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Bounds on the growth rate of time-invariant switching max-min-plus-scaling discrete-event systems

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Abstract: We consider the growth rate of a switching max-min-plus-scaling (S-MMPS) system in a discrete-event framework. We show that an explicit, time-invariant, monotone, and arbitrarily switching MMPS system has a bounded growth rate. Further, we propose a mixed-integer linear programming problem to calculate the estimates of the smallest upper bound and the largest lower bound of the growth rate of an S-MMPS system.

Keywords: Max-plus algebra, Switching max-min-plus-scaling systems, Algebraic Systems Theory, Discrete Event Systems, Nonlinear Systems and Control

1. INTRODUCTION

Max-min-plus-scaling (MMPS) systems are algebraic models that can represent both linear and nonlinear discrete-event systems. These systems act as a nonlinear generalization for models based on max-plus or min-plus algebra (Baccelli et al., 1992) such as max-plus linear system, min-plus linear system, max-min-plus system etc. The basic operations in this system include maximization (synchronization or sequential operation), minimization (competition), addition (operation time), and scaling (state-dependent processing time) (van den Boom et al., 2023; Markkassery et al., 2024).

Switching max-min-plus-scaling (S-MMPS) systems are discrete-event systems that can switch between various modes of operation. Each mode of operation is described by an MMPS system. The mode is either determined by the present state, previous mode, and some external signals, or the mode is arbitrary and cannot be determined in advance. This is analogous to the principle of switching max-plus-linear systems, where, in each switching mode, the dynamics of the discrete-event system is represented by a max-plus-linear system (Kersbergen et al., 2016; Segovia et al., 2022; Lopes et al., 2014; van den Boom and De Schutter, 2012). An S-MMPS system can model structural changes in a discrete-event system due to some external factors. Examples of switching MMPS systems are flexible production systems, telecommunications networks and traffic light-controlled urban traffic networks, where switching between different modes can occur due to changing production recipes, customer or traffic demand, or failures in production units, transmission lines or traffic connections.

The growth rate or the additive eigenvalue of max-plus/min-plus algebraic models has been studied widely in literature (Van Der Woude and Heidergott, 2006; Zhao et al., 2001; Markkassery et al., 2024). For a discrete-event system to be stable, all the states should have equal asymptotic growth rates (Gupta et al., 2020). The max-plus-linear systems, min-plus linear systems and max-min-plus systems have a unique growth rate (if the growth rate exists) (Zhao et al., 2001). A general MMPS system is non-monotonic and can have multiple growth rates (Markkassery et al., 2024). However, when the function governing the MMPS system is time-invariant, monotonic and non-expansive, the system has a unique growth rate.

The class of systems that are non-monotonic was considered in (Plus, 1999), in which the issue of analyzing the asymptotic behavior in general (without the monotonicity condition) was stated as an open problem. The case of monotonic systems satisfying a homogeneity condition has been studied in (Cohen et al., 1995, 1998) and more recently in (Allamigeon et al., 2021). In general, the monotonic, time invariant MMPS system dynamics are equivalent to the dynamic programming equations of turn based stochastic games, which have been widely studied in the game literature (Akian et al., 2023). This provides further motivation for the study of this class of systems.

The main contributions of this paper are as follows. We analyze the growth rate of an autonomous, explicit, time-invariant, and monotone S-MMPS system. We show that the growth rate of such a system is bounded. Further, an algorithm is presented to calculate the least upper bound and greatest lower bound of the growth rate of an S-MMPS system.

The paper is organized as follows. Section 2 presents the mathematical preliminaries and notations. Section 3 introduces the structural properties of a time-invariant and monotone S-MMPS system. In Section 4, the proof

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for boundedness of the growth rate of a time-invariant and monotone S-MMPS system is given. Section 5 shows the derivation of a mixed-integer linear programming problem for finding the upper and lower bounds of the growth rate of an S-MMPS system. The simulations in support of these results are provided in Section 6 and the conclusions are given in Section 7.

2. PRELIMINARIES

In this section, we introduce some basic concepts and definitions from max-plus linear algebra and min-plus linear algebra based on Heidergott et al. (2006). Let $\top = \infty$, $\varepsilon = -\infty$, $\mathbb{R}_\top = \mathbb{R} \cup \{\infty\}$, $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{-\infty\}$, and $\mathbb{R}_c = \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$, where $\mathbb{R}$ is the set of real numbers. Further, we introduce the convention $\top + \varepsilon = 0$. Often we use the set $\mathcal{R}$, which can be either $\mathbb{R}$, $\mathbb{R}_\varepsilon$, $\mathbb{R}_\top$ or $\mathbb{R}_c$. We define $\mathbb{0}_n = [\,0\ 0\ \cdots\ 0\,]^\intercal \in \mathbb{R}^n$ and $\mathbb{1}_n = [\,1\ 1\ \cdots\ 1\,]^\intercal \in \mathbb{R}^n$. The notation $\mathbb{Z}^+$ denotes the set of positive integers. The matrix-vector product of a matrix $C \in \mathbb{R}^{p\times n}$ and a vector $x \in \mathbb{R}^n$ is denoted as, $C \cdot x$ and the scalar multiplication of a scalar $\mu \in \mathbb{R}$ and a vector x is denoted as μx. The d-norm of a vector $x \in \mathbb{R}^n$ is denoted as $||x||_d = \sum_{i=1}^n |x_i|^{1/d}$, $d \geq 1$. Let $a, b \in \mathbb{R}_c$. Then, the operations $a \oplus b = \max(a, b)$, $a \otimes b = a + b$, $a \oplus' b = \min(a, b)$, and $a \otimes' b = a + b$ represent the max-plus addition, max-plus multiplication, the min-plus addition, and min-plus multiplication, respectively (Heidergott et al., 2006).

From max-plus algebra, we adopt the following operations for matrices $A, B \in \mathbb{R}_\varepsilon^{m\times n}$ and $C \in \mathbb{R}^{n\times p}$:

$$[A \oplus B]_{i,j} = \max([A]_{i,j}, [B]_{i,j})$$

$$[A \otimes C]_{i,j} = \bigoplus_{k=1}^{n} \left([A]_{i,k} \otimes [C]_{k,j}\right) = \max_k \left([A]_{i,k} + [C]_{k,j}\right)$$

From min-plus algebra, we adopt the following operations for matrices $A, B \in \mathbb{R}_\top^{m\times n}$ and $C \in \mathbb{R}^{n\times p}$:

$$[A \oplus' B]_{i,j} = \min\left([A]_{i,j}, [B]_{i,j}\right)$$

$$[A \otimes' C]_{i,j} = {\bigoplus_{k=1}^{n}}' \left([A]_{i,k} \otimes' [C]_{k,j}\right) = \min_k \left([A]_{i,k} + [C]_{k,j}\right)$$

Given the vector $v \in \mathbb{R}^n$, we define a max-plus diagonal matrix $\mathrm{d}_\otimes(v)$ and min-plus diagonal matrix $\mathrm{d}_{\otimes'}(v)$,

$$\mathrm{d}_\otimes(v) = \begin{bmatrix} v_1 & \varepsilon & \cdots & \varepsilon \\ \varepsilon & v_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ \varepsilon & \cdots & \cdots & v_n \end{bmatrix}, \quad \mathrm{d}_{\otimes'}(v) = \begin{bmatrix} v_1 & \top & \cdots & \top \\ \top & v_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ \top & \cdots & \cdots & v_n \end{bmatrix}$$

The inverse max-plus diagonal matrix is $[\mathrm{d}_\otimes(v)]^{-1} = \mathrm{d}_\otimes(-v)$ and the inverse min-plus diagonal matrix is $[\mathrm{d}_{\otimes'}(v)]^{-1} = \mathrm{d}_{\otimes'}(-v)$. Further, we define the following bounds:

$$\{x(k)\}_{\max} = \max_i x_i(k), \quad \{x(k)\}_{\min} = \min_i x_i(k)$$

Finally, we define the following limits for a sequence $x(n)$:

$$\limsup_{n\to\infty} x(n) = \lim_{n\to\infty} \sup\{x(n); n \geq \infty\}$$

$$\liminf_{n\to\infty} x(n) = \lim_{n\to\infty} \inf\{x(n); n \geq \infty\}$$

3. TIME-INVARIANT, MONOTONE, SWITCHING MMPS SYSTEM

Definition 1. ((Markkassery et al., 2024)MMPS system in canonical form) Consider the following system:

$$x(k) = A \otimes (B \otimes' (C \cdot x(k-1))) \tag{1}$$

where $A \in \mathbb{R}_\varepsilon^{n\times m}$, $B \in \mathbb{R}_\top^{m\times p}$, and $C \in \mathbb{R}^{p\times n}$ and the index $k \in \mathbb{Z}^+$ is the event counter. This system is called an MMPS system in the ABC canonical form.

In Markkassery et al. (2024), it has been shown that any MMPS system can be written in the ABC canonical form (1). We extend the system (1) to include switching as follows:

Definition 2. (Switching max-min-plus-scaling system)

Consider the following system:

$$x(k) = A(\ell(k)) \otimes (B(\ell(k)) \otimes' (C \cdot x(k-1))) \tag{2}$$

in which the matrices $A(\ell) \in \mathbb{R}_\varepsilon^{n\times m}$, $B(\ell) \in \mathbb{R}_\top^{m\times p}$, $C \in \mathbb{R}^{p\times n}$ are the system matrices for the ℓ-th mode, $\ell \in \{1, \ldots, n_L\}$. This system is called a switching MMPS system.

We assume that there are n_L possible modes for the system (2). For S-MMPS systems, the state $x(k)$ typically contains the time instants at which the internal events occur for the k-th time, and the mode $\ell(k)$ determines which MMPS model is valid during the k-th event.

Remark 1. Note that in Definition 2 the C-matrix does not depend on $\ell(k)$. This is done for the brevity of proofs in Section 4. Consider a system with system matrices $A(\ell(k))$, $B(\ell(k))$, $C(\ell(k))$ where the mode is determined as in Definition 2. Then we can choose new matrices

$$\bar{B}(\ell(k)) = [\,P_1(\ell(k))\ P_2(\ell(k))\ \cdots\ P_{n_L}(\ell(k))\,]$$

$$\bar{C}^T = \left[\,C^T(1)\ C^T(2)\ \cdots\ C^T(n_L)\,\right]$$

where $P_i(\ell(k)) \in \mathbb{R}_\top^{m\times p}$ with

$$P_i(\ell(k)) = \begin{cases} B(i) & \text{if } i = \ell(k) \\ \top^{m\times p} & \text{if } i \neq \ell(k) \end{cases}$$

such that,

$$x(k) = A(\ell(k)) \otimes (\bar{B}(\ell(k)) \otimes' (\bar{C} \cdot x(k-1))).$$

Hence, the matrix C in (2) can always be chosen to be independent of mode $\ell(k)$.

Definition 3. (Homogeneous, monotone and non-expansive system (Gunawardena, 2003)) Consider a system,

$$x(k+1) = f(x(k)) \tag{3}$$

The system is called homogeneous if

$$f(x + \alpha\mathbb{1}) = f(x) + \alpha\mathbb{1}$$

for any $\alpha \in \mathbb{R}$. As the states of an MMPS system has the dimension of time, time-invariance in these systems is the same as homogeneity. I.e. if we shift all the events by the same amount of time, the dynamics of the system is not altered. The system (3) is called monotone if $x \leq y$, then $f(x) \leq f(y)$. The system (3) is called non-expansive in the d-norm, with $d \geq 1$ if $\|f(x) - f(y)\|_d \leq \|x - y\|_d$. When the system (3) is time-invariant and monotonic, it is also non-expansive (Gunawardena, 2003). Similar to the MMPS system (1), the switching MMPS system (2) is time-invariant if and only if $\sum_j C_{i,j} = 1,\ \forall i$ where $C_{i,j}$

are the components of the matrix C (Markkassery et al., 2024). A switching MMPS system as defined in Definition 2 is monotone if $C_{i,j} \geq 0,\ \forall i,j$. A switching MMPS system is non-expansive if it is time-invariant and monotone. Therefore, from monotonicity and time-invariance we get $C_{i,j} \leq 1,\ \forall i,j$. The proof for these results can be found in (Markkassery et al., 2024).

Definition 4. (Markkassery et al. (2024)). The time invariant MMPS system, $x(k) = f(x(k-1))$, $x \in \mathcal{R}^n$ and $f : \mathcal{R}^n \to \mathcal{R}^n$ is said to have an additive eigenvalue/growth rate if there exists a real number $\lambda \in \mathcal{R}$ and a vector $v \in \mathbb{R}^n$ such that

$$f(v) = v + \lambda \mathbb{1}_n.$$

The scalar λ is then called an additive eigenvalue/growth rate and the vector v is called a corresponding additive eigenvector.

Proposition 1. A time-invariant, monotonic and non expansive MMPS system has a unique growth rate, if it exists

Proof: This can be proved using the non-expansive property of the system as in Lemma 1.2 (Cochet-Terrasson et al., 1997) □ End Proof

Definition 5. A switching MMPS system $x(k) = f(x(k-1))$ is well-defined in $\mathbb{R}^n$ if the following holds:

$$x(k-1) \in \mathbb{R}^n \quad \Longrightarrow \quad x(k) \in \mathbb{R}^n$$

A sufficient condition for the MMPS system (1) to be well-defined is that the matrices A and B have at least one finite element in each row (Heidergott et al., 2006) and C has all elements in $\mathbb{R}$.

Assumption 1. We assume the switching MMPS system (2) to be well-defined, time-invariant and monotone in the rest of the analysis.

Assumption 2. We assume that each MMPS system (5), between which the system (2) switches, has a unique growth rate.

4. BOUNDS ON GROWTH RATE

Definition 6. Given a system (2) with arbitrary switching modes $\ell(k)$. Then, the asymptotic maximum growth rate $\sigma_{\max}$ and asymptotic minimum growth rate $\sigma_{\min}$ are defined as

$$\begin{aligned} \sigma_{\max} &= \limsup_{n\to\infty} \frac{1}{n}\{x(n)\}_{\max} \\ \sigma_{\min} &= \liminf_{n\to\infty} \frac{1}{n}\{x(n)\}_{\min} \end{aligned} \tag{4}$$

regardless of any finite initial state $x(0)$. A max-plus algebraic system is stable when all the states with the dimension of time evolve with the same growth rate. For stable max-plus-linear systems the asymptotic maximum growth rate is equal to the largest eigenvalue, and the same holds for stable max-min-plus systems and stable max-min-plus-scaling systems (Heidergott et al., 2006; Gunawardena, 1994b,a; Markkassery et al., 2024). For switching max-plus-linear systems, the asymptotic maximum growth rate is larger or equal to the largest eigenvalue of all system matrices (van den Boom and De Schutter, 2012).

Lemma 1. Consider a time-invariant, monotone S-MMPS system (2). Let

$$\begin{aligned} \rho_{\max} &= \max_{\ell} \left(\mathbb{0}_n^T \otimes (A(\ell) \otimes (B(\ell) \otimes' \mathbb{0}_p))\right) \\ &= \max_{\ell} \max_{i} \max_{j} (A_{i,j}(\ell) + \min_{q} B_{j,q}(\ell)), \end{aligned} \tag{5}$$

$$\begin{aligned} \rho_{\min} &= \min_{\ell} \left(\mathbb{0}_n^T \otimes' (A(\ell) \otimes (B(\ell) \otimes' \mathbb{0}_p))\right) \\ &= \min_{\ell} \min_{i} \max_{j} (A_{i,j}(\ell) + \min_{q} B_{j,q}(\ell)), \end{aligned} \tag{6}$$

then

$$\begin{aligned} \{x(k)\}_{\max} - \{x(k-1)\}_{\max} &\leq \rho_{\max} \\ \{x(k)\}_{\min} - \{x(k-1)\}_{\min} &\geq \rho_{\min} \end{aligned}$$

Proof: Consider the switching MMPS system

$$\begin{aligned} z(k) &= C \cdot x(k-1) \\ y(k) &= B(\ell) \otimes' z(k) \\ x(k) &= A(\ell) \otimes y(k) \end{aligned}$$

From time-invariance and monotonicity property of S-MMPS system, we have $C_{i,j} \leq 1$. So, it follows that

$$\{x(k-1)\}_{\min} \leq \{z(k)\}_{\max} \leq \{x(k-1)\}_{\max}$$

Furthermore,

$$\begin{aligned} y(k) &= B(\ell) \otimes' z(k) \\ &\leq B(\ell) \otimes' (\mathbb{0}_p + \{z(k)\}_{\max} \mathbb{1}_p) \\ &\leq (B(\ell) \otimes' \mathbb{0}_p) + \{x(k-1)\}_{\max} \mathbb{1}_p \\ &\leq b(\ell) + \{x(k-1)\}_{\max} \mathbb{1}_p \end{aligned}$$

and

$$\begin{aligned} y(k) &= B(\ell) \otimes' z(k) \\ &\geq B(\ell) \otimes' (\mathbb{0}_p + \{z(k)\}_{\min} \mathbb{1}_p) \\ &\geq (B(\ell) \otimes' \mathbb{0}_p) + \{x(k-1)\}_{\min} \mathbb{1}_p \\ &\geq b(\ell) + \{x(k-1)\}_{\min} \mathbb{1}_p \end{aligned}$$

where $b(\ell) = B(\ell) \otimes' \mathbb{0}_p$. For $x(k)$ we find

$$\begin{aligned} x(k) &= A(\ell) \otimes y(k) \\ &= A(\ell) \otimes (B(\ell) \otimes' z(k)) \\ &\leq A(\ell) \otimes (b(\ell) + \{x(k-1)\}_{\max} \mathbb{1}_p) \\ &\leq (A(\ell) \otimes b(\ell)) + \{x(k-1)\}_{\max} \mathbb{1}_p \\ &\leq a(\ell) + \{x(k-1)\}_{\max} \mathbb{1}_p \end{aligned} \tag{7}$$

and

$$\begin{aligned} x(k) &= A(\ell) \otimes y(k) \\ &= A(\ell) \otimes (B(\ell) \otimes' z(k)) \\ &\geq A(\ell) \otimes (b(\ell) + \{x(k-1)\}_{\min} \mathbb{1}_p) \\ &\geq (A(\ell) \otimes b(\ell)) + \{x(k-1)\}_{\min} \mathbb{1}_p \\ &\geq a(\ell) + \{x(k-1)\}_{\min} \mathbb{1}_p \end{aligned} \tag{8}$$

where $a(\ell) = A(\ell) \otimes (B(\ell) \otimes' \mathbb{0}_p)$. Finally, using (7) and (8), we have

$$\begin{aligned} \{x(k)\}_{\max} &\leq \max_{\ell} \max_{i} \left(a_i(\ell) + \{x(k-1)\}_{\max}\right) \\ &\leq \max_{\ell} \max_{i} \max_{j} (A_{i,j}(\ell) + \min_{q} B_{j,q}(\ell)) \\ &\qquad + \{x(k-1)\}_{\max} \\ &\leq \rho_{\max} + \{x(k-1)\}_{\max} \\ \{x(k)\}_{\min} &\geq \min_{\ell} \min_{i} \left(a_i(\ell) + \{x(k-1)\}_{\min}\right) \\ &\geq \min_{\ell} \min_{i} \max_{j} (A_{i,j}(\ell) + \min_{q} B_{j,q}(\ell)) \\ &\qquad + \{x(k-1)\}_{\min} \\ &\geq \rho_{\min} + \{x(k-1)\}_{\min} \end{aligned} \tag{9}$$

where $\rho_{\max}$ and $\rho_{\min}$ are defined in (5)-(6).

□ End Proof

Proposition 2. For time-invariant and monotone S-MMPS systems,

A. the asymptotic maximum growth rate is larger or equal to the largest eigenvalue among all the MMPS subsystems.
B. the asymptotic minimum growth rate is smaller or equal to the smallest eigenvalue among all the MMPS subsystems.

Proof: The time-invariant, monotone MMPS system has a unique eigenvalue/growth rate (when an eigenvalue exists (Proposition 1)). Each MMPS system grows at a rate equal to its eigenvalue. Let $\ell(k)$ be a constant, and the MMPS system corresponding to this mode is the one with the maximum eigenvalue. Then this S-MMPS system has a growth rate equal to the maximum eigenvalue among all the MMPS systems within which the S-MMPS system is switching. So, the maximum growth rate of the arbitrarily switching S-MMPS system should be greater than or equal to the largest eigenvalue among all the MMPS systems. Following the same argument, we can say that the minimum growth rate of an S-MMPS system with arbitrary switching is lesser or equal to the smallest eigenvalue among the MMPS systems within which the S-MMPS system is switching. Hence, we have $\rho_{\max} \geq \lambda_{\max}$ and $\rho_{\min} \leq \lambda_{\min}$, where $\lambda_{\max}$ and $\lambda_{\min}$ are the largest and smallest eigenvalues, respectively, among all the MMPS systems. □ End Proof

5. TIGHTER BOUNDS FOR THE ASYMPTOTIC GROWTH RATE

The bounds obtained in Section 4 are analytical and conservative in nature. In this section, we present a methodology to calculate tighter bounds by using a shifted S-MMPS system.

Let $r \in \mathbb{R}^n$, $s \in \mathbb{R}^m$, and $t = C \cdot r \in \mathbb{R}^p$. Furthermore, let $R = \mathrm{d}_\otimes(r)$, $S = \mathrm{d}_\otimes(s)$, $S' = \mathrm{d}_{\otimes'}(s)$ and $T' = \mathrm{d}_{\otimes'}(t)$, and

$$\tilde{A}(\ell, r, s) = R \otimes A(\ell) \otimes S^{-1}\ ,\quad \tilde{B}(\ell, s, t) = S' \otimes' B(\ell) \otimes' T'^{-1}$$

where $S^{-1} = \mathrm{d}_\otimes(-s)$ and $T'^{-1} = \mathrm{d}_{\otimes'}(-t)$. For brevity, we follow the notations $\tilde{A}(\ell)$ for $\tilde{A}(\ell, r, s)$ and $\tilde{B}(\ell)$ for $\tilde{B}(\ell, s, t)$. Finally, define

$$\tilde{x}(k) = x(k) + r,\quad \tilde{y}(k) = y(k) + s,\quad \tilde{z}(k) = z(k) + t \tag{10}$$

Then we have

$$\begin{aligned}
\tilde{z}(k) &= z(k) + t = C \cdot x(k-1) + t \\
&= C \cdot (x(k-1) + r) = C \cdot \tilde{x}(k-1) \\
\tilde{y}(k) &= y(k) + s = S' \otimes' y(k) = S' \otimes' B(\ell) \otimes' z(k) \\
&= S' \otimes' B(\ell) \otimes' T'^{-1} \otimes' \tilde{z}(k) = \tilde{B}(\ell) \otimes' \tilde{z}(k) \\
\tilde{x}(k) &= x(k) + r = R \otimes x(k) = R \otimes A(\ell) \otimes y(k) \\
&= R \otimes A(\ell) \otimes S^{-1} \otimes \tilde{y}(k) = \tilde{A}(\ell) \otimes \tilde{y}(k)
\end{aligned}$$

Hence, for any, r, s, t we can write:

$$\tilde{x}(k) = \tilde{A}(\ell) \otimes (\tilde{B}(\ell) \otimes' (C \cdot \tilde{x}(k-1))) \tag{11}$$

The system (11) is called a shifted switching MMPS system corresponding to (2).

Consider the system (11) and define $\rho_{\max}$ (5) and $\rho_{\min}$ (6) for this system:

$$\tilde{\rho}_{\max} = \max_\ell \max_i \max_j (\tilde{A}_{i,j}(\ell) + \min_q \tilde{B}_{j,q}(\ell)),$$
$$\tilde{\rho}_{\min} = \min_\ell \min_i \max_j (\tilde{A}_{i,j}(\ell) + \min_q \tilde{B}_{j,q}(\ell))$$

Using (9), we derive

$$\{\tilde{x}(1)\}_{\max} \leq \tilde{\rho}_{\max} + \{\tilde{x}(0)\}_{\max}$$
$$\{\tilde{x}(1)\}_{\min} \geq \tilde{\rho}_{\min} + \{\tilde{x}(0)\}_{\min}$$

and therefore using successive substitution, we get

$$\{\tilde{x}(n)\}_{\max} \leq n \cdot \tilde{\rho}_{\max} + \{\tilde{x}(0)\}_{\max}$$
$$\{\tilde{x}(n)\}_{\min} \geq n \cdot \tilde{\rho}_{\min} + \{\tilde{x}(0)\}_{\min}$$

From (10), we have

$$\begin{aligned}
\{x(n)\}_{\max} &= \{\tilde{x}(n) - r\}_{\max} \leq \{\tilde{x}(n)\}_{\max} - \{r\}_{\min} \\
&\leq n \cdot \tilde{\rho}_{\max} + \{\tilde{x}(0)\}_{\max} - \{r\}_{\min} \\
&\leq n \cdot \tilde{\rho}_{\max} + \{x(0) + r\}_{\max} - \{r\}_{\min} \\
&\leq n \cdot \tilde{\rho}_{\max} + \{x(0)\}_{\max} + \{r\}_{\max} - \{r\}_{\min}
\end{aligned} \tag{12}$$

and for $\{x(n)\}_{\min}$:

$$\begin{aligned}
\{x(n)\}_{\min} &= \{\tilde{x}(n) - r\}_{\min} \geq \{\tilde{x}(n)\}_{\min} - \{r\}_{\max} \\
&\geq n \cdot \tilde{\rho}_{\min} + \{\tilde{x}(0)\}_{\min} - \{r\}_{\max} \\
&\geq n \cdot \tilde{\rho}_{\min} + \{x(0) + r\}_{\min} - \{r\}_{\max} \\
&\geq n \cdot \tilde{\rho}_{\min} + \{x(0)\}_{\min} - \{r\}_{\max} + \{r\}_{\min}
\end{aligned} \tag{13}$$

Theorem 1. The asymptotic maximum growth rate $\sigma_{\max}$ and asymptotic minimum growth rate $\sigma_{\min}$ are bounded as follows:

$$\sigma_{\max} \leq \tilde{\rho}_{\max},\quad \sigma_{\min} \geq \tilde{\rho}_{\min}$$

Proof: From (4), (12), and (13), we have

$$\sigma_{\max} = \limsup_{n\to\infty} \frac{1}{n}\{x(n)\}_{\max} \leq \tilde{\rho}_{\max}$$
$$\sigma_{\min} = \liminf_{n\to\infty} \frac{1}{n}\{x(n)\}_{\max} \geq \tilde{\rho}_{\min}$$

□ End Proof

Let,

$$\tilde{b}(\ell, r, s, t) = \tilde{B}(\ell) \otimes' \mathbb{0}_p,\quad \tilde{a}(\ell, r, s, t) = \tilde{A}(\ell) \otimes \tilde{b}(\ell, r, s, t)$$

Then,

$$\begin{aligned}
\tilde{b}_j(\ell, r, s, t) &= \min_q \tilde{B}_{j,q}(\ell) \\
&= \min_q [S' \otimes' B(\ell) \otimes' T^{-1}]_{j,q} \\
&= \min_q [\mathrm{d}_\otimes(s) \otimes' B(\ell) \otimes' \mathrm{d}_\otimes(-t)]_{j,q} \\
&= \min_q (B_{j,q}(\ell) + s_j - t_q) \\
\tilde{a}_i(\ell, r, s, t) &= \max_j (\tilde{A}_{i,j}(\ell) + \tilde{b}_j(\ell, r, s, t)) \\
&= \max_j \Big([R \otimes' A(\ell) \otimes' S^{-1}]_{i,j} + \tilde{b}_j(\ell, r, s, t)\Big) \\
&= \max_j \Big(A_{i,j}(\ell) + r_i - s_j + \min_q (B_{j,q}(\ell) + s_j - t_q)\Big) \\
&= \max_j \Big(A_{i,j}(\ell) + r_i + \min_q (B_{j,q}(\ell) - t_q)\Big)
\end{aligned}$$

Note that $\tilde{a}_i$ does not depend on the vector s. Now, finding the smallest upper bound for the asymptotic maximum growth rate $\sigma_{\max}$ is the same as minimizing $\{\tilde{a}\}_{\max}$ over r:

$$\tilde{\rho}_{\max} = \min_r \max_\ell \max_i \tilde{a}_i(\ell, r, t)$$
$$= \min_r \max_\ell \max_i \max_j \Big(A_{i,j}(\ell) + r_i + \min_q (B_{j,q}(\ell) - t_q)\Big)$$

This optimization problem can be recast as a mixed-integer linear programming problem as follows:

$$\min_{r,t,\ell,\mu} \tilde{\rho}_{\max}$$

subject to

$$\begin{aligned} -\tilde{\rho}_{\max} + r_i + b_{j,\ell} &\leq -A_{i,j}(\ell) & \forall i,j,\ell \\ b_{j,\ell} + t_q &\leq B_{j,q}(\ell) & \forall j,q,\ell \\ -b_{j,\ell} + M\,\mu_{j,q,\ell} - t_q &\leq -B_{j,q}(\ell) + M & \forall j,q,\ell \\ -\sum_q \mu_{j,q,\ell} &\leq -1 & \forall j,\ell \\ t &= C\,r \end{aligned} \tag{14}$$

where μ is a binary tensor variable and M is a large positive number. Here μ and M are used to make sure that in one of the indices q, $\forall j, l$, the value of $b_{j,l}$ hit the minimum. This is because, we have to maximize $\tilde{a}(\ell, r, t)$, which contains a 'min' expression. The details of this technique can be seen in Bemporad and Morari (1999). Similarly, finding the largest lower bound for the asymptotic minimum growth rate $\sigma_{\min}$ is the same as maximizing $\{\tilde{a}\}_{\min}$ over r:

$$\tilde{\rho}_{\min} = \max_r \min_\ell \min_i a_i(\ell, r, t)$$
$$= \max_r \min_\ell \min_i \max_j \Big(A_{i,j}(\ell) + r_i + \min_q (B_{j,q}(\ell) - t_q)\Big)$$

Also, this optimization problem can be recast as the following mixed-integer linear programming problem.

$$\min_{r,t,\ell,\mu,\nu} (-\tilde{\rho}_{\min})$$

subject to

$$\begin{aligned} -a_i(\ell) + r_i + b_{j,\ell} &\leq -A_{i,j}(\ell) & \forall i,j,\ell \\ a_i(\ell) - r_i - b_{j,\ell} + M\,\nu_{i,j,\ell} &\leq A_{i,j}(\ell) + M & \forall i,j,\ell \\ \tilde{\rho}_{\min} - a_i(\ell) &\leq 0 & \forall i,j,\ell \\ b_{j,\ell} + t_q &\leq B_{j,q}(\ell) & \forall j,q,\ell \\ -b_{j,\ell} - t_q + M\,\mu_{j,q,\ell} &\leq -B_{j,q}(\ell) + M & \forall j,q,\ell \\ -\sum_q \mu_{j,q,\ell} \leq -1 \quad \forall j,\ell, \quad -\sum_j \nu_{i,j,\ell} &\leq -1 & \forall i,\ell \\ t &= C\,r \end{aligned} \tag{15}$$

where μ, ν are binary tensors, which are associated with a 'max min' expression as explained for the problem (14).

6. SIMULATIONS

In this simulation, we consider 250 well-defined, time-invariant, monotone, and arbitrary S-MMPS systems of the form (2) with $n = m = p = 5$ and $n_L = 4$ switching modes per S-MMPS system. We set 50% of the variables of the matrix A as ε, and 50% of the variable of the matrix B as $\top$, arbitrarily with a uniform distribution. The finite values of A and B are set to an integer value between 0 and 5 arbitrarily. The bounds $\rho_{\max}$ and $\rho_{\min}$ are computed using (5) and (6). The bounds $\tilde{\rho}_{\max}$ and $\tilde{\rho}_{\min}$ are computed using (14) and (15) in Matlab using YALMIP. Now, we perform 100 simulations for each generated S-MMPS system and compute the maximum value $\{x(N)\}_{\max}$, $N = 200$ over all 100 simulations, for arbitrary initial states $x(0)$ drawn from a normal distribution and arbitrary modes $\ell(k) \in \{1, \ldots, n_L\}$. Similarly, we also compute the minimum value $\{x(N)\}_{\min}$, $N = 200$ over all 100 simulations. We calculate the estimates of $\sigma_{\max}$ and $\sigma_{\min}$ (denoted as $\hat{\sigma}_{\max}$ and $\hat{\sigma}_{\min}$, respectively) as follows:

$$\hat{\sigma}_{\max} = \max_{i \in \{1,\ldots,100\}} \frac{\{x^{(i)}(N)\}_{\max}}{N}$$
$$\hat{\sigma}_{\min} = \min_{i \in \{1,\ldots,100\}} \frac{\{x^{(i)}(N)\}_{\min}}{N}$$

where $x^{(i)}(N)$ is the state value at i-th simulation.

Figure 1 plots $\rho_{\max}/\hat{\sigma}_{\max}$ and $\tilde{\rho}_{\max}/\hat{\sigma}_{max}$ whereas Figure 2 plots $\rho_{\min}/\hat{\sigma}_{\min}$ and, $\tilde{\rho}_{\min}/\hat{\sigma}_{min}$ for all the S-MMPS systems (250 systems) along with their corresponding box plots. The box plots show the average range of the data. The center line indicates the median. The markers that extend in y axis are the extreme points.

It can be observed from the box plots that the mixed-integer linear programming problem results in a tighter upper and lower bound as compared to the analytical bounds calculated using (5) and (6). The average improvement of the upper bound $(\rho_{\max} - \tilde{\rho}_{\max})/\hat{\sigma}_{\max}$ is 18.31% and the average improvement of the lower bound $(\tilde{\rho}_{\min} - \rho_{\min})/\hat{\sigma}_{\min}$ is 22.67%.

7. CONCLUSIONS

In this paper, we have considered switching max-min-plus-scaling systems, which is a subclass of discrete-event systems in which the system can switch between different modes of operation. We have proved that the growth rate of a time-invariant, and monotone S-MMPS system is bounded. We have also derived a mixed-integer linear programming problem to find the estimates of the smallest upper bound and largest lower bound of the growth rate. We found that the mixed-integer linear programming problem gives a tighter bound on the growth rate of the S-MMPS system.

In future research, we plan to relax the condition of monotonicity of S-MMPS systems and analyze the conditions for existence of bounds on growth rate. Further, formulate an optimization problem for this case to get optimum bounds.

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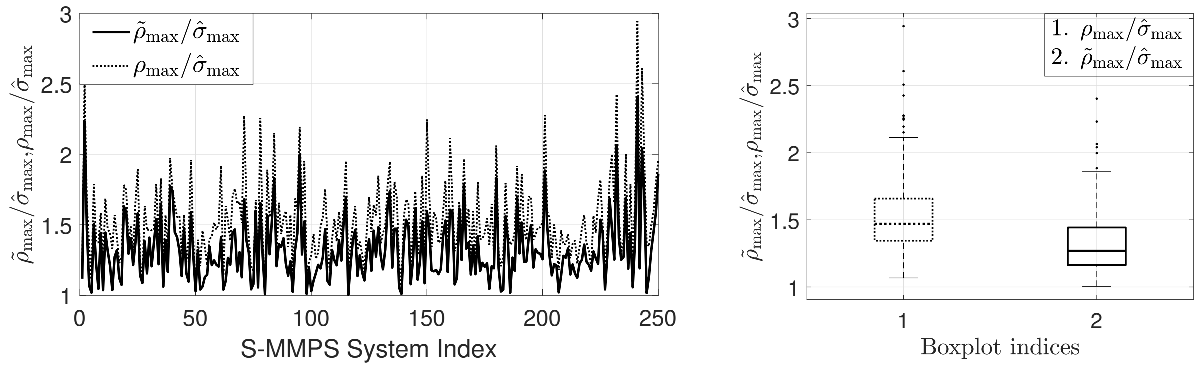

Fig. 1. Ratio of smallest upper bound to the growth rate of 250 well-defined, arbitrarily switching MMPS systems

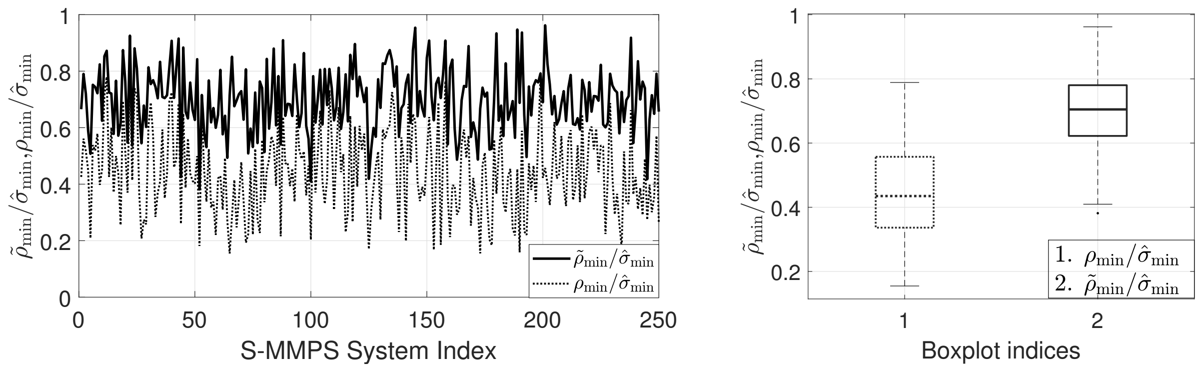

Fig. 2. Ratio of largest lower bound to the growth rate of 250 well-defined and arbitrarily switching MMPS systems

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