Testing nonlocality in quantum networks using iteratively obtained Bell inequalities

by

Rik van Heerden

to obtain the degree of Bachelor of Science
at the Delft University of Technology,
to be defended publicly on Monday July 9, 2018 at 3:30 PM.

Student number:	4487826	
Project duration:	September 1, 2017 – July 9, 2018	
Thesis committee:	Dr. J. L. A. Dubbeldam,	TU Delft, supervisor
	Dr. T. H. Taminiau,	TU Delft, supervisor
	Dr. M. Blaauboer,	TU Delft
	Dr. W. G. M. Groenevelt,	TU Delft

TU Delft

Abstract

Quantum networks play an important role in the fields of quantum information and quantum computation. One of the current problems for these networks concerns nonlocality. Characterizing and detecting nonlocality is relevant to the implementation of quantum networks and quantum repeaters, where Bell inequalities can be used to test if configurations are prepared correctly.
This report contains an overview of an iterative method to find Bell inequalities for networks. Starting from a given network, this method constructs a new Bell inequality for a network containing one additional source and one additional party. We use this procedure to find new Bell inequalities for specific network structures and analyse how these can be used to detect nonlocality within a network.

In the first part the Bell inequalities are considered from a more theoretical point of view. We focus on star-shaped networks and discuss violations predicted by quantum mechanics. We look for quantitative bounds describing a set of states that lead to violation, giving an indication of the required quality of the sources.

Finally this method is applied to a setup similar to the one described by Bernien et al. [2]. We consider a network consisting of three parties and two sources. The effect of errors during preparation of an entangled pair of photons on the ability to detect nonlocality is evaluated. The same is done for the effect of measurement errors. Using numerical computations we show that violation of the Bell inequality can be improved by choosing different measurement angles.

Contents

1

Introduction

At the beginning of the twentieth century quantum mechanics arose from theories explaining observations that could not be explained by classical physics. Quantum mechanics has some very strange aspects and as a result there were numerous interpretations to explain the theory. Even the strange point of view arose that an unobserved particle does not possess physical properties that exist independent of observation. Several physicists rejected this view on nature, including Albert Einstein. In 1935 Einstein, Podolsky and Rosen came up with a thought experiment in quantum mechanics [8] to oppose this view (this is often referred to as the EPR paper). Their reasoning depends on the existence of *'elements of reality'* and within a complete physical theory *'every element of the physical reality must have a counterpart in the physical theory'* according to them. This basically means that with a complete description of a physical system the outcomes of all measurements should be predictable. In contrast to classical physics, where physical systems have predefined properties, quantum mechanics only gives a description in terms of probabilities to measure a certain quantity. With a simple thought experiment (see [8]), Einstein showed that quantum mechanics is not complete in the sense that it describes all *'elements of reality'*. Einstein believed however that it would be possible to develop a theory that is complete in his sense.

In 1964 John Bell proposed a test to verify whether or not the view of EPR is right. This test is in the form of an Bell inequality, named after its inventor. The inequality is based on two assumptions, known as local realism [12]:

1. The assumption that physical properties/quantities have definite values which exist independent of observation. This is sometimes known as the assumption of realism.
2. The assumption that performing a measurement on one place does not influence the result of a measurement on another place. This is sometimes known as the assumption of locality.

It is hard to give a precise physical description of realism, since it is a somewhat intuitive concept. A local, hidden variables theory gives an interpretation of local realism. Such a theory attempts to explain the probabilistic features of quantum mechanics by a mechanism of hidden variables (the *'elements of reality'*), accounting for the assumption of realism. The key purpose of Bells inequality was to show that the idea of a local, hidden variables theory is incompatible with the statistical predictions of quantum mechanics. A local, hidden variables theory is also referred to as a local model. In general, all inequalities that are necessarily satisfied by any local model, but which can be violated by quantum measurements, are called Bell inequalities.

1.1. Clauser-Horne-Shimony-Holt (CHSH) inequality

The best-known Bell inequality is the one described by John Clauser, Michael Horne, Abner Shimony, and Richard Holt [6]. It is also called the CHSH inequality, named after its inventors. The setting is fairly simple. We have two parties, usually called Alice and Bob, that are both equipped with two

different measurement apparatuses. For simplicity, we suppose that the measurement outcomes are binary (denoted by ± 1). Furthermore, Alice and Bob are totally free to perform any of their two available measurements apparatuses. This means that we can consider their choice to perform a specific measurement as flipping a fair coin.

The next thing we need is a source, which is usually called Charlie, that prepares two particles. After preparation, one of the particles is sent to Alice and the other one to Bob. This is when Alice and Bob come into play. They both choose one of their two possible measurements and they perform this measurement. As a result of this, they find a binary value (± 1) as their measurement outcome.

Suppose Alice can measure one of two properties (for example the electron spin in the z-direction or the electron spin in the x-direction). We use a_0 and a_1 to denote respectively the 'value' of the first and second property. Each of these values will be either $+1$ or -1, corresponding to the measurement outcome. For Bob we have an analogous situation, however his measurement outcomes are denoted by b_0 and b_1. The correlations between the measurement outcomes of Alice and Bob can tell us whether the mechanism can be described by a local model. To see this, we follow a simplified derivation from Nielsen and Chuang [12].

We start by assuming local realism as described above. Let us consider the quantity

$$a_0 b_0 + a_0 b_1 + a_1 b_0 - a_1 b_1 \tag{1.1}$$

At first hand this expression seems to make no sense. If Alice decides to measure a_0, then it might not even be possible to determine the value a_1. For example, Alice's particle could have been destroyed by the measurement. But the assumption of realism assures us that all properties already have a value before the measurement is performed. So both a_0 and a_1 are well defined in this model and the same holds for b_0 and b_1. We can rewrite the quantity as

$$(a_0 + a_1)\, b_0 + (a_0 - a_1)\, b_1 \tag{1.2}$$

Note that a_0, a_1, b_0 and b_1 have some predefined value (their values are determined during the preparation, or at least before the measurement takes place), so we can consider two different cases. The first one is $a_0 = a_1 = \pm 1$, which gives $(a_0 - a_1) = 0$ and therefore we find using equation 1.2 $a_0 b_0 + a_0 b_1 + a_1 b_0 - a_1 b_1 = \pm 2$. The other case is $a_0 = -a_1 = \pm 1$, which leads to $(a_0 + a_1) = 0$ and again we find $a_0 b_0 + a_0 b_1 + a_1 b_0 - a_1 b_1 = \pm 2$. So in all cases we have

$$a_0 b_0 + a_0 b_1 + a_1 b_0 - a_1 b_1 = \pm 2 \tag{1.3}$$

Let $P(\alpha_0, \alpha_1, \beta_0, \beta_1)$ be the probability that $a_0 = \alpha_0,\ a_1 = \alpha_1,\ b_0 = \beta_0$ and $b_1 = \beta_1$. The CHSH inequality can now be found by taking the expectation value of this quantity:

$$\mathbb{E}\,(a_0 b_0 + a_0 b_1 + a_1 b_0 - a_1 b_1) = \sum_{\alpha_0,\alpha_1,\beta_0,\beta_1} P(\alpha_0, \alpha_1, \beta_0, \beta_1)\,(\alpha_0\beta_0 + \alpha_0\beta_1 + \alpha_1\beta_0 - \alpha_1\beta_1) \tag{1.4}$$

$$\leq \sum_{\alpha_0,\alpha_1,\beta_0,\beta_1} P(\alpha_0, \alpha_1, \beta_0, \beta_1) \cdot 2 \tag{1.5}$$

$$= 2 \tag{1.6}$$

where we used that $(\alpha_0\beta_0 + \alpha_0\beta_1 + \alpha_1\beta_0 - \alpha_1\beta_1) \leq 2$ since it only takes the values $+2$ and -2. We also used that the sum of the probabilities of all possible outcomes equals 1. Using the linearity of the expectation operator yields the CHSH inequality:

$$\mathbb{E}\,(a_0 b_0) + \mathbb{E}\,(a_0 b_1) + \mathbb{E}\,(a_1 b_0) - \mathbb{E}\,(a_1 b_1) \leq 2 \tag{1.7}$$

Quantum mechanics predicts that this inequality can be violated. Using a source that creates an entangled state and the right measurement setup (see for example [12]), one is able to find the value $2\sqrt{2}$ for the left hand side of the inequality. That is clearly a violation. Real experiments have been performed in agreement with the quantum mechanical prediction. However, the rather strict nature of our assumptions result in possible loopholes [5]. In October 2015, Hensen et al. were the first to realize a Bell-test closing simultaneously the detection loophole and the locality loophole [11]. This gives a very strong indication that, however counter-intuitive it may seem, our world cannot be described by any local model.

1.2. Quantum networks

The concept of nonlocality can also be applied to quantum networks. A network is an abstract representation of relationships (represented by edges) between parts of a system (represented by nodes) [3]. The quantum networks we consider are systems whose connections are represented by entangled states. Characterizing and detecting nonlocality for these networks is relevant to the implementation of quantum networks and quantum repeaters, where Bell inequalities can be used to test if configurations are prepared correctly. One such application is quantum cryptography. In 1991 Ekert demonstrated how the CHSH inequality can be used for cryptography [9]. The details of this procedure are beyond the scope of this thesis, but the interested reader can find a short description in [1]. This application can likely be extended to networks, where cryptography is useful to establish secure and encrypted communication between the nodes.

1.3. Thesis overview

This thesis is organized as follows. In the next chapter we treat the necessary theory, where nonlocality for networks is defined and some relevant aspects from quantum mechanics are treated. We also introduce a theorem by Rosset et al., describing a procedure to find new Bell inequalities in an iterative way. In chapter 3 several results are derived based on the framework layed out by Rosset et al. [14]. We look in particular to star-shaped networks and consider quantum violations. In chapter 4 we discuss how Bell inequality violations can be detected in a real experiment. The effect of distinguishable photons, causing non-entangled states, and measurement errors are analysed. We also optimize the measurement angle showing that nonlocality can be detected in cases where the standard measurement setup would not suffice. In the last chapter the conclusions are summarized and we give a short outlook on possible further research.

2

Theory

In this chapter the necessary theory is treated. In the first section a mathematical description of nonlocality is given. We also introduce the theorem to create new Bell inequalities, provided with an elaborate proof. The remaining sections treat relevant parts of quantum mechanics, the trace and the tensor product. The last section gives a short description of the four Bell states, which are frequently used for Bell inequality violations.

2.1. Creating Bell inequalities for networks

Our starting point is a network $\mathcal{N}$ and a given Bell inequality for it. An example of a network is shown in figure 2.1. The network $\mathcal{N}$ consists of N sources $\mathcal{S}_1, \ldots, \mathcal{S}_N$ and M parties denoted by $\mathcal{A}^1, \ldots, \mathcal{A}^M$. Each source sends physical systems to all parties connected to it. Each party $\mathcal{A}^j$ thus receives a set of systems on which it will perform a measurement. However, the party $\mathcal{A}^j$ can of course perform any possible measurement. An input parameter x^j determines which measurement is performed. The value of this parameter can for example be determined by a coin toss or a random number generator. Another example is that the value is chosen by humans [7]. The output of the measurement performed by $\mathcal{A}^j$ is denoted by $a^j_{x^j}$. In the following we consider only measurements with binary outputs $a^j_{x^j} = \pm 1$.

The network $\mathcal{N}$ can be extended with a party $\mathcal{A}^{M+1}$ and a source $\mathcal{S}_{N+1}$ that connects the parties $\mathcal{A}^M$ and $\mathcal{A}^{M+1}$. An example of such an extension is also shown in figure 2.1. The new party gets a binary input $x^{M+1} = 0, 1$ and a binary output $a^{M+1}_{x^{M+1}} = \pm 1$. The resulting network that is obtained this way is denoted by $\mathcal{N}'$. Rosset, Denis, et al. [14] have described a method to obtain a Bell inequality for network $\mathcal{N}'$ given a Bell inequality for network $\mathcal{N}$. This method is elaborated in this section.

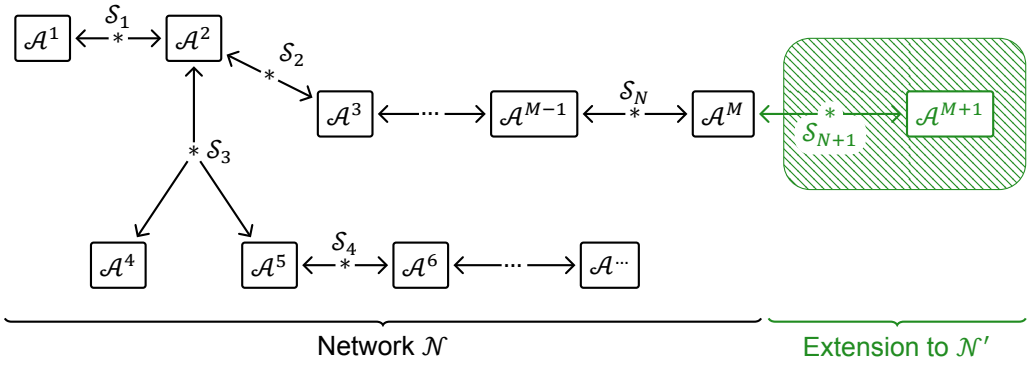


Figure 2.1: A schematic view of a sample network $\mathcal{N}'$. The network consists of network $\mathcal{N}$ extended with a party $\mathcal{A}^{M+1}$ and a source $\mathcal{S}_N$ that connects the parties $\mathcal{A}^M$ and $\mathcal{A}^{M+1}$. The arrows represent the sending of a physical system from a source to a party, or equivalently, the sending of a random local variable λ_i.

Mathematically the setting can be described using random local variables λ_i. A local variable λ_i is attached to each source S_i and is sent to all parties connected to S_i. Every party $\mathcal{A}^j$ thus receives a list of variables $\vec{\lambda}_{\mathcal{A}^j}$. This list contains precisely all random variables corresponding to the sources connected to $\mathcal{A}^j$.

In order to obtain a Bell inequality, we now have to make two key assumptions. The first one is that the measurement output $a^j_{x^j}$ only depends on the input x^j and $\vec{\lambda}_{\mathcal{A}^M}$. The second assumption is that all λ_i are pairwise independent, i.e. $\rho(\lambda_1, \dots, \lambda_N) = \prod_{i=1} \rho_i(\lambda_i)$. Both assumptions can be combined by the assumption of $\mathcal{N}$**-locality** [14].

Definition 2.1.1. *Let ρ_i denote the probability density function of λ_i. We say that the joint probability distribution function $P\left(a^1_{x^1}, \dots, a^M_{x^M} | x^1, \dots, x^M\right)$ is $\mathcal{N}$**-local** if and only if it can be decomposed as*

$$P\left(a^1_{x^1}, \dots, a^M_{x^M} \middle| x^1, \dots, x^M\right) = \int_{\Lambda_1} d\lambda_1 \rho_1(\lambda_1) \dots \int_{\Lambda_N} d\lambda_N \rho_N(\lambda_N) \times P\left(a^1_{x^1} \middle| x^1, \vec{\lambda}_{\mathcal{A}^1}\right) \dots P\left(a^M_{x^M} \middle| x^M, \vec{\lambda}_{\mathcal{A}^M}\right) \quad (2.1)$$

where each $P(a^i | x^j, \vec{\lambda}_{\mathcal{A}^j})$ is a probability distribution.

It will also be useful to consider correlators. The probability distributions $P(a^i_{x^j} | x^j, \vec{\lambda}_{\mathcal{A}^j})$ in definition 2.1.1 can be assumed to be deterministic, since any randomness used locally can be included in the variables λ_i. Deterministic means in this case that the result $a^i_{x^j}$ is always the same for given x^j and $\vec{\lambda}_{\mathcal{A}^j}$. This allows us to write the output $a^j_{x^j}$ as a function of $\vec{\lambda}_{\mathcal{A}^j}$. Therefore, the correlators in an $\mathcal{N}$-local model can be written as

$$\left\langle a^1_{x^1} \cdots a^M_{x^M} \right\rangle = \int_{\Lambda_1} d\lambda_1 \rho_1(\lambda_1) \dots \int_{\Lambda_N} d\lambda_N \rho_N(\lambda_N) \times a^1_{x^1}\left(\vec{\lambda}_{\mathcal{A}^1}\right) \dots a^M_{x^M}\left(\vec{\lambda}_{\mathcal{A}^M}\right) \quad (2.2)$$

From the paper by Rosset, Denis, et al. [14] we retrieve an interesting theorem that gives a recipe to find new Bell inequalities for the extended network $\mathcal{N}'$.

Theorem 2.1.1. *Suppose that the correlators $\left\langle a^1_{x^1} \cdots a^M_{x^M} \right\rangle$ in any $\mathcal{N}$-local model satisfy a Bell inequality of the form*

$$\sum_{x^1, \dots, x^M} \beta_{x^1, \dots, x^M} \left\langle a^1_{x^1} \cdots a^M_{x^M} \right\rangle \leq 1 \quad (2.3)$$

where $\beta_{x^1, \dots, x^M}$ are real coefficients depending on the inputs $x^1, \dots, x^M$. Then $\mathcal{N}'$-local correlations satisfy the following constraint: either there exists $q \in (0, 1)$ such that for any partition of the set of party $\mathcal{A}^M$'s inputs into two disjoint subsets χ^M_+ and χ^M_- the inequality

$$\frac{1}{q}\Sigma_{\chi_+} + \frac{1}{1-q}\Sigma_{\chi_-} \leq 1 \quad (2.4)$$

holds, or

$$\Sigma_{\chi_-} = 0 \quad \textit{and} \quad \Sigma_{\chi_+} \leq 1 \quad \textit{for all} \quad \chi^M_\pm, \quad (2.5)$$

or

$$\Sigma_{\chi_+} = 0 \quad \textit{and} \quad \Sigma_{\chi_-} \leq 1 \quad \textit{for all} \quad \chi^M_\pm. \quad (2.6)$$

Here we use the notation

$$\Sigma_{\chi_\pm} = \sum_{\substack{x^1, \dots, x^{M-1} \\ x^M \in \chi^M_\pm}} \beta_{x^1, \dots, x^M} \left\langle a^1_{x^1} \cdots a^M_{x^M} \frac{a^{M+1}_0 \pm a^{M+1}_1}{2} \right\rangle \quad (2.7)$$

Proof. Consider an $\mathcal{N}'$-local model with M parties $\mathcal{A}^j$ connected to N independent sources $\mathcal{S}_i$. In addition, consider a party $\mathcal{A}^{M+1}$ with binary input ($x^{M+1} = \pm 1$) and output ($a^{M+1}_{x^{M+1}} = \pm 1$) that is connected to a source $\mathcal{S}_{N+1}$. The source $\mathcal{S}_{N+1}$ is also connected to party $\mathcal{A}^M$. The sources $\mathcal{S}_i$ with $1 \leq i \leq N$ have a local variable $\lambda_i \in \Lambda_i$ attached to it. The local variable attached to source $\mathcal{S}_{N+1}$ will be denoted

by $\mu \in \Omega$ and is distributed according to $\rho_\Omega(\mu)$. The model thus consist of a network N extended with a party $\mathcal{A}^M$ and a source $\mathcal{S}_{N+1}$, like the network in figure 2.1. Assume further that the outputs have binary response functions $a^j_{x^j}\left(\vec{\lambda}_{\mathcal{A}^j}\right) = \pm 1$, $a^M_{x^M}\left(\vec{\lambda}_{\mathcal{A}^M}, \mu\right) = \pm 1$ and $a^{M+1}_{x^{M+1}}(\mu) = \pm 1$ for parties $\mathcal{A}^j$ with $1 \leq j \leq M-1$, $\mathcal{A}^M$ and $\mathcal{A}^{M+1}$ respectively. Now we split the sample space of μ into two pieces as follows:

$$\Omega_\pm = \left\{\mu \in \Omega \middle| a_0^{M+1}(\mu) = \pm a_1^{M+1}(\mu)\right\} \tag{2.8}$$

We thus distinguish the cases where the value of μ results in the same outcomes for $a_0^{M+1}(\mu)$ and $a_1^{M+1}(\mu)$ and the cases where the value of μ results in opposite outcomes for $a_0^{M+1}(\mu)$ and $a_1^{M+1}(\mu)$. It is convenient to introduce the following notation:

$$q_\pm = \int_{\Omega_\pm} d\mu \rho_\Omega(\mu) \tag{2.9}$$

$$\rho_{\Omega_\pm}(\mu) = \frac{\rho_\Omega(\mu)}{q_\pm} \qquad \text{for } q_\pm \neq 0 \tag{2.10}$$

For $q_\pm \neq 0$ it is clear from this expressions that $q_+ + q_- =$ and $\rho_{\Omega_\pm}$ therefore defines a normalized measure on $\Omega_\pm$. If $q_\pm = 0$, then let $\rho_{\Omega_\pm}$ be any normalized measure on $\Omega_\pm$.

Note that we have

$$\frac{a_0^{M+1}(\mu) + a_1^{M+1}(\mu)}{2} = a_0^{M+1}(\mu) \quad \text{and} \quad \frac{a_0^{M+1}(\mu) - a_1^{M+1}(\mu)}{2} = 0 \quad \text{for all} \quad \mu \in \Omega_+ \tag{2.11}$$

Similarly it holds that

$$\frac{a_0^{M+1}(\mu) + a_1^{M+1}(\mu)}{2} = 0 \quad \text{and} \quad \frac{a_0^{M+1}(\mu) - a_1^{M+1}(\mu)}{2} = a_0^{M+1}(\mu) \quad \text{for all} \quad \mu \in \Omega_- \tag{2.12}$$

This yields the following identity:

$$\begin{aligned}\int_\Omega d\mu\, \rho_\Omega(\mu) \frac{a_0^{M+1}(\mu) \pm a_1^{M+1}(\mu)}{2} &= \int_{\Omega_\pm} d\mu\, \rho_\Omega(\mu) \frac{a_0^{M+1}(\mu) \pm a_1^{M+1}(\mu)}{2} \\ &\quad + \int_{\Omega_\mp} d\mu\, \rho_\Omega(\mu) \frac{a_0^{M+1}(\mu) \pm a_1^{M+1}(\mu)}{2} \\ &= \int_{\Omega_\pm} d\mu\, \rho_\Omega(\mu)\, a_0^{M+1}(\mu) + \int_{\Omega_\mp} d\mu\, \rho_\Omega(\mu) \cdot 0 \\ &= q_\pm \int_{\Omega_\pm} d\mu\, \rho_{\Omega_\pm}(\mu)\, a_0^{M+1}(\mu)\end{aligned} \tag{2.13}$$

Using this identity, we find for the following $\mathcal{N}'$-local correlator that

$$\begin{aligned}\left\langle a^1_{x^1} \cdots a^M_{x^M} \frac{a_0^{M+1} \pm a_1^{M+1}}{2} \right\rangle &= \int_{\Lambda_1} d\lambda_1 \rho_1(\lambda_1) \ldots \int_{\Lambda_N} d\lambda_N \rho_N(\lambda_N) \int_\Omega d\mu\, \rho_\Omega(\mu) \\ &\quad \times a^1_{x^1}\left(\vec{\lambda}_{\mathcal{A}^1}\right) \ldots a^{M-1}_{x^{M-1}}\left(\vec{\lambda}_{\mathcal{A}^{M-1}}\right) a^M_{x^M}\left(\vec{\lambda}_{\mathcal{A}^M}, \mu\right) \frac{a_0^{M+1}(\mu) \pm a_1^{M+1}(\mu)}{2} \\ &= q_\pm \int_{\Lambda_1} d\lambda_1 \rho_1(\lambda_1) \ldots \int_{\Lambda_N} d\lambda_N \rho_N(\lambda_N) \int_{\Omega_\pm} d\mu\, \rho_{\Omega_\pm}(\mu) \\ &\quad \times a^1_{x^1}\left(\vec{\lambda}_{\mathcal{A}^1}\right) \ldots a^{M-1}_{x^{M-1}}\left(\vec{\lambda}_{\mathcal{A}^{M-1}}\right) a^M_{x^M}\left(\vec{\lambda}_{\mathcal{A}^M}, \mu\right) a_0^{M+1}(\mu)\end{aligned} \tag{2.14}$$

The next step is to look at our network differently. We want to write this expression as an integral over $\mathcal{N}$-local correlators. Therefore we need to alter the network structure. We have already eliminated party $\mathcal{A}^{M+1}$'s input parameter. Indeed only the term a_0^{M+1} is left and this does not depend on $\mathcal{A}^{M+1}$'s input x^{M+1}. This suggests to replace party $\mathcal{A}^M$, source $\mathcal{S}_{N+1}$ and $\mathcal{A}^{M+1}$ by a new party $\tilde{\mathcal{A}}^M$ as illustrated in figure 2.2.

We will define the output of our new party $\tilde{\mathcal{A}}^M$ to be $\tilde{a}^M_{x^M,\mu}\left(\vec{\lambda}_{\mathcal{A}^M}\right) = a^M_{x^M}\left(\vec{\lambda}_{\mathcal{A}^M}, \mu\right) a_0^{M+1}(\mu)$.

Substituting $\tilde{\mathcal{A}}^M$'s output into equation 2.14 yields

$$\begin{aligned}\left\langle a^1_{x^1} \cdots a^M_{x^M} \frac{a_0^{M+1} \pm a_1^{M+1}}{2} \right\rangle &= q_\pm \int_{\Lambda_1} d\lambda_1 \rho_1(\lambda_1) \ldots \int_{\Lambda_N} d\lambda_N \rho_N(\lambda_N) \int_{\Omega_\pm} d\mu\ \rho_{\Omega_\pm}(\mu) \\ &\quad \times a^1_{x^1}\left(\vec{\lambda}_{\mathcal{A}^1}\right) \ldots a^{M-1}_{x^{M-1}}\left(\vec{\lambda}_{\mathcal{A}^{M-1}}\right) a^M_{x^M}\left(\vec{\lambda}_{\mathcal{A}^M}, \mu\right) a_0^{M+1}(\mu) \\ &= q_\pm \int_{\Lambda_1} d\lambda_1 \rho_1(\lambda_1) \ldots \int_{\Lambda_N} d\lambda_N \rho_N(\lambda_N) \int_{\Omega_\pm} d\mu\ \rho_{\Omega_\pm}(\mu) \\ &\quad \times a^1_{x^1}\left(\vec{\lambda}_{\mathcal{A}^1}\right) \ldots a^{M-1}_{x^{M-1}}\left(\vec{\lambda}_{\mathcal{A}^{M-1}}\right) \tilde{a}^M_{x^M,\mu}\left(\vec{\lambda}_{\mathcal{A}^M}\right) \\ &= q_\pm \int_{\Omega_\pm} d\mu\ \rho_{\Omega_\pm}(\mu) \left\langle a^1_{x^1} \ldots a^{M-1}_{x^{M-1}} \tilde{a}^M_{x^M,\mu} \right\rangle\end{aligned} \tag{2.15}$$

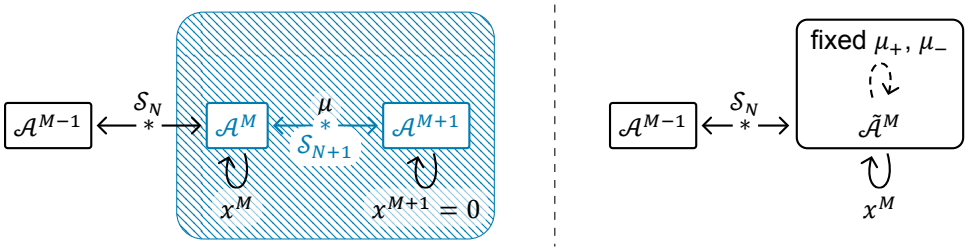


Figure 2.2: A schematic view of how party $\mathcal{A}^M$, source $\mathcal{S}_{N+1}$ and party $\mathcal{A}^{M+1}$ are replaced by a single party $\tilde{\mathcal{A}}^M$. All arrows represent sending a physical system or information. The ones pointing from a party to itself can thus be regarded as input parameters. The dashed arrow represents the fixed parameters μ_+ and μ_-. For fixed μ_+ and μ_- these are no longer input parameters and they can be regarded as being part of the measurement process.

It should be noticed that party $\mathcal{A}^M$ now has an additional input parameter μ. We cannot yet apply inequality 2.3, since there we have only input parameter x^M. We need to find a way to describe the outcome $a^1_{x^1} \ldots a^{M-1}_{x^{M-1}} \tilde{a}^M_{x^M,\mu}$ in terms of a model that uses x^M as the only input of party $\tilde{\mathcal{A}}^M$.

Now consider any partition of the set of party $\mathcal{A}^M$'s inputs into two disjoint subsets χ^M_+ and χ^M_-. Substituting equation 2.15 into the definition of $\Sigma_{\chi_\pm}$ (equation 2.7) shows that

$$\Sigma_{\chi_\pm} = \sum_{\substack{x^1,\ldots,x^{M-1} \\ x^M \in \chi^M_\pm}} \beta_{x^1,\ldots,x^M} q_\pm \int_{\Omega_\pm} d\mu\ \rho_{\Omega_\pm}(\mu) \left\langle a^1_{x^1} \ldots a^{M-1}_{x^{M-1}} \tilde{a}^M_{x^M,\mu} \right\rangle \tag{2.16}$$

Our goal is to find an inequality with Σ_{χ_+} and Σ_{χ_-}. Taking a closer look at the above equation, we see that it only contains correlators for which either $x^M \in \chi_+$ and $\mu \in \Omega_+$ or $x^M \in \chi_-$ and $\mu \in \Omega_-$. This means that x^M and μ can be grouped into a single input, as will be explained next.

If we fix $\mu_+ \in \Omega_+$ and $\mu_- \in \Omega_-$, then we can describe the generation of party $\tilde{\mathcal{A}}^M$'s output using only one input parameter. We do this by letting x^M be the input parameter. When $x^M \in \chi_+$, the output is generated by a measurement depending only on $\vec{\lambda}_{\mathcal{A}^M}$ and μ_+. Similarly, when $x^M \in \chi_-$, the output is generated by a measurement depending only on $\vec{\lambda}_{\mathcal{A}^M}$ and μ_-. Party $\tilde{\mathcal{A}}^M$ has only one input parameter left (namely x^M). The other parameters μ_+ and μ_- can be seen as variables that determine how the measurement is performed. For clarity in notation we add $\mu_\pm$ to party $\tilde{\mathcal{A}}^M$'s inputs, but be aware that it is not an actual input. This is also shown in figure 2.2 with a dashed arrow fully inside party $\tilde{\mathcal{A}}^M$.

In this way get an $\mathcal{N}$-local model, since the structure is the same as in network $\mathcal{N}$. We can now apply inequality 2.3. We have for all $\mu_+ \in \Omega_+$ and $\mu_- \in \Omega_-$ that

$$\sum_{\substack{x^1,\ldots,x^{M-1} \\ x^M \in \chi^M_+}} \beta_{x^1,\ldots,x^M} \left\langle a^1_{x^1} \ldots a^{M-1}_{x^{M-1}} \tilde{a}^M_{x^M,\mu_+} \right\rangle + \sum_{\substack{x^1,\ldots,x^{M-1} \\ x^M \in \chi^M_-}} \beta_{x^1,\ldots,x^M} \left\langle a^1_{x^1} \ldots a^{M-1}_{x^{M-1}} \tilde{a}^M_{x^M,\mu_-} \right\rangle \leq 1 \tag{2.17}$$

To ease notation, we define

$$S_{\chi_\pm,\mu_\pm} = \sum_{\substack{x^1,\ldots,x^{M-1} \\ x^M \in \chi_\pm^M}} \beta_{x^1,\ldots,x^M} \left\langle a^1_{x^1} \cdots a^{M-1}_{x^{M-1}} \tilde{a}^M_{x^M,\mu_\pm} \right\rangle \tag{2.18}$$

For all $\mu_+ \in \Omega_+$ and $\mu_- \in \Omega_-$, equation 2.17 holds. This gives

$$S_{\chi_+,\mu_+} + S_{\chi_-,\mu_-} \leq 1 \tag{2.19}$$

Furthermore, substituting equation 2.18 into equation 2.16 and interchanging summation and integration yields

$$\Sigma_{\chi_\pm} = q_\pm \int_{\Omega_\pm} d\mu \; \rho_{\Omega_\pm}(\mu) \, S_{\chi_\pm,\mu_\pm} \tag{2.20}$$

Consider first the case that $q_+ \in (0,1)$. Equation 2.4 can be recovered by averaging inequality 2.19 over $\mu_+ \in \Omega_+$ and $\mu_- \in \Omega_-$, dividing it by q_+ and substituting equation 2.20. More formally, we use $q_- = 1 - q_+$ and obtain that

$$\begin{aligned} \frac{1}{q_+}\Sigma_{\chi_+} + \frac{1}{1-q_+}\Sigma_{\chi_-} &= \int_{\Omega_+} d\mu \; \rho_{\Omega_+}(\mu) \, S_{\chi_+,\mu_+} + \int_{\Omega_-} d\mu \; \rho_{\Omega_-}(\mu) \, S_{\chi_-,\mu_-} \\ &\leq \max_{\mu\in\Omega_+}\{S_{\chi_+,\mu_+}\} + \max_{\mu\in\Omega_-}\{S_{\chi_-,\mu_-}\} \\ &\leq 1 \end{aligned} \tag{2.21}$$

where the last step follows from equation 2.19. Herewith we arrived at the required inequality.

Next consider the case that $q_+ = 1$ (and $q_- = 0$). For any $\mathcal{N}$-local model we can transform inequality 2.3 slightly. Suppose party $\mathcal{A}^M$ flips its output (multiplies with -1) whenever $x^M \in \chi_-^M$. Then this is of course still an $\mathcal{N}$-local model and by using the original notation (thus with unflipped outputs), we obtain

$$\sum_{\substack{x^1,\ldots,x^{M-1} \\ x^M \in \chi_+^M}} \beta_{x^1,\ldots,x^M} \left\langle a^1_{x^1} \cdots a^M_{x^M} \right\rangle - \sum_{\substack{x^1,\ldots,x^{M-1} \\ x^M \in \chi_-^M}} \beta_{x^1,\ldots,x^M} \left\langle a^1_{x^1} \cdots a^M_{x^M} \right\rangle \leq 1 \tag{2.22}$$

Summing up inequality 2.3 and inequality 2.22 and dividing by 2 yields

$$\sum_{\substack{x^1,\ldots,x^{M-1} \\ x^M \in \chi_+^M}} \beta_{x^1,\ldots,x^M} \left\langle a^1_{x^1} \cdots a^M_{x^M} \right\rangle \leq 1 \tag{2.23}$$

Analogous to the way we obtained equation 2.19, for all $\mu_+ \in \Omega_+$ we get

$$S_{\chi_+,\mu_+} \leq 1 \tag{2.24}$$

Using equation 2.20 it clearly follows that $\Sigma_{\chi_-} = 0$, since $q_- = 0$. The required inequality also follows easily:

$$\Sigma_{\chi_+} = \frac{1}{q_+}\Sigma_{\chi_+} = \int_{\Omega_+} d\mu \; \rho_{\Omega_+}(\mu) \, S_{\chi_+,\mu_+} \leq \max_{\mu\in\Omega_+}\{S_{\chi_+,\mu_+}\} \leq 1 \tag{2.25}$$

The case that $q_+ = 0$ (and $q_- = 1$) is treated similarly. In this case one arrives at $\Sigma_{\chi_+} = 0$ and $\Sigma_{\chi_+} \leq 1$. □

From now on, we let $q \in [0,1]$. With $q = 1$ we actually mean $q \to 1$, corresponding to the case of equation 2.5. Similarly $q = 0$ means $q \to 0$, corresponding to the case of equation 2.6.
Moreover, a more detailed version of this theorem can be found in the work of Rosset, Denis, et al. [14].

2.2. Quantum states

According to quantum mechanics any isolated physical system is associated to a Hilbert space. This is an inner product space which is complete under the norm induced by its inner product. We also call this the state space of the system. Within this space the system can be completely described by a state vector, which is a unit vector in the state space.

Since all operators act on state vectors as linear transformations, quantum mechanics can be described in the language of linear algebra. It is useful to introduce the Dirac notation (also known as bra-ket notation) that is commonly used in quantum mechanics. Using this notation, a state vector is written as $|\alpha\rangle$ (this is a 'ket') and its Hermitian conjugate is written as $\langle\alpha|$ (this is a 'bra'). The inner product between two state vectors $|\alpha\rangle$ and $|\beta\rangle$ is written as $\langle\alpha|\beta\rangle$. The state vectors must be normalized, which is expressed as the requirement that $\langle\alpha|\alpha\rangle = 1$ for every state $|\alpha\rangle$.

2.3. Measurements

In order to verify an inequality, one needs to perform measurements. Nielsen and Chuang describe in [12] the operation of measurements in quantum mechanics. The most general measurement is described by a collection of measurement operators $\{\hat{M}_m\}$. These operators act on the quantum system that is being measured and the index m refers to the possible measurement outcomes. So for each possible measurement outcome there is one such operator and the probability that the measurement yields outcome m is given by

$$p(m) = \langle\psi|\, \hat{M}_m^\dagger \hat{M}_m \,|\psi\rangle \tag{2.26}$$

Given that the measurement outcome is m, the state after measurement is given by

$$\frac{M_m \,|\psi\rangle}{\sqrt{p(m)}} \tag{2.27}$$

Since probabilities measures need to satisfy certain criteria, it is clear that there has to be a restriction on the collection of measurement operators. Indeed the collection of measurement operator must satisfy the completeness equation:

$$\sum_m \hat{M}_m^\dagger \hat{M}_m = I \tag{2.28}$$

The completeness equation is equivalent to the statement that the probabilities of all possible measurement outcomes sum up to one, as required:

$$\sum_m p(m) = \sum_m \langle\psi|\, \hat{M}_m^\dagger \hat{M}_m \,|\psi\rangle = \langle\psi| \sum_m \left(\hat{M}_m^\dagger \hat{M}_m\right) |\psi\rangle = \langle\psi|\psi\rangle = 1 \tag{2.29}$$

2.3.1. Projective Measurements

Projective measurements are an important class of measurements. Also spin measurements fall into this category. A projective measurement is described by an observable $\hat{M}$. The observable is represented by a Hermitian operator with a spectral decomposition

$$\hat{M} = \sum_m m P_m \tag{2.30}$$

Here P_m is the projector onto the eigenspace of $\hat{M}$ with eigenvalue m. The possible outcomes of the measurement are the eigenvalues m of the operator. Given a state $|\psi\rangle$, the probability of getting result m is given by

$$p(m) = \langle\psi|\, P_m \,|\psi\rangle \tag{2.31}$$

Immediately after a measurement with outcome m, the state is given by

$$\frac{P_m \,|\psi\rangle}{\sqrt{p(m)}} \tag{2.32}$$

A projective measurement thus really projects the state onto the eigenspace corresponding to the outcome. It can be shown that projective measurements are a special case of general measurements. Let $\hat{M}_m = P_m$ for all eigenvalues m of $\hat{M}$. Since P_m are orthogonal projectors, we have $P_m^\dagger = P_m$ and $P_m P_m = P_m$. So indeed,

$$p(m) = \langle\psi| \hat{M}_m^\dagger \hat{M}_m |\psi\rangle = \langle\psi| P_m |\psi\rangle$$

and the state after measurement given by equation 2.32 corresponds to that given by equation 2.27. The completeness equation is also satisfied, since the identity $\sum_m P_m = I$ holds and therefore

$$\sum_m \hat{M}_m^\dagger \hat{M}_m = \sum_m P_m^\dagger P_m = \sum_m P_m P_m = \sum_m P_m = I$$

Projective measurements have the nice property that their expectation value can be easily calculated. The expectation value of a projective measurement $\hat{M}$ is given by

$$\begin{aligned} \mathbb{E}(M) &= \sum_m m\, p(m) \\ &= \sum_m m \langle\psi| P_m |\psi\rangle \\ &= \langle\psi| \left(\sum_m m P_m \right) |\psi\rangle \\ &= \langle\psi| \hat{M} |\psi\rangle \end{aligned} \tag{2.33}$$

It is customary to write the expectation value $\mathbb{E}(M)$ as $\langle M\rangle \equiv \langle\psi| M |\psi\rangle$ [12].

2.3.2. Spin measurements

An important class of projective measurements are spin measurements. They are described by the Pauli matrices, which are denoted by $\hat{\sigma}_x$, $\hat{\sigma}_y$ and $\hat{\sigma}_z$:

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.34}$$

The quantum states we are dealing with in the next sections are spin qubits. These are two-state systems, which means that spin measurements performed on them have two possible outcomes. A spin qubit can effectively be written as

$$|\psi\rangle = \cos\frac{\theta}{2} |0\rangle + e^{i\varphi} \sin\frac{\theta}{2} |1\rangle \tag{2.35}$$

where $|0\rangle$ and $|1\rangle$ are the eigenstates of $\hat{\sigma}$ with respectively eigenvalue 1 and -1 (see also [12]). We can identify with the state $|\psi\rangle$ a point on the so called Bloch sphere shown in figure 2.3. The position on the Bloch sphere is determined by spherical coordinates. The parameter θ is used as the polar angle, ϕ is used as the azimuthal angle and radial distance is taken to be unity. In this way we can associate a direction with all spin states. This will be a useful tool to choose measurement operators that lead to violations of Bell inequalities.

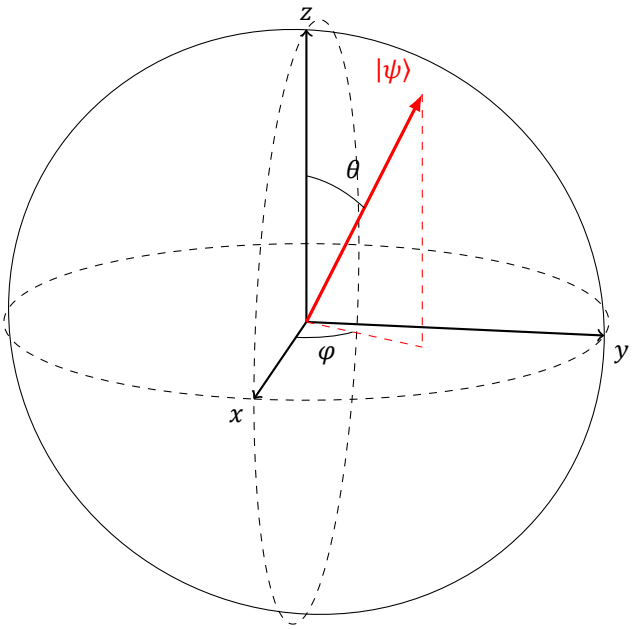


Figure 2.3: Bloch sphere representation of a spin qubit $|\psi\rangle$.

Inspired by the Bloch sphere, we can define a spin measurement $\hat{\sigma}_\phi$ along an angle ϕ by

$$\hat{\sigma}_\phi = \cos\phi\hat{\sigma}_z + \sin\phi\hat{\sigma}_x \tag{2.36}$$

The eigenstates of $\hat{\sigma}_\phi$ are

$$\begin{pmatrix} \cos\frac{\phi}{2} \\ \sin\frac{\phi}{2} \end{pmatrix} \text{ with eigenvalue } 1; \quad \begin{pmatrix} -\sin\frac{\phi}{2} \\ \cos\frac{\phi}{2} \end{pmatrix} \text{ with eigenvalue } -1 \tag{2.37}$$

Now suppose we have a state $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle$ and we measure its spin along an angle ϕ. According to equation 2.31 the probability to measure 1 ('spin up') is

$$p(1) = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} \cos^2\frac{\phi}{2} & \cos\frac{\phi}{2}\sin\frac{\phi}{2} \\ \cos\frac{\phi}{2}\sin\frac{\phi}{2} & \sin^2\frac{\phi}{2} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix} = \cos^2\left(\frac{\theta-\phi}{2}\right) \tag{2.38}$$

where we used trigonometric identities. The probability to measure 1 is thus determined by the difference between the spin angle θ and the measurement angle ϕ. The better the spin and measurement directions are aligned, the higher this probability is. In case of perfect alignment, the probability is 1. When the angle difference is $\pi/2$, we have equal probabilities to measure -1 and 1.

2.4. Density Operator

If the total state of a quantum system is not completely known, we can describe this system using a density operator. We often know the probabilities p_i that the system is in one of a number of states $|\psi_i\rangle$, which invites us to describe the system with an ensemble of pure states $\{p_i, |\psi_i\rangle\}$. For such a system the density operator is defined as

$$\rho \equiv \sum_i p_i |\psi_i\rangle\langle\psi_i| \tag{2.39}$$

This operator is also referred as the density matrix and sometimes even as state. We distinguish two types of quantum systems: pure states and mixed states. A quantum state whose state is known exactly (so $p_i = 1$ for a certain $|\psi_i\rangle$) is in a pure state. All other states are in a mixed state.

The density operator is very useful in cases when a quantum system is prepared in the state ρ_i with probability p_i. This can occur for example because of errors during the preparation of the system. Suppose that ρ_i corresponds to the ensemble $\{p_{ij}, |\psi_{ij}\rangle\}$. The probability that the total system is in the state $|\psi_{ij}\rangle$ is then given by $p_i p_{ij}$ (if there are n, m such that $n \neq i$ or $m \neq j$ for which $|\psi_{ij}\rangle = |\psi_{nm}\rangle$, we can just add the corresponding probabilities). The density operator for the total system is thus given by

$$\rho = \sum_i p_i p_{ij} |\psi_{ij}\rangle \langle\psi_{ij}| = \sum_i p_i \rho_i \tag{2.40}$$

A density operator of this type is called a mixtures of the states ρ_i with probabilities p_i.

2.5. Trace

The trace of an $n \times n$ matrix A is defined as

$$\mathrm{Tr}\{A\} \equiv \sum_{i=1}^{n} A_{ii} \tag{2.41}$$

The question arises whether the trace depends on our choice of a basis. In order to answer this question, we use that the trace is cyclic. Let $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{m\times n}$. It follows that

$$\mathrm{Tr}\{AB\} = \sum_{i=1}^{n} (AB)_{ii} = \sum_{i=1}^{n}\sum_{j=1}^{m} A_{ij}B_{ji} = \sum_{j=1}^{m}\sum_{i=1}^{n} B_{ji}A_{ij} = \sum_{i=1}^{m} (AB)_{jj} = \mathrm{Tr}\{BA\} \tag{2.42}$$

Now it can be easily concluded that taking the trace is invariant with respect to a change of basis. If A is expressed in a different basis, we can write this new matrix as $A' = PAP^{-1}$. We have

$$\mathrm{Tr}\{A'\} = \mathrm{Tr}\{PAP^{-1}\} = \mathrm{Tr}\{P^{-1}PA\} = \mathrm{Tr}\{A\} \tag{2.43}$$

This ensures that the trace of an operator is well defined. Furthermore, we can calculate the trace of A in terms of any basis $\{|1\rangle, |2\rangle, |3\rangle, \dots\}$. We have

$$\mathrm{Tr}\{A\} = \sum_i \langle i| A |i\rangle \tag{2.44}$$

The trace is a linear operator. Let A and B be $n \times n$ matrices and α and β be constants. Then

$$\begin{aligned} \mathrm{Tr}\{\alpha A + \beta B\} &= \sum_{i=1}^{n} (\alpha A + \beta B)_{ii} = \sum_{i=1}^{n} (\alpha A_{ii} + \beta B_{ii}) \\ &= \sum_{i=1}^{n} \alpha A_{ii} + \sum_{i=1}^{n} \beta B_{ii} = \alpha\,\mathrm{Tr}\{A\} + \beta\,\mathrm{Tr}\{B\} \end{aligned} \tag{2.45}$$

2.5.1. Application in quantum mechanics

The trace is a very useful tool to compute the expected value of a quantum measurements. Consider a projective measurement $\hat{M}$ and its projectors P_m onto the eigenspace corresponding to eigenvalue m. Let $|\psi\rangle$ be a state vector and let $\{|1\rangle, |2\rangle, |3\rangle, \dots\}$ be an orthonormal basis with $|1\rangle = |\psi\rangle$. Such a basis can indeed be found using the Gram-Schmidt process. It follows that

$$\mathrm{Tr}\{\hat{M} \langle\psi|\psi\rangle\} = \sum_i \langle i| \hat{M} |\psi\rangle \langle\psi|i\rangle = \langle\psi| \hat{M} |\psi\rangle \tag{2.46}$$

where the first step makes use of equation 2.44 and the second step follows from orthonormality of the basis vectors. This property can be used to express $p(m|i)$ in terms of the trace for a system specified by the density operator ρ. Given an initial state $|\psi_i\rangle$, the probability of getting result m is given by

$$p(m|i) = \langle\psi_i| P_m |\psi_i\rangle \tag{2.47}$$

Invoking equation 2.46 yields

$$p(m|i) = \mathrm{Tr}\{P_m |\psi_i\rangle \langle\psi_i|\}, \tag{2.48}$$

Using this result and the linearity of the trace (equation 2.45), we find

$$\begin{aligned} p(m) &= \sum_i p(m|i)p_i \\ &= \sum_i \mathrm{Tr}\{P_m |\psi_i\rangle \langle\psi_i|\}p_i \\ &= \mathrm{Tr}\{P_m\rho\} \end{aligned} \tag{2.49}$$

Now we can use this to evaluate the expectation value $\langle M\rangle$. Linearity of the trace and distributivity of the matrix product yield

$$\begin{aligned} \langle M\rangle &= \sum_m m\ p(m) \\ &= \sum_m m\,\mathrm{Tr}\{P_m\rho\} \\ &= \mathrm{Tr}\left\{\sum_m mP_m\rho\right\} \\ &= \mathrm{Tr}\left\{\left(\sum_m mP_m\right)\rho\right\} \\ &= \mathrm{Tr}\{M\rho\} \end{aligned} \tag{2.50}$$

2.6. Tensor product

Tensor products are used to describe quantum systems consisting of multiple subsystems. The tensor product of a matrix $A \in \mathbb{R}^{p\times q}$ with a matrix $B \in \mathbb{R}^{r\times s}$ is called the Kronecker product, which is defined as

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1q}B \\ \vdots & & \vdots \\ a_{p1}B & \dots & a_{pq}B \end{bmatrix} \tag{2.51}$$

The Kronecker product represents the tensor products of two matrices with respect to a particular basis. It is very useful to describe combinations of quantum systems, since it allows us to keep track of an entire system using a single state vector or density matrix.

A very useful identity is the so called mixed-product property. Let $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{q\times r}$, $C \in \mathbb{R}^{n\times p}$ and

$D \in \mathbb{R}^{r\times s}$. Then it follows that

$$\begin{aligned}(A \otimes B)(C \otimes D) &= \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix}\begin{bmatrix} c_{11}D & \dots & c_{1p}D \\ \vdots & & \vdots \\ c_{n1}D & \dots & c_{np}D \end{bmatrix} \\ &= \begin{bmatrix} \sum_{i=1}^{n} a_{1i}c_{i1}BD & \dots & \sum_{i=1}^{n} a_{1i}c_{ip}BD \\ \vdots & & \vdots \\ \sum_{i=1}^{n} a_{mi}c_{i1}BD & \dots & \sum_{i=1}^{n} a_{mi}c_{ip}BD \end{bmatrix} \\ &= \begin{bmatrix} \sum_{i=1}^{n} a_{1i}c_{i1} & \dots & \sum_{i=1}^{n} a_{1i}c_{ip} \\ \vdots & & \vdots \\ \sum_{i=1}^{n} a_{mi}c_{i1} & \dots & \sum_{i=1}^{n} a_{mi}c_{ip} \end{bmatrix} \otimes BD \\ &= (AC) \otimes (BD)\end{aligned} \tag{2.52}$$

Another useful property is related to the trace. Let A be a square $n \times n$-matrix and B a square $m \times m$-matrix. Then we have

$$\mathrm{Tr}\{A \otimes B\} = \sum_{i=1}^{n}\left(a_{ii}\sum_{j=1}^{m} b_{jj}\right) = \sum_{i=1}^{n} a_{ii}\sum_{j=1}^{m} b_{jj} = \mathrm{Tr}\{A\}\,\mathrm{Tr}\{B\} \tag{2.53}$$

2.6.1. Application in quantum mechanics

Consider two measurement operators represented by $\hat{A} \in \mathbb{R}^{m\times m}$ and $\hat{B} \in \mathbb{R}^{n\times n}$. Suppose $|\psi_A\rangle$ is an eigenstate of $\hat{A}$ with eigenvalue λ_A and $|\psi_B\rangle$ is an eigenstate of $\hat{B}$ with eigenvalue λ_B. Then $|\psi_A\rangle \otimes |\psi_B\rangle$ is an eigenvector of $\hat{A} \otimes \hat{B}$ with eigenvalue $\lambda_A\lambda_B$, since invoking equation 2.52 yields:

$$\left(\hat{A} \otimes \hat{B}\right)(|\psi_A\rangle \otimes |\psi_B\rangle) = \left(\hat{A}\,|\psi_A\rangle\right) \otimes \left(\hat{B}\,|\psi_B\rangle\right) = (\lambda_A\,|\psi_A\rangle) \otimes (\lambda_B\,|\psi_B\rangle) = \lambda_A\lambda_B\,(|\psi_A\rangle \otimes |\psi_B\rangle) \tag{2.54}$$

Furthermore, using this procedure we can find an orthonormal set of eigenstates of $\hat{A} \otimes \hat{B}$ spanning the entire Hilbert space. To see this, we only need to show that orthogonality is preserved under the tensor product. Let $|a\rangle \in \mathbb{R}^n$ and $|b\rangle \in \mathbb{R}^n$ be two orthonormal vectors, i.e. $\langle a|b\rangle = 0$, and let $|c\rangle \in \mathbb{R}^m$ and $|d\rangle \in \mathbb{R}^m$. We see that

$$(\langle a| \otimes \langle c|)(|b\rangle \otimes |d\rangle) = \begin{bmatrix} a_1\,|c\rangle \\ \vdots \\ a_n\,|c\rangle \end{bmatrix} \cdot \begin{bmatrix} b_1\,|d\rangle \\ \vdots \\ b_n\,|d\rangle \end{bmatrix} = \sum_{i=1}^{n} a_i b_i\,\langle c|d\rangle = \langle a|b\rangle \cdot \langle c|d\rangle = 0 \tag{2.55}$$

So $|a\rangle \otimes |c\rangle$ and $|b\rangle \otimes |d\rangle$ are also orthogonal. We also have

$$(\langle a| \otimes \langle b|)(|a\rangle \otimes |b\rangle) = \begin{bmatrix} a_1\,|b\rangle \\ \vdots \\ a_n\,|b\rangle \end{bmatrix} \cdot \begin{bmatrix} a_1\,|b\rangle \\ \vdots \\ a_n\,|b\rangle \end{bmatrix} = \sum_{i=1}^{n} a_i a_i\,\langle b|b\rangle = \langle a|a\rangle \cdot \langle b|b\rangle = 1 \tag{2.56}$$

This means that the tensor product of two normalized states is still normalized. Because the tensor product transforms two eigenstates into a new one, and in addition it preserves orthogonality and normalization, the projector onto the eigenspace can also be obtained using a tensor product of projectors. This works as follows. Suppose $\hat{A}$ is a measurement operator acting on state $|\psi_A\rangle$ and $\hat{B}$ is a measurement operator acting on state $|\psi_B\rangle$. The measurement operators have respectively $P_{A,m}$ and $P_{B,n}$ as their projectors onto the eigenspace with eigenvalue m (for $\hat{A}$) and n (for $\hat{B}$). Both $P_{A,m}$ and $P_{B,n}$ can be expressed in terms of an orthonormal set of eigenstates. We thus have:

$$P_{A,m} = \sum_{i} |\psi_{m,i}\rangle\langle\psi_{m,i}| \tag{2.57}$$

and similarly

$$P_{B,n} = \sum_{j} |\psi_{n,j}\rangle\langle\psi_{n,j}| \tag{2.58}$$

We see that

$$\begin{aligned} P_{A,m} \otimes P_{B,n} &= \sum_i \sum_j |\psi_{m,i}\rangle \langle \psi_{m,i}| \otimes |\psi_{n,j}\rangle \langle \psi_{n,j}| \\ &= \sum_i \sum_j \left(|\psi_{m,i}\rangle \otimes |\psi_{n,j}\rangle\right) \left(\langle \psi_{m,i}| \otimes \langle \psi_{n,j}|\right) \end{aligned} \tag{2.59}$$

The second step follows directly from equation 2.52. Notice that this is a sum over precisely all the eigenstates $|\psi_{m,i}\rangle \otimes |\psi_{n,j}\rangle$ of $\hat{A} \otimes \hat{B}$ for which measurements would yield the outcomes $A = m$ and $B = n$. This means that $P_{A,m} \otimes P_{B,n}$ is actually the projector of $\hat{A} \otimes \hat{B}$ onto the subspace corresponding to $A = m$ and $B = n$. The operator $\hat{A} \otimes \hat{B}$ can therefore be seen as a measurement operator acting on the entire system which has as outcome the product of all seperate measurement. Summarized, it holds that

$$\langle AB \rangle = \langle \psi | (A \otimes B) | \psi \rangle = \mathrm{Tr}\{\rho (A \otimes B)\} \tag{2.60}$$

where $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$ is the quantum state of the entire system. This result generalizes naturally to systems consisting of more quantum particles. For this reason combinations of quantum states are described using the tensor product. Often the notation $|ab\rangle \equiv |a\rangle \otimes |b\rangle$ is used to describe such combinations.

2.7. Bell states

In order to violate Bell inequalities one often uses Bell states. They describe a system consisting of two qubits (each with eigenstates $|0\rangle$ and $|1\rangle$). The common notation for the four Bell states is:

$$\begin{aligned} |\Phi^+\rangle &= \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \\ |\Phi^-\rangle &= \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle) \\ |\Psi^+\rangle &= \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle) \\ |\Psi^-\rangle &= \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle) \end{aligned} \tag{2.61}$$

The Bell states all have in common that they are maximally entangled. Entanglement means that the combined state cannot be written as a product of the form $|\psi\rangle_A \otimes |\psi\rangle_B$, where $|\psi\rangle_A$ is the state of qubit A and $|\psi\rangle_B$ the state of qubit B. This entanglement is the main reason for nonlocal correlations, as will be shown in the next chapter. Measurements on qubits A and B in a Bell state are strongly correlated. We show this now for the state $|\Psi^-\rangle$, with $|0\rangle$ and $|1\rangle$ being the eigenstates of $\hat{\sigma}_z$. Suppose $|a\rangle$ and $|b\rangle$ are eigenstates of a spin measurement along an angle ϕ with respectively eigenvalues 1 and -1 (see equation 2.37). We can write

$$\begin{aligned} |0\rangle &= \cos\frac{\phi}{2} |a\rangle - \sin\frac{\phi}{2} |b\rangle \\ |1\rangle &= \sin\frac{\phi}{2} |a\rangle + \cos\frac{\phi}{2} |b\rangle \end{aligned} \tag{2.62}$$

Substituting this into the definition of $|\Psi^-\rangle$ we find

$$\begin{aligned} |\Psi^-\rangle &= \frac{1}{\sqrt{2}} \left(\cos^2\frac{\phi}{2} |ab\rangle + \sin^2\frac{\phi}{2} |ab\rangle - \sin^2\frac{\phi}{2} |ba\rangle - \cos^2\frac{\phi}{2} |ba\rangle \right) \\ &= \frac{1}{\sqrt{2}} (|ab\rangle - |ba\rangle) \end{aligned} \tag{2.63}$$

A measurement on qubit A thus projects qubit B onto the opposite state that is measured by A and vice versa. For example, if both qubits are measured in the same direction, the results are perfectly anti-correlated. We will use this to find violations of Bell inequalities.

3

Results

In this chapter several results are derived based on the framework layed out by Rosset et al. [14]. The derivations in sections 3.1 and 3.2 are results found by respectively Rosset et al. [14] and Brunner et al. [5]. Because these results are fundamental and used later in the text, the derivation is also included here. The last two sections are new results, showing the applicability of theorem 2.1.1 to star-shaped networks and proving that it is possible to violate the new inequalities using quantum mechanics.

3.1. Entanglement

If we want to find nonlocal correlations from measurements on a quantum state, it is necessary that there is entanglement. We use the following definition, which is taken from [5].

Definition 3.1.1. *A quantum state is called entangled if the state cannot be written in the separable form*

$$\rho_{AB} = \sum_{\lambda} p_\lambda \rho_A^\lambda \otimes \rho_B^\lambda \tag{3.1}$$

where ρ_A^λ and ρ_A^λ are density operators of subsystems A and B.

In this definition subsystems A and B can be seen as two parties connected to a single source $\mathcal{S}$ with a random local variable λ associated to it. It can be shown that the Bell states are examples of entanglement. Brunner shows in [5] that entanglement is a necessary condition for observing nonlocal correlations. If a state can be written in the separable form, then

$$\begin{aligned} P(a,b|x,y) &= \mathrm{Tr}\left\{\sum_{\lambda} p_\lambda \left(\rho_A^\lambda \otimes \rho_B^\lambda\right) \hat{a}_x \otimes \hat{b}_y\right\} \\ &= \sum_{\lambda} p_\lambda \,\mathrm{Tr}\{\rho_A^\lambda \hat{a}_x\}\,\mathrm{Tr}\{\rho_B^\lambda \hat{b}_y\} \\ &= \sum_{\lambda} P_\lambda P(a|x,\lambda) p(b|y,\lambda) \end{aligned} \tag{3.2}$$

which is of the local form (definition 2.1.1). So indeed there should be entanglement in order to obtain nonlocal correlations. This argument easily generalizes to a network configuration, where it means that non-$\mathcal{N}$-local behaviour can only be observed if there is entanglement between subsystems in the network.

3.2. Alternative derivation of the CHSH inequality

In the introduction the famous CHSH inequality has been described. However, the proof of theorem 2.1.1 gives an another way of deriving it. This derivation is also done by Rosset et al. [14]. Consider a network $\mathcal{N}$ consisting of a single party $\mathcal{A}^1$ with binary input. We obtain network $\mathcal{N}'$ by adding a source that is connected to party $\mathcal{A}^1$ and a new party $\mathcal{A}^2$, again with binary input. According to equation 2.15 $\mathcal{N}'$-local correlations satisfy

$$\left\langle a^1_{x^1} \frac{a^2_0 \pm a^2_1}{2} \right\rangle = q_\pm \int_{\Omega_\pm} d\mu\ \rho_{\Omega_\pm}(\mu) \left\langle \tilde{a}^1_{x^1,\mu} \right\rangle \tag{3.3}$$

Since $\tilde{a}^1_{x^1,\mu} = \pm 1$, it clearly holds that $-1 \leq \left\langle \tilde{a}^1_{x^1,\mu} \right\rangle \leq 1$. The integral in this equation just averages the quantity $\left\langle \tilde{a}^1_{x^1,\mu} \right\rangle$ over $\Omega_\pm$ and therefore we have

$$\left| \int_{\Omega_\pm} d\mu\ \rho_{\Omega_\pm}(\mu) \left\langle \tilde{a}^1_{x^1,\mu} \right\rangle \right| \leq \int_{\Omega_\pm} d\mu\ \rho_{\Omega_\pm}(\mu) \left| \left\langle \tilde{a}^1_{x^1,\mu} \right\rangle \right| \leq \int_{\Omega_\pm} d\mu\ \rho_{\Omega_\pm}(\mu) = 1 \tag{3.4}$$

So we have

$$\left| \left\langle a^1_0 \frac{a^2_0 + a^2_1}{2} \right\rangle + \left\langle a^1_1 \frac{a^2_0 - a^2_1}{2} \right\rangle \right| \leq \left| \left\langle a^1_0 \frac{a^2_0 + a^2_1}{2} \right\rangle \right| + \left| \left\langle a^1_1 \frac{a^2_0 - a^2_1}{2} \right\rangle \right| \leq q_+ + q_- \tag{3.5}$$

Invoking now that $q_+ + q_- = 1$, we retreive the CHSH inequality

$$\left\langle a^1_0 \frac{a^2_0 + a^2_1}{2} \right\rangle + \left\langle a^1_1 \frac{a^2_0 - a^2_0}{2} \right\rangle \leq 1 \tag{3.6}$$

3.3. Star network with four parties

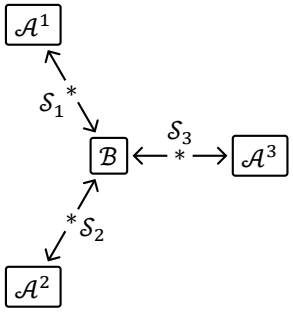


Figure 3.1: A schematic view of the star network with central party $\mathcal{B}$ and three parties $\mathcal{A}^1$, $\mathcal{A}^2$ and $\mathcal{A}^3$ surrounding it. Party $\mathcal{B}$ is connected to each party $\mathcal{A}^i$ via source $\mathcal{S}_i$.

Consider a star network with four parties as shown in figure 3.1. The network consists of a central party $\mathcal{B}$ that is connected with parties $\mathcal{A}^i$ via the corresponding sources $\mathcal{S}_i$ with $i \in \{1, 2, 3\}$). We can find a Bell inequality for this network in two steps. We will start with the CHSH inequality (equation 3.6) from the previous section.

Our starting point is a network $\mathcal{N}$ with parties $\mathcal{A}^1$ and $\mathcal{B}$ with binary input, both connected to a single source $\mathcal{S}_1$. The $\mathcal{N}$-local correlations satisfy the CHSH inequality (with $a^2_{x^2}$ replaced by b_y). We obtain network $\mathcal{N}'$ by adding a source $\mathcal{S}_2$ connected to party $\mathcal{B}$ and a new party $\mathcal{A}^2$ with binary input. Applying

theorem 2.1.1 with $\chi_+ = \{0\}$ and $\chi_- = \{1\}$, we find that $\mathcal{N}'$-local correlations satisfy

$$\begin{aligned} &\exists q \in [0,1] \text{ such that} \\ &\frac{1}{q}\left\langle \frac{a_0^1 + a_1^1}{2}\frac{a_0^2 + a_1^2}{2} b_0 \right\rangle \\ &+ \frac{1}{1-q}\left\langle \frac{a_0^1 - a_1^1}{2}\frac{a_0^2 - a_1^2}{2} b_1 \right\rangle \leq 1 \end{aligned} \tag{3.7}$$

Adding another source $\mathcal{S}_3$ connected to party $\mathcal{B}$ and a new party $\mathcal{A}^3$ creates a network $\mathcal{N}''$. This time we apply theorem 2.1.1 on the above inequality with $\chi_+ = \{0\}$ and $\chi_- = \{1\}$ and find that $\mathcal{N}''$-local correlations satisfy

$$\begin{aligned} &\exists q, r \in [0,1] \text{ such that} \\ &\frac{1}{q}\frac{1}{r}\left\langle \frac{a_0^1 + a_1^1}{2}\frac{a_0^2 + a_1^2}{2}\frac{a_0^3 + a_1^3}{2} b_0 \right\rangle \\ &+ \frac{1}{1-q}\frac{1}{1-r}\left\langle \frac{a_0^1 - a_1^1}{2}\frac{a_0^2 - a_1^2}{2}\frac{a_0^3 - a_1^3}{2} b_1 \right\rangle \leq 1 \end{aligned} \tag{3.8}$$

Inspired by the violation of the CHSH inequality, we can show that inequality 3.8 can indeed be violated by quantum mechanics. Assume that each source $\mathcal{S}_i$ (with $i \in \{0,1,2\}$) emits two particles in the 2-qubit Werner state $\rho = v_i |\Phi^+\rangle\langle\Phi^+| + (1-v_i)\mathbb{1}/4$ with $v_i \in [0,1]$. Here $|\Phi^+\rangle = 1/\sqrt{2}\,(|00\rangle + |11\rangle)$. The state $\mathbb{1}/4$ is also referred to as the completely mixed state [12].

The parties $\mathcal{A}^1$, $\mathcal{A}^2$ and $\mathcal{A}^3$ will perform single qubit measurements given by operators $\hat{a}_0^1 = \hat{a}_0^2 = \hat{a}_0^3 = (\hat{\sigma}_z + \hat{\sigma}_x)/\sqrt{2}$ and $\hat{a}_1^1 = \hat{a}_1^2 = \hat{a}_1^3 = (\hat{\sigma}_z - \hat{\sigma}_x)/\sqrt{2}$. Party $\mathcal{B}$ will perform three-qubit measurements given by operators $\hat{b}_0 = \hat{\sigma}_z \otimes \hat{\sigma}_z \otimes \hat{\sigma}_z$ and $\hat{b}_1 = \hat{\sigma}_x \otimes \hat{\sigma}_x \otimes \hat{\sigma}_x$. We then find

$$\left\langle a_{x^1}^1 a_{x^2}^2 a_{x^3}^3 b_y \right\rangle = \mathrm{Tr}\left\{\rho\left(\hat{a}_{x^1}^1 \otimes \hat{a}_{x^2}^2 \otimes \hat{a}_{x^3}^3 \otimes \hat{b}_y\right)\right\} = (-1)^{x_1 y + x_2 y + x_3 y}\frac{\sqrt{2} v_1 v_2 v_3}{4} \tag{3.9}$$

The most straightforward way of finding this result is by writing the operators as a convex combination, e.g. $\hat{a}_{x^1}^1 = (1-x_1)(\hat{\sigma}_z + \hat{\sigma}_x)/\sqrt{2} + x^1(\hat{\sigma}_z - \hat{\sigma}_x)/\sqrt{2}$. We can eventually substitute all terms like $1 - 2x_1 y$ by $(-1)^{x_1 y}$ to arrive at equation 3.9. However, in the next section a cleaner method is used that also works in this case.

Using the linearity of the expectation value, it follows that

$$\left\langle \frac{a_0^1 + a_1^1}{2}\frac{a_0^2 + a_1^2}{2}\frac{a_0^3 + a_1^3}{2} b_0 \right\rangle = \left\langle \frac{a_0^1 - a_1^1}{2}\frac{a_0^2 - a_1^2}{2}\frac{a_0^3 - a_1^3}{2} b_1 \right\rangle = \frac{\sqrt{2}}{4} v_1 v_2 v_3 \tag{3.10}$$

Substituting this result into inequality 3.8, we see that the minimum of the left hand side occurs for those q and r that minimize

$$\frac{1}{q}\frac{1}{r} + \frac{1}{1-q}\frac{1}{1-r} \tag{3.11}$$

This is the case for $q = 1/2$ and $r = 1/2$. The proof for this uses partial derivatives to q and r and can be found in the next section. By substituting these values, the minimum of the left hand side of inequality 3.8 is found to be $2\sqrt{2} v_1 v_2 v_3$.

Defining $V = v_1 v_2 v_3$, we see that the inequality is violated, and hence the correlations are non-$\mathcal{N}$-local, for $V \geq \frac{\sqrt{2}}{4} \approx 0.35$. In the next section we will show that a similar result holds for general star-shaped networks.

3.4. General star network

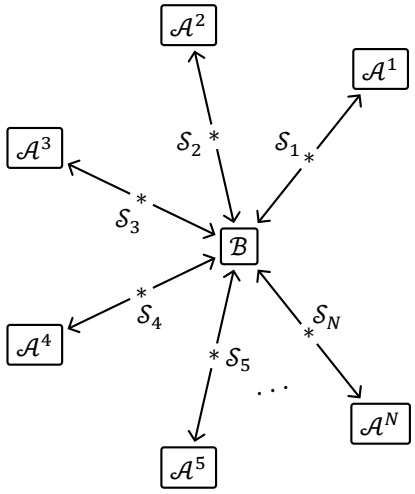


Figure 3.2: A schematic view of the star network with central party $\mathcal{B}$ and N parties $\mathcal{A}^i$ surrounding it. Party $\mathcal{B}$ is connected to each party $\mathcal{A}^i$ via source $\mathcal{S}_i$.

In this section we consider a star network $\mathcal{N}$ with a central party $\mathcal{B}$ and N parties $\mathcal{A}^i$ which are connected to the central party via a source $\mathcal{S}_i$. For $\mathcal{N}$-local correlations in this network we have the following inequality [14]:

$\exists q_1, \dots, q_{N-1} \in [0,1]$ such that

$$\frac{1}{q_1} \cdots \frac{1}{q_{N-1}} \left\langle \frac{a_0^1 + a_1^1}{2} \cdots \frac{a_0^N + a_1^N}{2} b_0 \right\rangle + \frac{1}{1-q_1} \cdots \frac{1}{1-q_{N-1}} \left\langle \frac{a_0^1 - a_1^1}{2} \cdots \frac{a_0^N - a_1^N}{2} b_1 \right\rangle \leq 1 \tag{3.12}$$

This inequality is obtained by repeatedly applying theorem 2.1.1, starting from the CHSH inequality and each time extending the network from party $\mathcal{B}$ with $\chi_+ = \{0\}$ and $\chi_- = \{1\}$. The left hand side of inequality 3.12 can be written as

$$\sum_{x^1,x^2,\cdots,x^N} \frac{1}{q_1} \cdots \frac{1}{q_{N-1}} 2^{-N} \left\langle a_{x^1}^1 a_{x^2}^2 \cdots a_{x^N}^N b_0 \right\rangle + \frac{1}{1-q_1} \cdots \frac{1}{1-q_{N-1}} 2^{-N} (-1)^{x_1+x_2+\cdots+x_N} \left\langle a_{x^1}^1 a_{x^2}^2 \cdots a_{x^N}^N b_1 \right\rangle \tag{3.13}$$

Next we will discuss quantum violations. Assume that each source $\mathcal{S}_i$ (with $i \in \{0,1,2\}$) emits two particles in the 2-qubit Werner state $\rho = v_i |\Phi^+\rangle\langle\Phi^+| + (1-v_i)\mathbb{1}/4$ with $v_i \in [0,1]$. Here $|\Phi^+\rangle = 1/\sqrt{2}\,(|00\rangle + |11\rangle)$ is one of the Bell states. Now let the measurements of central party $\mathcal{B}$ be defined as $\hat{b}_0 = \hat{\sigma}_z \otimes \cdots \otimes \hat{\sigma}_z$ and $\hat{b}_1 = \hat{\sigma}_x \otimes \cdots \otimes \hat{\sigma}_x$. Let the measurement of the surrounding parties $\mathcal{A}^i$ be defined as $\hat{a}_0^i = (\hat{\sigma}_z + \hat{\sigma}_x)/\sqrt{2}$ and $\hat{a}_1^i = (\hat{\sigma}_z - \hat{\sigma}_x)/\sqrt{2}$. We find using a simple calculation that

$$\mathrm{Tr}\left\{\rho_{\mathcal{S}_i} \left[a_{x^i}^i \otimes b_y^i\right]\right\} = \frac{1}{\sqrt{2}} \cdot (-1)^{x_i y} v_i \tag{3.14}$$

Using equation 3.22 we find that

$$\left\langle a_{x^1}^1 a_{x^2}^2 \cdots a_{x^N}^N b_y \right\rangle = 2^{-\frac{N}{2}} \cdot (-1)^{x_1 y + x_2 y + \cdots + x_N y} v_1 v_2 \cdots v_N \tag{3.15}$$

The reason why this holds can be found in the next section (3.4.1), where the evaluation of correlators $\langle a_{x^1}^1 a_{x^2}^2 \cdots a_{x^N}^N b_y \rangle$ is treated.

If we substitute equation 3.15 into equation 3.13, we can rewrite the left hand side of inequality 3.12 as

$$\sum_{x^1,x^2,\cdots,x^N} \frac{1}{q_1} \cdots \frac{1}{q_{N-1}} 2^{-\frac{3N}{2}} \cdot v_1 v_2 \cdots v_N + \frac{1}{1-q_1} \cdots \frac{1}{1-q_{N-1}} 2^{-\frac{3N}{2}} \cdot v_1 v_2 \cdots v_N \tag{3.16}$$

Since the expression we sum over does not depend on x^i for all $i = 1, \dots, N$, we can rewrite this as

$$2^{-\frac{N}{2}} \cdot v_1 v_2 \cdots v_N \left(\frac{1}{q_1} \cdots \frac{1}{q_{N-1}} + \frac{1}{1-q_1} \cdots \frac{1}{1-q_{N-1}} \right) \tag{3.17}$$

Now we use that the minimum of this expression occurs when $q_i = \frac{1}{2}$ for all $i = 1, \dots, N$ (this is explained in section 3.4.2). The minimum of the left hand side of equation 3.12 is given by

$$2^{\frac{N}{2}} \cdot v_1 v_2 \cdots v_N \tag{3.18}$$

Defining $V = v_1 v_2 \cdots v_N$, we see that inequality 3.12 is violated, and hence the correlations are non-$\mathcal{N}$-local, for $V > 2^{-\frac{N}{2}}$. It is immediately clear that this setup does not lead to violation if any of the sources produces always the completely mixed state ($v_i = 0$ for a certain i). Furthermore, 'bad' sources (with $v < 1/\sqrt{2}$) need to be compensated by 'good' sources (with $v > 1/\sqrt{2}$). Interestingly, this bound is equal to that for a chain network described in [14]. This raises the question what the importance of the network structure is. However, this is a complicated question and further analysis of this aspect would be beyond the scope of this thesis.

3.4.1. Evaluating correlators

In this section a method to evaluate $\langle a^1_{x^1} a^2_{x^2} \cdots a^N_{x^N} b_y \rangle$ is provided. For clarity, we adopt some new notation. Let A_i for $i = 1, \dots n$ be operators. Then we define

$$\bigotimes_{i=1}^{n} A_i \equiv A_1 \otimes A_2 \otimes \cdots \otimes A_n \tag{3.19}$$

In order to compute the expectation value, a description of the density matrix ρ of our system is needed. Since the system is generated by independent sources, the density matrix ρ can be decomposed into its subsystems created by the sources. So we have $\rho = \rho_{\mathcal{S}_1} \otimes \rho_{\mathcal{S}_2} \otimes \cdots \otimes \rho_{\mathcal{S}_N}$. Measurement operator $\hat{b}_y$ is a combination of measurement operators that act on a subsystem coming from a single source, so $\hat{b}_y = \hat{b}^1_y \otimes \hat{b}^2_y \otimes \cdots \otimes \hat{b}^N_y$. In this case the superscript denotes the corresponding source. For example, b^1_y is the measurement operator acting on the subsystem sent by source $\mathcal{S}_1$ to party B.

The expectation value can now be evaluated using the trace,

$$\left\langle a^1_{x^1} a^2_{x^2} \cdots a^N_{x^N} b_y \right\rangle = \mathrm{Tr}\{\rho \left[\hat{a}^1_{x^1} \otimes \hat{b}^1_y \otimes \hat{a}^2_{x^2} \otimes \hat{b}^2_y \otimes \cdots \otimes \hat{a}^N_{x^N} \otimes \hat{b}^N_y \right]\} = \mathrm{Tr}\left\{ \rho \left[\bigotimes_{i=1}^{N} \hat{a}^i_{x^1} \otimes \hat{b}^i_y \right] \right\} \tag{3.20}$$

Substituting $\rho = \otimes_{i=1}^{N} \rho_{\mathcal{S}_i}$ and using $N - 1$ times property 2.52, we find that

$$\rho \left[\bigotimes_{i=1}^{N} \hat{a}^i_{x^1} \otimes \hat{b}^i_y \right] = \left[\bigotimes_{i=1}^{N} \rho_{\mathcal{S}_i} \right] \left[\bigotimes_{i=1}^{N} \hat{a}^i_{x^1} \otimes \hat{b}^i_y \right] = \bigotimes_{i=1}^{N} \rho_{\mathcal{S}_i} \left[\hat{a}^i_{x^i} \otimes \hat{b}^i_y \right] \tag{3.21}$$

Substituting this back into equation 3.20 and using property 2.53, we get

$$\left\langle a^1_{x^1} a^2_{x^2} \cdots a^N_{x^N} b_y \right\rangle = \mathrm{Tr}\left\{ \bigotimes_{i=1}^{N} \rho_{\mathcal{S}_i} \left[\hat{a}^i_{x^i} \otimes \hat{b}^i_y \right] \right\} = \prod_{i=1}^{N} \mathrm{Tr}\left\{ \rho_{\mathcal{S}_i} \left[\hat{a}^i_{x^i} \otimes \hat{b}^i_y \right] \right\} \tag{3.22}$$

3.4.2. Minimization problem

In this section we will prove the following:

$$\min_{q_1,\dots,q_{N-1}} \left[\frac{1}{q_1} \cdots \frac{1}{q_{N-1}} + \frac{1}{1-q_1} \cdots \frac{1}{1-q_{N-1}} \right] = 2^N \quad \text{for } q_i \in (0,1) \tag{3.23}$$

We start by defining a function $f : (0,1) \times \cdots \times (0,1) \to \mathbb{R}$ given by

$$f(q_1, \dots, q_{N-1}) = \frac{1}{q_1} \cdots \frac{1}{q_{N-1}} + \frac{1}{1-q_1} \cdots \frac{1}{1-q_{N-1}} \tag{3.24}$$

Taking the partial derivative to q_i yields

$$\frac{\partial}{\partial q_i} f(q_1, \dots, q_{N-1}) = -\frac{1}{q_1} \cdots \frac{1}{q_i^2} \cdots \frac{1}{q_{N-1}} + \frac{1}{1-q_1} \cdots \frac{1}{(1-q_i)^2} \cdots \frac{1}{1-q_{N-1}} \tag{3.25}$$

Requiring the partial derivative to q_i to be zero yields

$$q_1 \cdots q_i^2 \cdots q_{N-1} = (1-q_1) \cdots (1-q_i)^2 \cdots (1-q_{N-1}) \tag{3.26}$$

All stationary points of f need to satisfy equation 3.26 for each i. By multiplication of all these equations, we find

$$q_1^N \cdots q_{N-1}^N = (1-q_1)^N \cdots (1-q_{N-1})^N \tag{3.27}$$

and therefore

$$q_1 \cdots q_{N-1} = (1-q_1) \cdots (1-q_{N-1}) \tag{3.28}$$

For $q_i \in (0,1)$, we can combine 3.26 and 3.28 to get

$$q_i = 1 - q_i \tag{3.29}$$

This means that the only stationary point is given by $(q_1, \dots, q_{N-1}) = \left(\frac{1}{2}, \dots, \frac{1}{2}\right)$. We see that the $(N-1) \times (N-1)$-Hessian matrix of f at this stationary point is given by

$$H_f\left(\frac{1}{2}, \dots, \frac{1}{2}\right) = 2^{N+2} \cdot \begin{pmatrix} 2 & 1 & 1 & \cdots & 1 \\ 1 & 2 & \ddots & \ddots & \vdots \\ 1 & \ddots & \ddots & \ddots & 1 \\ \vdots & \ddots & \ddots & 2 & 1 \\ 1 & \cdots & 1 & 1 & 2 \end{pmatrix} \tag{3.30}$$

Now we want to know if the Hessian matrix is positive definite. Sylvester's criterion tells us that the Hessian matrix is positive definite if the determinants associated with all upper-left submatrices of the Hessian matrix are positive. The determinant of a matrix does not change when multiples of rows and columns are added together. Adding all rows of the upper-left submatrix with dimension $(m-1)\times(m-1)$ to the first row and then subtracting the first column from the other columns yields:

$$\begin{vmatrix} 2 & 1 & 1 & \cdots & 1 \\ 1 & 2 & \ddots & \ddots & \vdots \\ 1 & \ddots & \ddots & \ddots & 1 \\ \vdots & \ddots & \ddots & 2 & 1 \\ 1 & \cdots & 1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} m & m & m & \cdots & m \\ 1 & 2 & 1 & \cdots & 1 \\ 1 & 1 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 2 & 1 \\ 1 & 1 & \cdots & 1 & 2 \end{vmatrix} = \begin{vmatrix} m & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 & 0 \\ 1 & 0 & \cdots & 0 & 1 \end{vmatrix} \tag{3.31}$$

The last matrix is triangular matrix with determinant m. So the upper-left submatrices of the Hessian matrix in equation 3.30 have determinant $m \cdot 2^{N+2} > 0$. The Hessian matrix is indeed positive definite, and therefore f has a local minimum at $(q_1, \dots, q_{N-1}) = \left(\frac{1}{2}, \dots, \frac{1}{2}\right)$. Since clearly f is continuous and $f \to \infty$ at the boundaries, we see that this is a global minimum. Therefore the minimum value of f is 2^N.

4

Detecting violations

As we have already seen in the previous chapter, Bell inequalities can be violated by quantum mechanics. We usually choose a maximally entangled state (these are the Bell states). This can be done by letting each source create two particles in the Bell state $|\psi^-\rangle$ and choose the measurements cleverly, as we have seen in the previous chapter. However, in a real experimental setup there are some other limitations. In the next sections two of these will be elaborated. First the effect of distinguishable photons will be discussed and then also measurement errors are taken into account.

4.1. Creating a network using nitrogen-vacancy centres

Nitrogen-vacancy centres could be used to create entangled qubit pairs for a quantum network. A nitrogen-vacancy centre (NV) is a defect in diamond. It consist of a nitrogen atom, which substitutes a carbon atom, and a lattice vacancy. The protocol to establish entanglement between two distant qubits is used and described in [2]. It concerns the setting of a source connecting two parties $\mathcal{A}$ (NV A) and $\mathcal{B}$ (NV B).

Within the NV centre we have a spin qubit, with eigenstates denoted by $|\uparrow\rangle$ and $|\downarrow\rangle$ (these are the eigenstates of $\hat{\sigma}_z$). We initially prepare both NV A and NV B in the superposition state $1/\sqrt{2}(|\uparrow\rangle + |\downarrow\rangle)$. Each NV centre will then be excited by a short laser pulse that is resonant to a transition of the $|\uparrow\rangle$ state to an optically excited state with the same spin projection as $|\uparrow\rangle$. Spontaneous emission will occur for this excited state and this entangles the qubit with the absence/presence of an emitted photon. The new state is described by $1/\sqrt{2}(|\uparrow 1\rangle + |\downarrow 0\rangle)$, where 1 denotes the presence of an emitted photon and 0 its absence. After this laser pulse, the qubit-photon state of both parties can thus be described by

$$\frac{1}{2}\left(|\uparrow_A\uparrow_B\rangle\,|1_A1_B\rangle + |\downarrow_A\downarrow_B\rangle\,|0_A0_B\rangle + |\uparrow_A\downarrow_B\rangle\,|1_A0_B\rangle + |\downarrow_A\uparrow_B\rangle\,|0_A1_B\rangle\right) \tag{4.1}$$

All created photons are directed to a beamsplitter where each photon will cause the detector to ‘click’. So we are able to count the number of photons (0, 1 or 2). However, because of the beamsplitter, any observed fluorescence could have originated from either NV A or NV B. The photons are thus indistinguishable and measuring a single photon would corresponds to measuring the photon state

$$\frac{1}{\sqrt{2}}\left(|1_A0_A\rangle + e^{i\varphi}\,|0_A1_B\rangle\right) \tag{4.2}$$

Here the phase factor φ depends on the optical path length [2]. Referring back to equation 4.1, we see that this measurement projects the qubits onto the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_A\downarrow_B\rangle + e^{i\varphi}\,|\downarrow_A\uparrow_B\rangle\right) \tag{4.3}$$

which is clearly an entangled state, a necessary property for nonlocal behaviour.

Due to detector inefficiencies, it is in general not possible to create the state from equation 4.3. However, some of these inefficiencies can be eliminated. Suppose that both NV centres emit a photon, but one of them gets lost before arriving at the beamsplitter. We would then detect only one photon yielding the wrong conclusion that the qubits are in the maximally entangled state (equation 4.3). In [2] the procedure is therefore slightly altered. We first perform the above steps, then flip both qubits and then perform the above steps again. The state $|\uparrow_A\uparrow_B\rangle$ might cause detection of a single photon the first time, but the second time (when the state is flipped to $|\downarrow_A\downarrow_B\rangle$) no photons will be emitted. In contrary, the states $|\uparrow_A\downarrow_B\rangle$ and $|\downarrow_A\uparrow_B\rangle$ will emit only one photon both the first and second time. We thus want to select a state that induces a photon detection in both rounds. This improved procedure is then robust against photon loss. An additional advantage is flipping the qubits eliminates the phase factor φ (see [2]), which leaves us with one of the two Bell states

$$|\psi^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_A\downarrow_B\rangle \pm |\downarrow_A\uparrow_B\rangle\right) \tag{4.4}$$

Also the sign can be deduced using two output detectors at the beam splitter. Both $|\psi^+\rangle$ and $|\psi^-\rangle$ are maximally entangled states and are therefore good candidates to violate Bell inequalities. We can use them as the physical system that is sent by each source. In the next sections we evaluate some conditions under which we can reveal nonlocal behaviour with the state $|\psi^-\rangle$.

4.2. Quantum violation in a network with three parties

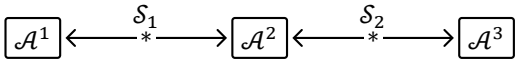


Figure 4.1: A schematic view of the network with three parties $\mathcal{A}^1$, $\mathcal{A}^2$ and $\mathcal{A}^3$ and two sources $\mathcal{S}_1$ and $\mathcal{S}_2$. It is the extension of the network used in the well-known CHSH inequality.

We consider a network $\mathcal{N}_3$ consisting of three parties $\mathcal{A}^1$, $\mathcal{A}^2$ and $\mathcal{A}^3$ with binary input and two sources $\mathcal{S}_1$ and $\mathcal{S}_2$. The first two parties $\mathcal{A}^1$ and $\mathcal{A}^2$ are connected to source $\mathcal{S}_1$, while parties $\mathcal{A}^2$ and $\mathcal{A}^3$ are connected to source $\mathcal{S}_2$.

For the network $\mathcal{N}$ that is equal to network $\mathcal{N}_3$, but with source $\mathcal{S}_2$ and party $\mathcal{A}^3$ removed, we know that $\mathcal{N}$-local correlations satisfy the CHSH inequality 3.6

$$\left\langle \frac{a_0^1 + a_1^1}{2} a_0^2 \right\rangle + \left\langle \frac{a_0^1 - a_1^1}{2} a_1^2 \right\rangle \leq 1 \tag{4.5}$$

Network $\mathcal{N}_3$ is then the extended network of $\mathcal{N}$. We choose $\chi_+^2 = \{0\}$ and $\chi_-^2 = \{1\}$. Theorem 2.1.1 now gives the following inequality that $\mathcal{N}_3$-local correlations need to satisfy:

$$\begin{aligned} &\exists q \in [0,1] \text{ such that} \\ &\frac{1}{q}\left\langle \frac{a_0^1 + a_1^1}{2} a_0^2 \frac{a_0^3 + a_1^3}{2} \right\rangle + \frac{1}{1-q}\left\langle \frac{a_0^1 - a_1^1}{2} a_1^2 \frac{a_0^3 - a_1^3}{2} \right\rangle \leq 1 \end{aligned} \tag{4.6}$$

This inequality can be violated by quantum mechanics. We let each source create two particles in the state $|\psi^-\rangle$ (equation 4.4) and the next step is to find measurement settings that yield violation.

4.2.1. Minimization

Inequality 4.6 contains an existential quantifier. So violation corresponds with the non-existence of any q for which this inequality holds. We therefore need to minimize over q. For clarity in notation, let $A = \left\langle \frac{a_0^1+a_1^1}{2} a_0^2 \frac{a_0^3+a_1^3}{2} \right\rangle$ and $B = \left\langle \frac{a_0^1-a_1^1}{2} a_1^2 \frac{a_0^3-a_1^3}{2} \right\rangle$. The minimization problem is then:

$$\min_{q\in[0,1]} \left[\frac{1}{q} A + \frac{1}{1-q} B \right] \tag{4.7}$$

where we still use the notation $q = 0$ for $q \to 0$ and $q = 1$ for $q \to 1$.

Note that for $A = 0$ or $B = 0$ respectively $q = 0$ and $q = 1$ yield a minimum. For $A < 0$ or $B < 0$ there clearly is no minimum. The case $A > 0$ and $B > 0$ needs more attention.

So let us assume $A > 0$ and $B > 0$. Since $\frac{1}{q}A + \frac{1}{1-q}B$ gives rise to a continuous function of q on the interval $(0, 1)$, all local minima need to satisfy

$$\frac{d}{dq}\left[\frac{1}{q}A + \frac{1}{1-q}B\right] = -\frac{1}{q^2}A + \frac{1}{(1-q)^2}B = 0 \tag{4.8}$$

and

$$\frac{d^2}{dq^2}\left[\frac{1}{q}A + \frac{1}{1-q}B\right] = \frac{2}{q^3}A + \frac{2}{(1-q)^3}B > 0 \tag{4.9}$$

In case $A \neq B$ we find that the only local minimum is given by $q = \frac{A-\sqrt{AB}}{A-B}$. In case $A = B$ we find that $q = \frac{1}{2}$ is the only local minimum. For these two cases we see that $\frac{1}{q}A + \frac{1}{1-q}B \to \infty$ for both $q \to 0$ and $q \to 1$. Therefore the local minima we found are all global minima.

The case $A \neq B$ with $A, B > 0$ yields an easy formula for the minimum value:

$$\begin{aligned}
\min_{q\in[0,1]}\left[\frac{1}{q}A + \frac{1}{1-q}B\right] &= \frac{1}{\frac{A-\sqrt{AB}}{A-B}}A + \frac{1}{1-\frac{A-\sqrt{AB}}{A-B}}B \\
&= \frac{A^2 - AB}{A-\sqrt{AB}} + \frac{AB-B^2}{\sqrt{AB}-B} \\
&= \frac{A\sqrt{AB} - B\sqrt{AB} + AB - B^2}{\sqrt{AB}-B} \\
&= 2\sqrt{AB} + A + B \\
&= \left(\sqrt{A} + \sqrt{B}\right)^2
\end{aligned} \tag{4.10}$$

It can be easily seen that the formula also holds for the cases $A = B$, $A = 0$ and $B = 0$.

4.2.2. Choice of measurement operators

Violation of a Bell inequality requires a clever choice of quantum states and measurements. We definitely want the quantum system that is distributed by the source to be maximally entangled, since entanglement is required for nonlocality. For this reason, we let both source $\mathcal{S}_1$ and source $\mathcal{S}_2$ create two particles in the state $|\psi^-\rangle$ from equation 4.4. However, finding measurement operators that lead to violation is a more challenging problem. We need to find operators $a^j_{x^j}$ for each party $\mathcal{A}^j$ that satisfy $\sqrt{A} + \sqrt{B} > 1$ subject to the condition that $A, B \geq 0$. Let us take a closer look at the expressions for A and B:

$$\begin{aligned}
A &= \frac{1}{4}\left[\langle a^1_0 a^2_0 a^3_0\rangle + \langle a^1_0 a^2_0 a^3_1\rangle + \langle a^1_1 a^2_0 a^3_0\rangle + \langle a^1_1 a^2_0 a^3_1\rangle\right] \\
B &= \frac{1}{4}\left[\langle a^1_0 a^2_1 a^3_0\rangle - \langle a^1_0 a^2_1 a^3_1\rangle - \langle a^1_1 a^2_1 a^3_0\rangle + \langle a^1_1 a^2_1 a^3_1\rangle\right]
\end{aligned} \tag{4.11}$$

Equation 4.10 shows that the inequality can be violated when A and B are both larger than $1/4$. At first sight, the minus signs in the expression for B seems to be a problem. Nevertheless, it is possible to find a violation using spin measurements along different angles. Party $\mathcal{A}^2$ receives two particles, one from source $\mathcal{S}_1$ and one from source $\mathcal{S}_2$. We let $a^2_{x_2} = a^{2,1}_{x_2} a^{2,2}_{x_2}$ where $a^{2,1}_{x_2}$ is the outcome of the measurement on the particle from source $\mathcal{S}_1$ and $a^{2,2}_{x_2}$ is the outcome of the measurement on the particle from source $\mathcal{S}_2$.

We will use spin measurements along a certain angle, so we write $\hat{a}^1_{x_1} = \hat{\sigma}_{\phi_1}$, $\hat{a}^2_{x_2} = \hat{\sigma}_{\phi_2} \otimes \hat{\sigma}_{\phi_2}$ and $\hat{a}^3_{x_3} = \hat{\sigma}_{\phi_3}$.

Because of the symmetry in network $\mathcal{N}_3$, we first zoom in on the left side of the network. Source $\mathcal{S}_1$ creates two particles in the state $|\psi^-\rangle$. Party $\mathcal{A}_1$ performs a measurement on one of these particles and the other one is measured by party $\mathcal{A}_2$. From section 2.7 we know that a spin measurement by one of the parties projects the spin state of the other party in the opposite direction. The value of $a^1_{x_1} a^{2,1}_{x_2}$ therefore depends on the difference in measurement angles of $\hat{a}^1_{x_1}$ and $\hat{a}^2_{x_2}$. More precisely, using equation 2.38, the probability to find 1 is given by

$$P\left(a^1_{x_1} a^{2,1}_{x_2} = 1\right) = P\left(a^1_{x_1} = 1, a^{2,1}_{x_2} = 1\right) + P\left(a^1_{x_1} = -1, a^{2,1}_{x_2} = -1\right) = \sin^2\left(\frac{\phi_1 - \phi_2}{2}\right) \tag{4.12}$$

At the other side of the network (source $\mathcal{S}_2$ and parties $\mathcal{A}_2$ and $\mathcal{A}_3$) the same reasoning holds. The probability to find 1 for $a^1_{x_1} a^2_{x_2} a^3_{x_3}$ is thus given by

$$\begin{aligned} P\left(a^1_{x_1} a^2_{x_2} a^3_{x_3} = 1\right) &= P\left(a^1_{x_1} a^{2,1}_{x_2} = 1\right) P\left(a^{2,2}_{x_2} a^3_{x_3} = 1\right) \\ &\quad + P\left(a^1_{x_1} a^{2,1}_{x_2} = -1\right) P\left(a^{2,2}_{x_2} a^3_{x_3} = -1\right) \\ &= \sin^4\left(\frac{\phi_1 - \phi_2}{2}\right) + \cos^4\left(\frac{\phi_1 - \phi_2}{2}\right) \end{aligned} \tag{4.13}$$

We can find a possible measurement setup using an educated guess inspired by [14] and exploiting the symmetry of the network. Choosing angle differences of $\pi/4$ (or $3\pi/4$) between party $\mathcal{A}^1$'s measurement and party $\mathcal{A}^2$'s measurement and also between party $\mathcal{A}^2$'s measurement and party $\mathcal{A}^3$'s measurement, will yield a value of $1/2$ for each term $\langle a^1_{x_1} a^2_{x_2} a^3_{x_3} \rangle$.

It turns out that the following measurements settings will work. Let parties $\mathcal{A}^1$ and $\mathcal{A}^2$ perform different measurements depending or their inputs x^1 (for party $\mathcal{A}^1$) and x^3 (for party $\mathcal{A}^3$). For $x^1, x^3 = 0$, the corresponding measurement operators are $\hat{a}^1_0 = \hat{a}^3_0 = (\hat{\sigma}_z + \hat{\sigma}_x)/\sqrt{2}$ and for $x^1, x^3 = 1$, the measurement operators are given by $\hat{a}^1_1 = \hat{a}^3_1 = (\hat{\sigma}_z - \hat{\sigma}_x)\sqrt{2}$. Party $\mathcal{A}^2$ also performs different measurements depending on its input x^2. For $x^2 = 0$, the corresponding operator is given by $\hat{a}^2_0 = \hat{\sigma}_z \otimes \hat{\sigma}_z$ and for $x^2 = 1$, it is given by $\hat{a}^2_1 = \hat{\sigma}_x \otimes \hat{\sigma}_x$.

This combination of quantum states (both sources create two particles in the state $|\psi^-\rangle$) and measurement operators yield $A = 1/2$ and $B = 1/2$, thus giving a violation of inequality 4.6 according to equation 4.10.

4.3. Effect of distinguishable photons

Unfortunately, the photons created at different parties are not fully indistinguishable. This is probably caused by phonon-induced transitions between the excited states [2]. We use a parameter $v \in [0, 1]$ to model the distinguishability of the photons ($v = 1$ means fully indistinguishable and $v = 0$ fully distinguishable). If the photons are distinguishable, this would result in a state $|{\downarrow\uparrow}\rangle$ or $|{\uparrow\downarrow}\rangle$. It is assumed that other imperfections are negligible compared to the photon indistinguishability, so this leads to the following model [13] of the system distributed by each source $\mathcal{S}_i$:

$$\rho(v) = v\,|\Psi^-\rangle\langle\Psi^-| + \frac{1-v}{2}\left(|{\downarrow\uparrow}\rangle\langle{\downarrow\uparrow}| + |{\uparrow\downarrow}\rangle\langle{\uparrow\downarrow}|\right) \tag{4.14}$$

where $|\Psi^-\rangle$ is one of the Bell states, given by

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}\left(|{\downarrow\uparrow}\rangle - |{\uparrow\downarrow}\rangle\right) \tag{4.15}$$

The main difference between this model and the Werner state (see sections 3.3 and 3.4) is that all states occuring in equation 4.14 show strong correlations (if one state is 'spin up', the other is 'spin down'). In contrast, the measurement outcomes on two particles in a completely mixed state are uncorrelated.

Each source $\mathcal{S}_i$ thus creates two particles in the state $\rho(v_i)$ with $i \in \{0, 1\}$ as given in equation 4.14. Since the sources are independent, we can describe the density matrix of the entire network using

$$\rho_{\mathcal{N}_3} = \rho_{v_1} \otimes \rho_{v_2} \tag{4.16}$$

The terms $\left\langle \frac{a_0^1 + a_1^1}{2} a_0^2 \frac{a_0^3 + a_1^3}{2} \right\rangle$ and $\left\langle \frac{a_0^1 - a_1^1}{2} a_1^2 \frac{a_0^3 - a_1^3}{2} \right\rangle$ are computed by first evaluating all correlators $\langle a_{x^1}^1 a_{x^2}^2 a_{x^3}^3 \rangle$ and then using the linearity of the expectation value. The correlators can be easily computed using the trace:

$$\left\langle a_{x^1}^1 a_{x^2}^2 a_{x^3}^3 \right\rangle = \mathrm{Tr}\left\{ \rho_{\mathcal{N}_3} \left[a_{x^1}^1 \otimes a_{x^2}^2 \otimes a_{x^3}^3 \right] \right\} \tag{4.17}$$

If we use our standard choice of measurement operators from the previous section, we obtain figure 4.2 using a numerical computation. It shows a plot the left hand side of Bell inequality 4.6 for all values of v_1 and v_2. A green color indicates a violation.

Suppose that both sources create photons with the same indistinguishability v. Figure 4.2 shows that violation in that case occurs for $v > 0.42$. We also see that if one source produces fully indistinguishable photons, the indistinguishability of the other pair of photons should at least be $v = 0.18$.

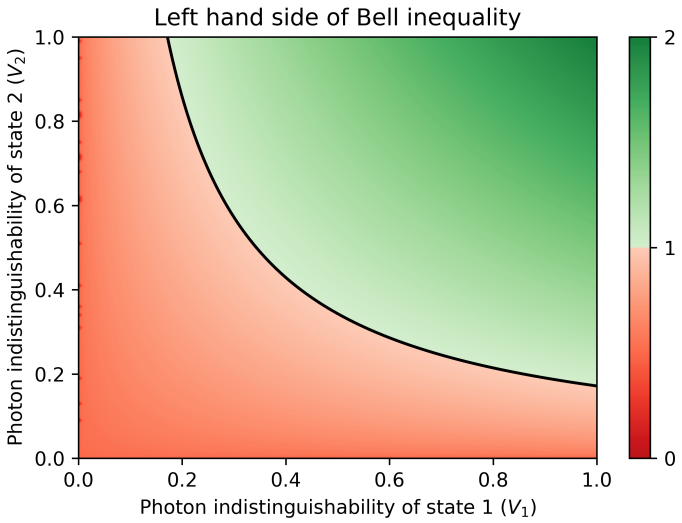


Figure 4.2: A plot showing the minimal value of the left hand side of inequality 4.6 for all possible values of v_1 and v_2. Violation of inequality 4.6 is indicated with a green color, while a red color indicates that there is no violation. The black line indicates the boundary between violation and no violation.

4.4. Modeling measurement errors

In this section we analyse the effects of measurement errors on the violation of our Bell-inequalities. We start by writing the state as a linear combination of eigenstates. For simplicity, we first consider a single spin state and a measurement along the z-axis. The general case is easily obtained by using the desired eigenstates. The quantum state can be written in terms of the eigenstates:

$$|\psi\rangle = c_\uparrow |\uparrow\rangle + c_\downarrow |\downarrow\rangle \tag{4.18}$$

The coefficients $c_\uparrow = \langle\uparrow|\psi\rangle$ and $c_\downarrow = \langle\downarrow|\psi\rangle$ determine the probabilities to measure the eigenvalue corresponding to that particular eigenstate. However, due to measurement errors, the probabilities $p_\uparrow$ and $p_\downarrow$ to measure respectively spin up and spin down are altered.

We will use a simple model for the measurement error from [2]. For a given eigenstate, the measurement gives the correct outcome with a certain probability. We will call this probability the fidelity of measuring a specific eigenstate correctly. These probabilities are denoted by $\mathcal{F}_\uparrow$ and $\mathcal{F}_\downarrow$, which we also call the fidelity for reading out $|\uparrow\rangle$ and $|\downarrow\rangle$ correctly.

Performing a measurement on the state $|\uparrow\rangle$ now yields the outcome 'spin up' with probability $\mathcal{F}_\uparrow$ and the outcome 'spin down' with probability $1 - \mathcal{F}_\uparrow$. Similarly, the outcome of a measurement on the state $|\downarrow\rangle$ is 'spin down' with probability $\mathcal{F}_\downarrow$ and 'spin up' with probability $1 - \mathcal{F}_\uparrow$. The probabilities according to this model are thus given by:

$$\begin{pmatrix} p_\uparrow \\ p_\downarrow \end{pmatrix} = E \begin{pmatrix} |c_\uparrow|^2 \\ |c_\downarrow|^2 \end{pmatrix} \tag{4.19}$$

where the matrix E is given by

$$E = \begin{bmatrix} \mathcal{F}_\uparrow & 1 - \mathcal{F}_\downarrow \\ 1 - \mathcal{F}_\uparrow & \mathcal{F}_\downarrow \end{bmatrix} \tag{4.20}$$

For perfect measurements we just have $\mathcal{F}_\uparrow = \mathcal{F}_\downarrow = 1$, resulting in the matrix $E = I$.

Let us now consider two spin measurements along the z-axis, one on a particle A and the other one on the second particle B. This can be described by measurement operator $\sigma_z \otimes \sigma_z$. The two-spin state of these particles can be written in terms of the eigenstates:

$$|\psi\rangle = c_{\uparrow\uparrow} |\uparrow\uparrow\rangle + c_{\uparrow\downarrow} |\uparrow\downarrow\rangle + c_{\downarrow\uparrow} |\downarrow\uparrow\rangle + c_{\downarrow\downarrow} |\downarrow\downarrow\rangle \tag{4.21}$$

Our measurement model is now given by

$$\begin{pmatrix} p_{\uparrow\uparrow} \\ p_{\uparrow\downarrow} \\ p_{\downarrow\uparrow} \\ p_{\downarrow\downarrow} \end{pmatrix} = E \begin{pmatrix} |c_{\uparrow\uparrow}|^2 \\ |c_{\uparrow\downarrow}|^2 \\ |c_{\downarrow\uparrow}|^2 \\ |c_{\downarrow\downarrow}|^2 \end{pmatrix} \tag{4.22}$$

The matrix E can be expressed in terms of the measurement fidelities $\mathcal{F}_{A,\uparrow}$, $\mathcal{F}_{A,\downarrow}$, $\mathcal{F}_{B,\uparrow}$ and $\mathcal{F}_{B,\downarrow}$. For example $\mathcal{F}_{A,\uparrow}$ is the probability that the outcome of a measurement on particle A in a state $|\downarrow\rangle$ is 'spin down'. We thus have

$$E = \begin{bmatrix} \mathcal{F}_{A,\uparrow}\mathcal{F}_{B,\uparrow} & \mathcal{F}_{A,\uparrow}(1-\mathcal{F}_{B,\downarrow}) & (1-\mathcal{F}_{A,\downarrow})\mathcal{F}_{B,\uparrow} & (1-\mathcal{F}_{A,\downarrow})(1-\mathcal{F}_{B,\downarrow}) \\ \mathcal{F}_{A,\uparrow}(1-\mathcal{F}_{B,\uparrow}) & \mathcal{F}_{A,\uparrow}\mathcal{F}_{B,\downarrow} & (1-\mathcal{F}_{A,\downarrow})(1-\mathcal{F}_{B,\uparrow}) & (1-\mathcal{F}_{A,\downarrow})\mathcal{F}_{B,\downarrow} \\ (1-\mathcal{F}_{A,\uparrow})\mathcal{F}_{B,\uparrow} & (1-\mathcal{F}_{A,\uparrow})(1-\mathcal{F}_{B,\downarrow}) & \mathcal{F}_{A,\downarrow}\mathcal{F}_{B,\uparrow} & \mathcal{F}_{A,\downarrow}(1-\mathcal{F}_{B,\downarrow}) \\ (1-\mathcal{F}_{A,\uparrow})(1-\mathcal{F}_{B,\uparrow}) & (1-\mathcal{F}_{A,\uparrow})\mathcal{F}_{B,\downarrow} & \mathcal{F}_{A,\downarrow}(1-\mathcal{F}_{B,\uparrow}) & \mathcal{F}_{A,\downarrow}\mathcal{F}_{B,\downarrow} \end{bmatrix} \tag{4.23}$$

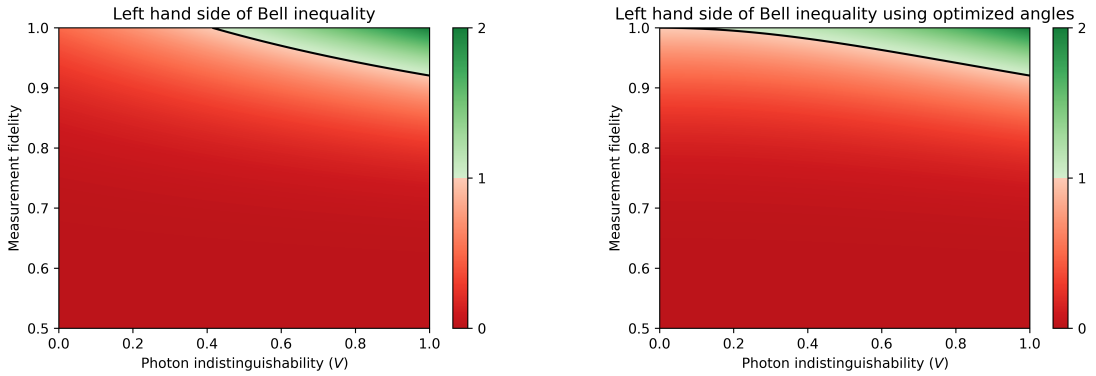
The matrix E can be written as $E = E_1 \otimes E_2$, with E_A and E_B being matrices that describe the measurement error of a single spin measurement. This model generalizes to an arbitrary number of particles, which becomes clear from applying definition 2.51 of the Kronecker product inductively.

This model can also be applied in the case of network $\mathcal{N}_3$. The total system consists of four particles, two coming from source $\mathcal{S}_1$ and two coming from source $\mathcal{S}_2$. Parties $\mathcal{A}^1$ and $\mathcal{A}^3$ both perform a single measurement on one particle. Party $\mathcal{A}^2$ performs two measurements, one on the particle coming from source $\mathcal{S}_1$ and one on that from source $\mathcal{S}_2$. All these measurements can be modelled with their own parameters $\mathcal{F}_\uparrow$ and $\mathcal{F}_\downarrow$.

Given a fixed set of measurement inputs/settings (x^1, x^2 and x^3) for all parties, the system can be described in terms of eigenstates of $\hat{a}^1_{x^1}$, $\hat{a}^2_{x^2}$ and $\hat{a}^3_{x^3}$. Since party $\mathcal{A}^2$'s measurement actually consists of two measurements, the eigenstate of $\hat{a}^2_{x^2}$ can be described as a tensor product of two eigenstates. Each eigenstate of the total system can thus be described by the tensor product of four eigenstates. So there are 2^4 eigenstates for a fixed set of measurement settings.

The probabilities that all measurement outcomes correspond to a specific eigenstate of the entire system can be computed using the trace (equation 2.49). Such an eigenstates has a fixed value ± 1 for the quantity $a^1_{x^1} a^2_{x^2} a^3_{x^3}$. To take measurement errors into account, the matrix E (constructed as above, but now with four particles) is used to 'adjust' the probabilities. The correlator $\langle a^1_{x^1} a^2_{x^2} a^3_{x^3} \rangle$ can then be numerically computed using the definition of an expectation value.

Figure 4.3a shows a plot of the left hand side of equality 4.6 when all parties have the same measurement fidelities (and also $\mathcal{F}_\uparrow = \mathcal{F}_\downarrow$). The state created by each source is again modeled using equation 4.14 (all with the same parameter v). As expected from the model without measurement errors, the figure shows that for perfect measurements ($\mathcal{F}_\uparrow = \mathcal{F}_\downarrow = 1$) the parameter v should be greater than 0.42 in order to have violation. A more interesting aspect from this plot is that the measurement fidelity must at least be above 0.93 (for fully indistinguishable photons).



(a) Using the standard measurement setup.

(b) Using an optimized measurement setup.

Figure 4.3: Plots showing the left hand side of inequality 4.6 for different photon (in)distinguishabilities and measurement fidelities. Violation of inequality 4.6 is indicated with a green color, while a red color indicates that there is no violation. The black line indicates the boundary between violation and no violation.

4.5. Optimizing measurement angles

Distinguishable photons cause stronger correlations in the z-direction. For this reason it can be beneficial to rotate the measurement angles of parties $\mathcal{A}_1$ and $\mathcal{A}_3$ (slightly) towards the z-direction [11]. The new measurement angles become: $\pi/4 - \epsilon$ for $\hat{a}^1_0$, $-\pi/4 + \epsilon$ for $\hat{a}^1_1$, 0 for $\hat{a}^2_0$ (both measurements), $\pi/2$ for $\hat{a}^2_1$ (both measurements), $\pi/4 - \epsilon$ for $\hat{a}^3_0$ and $-\pi/4 + \epsilon$ for $\hat{a}^3_1$. A numerical optimization yields the optimal angles shown in figure 4.4. Along the vertical axis the angle offset ϵ is plotted.

Apparently the angles should be altered significantly when the photons distinguishability becomes large (corresponding to low values of v). For $v = 0$ the optimal angles are even totally aligned in the z-direction. Figure 4.3b shows a plot of the left hand side of Bell inequality 4.6 for different degrees of measurement fidelities and photon indistinguishabilities, similar to figure 4.3a. However this time the optimal angles are used. We see that violation can even occur for very low photon indistinguishabilities, particularly for high measurement fidelities. Only for $v = 0$ no violation can occur, since then there is

no entanglement at all. Note that this would not be possible using the Werner state, because the measurement outcomes on two particles in a completely mixed state are uncorrelated and in particular do not show stronger correlations in a specific direction.

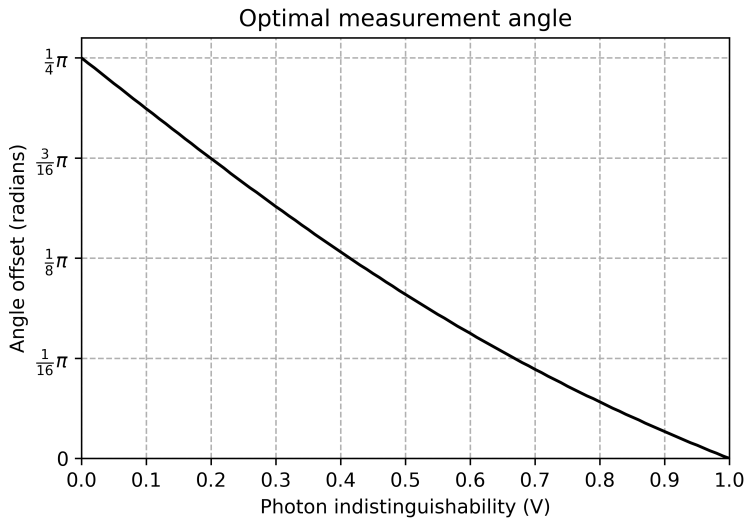


Figure 4.4: A plot showing the optimal measurement angle against photon distinguishability. The angles are numerically determined. The angle is expressed as the offset from the standard measurement setup, as described in the main text.

5

Conclusions and outlook

We have seen that the framework of Rosset et al. to find Bell inequalities is applicable to various network structures. For a general star-shaped network we have proven that nonlocal behaviour can be revealed using the Bell inequality obtained by this iterative procedure.
In chapter 4 we analysed how the setup used by Bernien et al. [2] can be used to demonstrate nonlocal behaviour in a network consisting of three parties and two sources. We found that a measurement setup similar to the one used in a standard Bell test can be used for this purpose and yields significant violations.
Also several practical limitations have been highlighted. Due to distinguishable photons the Bell inequality violation becomes less and in worse scenarios there is no violation at all. We have shown that optimizing the measurement angles yield better results. Especially for low photon indistinguishabilities a change of measurement angles is important and can make the difference between violation and no violation. Anoter interesting result is that the measurement fidelity must at least be above 0.93 (for fully indistinguishable photons) to detect nonlocal behaviour in this network.

5.1. Further research

It would be interesting to create Bell inequalities for networks featuring loops. The method used in this report does not allow creating a cycle in the new network. Branciard notes in [4] that deriving a Bell inequality for a simple loop where three parties receive states from three sources is already a challenging problem. In this light it is also worth mentioning that a special class of ring networks is treated by Frey [10].

Another interesting question concerns the optimality of our Bell inequalities and their violations. We have shown that nonlocality can be demonstrated using classes of quantum states and the right measurement setups, but there might be different Bell inequalities or different measurement choices leading to more significant violations. This can be of importance in a practical application, since greater violations are more easily shown due to the statistical character of these type of Bell inequalities.

Bibliography

[1] John S Bell. On the einstein podolsky rosen paradox. In *John S Bell On The Foundations Of Quantum Mechanics*, pages 7–12. World Scientific, 2001.

[2] Hannes Bernien, Bas Hensen, Wolfgang Pfaff, Gerwin Koolstra, MS Blok, Lucio Robledo, TH Taminiau, Matthew Markham, DJ Twitchen, Lilian Childress, et al. Heralded entanglement between solid-state qubits separated by three metres. *Nature*, 497(7447):86, 2013.

[3] Jacob Biamonte, Mauro Faccin, and Manlio De Domenico. Complex networks: from classical to quantum. *arXiv preprint arXiv:1702.08459*, 2017.

[4] Cyril Branciard, Denis Rosset, Nicolas Gisin, and Stefano Pironio. Bilocal versus nonbilocal correlations in entanglement-swapping experiments. *Physical Review A*, 85(3):032119, 2012.

[5] Nicolas Brunner, Daniel Cavalcanti, Stefano Pironio, Valerio Scarani, and Stephanie Wehner. Bell nonlocality. *Reviews of Modern Physics*, 86(2):419, 2014.

[6] John F Clauser, Michael A Horne, Abner Shimony, and Richard A Holt. Proposed experiment to test local hidden-variable theories. *Physical review letters*, 23(15):880, 1969.

[7] BIG Bell Test Collaboration et al. Challenging local realism with human choices. *arXiv preprint arXiv:1805.04431*, 2018.

[8] Albert Einstein, Boris Podolsky, and Nathan Rosen. Can quantum-mechanical description of physical reality be considered complete? *Physical review*, 47(10):777, 1935.

[9] Artur K Ekert. Quantum cryptography based on bell's theorem. *Physical review letters*, 67(6):661, 1991.

[10] Michael Frey. A bell inequality for a class of multilocal ring networks. *Quantum Information Processing*, 16(11):266, 2017.

[11] Bas Hensen, Hannes Bernien, Anaïs E Dréau, Andreas Reiserer, Norbert Kalb, Machiel S Blok, Just Ruitenberg, Raymond FL Vermeulen, Raymond N Schouten, Carlos Abellán, et al. Loophole-free bell inequality violation using electron spins separated by 1.3 kilometres. *Nature*, 526(7575): 682–686, 2015.

[12] Michael A Nielsen and Isaac Chuang. Quantum computation and quantum information, 2002.

[13] Wolfgang Pfaff, BJ Hensen, Hannes Bernien, Suzanne B van Dam, Machiel S Blok, Tim H Taminiau, Marijn J Tiggelman, Raymond N Schouten, Matthew Markham, Daniel J Twitchen, et al. Unconditional quantum teleportation between distant solid-state quantum bits. *Science*, 345(6196):532–535, 2014.

[14] Denis Rosset, Cyril Branciard, Tomer Jack Barnea, Gilles Pütz, Nicolas Brunner, and Nicolas Gisin. Nonlinear bell inequalities tailored for quantum networks. *Physical review letters*, 116(1): 010403, 2016.

6

Appendix

This code has been used for verifying the inequalities and optimizing the measurement angle:

```
import math
import numpy as np
import matplotlib.pyplot as plt
import matplotlib.colors as mcolors
import xlsxwriter

state_0 = np.matrix([
    [1],
    [0]
])

state_1 = np.matrix([
    [0],
    [1]
])

sigma_x = np.matrix([
    [0, 1],
    [1, 0]
])

sigma_z = np.matrix([
    [1, 0],
    [0, -1]
])

def spinmeasurement_op(angle):
    operator = math.cos(angle) * sigma_z + math.sin(angle) * sigma_x
    return operator

def spinmeasurement_ev(angle):
    eigenvectors = np.matrix([[math.cos(angle/2.), -math.sin(angle/2.)],[math.sin(angle/2.),
                                                    math.cos(angle/2.)]])
    eigenvalues = np.array([1,-1])
    return eigenvalues, eigenvectors

def evs_totalsystem(*angles):
    # COMPUTES EIGENVALUES AND EIGENVECTORS OF TOTAL SYSTEM
    eigenvalues = np.array([1])
    eigenvectors = np.array([1])
    for angle in angles:
        evalue, evector = spinmeasurement_ev(angle)
        eigenvalues = np.kron(eigenvalues, evalue)
        eigenvectors = np.kron(eigenvectors, evector)
    return eigenvalues, eigenvectors

def eigenstate_coefs(state, eigenvectors):
```

```
    # COMPUTES THE IDEAL PROBABILITIES OF FINDING THE STATE IN A CERTAIN EIGENSTATE
    coefs = np.array([])
    for eigenstate in eigenvectors.T:
        projector = np.matmul(eigenstate.T,eigenstate)
        fraction = np.trace(np.matmul(projector,state))
        coefs = np.append(coefs, fraction)
    return coefs

def error_matrix(*coefs):
    # COMPUTES THE ERROR MATRIX CORRESPONDING TO THE SPECIFIED CORRECT-MEASUREMENT-
                                                    PROBABILITIES
    # coefficients should be formatted as follows:
    # np.array([ pA(ev_1 correct), pA(ev_2 correct) ]), np.array([ pB(ev_1 correct), pB(ev_2
                                                    correct) ]), ...
    errormatrix = np.array([1])
    for p in coefs:
        errormatrix = np.kron(errormatrix, np.array([[p[0], 1-p[1]],[1-p[0], p[1]]]))
    return errormatrix

def generate_state(v1,v2):
    # GENERATES THE DENSITY MATRIX ACCORDING TO THE MODEL WITH PARAMETERS V1 AND V2
    psi = 1/math.sqrt(2)*(np.kron(state_0, state_1) - np.kron(state_1, state_0))
    state_error = 1./2*(np.matmul(np.kron(state_0, state_1),np.kron(state_0, state_1).T)+np.
                                                    matmul(np.kron(state_1, state_0),np.kron(
                                                    state_1, state_0).T))
    state = np.kron(v1 * np.matmul(psi, psi.T) + (1-v1) * state_error, v2 * np.matmul(psi,
                                                    psi.T) + (1-v2) * state_error)
    return state

def expv(state,errormatrix,angles,x1,x2,x3):
    # RETURNS THE EXPECTATION VALUE <a1*a2*a3>
    [theta10,theta11,theta20,theta21,theta30,theta31] = angles
    a1_angle = x1 * theta11 + (1 - x1) * theta10
    a2_angle = x2 * theta21 + (1 - x2) * theta20
    a3_angle = x3 * theta31 + (1 - x3) * theta30
    outcomes, eigenstates = evs_totalsystem(a1_angle, a2_angle, a2_angle, a3_angle)
    probs_ideal = eigenstate_coefs(state, eigenstates)
    probs_corr = np.dot(errormatrix,probs_ideal)
    exp_value = sum(outcomes*probs_corr)
    return exp_value

def expv_trace(state,angles,x1,x2,x3):
    # RETURNS THE EXPECTATION VALUE <a1*a2*a3>
    [theta10,theta11,theta20,theta21,theta30,theta31] = angles
    a1_angle = x1 * theta11 + (1 - x1) * theta10
    a2_angle = x2 * theta21 + (1 - x2) * theta20
    a3_angle = x3 * theta31 + (1 - x3) * theta30
    a1 = spinmeasurement_op(a1_angle)
    a2 = spinmeasurement_op(a2_angle)
    a3 = spinmeasurement_op(a3_angle)
    exp_value = np.trace(np.matmul(state,np.kron(np.kron(np.kron(a1,a2),a2),a3)))
    return exp_value

def verify_inequality(A,B):
    # what if A is just smaller than zero due to an error
    if A >= 0 and B>=0:
        return 2*math.sqrt(A*B)+A+B
    else:
        return 0

# INITIALIZING ARRAYS AND COLORMAP
Vlist = np.linspace(0,1,101)
MERRlist = np.linspace(0.5,1,101)
epslist = np.linspace(0,math.pi/4,1000)

colors1 = plt.cm.Reds_r(np.linspace(0.2, 0.8, 128))
colors2 = plt.cm.Greens(np.linspace(0.2, 0.8, 128))
colors = np.vstack((colors1, colors2))
mymap = mcolors.LinearSegmentedColormap.from_list('my_colormap', colors)
levels=np.linspace(0,2,1001)
```

```
# CREATE A PLOT FOR THE OPTIMAL ANGLE
validity_table = np.zeros(len(epslist))
optimal_eps = np.zeros(len(Vlist))

for v_i, v in enumerate(Vlist):
    rho = generate_state(v,v)

    for eps_i, eps in enumerate(epslist):
        angles = [math.pi/4-eps,-math.pi/4+eps,0,math.pi/2,math.pi/4-eps,-math.pi/4+eps]

        A = 1./4 * (expv_trace(rho,angles,0,0,0) + expv_trace(rho,angles,0,0,1) +
                    expv_trace(rho,angles,1,0,0) + expv_trace(rho,angles,1,0,1))
        B = 1./4 * (expv_trace(rho,angles,0,1,0) - expv_trace(rho,angles,0,1,1) -
                    expv_trace(rho,angles,1,1,0) + expv_trace(rho,angles,1,1,1))

        validity_table[eps_i] = verify_inequality(A,B)

    optimal_eps[v_i] = epslist[validity_table.argmax()]

plt.title('Optimal measurement angle')
plt.xlabel('Photon indistinguishability (V)')
plt.ylabel('Angle offset (radians)')
plt.grid(True,linestyle='dashed')
plt.plot(Vlist, optimal_eps, color='black')
plt.gca().set_xlim([0,1])
plt.gca().set_ylim(bottom=0)
plt.xticks(np.arange(0, 1.1, 0.1))
plt.yticks([0.0, 0.0625*np.pi, 0.125*np.pi, 0.1875*np.pi, 0.25*np.pi], ["$0$", r"$\frac{1}{16
                                                }\pi$", r"$\frac{1}{8}\pi$", r"$\frac{3}{16}\
                                                pi$", r"$\frac{1}{4}\pi$"])
plt.savefig('images/angle-optimization.png',dpi=600)
plt.show()

workbook = xlsxwriter.Workbook('images/angle-optimization.xlsx')
worksheet = workbook.add_worksheet()
array = np.append([Vlist], [optimal_eps], axis=0)
for col, data in enumerate(array):
    worksheet.write(0,0,"V")
    worksheet.write(0,1,"Angle")
    worksheet.write_column(1, col, data)
workbook.close()

# CREATE A PLOT OF THE VIOLATION FOR DIFFERENT STATES
validity_table = np.zeros((len(Vlist), len(Vlist)))
angles = [math.pi/4,-math.pi/4,0,math.pi/2,math.pi/4,-math.pi/4]

for v1_i, v1 in enumerate(Vlist):
    for v2_i, v2 in enumerate(Vlist):
        rho = generate_state(v1,v2)

        A = 1./4 * (expv_trace(rho,angles,0,0,0) + expv_trace(rho,angles,0,0,1) +
                    expv_trace(rho,angles,1,0,0) + expv_trace(rho,angles,1,0,1))
        B = 1./4 * (expv_trace(rho,angles,0,1,0) - expv_trace(rho,angles,0,1,1) -
                    expv_trace(rho,angles,1,1,0) + expv_trace(rho,angles,1,1,1))

        validity_table[v2_i, v1_i] = verify_inequality(A,B)

boundary_value=1.0
contour = plt.contour(Vlist, Vlist, validity_table, [boundary_value], colors='black')

contour_filled = plt.contourf(Vlist, Vlist, validity_table, levels, cmap=mymap)
plt.colorbar(contour_filled, ticks=[0, boundary_value, 2])
plt.title('Left hand side of Bell inequality')
plt.xlabel(r'Photon indistinguishability of state 1 ($V_1$)')
plt.ylabel(r'Photon indistinguishability of state 2 ($V_2$)')
plt.savefig('images/photondistinguishability.png',dpi=600)
plt.show()

workbook = xlsxwriter.Workbook('images/photondistinguishability.xlsx')
worksheet = workbook.add_worksheet()
array = validity_table
```

```
for row, data in enumerate(array):
    worksheet.write_row(0,1,Vlist)
    worksheet.write_column(1,0,Vlist)
    worksheet.write_row(row+1, 1, data)
workbook.close()

# CREATE A PLOT OF THE VIOLATION FOR STATES AND MEASUREMENT ERRORS
validity_table = np.zeros((len(MERRlist), len(Vlist)))
angles = [math.pi/4,-math.pi/4,0,math.pi/2,math.pi/4,-math.pi/4]

for v_i, v in enumerate(Vlist):
    for merr_i, merr in enumerate(MERRlist):
        rho = generate_state(v,v)

        errormatrix = error_matrix(np.array([merr, merr]), np.array([merr, merr]), np.array([
                                                    merr, merr]), np.array([merr, merr]))

        A = 1./4 * (expv(rho,errormatrix,angles,0,0,0) + expv(rho,errormatrix,angles,0,0,1) +
                    expv(rho,errormatrix,angles,1,0,0) + expv(rho,errormatrix,angles,1,0,1))
        B = 1./4 * (expv(rho,errormatrix,angles,0,1,0) - expv(rho,errormatrix,angles,0,1,1) -
                    expv(rho,errormatrix,angles,1,1,0) + expv(rho,errormatrix,angles,1,1,1))

        validity_table[merr_i, v_i] = verify_inequality(A,B)

boundary_value=1.0
contour = plt.contour(Vlist, MERRlist, validity_table, [boundary_value], colors='black')

contour_filled = plt.contourf(Vlist, MERRlist, validity_table, levels, cmap=mymap)
plt.colorbar(contour_filled, ticks=[0, boundary_value, 2])
plt.title('Left hand side of Bell inequality')
plt.xlabel(r'Photon indistinguishability ($V$)')
plt.ylabel('Measurement fidelity')
plt.savefig('images/measurementerror-photondistinguishability.png',dpi=600)
plt.show()

workbook = xlsxwriter.Workbook('images/measurementerror-photondistinguishability.xlsx')
worksheet = workbook.add_worksheet()
array = validity_table
for row, data in enumerate(array):
    worksheet.write_row(0,1,Vlist)
    worksheet.write_column(1,0,MERRlist)
    worksheet.write_row(row+1, 1, data)
workbook.close()

# CREATE A PLOT OF THE VIOLATION USING OPTIMIZED ANGLES
validity_table = np.zeros((len(MERRlist), len(Vlist)))

for v_i, v in enumerate(Vlist):
    eps = optimal_eps[v_i]
    angles = [math.pi/4-eps,-math.pi/4+eps,0,math.pi/2,math.pi/4-eps,-math.pi/4+eps]

    for merr_i, merr in enumerate(MERRlist):
        rho = generate_state(v,v)
        errormatrix = error_matrix(np.array([merr, merr]), np.array([merr, merr]), np.array([
                                                    merr, merr]), np.array([merr, merr]))

        A = 1./4 * (expv(rho,errormatrix,angles,0,0,0) + expv(rho,errormatrix,angles,0,0,1) +
                    expv(rho,errormatrix,angles,1,0,0) + expv(rho,errormatrix,angles,1,0,1))
        B = 1./4 * (expv(rho,errormatrix,angles,0,1,0) - expv(rho,errormatrix,angles,0,1,1) -
                    expv(rho,errormatrix,angles,1,1,0) + expv(rho,errormatrix,angles,1,1,1))

        validity_table[merr_i, v_i] = verify_inequality(A,B)

boundary_value=1.0
contour = plt.contour(Vlist, MERRlist, validity_table, [boundary_value], colors='black')

contour_filled = plt.contourf(Vlist, MERRlist, validity_table, levels, cmap=mymap)
plt.colorbar(contour_filled, ticks=[0, boundary_value, 2])
plt.title('Left hand side of Bell inequality using optimized angles')
plt.xlabel(r'Photon indistinguishability ($V$)')
plt.ylabel('Measurement fidelity')
```

```
plt.savefig('images/measurementerror-photondistinguishability-optimal.png',dpi=600)
plt.show()

workbook = xlsxwriter.Workbook('images/measurementerror-photondistinguishability-optimal.xlsx
                                                        ')
worksheet = workbook.add_worksheet()
array = validity_table
for row, data in enumerate(array):
    worksheet.write_row(0,1,Vlist)
    worksheet.write_column(1,0,MERRlist)
    worksheet.write_row(row+1, 1, data)
workbook.close()
```