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Control of fluid flows using multivariate spline reduced order models

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This paper presents a study on control of fluid flows using multivariate spline reduced order models. A new approach is presented for model reduction of the incompressible Navier-Stokes equations using multivariate splines defined on triangulations. State space descriptions are derived that can be used for control design. This paper considers the linearised Navier-Stokes equations in velocity-pressure formulation. The pressure is eliminated from the equations by using a space of velocity fields which are divergence free. The divergence free condition along with the smoothness across the domain and the boundary conditions are imposed as a linear system of side constraints. The projection of the system on the null space of these constraints significantly reduces the dimension of the model while satisfying these constraints. The reduction method is applied to design and implement feedback controllers for stabilization of disturbances in a Poiseuille flow. It is shown that effective feedback stabilization can be achieved using low order control models.

I. Introduction

Control system approaches to active flow control (AFC) are considered as a viable route towards aerodynamic and acoustic improvements of aerodynamic bodies. A systems theory framework for flow control was first presented in [1] and since then, AFC has received a lot of attention from both the control community and fluid community. An account of research developments can be found in [2,3]. Control system approaches are often based on reduced order models of fluid flows. This paper presents a new approach for reduced order modeling and control of fluid flows using multivariate splines defined on triangulation.[4–6] In this initial study the method is applied to design and implement spatially localized output feedback controllers for a non-periodic Poiseuille flow. This method can in principle be applied to general geometries and allows for a straightforward extension to non-parallel flows.

Model reduction is the process of reducing the infinite dimensional Navier-Stokes (NS) equations (PDEs) to a set of finite dimensional ODEs that can be used for control design. By selecting an appropriate set of spatial basis functions the flow field can be written as a truncated series expansion with time-varying coefficients and an appropriate model reduction method can be applied to obtain a finite set of ODEs.[7] Galerkin projection is most commonly applied and in this method, one obtains a lower dimensional approximation of the flow by projecting the Navier-Stokes equations onto the set of spatial basis functions. Selection of the spatial basis functions is critical and has a great impact to the modeling performance and practical feasibility for control applications.

For wall bounded parallel flows Fourier spectral basis functions are commonly used as basis in the Galerkin projection.[1, 2, 8–12] Fourier Galerkin methods often exploit the spatial invariance property of parallel flows. In case of spatial invariance one assumes that the dynamics are invariant with respect to translations in one or more spatial coordinates and that the sensors and actuators are fully distributed along this coordinate in the domain. By using a Fourier expansion along the spatially invariant coordinates the system can be block diagonalized and decoupled in terms of discrete sets of wavenumbers that replace the spatial invariant coordinates.[1, 8] In this way analysis and design of the controller can be carried out on a parameterized

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lower-dimensional system and later reconstructed in physical space.[12,13] Analogous formulations can be constructed in the fully continuous (PDE) setting.[13,14]

Many practical flow control applications consider non-parallel flows such as spatially developing boundary layers, or require that the actuators and sensors are spatially localized. This led to the use of modeling methods which make no assumptions on the flow geometry, and the shape and distribution of the actuators and sensors.[15–19] These methods often rely on an initial accurate high order validation model such as a full CFD discretization. One such an approach is the Proper Orthogonal Decomposition (POD) method and in this methods a reduced order basis is obtained from experimental or simulation data of the high order system. We refer to [20] for an in-depth coverage of the application of the POD Galerkin method for flow control and references therein. The POD method extracts the most energetic modes from a set of snapshots and therefore leads to low order expansions. However POD-Galerkin methods are not robust and have limitations for describing the input-output behavior of the system.[21,22] A more accurate approach for capturing the input-output behavior is balanced truncation.[23] This method is commonly applied in the control community and extracts the most controllable and observable modes of the system. This involves the computation of the controllability and observability Gramians of the high order model. These Gramians are usually obtained by solving a set of Lyapunov equations which becomes computationally intractable for high order systems (e.g. 10000 states or more). To lower the complexity an approximate method is proposed in [22], called balanced POD (BPOD), in which empirical Gramians are computed directly from a set of snapshots. This method is suited for large systems as it avoids the direct computation of the Gramians. It is shown that BPOD results in models that are equivalent to those obtained through balanced truncation. In either case an initial model of the flow in state space format is required.

In this paper a new approach is introduced to derive state space descriptions of fluid flows using multivariate splines defined on triangulations.[4–6] These models can subsequently be used in combination with (approximate) balanced truncation to construct reduced order models suitable for real-time application. This method is similar to the finite element method (FEM) in the sense that it uses piecewise defined polynomials defined on triangulations as basis in the Galerkin projection. The main features are: arbitrary degree of the basis polynomials, arbitrary smoothness between the elements and reduced order of the model. This method considers the linearized Navier-Stokes equations in velocity-pressure formulation. The pressure is eliminated from the equations by using a space of velocity fields which are divergence free. The divergence free condition along with any desired smoothness conditions across the domain and the boundary conditions are imposed as a linear system of side constraints. The projection of the system on the null space of these constraints significantly reduces the dimension of the model while satisfying these constraints. This reduction makes the application of control theoretic tools computational tractable.

The multivariate B-spline has been used in the past to find numerical solutions of elliptic PDEs[24,25] and steady Navier-Stokes equations[26,27] based on energy methods. This work departs from the followed approach in the sense that is does not find explicit numerical solutions for PDEs. Instead the NS equations are spatially discretized using the numerical techniques from [24,26,27] and converted to a state space representation that can be used for control design.

The outline of the paper is as follows. In Section II the control problem is formulated. In Section III the variational formulation is introduced through which the spline approximation is determined. In Section IV the weak formation is used to derive state space descriptions of the NS equations. In Section V the state space models are used to design and implement feedback controllers to stabilize convective disturbances in a Poiseuille flow followed by conclusions in Section VI.

II. Problem statement

This papers considers stabilization of laminar viscous flows modeled by the incompressible Navier-Stokes (NS) equations

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\, \mathbf{u} + \nabla p &= \mathbf{f} && \text{in } \Omega && \text{(1a)} \\ \nabla \cdot \mathbf{u} &= 0 && \text{in } \Omega && \text{(1b)} \\ \mathbf{u} &= \mathbf{g} && \text{on } \Gamma_{in} && \text{(1c)} \\ -p\mathbf{n} + \nu\, (\mathbf{n} \cdot \nabla)\, \mathbf{u} &= 0 && \text{on } \Gamma_N && \text{(1d)} \end{aligned}$$

With $\mathbf{u}(\mathbf{x},t)$ the velocity vector $p(\mathbf{x},t)$ the kinematic pressure, $\mathbf{f}(\mathbf{x},t)$ a body force field per unit mass (typically used for applying control) and ν the kinematic viscosity. When $\mathbf{u}$ and p are nondimensionalized, ν is the inverse of the Reynolds number Re. The system is closed by the boundary conditions (1c)-(1d) where Γ_D is the Dirichlet part of the boundary, Γ_N the Neumann outflow part of the boundary and $\mathbf{g}(\mathbf{x})$ a prescribed velocity profile. The outflow boundary condition (1d) is also known as a "do nothing" boundary conditions and has proven to be the well suited for unidirectional outflows.[28] It naturally occurs in the variational formulation if nothing is prescribed at the outflow as will be seen in Section III. The body force $\mathbf{f}$ source term represents the actuator model to control the flow and is assumed to be known. The effect of the actuators are modeled as a linear distributed force field

$$\mathbf{f}(\mathbf{x},t) = \sum_{i=1}^{m} \mathbf{f}_i(\mathbf{x})\delta_i(t) = \begin{bmatrix} \mathbf{f}_1(\mathbf{x}) & \mathbf{f}_2(\mathbf{x}) & \cdots & \mathbf{f}_m(\mathbf{x}) \end{bmatrix} \begin{bmatrix} \delta_1(t) \\ \delta_2(t) \\ \vdots \\ \delta_m(t) \end{bmatrix} = F(\mathbf{x})\boldsymbol{\delta}(t) \tag{2}$$

with $\mathbf{f}_i(x)$ a vector function that describes how the physical control action $\delta_i(t)$ from the i-th actuator is distributed in the domain Ω. Physical examples of such distributions are models of plasma actuators,[29] electromagnetic actuators and magneto-hydrodynamic actuators. We refer to [30] for a recent review on actuators for flow control. The goal is to stabilize (1a) around the equilibrium profile. Let $(\mathbf{u}_0, p_0)$ be a steady state solution of (1a) without control, that is a solution of (1) with $\frac{\partial \mathbf{u}}{\partial t} = 0,\ \mathbf{f} = 0$. Defining the perturbations around the equilibrium $(\mathbf{u}_0, p_0)$ as

$$\mathbf{u}'(\mathbf{x},t) = \mathbf{u}(\mathbf{x},t) - \mathbf{u}_0(\mathbf{x}) \tag{3}$$

$$p'(\mathbf{x},t) = p(\mathbf{x},t) - p_0(\mathbf{x}) \tag{4}$$

Assuming that the perturbations are small, the dynamics of the perturbations $\mathbf{u}'(\mathbf{x},t)$ is governed by the NS equations linearized around the base flow. Substituting (3) and (4) in (1), subtracting the steady equations from (1) and dropping the nonlinear term gives the linearized NS equations

$$\frac{\partial \mathbf{u}'}{\partial t} - \nu\Delta\mathbf{u}' + (\mathbf{u}_0 \cdot \nabla)\,\mathbf{u}' + (\mathbf{u}' \cdot \nabla)\,\mathbf{u}_0 + \nabla p' = \mathbf{f} \quad \text{in } \Omega \tag{5a}$$

$$\nabla \cdot \mathbf{u}' = 0 \quad \text{in } \Omega \tag{5b}$$

$$\mathbf{u}' = 0 \quad \text{on } \Gamma_D \tag{5c}$$

$$-p'\mathbf{n} + \nu\,(\mathbf{n} \cdot \nabla)\,\mathbf{u}' = 0 \quad \text{on } \Gamma_{out} \tag{5d}$$

(5a) is commonly used in hydrodynamic stability theory[31] and linear system approaches to flow control.[2,32] Controllers which stabilize (5), stabilize the original system (1) around the base flow. Such controllers can then be used for example to prevent or delay transition in laminar boundary layer flows[3,8,10] or to suppress the vortex shedding of bluff body flows.[33] The system (5) is the point of departure for constructing a reduced order model that can be used for control design. The objective is to reduce (5) to a finite set of ordinary differential equations. This requires both a solution for the mean flow $\mathbf{u}_0$ and the spatial discretization of (5). In this study it is assumed that the mean flow can be calculated on beforehand using either analytic solutions or numerical solutions of the steady NS equations. We only consider the spatial discretization of (5) using multivariate splines.

III. Galerkin-type variational formulation

In this section the variational formulation is introduced through which the spline approximation is determined. In order to introduce the variational formulation some functions spaces need to be defined. Let $L^2(\Omega)$ be the space of square-integrable function over Ω. We define the following Sobolev space

$$H^1(\Omega) = \left\{ u \in L^2(\Omega) : \frac{\partial u}{\partial x_i} \in L^2(\Omega) \text{ for } i = 1, \cdots, n \right\} \tag{6}$$

and the subspace in which all functions vanish at the Dirichlet boundary

$$H^1_0(\Omega) = \left\{ u \in H^1(\Omega) : u|_{\Gamma_D} = 0 \right\} \tag{7}$$

For vector valued functions we use

$$\mathbf{H}^1(\Omega) = H^1(\Omega)^n = \{\mathbf{u} : u_i \in H^1(\Omega) \quad \text{for } i = 1, \cdots, n\} \tag{8}$$

We define the bilinear form

$$a(\mathbf{v}, \mathbf{u}) = \int_\Omega \nabla\mathbf{v} : \nabla\mathbf{u} \, d\Omega = \int_\Omega \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j} \, d\Omega \qquad \forall \, \mathbf{v}, \mathbf{u} \in \mathbf{H}^1(\Omega) \tag{9}$$

and the trilinear form

$$b(\mathbf{v}, \mathbf{u}, \mathbf{w}) = \int_\Omega \mathbf{v} \cdot (\mathbf{u} \cdot \nabla)\, \mathbf{w} \, d\Omega = \int_\Omega \sum_{i=1}^{n}\sum_{j=1}^{n} v_i u_j \frac{\partial w_i}{\partial x_j} \, d\Omega \qquad \forall \, \mathbf{v}, \mathbf{u}, \mathbf{w} \in \mathbf{H}^1(\Omega) \tag{10}$$

Also the inner product for functions belonging to $\mathbf{L}^2(\Omega)$ is given by

$$(\mathbf{v}, \mathbf{u}) = \int_\Omega \mathbf{v} \cdot \mathbf{u} \, d\Omega \tag{11}$$

Equation (5) has no dynamic equation for the pressure that can be utilized for control. Therefore the pressure is eliminated from the variational formulation. This is done by using a space of velocity fields which are exactly divergence free

$$\mathbf{V}_0 = \{\mathbf{u} \in \mathbf{H}^1_0(\Omega) : \nabla \cdot \mathbf{u} = 0\} \tag{12}$$

The weak form of (5) can be obtained by multiplying the equation by a test function $\mathbf{v} \in \mathbf{V}_0(\Omega)$ and integrating over the domain

$$\int_\Omega \left\{ \mathbf{v} \cdot \frac{\partial \mathbf{u}}{\partial t} - \mathbf{v} \cdot \nu\Delta\mathbf{u} + \mathbf{v} \cdot (\mathbf{u}_0 \cdot \nabla)\,\mathbf{u} + \mathbf{v} \cdot (\mathbf{u} \cdot \nabla)\,\mathbf{u}_0 + \mathbf{v} \cdot \nabla p \right\} \, d\Omega = \int_\Omega \mathbf{v} \cdot \mathbf{f} \, d\Omega \tag{13}$$

The prime superscript to denote that the flow field is a perturbation field is omitted in (13) to improve readability. Applying integration parts and the divergence theorem to the diffusion term and the pressure gradient term gives

$$\begin{aligned} &\int_\Omega \left\{ \mathbf{v} \cdot \frac{\partial \mathbf{u}}{\partial t} + \nu\nabla\mathbf{v} : \nabla\mathbf{u} + \mathbf{v} \cdot (\mathbf{u} \cdot \nabla)\,\mathbf{u} + \mathbf{v} \cdot (\mathbf{u} \cdot \nabla)\,\mathbf{u}_0 - p\,(\nabla \cdot \mathbf{v}) \right\} \, d\Omega \\ - &\int_{\Gamma_D} \nu\mathbf{v} \cdot (\mathbf{n} \cdot \nabla)\,\mathbf{u} \, d\Gamma + \int_{\Gamma_D} \mathbf{v} \cdot \mathbf{n}p \, d\Gamma - \int_{\Gamma_N} \mathbf{v} \cdot (-p\mathbf{n} + \nu\,(\mathbf{n} \cdot \nabla)\,\mathbf{u}) \, d\Gamma = \int_\Omega \mathbf{v} \cdot \mathbf{f} \, d\Omega \end{aligned} \tag{14}$$

The Neumann outflow boundary condition (5d) occurs in (14) as a boundary integral term and can therefore naturally be imposed by setting it to zero. Furthermore, $\nabla \cdot \mathbf{v} = 0$ and $\mathbf{v}|_{\Gamma_D} = 0$ for all $\mathbf{v} \in \mathbf{V}_0$. The variational formulation of (5a) under the incompressibility condition (5b) and boundary conditions Eqs. (5c)-(5d) can be stated as: Given $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and $\mathbf{u}_0 \in \mathbf{V}$, find $\mathbf{u} \in \mathbf{V}_0$ such that

$$\left(\mathbf{v}, \frac{\partial \mathbf{u}}{\partial t}\right) + a\,(\mathbf{v}, \mathbf{u}) + b\,(\mathbf{v}, \mathbf{u}_0, \mathbf{u}) + b\,(\mathbf{v}, \mathbf{u}, \mathbf{u}_0) = (\mathbf{v}, \mathbf{f}) \qquad \forall \, \mathbf{v} \in \mathbf{V}_0 \tag{15}$$

IV. State space formulation for the linearized Navier-Stokes equations

In this section finite dimensional state space descriptions of (15) are constructed using multivariate splines. We use the methodology from [26] in which a numerical scheme is presented to approximate steady Navier-Stokes equations in velocity pressure formulation using multivariate splines. To keep this paper short and accessible, no detailed preliminaries on multivariate splines are given. Instead we outline the methodology from [26] and how this methodology is adapted to construct state space models for the linearized Navier-Stokes equations. Only the essential properties of spline spaces and spline functions necessary for the treatment of the spline model reduction framework is discussed. We refer to [6] for a complete coverage on the multivariate spline theory.

A. Spatial Discretization

First the approximating spline space for the velocity field is chosen. Let $\mathcal{T}$ be the triangulation of Ω if Ω is a polygonal domain. Otherwise we chose the vertices on the boundary Γ such that $\mathcal{T}$ becomes the approximation of Ω. The spline space is the space of all smooth piecewise polynomial functions of degree d and smoothness r over $\mathcal{T}$ with $0 \leq r < d$

$$S_d^r(\mathcal{T}) := s \in C^r(\Omega) : s|_t \in \mathcal{P}_d, \ \forall t \in \mathcal{T} \tag{16}$$

With $\mathcal{P}_d$ the space of all polynomials of total degree d and t denotes a triangle. Let d and r be two positive integers with $d > r$, $r \geq 0$ and let $\mathbf{S}_0$ be a spline subspace consisting of divergence free spline vector functions which are C^r inside Ω and vanish at the Dirichlet boundary

$$\mathbf{S}_0 = \{\mathbf{s} : s_i \in S_d^r(\mathcal{T}), \ s_i|_{\Gamma_D} = 0, \ \nabla \cdot \mathbf{s} = 0 \quad \text{for } i = 1, \cdots, n\} \tag{17}$$

To approximate the velocity vector $\mathbf{u} = (u_1, u_2, u_3)$ we let $\mathbf{s_u} = (s_1, s_2, s_3) \in \mathbf{S}_0$ be the spline approximating vector. Thus we choose $\mathbf{S}_0 \subset \mathbf{V}_0$ as the approximating space and seek $\mathbf{s_u}(\cdot, t) \in \mathbf{S}_0$ such that

$$\left(\mathbf{s_v}, \frac{\partial \mathbf{s_u}}{\partial t}\right) + a(\mathbf{s_v}, \mathbf{s_u}) + b(\mathbf{s_v}, \mathbf{u}_0, \mathbf{s_u}) + b(\mathbf{s_v}, \mathbf{s_u}, \mathbf{u}_0) = (\mathbf{s_v}, \mathbf{f}) \qquad \forall\, \mathbf{s_v} \in \mathbf{S}_0 \tag{18}$$

After choosing a spline basis for $\mathbf{S}_0$, (18) is equivalent to a system of ordinary differential equations. The main problem with implementing (18) is the construction of a basis for the appropriate divergence free spline subspace $\mathbf{S}_0$. In [26] Awanou & Lai streamlined this process by skipping the construction of smooth divergence free finite elements. Instead, they used discontinuous piecewise polynomial functions over a triangulation and treated desired smoothness properties together with the boundary conditions and the incompressibility condition as side constraints. This approach is also used in this study and adapted for the time dependent problem (18).

The multivariate spline function is represented using the B-form of splines.[4,5] We use the vector formulation from [34]

$$s(\mathbf{x}, t) = B^d(\mathbf{x})\mathbf{c}(t) \tag{19}$$

with $B^d(\mathbf{x}) \in \mathbb{R}^{1 \times \cdot n_e \hat{d}}$ the global vector of basis polynomials, n_e the number of elements in $\mathcal{T}$ and $\hat{d} = \binom{n}{n+d}$ the number of basis polynomials per element. The spline function is identified by its B-coefficients vector $\mathbf{c}(t) \in \mathbb{R}^{n_e \hat{d} \times 1}$ which are used as the time-varying expansion coefficients. Since s has a certain smoothness, the smoothness conditions can be expressed by a linear system. That is $s \in C^r$ if and only if

$$H\mathbf{c} = 0. \tag{20}$$

Constructing H is not trivial and we refer to [34] for a general formulation of the continuity conditions and the procedure to derive them. The Dirichlet boundary condition (5c) provides additional constraints on the B-coefficient vector. These constraints can also be given by a linear system[24,35]

$$R\mathbf{c} = 0 \tag{21}$$

The function $s \in C^r(\Omega)$ is guaranteed to be r-times continuously differentiable on the domain Ω. To approximate the variational formulation only first order derivatives are required. It is shown in [26, 36] that there exist matrices D_i which map the B-coefficient vector of any spline function $s \in S_d^r(\mathcal{T})$ to the B-coefficient vector of $\frac{\partial}{\partial x_i} s$, that is

$$\frac{\partial}{\partial x_i}\left[B^d(\mathbf{x})\mathbf{c}\right] = B^{d-1}(\mathbf{x}) D_i \mathbf{c} \tag{22}$$

To approximate the velocity vector $\mathbf{u} = (u_1, u_2, u_3)$ we use $\mathbf{s_u} = (s_1, s_2, s_3)$ which are identified with B-coefficients $\mathbf{c} = (\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3)$. Hence the discrete analog of $\nabla \cdot \mathbf{u} = 0$ is given by[24]

$$D_1\mathbf{c}_1 + D_2\mathbf{c}_2 + D_3\mathbf{c}_3 = \begin{bmatrix} D_1 & D_2 & D_3 \end{bmatrix} \mathbf{c} = \bar{D}\mathbf{c} = 0 \tag{23}$$

Let $\bar{H}$ and $\bar{R}$ be the matrices that encode the smoothness conditions and the boundary conditions for the complete discrete velocity field. Furthermore let

$$L = \begin{bmatrix} \bar{H}^T & \bar{R}^T & \bar{D}^T \end{bmatrix}^T \tag{24}$$

then for all spline vector functions $\mathbf{s} = (s_1, s_2, s_3)$ with B-coefficient $\mathbf{c} = (\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3)$ satisfying

$$L\mathbf{c} = 0 \tag{25}$$

we have that $\mathbf{s} \in \mathbf{S}_0$, and can thus be used to approximate the variational formulation.

B. Transformation to state space

In this section the discrete system resulting from the spatial discretization defined by (18) is tranformed to state space format. Let $\mathbf{d}$ denote the B-coefficient vector of the test function $\mathbf{s_v}$ satisfying $L\mathbf{d} = 0$. (18) results in the following system of equations

$$\mathbf{d}^T M \frac{d}{dt}\mathbf{c} + \mathbf{d}^T K \mathbf{c} = \mathbf{d}^T F \boldsymbol{\delta} \tag{26}$$

$$L\mathbf{c} = 0 \tag{27}$$

where M is a velocity mass matrix and $\mathbf{d}^T K \mathbf{c}$ denotes the discretization of the linear diffusion term and the two linear convective terms. The right hand side matrix F contains the contribution of the actuator model defined in (2) and $\boldsymbol{\delta}(t)$ denotes the vector of temporal control inputs. We refer to [26] for details regarding the constructive aspects and assembling of the matrices in (26). In [26] the side constraints were enforced through Lagrange multipliers. In this paper we propose a null space approach which significantly reduces the size of the system with rank of the side constraints, making the state space models more suitable for control applications. Let V be the basis of $null(L)$ such that $LV = \mathbf{0}$. The general solution set for (27) can be written as

$$\mathbf{c} = V\tilde{\mathbf{c}} \tag{28}$$

with $\tilde{\mathbf{c}} \in \mathbb{R}^{N-R^*}$ the coordinate vector of $\mathbf{c}$ relative to the basis for $null(L)$ and with R^* the rank of L. Since (26) must hold for all $\mathbf{d}$ satisfying $L\mathbf{d} = 0$, the solution set for $\mathbf{d}$ can also be written as

$$\mathbf{d} = V\tilde{\mathbf{d}} \tag{29}$$

Substituting (28) for $\mathbf{c}$ and (29) for $\mathbf{d}$ in (26) gives

$$\tilde{\mathbf{d}}^T V^T M V \frac{d}{dt}\tilde{\mathbf{c}} + \tilde{\mathbf{d}}^T V^T K V \tilde{\mathbf{c}} = \tilde{\mathbf{d}}^T V^T F \boldsymbol{\delta} \tag{30}$$

a reduced order system projected on the null space of the side constraints. Since (30) must hold for all $\tilde{\mathbf{d}}$, (30) is equivalent to

$$V^T M V \frac{d}{dt}\tilde{\mathbf{c}} + V^T K V \tilde{\mathbf{c}} = V^T F \boldsymbol{\delta} \tag{31}$$

Assuming that the reduced order system is non-singular the system can be solved for $\frac{d}{dt}\tilde{\mathbf{c}}$

$$\frac{d}{dt}\tilde{\mathbf{c}} = \left(V^T M V\right)^{-1} \left(-V^T K V \tilde{\mathbf{c}} + V^T F \boldsymbol{\delta}\right) \tag{32}$$

Finally (32) can be written in state space format

$$\frac{d}{dt}\tilde{\mathbf{c}} = A\tilde{\mathbf{c}} + B\boldsymbol{\delta} \tag{33}$$

with A the system matrix B the input matrix given by

$$A = \left(V^T M V\right)^{-1} V^T K V \tag{34}$$

$$B = \left(V^T M V\right)^{-1} V^T F \tag{35}$$

By projecting the set of ODEs (26) on the null space of the side constraints (27) the dimension of the model is reduced with rank R^* of these constraints. The elimination of only the incompressibility constraints in (23) already reduces the size of the system with the dimension of the spatial basis for the first order derivative, that is the dimension of $B^{d-1}(\mathbf{x})$. Assuming that the dimension of the basis $B^{d-1}(\mathbf{x})$ is approximately equal to the dimension of the original basis $B^d(\mathbf{x})$, which is the case for a high degree basis. Then, with this reduction it is as if only $n-1$ equations need to be approximated. So in the 3-D case it is as if only 2 equation need to be approximated.

To complete the state space system the measurement equation is converted to linear output equations for the B-coefficients. In this study it is assumed that point shear measurements from K boundary sensors are available for feedback which are modeled as

$$z_i(t) = \frac{\partial}{\partial \mathbf{n}}\mathbf{u}(\mathbf{x}_i, t), \quad i = 1, 2, \cdots, K \tag{36}$$

with $\mathbf{x}_i$ points on the boundary. The discrete analog of (36) is given by

$$z_i^N(t) = B^{d-1}(\mathbf{x}_i) D_{\mathbf{n}} \mathbf{c}(t) \tag{37}$$

where $D_{\mathbf{n}}$ maps the B-coefficient vector of s to the B-coefficient vector of $\frac{\partial}{\partial \mathbf{n}} s$. Substituting the solution set (28) for $\mathbf{c}(t)$ in (38) gives

$$z_i^N(t) = B^{d-1}(\mathbf{x}_i) D_{\mathbf{n}} V \tilde{\mathbf{c}} = C_i \tilde{\mathbf{c}}(t) \tag{38}$$

V. Feedback stabilization of convective instabilities in a Poiseuille flow

Stabilization of convective instabilities in a two-dimensional non-periodic Poiseuille flow is considered as a demonstration test case for the developed technique. The model reduction framework is applied to design and implement linear output feedback controllers. An accurate high order model is assumed to be the 'true' system that is used for simulating the response and a lower order balanced truncation of the simulation model is used for controller and observer design. The geometry of the channel, the control layout along with the triangulation used in this study is shown in Figure 1. All body forces are modeled as a horizontal body force

$$\mathbf{f}(\mathbf{x}, t) = \begin{bmatrix} f(\mathbf{x}) & 0 \end{bmatrix}^T \delta(t) \tag{39}$$

with $f(\mathbf{x})$ the force distribution as shown in Figure 1 and $\delta(t)$ the manipulated input. The control objective is to stabilize the perturbations around the steady-state parabolic velocity profile $\mathbf{u}_0 = \left[1 - y^2,\ 0\right]^T$ where the velocity field quantities are non-dimensionalized by the channel half-height and the centerline velocity with corresponding Reynolds number $Re = 2000$. To demonstrate the controller, a disturbance is introduced in the domain using the first upstream volume force $f_w(\mathbf{x})$ which is convected downstream by the flow. The goal is to cancel the disturbance using the two downstream control actuators modeled by $f_{c_1}(\mathbf{x})$ and $f_{c_2}(\mathbf{x})$. The streamwise component of shear at two points ($z_{m_i}(\mathbf{x}_i, t) = \partial u_1(\mathbf{x}_i, t)/\partial y$) at the lower wall are used as measurements for feedback. The shear at a single point downstream ($z_c(\mathbf{x}_i, t)$) is defined as the controlled output which is used to define the control objective later in this section. The complete system is transformed to state space using a fifth order continuous spline bases $\left(s \in S_5^1(\mathcal{T}_{480})\right)$ and takes the form

$$\begin{aligned} \dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B_c \boldsymbol{\delta}(t) + B_w w(t) \\ z_m(t) &= C_m \mathbf{x}(t) \\ z_c(t) &= C_c \mathbf{x}(t) \end{aligned} \tag{40}$$

where B_w represents the disturbance model and $w(t)$ the disturbance input. The dimension of the state space system is given by $N - R^* = 20160 - 18254 = 1906$. This shows the significance of the size reduction R^* resulting from the null space projection. By applying the state transformation (28) the $S_5^1(\mathcal{T}_{480})$ model is reduced from 20160 states to 1906 states. This initial reduction makes the application of balanced truncation computational tractable.

A reduced order model (ROM) is constructed by first creating a balanced realization of the system (40) after which the states corresponding to small Hankel singular values are discarded. Figure 2 shows the Hankel singular values of the balanced realization. These values indicate that the input-output behavior of the system can be captured by low order models. We construct the balanced truncation (A^r, B^r, C^r) by selecting the states corresponding to the first 50 singular values. The input-output behaviour of the reduced order model is compared to the full system in Figure 3. This figure shows the impulse response (a) and the magnitude frequency response (b) from the disturbance w to the first output z_{m_1} and from the first input δ_1 to the controlled output z_c. Similar results hold for the other transfer functions. There is a good agreement between the full system and the ROM. The ROM only deviates at higher frequencies where the system has a low gain.

Next, the ROM is used to design the controller. A classical LQR optimal control problem is considered were the controlled output is used to build the objective function

$$\mathcal{J} = \int_0^\infty z_c^T z_c + \boldsymbol{\delta}^T R \boldsymbol{\delta}\ dt = \int_0^\infty (\mathbf{x}^r)^T (C_c^r)_c^T C_c^r \mathbf{x}^r + \boldsymbol{\delta}^T R \boldsymbol{\delta}\ dt \tag{41}$$

The state feedback $\boldsymbol{\delta}(t) = -K_c \mathbf{x}^r$ that minimizes (41) can be computed by solving the associated algebraic Riccati equation for (41). An output feedback controller is obtained by combining the state feedback with a state observer and takes the form

$$\begin{aligned} &\boldsymbol{\delta}(t) = -K_c \hat{\mathbf{x}}^r(t) \\ &\dot{\hat{\mathbf{x}}}^r(t) = A^r \hat{\mathbf{x}}^r + B_c^r \boldsymbol{\delta}(t) + K_o \Big(z_m(t) - \hat{z}_m(t) \Big) \\ &\hat{z}_m(t) = C_m^r \hat{\mathbf{x}}^r(t), \qquad \hat{\mathbf{x}}^r(0) = 0 \end{aligned} \tag{42}$$

where the observer gain K_o is tuned such that $A^r - K_o C_m^r$ has desired stability margins.

To demonstrate the controller a broad frequency spectrum is excited using a short Dirac forcing pulse for the upstream disturbance, that is $w = \{1 \text{ if } t = t_0,\ 0 \text{ if } t \neq t_0\}$. Figure 4 shows the performance of the controller, Figure 5 shows the control inputs and Figure 6 visualizes the evolution of the streamwise perturbation field. As the perturbation convects downstream towards the actuator region (time-interval $2 \leq t \leq 8$), the actuators respond and effectively cancel the perturbation. After $t = 10s$ the perturbation starts leaving the computational domain which explains the energy decrease after $t = 10s$ in Figure 4. The controller is designed to minimize the shear ($z_c = \partial u_1 / \partial y$) at a single point downstream which is shown in the right plot of Figure 4. It can be observed that even with this simple performance objective the controller effectively damps the perturbation and achieves a significant reduction of the total perturbation energy in the domain.

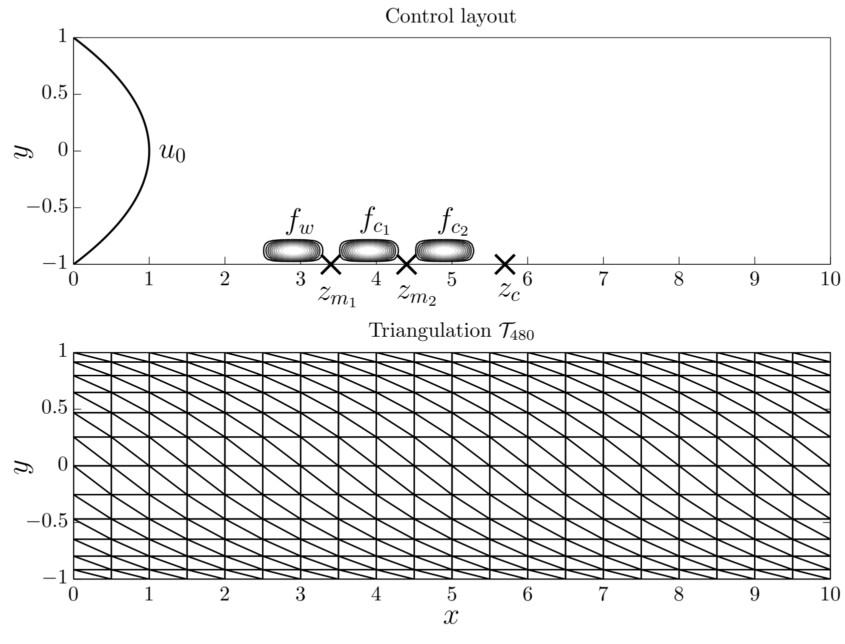


Figure 1. Control layout and triangulation. Top: domain, actuator distribution contours $f_i(x)$ and sensor locations z_i. Bottem: Triangulation with 480 simplices used to construct the state space model

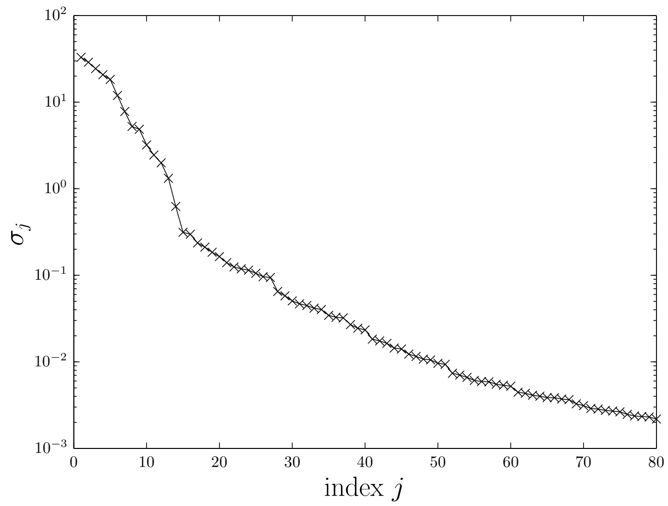


Figure 2. First 80 Hankel singular values of the balanced realization of the full system

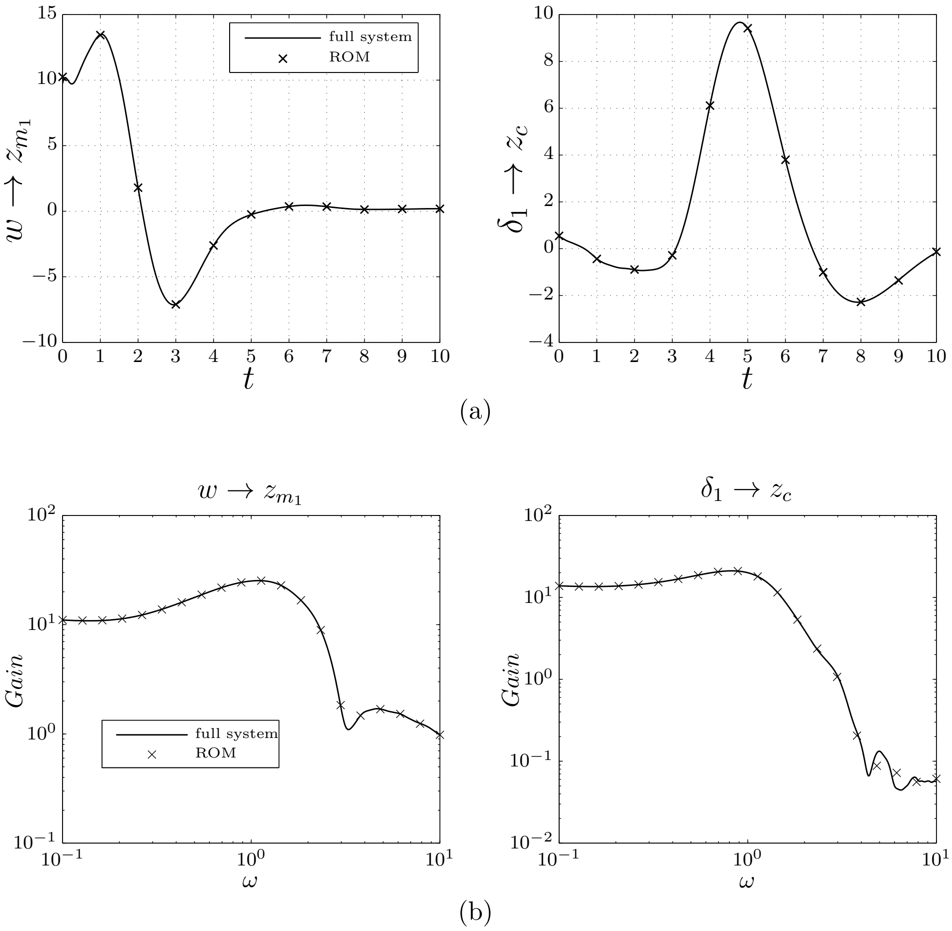


Figure 3. Performance of the reduced order model (ROM) compared with the full system: (a) Impulse response, (b) magnitude frequency response from disturbance w to output z_{m_1} and from input δ_1 to the controlled output z_c.

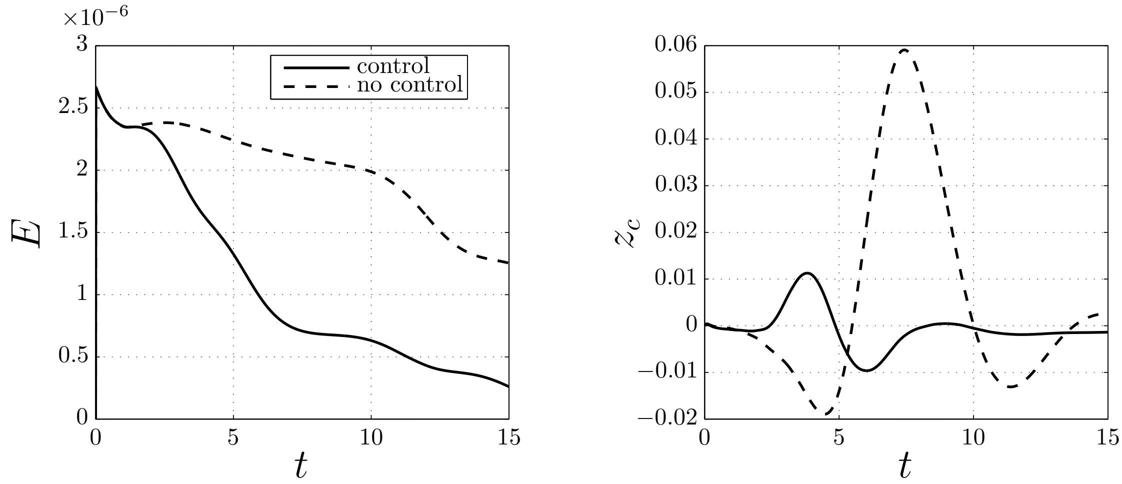


Figure 4. Performance of the controller. The total perturbation energy $E = \|\mathbf{u}'\|^2_{L_2}$ and the controlled output

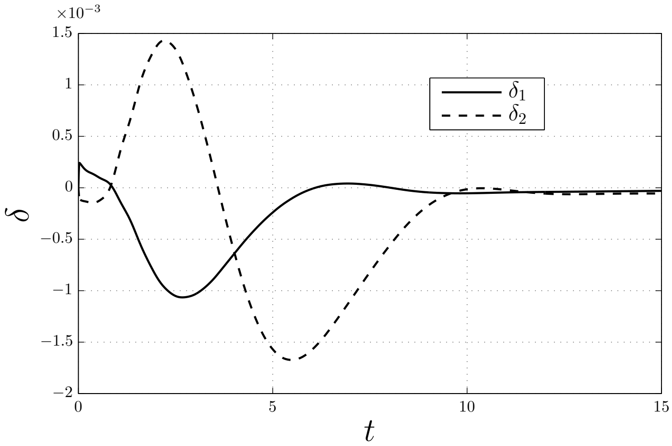
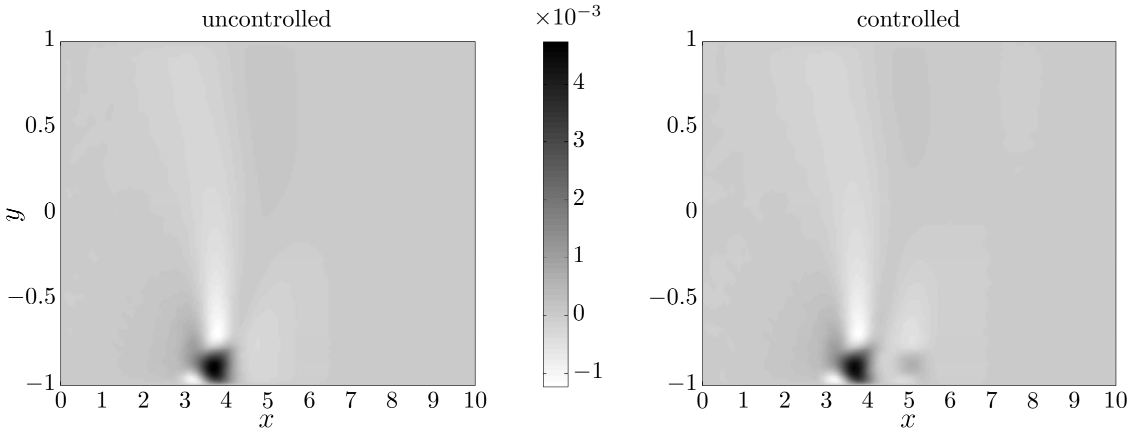


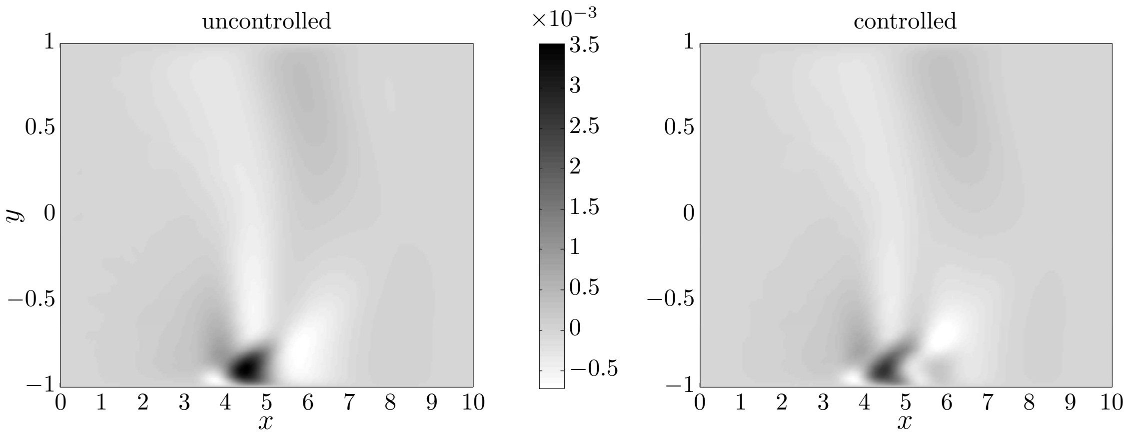
Figure 5. Control input

VI. Conclusion

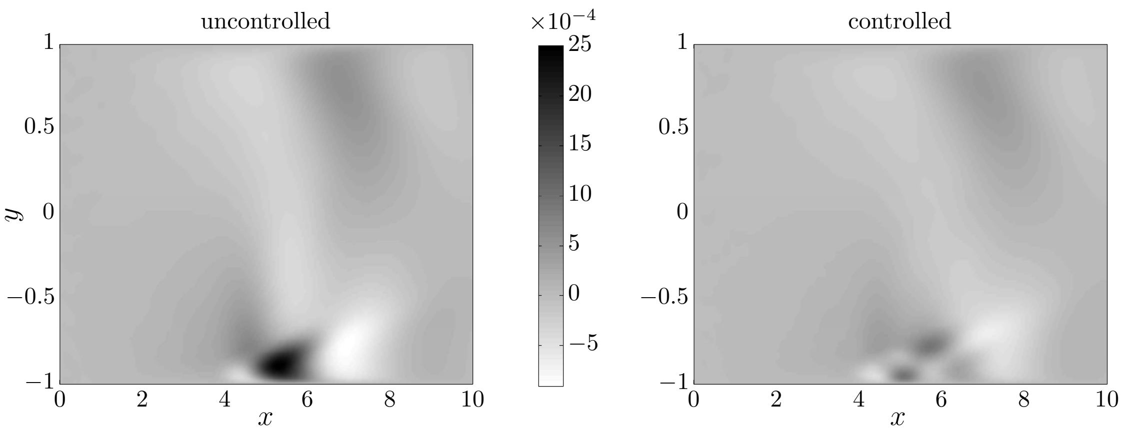
This paper presented a new approach to derive state description for fluid flows using multivariate splines. Multivariate splines are very user-friendly for creating control models. First, they are defined on triangular meshes allowing to approximate any domain and to use local refinements in regions of interest. Secondly, they are general in terms of smoothness and degree allowing for a high "spectral like" resolving power. The degree and order of continuity of splines are simply input variables for creating the state space models. The method is used in combination with balanced truncation to construct reduced order models suitable for real time application and is successfully applied to design and implement feedback controllers for a non-periodic Poiseuille flow using a set of spatially localized sensors and actuators. It is shown that effective feedback stabilization of convective instabilities can be achieved using low order models. This method can in principle be applied to general geometries and allows for a straightforward extension to non-parallel flows. Future work will focus on the testing of the controllers in a full nonlinear Navier-Stokes (CFD) solver and on the application of this method to more complex flow configurations such as spatially developing boundary layer flows and open cavity flows.



a) $t = 2.5s$



b) $t = 5s$



c) $t = 7.5s$

Figure 6. The streamwise perturbation field at three time instants

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