices, one containing the vertices with an even number of minus signs and one set containing the vertices with an odd number of minus signs. Those two sets both form the vertices of a 16-cell, which we will soon show. The 16-cell is the dual of the tesseract. In Figure 33, the alternation of the vertices of the tesseract together with the orthographic projection of a 16-cell is given.

To show that the demi-hypercubes are 16-cells, we show that there exist an isomormphism between the vertices of a standard 16-cell and the vertices obtained from the alternation labeling of vertices of the unit tesseract described earlier. It then follows that the alternation labeling of any unit tesseract gives two demi-tesseracts that are 16-cells isomorphic to the standard 16-cell.
The vertices of a standard 16-cell are given by the coordinates $(\pm1,0,0,0)$. The orientation preserving matrix $\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$ maps these vertices to the vertices of the demi-tesseract with an odd number of minus signs described earlier. Thus, this demi-tesseract is isomorphic to the standard 16-cell. In a same way is the demi-tesseract described by the vertices with an even number of minus signs isomorphic to the standard 16-cell via the orientation preserving map described by the matrix $\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$. As the demi-hypercubes are 16-cells, we make



(a) Tesseract with alternating vertices **(b) 16-cell**

Figure 33: Alternated labeling of the vertices of a tesseract. The black and the white vertices form two 16-cells.

the convention to call a demi-hypercube just a 16-cell.

4.2.3 Inscription of 16-cells in a tesseract and the circumscription of 16-cells by a tesseract

A tetrahedron uniquely determines the cube it inscribes and its opposite tetrahedron inscribed in the same cube. However, this analogue cannot be lifted to the 16-cell inscribing a tesseract. Namely, each 16-cell is contained in exactly two tesseracts. We will show that one the one hand each 16-cell describes two tesseracts and, on the on the other hand, any tesseract is described by two unique 16-cells. Define

- T to be tesseract,
- C to be a 16-cell,
- $G_{\mathrm{T}} = \{g \in \mathrm{O}(4) \mid g(\mathrm{T}) = \mathrm{T}\}$
- $G_C = \{g \in \mathrm{O}(4) \mid g(\mathrm{C}) = \mathrm{C}\}$.

Suppose we have a fixed tesseract and we consider all 16-cells that are inscribed in that particular tesseract. The stabilizer group $\mathrm{Stab}_T(C)$ in G_{T} is precisely the set of all orthogonal transformations in O(4) that preserves both the tesseract and the 16-cell. From the signed cycles in Figure 12 and Figure 14 it follows that exactly half of the symmetries of the tesseract swaps the two 16-cells inscribed. Indeed, all transformations with $\epsilon_1\epsilon_2\epsilon_3\epsilon_4 = -1$ do so. For example, the 16-cell described by $(\pm 1, \pm 1, \pm 1, \pm 1)$ with an even number of the minus signs in the coordinates gets mapped to the vertices with an odd number of minus signs whenever $\epsilon_1\epsilon_2\epsilon_3\epsilon_4 = -1$. The other half of the rotations permutes the vertices of a 16-cell, preserving the vertices having an even or odd number of minus signs. Since any tesseract is similar to the tesseract with vertices $(\pm 1, \pm 1, \pm 1, \pm 1)$, it follows that any any 16-cell is either mapped to itself or the other inscribed 16-cell in the fixed tesseract. Thus, the length of the orbit of a 16-cell in a fixed tesseract is 2. As a consequence, the index of the $\mathrm{Stab}_T(C)$ is 2. Since there are two 16-cells inscribed in a fixed tesseract and these cells lie in the same orbit, the action of symmetry group of the tesseract on the 16-cell is transitive.

Suppose now we have a fixed 16-cell and we consider all those tesseracts that are acted upon by the symmetry group of this fixed 16-cell. Since the $\mathrm{Stab}_C(\mathrm{T})$ in G_C consists of the orthogonal transformation in O(4) that preserve both the 16-cell and the tesseract, it equals $\mathrm{Stab}_{\mathrm{T}}(C)$ in G_{T}. Since the symmetry group of the tesseract is the same as the symmetry group of the 16-cell by duality, it follows that the stabilizer group of the tesseract in G_C has index 2.

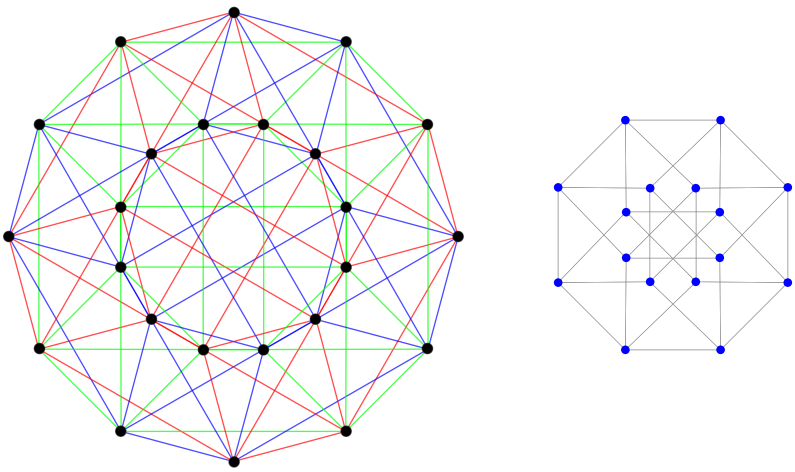
We now show that there are exactly two tesseracts containing a given 16-cell. In other words, we will show that the full symmetry group G_C of C acts transitively on the set of tesseracts containing C. Let a fixed a 16-cell be given and call it Δ. Let also two different arbitrary tesseracts sharing half of its vertices with the 16-cell Δ be given. Those tesseracts will be called T_1 and T_2. Since T_1 and T_2 are isomorphic, there exist a $g \in \mathrm{O}(4)$ such that $g(T_1) = T_2$ and vice versa. Since T_1 and T_2 are tesseracts, it follows from previous reasoning that they both contain two 16-cells of which one is Δ by construction. Call the other two 16-cells Δ_1 and Δ_2, contained in T_1 and in T_2 respectively. The orthogonal transformation g maps Δ_1 to either Δ or Δ_2 and the same holds for Δ. In case Δ is mapped to Δ and Δ_1 to Δ_2, we are done. Indeed, it follows that two arbitrary tesseracts lie in the same orbit, thus the action of the 16-cell on a tesseract containing Δ is transitive. In the latter case, that is, g maps Δ to Δ_2, we can take an orthogonal transformation g' that preserves T_1, but swaps the 16-cells inscribed in it. Any symmetry isomorphic to the symmetry of the standard tesseract with $\epsilon_1\epsilon_2\epsilon_3\epsilon_4 = -1$ will work. The composition of g and g' is an orthogonal transformation again. Since this orthogonal transformation preserves Δ, it is also an orthogonal transformation of Δ. It follows that the symmetry group of a 16-cell acts transitively on the two tesseracts it inscribes.

4.2.4 Inscription of a tesseract and 16-cell in a 24-cell

The quaternions in $2T$ can be split in three disjoint sets such that the elements in each set are the vertices of 16-cell. The first set is the normal subgroup Q_8, describing the standard 16-cell. As we saw in Section 4.2.2, the conjugacy classes C and E from Figure 10 and the conjugacy classes D and F both form a 16-cell as well.

We have shown that at least three 16-cells are inscribed in a 24-cell. This can also be seen from the construction of the 24-cell from one tesseract and a 16-cell as done in [10]. However, we can show that there are no more than three 16-cells inscribed in a 24-cell in a geometrical way. We work with the 24-cell whose vertices are the quaternions from $2T$. We thus know that the edges of the inscribed 16-cells have length $\sqrt{2}$. Furthermore, the distance between two opposite vertices, for example between 1 and -1 is 2. We show that there is only one 16-cell in $2T$ containing the quaternion 1. It then follows from symmetry that each vertex lies in a single 16-cells and thus that there are only $\frac{24}{8} = 3$ 16-cells inscribed in each 24-cell. We have that -1 lies 2 away from 1, while the quaternions $(\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$ all have distance 1 to the quaternion 1 and the quaternions $(-\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}$ have distance $\sqrt{3}$ to the quaternion 1. It thus follows that there are only 6 quaternions that lie $\sqrt{2}$ away from 1 and can lie in the same 16-cell. It thus follows that there is only a single 16-cell described at each fixed vertex of the 24-cell. Hence, it follows that only three 16-cells inscribe the 24-cell.

As a consequence, we also know that there are exactly three tesseracts inscribed in the 24-cell. We have already seen that each 16-cell gives rise to precisely two tesseracts and vice versa. Each pair of 16-cells in the 24-cells thus forms a tesseract in the 24-cell. There cannot be inscribed any more than those three tesseracts, as that would imply that there was another 16-cell inscribed.



(a) 24-cell **(b) Tesseract**

Figure 34: An orthographic projection of the inscription of three tesseracts in green, red and blue in the 24-cell and of a tesseract.

4.2.5 The compound of a 24-cell and its dual 24-cell

The quaternions in the group $2O$ are given by:

$$(\pm 1, 0, 0, 0), \quad (\pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}). \quad (\pm 1, \pm 1, 0, 0).$$

These quaternions are the vertices of the standard 24-cell and its unscaled dual and form the roots in the root system F_4 [22]. The vertices of the standard 24-cell with the vertices of its scaled dual form the vertices of the compound of two 24-cells.

One could verify that the dual of the standard 24-cell indeed has as vertices the permutations of $(\pm 1, \pm 1, 0, 0)$. In Table 19, the coordinates of a so-called dual 24-cell with vertices $(\pm 1, \pm 1, 0, 0)$ are summarized. The cell-centres of this 24-cell are precisely the vertices of the standard 24-cell. As an example to the table, two octahedra with vertices of the dual 24-cell are drawn in Figure 36.

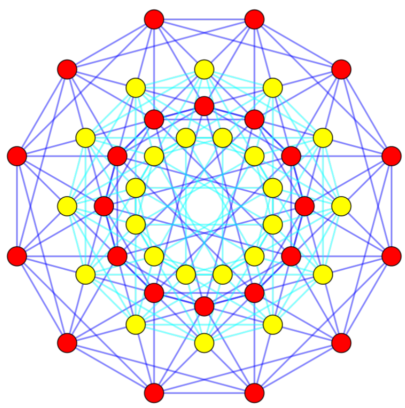


Figure 35: The root system F_4 represented by the 48 vertices of the 24-cell and its unscaled dual. [1]

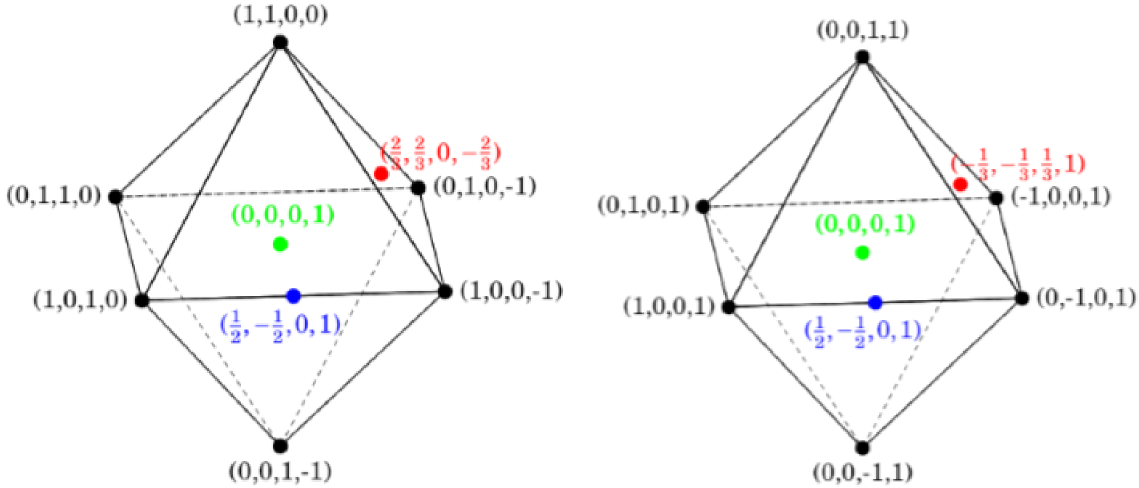


Figure 36: Two possible octahedrons inscribed in a 24-cell with vertices $(\pm1,\pm1,0,0)$. The vertices, a edge-centre, face-centre and the centre of the octahedron are denoted by coordinates.

24-cell		
	Coordinates	Number
Vertices	$(\pm1,\pm1,0,0)$	$\binom{4}{2}\cdot 2^2 = 24$
Edge-centres	$(\pm\frac{1}{2},\pm\frac{1}{2},1,0)$	$\binom{4}{1}\cdot\binom{3}{1}\cdot 2^3 = 96$
Face-centres	$(\pm\frac{1}{3},\pm\frac{1}{3},\pm\frac{1}{3},\pm1)$	$\binom{4}{1}\cdot 2^4 = 64$
	$(\pm\frac{2}{3},\pm\frac{2}{3},\pm\frac{2}{3},0)$	$\binom{4}{1}\cdot 2^3 = 32$
Cell-centre	$(\pm1,0,0,0)$	8
	$(\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2})$	$8+2^4 = 24$

Table 19: The vertices, edge-centres, face-centres and cell-centres of the 24-cell with vertices given by the permutations of the coordinates $(\pm1,\pm1,0,0)$.

4.3 The 600-cell

4.3.1 Definition of the 600-cell

The quaternions from $2I$ describe the vertices of a 600-cell. We make the convention to refer to the 600-cell with vertices the quaternions from $2I$ as defined in Section 3.2, by *the* 600-*cell* $2I$. At each vertex of a 600-cell, 20 tetrahedra meet. In total, a 600-cell contains 600 tetrahedra. The 600-cell is the four-dimensional analogue of the icosahedron. That is, in the icosahedron at each vertex 5 triangles meet, while in the 600-cell at each edge 5 tetrahedra meet.

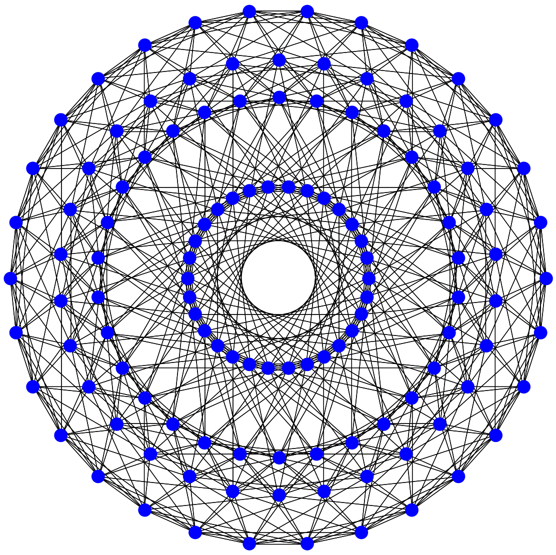


Figure 37: An orthographic projection of the 600-cell.

4.3.2 The inscription of 24-cells in a 600-cell

In this section, we will show that there are precisely twenty-five 24-cells inscribed in a 600-cell. Throughout this entire section, we write *decomposition of the* 600-*cell* for the decomposition of the 600-cell into five disjoint 24-cells. This decomposition is thus a partition of the 120 vertices of the 600-cell in five subsets that each contain 24 vertices. The main result of this section is that there are are at least and at most twenty-five 24-cells embedded in a 600-cell.

Theorem 4.17. *There are precisely* 25 *ways to embed a* 24-*cell in a* 600-*cell.*

To prove this theorem, we proceed as follows. First we show that there are at least twenty-five 24-cells inscribed in the 600-cell $2I$ in Theorem 4.18. Then we show that each vertex of the 600-cell lies in at most five 24-cells in Theorem 4.19.

Theorem 4.18. *There are 25 left cosets and 25 right cosets of $H = 2T$ incribed in the 600-cell $2I$. However, the vertices in the 25 left cosets and the 25 right are the vertices of the same twenty-five 24-cells.*

Proof. The idea of the proof is as follows. We use the fact that there are five conjugate ways to embed A_4 in A_5. As a result, the left cosets of these conjugates of $2T$ in $2I$ are 24-cells. Using Mathematica, we show that those 25 left cosets are different.

First of all, the five conjugate subgroups of A_4 in A_5 each fix a different element of A_5. In Figure 20 these conjugate classes of A_4 are represented. Each of these conjugates of A_4 are subgroups of A_5.

A_4	id,(12)(34),(13)(24),(14)(23),(123),(132),(124),(142),(134),(143),(234),(243)
$(123)A_4(132)$	id,(23)(45),(24)(35),(25)(34),(234),(243),(235),(253),(245),(254),(345),(354)
$(234)A_4(243)$	id,(13)(45),(14)(35),(15)(34),(134),(143),(135),(153),(145),(154),(345),(354)
$(345)A_4(354)$	id,(12)(45),(14)(25),(15)(24),(124),(142),(125),(152),(145),(154),(245),(254)
$(423)A_4(432)$	id,(12)(35),(13)(25),(15)(23),(123),(132),(125),(152),(135),(153),(235),(253)

Table 20: Conjugate subgroups of A_4 fixing the element 5,1,2,3,4 respectively.

The different embeddings of A_4 in A_5 embed $2T$ in $2I$ in five different ways via the map from Theorem 3.14. The conjugates of $2T$ can be obtained by taking an element $c \in 2I$ with order 5 and computing the conjutate groups $c^i 2T c^{-i}$. For notational convenience, denote $2T$ by H. In the code found in Section B.2.1, we use the quaternion $c = -\left(\frac{\phi}{2}, 0, \frac{1}{\phi}, -\frac{1}{2}\right)$ as quaternion of order 5 to compute the conjugate subgroups of H. In Table 21 it can be read off that the left and right cosets of these conjugate subgroups of H overlap.[1] In the left table the left cosets are represented, while in the right table the right cosets are represented and the colors denote which left and right cosets are the same. The fact that none of the 25 left cosets found in this way are the same, can be found in Section B.2.1. □

We now know that there are twenty-five different 24-cells inscribed in a 600-cell. However, a priori, it might be the case that there are more than twenty-five 24-cells inscribed in any 600-cell. We show that each vertex of a 600-cell lies in only five 24-cells.

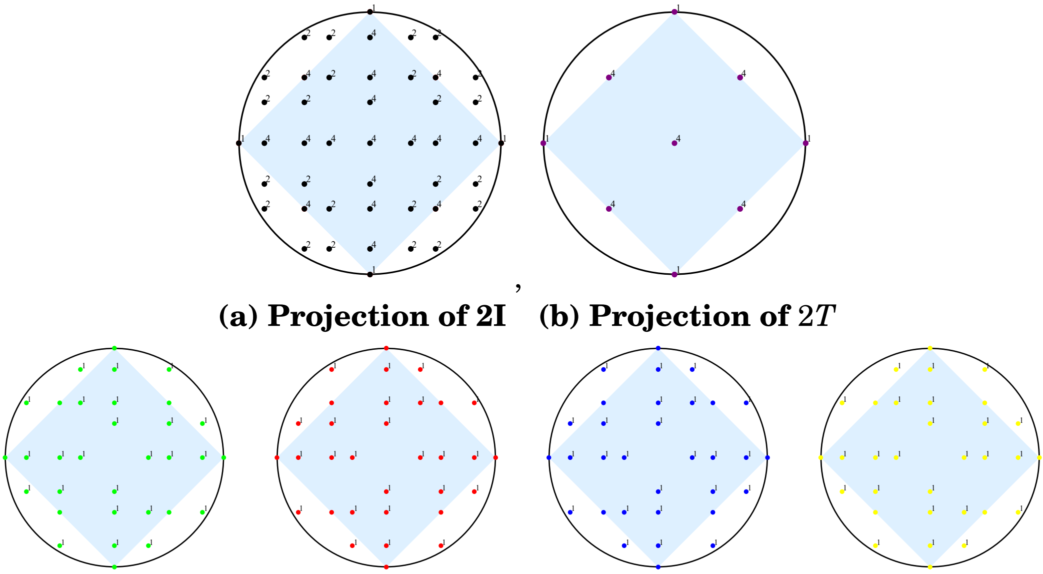
Theorem 4.19. *Each vertex of the 600-cell $2I$ lies in at most 5 distinct 24-cells.*

[1]The conjugate subgroups of $2T$ by the left action of $2I$ on $2T$ give the conjugates $c^i H c^{-i}$ and the left cosets of these conjugates in $2I$. However, the right action of $2I$ on $2T$ give the conjugates $c^{-i} H c^i$ and the right cosets of these conjugates in $2I$. In the implementation in Appendix B.2.1 this subtle yet important difference has been used.

H	cHc^{-1}	c^2Hc^{-2}	c_3Hc^{-3}	c_4Hc^{-4}
cH	c^2Hc^{-1}	c^3Hc^{-2}	c^4Hc^{-3}	Hc^{-4}
c^2H	c^3Hc^{-1}	c^4Hc^{-2}	Hc^{-3}	cHc^{-4}
c^3H	c^4Hc^{-1}	Hc^{-2}	cHc^{-3}	c^2Hc^{-4}
c^4H	Hc^{-1}	cHc^{-2}	c^2Hc^{-3}	c^3Hc^{-4}

H	$c^{-1}Hc$	$c^{-2}Hc^2$	$c^{-3}Hc^3$	$c^{-4}Hc^4$
Hc	$c^{-1}Hc^2$	$c^{-2}Hc^3$	$c^{-3}Hc^4$	$c^{-4}H$
Hc^2	$c^{-1}Hc^3$	$c^{-2}Hc^4$	$c^{-3}H$	$c^{-4}Hc$
Hc^3	$c^{-1}Hc^4$	$c^{-2}H$	$c^{-3}Hc$	$c^{-4}Hc^2$
Hc^4	$c^{-1}H$	$c^{-2}Hc$	$c^{-3}Hc^2$	$c^{-4}Hc^3$

Table 21: Overlap in left and right cosets of $2T$ in $2I$ where the left cosets are marked in the right cosets and vice versa.



(a) Projection of 2I , **(b) Projection of $2T$**

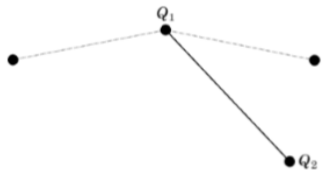
(c) Projection of the left cosets of $2T$ in $2I$

Figure 38: Projection of $2I$, $2T$ and the left cosets of $2T$ in $2I$ on the rx-plane.

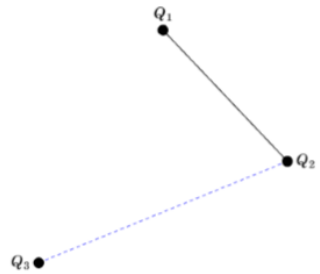
Proof. In this proof we use Mathematica calculations found in Section 4.3. The idea is to show that there are at most 5 ways to construct a different 24-cell containing the vertex 1 in the 600-cell $2I$. We do so by using the distances between edges in a 24-cell and the distances between all of the vertices in the 600-cell $2I$.

Without loss of generality, we may show that the quaternion $Q_1 = 1$ is contained in at most five 24-cells in the 600-cell $2I$. a quaternion in $2I$ and call it Q_1. We want to find 8 vertices that can be connected to Q_1 by an edge to construct a 24-cell. The distance between two vertices of the 24-cell that are connected by an edge is 1. Thus, we start looking for 4 vertices Q_2, Q_3, Q_4, Q_5 that can be connected to Q_1 by an edge. The vertices Q_2 and Q_4 need to satisfy additionally that they have distance $\sqrt{2}$ to each other and will thus not be connected by an edge. The same requirement needs to hold for Q_3 and Q_5. From the calculations in Mathematica, it follows that there are 20 ways to choose a vertex Q_2 with distance 1 to Q_1. Then, there are six possibile choices for a vertex Q_3 with distance 1 and $\sqrt{2}$ to Q_1 and Q_2 respectively. The vertices Q_3 and Q_5 are then fixed. That is, there are only two vertices that have distance 1 to Q_1, Q_2 and Q_4, but lie $\sqrt{2}$ apart from each other.

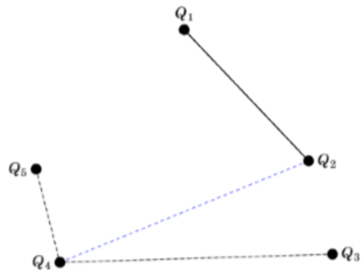
However, there are four combinations of possible vertices Q_2 and Q_3 that describe the same part of an octahedron containing Q_1. Furthermore, in each vertex of a 24-cells 6 octahedra meet. So the extension to a 24-cell containing Q_1 from a single octahedron is the same for at least 6 different otahedra. It follows that at least $\frac{20\cdot 6}{4\cdot 6} = 5$ 24-cells in a 600-cell contain a fixed vertex Q_1. □



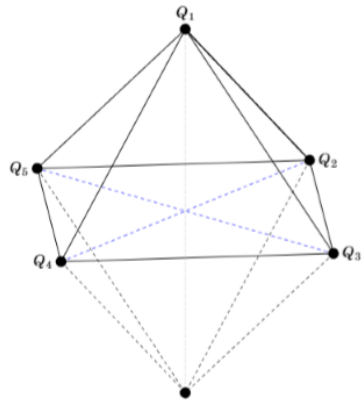
(a) 20 **possible neighbors** Q_2



(b) 6 **possible opposites** Q_3



(c) 2 **possible neighbors** Q_3 **and** Q_5



(d) Octahedron defined by Q_1, Q_2, Q_3, Q_4

Figure 39: Construction of an octahedron of a 24-cell in a 600-cell.

Using the results from Theorem 4.18 and Theorem 4.19, we can show that only twenty-five 24-cells inscribe a 600-cell.

Proof Theorem 4.17. Each vertex of a 600-cell lies in five 24-cells. Furthermore, each of these 24-cells has 24 edges. Thus, the total number of 24-cells in a 600-cell is $\frac{120 \cdot 5}{24} = 25$. □

4.3.3 Decompositions of the 600-cell into five disjoint 24-cells

In this section, we will show that there are precisely $5+5$ different ways to embed five 24-cells in a 600-cell such that each vertex belongs to exactly one of these 24-cells. The decompositions are the $5+5$ columns in the two Tables in Table 22. However, a priori, it is not immediately clear that these decompositions are, on the one hand, all distinct and, on the other hand, all the possible decompositions.

Theorem 4.20. *There are precisely* $5+5$ *ways to decompose the* 600*-cell into five disjoint* 24*-cells.*

Proof. This is proved by a computation in Mathematica. The code is found in Appendix B.2.1. □

The Mathematica code is not entirely self-explanatory nor efficient. This code tries every combination of five 24-cells and checks whether all quaternions from $2I$ lie in these five 24-cells. From the code it follows that there are 1200 ways to take five of the twenty-five 24-cells that form a partition the vertices of the 600-cell. Since each combination of five 24-cells is counted 5! = 120 times, it follows that the total number of decompositions is 10.

Since the 24-cells in the table for the left and right cosets overlap, it might be insightful to make one table that both contains all twenty-five 24-cells inscribed in a 600-cell as an easy way of reading of the 10 decompositions of the 600-cell. We make a table whose first row and column are the the two decompositions containing $H = 2T$. Furthermore, we denote the 24-cells obtained by conjugation and left multiplication with c^j and c^i respectively by $\langle i+j, -j\rangle$. In this way, the left and right cosets of the conjugates of H are given by the columns and rows of the table and represent all decompositions of the 600-cell into five disjoint 24-cells.

$\langle 0,0\rangle$	$\langle 0,1\rangle$	$\langle 0,2\rangle$	$\langle 0,3\rangle$	$\langle 0,4\rangle$
$\langle 1,0\rangle$	$\langle 1,1\rangle$	$\langle 1,2\rangle$	$\langle 1,3\rangle$	$\langle 1,4\rangle$
$\langle 2,0\rangle$	$\langle 2,1\rangle$	$\langle 2,2\rangle$	$\langle 2,3\rangle$	$\langle 2,4\rangle$
$\langle 3,0\rangle$	$\langle 3,1\rangle$	$\langle 3,2\rangle$	$\langle 3,3\rangle$	$\langle 3,4\rangle$
$\langle 4,0\rangle$	$\langle 4,1\rangle$	$\langle 4,2\rangle$	$\langle 4,3\rangle$	$\langle 4,4\rangle$

Table 22: The twenty-five 24-cells in the 600-cell. Each row and column represents a distinct decomposition of the 600-cell.

4.3.4 Quaternions with real part $\frac{1}{2}$ in each 24-cell in the 600-cell $2I$

In this subsection, we use the notation $c^{i+j}Hc^i$ and c^iHc^{i+j} for the left and right cosets of the group $H = 2T$ in $2I$. Furthermore, if we say that a quaternion from $2I$ lies in the dodecahedron, we mean that its imaginary part describe the coordinate representation of a vertex in the dodecahedron.

We consider the intersection of each 24-cell in a decomposition of the 600-cell with the set $\{r = \frac{1}{2}\}$. A priori, one could think and hope the 20 quaternions lying in this intersection would divide itself over the 5 subgroups of $2I$ and its left and right cosets such that each left cosets describes one of the tetrahedra in the left-handed compound of 5 tetrahedra in the dodecahedron, while each right cosets represents one of the tetrahedra from the right-handed compound of tetrahedra. Although this would relate the 10 inscribed tetrahedra in a dodecahedron beautifully to the 10 decompositions, it is not what occurs in the group $2I$. In this thesis, this fact was discoverd in Mathematica.

Theorem 4.21. *Take a decomposition of the* 600*-cell* $2I$ *into* 5 *disjoint* 24*-cells and intersect with* $\{r = \frac{1}{2}\}$*. This decomposition does not decompose the dodecahedron* $2I \cap \{r = \frac{1}{2}\}$ *into* 5 *disjoint tetrahedra.*

This can be read of from the Mathematica code included in Appendix B.2.2. Interestingly what actually occurs is that intersection of a 24-cell in the 600-cell $2I$ and the set $\{r = \frac{1}{2}\}$ contains either 3 or 8 vertices of the 24-cell. Only five of the twenty-five 24-cells in a 600-cell share 8 vertices with the set $\{r = \frac{1}{2}\}$. Those 8 vertices are the vertices of cube.

Theorem 4.22. *Each distinct binary tetrahedral subgroup of* $2I$ *intersects* $\{r = \frac{1}{2}\}$ *in a distinct cube.*

This result can be seen as a consequence of the fact that there are five ways to embed $A_4 \cong O$ in $A_5 \cong I$. Each embedding fixes an element in A_5, in this case a cube. Since $2I$ is a double cover of I, the rotations that fix a particular cube in the dodecahedron lie in $2I$ twice. It is therefore certainly true that the intersection of a distinct binary tetrahedral subgroup of $2I$ and $\{r - \frac{1}{2}\}$ is a cube as well.

The cube in $H = 2T$ is given by $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$. Since the real part of a unit quaternion is preserved under conjugation in $\mathbb{H}_1$ (Theorem 3.10), it follows that the cubes in the distinct binary tetrahedral subgroups of $2I$ are given by $\{c^m \left(\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right) c^{-m}\}$ where $m \in \{1,2,3,4,5\}$ and $c \in 2I$ with order 5.

The question that remains unanswered is what the intersection of a non-trivial coset of a binary tetrahedral subgroup of $2I$ and the set $\{\frac{1}{2}\}$ describes. We already mentioned that the 12 quaternions with $r = \frac{1}{2}$ are divided over the left cosets of such a binary tetrahedral subgroup in sets of 3. Surely, the intersection of left cosets of one binary tetrahedral subgroup of $2I$ is empty. What is interesting is that these remaining 12 vertices with $r = \frac{1}{2}$ describe three orthogonal frames in the dodecahedron. It furthermore appears that the quaternions with $r = \frac{1}{2}$ in each coset describe one vertex of each of these orthogonal frames. The division of vertices of the dodecahedron by the quaternions with $r = \frac{1}{2}$ is visualized in Figure 41 for the subgroup $H = 2T$ together with its left coset cH, c^2H, c^3H and c^4H.

We can also examine what the difference is between the quaternions with real part $r = \frac{1}{2}$ in the left cosets of H and in the right cosets. These quaternions in the left and right cosets of $H = 2T$ are represented by the vertices of the dodecahedron

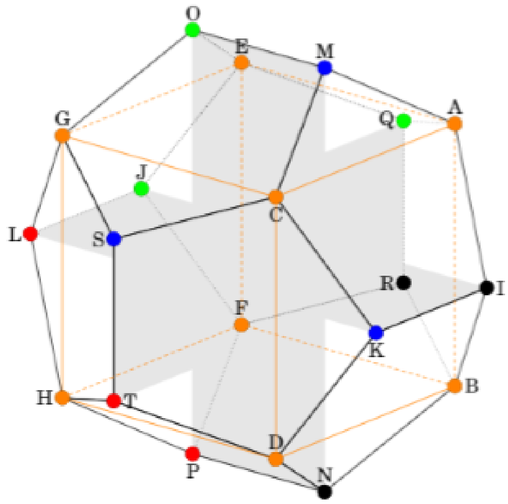


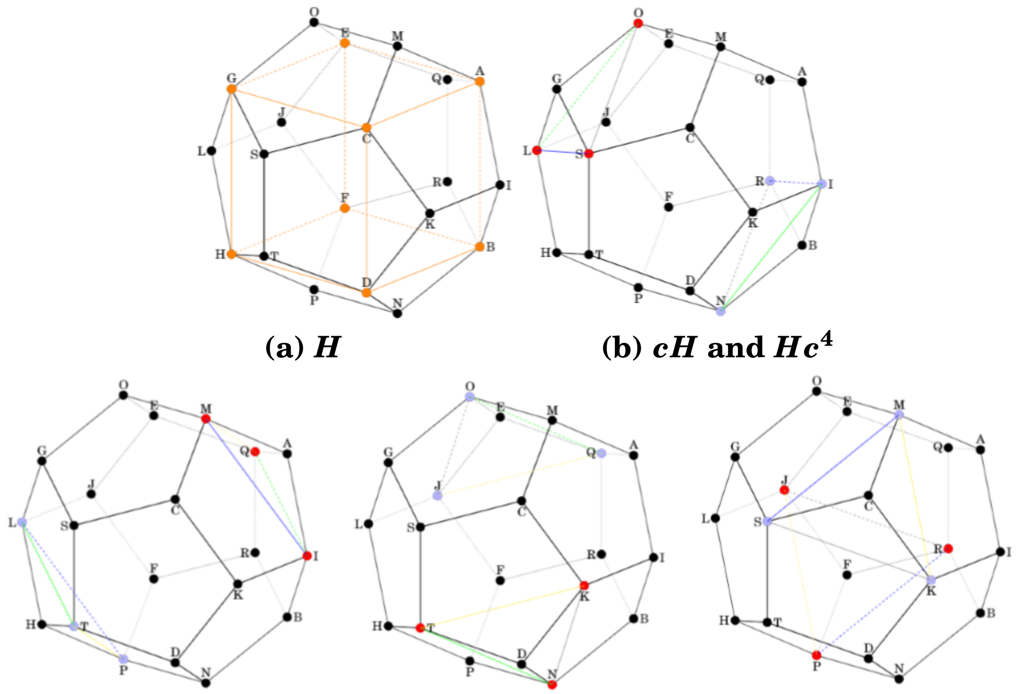
Figure 40: The three orthogonal frames obtained from the vertices represented by the imaginary part of the quaternions with real part $r = \frac{1}{2}$ in the left cosets of H. The vertices from H, cH, c^2H, c^3H and c^4H are represented by orange, black, red, green and blue vertices of the dodecahedron respectively.

in Figure 41. The left cosets are represented by the red vertices and the right cosets by the blue ones.

The three quaternions with $\frac{1}{2}$ in a non-trivial coset of a binary tetrahedral subgroup are not vertices of a single tetrahedron or cube. Two of these three vertices do rather represent a cube inside the dodecahedron. In this way, these three vertices describe three distinct cubes. These cubes are represented by one of their edges, drawn in the Figure 41 by an edge in the color of the cube as given in Figure 13.

What is particularly striking is that none of the cosets contains three of the same cubes. Furthermore, each cube occurs in precisely three of the four cosets that are no subgroup. There are only $\binom{4}{3} = 4$ ways to combine three cubes from a total of four cubes. The cube fixed by the subgroup is never described by two of the three quaternions with $r = \frac{1}{2}$ in a coset.

What can be seen from Figure 41 as well, is that the left and right cosets corresponding to the same subgroup of $2I$ describe the cosets in a very similar way. The quaternions with $r = \frac{1}{2}$ lying in the left cosets are precisely the opposites of the quaternions with $r = \frac{1}{2}$ lying in the right cosets. In fact, the cosets cH and Hc contain the same vertices of the dodecahedron and icosidodecahedron and midpoints of edges of the dodecahedron, but they differ in the sign of the real part. That is, the quaternions describe the opposite rotation of that vertex, edge or midpoint of the face of the dodecahedron.



(a) $\boldsymbol{H}$ **(b)** $\boldsymbol{cH}$ **and** $\boldsymbol{Hc^4}$

(c) $\boldsymbol{c^2H}$ **and** $\boldsymbol{Hc^3}$ **(d)** $\boldsymbol{c^3H}$ **and** $\boldsymbol{Hc^2}$ **(e)** $\boldsymbol{c^4H}$ **and** $\boldsymbol{Hc}$

Figure 41: The division of the quaternions with $r = \frac{1}{2}$ into the left and right cosets of H. The blue vertices represent the imaginary part of the quaternions with $r = \frac{1}{2}$ from the left cosets, while the red vertices are the quaternions with $r = \frac{1}{2}$ in the right cosets.

We conclude that the left and right cosets of a binary tetrahedral subgroup differ to the extent that the quaternions with $r = \frac{1}{2}$ in a right coset describe the inverses of the rotations in the left coset containing these quaternions with the same imaginary part.

4.3.5 The two 600-cells circumscribing a 24-cell

In Section 4.2.3 we showed that each tesseract is described by two unique 16-cell and that each 16-cell is inscribed in two unique tesseracts. We will now show that there are two 600-cells that contain a fixed 24-cell. We already showed in Theorem 4.17 that each fixed 600-cell contains twenty-five 24-cells and we will use that result. We argue as we did for the tesseract and 16-cell.

Let a 600-cell be given and let a 24-cell lying in this 600-cell be given as well. Denote the 600-cell by X_1 and the 24-cell by X_2. Let G_1 and G_2 denote the symmetry group of respectively X_1 and X_2. That is,

$$G_1 = \{R \in \mathrm{O}(4) \mid R(X_1) = X_1\}, \qquad G_2 = \{R \in O(4) \mid R(X_2) = X_2\}.$$

Define $H = G_1 \cap G_2$. H is the stabilizer of X_1, where X_1 is seen as a point in the set of subgroups of G_2 but also the stabilizer of X_2, where X_2 is seen as point in the set of subgroups of G_1. Thus, any orthogonal transformation in H preserves both X_1 as X_2. Since H is a group and a subgroup of both G_1 and G_2, it should divide the group order of both G_1 and G_2. We have already seen that $|G_2| = 1152$. Since the 600-cell contains 600 tetrahedra, we have that $|G_1| = 600 \cdot 24 = 14400$. Thus, the order of H divides $\mathrm{ggd}(1152, 14400) = 576$.

We denote the order of H by $\frac{576}{k}$ for a $k \in \mathbb{N}$. The index of H in G_1 is then $25k$ and the order of H in G_2 is $2k$. Thus, the length of the orbit of X_1 under G_2 is $2k$ and the the length of the orbit of X_2 under G_1 is $25k$. However, from Theorem 4.17 we know that there are twenty-five 24-cells in a 600-cell. It follows that $k = 1$. Apparently is the action of the symmetry group G_1 of the 600-cell on the inscribed 24-cells transitive.

We still need to show that G_2 works transitively on the two 600-cells containing X_2 as well. A same argument as used for the tesseract and 16-cells works here. Suppose we are given a fixed 24-cell that we call Δ_1 and two 600-cells Y_1 and Y_2 containing Δ_1. There exist an orthogonal transformation R that maps Y_1 to Y_2. If R maps Δ_1 to Δ_1 we are done. If not, Δ_1 is mapped to another 24-cell Δ_2 in Y_2. Then we take a symmetry that preserves Y_2 but maps Δ_2 to Δ_1. Hence, the symmetry group of a 24-cell works transitively on the two 600-cells containing a given 24-cell.

4.3.6 Action symmetry group of the 600-cell on the set of its 10 decompositions

Definition 4.23 (Chiral objects). An object $X \in \mathbb{R}^n$ is chiral if it cannot be made incident with any of its mirror images by translations and rotations of $\mathbb{R}^n$. Such an object is said to have chirality (handedness). [28], [35]

However, there are infinitely many hyperplanes in $\mathbb{R}^n$ and thus infinetely many reflections in hyperplanes. In what sense does chirality depend on this choice of the reflection hyperplane? The answer is that if an object is chiral with reflect to one reflection hyperplane, it is chiral with respect to all reflections hyperplanes. [35].

It would be beautiful if the orientation reversing symmetries of the symmetry group of W would reverse chirality of the decompositions. In that case the 600-cell would contain two sets both consisting of 5 compounds of 5 24-cells differing in chirality like the two chiral compounds of 5 tetrahedra contained in the dodecahedron. It appears that the left-handed decompositions and right-handed decompositions are indeed chiral. This result is formalized in Theorem 4.24.

Theorem 4.24. *An element of the full symmetry group of the 600-cell preserves chirality of decompositions, that is, it maps left decompositions to left decompositions and right decompositions to right decompositions, if and only if it preserves the orientation.*

We only show here that the four generators of the reflection group of the 600-cell does not preserve chirality of the decompositions. We first state the relations between the four generators of the reflection group $W = [3,3,5]$, which can be read of from the Coxeter diagram. We then make a choice for these four generators and compute using Mathematica that each reflection does not preserve the chirality of the decompositions. Afterwards, we check whether the relations the generators have to satisfy hold for our particular choice of generators. The implementation in Mathematica can be found in Appendix B.2.2.

We start with the relations the four generators of the reflection group of the 600-cell have to satisfy. Those relations were not studied by myself, but explained by my supervisors. [35]. These generators of the symmetry group $W = [3,3,5]$ can found using the Coxeter diagram. This diagram consist of 4 vertices representing simple roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ of length 1 in $2I$ with

$$\alpha_1 \cdot \alpha_3 = \alpha_1 \cdot \alpha_4 = \alpha_2 \cdot \alpha_4 = 0,$$

$$\alpha_1 \cdot \alpha_2 = \alpha_2 \cdot \alpha_3 = -\tfrac{1}{2},$$

and

$$\alpha_3 \cdot \alpha_4 = -\tfrac{1}{2}\phi,$$

where $\phi = \frac{1=\sqrt{5}}{2}$ is the golden ratio. The full symmetry group $W = [3,3,5]$ can be defined as

$$[3,3,5] = \langle s_1, s_2, s_3.s_4 | R\rangle,$$

where R are the relations generated by

$$s_1^2 = s_2^2 = s_3^2 = s_4^2 = e$$

$$(s_1 s_3)^2 = (s_1 s_4)^2 = (s_2 s_4)^2 = e$$

and

$$(s_1 s_2)^3 = (s_2 s_3)^3 = (s_3 s_4)^5 = e.$$

We take $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ to be

$$\begin{aligned} \alpha_1 &= (1,0,0,0) \\ \alpha_2 &= -\tfrac{1}{2}(1,1,1,1) \\ \alpha_3 &= (0,0,0,1) \\ \alpha_4 &= \tfrac{1}{2}(0,-1,-\tfrac{1}{\phi},\phi). \end{aligned}$$

These roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ generate reflections s_1, s_2, s_3, s_4 as described already in Definition 2.10 in Section 2.3.2. The action of the reflections s_1, s_2, s_3, s_4 on the sets of five 24-cells in each decomposition can then be visualized as in Table 23**b-e**. In Table 23**a**, the identity transformation is once more denoted. The columns of Tables 23**b-e** are the right decompositions obtained from the reflection s_i of the left decompositions in the columns of Table 23**a**.

id	$\langle\bullet,0\rangle$	$\langle\bullet,1\rangle$	$\langle\bullet,2\rangle$	$\langle\bullet,3\rangle$	$\langle\bullet,4\rangle$
$\langle 0,\bullet\rangle$	$\langle 0,0\rangle$	$\langle 0,1\rangle$	$\langle 0,2\rangle$	$\langle 0,3\rangle$	$\langle 0,4\rangle$
$\langle 1,\bullet\rangle$	$\langle 1,0\rangle$	$\langle 1,1\rangle$	$\langle 1,2\rangle$	$\langle 1,3\rangle$	$\langle 1,4\rangle$
$\langle 2,\bullet\rangle$	$\langle 2,0\rangle$	$\langle 2,1\rangle$	$\langle 2,2\rangle$	$\langle 2,3\rangle$	$\langle 2,4\rangle$
$\langle 3,\bullet\rangle$	$\langle 3,0\rangle$	$\langle 3,1\rangle$	$\langle 3,2\rangle$	$\langle 3,3\rangle$	$\langle 3,4\rangle$
$\langle 4,\bullet\rangle$	$\langle 4,0\rangle$	$\langle 4,1\rangle$	$\langle 4,2\rangle$	$\langle 4,3\rangle$	$\langle 4,4\rangle$

(a) No reflection

s_1	$\langle\bullet,0\rangle$	$\langle\bullet,1\rangle$	$\langle\bullet,2\rangle$	$\langle\bullet,3\rangle$	$\langle\bullet,4\rangle$
$\langle 0,\bullet\rangle$	$\langle 0,0\rangle$	$\langle 4,0\rangle$	$\langle 3,0\rangle$	$\langle 2,0\rangle$	$\langle 1,0\rangle$
$\langle 1,\bullet\rangle$	$\langle 0,4\rangle$	$\langle 4,4\rangle$	$\langle 3,4\rangle$	$\langle 2,4\rangle$	$\langle 1,4\rangle$
$\langle 2,\bullet\rangle$	$\langle 0,3\rangle$	$\langle 4,3\rangle$	$\langle 3,3\rangle$	$\langle 2,3\rangle$	$\langle 1,3\rangle$
$\langle 3,\bullet\rangle$	$\langle 0,2\rangle$	$\langle 4,2\rangle$	$\langle 3,2\rangle$	$\langle 2,2\rangle$	$\langle 1,2\rangle$
$\langle 4,\bullet\rangle$	$\langle 0,1\rangle$	$\langle 4,1\rangle$	$\langle 3,1\rangle$	$\langle 2,1\rangle$	$\langle 1,1\rangle$

(b) Reflection s_1

s_2	$\langle\bullet,0\rangle$	$\langle\bullet,1\rangle$	$\langle\bullet,2\rangle$	$\langle\bullet,3\rangle$	$\langle\bullet,4\rangle$
$\langle 0,\bullet\rangle$	$\langle 0,0\rangle$	$\langle 4,0\rangle$	$\langle 1,0\rangle$	$\langle 3,0\rangle$	$\langle 2,0\rangle$
$\langle 1,\bullet\rangle$	$\langle 0,2\rangle$	$\langle 4,2\rangle$	$\langle 1,2\rangle$	$\langle 3,2\rangle$	$\langle 2,2\rangle$
$\langle 2,\bullet\rangle$	$\langle 0,4\rangle$	$\langle 4,4\rangle$	$\langle 1,4\rangle$	$\langle 3,4\rangle$	$\langle 2,4\rangle$
$\langle 3,\bullet\rangle$	$\langle 0,3\rangle$	$\langle 4,3\rangle$	$\langle 1,3\rangle$	$\langle 3,3\rangle$	$\langle 2,3\rangle$
$\langle 4,\bullet\rangle$	$\langle 0,1\rangle$	$\langle 4,1\rangle$	$\langle 1,1\rangle$	$\langle 3,1\rangle$	$\langle 2,1\rangle$

(c) Reflection s_2

s_3	$\langle\bullet,0\rangle$	$\langle\bullet,1\rangle$	$\langle\bullet,2\rangle$	$\langle\bullet,3\rangle$	$\langle\bullet,4\rangle$
$\langle 0,\bullet\rangle$	$\langle 0,0\rangle$	$\langle 3,0\rangle$	$\langle 4,0\rangle$	$\langle 1,0\rangle$	$\langle 2,0\rangle$
$\langle 1,\bullet\rangle$	$\langle 0,3\rangle$	$\langle 3,3\rangle$	$\langle 4,3\rangle$	$\langle 1,3\rangle$	$\langle 2,3\rangle$
$\langle 2,\bullet\rangle$	$\langle 0,2\rangle$	$\langle 3,4\rangle$	$\langle 4,4\rangle$	$\langle 1,4\rangle$	$\langle 2,4\rangle$
$\langle 3,\bullet\rangle$	$\langle 0,1\rangle$	$\langle 3,1\rangle$	$\langle 4,1\rangle$	$\langle 1,1\rangle$	$\langle 2,1\rangle$
$\langle 4,\bullet\rangle$	$\langle 0,2\rangle$	$\langle 3,2\rangle$	$\langle 4,2\rangle$	$\langle 1,2\rangle$	$\langle 2,2\rangle$

(d) Reflection s_3

s_4	$\langle\bullet,0\rangle$	$\langle\bullet,1\rangle$	$\langle\bullet,2\rangle$	$\langle\bullet,3\rangle$	$\langle\bullet,4\rangle$
$\langle 0,\bullet\rangle$	$\langle 4,1\rangle$	$\langle 0,1\rangle$	$\langle 2,1\rangle$	$\langle 3,1\rangle$	$\langle 1,1\rangle$
$\langle 1,\bullet\rangle$	$\langle 4,4\rangle$	$\langle 0,4\rangle$	$\langle 2,4\rangle$	$\langle 3,4\rangle$	$\langle 1,4\rangle$
$\langle 2,\bullet\rangle$	$\langle 4,2\rangle$	$\langle 0,2\rangle$	$\langle 2,2\rangle$	$\langle 3,2\rangle$	$\langle 1,2\rangle$
$\langle 3,\bullet\rangle$	$\langle 4,3\rangle$	$\langle 0,3\rangle$	$\langle 2,3\rangle$	$\langle 3,3\rangle$	$\langle 1,3\rangle$
$\langle 4,\bullet\rangle$	$\langle 4,0\rangle$	$\langle 0,0\rangle$	$\langle 2,0\rangle$	$\langle 3,0\rangle$	$\langle 1,0\rangle$

(e) Reflection s_4

Table 23: The image of each of the 25 inscribed 24-cells in a 600-cell after reflection by the the generating reflections s_1, s_2, s_3, s_4 of the 600-cell. Each entry of a table denotes $s_i(\langle j,k\rangle)$ for $i \in \{1,2,3,4\}$ and $j,k \in \{0,1,2,3,4\}$.

The action of the reflections on the $5+5$ decompositions can also be visualized in a graph. Such a graph is given in Figure 42, where $5+5$ nodes denote the $5+5$ decompositions. An edge is drawn if two decompositions are mapped to each other by a reflections s_i. To distinguish between the actions of the four reflections, each reflection has been given a different color. The edges drawn between decompositions are given the color of the reflection that maps one to the other.

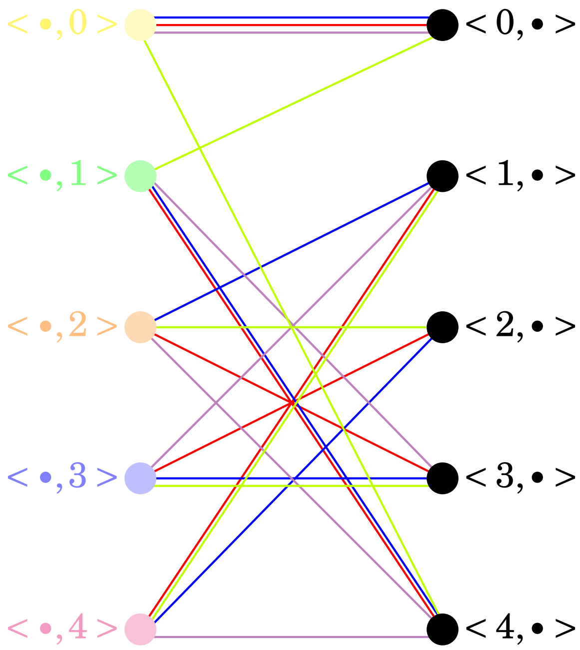


Figure 42: Graph showing to which decompositions the left and right decompositions of the 600-cell are mapped after performing a generating reflection s_1, s_2, s_3 and s_4 of the 600-cell to each decomposition.

We should remark here that all reflections in Figure 42 have been checked to satisfy the relations from the Coxeter diagram defined at the beginning of this section.

In the last part of this section, we formalize the action of the symmetry group $W = [3,3,5]$ on the decompositions of the 600-cell. To make the description of this action easier, we will use another notation for the left and right decompositions.

Definition 4.25. Define L_j to be the left decomposition $\langle i,j \rangle$ for a fixed $j \in \{0,1,2,3,4\}$. Define R_i to be the right decomposition $\langle i,j \rangle$ for a fixed $i \in \{0,1,2,3,4\}$.

Denote a symmetry $w \in W$ by $(\sigma,\tau;\pm 1)$ with $\sigma,\tau \in S_5$. The element $(\sigma,\tau,1)$ is the permutation of $\{L_0,\ldots,L_4\} \cup \{R_0,\ldots,R_4\}$ that maps

$$L_j \mapsto L_{\sigma(j)}, \qquad R_i \mapsto R_{\tau(i)}.$$

The element $(\sigma,\tau;-1)$ is the permutation of $\{L_0,\ldots,L_4\} \cup \{R_0,\ldots,R_4\}$ that maps

$$L_j \mapsto R_{\sigma(j)}, \qquad R_i \mapsto L_{\tau(i)}.$$

The product of two elements $(\sigma,\tau;\pm1)$ and $(\sigma',\tau';\pm1)$ is given by the following formulas:

$$\begin{aligned}
(\sigma,\tau;1)(\sigma',\tau';1) &= (\sigma\sigma',\tau\tau';1)\\
(\sigma,\tau;-1)(\sigma',\tau';-1) &= (\tau'\sigma,\sigma'\tau;1)\\
(\sigma,\tau;-1)(\sigma',\tau';1) &= (\sigma'\sigma,\tau'\tau;-1)\\
(\sigma,\tau;1)(\sigma',\tau';-1) &= (\tau'\sigma,\sigma'\tau;-1).
\end{aligned}$$

Now we can write down the actions of the four generators s_1,s_2,s_3,s_4 on the left and right decompositions L_j,R_i as follows:

$$\begin{aligned}
s_1' &= ((14)(23),(14)(23);-1)\\
s_2' &= ((142),(124);-1)\\
s_3' &= ((13)(24),(13)(24);-1)\\
s_4' &= ((041),(014);-1)
\end{aligned}$$

Next, the generators s_is_j of W are rotations of the 600-cell. Indeed, applying a reflection twice reverses the orientation twice and thus preserves the orientation. The generators s_is_j of W act on the decompositions L_j and R_i in the following way:

$$\begin{aligned}
s_1's_2' &= ((243),(132);1)\\
s_1's_3' &= ((12)(34),(12)(34);1)\\
s_1's_4' &= ((01)(23),(04)(23);1)\\
s_2's_3' &= ((132),(134);1)\\
s_2's_4' &= ((01)(24),(04)(12);1)\\
s_3's_4' &= ((02431),(03124);1)).
\end{aligned}$$

Thus, these rotations have order $3,2,2,3,2$ and 5 respectively. Those are precisely the orders of elements s_is_j of W as described earlier.

From Table 23 and Figure 42 it can be seen that the rotation group $W^+=[3,3,5]^+$ acts on the $5+5$ decompositions by permutations from $A_5\times A_5$. The reflections in $W=[3,3,5]$ also permute the $5+5$ decompositions by a permutation in $A_5\times A_5$, but it additionally swaps the 5 left-handed and 5 right-handed decompositions. We formalize this result in the following theorem.

Theorem 4.26. *The action of the symmetry group $W=[3,3,5]$ on the set of $5+5$ decompositions of the a 600-cell defines a map*

$$f:[3,3,5]\to A_5\times A_5\rtimes\{\pm1\}\subset S_{5+5},$$

where -1 acts on the normal subgroup $A_5\times A_5$ by swapping the factors.

However, there are only $(5\cdot4\cdot3)^2\cdot2 = 7200$ symmetries of the 600-cell described in this way. Indeed, $\mathrm{id}\times\mathrm{id}\rtimes -1$ acts trivially on the sets of $5+5$ decompositions, although it does not act trivially on the vertices of the 600-cell itself. I conjecture that -1 is the only non-trivial element in the kernel of the map in Theorem 4.26.

Conjecture 4.27. *The kernel of* $f : [3,3,5] \to A_5 \times A_5 \rtimes \{\pm1\} \subset S_{5+5}$ *is* $\{\pm1\}$.

In particular, f is surjective as there are $2\cdot7200 = 14400$ symmetries of the 600-cell described in this way. I have not proved this conjecture, but a proof has been given as mentioned in the Introduction.

5 Recommendations

In this thesis, the group $2I$ was established as a double cover of the rotation group I of the icosahedron and as the vertices of a 600-cell. The arguments used to describe the rotations group of the regular polytopes were mainly algebraic or were based on calculations in Mathematica. However, the limitations of these types of arguments are of course that the larger the size of a symmetry group, the more time-consuming and complicated the investigation of the symmetry group becomes. For example, an explicit investigation of the symmetries in the symmetry group of the 600-cell seems quite cumbersome in Mathematica. However, the description by the four generators of the reflection group of the 600-cell used in Section 4.3.6 are highly effective and can be read of from a simple Coxeter diagram.

Since the symmetry group of the 600-cell seems to be rich and filled with beautiful symmetries that are not yet fully explored, it would be a great learning experience to investigate the symmetry group of the 600-cell starting with the Coxeter diagrams, root systems and Weyl groups that are related to the 600-cell. Furthermore, there are questions that have remained unanswered during this thesis and are waiting to be formally investigated and proved.

First of all would it be interesting to continue studying the decompositions of the 600-cell. A next step in this research could be to describe the action of the symmetry group of the 600-cell on the twenty-five 24-cells and especially on its vertices. Some questions that could be researched are summarized below.

- What is the action of the rotation group on the vertices of the individual twenty-five 24-cells in the 600-cell? For example, which rotations do those rotations of the 600-cell that map a particular 24-cell back to itself describe?
- What are the orders and conjugacy classes of the symmetries of W, the symmetry group of the 600-cell? This question should not be too hard to answer if the first two are formally proved.
- Using root systems and Weyl groups, one can show that $(2I \times 2I)/\{\pm 1\} \rtimes \{\pm 1\}$ is isomorphic to $W(H_4)$, where $W(H_4)$ is the Weyl group of the root system H_4 and describes the symmetry group of the 600-cell as well. What does this tell us about the action on the decompositions, individual 24-cells and the vertices of these 24-cells?

Furthermore, it would be interesting to investigate the relation between the 120 elementary particles in the Standard Model and the group $2I$. This bachelor thesis is a good introduction to the group $2I$ and the geometry of the 600-cell. The investigation of root systems and theorems about these systems, might lead to new insights, proofs and calculations. Hopefully, this investigation leads to new insights in the 600-cell or in the relation between the 600-cell and the particles in the Standard Model. However, roots systems find its application in other part of physics as well. For example, the classification of symmetric spaces can be done using Lie algebras and root systems. These symmetric spaces are a special topic in Riemannian geometry and quantum mechanics. The Lie algebras are used to classify the particles in quantum physics. For example, Murray Gell-Mann found could explain both the appearance and connection between many particles using the Lie Group $SU(3)$.

All in all would it be interesting and insightful to continue studying the 600-cell using root systems. The study of these root systems might lead to new theorems and proofs, possibly for the 600-cell, or for Lie algebras in general or maybe for the particles in the Standard Model.

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Appendices

A Groups and Algebra

A.1 Prerequisites

Definition A.1 (Symmetry group). The symmetry group on a set $X \subset \mathbb{R}^n$ is the group of all orthogonal transformations of $\mathbb{R}^n$ that map X to X.

Definition A.2 (Index of subgroup). For a subgroup H of G the index $[G:H]$ is defined as the cardinality of the set of left cosets of H in G.

Definition A.3. (Conjugacy classes) The conjugacy class of an element $a \in G$ is given by

$$[a] = \{b \in G \mid \exists g \in G : b = gag^{-1}\}$$

Definition A.4. (Normal subgroup) A subgroup N of a group G is called a normal subgroup of G or normal if for all $g \in G$ and $h \in N$ we have that $ghg^{-1} \in N$. We use that notation $N \trianglelefteq G$ to indicate that N is a normal subgroup of G.

Theorem A.5. *Let $f : G_1 \to G_2$ be a homomorphism from a group G_1 to a group G_2. Then* $\ker(f)$ *is a normal subgroup of G_1.*

Proof. For any $h \in \ker(f)$ and $g \in G$ we need to show that $ghg^{-1} \in \ker(f)$. Using that f is a homomorphism it follows that:

$$f(ghg^{-1}) = f(g)f(h)f(g^{-1}) = f(g)f(g)^{-1} = e$$

where e denotes the identity element of G_1. □

Theorem A.6. *Let G_1 and G_2 be isomorphic groups and let $f : G_1 \to G_2$ be the corresponding isomorphism. Then the order of $f(g)$ in G_1 is equal to the order of g in G_1.*

Proof. Denote the identity in G_1 by id and the identity in G_1 by e. Let $g \in G_1$. Suppose $\alpha \in \mathbf{Z}_+$ is the smallest integer such that $g^\alpha = \text{id}$. Then $f(g)^\alpha) = f(g^\alpha) = f(\text{id}) = e$.

Now let $h \in G_2$ be arbitrary. Suppose that $\beta \in \mathbb{Z}_+$ is the smallest integer such that $h^\beta = e$. Since $f^{-1}(h)$ exists, it follows that $f(f^{-1}(h)^\beta) = h^\beta = e$. Since $\ker(f) = \{\text{id}\}$, it follows that $f^{-1}(h)^\beta = \text{id}$.

Now suppose there exists an integer $\gamma < \beta$ with $f^{-1}(h)^\gamma = \text{id}$. Then $e = f(f^{-1}(h)^\gamma = h^\gamma$ which contradicts our choice of β. In a similar way one can show that α must be the order of $f(g)$. □

Theorem A.7. *A subgroup of group G is normal if and only if it is the union of conjugacy classes of G.*

Proof. Suppose that N is a normal subgroup of G. For an arbitrary $n \in N$, its conjugacy class is given by $[n] = \{b \in G | \exists g \in G : b = gng^{-1}\}$. For any $g \in G$ it holds that $gng^{-1} \in N$. Hence, $N = \bigcup_{n \in N}[n]$.
Conversely, suppose that H is a subgroup of G satisfying $H = \bigcup_{n \in I}[n]$ for some $I \subset G$. Take $g \in G$ and $h \in H$ arbitrarily. Then $ghg^{-1} \in [n]$ for some $n \in H$ and $n \in I$ by definition of H. It follows that $ghg^{-1} \in H$. □

Theorem A.8. *A subgroup H of index 2 of a group G is normal.*

Proof. Let $g \in G$ and $h \in H$. We want to show $ghg^{-1} \in H$. If $g \in H$ this follows from the fact that H is a subgroup of G. Suppose $g \notin H$. Then $g^{-1} \notin H$ and thus $N \neq g^{-1}H$. It thus follows that $G = H \cup g^{-1}H$. The fact that $g^{-1} \notin H$ shows that $hg^{-1} \notin H$ either. Thus, $hg^{-1} \in g^{-1}H$ as it is an element of G. It thus follows that there is a $g^{-1}h' \in g^{-1}N$ such that $hg^{-1} = gh'$ or, equivalently, such that $ghg^{-1} = h'$. Hence, it follows that H is normal. [18] □

Theorem A.9. *The intersection $N_1 \cap N_2$ of two normal subgroups N_1 and N_2 of a group G is normal.*

Definition A.10 (Left coset). Given a $g \in G$ and a subgroup H of G, a left coset of H consist of all $n \in G$ such that $gh = n$ for a fixed $g \in G$ and $h \in H$. The left coset, denoted gH is thus given by:

$$gH := \{gh | h \in H\}$$

Theorem A.11. *A subgroup N of a group G is a normal if and only if $gN = Ng$ for all $g \in G$. That is, the left and right cosets overlap.*

Theorem A.12. *All left cosets of a subgroup H of a group G have the same cardinality.*

Proof. Two cosets aH and bH have same cardinality if there exist a bijection $f : aH \to bH$. Define f by $f(ah) = bh$ such that f multiplies the left coset aH with ba^{-1} from the left. This map is clearly bijective. □

Theorem A.13 (Lagrange theorem)**.** *For any subgroup H of a group G:*

$$|G| = |H| \cdot [G : H]$$

where $[G : N]$ is the index of the subgroup N in G and $|H|$ denotes the number of elements in the left cosets H in G.

Definition A.14 (Action of group G on set X). Let G be a group, $e \in G$ its identity element and let X be a set. We say that G acts on X from the left if for every $g \in G$ and every $x \in X$ an element $gx \in X$ is given such that

- $ex = x$ for all $x \in X$
- $(gh)x = g(hx)$ for all $g, h \in G$ and $x \in X$.

If G acts on X then the map $G \times X \to X$, given by $(g, x) \mapsto gx$, is the left action of G on X. [18]

Remark A.15. *An action is often referred to as left action. A right action is map $G \times X \to X$, denoted by $(g, x) \mapsto xg$ such that:*

- $xe = x$
- $x(gh) = (xg)h$

for all $x \in X$ and $g, h \in G$.

Definition A.16 (Orbit of x). Let G be a group acting on a set X. The orbit of an element $x \in X$, denoted Gx is then given by

$$Gx := \{g \in G | \exists h \in G : h = gx\}.$$

Thus, the orbit of x consists of all possible elements of the set X to which x can be moved by the elements of G.

Definition A.17 (Stabilizer of x). Let G be a group acting on a set X. The stabilizer of an $x \in X$, denoted $\text{Stab}_G(x)$, is a group and defined by:

$$\text{Stab}_G(x) := \{g \in G | gx = x\}.$$

Definition A.18 (Normalizer of a subgroup). Let G be a group and let H be a subset, not necessarily subgroup, of G. The normalizer of H is then

$$N_G(H) := \{g \in G | ghg^{-1} \in H \text{ for all } h \in H\}.$$

Theorem A.19 (Orbit stabilizer theorem)**.** *Let G be a finite group acting on a set X and let $x \in X$. Then the order of the orbit of G_x is equal to the index of the stabiliser group in $Stab_G(x)$.*

Theorem A.20. *Let $H \subseteq G$ be a subgroup of a group G. Let $g \in G$ be given. Define $K := gHg^{-1}$. Then K is a subgroup of G. Furthermore, the left cosets gH of H in G correspond to the right cosets of K.*

Proof. To show that K is a subgroup of G, we show that the identity an products of elements of K and their inverses are contained in K. Since $\mathrm{id} \in K$ it follows that $g \,\mathrm{id}\, g^{-1} = \mathrm{id}$ is an element of K. Furthermore, for two $gh^{-1}g^{-1}, gh_2g^{-1} \in K$ it follows that their product $gh-1g^{-1}gh_2g^{-1} = gh_1h_2g^{-1}$ lies in K. Indeed h_1h_2 lies in H as it is a subgroup of G. Lastly, the inverse of a $ghg^{-1} \in K$ is given by $gh^{-1}g^{-1}$. Since $h^{-1} \in H$, $gh^{-1}g^{-1} \in K$.

It thus follows that the right cosets of K are equal to the left cosets of H. Indeed,

$$K = gHg^{-1} \quad \implies Kg = gH.$$

□

Definition A.21 (Quotient group). Let N be a normal subgroup of a group G. Then the quotient is defined to be $G/N := \{aN : a \in G\}$, to be the set of all left cosets of N in G.

Since N is a normal group, the definition of G/N could have been defined to be the set of right cosets of N in G as well.

Definition A.22 (Orthogonal group O(n)). $\mathrm{O}(n)$ is the group of $n \times n$ orthogonal matrices. More specifically, $\mathrm{O}(n)$ is the group of matrices satisfying

$$AA^T = AA^{-1} = I \qquad \text{for all } A \in \mathrm{O}(n).$$

Definition A.23 (Special unitary group SU(n)). The special unitary group of the group of $n \times n$ unitary matrices. More specifically, $\mathrm{SU}(n)$ is the group of matrices satisfying

$$AA^* = A^*A = I \qquad \text{for all } A \in \mathrm{SO}(n),$$

where A^* denotes the Hermetian transpose of A.

Definition A.24 ($\mathbf{S_n}$). S_n is the group of all permutations on a set of n elements.

As a matter of fact, the group S_n consists of $n!$ elements.

Definition A.25 ($\mathbf{A_n}$). A_n is the subgroup of S_n and consists of all even permutations.

There are exactly $\frac{n!}{2}$ even permutations in S_n. Hence, A_n has group order $\frac{n!}{2}$.

Theorem A.26. *Let σ and τ be two elements of S_n. Suppose that*

$$\sigma = (a_1, a_2, \cdots, a_k)(b_1, b_2, \cdots, b_l)\cdots$$

is the cycle decomposition of σ. Then,

$$(\tau(a_1), \tau(a_2), \cdots, \tau(a_k))(\tau(b_1), \tau(b_2), \cdots, \cdots, \tau(b_l))$$

is the cycle decomposition of $\tau\sigma\tau^{-1}$. [18]

Theorem A.27. *The equivalence classes of the symmetric group S_n are precisely given by the cycle types. That is, two permutation σ and σ' are conjugate in S_n iff they have the same number of cycles and those are of same length in the disjoint cycle decomposition.*

Representative	Order	Number of cycles	Even or odd
(1)(2)(3)(4)(5)	1	1	even
(12)(34)	2	15	even
(123)	3	20	even
(12345)	5	12	even
(12354)	5	12	even
(1234)	4	30	odd
(12)	2	10	odd
(123)(45)	6	20	odd
Total		60 +60 = 120	

Theorem A.28. *The group A_5 is the unique subgroup of S_5 with index* 2.

Proof. Suppose that there exist another subgroup H of S_5 with index 2. Then H is a normal subgroup by Theorem (A.8). Since $H \cap A_5$ is normal by Theorem (A.9), it follows by the simpleness of A_5 that $H \cap A_5 = \text{id}$ or $H \cap A_5 = A_5$. Hence, H is either the trivial subgroup or the whole group S_5 or A_5. Since S_5 and id do not have index 2, the only subgroup of S_5 with index 2 is A_5. □

Lemma A.29. *The group A_n is generated by* 3*-cycles.*

Proof. This is a consequence of the fact that $(abc) = (ab)(bc)$ together with the fact that all even permutations can be written as a product of an even number of 2-cycles. □

Theorem A.30. *The group A_5 is simple.*

Proof. This follows from the fact there is no combination of conjugation classes of A_5 which include the identity such that the order of the group divides the group order 60. □

A.2 Direct and semi-direct product

Although we already introduced the definition of a direct product in Section 2.4.4, we include it here as well to stress the difference between a direct and semi-direct product.

Definition A.31 (Direct product). If G_1 and G_2 are groups, then the direct product of G_1 and G_2 is the set

$$G_1 \times G_2 := \{(g_1, g_2) : g_1 \in G_1, g_2 \in G_2\}$$

with the operation

$$(g_1, g_2) \cdot (h_1, h_2) = (g_1 h_1, g_2, h_2)$$

for all $g_1, h_2 \in G_1$ and $g_2, h_2 \in G_2$. In other words, the operation is componentwise multiplication. [18]

Theorem A.32. *Let G be a group and let $H_1, H_2 \subset G$ be subgroups and $e \in G$ the identify element. Suppose the following properties hold:*

1. $h_1 h_2 = h_2 h_1$ *for all* $h_1 \in H_1$ *and* $h_2 \in H_2$*;*
2. $H_1 \cap H_2 = \{e\}$*;*
3. *Every* $g \in G$ *can be written as* $g = h_1 h_2$*, with* $h_1 \in H_1$ *and* $h_2 \in H_2$*.*

Then $G \cong H_1 \times H_2$. [18]

Definition A.33 (Semi-direct product). Let G be a group with normal subgroup N and subgroup H. Then G is the semi-direct product of N and H, denoted $G = N \rtimes H$, if the following requirements hold:

1. $N \cap H = \{\text{id}\}$
2. $G = NH$

Furthermore, we may define $\rho : H \to \text{Aut}(N)$ where $\rho(h)(n) = hnh^{-1}$. Then, the group action of G is defined to be:

$$\begin{aligned} (N \rtimes_\rho H)(N \rtimes_\rho H) &\rightarrow N \rtimes_\rho H \\ (n_1, h_1)(n_2, h_2) &\mapsto (n_1 \rho(h_1)(n_2), h_1 h_2). \end{aligned}$$

A.3 Roots systems and reflection groups

Definition A.34 (Root system). A root system, denoted Φ, is finite set of nonzero vectors in V satisfying

- $\Phi \cap \mathbf{R}\alpha = \{\alpha, -\alpha\}$
- $s_\alpha \Phi = \Phi$ for all $\alpha \in \Phi$. [22]

Definition A.35 (Finite reflection group). A finite reflection group of a Euclidean space V, denoted by the letter W, is the group generated by reflections s_α. W is a finite subgroup of the group of all orthogonal transformations of V. [22]

Hence, there is a close connection between a root system Φ and the finite reflection group W. In fact, any root system gives rise to a finite reflection group W. However, there are many root systems that give rise to the same reflection group W.

Definition A.36 (Simple system). A subset Δ of a root system Φ is a simple system if

1. it is a vector space basis for the subspace of V spanned by Φ
2. each $\alpha \in \Phi$ is a linear combination of Δ with all its coefficients nonnegative or nonpositive. [22]

Theorem A.37. *Fix a simple system Δ in Φ. Then W is generated by the set $S := \{s_\alpha, \alpha \in \Delta\}$, subject to the relations:*

$$s_\alpha s_\beta^{m(\alpha,\beta)} = 1, \qquad (\alpha, \beta \in \Delta),$$

where $m(\alpha,\beta)$ denotes the order of $s_\alpha s_\beta$ in W. [22]

A.4 Additional proofs

A.4.1 Conjugacy classes A_5

Theorem A.38. *A_5 consists of 5 conjugacy classes of size $1, 12, 12, 15, 20$.*

Proof. We prove that the even cycles of order 2 and the 3-cycles lie in an orbit of length 15 and 20 respectively. We show that the rotations of order 5 are separated into two conjugacy classes.

First, we consider the rotations of order 5. Take $g = (12345) \in I$. There are 24 elements of order 5, so it directly follows that these rotations of order 5 have to split up in at least two conjugacy classes as 24 does not divide 60. The possible sizes for the conjugacy classes are given as follows:

- 2 classes of size 12
- 4 classes of size 6
- 6 classes of size 4
- 12 classes of size 2
- 24 classes of size 1

If we can find 7 cycles conjugate to (12345), we found that there must be 2 conjugacy classes of size 12. We begin with (12345) and take $(123),(234),(235) \in A_5$.

$$\begin{aligned}
(132)(12345)(123) &= (12453)\\
(132)(12453)(123) &= (14523)\\
(243)(12345)(234) &= (14235)\\
(243)(14235)(234) &= (13425)\\
(243)(14523)(234) &= (13542)\\
(253)(12345)(235) &= (15243)\\
(253)(12453)(235) &= (15432)
\end{aligned}$$

That is, we found 7 different 5-cycles which are conjugate in A_5. Hence, the 5-cycles lie in two conjugacy classes of length 12 in A_5. In fact, the conjugacy classes of these cycles are as given in Figure 24.

Conjugacy class [12345]	Conjugacy class [12354)]
(12345)	(12354)
(12453)	(12435)
(12534)	(12543)
(13254)	(13245)
(13425)	(13452)
(13542)	(13542)
(14235)	(14253)
(14352)	(14325)
(14523)	(14532)
(15243)	(15234)
(15324)	(15342)

Table 24: Conjugacy classes of 5-cycles in A_5

Secondly we consider the 3-cycles in A_5. To show that those cycles are all conjugate, we consider two cases:

1. the two cycles have one element in common, for example (123) and (345)
2. the two cycles have two elements in common, for example (123) and (125)

In the first case we can take two 2-cycles that swap the different elements in the two cycles. For example:

$$(123) = ((14)(25))^{-1}(345)(14)(25)$$

Then, if the two 3-cycles agreeing in two elements, we construct a cycle with as first element the element that is in the first but not in the second cycle. For the cycles (123) and (234) this would be 1. As second element we take the element that is the second, but not in the first. As third we choose the element that is is neither of the cycles. Letting this cycle act on the first from the left and from the right on the second, we get for the example with (123) and (234):

$$(145)(123) = (12345) = (234)(145)$$

Hence, all 3-cycles are conjugate.

Lastly, we take $(12)(34) \in A_5$. There are 15 even permutations of this form. That means that there are three possibilities for the conjugacy classes:

- 1 class of length 15
- 3 classes of length 5
- 5 classes of length 3
- 15 classes of length 1

Performing some calculations, it follows that at least 6 cycles are conjugate with (12)(34). That means, the conjugacy class has to consists of all 15 double 2-cycles.

$$\begin{aligned}
(321)(12)(34)(123) &= (13)(24) \\
(421)(12)(34)(124) &= (14)(23) \\
(521)(12)(34)(125) &= (15)(43) \\
(432)(12)(34)(234) &= (14)(32) \\
(532)(12)(34)(235) &= (15)(24) \\
(531)(12)(34)(135) &= (14)(25)
\end{aligned}$$

In conclusion, A_5 consists of 5 conjugacy classes of sizes $1, 12, 12, 15, 20$. □

A.4.2 Double covering map between the unit quaternions $\mathbb{H}_1$ and the special orthogonal group SO(3)

Theorem A.39. *The rotation qvq^{-1} of a purely imaginary quaternion $\mathbf{v} = (a, b, c)$ by a unit quaternion $q = r + xi + yj + zk$ can be represented by a matrix $O \in \mathrm{SO}(3)$.*

Proof.

$$\begin{aligned}
qvq^{-1} =&(r+xi+yj+zk)(ai+bj+ck)(r-xi-yj+zk)\\
=& \ [(-yz+yz+xr-xr)a+(xz+ry-xz-ry)b+(rz+-xy+xy-rz)c]+\\
& i\cdot\left[(r^2+x^2-y^2-z^2)a+(xy-rz-rz+xy)b+(ry+xz+ry+xz)c\right]\\
& j\cdot\left[((xy+rz+rz+xy)a+(-x^2+r^2-z^2+y^2)b+(-rx-rx+yz+yz)c\right]\\
& k\cdot\left[(-ry-xz-ry+xz)a+(rx+rx+yz+yz)b+(r^2-x^2-y^2+z^2)c\right]=\\
=& \ i\left[(r^2+x^2-y^2-z^2)a+(2xy-2rz)b+(2ry+2xz)c\right]\\
& j\cdot\left[((2xy+2rz)a+(-x^2+r^2-z^2+y^2)b+(-2rx+2yz)c\right]\\
& k\cdot\left[(-2ry-2xz)a+(2rx+2yz)b+(r^2-x^2-y^2+z^2)c\right]
\end{aligned}$$

This equation is equivalent to the matrix vector multiplication given by:

$$\begin{pmatrix} r^2+x^2-y^2-z^2 & 2xy-2rz & 2ry+2xz \\ 2xy+2rz & r^2-x^2+y^2-z^2 & -2rx+2yz \\ -2ry+2xz & 2rx+2yz & r^2-x^2-y^2+z^2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{11}$$

What remains to check is whether the matrix in Equation (11) is indeed an element of SO(3). We check that its determinant equals 1. The determinant of the rotation matrix can be calculated in Maple or Mathematica. This calculation in Maple can be found in Figure 43. Using that q is a unit quaternion and thus $\sqrt{r^2+x^2+y^2+z^2}=1$ this implies that $|q|^6=1$ and thus that the determinant of the rotation matrix equals 1. □

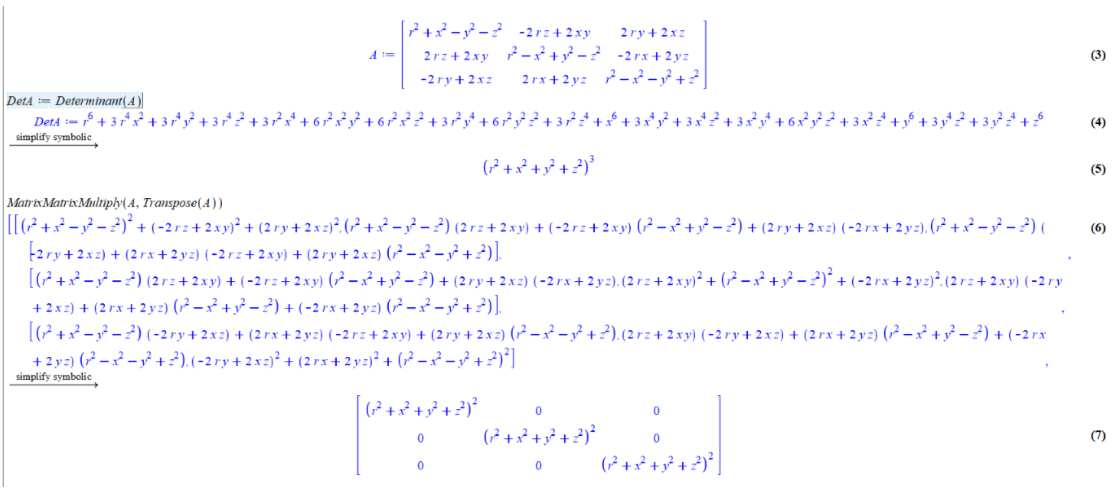


Figure 43: Determinant of the rotation matrix and the matrix obtained by multiplication with its inverse.

Theorem A.40. *The map $f : \mathbb{H}_1 \to \mathrm{SO}(3)$ defined by $q \mapsto qvq^{-1}$ is surjective and $\mathbb{H}_1$ is a double cover of $SO(3)$. That is, the map $f : \mathbb{H}_1 \to SO(3)$ is two-to-one.*

Proof. From Theorem A.39 it follows that any quaternion q gives rise to a rotation matrix $R \in SO(3)$. To show that any rotation matrix in SO(3) can be obtained by a $p \in \mathbb{H}_1$, we describe the rotation by a unit vector $v = (a, b, c) \in \mathbb{R}^3$ and a rotation angle $\theta \in [0, \pi]$. Note here that a rotation of angle $\theta \in (\pi, 2\pi)$ is the same a rotation about the negative unit vector about an angle $-\theta$.

Define the quaternion $p = \cos(\theta) + \sin(\theta)(ui + vj + wk)$. The rotation matrix belonging to p is given by

$$\begin{pmatrix} \cos\left(\frac{1}{2}\theta\right)^2 + \sin\left(\frac{1}{2}\theta\right)(u^2 - v^2 - w^2) & 2\sin^2(\frac{1}{2}\theta)uv - 2\cos\left(\frac{1}{2}\theta\right)\sin\left(\frac{1}{2}\theta\right)w & 2\cos\left(\frac{1}{2}\theta\right)\sin\left(\frac{1}{2}\theta\right)v + 2\sin^2(\frac{1}{2}\theta)uw \\ 2\sin^2(\frac{1}{2}\theta)uv + 2\cos\left(\frac{1}{2}\theta\right)\sin\left(\frac{1}{2}\theta\right)w & \cos\left(\frac{1}{2}\theta\right)^2 - \sin\left(\frac{1}{2}\theta\right)(u^2 - v^2 + w^2) & -2\cos\left(\frac{1}{2}\theta\right)\sin\left(\frac{1}{2}\theta\right)u + 2\sin^2(\frac{1}{2}\theta)vw \\ -2\cos\left(\frac{1}{2}\theta\right)\sin\left(\frac{1}{2}\theta\right)v + 2\sin^2(\frac{1}{2}\theta)uw & 2\cos\left(\frac{1}{2}\theta\right)\sin\left(\frac{1}{2}\theta\right)u + 2\sin^2(\frac{1}{2}\theta)vw & \cos\left(\frac{1}{2}\theta\right)^2 - \sin\left(\frac{1}{2}\theta\right)(u^2 + v^2 - w^2) \end{pmatrix}$$

Its determinant can be calculated and equals $(\cos^2(\theta) + \sin^2(\theta))^3 = 1$. So the rotation matrix corresponding to the quaternion p belongs to the rotation with angle $\theta = 2\arccos(\cos(\theta))$ around x. This proves the surjectivity of f.

The fact that f is at least two-to-one can be proved using Equation (11). If two rotation matrices in SO(3) equal, all the nine entries of those matrices have to agree. The following system of equations in Equations (13),(14),(15) are obtained by setting the first, fourth and ninth entries of two equal rotation matrices equal to each other. For the first rotation matrices we describe the rotation by a vector (x, y, z) and angle r. The second rotation matrix is described by a vector (u, v, w) and angle θ. Equation (15) is obtained from the fact that (x, y, z) and (u, v, w) are unit vectors.

$$\begin{cases} r^2 + x^2 - y^2 - z^2 = \theta^2 + u^2 - v^2 - w^2 & (12) \\ r^2 - x^2 + y^2 - z^2 = \theta^2 - u^2 + v^2 - w^2 & (13) \\ r^2 - x^2 - y^2 + z^2 = \theta^2 - u^2 - v^2 + w^2 & (14) \\ x^2 + y^2 + z^2 = u^2 + v^2 + w^2 & (15) \end{cases}$$

Adding Equation (13) and (14) and adding Equation (13) and (15), while substracting Equation (14) from (13) and substracting Equation (15) from (13), we obtain the following system of equations:

$$\begin{cases} r^2 - z^2 = \theta^2 - w^2 \implies r^2 - \theta^2 = z^2 - w^2 & (16) \\ r^2 - y^2 = \theta^2 - v^2 \implies r^2 - \theta^2 = y^2 - v^2 & (17) \\ x^2 - y^2 = u^2 - v^2 \implies x^2 - u^2 = y^2 - v^2 & (18) \\ x^2 - z^2 = u^2 - w^2 \implies x^2 - u^2 = z^2 - w^2 & (19) \end{cases}$$

From Equation (17), (18), (19) and (19) it follows that

$$r^2 - \theta^2 = x^2 - u^2 = y^2 - v^2 = z^2 - w^2 \tag{20}$$

Adding the equations in Equation (20) results in

$$\begin{aligned} 4(r^2-\theta^2) &= r^2 + x^2 + y^2 + z^2 - \theta^2 - u^2 - v^2 \\ &= w^2 = r^2 - \theta^2 + (x^2+y^2+z^2) - (u^2+v^2+w^2) \\ &= r^2 - \theta^2. \end{aligned} \tag{21}$$

The only solution to Equation (21) is $r^2 = \theta^2$. This implies that

$$x^2 = u^2, \quad y^2 = v^2, \quad z^2 = w^2.$$

So x and u, for example, agree up to a minus sign. That is no problem, since it was already shown in Theorem 3.13 shows that the positive and negative of the same rotation vector and rotation angle describe the same rotation. It is good to note that once we have chosen the sign of one of z and w, the signs of y and z and v and w are fixed. The second and fourth entry of the rotation matrices, for instance, require that $2xy - 2rz = 2uv - 2\theta z$ and $2xy + 2rz = 2uv + 2\theta z$ such that $rz = 2\theta w$. Thus, the signs of r and θ follow. In a similar way, the signs of x, y and u, v are fixed as well.

Hence, $f : \mathbb{H}_1 \to \mathrm{SO}(3)$ is at most $2:1$. □

A.4.3 Finite subgroups of SO(3) and $\mathbb{H}_1$

The aim of this section is to classify all finite subgroups of $\mathbb{H}_1$. The finite subgroups of SO(3) can be classified, which we will do in Theorem A.41. Using the map from SO(3) to $\mathbb{H}_1$ from Theorem 3.14 once more, one finds the classification of the finite subgroups of $\mathbb{H}_1$.

Theorem A.41. *Every finite subgroup of* SO(3) *is isomorphic to one of the groups:*

- *the cyclic group C_n*
- *the dihedral group D_n*
- *the tetrahedral group T*
- *the octahedral group O*
- *the icosahedral group I.*

Proof. First of all, this proof will use poles defined by the rotations in SO(3). That is,each rotation is described by some rotation axis and rotation angle. The rotation axis intersects the unit sphere twice. The points of intersection are called the poles of the rotation. Furthermore, we will only show what the action is of a finite group G on the poles it describes. How to relate the description of the orbits to the five finite groups of SO(3) can be found in [7].

Consider the action of a non-trivial and non-empty finite group $G \subset \mathrm{SO}(3)$ on the set X consisting of sets of two poles of G. This action divides the set X into orbits. We use that

$$\sum_{g\in G} |\{g \in G | gx = x\}| = |\{(g,x) \in G \times X | gx = x\}| = \sum_{x\in X} |\{g \in G : gx = x\}| = \sum_{x\in X} |\mathrm{Stab}_G(x)|.$$

Since each non-trivial $g \in G$ fixes precisely two poles of X and the identity fixes all poles, we get the equality:

$$\sum_{x\in X} |\mathrm{Stab}_G(x)| = 2(|G|-1)+|X|. \tag{22}$$

However, the number of poles in X equals the total number of elements in all orbits under the action of G. If we let P be the set containing one pole of each orbit, it follows that

$$\sum_{p\in P} G(p) = |X|. \tag{23}$$

Taking X to the other side of the equality in Equation (22) we obtain:

$$\sum_{x\in X} |\mathrm{Stab}_G(x)| - |X| = 2(|G|-1). \tag{24}$$

We can take X into the sum into two ways. The first way is as follows:

$$\sum_{x\in X} (|\mathrm{Stab}_G(x)|-1) = 2(|G|-1). \tag{25}$$

Another way is to rewrite the sum over x as a sum over p as follows:

$$\sum_{x\in X} |\mathrm{Stab}_G(x)| = \sum_{p\in P} |\mathrm{Stab}_G(p)||G(p)|, \tag{26}$$

Equation (22) can then be rewritten as:

$$\sum_{p\in P} |G(p)|(\mathrm{Stab}_G(p)-1) = 2(|G|-1). \tag{27}$$

Using to the Orbit-Stabilizer theorem (Theorem A.19) and dividing both sides by $|G|$, Equation (27) can be written as:

$$\begin{aligned} \sum_{p\in P} |G| - |G(p)| &= 2(|G|-1) \\ \sum_{p\in P} 1 - \frac{1}{\mathrm{Stab}_G(p)} &= 2 - \frac{2}{|G|} \end{aligned} \tag{28}$$

We now show that it follows from Equation 28 that $|X| \not\in \{2,3\}$ leads to a contradiction. First, suppose $|X| = 1$. The length of $\text{Stab}_G(x)$ equals $|G|$. Substitution of these values into Equation (28), gives

$$1 - \frac{1}{|G|} = 2(1 - \frac{1}{|G|}) \qquad \Longrightarrow \qquad |G| = 1.$$

Since we assumed that G was a non-trivial subgroup of SO(3), it follows that $|X| \neq 1$.

Then we show that $|X| < 4$. We know that $\frac{1}{2} \leq \frac{1}{\text{Stab}_G(x)} < 1$ for any $x \in X$, as both the rotation through the poles x and the identity are elements of Stab_x. Furthermore, we know that $1 \leq 2 - \frac{2}{|G|} \leq 2$. Thus we know that:

$$|X|(1 - \frac{1}{|\text{Stab}_G(x)|}) < 2.$$

We thus obtain that:

$$\frac{1}{2}|X| < 2 \qquad \Longrightarrow \qquad |X| < 4.$$

Thus, it follows that the orbits of X under the rotations in G must have length 2 or 3. If there are only two orbits, that is $|X| = 2$, there is only one group that satisfies Equation (28). If $|X| = 3$, there are four groups that satisfy this equation.

If $|X| = 2$, we get from Equation (28) the equality:

$$\frac{1}{|\text{Stab}_G(x)|} + \frac{1}{|\text{Stab}_G(y)|} = \frac{2}{|G|}.$$

The group corresponding to these orbits is the cyclic group $C_n \subset \text{SO}(2)$. [26]

For $|X| = 3$, there are 4 groups that satisfy Equation (28).

$$\begin{array}{lllcl}
|G(x)| = 2, & |G(y)| = 2, & |G(z)| = n & \longrightarrow & D_n \\
|G(x)| = 2, & |G(y)| = 3, & |G(z)| = 3 & \longrightarrow & T \\
|G(x)| = 2, & |G(y)| = 3, & |G(z)| = 4 & \longrightarrow & O \\
|G(x)| = 2, & |G(y)| = 3, & |G(z)| = 5 & \longrightarrow & I
\end{array}$$

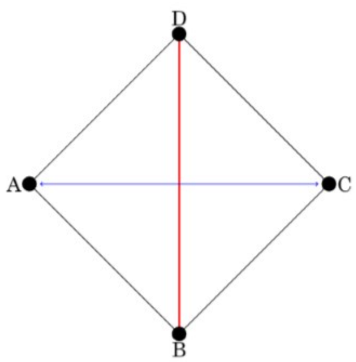
The full proof of how to obtain the groups corresponding to these orbit sizes can be found in [7]. □

Now we can prove that $2I$ is the largest finite subgroup in $2I$ such that its image under the map from Theorem 3.14 is not a cyclic of dihedral group of $\text{SO}(2) \subset \text{SO}(3)$.

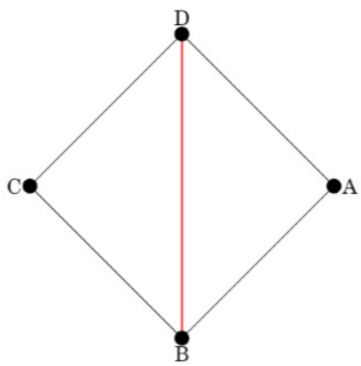
Theorem A.42. *The binary icosahedral group* $2I$ *is the largest finite subgroup of* $\mathbb{H}_1$ *whose rotation group is a subset of* SO(3) *but not of* SO(2).

Proof. Suppose we are given a finite subgroup $G \subset \mathbb{H}_1$. Consider $f(G)$ where f is as defined as in Theorem 3.14. The image under f is a finite subgroup $G' \subset \mathrm{SO}(3)$ as found in Theorem A.41.
First of all, suppose $f(G)$ equals C_n. Since f maps to SO(3) one-to-one or two-to-one, we distinguish between two cases. Either $-1 \in G$ or $-1 \not\in G$. In the first case, $|G| = 2|G'|$ and the group is the pullback of the two-to-one map from $\mathbb{H}_1$ to SO(3). In the second case the pre-image in $\mathbb{H}_1$ of C_n is isomorphic to C_n. The pre-image in $\mathbb{H}_1 1$ contains precisely the n roots of unity of the equation $z^n = -1$. This group of quaternions and can be made arbitrarly large, yet still finite, as n can be any natural number.

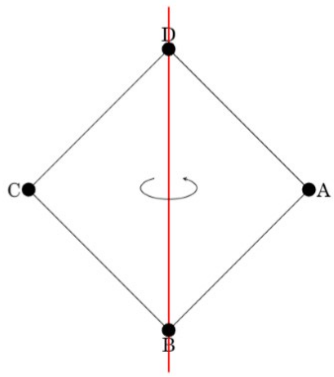
Suppose then, we are given a finite subgroup $G \subset \mathbb{H}_1$ such that $f(G) = D_n$. The dihedral group has order $2n$ and contains as a subgroup C_n, which we already considered. For $n \geq 2, D_n$ contains more than one element of order 2. It thus follows that the the pre-image of D_n in $\mathbb{H}_1$ cannot be isomormphic to D_n itself. Thus, the pre-image is the double cover of the rotations in the dihedral group. This group consists of the quaternions $e^{\frac{2\pi i}{n}}$ together with the n rotation axes representing the reflections in two-dimensional space. For example, the reflection of a square can be represented by a rotation in three-dimensional space (Figure 44). The rotation axis is obtained from the reflection line in two-dimensional space.



(a) Reflection square (2D)



(b) Square



(c) Rotation square (3D)

Figure 44: The reflection of a square in two-dimensional space represented by a rotation in three-dimensional space.

Suppose we have a group $G \subset \mathbb{H}_1$ whose image $f(G)$ is not a cyclic or dihedral group. It then follows that the image is either the tetraheral group, the octahedral group or the icosahedral group. The double cover of those groups are $2T, 2O$ and $2O$ as described in Sections 3.2, 3.3 and 3.4. We also established in these sections that there are no subgroups of $2T, 2O$ or $2I$ isomorphic to T , O or I respectively.

All in all, we found that the finite subgroups of $\mathbb{H}_1$ are the roots of the equation $z^n = -1$, the double cover of the dihedral groups and one of the groups $2T$, $2O$ and $2I$. Since $2I$ is the largest subgroup of $\mathbb{H}_1$ whose rotation group does not lie in SO(2), we are done. □

A.4.4 Quaternion conjugation in the special unitary group SU(2)

Theorem A.43. *Two matrices* $A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ -\overline{a_{1,2}} & \overline{a_{1,1}} \end{pmatrix}$ *and* $B = \begin{pmatrix} b_{1,1} & b_{1,2} \\ -\overline{b_{1,2}} & \overline{b_{1,1}} \end{pmatrix}$ *with nonzero determinant and equal distinct eigenvalues are conjugate. In particular, the matrix that conjugates A and B can be chosen to represent a quaternion.*

Proof. Since A and B have equal distinct eigenvalues and nonzero determinant, the can be diagonalized with the same eigenvalues on the diagonal of the diagonal matrix. Suppose $A = PDP^{-1}$ and $B = QDQ^{-1}$ where D is the diagonal matrix. Then $A = PQ^{-1}DQP^{-1}$. However, PQ^{-1} need not be an matrix that represents a quaternion.
We show that each matrix of eigenvalues can be written as a matrix of eigenvectors that represents a quaternion. It then follows that PQ^{-1} is a quaternion, as the product of quaternions is again a quaternion.
Suppose $P = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ and choose $P' = \begin{pmatrix} x\cdot\alpha & y\cdot\beta \\ x\cdot\gamma & y\cdot\delta \end{pmatrix}$ with $x, y \in \mathbb{C}$.
To represent a quaternion, P' needs to satisfy:

$$x\cdot\alpha = \overline{y\cdot\delta} \tag{29}$$

$$y\cdot\beta = -\overline{x\cdot\gamma} \tag{30}$$

Furthermore, if we choose x and y such that P' represents a unit quaternion, we need that

$$xy(\alpha\cdot\delta - \beta\cdot\gamma) = 1 \tag{31}$$

Substitution of Equation (31) into Equation (29) determines the complex numbers x and y up to a factor $e^{i\phi}$ with $\phi \in [0, 2\pi)$.

$$x = \frac{\overline{y\cdot\delta}}{\alpha} = \frac{\overline{\delta}}{\alpha\cdot x\cdot\overline{(\alpha\delta - \beta\gamma)}} \qquad \Longleftrightarrow \qquad |x|^2 = \frac{\overline{\delta}}{\alpha\cdot\overline{(\alpha\delta - \beta\gamma)}}$$

$$y = \overline{\left(\frac{x\alpha}{\overline{\delta}}\right)} = \frac{1}{\overline{x}\cdot(\alpha\delta - \beta\gamma)}$$

□

B Mathematica code

B.1 Binary groups

B.1.1 Binary icosahedral group $2I$

```
(* Construction 2I *)

{one, aai, jay, kay} = IdentityMatrix[4];
Nsub = phi → (Sqrt[5] + 1) / 2;
sub = Sqrt[5] → 2 phi - 1;

In[14]:= Quatermult[x_, y_] := Module[{
     xQ = {{x[[1]] + I x[[2]], I x[[4]] + x[[3]]}, {I x[[4]] - x[[3]], x[[1]] - I x[[2]]}},
     yQ = {{y[[1]] + I y[[2]], I y[[4]] + y[[3]]}, {I y[[4]] - y[[3]], y[[1]] - I y[[2]]}}
    },
    zQ = xQ.yQ;
    Simplify[{Re[zQ[[1, 1]]], Im[zQ[[1, 1]]], Re[zQ[[1, 2]]], Im[zQ[[1, 2]]]}]
   ]
Quaterconj[x_] := Simplify[{x[[1]], -x[[2]], -x[[3]], -x[[4]]}]

In[22]:= icosians = Expand[{one + aai + jay + kay, -aai + jay (phi - 1) + kay phi} / 2];
done = 0;
length = Length[icosians];
For[i = 1, done < length, i++,
  For[j = done + 1, j ≤ length, j++,
   x = icosians[[j]];
   For[k = 1, k ≤ j, k++,
    y = icosians[[k]];
    z = Expand[Expand[Quatermult[x, y] /. Nsub] /. sub];
    AppendTo[icosians, z];
    z = Expand[Expand[Quatermult[y, x] /. Nsub] /. sub];
    AppendTo[icosians, z]
   ]
  ];
  icosians = DeleteDuplicates[icosians];
  done = length;
  length = Length[icosians];
  If[length > 1136, Break[]]
]
length

Out[26]= 120

(* Conjugacy classes 2I *)
```

In[27]:=
```
icosians[[1]]
icosians[[2]]
icosians[[3]]
icosians[[6]]
icosians[[14]]
icosians[[110]]
icosians[[22]]
icosians[[27]]
```

Out[27]= $\left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}$

Out[28]= $\left\{0, -\frac{1}{2}, -\frac{1}{2}+\frac{\text{phi}}{2}, \frac{\text{phi}}{2}\right\}$

Out[29]= $\left\{-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}$

Out[30]= $\{-1, 0, 0, 0\}$

Out[31]= $\left\{-\frac{\text{phi}}{2}, -\frac{1}{2}+\frac{\text{phi}}{2}, -\frac{1}{2}, 0\right\}$

Out[32]= $\left\{\frac{\text{phi}}{2}, \frac{1}{2}-\frac{\text{phi}}{2}, -\frac{1}{2}, 0\right\}$

Out[33]= $\left\{-\frac{1}{2}+\frac{\text{phi}}{2}, -\frac{\text{phi}}{2}, 0, \frac{1}{2}\right\}$

Out[34]= $\left\{\frac{1}{2}-\frac{\text{phi}}{2}, \frac{\text{phi}}{2}, 0, \frac{1}{2}\right\}$

In[35]:=
```
icosians05conj = {};
For[i = 0, i < 120, i++;
 y = icosians[[i]];
 AppendTo[icosians05conj, Expand[
   Expand[Quatermult[Expand[Quatermult[y, icosians[[1]]]], Quaterconj[y]] /. Nsub] /.
     Nsub /. sub]];
 icosians05conj = DeleteDuplicates[icosians05conj];
]
```

```
Length[icosians05conj]
Sort[icosians05conj]
```

20

$$\left\{\left\{\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right\}, \left\{\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right\}, \left\{\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\}, \left\{\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\},\right.$$
$$\left\{\frac{1}{2}, 0, -\frac{\text{phi}}{2}, \frac{1}{2}-\frac{\text{phi}}{2}\right\}, \left\{\frac{1}{2}, 0, -\frac{\text{phi}}{2}, -\frac{1}{2}+\frac{\text{phi}}{2}\right\}, \left\{\frac{1}{2}, 0, \frac{\text{phi}}{2}, \frac{1}{2}-\frac{\text{phi}}{2}\right\},$$
$$\left\{\frac{1}{2}, 0, \frac{\text{phi}}{2}, -\frac{1}{2}+\frac{\text{phi}}{2}\right\}, \left\{\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right\}, \left\{\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right\}, \left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\},$$
$$\left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}, \left\{\frac{1}{2}, \frac{1}{2}-\frac{\text{phi}}{2}, 0, -\frac{\text{phi}}{2}\right\}, \left\{\frac{1}{2}, \frac{1}{2}-\frac{\text{phi}}{2}, 0, \frac{\text{phi}}{2}\right\},$$
$$\left\{\frac{1}{2}, -\frac{1}{2}+\frac{\text{phi}}{2}, 0, -\frac{\text{phi}}{2}\right\}, \left\{\frac{1}{2}, -\frac{1}{2}+\frac{\text{phi}}{2}, 0, \frac{\text{phi}}{2}\right\}, \left\{\frac{1}{2}, -\frac{\text{phi}}{2}, \frac{1}{2}-\frac{\text{phi}}{2}, 0\right\},$$
$$\left.\left\{\frac{1}{2}, -\frac{\text{phi}}{2}, -\frac{1}{2}+\frac{\text{phi}}{2}, 0\right\}, \left\{\frac{1}{2}, \frac{\text{phi}}{2}, \frac{1}{2}-\frac{\text{phi}}{2}, 0\right\}, \left\{\frac{1}{2}, \frac{\text{phi}}{2}, -\frac{1}{2}+\frac{\text{phi}}{2}, 0\right\}\right\}$$

In[39]:=
```
icosians0conj = {};
For[i = 0, i < 120, i ++;
 y = icosians[[i]];
 AppendTo[icosians0conj,
     Expand[Expand[Quatermult[Expand[Quatermult[y, icosians[[2]]]], Quaterconj[y]] /.
          Nsub] /. Nsub /. sub]] /. Nsub /. sub // N;
 icosians0conj = DeleteDuplicates[icosians0conj];
]
```

In[41]:=
```
Length[icosians0conj]
Sort[icosians0conj]
```

Out[41]= 30

Out[42]=

$$\left\{\{0, -1, 0, 0\}, \left\{0, -\frac{1}{2}, \frac{1}{2}-\frac{\text{phi}}{2}, -\frac{\text{phi}}{2}\right\}, \left\{0, -\frac{1}{2}, \frac{1}{2}-\frac{\text{phi}}{2}, \frac{\text{phi}}{2}\right\},\right.$$
$$\left\{0, -\frac{1}{2}, -\frac{1}{2}+\frac{\text{phi}}{2}, -\frac{\text{phi}}{2}\right\}, \left\{0, -\frac{1}{2}, -\frac{1}{2}+\frac{\text{phi}}{2}, \frac{\text{phi}}{2}\right\}, \{0, 0, -1, 0\}, \{0, 0, 0, -1\},$$
$$\{0, 0, 0, 1\}, \{0, 0, 1, 0\}, \left\{0, \frac{1}{2}, \frac{1}{2}-\frac{\text{phi}}{2}, -\frac{\text{phi}}{2}\right\}, \left\{0, \frac{1}{2}, \frac{1}{2}-\frac{\text{phi}}{2}, \frac{\text{phi}}{2}\right\},$$
$$\left\{0, \frac{1}{2}, -\frac{1}{2}+\frac{\text{phi}}{2}, -\frac{\text{phi}}{2}\right\}, \left\{0, \frac{1}{2}, -\frac{1}{2}+\frac{\text{phi}}{2}, \frac{\text{phi}}{2}\right\}, \{0, 1, 0, 0\}, \left\{0, \frac{1}{2}-\frac{\text{phi}}{2}, -\frac{\text{phi}}{2}, -\frac{1}{2}\right\},$$
$$\left\{0, \frac{1}{2}-\frac{\text{phi}}{2}, -\frac{\text{phi}}{2}, \frac{1}{2}\right\}, \left\{0, \frac{1}{2}-\frac{\text{phi}}{2}, \frac{\text{phi}}{2}, -\frac{1}{2}\right\}, \left\{0, \frac{1}{2}-\frac{\text{phi}}{2}, \frac{\text{phi}}{2}, \frac{1}{2}\right\},$$
$$\left\{0, -\frac{1}{2}+\frac{\text{phi}}{2}, -\frac{\text{phi}}{2}, -\frac{1}{2}\right\}, \left\{0, -\frac{1}{2}+\frac{\text{phi}}{2}, -\frac{\text{phi}}{2}, \frac{1}{2}\right\}, \left\{0, -\frac{1}{2}+\frac{\text{phi}}{2}, \frac{\text{phi}}{2}, -\frac{1}{2}\right\},$$
$$\left\{0, -\frac{1}{2}+\frac{\text{phi}}{2}, \frac{\text{phi}}{2}, \frac{1}{2}\right\}, \left\{0, -\frac{\text{phi}}{2}, -\frac{1}{2}, \frac{1}{2}-\frac{\text{phi}}{2}\right\}, \left\{0, -\frac{\text{phi}}{2}, -\frac{1}{2}, -\frac{1}{2}+\frac{\text{phi}}{2}\right\},$$
$$\left\{0, -\frac{\text{phi}}{2}, \frac{1}{2}, \frac{1}{2}-\frac{\text{phi}}{2}\right\}, \left\{0, -\frac{\text{phi}}{2}, \frac{1}{2}, -\frac{1}{2}+\frac{\text{phi}}{2}\right\}, \left\{0, \frac{\text{phi}}{2}, -\frac{1}{2}, \frac{1}{2}-\frac{\text{phi}}{2}\right\},$$
$$\left.\left\{0, \frac{\text{phi}}{2}, -\frac{1}{2}, -\frac{1}{2}+\frac{\text{phi}}{2}\right\}, \left\{0, \frac{\text{phi}}{2}, \frac{1}{2}, \frac{1}{2}-\frac{\text{phi}}{2}\right\}, \left\{0, \frac{\text{phi}}{2}, \frac{1}{2}, -\frac{1}{2}+\frac{\text{phi}}{2}\right\}\right\}$$

In[43]:=
```
icosians05negconj = {};
For[i = 0, i < 120, i++;
 y = icosians[[i]];
 AppendTo[icosians05negconj,
    Expand[Expand[Quatermult[Expand[Quatermult[y, icosians[[3]]]], Quaterconj[y]] /.
         Nsub] /. Nsub /. sub]] /. Nsub /. sub;
 icosians05negconj = DeleteDuplicates[icosians05negconj];
]
```

In[45]:=
```
Length[icosians05negconj]
Sort[icosians05negconj]
```

Out[45]= 20

Out[46]= $\left\{\left\{-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right\}, \left\{-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right\}, \left\{-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\}, \left\{-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\},\right.$
$\left\{-\frac{1}{2}, 0, -\frac{phi}{2}, \frac{1}{2} - \frac{phi}{2}\right\}, \left\{-\frac{1}{2}, 0, -\frac{phi}{2}, -\frac{1}{2} + \frac{phi}{2}\right\}, \left\{-\frac{1}{2}, 0, \frac{phi}{2}, \frac{1}{2} - \frac{phi}{2}\right\},$
$\left\{-\frac{1}{2}, 0, \frac{phi}{2}, -\frac{1}{2} + \frac{phi}{2}\right\}, \left\{-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right\}, \left\{-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right\}, \left\{-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\},$
$\left\{-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}, \left\{-\frac{1}{2}, \frac{1}{2} - \frac{phi}{2}, 0, -\frac{phi}{2}\right\}, \left\{-\frac{1}{2}, \frac{1}{2} - \frac{phi}{2}, 0, \frac{phi}{2}\right\},$
$\left\{-\frac{1}{2}, -\frac{1}{2} + \frac{phi}{2}, 0, -\frac{phi}{2}\right\}, \left\{-\frac{1}{2}, -\frac{1}{2} + \frac{phi}{2}, 0, \frac{phi}{2}\right\}, \left\{-\frac{1}{2}, -\frac{phi}{2}, \frac{1}{2} - \frac{phi}{2}, 0\right\},$
$\left.\left\{-\frac{1}{2}, -\frac{phi}{2}, -\frac{1}{2} + \frac{phi}{2}, 0\right\}, \left\{-\frac{1}{2}, \frac{phi}{2}, \frac{1}{2} - \frac{phi}{2}, 0\right\}, \left\{-\frac{1}{2}, \frac{phi}{2}, -\frac{1}{2} + \frac{phi}{2}, 0\right\}\right\}$

In[47]:=
```
icosiansphinegconj = {};
For[i = 0, i < 120, i++;
 y = icosians[[i]];
 AppendTo[icosiansphinegconj,
    Expand[Expand[Quatermult[Expand[Quatermult[y, icosians[[14]]]], Quaterconj[y]] /.
         Nsub] /. Nsub /. sub]] /. Nsub /. sub;
 icosiansphinegconj = DeleteDuplicates[icosiansphinegconj];
]

Length[icosiansphinegconj]
Sort[icosiansphinegconj]
```

12

```
Length[icosiansphiconj]
Sort[icosiansphiconj]
```

12

```
In[51]:= icosiansphiinvnegconj = {};
For[i = 0, i < 120, i++;
 y = icosians[[i]];
 AppendTo[icosiansphiinvnegconj,
   Expand[Expand[Quatermult[Expand[Quatermult[y, icosians[[27]]]], Quaterconj[y]] /.
       Nsub] /. Nsub /. sub]] /. Nsub /. sub;
 icosiansphiinvnegconj = DeleteDuplicates[icosiansphiinvnegconj];
]

Length[icosiansphiinvnegconj]
Sort[icosiansphiinvnegconj]
```

12

$$\left\{\left\{\frac{1}{2}-\frac{phi}{2},-\frac{1}{2},-\frac{phi}{2},0\right\},\left\{\frac{1}{2}-\frac{phi}{2},-\frac{1}{2},\frac{phi}{2},0\right\},\left\{\frac{1}{2}-\frac{phi}{2},0,-\frac{1}{2},-\frac{phi}{2}\right\},\left\{\frac{1}{2}-\frac{phi}{2},0,-\frac{1}{2},\frac{phi}{2}\right\},\left\{\frac{1}{2}-\frac{phi}{2},0,\frac{1}{2},-\frac{phi}{2}\right\},\left\{\frac{1}{2}-\frac{phi}{2},0,\frac{1}{2},\frac{phi}{2}\right\},\left\{\frac{1}{2}-\frac{phi}{2},\frac{1}{2},-\frac{phi}{2},0\right\},\left\{\frac{1}{2}-\frac{phi}{2},\frac{1}{2},\frac{phi}{2},0\right\},\left\{\frac{1}{2}-\frac{phi}{2},-\frac{phi}{2},0,-\frac{1}{2}\right\},\left\{\frac{1}{2}-\frac{phi}{2},-\frac{phi}{2},0,\frac{1}{2}\right\},\left\{\frac{1}{2}-\frac{phi}{2},\frac{phi}{2},0,-\frac{1}{2}\right\},\left\{\frac{1}{2}-\frac{phi}{2},\frac{phi}{2},0,\frac{1}{2}\right\}\right\}$$

```
In[53]:= icosiansphiinvconj = {};
For[i = 0, i < 120, i++;
 y = icosians[[i]];
 AppendTo[icosiansphiinvconj,
   Expand[Expand[Quatermult[Expand[Quatermult[y, icosians[[22]]]], Quaterconj[y]] /.
       Nsub] /. Nsub /. sub]] /. Nsub /. sub;
 icosiansphiinvconj = DeleteDuplicates[icosiansphiinvconj];
]

Length[icosiansphiinvconj]
Sort[icosiansphiinvconj]
```

12

$$\left\{\left\{-\frac{1}{2}+\frac{phi}{2},-\frac{1}{2},-\frac{phi}{2},0\right\},\left\{-\frac{1}{2}+\frac{phi}{2},-\frac{1}{2},\frac{phi}{2},0\right\},\left\{-\frac{1}{2}+\frac{phi}{2},0,-\frac{1}{2},-\frac{phi}{2}\right\},\left\{-\frac{1}{2}+\frac{phi}{2},0,-\frac{1}{2},\frac{phi}{2}\right\},\left\{-\frac{1}{2}+\frac{phi}{2},0,\frac{1}{2},-\frac{phi}{2}\right\},\left\{-\frac{1}{2}+\frac{phi}{2},0,\frac{1}{2},\frac{phi}{2}\right\},\left\{-\frac{1}{2}+\frac{phi}{2},\frac{1}{2},-\frac{phi}{2},0\right\},\left\{-\frac{1}{2}+\frac{phi}{2},\frac{1}{2},\frac{phi}{2},0\right\},\left\{-\frac{1}{2}+\frac{phi}{2},-\frac{phi}{2},0,-\frac{1}{2}\right\},\left\{-\frac{1}{2}+\frac{phi}{2},-\frac{phi}{2},0,\frac{1}{2}\right\},\left\{-\frac{1}{2}+\frac{phi}{2},\frac{phi}{2},0,-\frac{1}{2}\right\},\left\{-\frac{1}{2}+\frac{phi}{2},\frac{phi}{2},0,\frac{1}{2}\right\}\right\}$$

In[130]:=
```
(* Orders quaternions in 2I *)
(* Real part 1/2: order 6 *)
z1 =
 Quatermult[Expand[Quatermult[icosians05conj[[1]], icosians05conj[[1]]] /. Nsub /.
      sub], icosians05conj[[1]]] /. Nsub /. sub
z2 = Quatermult[
  z1,
  z1]
```

Out[130]= {-1, 0, 0, 0}

Out[131]= {1, 0, 0, 0}

In[132]:=
```
(* Real part -1/2: order 3 *)
Quatermult[
   Expand[Quatermult[icosians05negconj[[1]], icosians05negconj[[1]]] /. Nsub /. sub],
   icosians05negconj[[1]]] /. Nsub /. sub
```

Out[132]= {1, 0, 0, 0}

In[116]:=
```
(* Real part phi: order 10 *)
x2 = Expand[Quatermult[icosiansphiconj[[1]], icosiansphiconj[[1]]] /. sub] /. Nsub /.
   sub;
x3 = Expand[Quatermult[icosiansphiconj[[1]], x2] /. Nsub] /. Nsub /. sub;
x4 = Expand[Quatermult[icosiansphiconj[[1]], x3] /. Nsub] /. Nsub /. sub;
x5 = Expand[Quatermult[icosiansphiconj[[1]], x4] /. Nsub] /. Nsub /. sub
x6 = Expand[Quatermult[icosiansphiconj[[1]], x5] /. Nsub] /. Nsub /. sub;
x7 = Expand[Quatermult[icosiansphiconj[[1]], x6] /. Nsub] /. Nsub /. sub;
x8 = Expand[Quatermult[icosiansphiconj[[1]], x7] /. Nsub] /. Nsub /. sub;
x9 = Expand[Quatermult[icosiansphiconj[[1]], x8] /. Nsub] /. Nsub /. sub;
x10 = Expand[Quatermult[icosiansphiconj[[1]], x9] /. Nsub] /. Nsub /. sub
```

Out[119]= {-1, 0, 0, 0}

Out[124]= {1, 0, 0, 0}

```
(* Real part -phi: order 5 *)
y2 = Expand[Quatermult[icosiansphinegconj[[1]], icosiansphinegconj[[1]]] /. sub] /.
    Nsub /. sub;
y3 = Expand[Quatermult[icosiansphinegconj[[1]], y2] /. Nsub] /. Nsub /. sub;
y4 = Expand[Quatermult[icosiansphinegconj[[1]], y3] /. Nsub] /. Nsub /. sub;
y5 = Expand[Quatermult[icosiansphinegconj[[1]], y4] /. Nsub] /. Nsub /. sub
```

{1, 0, 0, 0}

```
(* Real part 1/phi: order 5*)
z2 = Expand[Quatermult[icosiansphiinvconj[[1]], icosiansphiinvconj[[1]]] /. sub] /.
    Nsub /. sub;
z3 = Expand[Quatermult[icosiansphiinvconj[[1]], z2] /. Nsub] /. Nsub /. sub;
z4 = Expand[Quatermult[icosiansphiinvconj[[1]], z3] /. Nsub] /. Nsub /. sub;
z5 = Expand[Quatermult[icosiansphiinvconj[[1]], z4] /. Nsub] /. Nsub /. sub
```

{1, 0, 0, 0}

{1, 0, 0, 0}

B.1.2 Binary octahedral group $2O$

```
(*Definition multiplication, conjugation,
standard basis vectors and generation of the group 2O *)

    (* Define multiplication and conjugation of quaternions *)
Quatermult[x_, y_] := Module[{
   xQ = {{x[[1]] + I x[[2]], I x[[4]] + x[[3]]}, {I x[[4]] - x[[3]], x[[1]] - I x[[2]]}},
   yQ = {{y[[1]] + I y[[2]], I y[[4]] + y[[3]]}, {I y[[4]] - y[[3]], y[[1]] - I y[[2]]}}
  },
  zQ = xQ.yQ;
  Simplify[{Re[zQ[[1, 1]]], Im[zQ[[1, 1]]], Re[zQ[[1, 2]]], Im[zQ[[1, 2]]]}]
 ]

Quaterconj[x_] := Simplify[{x[[1]], -x[[2]], -x[[3]], -x[[4]]}]


    (* Standard basis vectors in R4 *)
{one, aai, jay, kay} = IdentityMatrix[4];

    (* Construction octahedral group by two generators *)
octahedralgroup = Expand[{one + aai + jay + kay} / 2];
AppendTo[octahedralgroup, (1 / Sqrt[2]) * (aai + jay)];
AppendTo[octahedralgroup, (1 / Sqrt[2]) * (one + aai)];
done = 0;
length = Length[octahedralgroup];
For[i = 1, done < length, i++,
 For[j = done + 1, j ≤ length, j++,
  x = octahedralgroup[[j]];
  For[k = 1, k <= j, k++,
   y = octahedralgroup[[k]];
   z = Expand[Expand[Quatermult[x, y]]];
   AppendTo[octahedralgroup, z];
   z = Expand[Expand[Quatermult[y, x]]];
   AppendTo[octahedralgroup, z];
  ]
 ];
 octahedralgroup = DeleteDuplicates[octahedralgroup];
 done = length;
 length = Length[octahedralgroup];
 If[length > 500, Break[]];
]
length
Osort = Sort[octahedralgroup]
```

Out[230]= 48

Out[231]=

$$\Big\{\{-1,0,0,0\},\ \left\{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right\},\ \left\{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\},\ \left\{-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\},$$

$$\left\{-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\},\ \left\{-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right\},\ \left\{-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\},\ \left\{-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\},$$

$$\left\{-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\},\ \{0,-1,0,0\},\ \{0,0,-1,0\},\ \{0,0,0,-1\},\ \{0,0,0,1\},$$

$$\{0,0,1,0\},\ \left\{0,0,-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right\},\ \left\{0,0,-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right\},\ \left\{0,0,\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right\},$$

$$\left\{0,0,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right\},\ \{0,1,0,0\},\ \left\{0,-\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}}\right\},\ \left\{0,-\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right\},$$

$$\left\{0,-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0\right\},\ \left\{0,-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right\},\ \left\{0,\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}}\right\},\ \left\{0,\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right\},$$

$$\left\{0,\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0\right\},\ \left\{0,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right\},\ \left\{\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right\},\ \left\{\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\},$$

$$\left\{\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\},\ \left\{\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\},\ \left\{\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right\},\ \left\{\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\},$$

$$\left\{\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\},\ \left\{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\},\ \{1,0,0,0\},\ \left\{-\frac{1}{\sqrt{2}},0,0,-\frac{1}{\sqrt{2}}\right\},$$

$$\left\{-\frac{1}{\sqrt{2}},0,0,\frac{1}{\sqrt{2}}\right\},\ \left\{-\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}},0\right\},\ \left\{-\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}},0\right\},$$

$$\left\{-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0,0\right\},\ \left\{-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0,0\right\},\ \left\{\frac{1}{\sqrt{2}},0,0,-\frac{1}{\sqrt{2}}\right\},\ \left\{\frac{1}{\sqrt{2}},0,0,\frac{1}{\sqrt{2}}\right\},$$

$$\left\{\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}},0\right\},\ \left\{\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}},0\right\},\ \left\{\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0,0\right\},\ \left\{\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0,0\right\}\Big\}$$

```
In[1235]:= (* 2O is a group *)
product48 = {};
For [i = 0, i < 48, i++;
  x = Osort[[i]];
  For[j = 0, j < 48, j++;
   y = Osort[[j]];
   AppendTo[product48, Quatermult[x, y]];
  ];
  product48 = DeleteDuplicates[product48];
 ];
Length[product48]
product48

Out[1237]= 48
```

Out[1238]=

$$\left\{\{1,0,0,0\},\left\{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\},\left\{\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\},\left\{\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\},\right.$$

$$\left\{\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right\},\left\{\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\},\left\{\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\},\left\{\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\},$$

$$\left\{\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right\},\{0,1,0,0\},\{0,0,1,0\},\{0,0,0,1\},\{0,0,0,-1\},$$

$$\{0,0,-1,0\},\left\{0,0,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right\},\left\{0,0,\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right\},\left\{0,0,-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right\},$$

$$\left\{0,0,-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right\},\{0,-1,0,0\},\left\{0,\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right\},\left\{0,\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}}\right\},$$

$$\left\{0,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right\},\left\{0,\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0\right\},\left\{0,-\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right\},\left\{0,-\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}}\right\},$$

$$\left\{0,-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right\},\left\{0,-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0\right\},\left\{-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\},\left\{-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\},$$

$$\left\{-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\},\left\{-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right\},\left\{-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\},\left\{-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\},$$

$$\left\{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\},\left\{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right\},\{-1,0,0,0\},\left\{\frac{1}{\sqrt{2}},0,0,\frac{1}{\sqrt{2}}\right\},$$

$$\left\{\frac{1}{\sqrt{2}},0,0,-\frac{1}{\sqrt{2}}\right\},\left\{\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}},0\right\},\left\{\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}},0\right\},\left\{\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0,0\right\},$$

$$\left\{\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0,0\right\},\left\{-\frac{1}{\sqrt{2}},0,0,\frac{1}{\sqrt{2}}\right\},\left\{-\frac{1}{\sqrt{2}},0,0,-\frac{1}{\sqrt{2}}\right\},\left\{-\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}},0\right\},$$

$$\left.\left\{-\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}},0\right\},\left\{-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0,0\right\},\left\{-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0,0\right\}\right\}$$

In[1150]:=
```
(* The conjugacy classes of 2O *)

    (* We make sets of quaternions with the same real part. Afterwards,
we calculate whether those sets fall apart in multiple conjugacy classes. *)
O05neg = Osort[[2 ;; 9]];
O1neg = Osort[[1]];
O0 = Osort[[10 ;; 27]];
O05 = Osort[[28 ;; 35]];
O1 = Osort[[36]];
Osqrt2neg = Osort[[37 ;; 42]];
Osqrt2 = Osort[[43 ;; 48]];
```

In[1157]:=
```
(* Calculation of sizes of the conjugacy classes *)
O05conj = {};
list = {};
For[i = 0, i < Length[O05], i++;
  For[j = 0, j < 48, j++;
   x = Osort[[j]];
   y = O05[[i]];
   z = Quatermult[Quatermult[x, y], Quaterconj[x]];
   AppendTo[list, z];
  ];
  list = DeleteDuplicates[list];
  AppendTo[O05conj, list];
  list = {};
 ];
Length[O05conj[[1]]]
O05conj[[1]]

O05negconj = {};
list = {};
For[i = 0, i < Length[O05neg], i++;
  For[j = 0, j < 48, j++;
   x = Osort[[j]];
   y = O05neg[[i]];
   z = Quatermult[Quatermult[x, y], Quaterconj[x]];
   AppendTo[list, z];
  ];
  list = DeleteDuplicates[list];
  AppendTo[O05negconj, list];
  list = {};
 ];
Length[O05negconj[[1]]]
O05negconj[[1]]

O0conj = {};
list = {};
For[i = 0, i < Length[O0], i++;
  For[j = 0, j < 48, j++;
   x = Osort[[j]];
```

```
    y = O0[[i]];
    z = Quatermult[Quatermult[x, y], Quaterconj[x]];
    AppendTo[list, z];
   ];
   list = DeleteDuplicates[list];
   AppendTo[O0conj, list];
   list = {};
  ];
Length[O0conj[[1]]]
O0conj[[1]]

Osqrt2conj = {};
list = {};
For[i = 0, i < Length[Osqrt2], i++;
   For[j = 0, j < 48, j++;
    x = Osort[[j]];
    y = Osqrt2[[i]];
    z = Quatermult[Quatermult[x, y], Quaterconj[x]];
    AppendTo[list, z];
   ];
   list = DeleteDuplicates[list];
   AppendTo[Osqrt2conj, list];
   list = {};
  ];
Length[Osqrt2conj[[1]]]
Osqrt2conj[[1]]

Osqrt2negconj = {};
list = {};
For[i = 0, i < Length[Osqrt2neg], i++;
   For[j = 0, j < 48, j++;
    x = Osort[[j]];
    y = Osqrt2neg[[i]];
    z = Quatermult[Quatermult[x, y], Quaterconj[x]];
    AppendTo[list, z];
   ];
   list = DeleteDuplicates[list];
   AppendTo[Osqrt2negconj, list];
   list = {};
  ];
Length[Osqrt2negconj[[1]]]
Osqrt2negconj[[1]]
```

Out[1160]= 8

Out[1161]= $\left\{\left\{\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right\}, \left\{\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}, \left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\}, \left\{\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right\}, \left\{\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right\}, \left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}, \left\{\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\}, \left\{\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right\}\right\}$

Out[1165]= 8

Out[1166]= $\{\{-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\}, \{-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}, \{-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\}, \{-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\}, \{-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\}, \{-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}, \{-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\}, \{-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\}\}$

Out[1170]= 6

Out[1171]= {{0, -1, 0, 0}, {0, 0, -1, 0}, {0, 0, 0, 1}, {0, 0, 0, -1}, {0, 0, 1, 0}, {0, 1, 0, 0}}

Out[1175]= 6

Out[1176]= $\{\{\frac{1}{\sqrt{2}}, 0, 0, -\frac{1}{\sqrt{2}}\}, \{\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, 0\}, \{\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0\}, \{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\}, \{\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, 0\}, \{\frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}}\}\}$

Out[1180]= 6

Out[1181]= $\{\{-\frac{1}{\sqrt{2}}, 0, 0, -\frac{1}{\sqrt{2}}\}, \{-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, 0\}, \{-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0\}, \{-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\}, \{-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, 0\}, \{-\frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}}\}\}$

In[1212]:=
```
(* Normal subgroups *)

    (* The conjugacy classes with real part 1/sqrt(2) are all conjugate. However,
any of those quaternions to the power its own order is the identity,
which does not lie in those conjugacy classes. Hence, it is not a grouop. *)
normal8sqrt2 = {};
For [i = 0, i < 48, i++;
  y = Osqrt2conj[[1]][[1]];
  x = Osort[[i]];
  z = Quatermult[Quatermult[x, y], Quaterconj[x]];
  AppendTo[normal8sqrt2, z]
 ];
For[i = 0, i < 6, i++;
  z = Quatermult[Osqrt2conj[[1]][[i]], Osqrt2conj[[1]][[i]]];
  AppendTo[normal8sqrt2, z];
 ];
normal8sqrt2 = DeleteDuplicates[normal8sqrt2]
Length[normal8sqrt2];

    (* Also this union of conjugacy classes is not a subgroup as the +- i,
j, k do not lie within this union of conjugacy classes,
although it is the product of quaternions in the union *)
normal8sqrt2neg = {};
For [i = 0, i < 48, i++;
  y = Osqrt2negconj[[1]][[1]];
  x = Osort[[i]];
  z = Quatermult[Quatermult[x, y], Quaterconj[x]];
  AppendTo[normal8sqrt2neg, z]
 ];
```

```
For[i = 0, i < 6, i++;
  z = Quatermult[Osqrt2negconj[[1]][[i]], Osqrt2negconj[[1]][[i]]];
  AppendTo[normal8sqrt2neg, z];
 ];
normal8sqrt2neg = DeleteDuplicates[normal8sqrt2neg]
Length[normal8sqrt2neg];


    (* This is a normal subgroup of order 8 *)
normal80 = {{1, 0, 0, 0}, {-1, 0, 0, 0}};
For [i = 0, i < 48, i++;
  y = O0conj[[1]][[1]];
  x = Osort[[i]];
  z = Quatermult[Quatermult[x, y], Quaterconj[x]];
  AppendTo[normal80, z]
 ];
For [i = 0, i < 48, i++;
  x = Osort[[i]];
  y1 = O1;
  y2 = O1neg;
  AppendTo[normal80, Quatermult[Quatermult[x, y1], Quaterconj[x]]];
  AppendTo[normal80, Quatermult[Quatermult[x, y2], Quaterconj[x]]];
 ];
Length[normal80]
normal80 = DeleteDuplicates[normal80]
```

Out[1215]= $\left\{\left\{\frac{1}{\sqrt{2}}, 0, 0, -\frac{1}{\sqrt{2}}\right\}, \left\{\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, 0\right\}, \left\{\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0\right\}, \left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\right\}, \left\{\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, 0\right\}, \left\{\frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}}\right\}, \{0, 0, 0, -1\}, \{0, -1, 0, 0\}, \{0, 0, 1, 0\}, \{0, 1, 0, 0\}, \{0, 0, -1, 0\}, \{0, 0, 0, 1\}\right\}$

Out[1220]= $\left\{\left\{-\frac{1}{\sqrt{2}}, 0, 0, -\frac{1}{\sqrt{2}}\right\}, \left\{-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, 0\right\}, \left\{-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0\right\}, \left\{-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\right\}, \left\{-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, 0\right\}, \left\{-\frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}}\right\}, \{0, 0, 0, 1\}, \{0, 1, 0, 0\}, \{0, 0, -1, 0\}, \{0, -1, 0, 0\}, \{0, 0, 1, 0\}, \{0, 0, 0, -1\}\right\}$

Out[1225]= 146

Out[1226]= {{1, 0, 0, 0}, {-1, 0, 0, 0}, {0, -1, 0, 0}, {0, 0, -1, 0}, {0, 0, 0, 1}, {0, 0, 0, -1}, {0, 0, 1, 0}, {0, 1, 0, 0}}

In[1227]:=
```
(* There is a normal subgroup of order 24 in 2O *)

    (* normal24 is the union of all quaternions with real part +-1/2, +-1, 0 *)
normal24 = {};
For [i = 0, i < 48, i++;
  If[Osort[[i]][[1]] == 1 / 2, AppendTo[normal24, Osort[[i]]], Continue];
  If[Osort[[i]][[1]] == -1 / 2, AppendTo[normal24, Osort[[i]]], Continue];
  If[Osort[[i]][[1]] == 1, AppendTo[normal24, Osort[[i]]], Continue];
  If[Osort[[i]][[1]] == -1, AppendTo[normal24, Osort[[i]]], Nothing];
 ];
For[i = 0, i < Length[O0conj[[1]]], i++;
  AppendTo[normal24, O0conj[[1]][[i]]];
 ];
Length[normal24]

    (* The products of all 24 quaternions lies within the set (subgroup) again *)
product24 = {};
For[i = 0, i < 24, i++;
  For [j = 0, j < 24, j++;
   x = normal24[[i]];
   y = normal24[[j]];
   AppendTo[product24, Quatermult[x, y]];
  ];
  product24 = DeleteDuplicates[product24];
 ];
Length[product24]
product24
```

Out[1230]= 24

Out[1233]= 24

Out[1234]=

$$\left\{\{1, 0, 0, 0\}, \left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}, \left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\}, \left\{\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right\}, \left\{\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right\}, \left\{\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}, \left\{\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\}, \left\{\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right\}, \left\{\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right\}, \left\{-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}, \left\{-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\}, \left\{-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right\}, \left\{-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right\}, \left\{-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}, \left\{-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\}, \left\{-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right\}, \left\{-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right\}, \{-1, 0, 0, 0\}, \{0, 1, 0, 0\}, \{0, 0, 1, 0\}, \{0, 0, 0, -1\}, \{0, 0, 0, 1\}, \{0, 0, -1, 0\}, \{0, -1, 0, 0\}\right\}$$

B.1.3 Binary tetrahedral group $2T$

In[1265]:=
```
(* Definition multiplication, conjugation,
standards basis vectors and generation of the group 2T *)
```

In[1266]:=
```
      (* Define multiplication and conjugation of quaternions *)
Quatermult[x_, y_] := Module[{
    xQ = {{x[[1]] + I x[[2]], I x[[4]] + x[[3]]}, {I x[[4]] - x[[3]], x[[1]] - I x[[2]]}},
    yQ = {{y[[1]] + I y[[2]], I y[[4]] + y[[3]]}, {I y[[4]] - y[[3]], y[[1]] - I y[[2]]}}
   },
   zQ = xQ.yQ;
   Simplify[{Re[zQ[[1, 1]]], Im[zQ[[1, 1]]], Re[zQ[[1, 2]]], Im[zQ[[1, 2]]]}]
  ]

Quaterconj[x_] := Simplify[{x[[1]], -x[[2]], -x[[3]], -x[[4]]}]
```

In[1268]:=
```
      (* Standard basis vectors in R4 *)
{one, aai, jay, kay} = IdentityMatrix[4];
```

In[1269]:=
```
    (* Construction tetrahedral group by two generators *)
tetrahedralgroup = Expand[{one + aai + jay + kay} / 2];
AppendTo[tetrahedralgroup, aai];
done = 0;
length = Length[tetrahedralgroup];
For[i = 1, done < length, i++,
 For[j = done + 1, j ≤ length, j++,
  x = tetrahedralgroup[[j]];
  For[k = 1, k <= j, k++,
   y = tetrahedralgroup[[k]];
   z = Expand[Expand[Quatermult[x, y]]];
   AppendTo[tetrahedralgroup, z];
   z = Expand[Expand[Quatermult[y, x]]];
   AppendTo[tetrahedralgroup, z];
  ]
 ];
 tetrahedralgroup = DeleteDuplicates[tetrahedralgroup];
 done = length;
 length = Length[tetrahedralgroup];
 If[length > 500, Break[]];
]
length
tetrahedralgroup
```

Out[1274]= 24

Out[1275]=

$$\left\{\left\{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\},\{0,1,0,0\},\left\{-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\},\left\{-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\},\left\{-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\},\right.$$

$$\{-1,0,0,0\},\left\{-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\},\left\{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\},\left\{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right\},$$

$$\left\{-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right\},\left\{-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\},\{0,-1,0,0\},\{0,0,0,-1\},\{0,0,-1,0\},$$

$$\left\{\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right\},\left\{\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\},\left\{\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\},\left\{\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\},\{1,0,0,0\},$$

$$\left.\left\{\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right\},\{0,0,0,1\},\left\{\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\},\{0,0,1,0\},\left\{\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\}\right\}$$

In[1341]:=
```
(*2T is a group*)
product = {};
For [i = 1, i < 25, i++,
  For [j = 1, j < 25, j++,
   x = tetrahedralgroup[[i]];
   y = tetrahedralgroup[[j]];
   z = Quatermult[x, y];
   AppendTo[product, z];
  ]
 ];
product = DeleteDuplicates[product];
Length[product]
product
```

Out[1344]= 24

Out[1345]=

$$\left\{\left\{-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}, \left\{-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\}, \{-1, 0, 0, 0\}, \left\{-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right\}, \left\{-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\},\right.$$
$$\left\{-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right\}, \{0, -1, 0, 0\}, \{0, 0, -1, 0\}, \left\{\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right\}, \{0, 0, 0, -1\},$$
$$\left\{-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right\}, \left\{\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right\}, \left\{\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\}, \left\{\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right\},$$
$$\{1, 0, 0, 0\}, \{0, 0, 0, 1\}, \left\{\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}, \left\{\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right\}, \left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\},$$
$$\left.\left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\}, \left\{-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right\}, \{0, 1, 0, 0\}, \left\{-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}, \{0, 0, 1, 0\}\right\}$$

In[1281]:=
```
(* Conjugacy classes of 2T. First,
we make sets with quaternions with same real part. Afterwards,
we calculate whether those sets fall apart in smaller conjugacy classes. *)
```

```
In[1346]:= tetra05 = {};
For [i = 0, i < 24, i++;
 If[ tetra[[i]][[1]] == 1 / 2, AppendTo[tetra05, tetra[[i]]],
 ]
]

tetra05neg = {};
For [i = 0, i < 24, i++;
 If[ tetra[[i]][[1]] == -1 / 2, AppendTo[tetra05neg, tetra[[i]]], Nothing]
]

tetra0 = {};
For [i = 0, i < 24, i++;
 If[ tetra[[i]][[1]] == 0, AppendTo[tetra0, tetra[[i]]], Nothing]
]

tetra1 = {};
For [i = 0, i < 24, i++;
 If[ tetra[[i]][[1]] == 1, AppendTo[tetra1, tetra[[i]]], Nothing]
]

tetra1neg = {};
For [i = 0, i < 24, i++;
 If[ tetra[[i]][[1]] == -1, AppendTo[tetra1neg, tetra[[i]]], Nothing]
]

In[1292]:=     (* Calculate whether quaternions with
 same real part lie in smaller conjugacy classes *)

In[1839]:= tetra0conj = {};
list = {};
For[i = 0, i < 6, i++;
  For[j = 0, j < 24, j++;
   x = tetra[[j]];
   y = tetra0[[i]];
   z = Quatermult[Quatermult[x, y], Quaterconj[x]];
   AppendTo[list, z];
  ];
  list = DeleteDuplicates[list];
  AppendTo[tetra0conj, list];
  list = {};
 ];
Length[tetra0conj[[1]]]
tetra0conj[[1]]

tetra1conj = {};
list = {};
For[i = 0, i < 1, i++;
  For[j = 0, j < 24, j++;
   list = {};
   x = tetra[[j]];
```

```
    y = tetra1[[i]];
    z = Quatermult[Quatermult[x, y], Quaterconj[x]];
    AppendTo[list, z];
   ];
   list = DeleteDuplicates[list];
   AppendTo[tetra1conj, list];
   list = {};
  ];
Length[tetra1conj[[1]]]

tetra1negconj = {};
list = {};
For[i = 0, i < 1, i++;
   For[j = 0, j < 24, j++;
    x = tetra[[j]];
    y = tetra1neg[[i]];
    z = Quatermult[Quatermult[x, y], Quaterconj[x]];
    AppendTo[tetra1negconj, z];
   ];
   tetra1negconj = DeleteDuplicates[tetra1negconj];
  ];
Length[tetra1negconj]

tetra05negconj = {};
list = {};
For[i = 0, i < 8, i++;
   For[j = 0, j < 24, j++;
    x = tetra[[j]];
    y = tetra05neg[[i]];
    z = Quatermult[Quatermult[x, y], Quaterconj[x]];
    AppendTo[list, z];
   ];
   list = DeleteDuplicates[list];
   AppendTo[tetra05negconj, list];
   list = {};
  ];
Length[tetra05negconj[[1]]]
tetra05negconj[[1]]
tetra05negconj[[2]]

tetra05conj = {};
list = {};
For[i = 0, i < 8, i++;
   For[j = 0, j < 24, j++;
    x = tetra[[j]];
    y = tetra05[[i]];
    z = Quatermult[Quatermult[x, y], Quaterconj[x]];
    AppendTo[list, z];
   ];
   list = DeleteDuplicates[list];
```

```
   AppendTo[tetra05conj, list];
   list = {};
  ];
Length[tetra05conj[[1]]]
tetra05conj[[1]]
tetra05conj[[2]]
```

Out[1842]= 6

Out[1843]= {{0, -1, 0, 0}, {0, 0, -1, 0}, {0, 0, 0, 1}, {0, 0, 0, -1}, {0, 0, 1, 0}, {0, 1, 0, 0}}

Out[1847]= 1

Out[1851]= 1

Out[1855]= 4

Out[1856]= $\left\{\left\{-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right\}, \left\{-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}, \left\{-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\}, \left\{-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right\}\right\}$

Out[1857]= $\left\{\left\{-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right\}, \left\{-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right\}, \left\{-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\}, \left\{-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}\right\}$

Out[1861]= 4

Out[1862]= $\left\{\left\{\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right\}, \left\{\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}, \left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\}, \left\{\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right\}\right\}$

Out[1863]= $\left\{\left\{\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right\}, \left\{\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right\}, \left\{\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\}, \left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}\right\}$

In[1315]:=
```
(* Normal subgroup order 8 *)
Normal8 = tetra0conj[[1]];
AppendTo[Normal8,  tetra1[[1]] ];
AppendTo[Normal8, tetra1neg[[1]]];
```

In[1875]:=
```
Normal8conj = {};
For[i = 0, i < 8, i++;
  For[j = 0, j < 24, j++;
   x = tetra[[j]];
   y = Normal8[[i]];
   z = Quatermult[Quatermult[x, y], Quaterconj[x]];
   AppendTo[Normal8conj, z];
  ];
  Normal8conj = DeleteDuplicates[Normal8conj];
 ];
For[i = 0, i < 8, i++;
  x = Normal8[[i]];
  For[j = 0, j < 8, j++;
   y = Normal8[[j]];
   AppendTo[Normal8conj, Quatermult[x, y]];
  ];
 ];
Normal8conj = DeleteDuplicates[Normal8conj]
```

Out[1878]=
```
{{0, -1, 0, 0}, {0, 0, -1, 0}, {0, 0, 0, 1}, {0, 0, 0, -1},
 {0, 0, 1, 0}, {0, 1, 0, 0}, {1, 0, 0, 0}, {-1, 0, 0, 0}}
```

In[1327]:=
```
list = {tetra05[[1]]};
list2 = {};
For [i = 0, i < 24, i++;
  x = tetra[[i]];
  y = Quatermult[tetra05[[1]], x];
  z = Quatermult[x, tetra05[[1]]];
  If [y == z, AppendTo[list, x], AppendTo[list2, x]];
 ];
list2
```

Out[1330]=

$$\left\{\left\{-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right\}, \left\{-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\}, \left\{-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}, \left\{-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right\}, \left\{-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right\}, \left\{-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\}, \{0, -1, 0, 0\}, \{0, 0, -1, 0\}, \{0, 0, 0, -1\}, \{0, 0, 0, 1\}, \{0, 0, 1, 0\}, \{0, 1, 0, 0\}, \left\{\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right\}, \left\{\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\}, \left\{\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}, \left\{\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right\}, \left\{\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right\}, \left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\}\right\}$$

```
(* Left cosets of 2T*)
      (* Left cosets gN where g = (1,0,0,0),
(1/2, -1/2, 1/2, 1/2), (-1/2, -1/2, 1/2, 1/2) *)
tetra05conj[[4]][[1]]
tetra05negconj[[4]][[1]]
nevenklas1 = {};
nevenklas2 = {};
For[i = 0, i < 8, i++;
  AppendTo[nevenklas1, Quatermult[tetra05conj[[1]][[1]], Normal8[[i]]]];
  AppendTo[nevenklas2, Quatermult[tetra05negconj[[4]][[1]], Normal8[[i]]]];
 ];
Sort[nevenklas1]
Sort[nevenklas2]
```

Out[1907]= $\left\{\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}$

Out[1908]= $\left\{-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}$

Out[1912]= $\left\{\left\{-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right\}, \left\{-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\}, \left\{-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right\}, \left\{-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}, \left\{\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right\}, \left\{\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}, \left\{\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right\}, \left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\}\right\}$

Out[1913]= $\left\{\left\{-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right\}, \left\{-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}, \left\{-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right\}, \left\{-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\}, \left\{\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right\}, \left\{\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\}, \left\{\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right\}, \left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}\right\}$

B.2 600-cell

B.2.1 The 24-cells in a 600-cell

```
In[3925]:= (* COPY TO RUN ENTIRE DOCUMENT *)

(* Construction 2I *)
{one, aai, jay, kay} = IdentityMatrix[4];
Nsub = phi → (Sqrt[5] + 1) / 2;
sub = Sqrt[5] → 2 phi - 1;

In[3928]:= Quatermult[x_, y_] := Module[{
    xQ = {{x[[1]] + I x[[2]], I x[[4]] + x[[3]]}, {I x[[4]] - x[[3]], x[[1]] - I x[[2]]}},
    yQ = {{y[[1]] + I y[[2]], I y[[4]] + y[[3]]}, {I y[[4]] - y[[3]], y[[1]] - I y[[2]]}}
   },
   zQ = xQ.yQ;
   Simplify[{Re[zQ[[1, 1]]], Im[zQ[[1, 1]]], Re[zQ[[1, 2]]], Im[zQ[[1, 2]]]}]
  ]
Quaterconj[x_] := Simplify[{x[[1]], -x[[2]], -x[[3]], -x[[4]]}]
icosians = Expand[{one + aai + jay + kay, -aai + jay (phi - 1) + kay phi} / 2];
done = 0;
length = Length[icosians];
For[i = 1, done < length, i++,
 For[j = done + 1, j ≤ length, j++,
  x = icosians[[j]];
  For[k = 1, k ≤ j, k++,
   y = icosians[[k]];
   z = Expand[Expand[Quatermult[x, y] /. Nsub] /. sub];
   AppendTo[icosians, z];
   z = Expand[Expand[Quatermult[y, x] /. Nsub] /. sub];
   AppendTo[icosians, z]
  ]
 ];
 icosians = DeleteDuplicates[icosians];
 done = length;
 length = Length[icosians];
 If[length > 1136, Break[]]
]
length
120

Out[3934]= 120

Out[3935]= 120

In[3936]:= icosians05conj = {};
For[i = 0, i < 120, i++;
 y = icosians[[i]];
 AppendTo[icosians05conj, Expand[
   Expand[Quatermult[Expand[Quatermult[y, icosians[[1]]]], Quaterconj[y]] /. Nsub] /.
     Nsub /. sub]];
 icosians05conj = DeleteDuplicates[icosians05conj];
]
```

```
In[3938]:= icosians0conj = {};
For[i = 0, i < 120, i++;
 y = icosians[[i]];
 AppendTo[icosians0conj,
     Expand[Expand[Quatermult[Expand[Quatermult[y, icosians[[2]]]], Quaterconj[y]] /.
          Nsub] /. Nsub /. sub]] /. Nsub /. sub // N;
 icosians0conj = DeleteDuplicates[icosians0conj];
]
```

```
In[3940]:= icosians05negconj = {};
For[i = 0, i < 120, i++;
 y = icosians[[i]];
 AppendTo[icosians05negconj,
    Expand[Expand[Quatermult[Expand[Quatermult[y, icosians[[3]]]], Quaterconj[y]] /.
         Nsub] /. Nsub /. sub]] /. Nsub /. sub;
 icosians05negconj = DeleteDuplicates[icosians05negconj];
]

icosiansphinegconj = {};
For[i = 0, i < 120, i++;
 y = icosians[[i]];
 AppendTo[icosiansphinegconj,
    Expand[Expand[Quatermult[Expand[Quatermult[y, icosians[[14]]]], Quaterconj[y]] /.
         Nsub] /. Nsub /. sub]] /. Nsub /. sub;
 icosiansphinegconj = DeleteDuplicates[icosiansphinegconj];
]

icosiansphiconj = {};
For[i = 0, i < 120, i++;
 y = icosians[[i]];
 AppendTo[icosiansphiconj,
    Expand[Expand[Quatermult[Expand[Quatermult[y, icosians[[110]]]],
          Quaterconj[y]] /. Nsub] /. Nsub /. sub]] /. Nsub /. sub;
 icosiansphiconj = DeleteDuplicates[icosiansphiconj];
]

icosiansphiinvnegconj = {};
For[i = 0, i < 120, i++;
 y = icosians[[i]];
 AppendTo[icosiansphiinvnegconj,
    Expand[Expand[Quatermult[Expand[Quatermult[y, icosians[[27]]]], Quaterconj[y]] /.
         Nsub] /. Nsub /. sub]] /. Nsub /. sub;
 icosiansphiinvnegconj = DeleteDuplicates[icosiansphiinvnegconj];
]

icosiansphiinvconj = {};
For[i = 0, i < 120, i++;
 y = icosians[[i]];
 AppendTo[icosiansphiinvconj,
    Expand[Expand[Quatermult[Expand[Quatermult[y, icosians[[22]]]], Quaterconj[y]] /.
         Nsub] /. Nsub /. sub]] /. Nsub /. sub;
 icosiansphiinvconj = DeleteDuplicates[icosiansphiinvconj];
]
```

In[3950]:=
```
(* 24-CELLS IN THE 600-CELL *)

(* Normalizer 2T in 2I *)
(* There are multiple examples of g in 2I such that for a h in 2T,
ghg^{-1} lies outside 2T *)
normalizer = {};
outside = {};
For[i = 0, i < 24, i++;
  For[j = 0, j < 120, j++;
   x = Expand[
     Quatermult[icosians[[j]] /. Nsub, Quatermult[tetrahedralgroup[[i]] /. Nsub,
         Quaterconj[icosians[[j]] /. Nsub]]] /. sub // FullSimplify];
   If [ContainsAny[icosians, {{x}}], AppendTo[normalizer, x], AppendTo[outside, x]];
  ];
  normalizer = DeleteDuplicates[normalizer];
  outside = DeleteDuplicates[outside];
 ];
Length[outside]
```

Out[3953]= 72

In[3954]:=
```
(* Left cosets and conjugation of the subgroup 2T *)
(* Definition c in 2I with c^5 = id. We use the notation c = g1,
c^2 = g2, c^3 = g3, c^4 = g4 *)
H = tetrahedralgroup;
g1 = Expand[icosiansphinegconj[[1]] /. sub // FullSimplify]
g2 = Expand[Quatermult[g1 /. Nsub, g1 /. Nsub] /. sub // FullSimplify]
g3  = Expand[Quatermult[g1 /. Nsub, g2 /. Nsub] /. sub // FullSimplify]
g4 = Expand[Quatermult[g2 /. Nsub, g2 /. Nsub] /. sub // FullSimplify]
Quatermult[g1 /. Nsub, g4 /. Nsub];
```

Out[3955]= $\left\{-\frac{\text{phi}}{2}, 0, -\frac{1}{2}+\frac{\text{phi}}{2}, -\frac{1}{2}\right\}$

Out[3956]= $\left\{-\frac{1}{2}+\frac{\text{phi}}{2}, 0, -\frac{1}{2}, \frac{\text{phi}}{2}\right\}$

Out[3957]= $\left\{-\frac{1}{2}+\frac{\text{phi}}{2}, 0, \frac{1}{2}, -\frac{\text{phi}}{2}\right\}$

Out[3958]= $\left\{-\frac{\text{phi}}{2}, 0, \frac{1}{2}-\frac{\text{phi}}{2}, \frac{1}{2}\right\}$

In[3960]:=
```
(* Computation of the conjjugate subgroups of H =
 2T and the left cosets of H = 2T and H = c^iHc^{-i}. *)
(* Notation: g1Hg1 represents the conjugate
  cHc^{-1} and g1g2Hg1 represents c^3Hc^{-1}. *)

g1Hg1 = {};
g2Hg2 = {};
g3Hg3 = {};
g4Hg4 = {};
For[i = 0, i < 24, i++;
  y = tetrahedralgroup2[[i]] /. Nsub;
  AppendTo[g1Hg1,
   Expand[Quatermult[Quatermult[g1 /. Nsub, y], Quaterconj[g1] /. Nsub] /. sub //
     FullSimplify]];
```

```
  AppendTo[g2Hg2, Expand[Quatermult[Quatermult[g2 /. Nsub, y],
       Quaterconj[g2] /. Nsub] /. sub // FullSimplify]];
  AppendTo[g3Hg3, Expand[Quatermult[Quatermult[g3 /. Nsub, y],
       Quaterconj[g3] /. Nsub] /. sub // FullSimplify]];
  AppendTo[g4Hg4, Expand[Quatermult[Quatermult[g4 /. Nsub, y],
       Quaterconj[g4] /. Nsub] /. sub // FullSimplify]];
 ];

g1H = {};
g2H = {};
g3H = {};
g4H = {};
For[i = 0, i < 24, i++;
  y = tetrahedralgroup[[i]] /. Nsub;
  AppendTo[g1H, Expand[Quatermult[g1 /. Nsub, y] /. sub // FullSimplify]];
  AppendTo[g2H, Expand[Quatermult[g2 /. Nsub, y] /. sub // FullSimplify]];
  AppendTo[g3H, Expand[Quatermult[g3 /. Nsub, y] /. sub // FullSimplify]];
  AppendTo[g4H, Expand[Quatermult[g4 /. Nsub, y] /. sub // FullSimplify]];
 ];

g1g1Hg1 = {};
g2g1Hg2 = {};
g3g1Hg3 = {};
g4g1Hg4 = {};
For[i = 0, i < Length[g1H], i++;
  AppendTo[g1g1Hg1, Expand[Quatermult[Quatermult[g1 /. Nsub, g1H[[i]] /. Nsub],
       Quaterconj[g1] /. Nsub] /. sub // FullSimplify]];
  AppendTo[g2g1Hg2, Expand[Quatermult[Quatermult[g2 /. Nsub, g1H[[i]] /. Nsub],
       Quaterconj[g2] /. Nsub] /. sub // FullSimplify]];
  AppendTo[g3g1Hg3, Expand[Quatermult[Quatermult[g3 /. Nsub, g1H[[i]] /. Nsub],
       Quaterconj[g3] /. Nsub] /. sub // FullSimplify]];
  AppendTo[g4g1Hg4, Expand[Quatermult[Quatermult[g4 /. Nsub, g1H[[i]] /. Nsub],
       Quaterconj[g4] /. Nsub] /. sub // FullSimplify]];
 ];

g1g2Hg1 = {};
g2g2Hg2 = {};
g3g2Hg3 = {};
g4g2Hg4 = {};
For[i = 0, i < Length[g1H], i++;
  AppendTo[g1g2Hg1, Expand[Quatermult[Quatermult[g1 /. Nsub, g2H[[i]] /. Nsub],
       Quaterconj[g1] /. Nsub] /. sub // FullSimplify]];
  AppendTo[g2g2Hg2, Expand[Quatermult[Quatermult[g2 /. Nsub, g2H[[i]] /. Nsub],
       Quaterconj[g2] /. Nsub] /. sub // FullSimplify]];
  AppendTo[g3g2Hg3, Expand[Quatermult[Quatermult[g3 /. Nsub, g2H[[i]] /. Nsub],
       Quaterconj[g3] /. Nsub] /. sub // FullSimplify]];
  AppendTo[g4g2Hg4, Expand[Quatermult[Quatermult[g4 /. Nsub, g2H[[i]] /. Nsub],
       Quaterconj[g4] /. Nsub] /. sub // FullSimplify]];
 ];

g1g3Hg1  = {};
g2g3Hg2 = {};
g3g3Hg3 = {};
g4g3Hg4 = {};
For[i = 0, i < Length[g1H], i++;
```

```
  AppendTo[g1g3Hg1, Expand[Quatermult[Quatermult[g1 /. Nsub, g3H[[i]] /. Nsub],
       Quaterconj[g1] /. Nsub] /. sub // FullSimplify]];
  AppendTo[g2g3Hg2, Expand[Quatermult[Quatermult[g2 /. Nsub, g3H[[i]] /. Nsub],
       Quaterconj[g2] /. Nsub] /. sub // FullSimplify]];
  AppendTo[g3g3Hg3, Expand[Quatermult[Quatermult[g3 /. Nsub, g3H[[i]] /. Nsub],
       Quaterconj[g3] /. Nsub] /. sub // FullSimplify]];
  AppendTo[g4g3Hg4, Expand[Quatermult[Quatermult[g4 /. Nsub, g3H[[i]] /. Nsub],
       Quaterconj[g4] /. Nsub] /. sub // FullSimplify]];
 ];

g1g4Hg1 = {};
g2g4Hg2 = {};
g3g4Hg3 = {};
g4g4Hg4 = {};
For[i = 0, i < Length[g1H], i++;
  AppendTo[g1g4Hg1, Expand[Quatermult[Quatermult[g1 /. Nsub, g4H[[i]] /. Nsub],
       Quaterconj[g1] /. Nsub] /. sub // FullSimplify]];
  AppendTo[g2g4Hg2, Expand[Quatermult[Quatermult[g2 /. Nsub, g4H[[i]] /. Nsub],
       Quaterconj[g2] /. Nsub] /. sub // FullSimplify]];
  AppendTo[g3g4Hg3, Expand[Quatermult[Quatermult[g3 /. Nsub, g4H[[i]] /. Nsub],
       Quaterconj[g3] /. Nsub] /. sub // FullSimplify]];
  AppendTo[g4g4Hg4, Expand[Quatermult[Quatermult[g4 /. Nsub, g4H[[i]] /. Nsub],
       Quaterconj[g4] /. Nsub] /. sub // FullSimplify]];
 ];


twentyfive = {H , g1Hg1, g2Hg2, g3Hg3, g4Hg4, g1H, g1g1Hg1, g2g1Hg2,
   g3g1Hg3, g4g1Hg4, g2H, g1g2Hg1, g2g2Hg2, g3g2Hg3, g4g2Hg4, g3H, g1g3Hg1,
   g2g3Hg2, g3g3Hg3, g4g3Hg4, g4H, g1g4Hg1, g2g4Hg2, g3g4Hg3, g4g4Hg4};
```

```
In[3991]:= (* All 25 cosets are represent different 24-cells. *)
list = {};
count = 0;
For[i = 0, i < 25, i++;
  For[j = 0, j < 25, j++;
   If[Length[Intersection[twentyfive[[i]], twentyfive[[j]]]] == 24,
    count = count + 1, Nothing];
   AppendTo[list, Length[Intersection[twentyfive[[i]], twentyfive[[j]]]]];
  ];
 ];
count (* Intersection[twentyfive[[i]],
 twentyfive[[i]] appears, which accounts for the number 25. *)

Out[3994]= 25
```

```
In[4059]:= (* Definition of the right cosets of H*)
rightg1Hg1 = {};
rightg2Hg2 = {};
rightg3Hg3 = {};
rightg4Hg4 = {};
For[i = 0, i < 24, i++;
  y = tetrahedralgroup2[[i]] /. Nsub;
  AppendTo[rightg1Hg1,
   Expand[Quatermult[Quatermult[Quaterconj[g1] /. Nsub, y], g1 /. Nsub] /. sub //
     FullSimplify]];
```

```
  AppendTo[rightg2Hg2, Expand[Quatermult[Quatermult[Quaterconj[g2] /. Nsub, y],
       g2 /. Nsub] /. sub // FullSimplify]];
  AppendTo[rightg3Hg3, Expand[Quatermult[Quatermult[Quaterconj[g3] /. Nsub, y],
       g3 /. Nsub] /. sub // FullSimplify]];
  AppendTo[rightg4Hg4, Expand[Quatermult[Quatermult[Quaterconj[g4] /. Nsub, y],
       g4 /. Nsub] /. sub // FullSimplify]];
 ];

Hg1 = {};
Hg2 = {};
Hg3 = {};
Hg4 = {};
For[i = 0, i < 24, i++;
  y = tetrahedralgroup[[i]] /. Nsub;
  AppendTo[Hg1, Expand[Expand[Quatermult[y, g1 /. Nsub]] /. sub // FullSimplify]];
  AppendTo[Hg2, Expand[Expand[Quatermult[y, g2 /. Nsub]] /. sub // FullSimplify]];
  AppendTo[Hg3, Expand[Expand[Quatermult[y, g3 /. Nsub]] /. sub // FullSimplify]];
  AppendTo[Hg4, Expand[Expand[Quatermult[y, g4 /. Nsub]] /. sub // FullSimplify]];
 ];
Length[DeleteDuplicates[Union[Union[Union[Union[Hg1, Hg2], Hg3], Hg4], H]]]

g1Hg1g1 = {};
g2Hg1g2 = {};
g3Hg1g3 = {};
g4Hg1g4 = {};
For[i = 0, i < Length[g1H], i++;
  AppendTo[g1Hg1g1,
   Expand[Quatermult[Quatermult[Quaterconj[g1] /. Nsub, Hg1[[i]] /. Nsub],
       g1 /. Nsub] /. sub // FullSimplify]];
  AppendTo[g2Hg1g2, Expand[Quatermult[Quatermult[Quaterconj[g2] /. Nsub,
        Hg1[[i]] /. Nsub], g2 /. Nsub] /. sub // FullSimplify]];
  AppendTo[g3Hg1g3, Expand[Quatermult[Quatermult[Quaterconj[g3] /. Nsub,
        Hg1[[i]] /. Nsub], g3 /. Nsub] /. sub // FullSimplify]];
  AppendTo[g4Hg1g4, Expand[Quatermult[Quatermult[Quaterconj[g4] /. Nsub,
        Hg1[[i]] /. Nsub], g4 /. Nsub] /. sub // FullSimplify]];
 ];

g1Hg2g1 = {};
g2Hg2g2 = {};
g3Hg2g3 = {};
g4Hg2g4 = {};
For[i = 0, i < Length[g1H], i++;
  AppendTo[g1Hg2g1,
   Expand[Quatermult[Quatermult[Quaterconj[g1] /. Nsub, Hg2[[i]] /. Nsub],
       g1 /. Nsub] /. sub // FullSimplify]];
  AppendTo[g2Hg2g2, Expand[Quatermult[Quatermult[Quaterconj[g2] /. Nsub,
        Hg2[[i]] /. Nsub], g2 /. Nsub] /. sub // FullSimplify]];
  AppendTo[g3Hg2g3, Expand[Quatermult[Quatermult[Quaterconj[g3] /. Nsub,
        Hg2[[i]] /. Nsub], g3 /. Nsub] /. sub // FullSimplify]];
  AppendTo[g4Hg2g4, Expand[Quatermult[Quatermult[Quaterconj[g4] /. Nsub,
        Hg2[[i]] /. Nsub], g4 /. Nsub] /. sub // FullSimplify]];
 ];

g1Hg3g1 = {};
g2Hg3g2 = {};
```

```
g3Hg3g3 = {};
g4Hg3g4 = {};
For[i = 0, i < Length[g1H], i++;
  AppendTo[g1Hg3g1,
   Expand[Quatermult[Quatermult[Quaterconj[g1] /. Nsub, Hg3[[i]] /. Nsub],
       g1 /. Nsub] /. sub // FullSimplify]];
  AppendTo[g2Hg3g2, Expand[Quatermult[Quatermult[Quaterconj[g2] /. Nsub,
        Hg3[[i]] /. Nsub], g2 /. Nsub] /. sub // FullSimplify]];
  AppendTo[g3Hg3g3, Expand[Quatermult[Quatermult[Quaterconj[g3] /. Nsub,
        Hg3[[i]] /. Nsub], g3 /. Nsub] /. sub // FullSimplify]];
  AppendTo[g4Hg3g4, Expand[Quatermult[Quatermult[Quaterconj[g4] /. Nsub,
          Hg3[[i]] /. Nsub] g4 /. Nsub] /. sub // FullSimplify]];
 ];

g1Hg4g1 = {};
g2Hg4g2 = {};
g3Hg4g3 = {};
g4Hg4g4 = {};
For[i = 0, i < Length[g1H], i++;
  AppendTo[g1Hg4g1,
   Expand[Quatermult[Quatermult[Quaterconj[g1] /. Nsub, Hg4[[i]] /. Nsub],
       g1 /. Nsub] /. sub // FullSimplify]];
  AppendTo[g2Hg4g2, Expand[Quatermult[Quatermult[Quaterconj[g2] /. Nsub,
        Hg4[[i]] /. Nsub], g2 /. Nsub] /. sub // FullSimplify]];
  AppendTo[g3Hg4g3, Expand[Quatermult[Quatermult[Quaterconj[g3] /. Nsub,
        Hg4[[i]] /. Nsub], g3 /. Nsub] /. sub // FullSimplify]];
  AppendTo[g4Hg4g4, Expand[Quatermult[Quatermult[Quaterconj[g4] /. Nsub,
        Hg4[[i]] /. Nsub], g4 /. Nsub] /. sub // FullSimplify]];
 ];
```

Out[4069]= 120

```
In[4105]:= twentyfive2 = {H, rightg1Hg1, rightg2Hg2, rightg3Hg3, rightg4Hg4, Hg1, g1Hg1g1,
   g2Hg1g2, g3Hg1g3, g4Hg1g4, Hg2, g1Hg2g1, g2Hg2g2, g3Hg2g3, g4Hg2g4, Hg3,
   g1Hg3g1, g2Hg3g2, g3Hg3g3, g4Hg3g4, Hg4, g1Hg4g1, g2Hg4g2, g3Hg4g3, g4Hg4g4};
```

```
In[4106]:= (* Counting the number of times the left and right 24-cells overlap *)
length = {};
overlap = {};
For[i = 0, i < 25, i++;
  For[j = 0, j < 25, j++;
   AppendTo[length, Length[Intersection[twentyfive[[i]], twentyfive2[[j]]]]];
   If[ Length[Intersection[twentyfive[[i]], twentyfive2[[j]]]] == 24,
    AppendTo[overlap, {i, j}], Nothing];
  ];
 ];
Count[length, 24]
overlap
```

Out[4109]= 24

Out[4110]= {{1, 1}, {2, 5}, {3, 4}, {4, 3}, {5, 2}, {6, 10}, {7, 9}, {8, 8}, {9, 7},
{10, 6}, {11, 14}, {12, 13}, {13, 12}, {14, 11}, {15, 15}, {16, 18}, {17, 17},
{18, 16}, {20, 19}, {21, 22}, {22, 21}, {23, 25}, {24, 24}, {25, 23}}

In[4111]:=
```
(* 5 24-cells at each vertex of the 600-cell *)
dist1 = {};
For[i = 0, i < 120, i++;
  id = {1, 0, 0, 0};
  x = icosians[[i]] /. Nsub;
  dist = Expand[Power[x[[1]] - id[[1]], 2] + Power[x[[2]] - id[[2]], 2] +
     Power[x[[3]] - id[[3]], 2] + Power[x[[4]] - id[[4]], 2]];
  If [dist == 1, AppendTo[dist1, x /. sub], Nothing];
 ];
Length[dist1]
```

Out[4113]= 20

In[4033]:=
```
dist1 = Sort[dist1];
list1 = {};
list2 = {};
distQ2Q3 = {};
distQ2Q4 = {};
lengthQ2Q3 = {};
lengthQ2Q4 = {};
For[i = 0, i < Length[dist1], i++;
  For[j = 0, j < 120, j++;
   y = dist1[[i]] /. Nsub;
   x = icosians[[j]] /. Nsub;
   dist = Expand[Power[x[[1]] - y[[1]], 2] + Power[x[[2]] - y[[2]], 2] +
      Power[x[[3]] - y[[3]], 2] + Power[x[[4]] - y[[4]], 2]];
   If [dist == 1, AppendTo[list1, x /. sub], Nothing];
   If[dist == 2, AppendTo[list2, x /. sub], Nothing];
  ];
  AppendTo[distQ2Q3, Intersection[dist1, list1]];
  (* distance 1 from Q2 and 1 from the quaternion 1 *)
  AppendTo[distQ2Q4, Intersection[dist1, list2]];
  (* distance 2 from Q2 and dist 1 from the quaternion 1 *)
  AppendTo[lengthQ2Q3, Length[distQ2Q3[[i]]]];
  AppendTo[lengthQ2Q4, Length[distQ2Q4[[i]]]];
  list1 = {};
  list2 = {};
 ];
lengthQ2Q3
lengthQ2Q4
```

Out[4041]= {6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6}

Out[4042]= {6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6}

In[4114]:=
```
(* Compute neighbours fixed Q4 that lie 1 away from both Q1 and Q2 as well*)
list = {};
Q3Q5 = {};
list2 = {};
For[i = 0, i < Length[dist1], i++;
  z = dist1[[i]];
  For[j = 0, j < Length[distQ2Q4[[1]]], j++;
   y = distQ2Q4[[i]][[j]];
   For[k = 0, k < 120, k++;
    x = icosians[[k]];
    dist = Expand[Power[x[[1]] - y[[1]], 2] + Power[x[[2]] - y[[2]], 2] +
       Power[x[[3]] - y[[3]], 2] + Power[x[[4]] - y[[4]], 2]];
    If[dist == 1, AppendTo[list, x /. sub], Nothing];
   ];
   AppendTo[list2, list];
   list = {};
  ];
  AppendTo[Q3Q5, list2];
  list2 = {};
 ];
Length[Intersection[Intersection[Q3Q5[[1]][[1]], distQ2Q3[[1]]]]]
```

Out[4118]= 2

In[4119]:=
```
(* Example construction 24-cell by choice of Q_2,
Q_4, Q_6 yields two different 24-cells *)
q = distQ2Q4[[1]];
distsqrt2 = {};
list = {};
octato24 = {};
For[i = 0, i < 6, i++;
  For[j = 0, j < 6, j++;
   For[k = 0, k < 6, k++;
    x = q[[i]] /. Nsub;
    y = q[[j]] /. Nsub;
    z = q[[k]] /. Nsub;
    dist1 = Expand[Power[x[[1]] - y[[1]], 2] + Power[x[[2]] - y[[2]], 2] +
       Power[x[[3]] - y[[3]], 2] + Power[x[[4]] - y[[4]], 2]];
    dist2 = Expand[Power[x[[1]] - z[[1]], 2] + Power[x[[2]] - z[[2]], 2] +
       Power[x[[3]] - z[[3]], 2] + Power[x[[4]] - z[[4]], 2]];
    dist3 = Expand[Power[z[[1]] - y[[1]], 2] + Power[z[[2]] - y[[2]], 2] +
       Power[z[[3]] - y[[3]], 2] + Power[z[[4]] - y[[4]], 2]];
    If[dist1 == dist2 == dist3 == 2, AppendTo[octato24, {i, j, k}], Nothing];
   ];
  ];
 ];
Length[octato24]
octato24
```

Out[4124]= 12

Out[4125]=
```
{{1, 3, 4}, {1, 4, 3}, {2, 5, 6}, {2, 6, 5}, {3, 1, 4}, {3, 4, 1},
 {4, 1, 3}, {4, 3, 1}, {5, 2, 6}, {5, 6, 2}, {6, 2, 5}, {6, 5, 2}}
```

In[4096]:=
```
(* Maximally 10 ways to inscribe 5 disjunt 24-cells in a 600-cell. *)
count = 0;
For[i = 0, i < 25, i++;
  For[j = 0, j < 25, j++;
   For[k = 0, k < 25, k++;
    For[l = 0, l < 25, l++;
     For[m = 0, m < 25, m++;
      If[Length[
         DeleteDuplicates[Union[twentyfive[[i]], twentyfive[[j]], twentyfive[[k]],
           twentyfive[[l]], twentyfive[[m]]]]] == 120, count = count + 1, Nothing];
     ];
    ];
   ];
  ];
 ];
count
```

Out[4098]= 1200

```
In[479]:= Sort[H][[16 ;; 23]]
Sort[g1Hg1][[16 ;; 23]]
Sort[g2Hg2][[16 ;; 23]]
Sort[g3Hg3][[16 ;; 23]]
Sort[g4Hg4][[16 ;; 23]]

Sort[g1H][[10 ;; 12]]
Sort[g1g2Hg1][[10 ;; 12]]
Sort[g2g1Hg2][[10 ;; 12]]
Sort[g3g1Hg3][[10 ;; 12]]
Sort[g4g1Hg4][[10 ;; 12]]

Sort[g2H][[10 ;; 12]];
Sort[g1g2Hg1][[10 ;; 12]];
Sort[g2g2Hg2][[10 ;; 12]];
Sort[g3g2Hg3][[10 ;; 12]];
Sort[g4g2Hg4][[10 ;; 12]];

Sort[g3H][[10 ;; 12]];
Sort[g1g3Hg1][[10 ;; 12]];
Sort[g2g3Hg2][[10 ;; 12]];
Sort[g3g3Hg3][[10 ;; 12]];
Sort[g4g3Hg4][[10 ;; 12]];

Sort[g2H][[10 ;; 12]];
Sort[g1g4Hg1][[10 ;; 12]];
Sort[g2g4Hg2][[10 ;; 12]];
Sort[g3g4Hg3][[10 ;; 12]];
Sort[g4g4Hg4][[10 ;; 12]];
```

Out[479]= $\{\{\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\}, \{\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\}, \{\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\}, \{\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}, \{\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\}, \{\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\}, \{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\}, \{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}\}$

Out[480]= $\{\{\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}, \{\frac{1}{2}, 0, -\frac{phi}{2}, -\frac{1}{2}+\frac{phi}{2}\}, \{\frac{1}{2}, 0, \frac{phi}{2}, \frac{1}{2}-\frac{phi}{2}\}, \{\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\}, \{\frac{1}{2}, \frac{1}{2}-\frac{phi}{2}, 0, -\frac{phi}{2}\}, \{\frac{1}{2}, -\frac{1}{2}+\frac{phi}{2}, 0, \frac{phi}{2}\}, \{\frac{1}{2}, -\frac{phi}{2}, \frac{1}{2}-\frac{phi}{2}, 0\}, \{\frac{1}{2}, \frac{phi}{2}, -\frac{1}{2}+\frac{phi}{2}, 0\}\}$

Out[481]= $\{\{\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\}, \{\frac{1}{2}, 0, -\frac{phi}{2}, \frac{1}{2}-\frac{phi}{2}\}, \{\frac{1}{2}, 0, \frac{phi}{2}, -\frac{1}{2}+\frac{phi}{2}\}, \{\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\}, \{\frac{1}{2}, \frac{1}{2}-\frac{phi}{2}, 0, \frac{phi}{2}\}, \{\frac{1}{2}, -\frac{1}{2}+\frac{phi}{2}, 0, -\frac{phi}{2}\}, \{\frac{1}{2}, -\frac{phi}{2}, \frac{1}{2}-\frac{phi}{2}, 0\}, \{\frac{1}{2}, \frac{phi}{2}, -\frac{1}{2}+\frac{phi}{2}, 0\}\}$

Out[482]= $\left\{\left\{\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right\}, \left\{\frac{1}{2}, 0, -\frac{phi}{2}, \frac{1}{2}-\frac{phi}{2}\right\}, \left\{\frac{1}{2}, 0, \frac{phi}{2}, -\frac{1}{2}+\frac{phi}{2}\right\}, \left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\}, \left\{\frac{1}{2}, \frac{1}{2}-\frac{phi}{2}, 0, -\frac{phi}{2}\right\}, \left\{\frac{1}{2}, -\frac{1}{2}+\frac{phi}{2}, 0, \frac{phi}{2}\right\}, \left\{\frac{1}{2}, -\frac{phi}{2}, -\frac{1}{2}+\frac{phi}{2}, 0\right\}, \left\{\frac{1}{2}, \frac{phi}{2}, \frac{1}{2}-\frac{phi}{2}, 0\right\}\right\}$

Out[483]= $\left\{\left\{\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right\}, \left\{\frac{1}{2}, 0, -\frac{phi}{2}, -\frac{1}{2}+\frac{phi}{2}\right\}, \left\{\frac{1}{2}, 0, \frac{phi}{2}, \frac{1}{2}-\frac{phi}{2}\right\}, \left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}, \left\{\frac{1}{2}, \frac{1}{2}-\frac{phi}{2}, 0, \frac{phi}{2}\right\}, \left\{\frac{1}{2}, -\frac{1}{2}+\frac{phi}{2}, 0, -\frac{phi}{2}\right\}, \left\{\frac{1}{2}, -\frac{phi}{2}, -\frac{1}{2}+\frac{phi}{2}, 0\right\}, \left\{\frac{1}{2}, \frac{phi}{2}, \frac{1}{2}-\frac{phi}{2}, 0\right\}\right\}$

Out[484]= $\left\{\left\{\frac{1}{2}, 0, -\frac{phi}{2}, \frac{1}{2}-\frac{phi}{2}\right\}, \left\{\frac{1}{2}, -\frac{1}{2}+\frac{phi}{2}, 0, -\frac{phi}{2}\right\}, \left\{\frac{1}{2}, \frac{phi}{2}, \frac{1}{2}-\frac{phi}{2}, 0\right\}\right\}$

Out[485]= $\left\{\left\{\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right\}, \left\{\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\}, \left\{\frac{1}{2}, -\frac{1}{2}+\frac{phi}{2}, 0, -\frac{phi}{2}\right\}\right\}$

Out[486]= $\left\{\left\{\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}, \left\{\frac{1}{2}, 0, \frac{phi}{2}, \frac{1}{2}-\frac{phi}{2}\right\}, \left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}\right\}$

Out[487]= $\left\{\left\{\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\}, \left\{\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}, \left\{\frac{1}{2}, -\frac{phi}{2}, \frac{1}{2}-\frac{phi}{2}, 0\right\}\right\}$

Out[488]= $\left\{\left\{\frac{1}{2}, 0, -\frac{phi}{2}, \frac{1}{2}-\frac{phi}{2}\right\}, \left\{\frac{1}{2}, \frac{1}{2}-\frac{phi}{2}, 0, -\frac{phi}{2}\right\}, \left\{\frac{1}{2}, -\frac{phi}{2}, \frac{1}{2}-\frac{phi}{2}, 0\right\}\right\}$

```
In[470]:= (* Comparison five left and right cosets *)
Sort[H][[16 ;; 23]]

Sort[g1H][[10 ;; 12]]
Sort[g2H][[10 ;; 12]]
Sort[g3H][[10 ;; 12]]
Sort[g4H][[10 ;; 12]]

Sort[Hg1][[10 ;; 12]]
Sort[Hg2][[10 ;; 12]]
Sort[Hg3][[10 ;; 12]]
Sort[Hg4][[10 ;; 12]]
```

Out[470]= $\left\{\left\{\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right\}, \left\{\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right\}, \left\{\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\}, \left\{\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}, \left\{\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right\}, \left\{\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right\}, \left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\}, \left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}\right\}$

Out[471]= $\left\{\left\{\frac{1}{2}, 0, -\frac{phi}{2}, \frac{1}{2}-\frac{phi}{2}\right\}, \left\{\frac{1}{2}, -\frac{1}{2}+\frac{phi}{2}, 0, -\frac{phi}{2}\right\}, \left\{\frac{1}{2}, \frac{phi}{2}, \frac{1}{2}-\frac{phi}{2}, 0\right\}\right\}$

Out[472]= $\left\{\left\{\frac{1}{2}, 0, \frac{phi}{2}, \frac{1}{2}-\frac{phi}{2}\right\}, \left\{\frac{1}{2}, \frac{1}{2}-\frac{phi}{2}, 0, -\frac{phi}{2}\right\}, \left\{\frac{1}{2}, -\frac{phi}{2}, -\frac{1}{2}+\frac{phi}{2}, 0\right\}\right\}$

Out[473]= $\left\{\left\{\frac{1}{2}, 0, -\frac{phi}{2}, -\frac{1}{2}+\frac{phi}{2}\right\}, \left\{\frac{1}{2}, \frac{1}{2}-\frac{phi}{2}, 0, \frac{phi}{2}\right\}, \left\{\frac{1}{2}, -\frac{phi}{2}, \frac{1}{2}-\frac{phi}{2}, 0\right\}\right\}$

Out[474]= $\left\{\left\{\frac{1}{2}, 0, \frac{phi}{2}, -\frac{1}{2}+\frac{phi}{2}\right\}, \left\{\frac{1}{2}, -\frac{1}{2}+\frac{phi}{2}, 0, \frac{phi}{2}\right\}, \left\{\frac{1}{2}, \frac{phi}{2}, -\frac{1}{2}+\frac{phi}{2}, 0\right\}\right\}$

Out[475]= $\left\{\left\{\frac{1}{2}, 0, -\frac{phi}{2}, \frac{1}{2}-\frac{phi}{2}\right\}, \left\{\frac{1}{2}, \frac{1}{2}-\frac{phi}{2}, 0, -\frac{phi}{2}\right\}, \left\{\frac{1}{2}, -\frac{phi}{2}, \frac{1}{2}-\frac{phi}{2}, 0\right\}\right\}$

Out[476]= $\left\{\left\{\frac{1}{2}, 0, \frac{phi}{2}, \frac{1}{2}-\frac{phi}{2}\right\}, \left\{\frac{1}{2}, -\frac{1}{2}+\frac{phi}{2}, 0, -\frac{phi}{2}\right\}, \left\{\frac{1}{2}, \frac{phi}{2}, -\frac{1}{2}+\frac{phi}{2}, 0\right\}\right\}$

Out[477]= $\left\{\left\{\frac{1}{2}, 0, -\frac{phi}{2}, -\frac{1}{2}+\frac{phi}{2}\right\}, \left\{\frac{1}{2}, -\frac{1}{2}+\frac{phi}{2}, 0, \frac{phi}{2}\right\}, \left\{\frac{1}{2}, \frac{phi}{2}, \frac{1}{2}-\frac{phi}{2}, 0\right\}\right\}$

Out[478]= $\left\{\left\{\frac{1}{2}, 0, \frac{phi}{2}, -\frac{1}{2}+\frac{phi}{2}\right\}, \left\{\frac{1}{2}, \frac{1}{2}-\frac{phi}{2}, 0, \frac{phi}{2}\right\}, \left\{\frac{1}{2}, -\frac{phi}{2}, -\frac{1}{2}+\frac{phi}{2}, 0\right\}\right\}$

B.2.2 Decompositions 600-cell

In[4582]:=
```
(* COPY TO RUN ENTIRE DOCUMENT *)

(* Construction 2I *)
{one, aai, jay, kay} = IdentityMatrix[4];
Nsub = phi → (Sqrt[5] + 1) / 2;
sub = Sqrt[5] → 2 phi - 1;

Quatermult[x_, y_] := Module[{
   xQ = {{x[[1]] + I x[[2]], I x[[4]] + x[[3]]}, {I x[[4]] - x[[3]], x[[1]] - I x[[2]]}},
   yQ = {{y[[1]] + I y[[2]], I y[[4]] + y[[3]]}, {I y[[4]] - y[[3]], y[[1]] - I y[[2]]}}
  },
  zQ = xQ.yQ;
  Simplify[{Re[zQ[[1, 1]]], Im[zQ[[1, 1]]], Re[zQ[[1, 2]]], Im[zQ[[1, 2]]]}]
 ]
Quaterconj[x_] := Simplify[{x[[1]], -x[[2]], -x[[3]], -x[[4]]}]
icosians = Expand[{one + aai + jay + kay, -aai + jay (phi - 1) + kay phi} / 2];
done = 0;

length = Length[icosians];
For[i = 1, done < length, i++,
 For[j = done + 1, j ≤ length, j++,
  x = icosians[[j]];
  For[k = 1, k ≤ j, k++,
   y = icosians[[k]];
   z = Expand[Expand[Quatermult[x, y] /. Nsub] /. sub];
   AppendTo[icosians, z];
   z = Expand[Expand[Quatermult[y, x] /. Nsub] /. sub];
   AppendTo[icosians, z]
  ]
 ];
 icosians = DeleteDuplicates[icosians];
 done = length;
 length = Length[icosians];
 If[length > 1136, Break[]]
]
length;
```

In[4592]:=
```
(* Left and right cosets H = 2T in 2I *)

(* Definition left cosets *)

H = tetrahedralgroup;
g1 = Expand[icosiansphinegconj[[1]] /. sub // FullSimplify];
g2 = Expand[Quatermult[g1 /. Nsub, g1 /. Nsub] /. sub // FullSimplify];
g3  = Expand[Quatermult[g1 /. Nsub, g2 /. Nsub] /. sub // FullSimplify];
g4 = Expand[Quatermult[g2 /. Nsub, g2 /. Nsub] /. sub // FullSimplify];
Quatermult[g1 /. Nsub, g4 /. Nsub];

g1Hg1 = {};
g2Hg2 = {};
g3Hg3 = {};
g4Hg4 = {};
For[i = 0, i < 24, i++;
  y = tetrahedralgroup[[i]] /. Nsub;
  AppendTo[g1Hg1,
```

```
    Expand[Quatermult[Quatermult[g1 /. Nsub, y], Quaterconj[g1] /. Nsub] /. sub //
      FullSimplify]];
  AppendTo[g2Hg2, Expand[Quatermult[Quatermult[g2 /. Nsub, y],
       Quaterconj[g2] /. Nsub] /. sub // FullSimplify]];
  AppendTo[g3Hg3, Expand[Quatermult[Quatermult[g3 /. Nsub, y],
       Quaterconj[g3] /. Nsub] /. sub // FullSimplify]];
  AppendTo[g4Hg4, Expand[Quatermult[Quatermult[g4 /. Nsub, y],
       Quaterconj[g4] /. Nsub] /. sub // FullSimplify]];
 ];

g1H = {};
g2H = {};
g3H = {};
g4H = {};
For[i = 0, i < 24, i++;
  y = tetrahedralgroup[[i]] /. Nsub;
  AppendTo[g1H, Expand[Quatermult[g1 /. Nsub, y] /. sub // FullSimplify]];
  AppendTo[g2H, Expand[Quatermult[g2 /. Nsub, y] /. sub // FullSimplify]];
  AppendTo[g3H, Expand[Quatermult[g3 /. Nsub, y] /. sub // FullSimplify]];
  AppendTo[g4H, Expand[Quatermult[g4 /. Nsub, y] /. sub // FullSimplify]];
 ];

g1g1Hg1 = {};
g2g1Hg2 = {};
g3g1Hg3 = {};
g4g1Hg4 = {};
For[i = 0, i < Length[g1H], i++;
  AppendTo[g1g1Hg1, Expand[Quatermult[Quatermult[g1 /. Nsub, g1H[[i]] /. Nsub],
       Quaterconj[g1] /. Nsub] /. sub // FullSimplify]];
  AppendTo[g2g1Hg2, Expand[Quatermult[Quatermult[g2 /. Nsub, g1H[[i]] /. Nsub],
       Quaterconj[g2] /. Nsub] /. sub // FullSimplify]];
  AppendTo[g3g1Hg3, Expand[Quatermult[Quatermult[g3 /. Nsub, g1H[[i]] /. Nsub],
       Quaterconj[g3] /. Nsub] /. sub // FullSimplify]];
  AppendTo[g4g1Hg4, Expand[Quatermult[Quatermult[g4 /. Nsub, g1H[[i]] /. Nsub],
       Quaterconj[g4] /. Nsub] /. sub // FullSimplify]];
 ];

g1g2Hg1 = {};
g2g2Hg2 = {};
g3g2Hg3 = {};
g4g2Hg4 = {};
For[i = 0, i < Length[g1H], i++;
  AppendTo[g1g2Hg1, Expand[Quatermult[Quatermult[g1 /. Nsub, g2H[[i]] /. Nsub],
       Quaterconj[g1] /. Nsub] /. sub // FullSimplify]];
  AppendTo[g2g2Hg2, Expand[Quatermult[Quatermult[g2 /. Nsub, g2H[[i]] /. Nsub],
       Quaterconj[g2] /. Nsub] /. sub // FullSimplify]];
  AppendTo[g3g2Hg3, Expand[Quatermult[Quatermult[g3 /. Nsub, g2H[[i]] /. Nsub],
       Quaterconj[g3] /. Nsub] /. sub // FullSimplify]];
  AppendTo[g4g2Hg4, Expand[Quatermult[Quatermult[g4 /. Nsub, g2H[[i]] /. Nsub],
       Quaterconj[g4] /. Nsub] /. sub // FullSimplify]];
 ];

g1g3Hg1  = {};
g2g3Hg2 = {};
g3g3Hg3 = {};
```

```
g4g3Hg4 = {};
For[i = 0, i < Length[g1H], i++;
  AppendTo[g1g3Hg1, Expand[Quatermult[Quatermult[g1 /. Nsub, g3H[[i]] /. Nsub],
       Quaterconj[g1] /. Nsub] /. sub // FullSimplify]];
  AppendTo[g2g3Hg2, Expand[Quatermult[Quatermult[g2 /. Nsub, g3H[[i]] /. Nsub],
       Quaterconj[g2] /. Nsub] /. sub // FullSimplify]];
  AppendTo[g3g3Hg3, Expand[Quatermult[Quatermult[g3 /. Nsub, g3H[[i]] /. Nsub],
       Quaterconj[g3] /. Nsub] /. sub // FullSimplify]];
  AppendTo[g4g3Hg4, Expand[Quatermult[Quatermult[g4 /. Nsub, g3H[[i]] /. Nsub],
       Quaterconj[g4] /. Nsub] /. sub // FullSimplify]];
 ];

g1g4Hg1 = {};
g2g4Hg2 = {};
g3g4Hg3 = {};
g4g4Hg4 = {};
For[i = 0, i < Length[g1H], i++;
  AppendTo[g1g4Hg1, Expand[Quatermult[Quatermult[g1 /. Nsub, g4H[[i]] /. Nsub],
       Quaterconj[g1] /. Nsub] /. sub // FullSimplify]];
  AppendTo[g2g4Hg2, Expand[Quatermult[Quatermult[g2 /. Nsub, g4H[[i]] /. Nsub],
       Quaterconj[g2] /. Nsub] /. sub // FullSimplify]];
  AppendTo[g3g4Hg3, Expand[Quatermult[Quatermult[g3 /. Nsub, g4H[[i]] /. Nsub],
       Quaterconj[g3] /. Nsub] /. sub // FullSimplify]];
  AppendTo[g4g4Hg4, Expand[Quatermult[Quatermult[g4 /. Nsub, g4H[[i]] /. Nsub],
       Quaterconj[g4] /. Nsub] /. sub // FullSimplify]];
 ];


twentyfive = {H , g1Hg1, g2Hg2, g3Hg3, g4Hg4, g1H, g1g1Hg1, g2g1Hg2,
   g3g1Hg3, g4g1Hg4, g2H, g1g2Hg1, g2g2Hg2, g3g2Hg3, g4g2Hg4, g3H, g1g3Hg1,
   g2g3Hg2, g3g3Hg3, g4g3Hg4, g4H, g1g4Hg1, g2g4Hg2, g3g4Hg3, g4g4Hg4};

(* Definition of the right cosets of H*)
rightg1Hg1 = {};
rightg2Hg2 = {};
rightg3Hg3 = {};
rightg4Hg4 = {};
For[i = 0, i < 24, i++;
  y = tetrahedralgroup[[i]] /. Nsub;
  AppendTo[rightg1Hg1,
   Expand[Quatermult[Quatermult[Quaterconj[g1] /. Nsub, y], g1 /. Nsub] /. sub //
     FullSimplify]];
  AppendTo[rightg2Hg2, Expand[Quatermult[Quatermult[Quaterconj[g2] /. Nsub, y],
       g2 /. Nsub] /. sub // FullSimplify]];
  AppendTo[rightg3Hg3, Expand[Quatermult[Quatermult[Quaterconj[g3] /. Nsub, y],
       g3 /. Nsub] /. sub // FullSimplify]];
  AppendTo[rightg4Hg4, Expand[Quatermult[Quatermult[Quaterconj[g4] /. Nsub, y],
       g4 /. Nsub] /. sub // FullSimplify]];
 ];

Hg1 = {};
Hg2 = {};
Hg3 = {};
Hg4 = {};
For[i = 0, i < 24, i++;
  y = tetrahedralgroup[[i]] /. Nsub;
```

```
  AppendTo[Hg1, Expand[Expand[Quatermult[y, g1 /. Nsub]] /. sub // FullSimplify]];
  AppendTo[Hg2, Expand[Expand[Quatermult[y, g2 /. Nsub]] /. sub // FullSimplify]];
  AppendTo[Hg3, Expand[Expand[Quatermult[y, g3 /. Nsub]] /. sub // FullSimplify]];
  AppendTo[Hg4, Expand[Expand[Quatermult[y, g4 /. Nsub]] /. sub // FullSimplify]];
 ];
Length[DeleteDuplicates[Union[Union[Union[Union[Hg1, Hg2], Hg3], Hg4], H]]]

g1Hg1g1 = {};
g2Hg1g2 = {};
g3Hg1g3 = {};
g4Hg1g4 = {};
For[i = 0, i < Length[g1H], i++;
  AppendTo[g1Hg1g1,
   Expand[Quatermult[Quatermult[Quaterconj[g1] /. Nsub, Hg1[[i]] /. Nsub],
      g1 /. Nsub] /. sub // FullSimplify]];
  AppendTo[g2Hg1g2, Expand[Quatermult[Quatermult[Quaterconj[g2] /. Nsub,
       Hg1[[i]] /. Nsub], g2 /. Nsub] /. sub // FullSimplify]];
  AppendTo[g3Hg1g3, Expand[Quatermult[Quatermult[Quaterconj[g3] /. Nsub,
       Hg1[[i]] /. Nsub], g3 /. Nsub] /. sub // FullSimplify]];
  AppendTo[g4Hg1g4, Expand[Quatermult[Quatermult[Quaterconj[g4] /. Nsub,
       Hg1[[i]] /. Nsub], g4 /. Nsub] /. sub // FullSimplify]];
 ];

g1Hg2g1 = {};
g2Hg2g2 = {};
g3Hg2g3 = {};
g4Hg2g4 = {};
For[i = 0, i < Length[g1H], i++;
  AppendTo[g1Hg2g1,
   Expand[Quatermult[Quatermult[Quaterconj[g1] /. Nsub, Hg2[[i]] /. Nsub],
      g1 /. Nsub] /. sub // FullSimplify]];
  AppendTo[g2Hg2g2, Expand[Quatermult[Quatermult[Quaterconj[g2] /. Nsub,
       Hg2[[i]] /. Nsub], g2 /. Nsub] /. sub // FullSimplify]];
  AppendTo[g3Hg2g3, Expand[Quatermult[Quatermult[Quaterconj[g3] /. Nsub,
       Hg2[[i]] /. Nsub], g3 /. Nsub] /. sub // FullSimplify]];
  AppendTo[g4Hg2g4, Expand[Quatermult[Quatermult[Quaterconj[g4] /. Nsub,
       Hg2[[i]] /. Nsub], g4 /. Nsub] /. sub // FullSimplify]];
 ];

g1Hg3g1 = {};
g2Hg3g2 = {};
g3Hg3g3 = {};
g4Hg3g4 = {};
For[i = 0, i < Length[g1H], i++;
  AppendTo[g1Hg3g1,
   Expand[Quatermult[Quatermult[Quaterconj[g1] /. Nsub, Hg3[[i]] /. Nsub],
      g1 /. Nsub] /. sub // FullSimplify]];
  AppendTo[g2Hg3g2, Expand[Quatermult[Quatermult[Quaterconj[g2] /. Nsub,
       Hg3[[i]] /. Nsub], g2 /. Nsub] /. sub // FullSimplify]];
  AppendTo[g3Hg3g3, Expand[Quatermult[Quatermult[Quaterconj[g3] /. Nsub,
       Hg3[[i]] /. Nsub], g3 /. Nsub] /. sub // FullSimplify]];
  AppendTo[g4Hg3g4, Expand[Quatermult[Quatermult[Quaterconj[g4] /. Nsub,
       Hg3[[i]] /. Nsub], g4 /. Nsub] /. sub // FullSimplify]];
 ];
```

```
g1Hg4g1 = {};
g2Hg4g2 = {};
g3Hg4g3 = {};
g4Hg4g4 = {};
For[i = 0, i < Length[g1H], i++;
  AppendTo[g1Hg4g1,
   Expand[Quatermult[Quatermult[Quaterconj[g1] /. Nsub, Hg4[[i]] /. Nsub],
       g1 /. Nsub] /. sub // FullSimplify]];
  AppendTo[g2Hg4g2, Expand[Quatermult[Quatermult[Quaterconj[g2] /. Nsub,
        Hg4[[i]] /. Nsub], g2 /. Nsub] /. sub // FullSimplify]];
  AppendTo[g3Hg4g3, Expand[Quatermult[Quatermult[Quaterconj[g3] /. Nsub,
        Hg4[[i]] /. Nsub], g3 /. Nsub] /. sub // FullSimplify]];
  AppendTo[g4Hg4g4, Expand[Quatermult[Quatermult[Quaterconj[g4] /. Nsub,
        Hg4[[i]] /. Nsub], g4 /. Nsub] /. sub // FullSimplify]];
 ];
```

In[4661]:=
```
(* Reflections of the 600-
 cell swap the 5 left decompositions with the 5 right decompositions*)

(* Definition of the 5 left decompositions and 5 right decompositions*)
leftcompounds = {H, g1H, g2H, g3H, g4H, g1Hg1, g1g1Hg1, g1g2Hg1, g1g3Hg1,
   g1g4Hg1, g2Hg2, g2g1Hg2, g2g2Hg2, g2g3Hg2, g2g4Hg2, g3Hg3, g3g1Hg3,
   g3g2Hg3, g3g3Hg3, g3g4Hg3, g4Hg4, g4g1Hg4, g4g2Hg4, g4g3Hg4, g4g4Hg4 };
rightcompounds = {H, Hg1, Hg2, Hg3, Hg4, rightg1Hg1, g1Hg1g1, g1Hg2g1, g1Hg3g1,
   g1Hg4g1, rightg2Hg2, g2Hg1g2, g2Hg2g2, g2Hg3g2, g2Hg4g2, rightg3Hg3, g3Hg1g3,
   g3Hg2g3, g3Hg3g3, g3Hg4g3, rightg4Hg4, g4Hg1g4, g4Hg2g4, g4Hg3g4, g4Hg4g4 };
```

In[4686]:=
```
(* Four generators of the refletions of the 600-cell *)
alpha1 = {1, 0, 0, 0}
alpha2 = -1/2 * {1, 1, 1, 1}
alpha3 = {0, 0, 0, 1}
alpha4 = 1/2 * {0, -1, -1/phi, phi} /. Nsub
```

Out[4686]= {1, 0, 0, 0}

Out[4687]= $\left\{-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right\}$

Out[4688]= {0, 0, 0, 1}

Out[4689]= $\left\{0, -\frac{1}{2}, -\frac{1}{1+\sqrt{5}}, \frac{1}{4}\left(1+\sqrt{5}\right)\right\}$

In[4690]:=
```
(* Reflection of alpha_i with all left cosets *)
alpha1left = {};
alpha2left = {};
alpha3left = {};
alpha4left = {};
list1 = {};
list2 = {};
list3 = {};
list4 = {};
wrong = {};
For[i = 0, i < 25, i++;
  For[j = 0, j < 24, j++;
   y = leftcompounds[[i]][[j]] /. Nsub;
   x1 = alpha1[[1]] * y[[1]] +
     alpha1[[2]] * y[[2]] + alpha1[[3]] * y[[3]] + alpha1[[4]] * y[[4]];
   x2 = alpha2[[1]] * y[[1]] + alpha2[[2]] * y[[2]] +
     alpha2[[3]] * y[[3]] + alpha2[[4]] * y[[4]];
   x3 = alpha3[[1]] * y[[1]] + alpha3[[2]] * y[[2]] +
     alpha3[[3]] * y[[3]] + alpha3[[4]] * y[[4]];
   x4 = alpha4[[1]] * y[[1]] + alpha4[[2]] * y[[2]] +
     alpha4[[3]] * y[[3]] + alpha4[[4]] * y[[4]];
   z1 = y - 2 * x1 * alpha1;
   z2 = y - 2 * x2 * alpha2;
   z3 = y - 2 * x3 * alpha3;
   z4 = y - 2 * x4 * alpha4;
   AppendTo[list1, Expand[z1 /. sub // FullSimplify ]];
   AppendTo[list2, Expand[z2 /. sub // FullSimplify ]];
   AppendTo[list3, Expand[z3 /. sub // FullSimplify ]];
   AppendTo[list4, Expand[Simplify[z4] /. sub // FullSimplify ]];
  ];
  AppendTo[alpha1left, list1];
  AppendTo[alpha2left, list2];
  AppendTo[alpha3left, list3];
  AppendTo[alpha4left, list4];
  list1 = {};
  list2 = {};
  list3 = {};
  list4 = {};
 ];
```

```
reflectionalpha1 = {};
reflectionalpha2 = {};
reflectionalpha3 = {};
reflectionalpha4 = {};
For[i = 0, i < 25, i++;
  For[j = 0, j < 25, j++;
   If[Sort[alpha1left[[i]]] == Sort[rightcompounds[[j]]],
    AppendTo[reflectionalpha1, {i, j}], Nothing];
   If[Sort[alpha2left[[i]]] == Sort[rightcompounds[[j]]],
    AppendTo[reflectionalpha2, {i, j}], Nothing];
   If[Sort[alpha3left[[i]]] == Sort[rightcompounds[[j]]],
    AppendTo[reflectionalpha3, {i, j}], Nothing];
   If[Length[Intersection[alpha4left[[i]], rightcompounds[[j]]]] > 19,
    AppendTo[reflectionalpha4, {i, j}], Nothing];
   (* Code is imprecise: it cannot compare 1/(2*phi) and -1/2 +
      phi/2 so the length of the intersections becomes 20 when it actually is 24.*)
  ];
 ];

(* Compare the 24-cells in a left decomposition to the 24-
  cells in right decomposition} *)
reflectionalpha1
reflectionalpha2
reflectionalpha3
reflectionalpha4
```

```
Out[4792]= {{1, 1}, {2, 5}, {3, 4}, {4, 3}, {5, 2}, {6, 21}, {7, 25}, {8, 24}, {9, 23},
  {10, 22}, {11, 16}, {12, 20}, {13, 19}, {14, 18}, {15, 17}, {16, 11}, {17, 15},
  {18, 14}, {19, 13}, {20, 12}, {21, 6}, {22, 10}, {23, 9}, {24, 8}, {25, 7}}

Out[4793]= {{1, 1}, {2, 3}, {3, 5}, {4, 4}, {5, 2}, {6, 20}, {7, 17}, {8, 16}, {9, 19},
  {10, 18}, {11, 13}, {12, 12}, {13, 15}, {14, 14}, {15, 11}, {16, 25}, {17, 23},
  {18, 22}, {19, 24}, {20, 21}, {21, 6}, {22, 10}, {23, 7}, {24, 9}, {25, 8}}

Out[4794]= {{1, 1}, {2, 4}, {3, 5}, {4, 2}, {5, 3}, {6, 16}, {7, 17}, {8, 19}, {9, 20},
  {10, 18}, {11, 21}, {12, 23}, {13, 24}, {14, 22}, {15, 25}, {16, 6}, {17, 7},
  {18, 10}, {19, 8}, {20, 9}, {21, 11}, {22, 14}, {23, 12}, {24, 13}, {25, 15}}

Out[4795]= {{1, 6}, {2, 9}, {3, 7}, {4, 8}, {5, 10}, {6, 21}, {7, 24}, {8, 25}, {9, 22},
  {10, 23}, {11, 11}, {12, 12}, {13, 14}, {14, 15}, {15, 13}, {16, 16}, {17, 18},
  {18, 19}, {19, 17}, {20, 20}, {21, 1}, {22, 2}, {23, 5}, {24, 3}, {25, 4}}
```