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L^p -Analysis of the Hodge–Dirac Operator Associated with Witten Laplacians on Complete Riemannian Manifolds

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Abstract We prove R -bisectoriality and boundedness of the H^∞ -functional calculus in L^p for all $1 < p < \infty$ for the Hodge–Dirac operator associated with Witten Laplacians on complete Riemannian manifolds with non-negative Bakry–Emery Ricci curvature on k -forms.

Keywords Witten Laplacian · Hodge–Dirac operator · R -bisectoriality · H^∞ -functional calculus · Bakry–Emery Ricci curvature

Mathematics Subject Classification Primary 47A60 · Secondary 58A10 , 58J35 , 58J60

1 Introduction

The Witten Laplacian was introduced by Witten [55] as a deformation of the Hodge Laplacian on a complete Riemannian manifold M and has been subsequently studied by many authors; see [9, 13, 15, 23, 26, 29, 30, 44–46, 56] and the references cited therein. The *Witten Laplacian* associated with a smooth strictly positive function $\rho : M \rightarrow \mathbb{R}$ is the operator

$$L_\rho : f \mapsto \Delta f - \nabla \log \rho \cdot \nabla f, \quad f \in C_c^\infty(M),$$

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where $\Delta = \nabla^* \nabla$ is the (negative) Laplace-Beltrami operator and ∇ is the gradient. Identifying functions with 0-forms, we have

$$L_\rho f = (d_\rho d_\rho^* + d_\rho^* d_\rho) f, \quad f \in C_c(M), \tag{1.1}$$

where d_ρ is the L^2 -realisation of the exterior derivative d with respect to the measure $m(dx) = \rho(x) dx$ on M , and d_ρ^* is the adjoint operator. The representation (1.1) can be used to define the Witten Laplacian for k -forms for $k \neq 0$. In the special case $M = \mathbb{R}^n$ and $\rho(x) = \exp(-\frac{1}{2}|x|^2)$, L_ρ corresponds to the Ornstein-Uhlenbeck operator.

Let $m(dx) = \rho(x) dx$ be the weighted volume measure on M . Generalising the celebrated Meyer inequalities for the Ornstein-Uhlenbeck operator, Bakry [9] proved boundedness of the Riesz transform $\nabla L_\rho^{-1/2}$ on $L^p(M, m)$ for all $1 < p < \infty$ under a curvature condition on M . An extension of this result to the corresponding L^p -spaces of k -forms is contained in the same paper. These results have been subsequently extended into various directions. As a sample of the extensive literature on this topic, we mention [15,44–46,56] (for the Witten Laplacian); see also [3,4,10,19,37,42,47,49,52,54] (for the Laplace-Beltrami operator), [17,31,51] (for the Hodge-de Rham Laplacian), and [11] (for sub-elliptic operators).

The aim of the present paper is to develop Bakry’s result along a different line by analysing the Hodge–Dirac operator

$$D_\rho = d_\rho + d_\rho^*$$

from the point of view of its functional calculus properties. Our main result can be stated as follows (the relevant definitions are given in the main body of the paper).

Theorem 1.1 *If M has non-negative Bakry–Emery Ricci curvature on k -forms for all $1 \leq k \leq n$, then the Hodge–Dirac operator D_ρ is R -bisectorial and admits a bounded H^∞ -calculus in $L^p(\Delta TM, m)$ for all $1 < p < \infty$.*

By standard arguments (cf. [8]), the boundedness of the H^∞ -calculus of D_ρ implies (by considering the operator $\text{sgn}(D_\rho)$, which is then well defined through the functional calculus) the boundedness of the Riesz transform $D_\rho L_\rho^{-1/2} = \text{sgn}(D_\rho)$. As such our results may be thought of as a strengthening of those in [9].

In the unweighted case $\rho \equiv 1$, the second assertion of Theorem 1.1 is essentially known, although we are not aware of a place where it is formulated explicitly or in some equivalent form. It can be pieced together from known results as follows: Firstly, [6, Theorem 5.12] asserts that the unweighted Hodge–Dirac operator D has a bounded H^∞ -calculus on the Hardy space $H^p(\Delta TM)$, even for $1 \leq p \leq \infty$, provided the volume measure has the so-called doubling property. By the Bishop comparison theorem (see [12]), this property is always satisfied if M has non-negative Ricci curvature. Secondly, for $1 < p < \infty$, this Hardy space is subsequently identified in [6, Theorem 8.5] to be the closure in $L^p(\Delta TM)$ of the range of D , provided the heat kernel associated with L satisfies Gaussian bounds on k -forms for all $0 \leq k \leq n$. When M has non-negative Ricci curvature, such bounds were proved in [43] for 0-forms, i.e. for functions on M . The bounds for k -forms then follow, under the curvature assumptions in the present paper, via pointwise domination of the heat kernel on k -forms by

the heat kernel for 0-forms (cf. (3.7) below). Modulo the kernel-range decomposition $L^p(\Lambda TM, m) = \mathbf{N}(D) \oplus \overline{\mathbf{R}}(D)$ (which follows from R -bisectorially proved in the present paper, but could also be established on the basis of other known results), this gives the boundedness of the H^∞ -calculus in $L^p(\Lambda TM, m)$ in the unweighted case.

In the weighted case, this approach cannot be pursued due to the absence of the doubling property and Gaussian bounds. Instead, our approach exploits the fact, proved in [56], that the non-negativity of the Bakry–Emery Ricci curvature implies, among other things, square function estimates on k -forms.

The analogue of Theorem 1.1 for the Hodge–Dirac operator associated with the Ornstein-Uhlenbeck operator has been established, in a more general formulation, in [48]. The related problem of the L^p -boundedness of the H^∞ -calculus of Hodge–Dirac operators associated with the Kato square root problem was initiated by the influential paper [8] and has been studied by many authors [7, 24, 32–34, 51].

The organisation of the paper is as follows: After a brief introduction to R -(bi)sectorial operators and H^∞ -calculi in Sect. 2, we introduce the Witten Laplacian L_ρ in Sect. 3 and recall some of its properties. Among others we prove that it is R -sectorial of angle less than $\frac{1}{2}\pi$ and admits a bounded H^∞ -calculus in L^p for $1 < p < \infty$. In Sect. 4 this result, together with the identity $D_\rho^2 = L_\rho$, is used to prove the corresponding assertions for the Hodge–Dirac operator D_ρ .

On some occasions, we will use the notation $a \lesssim b$ to signify that there exists a constant C such that $a \leq Cb$. To emphasise the dependence of C on parameters p_1, p_2, \dots , we shall write $a \lesssim_{p_1, p_2, \dots} b$. Finally, we write \asymp (respectively, $\asymp_{p_1, p_2, \dots}$) if both $a \lesssim b$ and $b \lesssim a$ (respectively, $a \lesssim_{p_1, p_2, \dots} b$ and $b \lesssim_{p_1, p_2, \dots} a$) hold.

2 R -(Bi)sectorial Operators and the H^∞ -functional Calculus

In this section, we present a brief overview of the various notions from operator theory used in this paper.

2.1 R -boundedness

Let X and Y be Banach spaces and let $(r_j)_{j \geq 1}$ be a sequence of independent *Rademacher variables* defined on a probability space (Ω, \mathbb{P}) , i.e. $\mathbb{P}(r_j = 1) = \mathbb{P}(r_j = -1) = \frac{1}{2}$ for each j .

A collection of bounded linear operators $\mathcal{T} \subseteq \mathcal{L}(X, Y)$ is said to be R -bounded if there exists a $C \geq 0$ such that for all $M = 1, 2, \dots$ and all choices of $x_1, \dots, x_M \in X$ and $T_1, \dots, T_M \in \mathcal{T}$ we have

$$\mathbb{E} \left\| \sum_{m=1}^M r_m T_m x_m \right\|^2 \leq C^2 \mathbb{E} \left\| \sum_{m=1}^M r_m x_m \right\|^2,$$

where \mathbb{E} denotes the expectation with respect to \mathbb{P} . By considering the case $M = 1$, one sees that every R -bounded family of operators is uniformly bounded. In Hilbert

spaces the converse holds, as is easy to see by expanding the square of the norm as an inner product and using that $\mathbb{E}r_m r_n = \delta_{mn}$.

Motivated by certain square function estimates in harmonic analysis, the theory of R -boundedness was initiated in [18] and has found widespread use in various areas of analysis, among them parabolic PDE, harmonic analysis and stochastic analysis. We refer the reader to [21,35,36,40] for detailed accounts.

2.2 Sectorial Operators

For $\sigma \in (0, \pi)$, we consider the open sector

$$\Sigma_\sigma^+ := \{z \in \mathbb{C} : z \neq 0, |\arg z| < \sigma\}.$$

A closed densely defined operator $(A, D(A))$ acting in a complex Banach space X is said to be *sectorial* of angle $\sigma \in (0, \pi)$ if $\sigma(A) \subseteq \overline{\Sigma_\sigma^+}$ and the set $\{\lambda(\lambda - A)^{-1} : \lambda \notin \overline{\Sigma_\vartheta^+}\}$ is bounded for all $\vartheta \in (\sigma, \pi)$. The least angle of sectoriality is denoted by $\omega^+(A)$. If A is sectorial of angle $\sigma \in (0, \pi)$ and the set $\{\lambda(\lambda - A)^{-1} : \lambda \notin \overline{\Sigma_\vartheta^+}\}$ is R -bounded for all $\vartheta \in (\sigma, \pi)$, then A is said to be *R -sectorial* of angle σ . The least angle of R -sectoriality is denoted by $\omega_R^+(A)$.

Remark 2.1 We wish to point out that most authors (including [21,36,40]) impose the additional requirements that A be injective and have dense range. In the setting considered here, this would be inconvenient: already in the special case of the Ornstein-Uhlenbeck operator, the kernel is non-empty. It is worth noting, however, (see [28, Proposition 2.1.1(h)]) that a sectorial operator A on a reflexive Banach space X induces a direct sum decomposition

$$X = N(A) \oplus \overline{R(A)}.$$

The part of A in $\overline{R(A)}$ is sectorial and injective and has dense range. Thus, A decomposes into a trivial part and a part that is sectorial in the more restrictive sense of [21,36,40]. Since we will be working with L^p -spaces in the reflexive range $1 < p < \infty$ the results of [21,36,40] can be applied along this decomposition.

The typical example of a sectorial operator is the realisation of the Laplace operator Δ in $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, and this operator is R -sectorial if $1 < p < \infty$. More general examples, including the Laplace-Beltrami operator, are discussed in [21,36,40].

2.3 Bisectorial Operators

The theory of sectorial operators has a bisectorial counterpart. We refer the reader to [1,5,22] for more information. For $0 < \sigma < \frac{1}{2}\pi$, we set $\Sigma_\sigma^- := -\Sigma_\sigma^+$ and

$$\Sigma_\sigma^\pm := \Sigma_\sigma^+ \cup \Sigma_\sigma^-.$$

The set Σ_σ^\pm is called the *bisector* of angle σ . A closed densely defined linear operator $(A, \overline{D(A)})$ acting in a complex Banach space X is called *bisectorial* of angle σ if $\sigma(A) \subseteq \overline{\Sigma_\sigma^\pm}$ and the set $\{\lambda(\lambda - A)^{-1} : \lambda \notin \overline{\Sigma_\vartheta^\pm}\}$ is bounded for all $\vartheta \in (\sigma, \frac{1}{2}\pi)$. The least angle of bisectoriality is denoted by $\omega^\pm(A)$. If A is bisectorial and the set $\{\lambda(\lambda - A)^{-1} : \lambda \notin \overline{\Sigma_\vartheta^\pm}\}$ is R -bounded for all $\vartheta \in (\sigma, \frac{1}{2}\pi)$, then A is said to be *R -bisectorial* of angle $\sigma \in (0, \frac{1}{2}\pi)$. The least angle of R -bisectoriality is denoted by $\omega_R^\pm(A)$.

Remark 2.2 If A is bisectorial (of angle ϑ), then iA is sectorial (of angle $\frac{1}{2}\pi + \vartheta$), and therefore Remark 2.1 applies to bisectorial operators as well.

Typical examples of bisectorial operators are $\pm i d/dx$ in $L^p(\mathbb{R})$ and the Hodge–Dirac operator $\begin{pmatrix} 0 & \nabla^* \\ \nabla & 0 \end{pmatrix}$ on $L^p(\mathbb{R}^n) \oplus L^p(\mathbb{R}^n; \mathbb{C}^n)$, $1 \leq p < \infty$. These operators are R -bisectorial if $1 < p < \infty$.

2.4 The H^∞ -Functional Calculus

In a Hilbert space setting, the H^∞ -functional calculus was introduced in [50]. It was extended to the more general setting of Banach spaces in [20]. For detailed treatments, we refer the reader to [21, 28, 36, 40].

Let $H^\infty(\Sigma_\sigma^+)$ be the space of all bounded holomorphic functions on Σ_σ^+ , and let $H^1(\Sigma_\sigma^+)$ denote the space of all holomorphic functions $\psi : \Sigma_\sigma^+ \rightarrow \mathbb{C}$ satisfying

$$\sup_{|\nu| < \sigma} \int_0^\infty |\psi(e^{i\nu}t)| \frac{dt}{t} < \infty.$$

If A is a sectorial operator and ψ is a function in $H^1(\Sigma_\sigma^+)$ with $0 < \omega^+(A) < \sigma < \pi$, we may define the bounded operator $\psi(A)$ on X by the Dunford integral

$$\psi(A)x := \frac{1}{2\pi i} \int_{\partial\Sigma_\nu^+} \psi(z)(z - A)^{-1}x \, dz, \quad x \in X,$$

where $\omega^+(A) < \nu < \sigma$ and $\partial\Sigma_\nu^+$ is parametrised counter-clockwise. By Cauchy’s theorem, this definition does not depend on the choice of ν .

A sectorial operator A on X is said to admit a *bounded $H^\infty(\Sigma_\sigma^+)$ -functional calculus*, or a *bounded H^∞ -calculus of angle σ* , if there exists a constant $C_\sigma \geq 0$ such that for all $\psi \in H^1(\Sigma_\sigma^+) \cap H^\infty(\Sigma_\sigma^+)$ and all $x \in X$, we have

$$\|\psi(A)x\| \leq C_\sigma \|\psi\|_\infty \|x\|,$$

where $\|\psi\|_\infty = \sup_{z \in \Sigma_\sigma^+} |\psi(z)|$. The infimum of all angles σ for which such a constant C exists is denoted by $\omega_{H^\infty}^+(A)$. We say that a sectorial operator A admits a *bounded H^∞ -calculus* if it admits a bounded $H^\infty(\Sigma_\sigma^+)$ -calculus for some $0 < \sigma < \pi$.

Typical examples of operators having a bounded H^∞ -calculus include the sectorial operators mentioned in Sect. 2.2. In fact, it requires quite some effort to construct sectorial operators without a bounded H^∞ -calculus, and to this date only rather artificial constructions of such examples are known.

Replacing the role of sectors by bisectors, the above definitions can be repeated for bisectorial operators. The examples of bisectorial operators mentioned in Sect. 2.3 have a bounded H^∞ -calculus.

2.5 R -(bi)sectorial Operators and Bounded H^∞ -functional Calculi

The following result is a straightforward generalisation of [5, Proposition 8.1] and [1, Sect. H] (see [36, Chapter 10] for the present formulation):

Proposition 2.3 *Suppose that A is an R -bisectorial operator on a Banach space of finite cotype. Then A^2 is R -sectorial, and for each $\omega \in (0, \frac{1}{2}\pi)$ the following assertions are equivalent:*

- (1) A admits a bounded $H^\infty(\Sigma_\omega^\pm)$ -calculus;
- (2) A^2 admits a bounded $H^\infty(\Sigma_{2\omega}^+)$ -calculus.

3 The Witten Laplacian

Let us begin by introducing some standard notations from differential geometry. For unexplained terminology, we refer to [27, 41].

Throughout this paper, we work on a complete Riemannian manifold (M, g) of dimension n . The exterior algebra over the tangent bundle TM is denoted by

$$\Lambda TM := \bigoplus_{k=0}^n \Lambda^k TM.$$

Smooth sections of $\Lambda^k TM$ are referred to as k -forms. We set

$$C_c^\infty(\Lambda TM) := \bigoplus_{k=0}^n C_c^\infty(\Lambda^k TM),$$

where $C_c^\infty(\Lambda^k TM)$ denotes the vector space of smooth, compactly supported k -forms. The inner product of two k -forms $dx^{i_1} \wedge \dots \wedge dx^{i_k}$ and $dx^{j_1} \wedge \dots \wedge dx^{j_k}$ is defined, in a coordinate chart (U, x) , as

$$(dx^{i_1} \wedge \dots \wedge dx^{i_k}) \cdot (dx^{j_1} \wedge \dots \wedge dx^{j_k}) := \det(g^{i_r j_s})_{r,s},$$

where (g^{ij}) is the inverse of the matrix (g_{ij}) representing g in the chart (U, x) . This definition extends to general k -forms by linearity. For smooth sections ω, η of ΛTM , say $\omega = \sum_{k=0}^n \omega^k$ and $\eta = \sum_{k=0}^n \eta^k$, we define

$$\omega \cdot \eta := \sum_{k=0}^n \omega^k \cdot \eta^k,$$

and we write $|\omega| := (\omega \cdot \omega)^{1/2}$.

We now fix a strictly positive function $\rho \in C^\infty(M)$ and consider the measure

$$m(dx) := \rho(x) dx$$

on M , where dx is the volume measure. For $1 \leq p < \infty$, we define $L^p(\Lambda^k TM, m)$ to be the Banach space of all measurable k -forms for which the norm

$$\|\omega\|_p := \left(\int_M |\omega|^p dm \right)^{1/p}$$

is finite, identifying two such forms when they agree m -almost everywhere on M . Equivalently, we could define this space as the completion of $C_c^\infty(\Lambda^k TM)$ with respect to the norm $\|\cdot\|_p$. Finally, we define

$$L^p(\Lambda TM, m) := \bigoplus_{k=0}^n L^p(\Lambda^k TM, m)$$

and endow this space with the norm $\|\cdot\|_p$ defined by $\|\omega\|_p = \sum_{k=0}^n \|\omega^k\|_p^p$, where $\omega = \sum_{k=0}^n \omega^k$ for k -forms ω^k . In the case of $p = 2$, we will denote the $L^2(\Lambda^k TM, m)$ inner product of two k -forms $\omega, \eta \in L^2(\Lambda^k TM, m)$ by

$$\langle \omega, \eta \rangle_\rho := \int_M \omega \cdot \eta dm.$$

Here, the subscript ρ indicates the dependence of the inner product on the function ρ . When considering the $L^2(\Lambda^k TM, dx)$ inner product, we will simply write $\langle \cdot, \cdot \rangle$.

The exterior derivative, defined a priori only on $C_c^\infty(\Lambda TM)$, is denoted by d . Its restriction as a linear operator from $C_c^\infty(\Lambda^k TM)$ to $C_c^\infty(\Lambda^{k+1} TM)$ is denoted by d_k . As a densely defined operator from $L^2(\Lambda^k TM, m)$ to $L^2(\Lambda^{k+1} TM, m)$, d_k is easily checked to be closable. With slight abuse of notation, its closure will again be denoted by d_k . Its adjoint is well defined as a closed densely defined operator from $L^2(\Lambda^{k+1} TM, m)$ to $L^2(\Lambda^k TM, m)$. We will denote this adjoint operator by δ_k . It maps $C_c^\infty(\Lambda^{k+1} TM)$ into $C_c^\infty(\Lambda^k TM)$.

Remark 3.1 It would perhaps be more accurate to follow the notation used in the Introduction and denote the operators d, d_k and δ_k by $d_\rho, d_{\rho,k}$ and $d_{\rho,k}^*$, respectively, to bring out their dependence on ρ , but this would unnecessarily burden the notation.

In Lemma 3.3 below, we will state an identity relating δ_k to the operator d_k^* , the adjoint of d_k with respect to the volume measure dx . For this purpose, we need the

following definition. Let $k \in \{1, \dots, n\}$. Let ω be a k -form and X a smooth vector field. We define $\iota(X)\omega$ as the $(k - 1)$ -form given by

$$\iota(X)\omega(Y_1, \dots, Y_{k-1}) = \omega(X, Y_1, \dots, Y_{k-1})$$

for smooth vector fields Y_1, \dots, Y_{k-1} . We refer to ι as the *contraction on the first entry with respect to X* . The next two lemmas are implicit in [9]; we include proofs for the reader's convenience.

Lemma 3.2 *For all smooth k -forms ω and $(k - 1)$ -forms ϵ and compactly supported smooth functions f on M , we have*

$$\omega \cdot (df \wedge \epsilon) = \iota(df^*) \omega \cdot \epsilon,$$

where df^* is the smooth vector field associated to the 1-form df by duality with respect to the Riemannian metric g .

Proof Working in a coordinate chart (U, x) , by linearity it suffices to prove the claim for $\omega = g dx^{i_1} \wedge \dots \wedge dx^{i_k}$ where $1 \leq i_1 < \dots < i_k \leq n$ and $\epsilon = h dx^{j_1} \wedge \dots \wedge dx^{j_{k-1}}$ where $1 \leq j_1 < \dots < j_{k-1} \leq n$. In that case, we find

$$\begin{aligned} &\omega \cdot (df \wedge \epsilon) \\ &= gh \left(dx^{i_1} \wedge \dots \wedge dx^{i_k} \right) \cdot \left(df \wedge dx^{j_1} \wedge \dots \wedge dx^{j_{k-1}} \right) \\ &= \sum_{r=1}^k (-1)^{r+1} gh \left(dx^{i_r} \cdot df \right) \left(dx^{i_1} \wedge \dots \wedge \widehat{dx^{i_r}} \wedge \dots \wedge dx^{i_k} \right) \\ &\quad \cdot \left(dx^{j_1} \wedge \dots \wedge dx^{j_{k-1}} \right) \\ &= \iota(df^*) \omega \cdot \epsilon. \end{aligned}$$

Here, the third line follows by recalling that the inner product can be seen as the determinant of a matrix, and that we can develop this determinant to the row of df . The last equality follows by simply expanding $\iota(df^*)\omega$. □

Lemma 3.3 *If ω is a k -form, then*

$$\delta_{k-1}\omega = d_{k-1}^*\omega - \iota((d \log \rho)^*) \omega,$$

where $d(\log \rho)^*$ is the smooth vector field associated to the 1-form $d(\log \rho)$ by duality with respect to the Riemannian metric g .

Proof Suppose that ω is a k -form. For any $(k - 1)$ -form ϵ , we have

$$\begin{aligned} \langle \epsilon, d_{k-1}^*\omega - \iota((d \log \rho)^*) \omega \rangle_\rho &= \langle \rho \epsilon, d_{k-1}^*\omega \rangle - \langle \rho \epsilon, \iota((d \log \rho)^*) \omega \rangle \\ &= \langle d_{k-1}(\rho \epsilon), \omega \rangle - \langle \epsilon, \iota(\rho (d \log \rho)^*) \omega \rangle \\ &= \langle \rho d_{k-1}\epsilon + d\rho \wedge \epsilon, \omega \rangle - \langle \epsilon, \iota((d\rho)^*) \omega \rangle \\ &= \langle d_{k-1}\epsilon, \omega \rangle_\rho, \end{aligned}$$

where we used that k -forms are linear over C^∞ functions to arrive at the second line. The last equality follows from the previous lemma. The claim now follows. \square

Definition 3.4 [Witten Laplacian] The *Witten Laplacian on k -forms* associated with ρ is the operator L_k defined on $C_c^\infty(\Lambda^k TM)$ as

$$L_k := d_{k-1}\delta_{k-1} + \delta_k d_k.$$

In the special case that $\rho \equiv 1$, we recover the Hodge-de Rham Laplacian

$$\Delta_k = d_{k-1}d_{k-1}^* + d_k^* d_k.$$

Using Lemma 3.3 for 1-forms, we obtain the following identity for the Witten Laplacian on functions:

$$L_0 = d_0^* d_0 - \iota((d \log \rho)^*) d_0 = \Delta_0 - d \log \rho \cdot d_0 = \Delta_0 - \nabla \log \rho \cdot \nabla,$$

where the second identity follows by duality via the Riemannian inner product. The *Bochner-Lichnérowicz-Weitzenböck formula* (cf. [9, Sect. 5]) asserts that

$$\frac{1}{2} \Delta_0 |\omega|^2 = \omega \cdot \Delta_k \omega - |\nabla \omega|^2 - \tilde{Q}_k(\omega, \omega), \tag{3.1}$$

where \tilde{Q}_k is a quadratic form which depends on the Ricci curvature tensor (see [9, Sect. 5]). Notice that in [9] there is an additional term $\frac{1}{k!}$, which comes from the fact that we define $|\nabla \omega|^2$ in a similar way as for k -forms, while [9] defines it in the sense of tensors.

An analogue of (3.1) may be derived for the Witten Laplacian as follows: Firstly, if we expand the above definitions using Lemma 3.3, we can express L_k in terms of Δ_k :

$$L_k \omega = \Delta_k \omega - d_k(\iota((d \log \rho)^*) \omega) - \iota((d \log \rho)^*) d_k \omega. \tag{3.2}$$

Obviously, when $k = 0$ the second term on the right-hand side vanishes, while for $k = n$ the last term vanishes. Inserting (3.2) into equation (3.1), we obtain the following variant of the Bochner-Lichnérowicz-Weitzenböck formula:

$$\frac{1}{2} L_0 |\omega|^2 = \omega \cdot L_k \omega - |\nabla \omega|^2 - Q_k(\omega, \omega), \tag{3.3}$$

where

$$Q_k(\omega, \omega) = \tilde{Q}_k(\omega, \omega) + \frac{1}{2} d|\omega|^2 \cdot d \log \rho - \omega \cdot d(\iota((d \log \rho)^*) \omega) - \omega \cdot \iota((d \log \rho)^*) d\omega. \tag{3.4}$$

As \tilde{Q}_k only depends on the Ricci curvature tensor, we see that Q_k only depends on the Ricci curvature tensor and the positive function ρ . One has $Q_0 = 0$, while for $k = 1$

one has $Q_1(\omega, \omega) = \text{Ric}(\omega^*, \omega^*) - \nabla \nabla \log \rho(\omega^*, \omega^*)$ (see [9]). The latter is usually referred to as the *Bakry–Emery Ricci curvature*. In what follows, we will refer to Q_k as the *Bakry–Emery Ricci curvature on k -forms*.

3.1 The Main Hypothesis

We are now ready to state the key assumption, which is a special case of the one in Bakry [9]:

Hypothesis 3.5 (Non-negative curvature condition) For all $k = 1, \dots, n$ the Bakry–Emery Ricci curvature on k -forms is non-negative, i.e. we have $Q_k(\omega, \omega) \geq 0$ for all k -forms ω .

We assume non-negativity of the Bakry–Emery Ricci curvature, rather than its boundedness from below (as done in [9]), as in the case of (negative) lower bounds one obtains inhomogeneous Riesz estimates only (see [9, Theorem 4.1,5.1]). Also note (see [9]) that to obtain boundedness of the Riesz transform on k -forms, not only does one need non-negativity of Q_k , but also of Q_{k-1} and Q_{k+1} .

As an example, we will show what this assumption means in the case of $M = \mathbb{R}^n$. The result of our computation is likely to be known, but for the reader’s convenience we provide the details of the computation. Note that, the case $k = 1$ is much easier due to the simple coordinate free expression for the Bakry–Emery Ricci curvature Q_1 . In particular, we will see that this assumption is satisfied in the case of the Ornstein–Uhlenbeck operator on \mathbb{R}^n .

Example 3.6 Let $M = \mathbb{R}^n$ with its usual Euclidean metric and consider a smooth strictly positive function ρ on \mathbb{R}^n . Let $k \in \{1, 2, \dots, n\}$. We will derive a sufficient condition on ρ so that $Q_k(\omega, \omega) \geq 0$ for all k -forms ω .

Since \mathbb{R}^n has zero curvature, $Q_k(\omega, \omega) = 0$ for all k -forms ω . Focussing on the remaining terms in (3.4), we will first show that Q_k has the ‘Pythagorean’ property described in (3.5) below. Suppose

$$\omega = \omega^{(1)} + \dots + \omega^{(N)},$$

where each $\omega^{(j)}$ is of the form $f^{(j)} dx^{i_1^{(j)}} \wedge \dots \wedge dx^{i_k^{(j)}}$ with $1 \leq i_1^{(j)} < \dots < i_k^{(j)} \leq n$, and write $I^{(j)} = \{i_1^{(j)}, \dots, i_k^{(j)}\}$. If the index sets $I^{(1)}, \dots, I^{(N)}$ are all different, then

$$Q_k(\omega, \omega) = Q_k(\omega_1, \omega_1) + \dots + Q_k(\omega_N, \omega_N). \tag{3.5}$$

To keep notations simple, we will prove (3.5) for the case $N = 2$; the reader will have no difficulty in generalising the argument to general N .

So let us take k -forms $\omega_1 = f dx^{i_1} \wedge \dots \wedge dx^{i_k}$, where $1 \leq i_1 < \dots < i_k \leq n$ and $\omega_2 = g dx^{j_1} \wedge \dots \wedge dx^{j_k}$, where $1 \leq j_1 < \dots < j_k \leq n$ and suppose that $(i_1, \dots, i_k) \neq (j_1, \dots, j_k)$. Now consider $\omega = \omega_1 + \omega_2$. Since the set of ‘elementary’

k-forms

$$\{dx^{i_1} \wedge \dots \wedge dx^{i_k} : 1 \leq i_1 < \dots < i_k \leq n\}$$

is an orthogonal basis for $\Lambda^k T\mathbb{R}^n$, we have $|\omega|^2 = |\omega_1|^2 + |\omega_2|^2$ and consequently,

$$d|\omega|^2 \cdot d(\log \rho) = d|\omega_1|^2 \cdot d(\log \rho) + d|\omega_2|^2 \cdot d(\log \rho).$$

Furthermore, for any smooth vector field X ,

$$\omega \cdot d(\iota(X)\omega) = \omega_1 \cdot d(\iota(X)\omega_1) + \omega_2 \cdot d(\iota(X)\omega_2) + \omega_1 \cdot d(\iota(X)\omega_2) + \omega_2 \cdot d(\iota(X)\omega_1)$$

and

$$\omega \cdot \iota(X) d\omega = \omega_1 \cdot \iota(X) d\omega_1 + \omega_2 \cdot \iota(X) d\omega_2 + \omega_1 \cdot \iota(X) d\omega_2 + \omega_2 \cdot \iota(X) d\omega_1.$$

Now

$$\begin{aligned} \iota(X) d\omega_1 &= \sum_{i=1}^n \partial_i f dx^i(X) dx^{i_1} \wedge \dots \wedge dx^{i_k} \\ &\quad + \sum_{i=1}^n \sum_{l=1}^k (-1)^l \partial_i f dx^{i_l}(X) dx^i \wedge dx^{i_1} \wedge \dots \wedge \widehat{dx^{i_l}} \wedge \dots \wedge dx^{i_k} \end{aligned}$$

and

$$d(\iota(X)\omega_1) = - \sum_{i=1}^n \sum_{l=1}^k (-1)^l \partial_i f dx^{i_l}(X) dx^i \wedge dx^{i_1} \wedge \dots \wedge \widehat{dx^{i_l}} \wedge \dots \wedge dx^{i_k}.$$

Consequently,

$$\iota(X) d\omega_1 + d(\iota(X)\omega_1) = \sum_{i=1}^n \partial_i f dx^i(X) dx^{i_1} \wedge \dots \wedge dx^{i_k}.$$

By orthogonality, we thus obtain that

$$\begin{aligned} &\omega_2 \cdot d(\iota((d \log \rho)^*)\omega_1) + \omega_2 \cdot \iota((d \log \rho)^*) d\omega_1 \\ &= \omega_2 \cdot (d(\iota((d \log \rho)^*)\omega_1) + \iota((d \log \rho)^*) d\omega_1) \\ &= \sum_{i=1}^n g \partial_i f dx^i((d \log \rho)^*)(dx^{i_1} \wedge \dots \wedge dx^{i_k}) \cdot (dx^{j_1} \wedge \dots \wedge dx^{j_k}) = 0. \end{aligned}$$

Obviously, the same holds if we interchange ω_1 and ω_2 . Putting everything together, we obtain $Q_k(\omega, \omega) = Q_k(\omega_1, \omega_1) + Q_k(\omega_2, \omega_2)$. This concludes the proof of (the case $N = 2$ of) (3.5).

Now consider a k -form ω of the form $f dx^{i_1} \wedge \dots \wedge dx^{i_k}$ with $1 \leq i_1 < \dots < i_k \leq n$. To simplify notations a bit, we shall suppose that $(i_1, \dots, i_k) = (1, \dots, k)$. We compute the three last terms on the right-hand side of (3.4).

As to the first term, from $|\omega|^2 = f^2$, we obtain

$$\frac{1}{2} d|\omega|^2 \cdot d(\log \rho) = \sum_{i=1}^n f \partial_i f \partial_i(\log \rho).$$

Turning to the second term,

$$\begin{aligned} \iota((d \log \rho)^*) d\omega &= \sum_{j=1}^n ((d \log \rho)^*)^j \iota(\partial_j) d\omega \\ &= \sum_{j=1}^n \sum_{i=k+1}^n \partial_i f \partial_j(\log \rho) \iota(\partial_j) dx^i \wedge dx^1 \wedge \dots \wedge dx^k \\ &= \sum_{i=k+1}^n \partial_i f \partial_i(\log \rho) dx^1 \wedge \dots \wedge dx^k \\ &+ \sum_{i=k+1}^n \sum_{j=1}^k (-1)^j \partial_i f \partial_j(\log \rho) dx^i \wedge dx^1 \wedge \dots \wedge \widehat{dx^j} \wedge \dots \wedge dx^k. \end{aligned}$$

Hence

$$\omega \cdot \iota((d \log \rho)^*) d\omega = \sum_{i=k+1}^n f \partial_i f \partial_i(\log \rho).$$

Computing the final term, we have

$$\begin{aligned} \iota((d \log \rho)^*) \omega &= f \sum_{j=1}^n ((d \log \rho)^*)^j \iota(\partial_j) dx^1 \wedge \dots \wedge dx^k \\ &= f \sum_{j=1}^k (-1)^j \partial_j(\log \rho) dx^1 \wedge \dots \wedge \widehat{dx^j} \wedge \dots \wedge dx^k. \end{aligned}$$

From this, it follows that

$$\begin{aligned} d(\iota((d \log \rho)^*) \omega) &= \sum_{j=1}^k \sum_{i=1}^n (-1)^j \partial_i (f \partial_j(\log \rho)) dx^i \wedge dx^1 \wedge \dots \wedge \widehat{dx^j} \wedge \dots \wedge dx^k \\ &= \sum_{j=1}^k \sum_{i=1}^n (-1)^j \partial_i f \partial_j(\log \rho) dx^i \wedge dx^1 \wedge \dots \wedge \widehat{dx^j} \wedge \dots \wedge dx^k \end{aligned}$$

$$+ \sum_{j=1}^k \sum_{i=1}^n (-1)^j f \partial_i \partial_j (\log \rho) dx^i \wedge dx^1 \wedge \dots \wedge \widehat{dx^j} \wedge \dots \wedge dx^k.$$

Noting that only the terms with $i = j$ can contribute a non-zero contribution to the inner product with ω , we obtain

$$\omega \cdot d(\iota((d \log \rho)^*)\omega) = \sum_{i=1}^k f \partial_i f \partial_i (\log \rho) + f^2 \partial_i^2 (\log \rho).$$

Collecting everything, we find that

$$\begin{aligned} Q_k(\omega, \omega) &= \tilde{Q}_k(\omega, \omega) + \frac{1}{2} d|\omega|^2 \cdot d \log \rho - \omega \cdot d(\iota((d \log \rho)^*)\omega) \\ &\quad - \omega \cdot \iota((d \log \rho)^*) d\omega \\ &= -f^2 \sum_{i=1}^k \partial_i^2 (\log \rho). \end{aligned}$$

We thus see that $Q_k(\omega, \omega) \geq 0$ precisely when $\sum_{i=1}^k \partial_i^2 (\log \rho) \leq 0$. Recalling the simplification for notational purposes, we conclude that $Q_k(\omega, \omega) \geq 0$ for all k -forms ω precisely if for all $1 \leq i_1 < \dots < i_k \leq n$ it holds that

$$\sum_{r=1}^k \partial_{i_r}^2 (\log \rho) \leq 0.$$

In the special case $\rho(x) = e^{-\frac{1}{2}|x|^2}$ which corresponds to the Ornstein-Uhlenbeck operator, this condition is clearly satisfied. Indeed, for any $j = 1, \dots, n$, we have $\partial_j^2 (\log \rho) = -1$.

We can use the previous example to consider a more general situation.

Example 3.7 Let (M, g) be a complete Riemannian manifold. Suppose the quadratic form \tilde{Q}_k depending solely on the Ricci curvature is bounded from below for all $1 \leq k \leq n$, i.e. there exist constants a_1, \dots, a_n such that for all k -forms ω , we have

$$\tilde{Q}_k(\omega, \omega) \geq a_k |\omega|^2.$$

Fix $k \in \{1, \dots, n\}$. In normal coordinates around a point $p \in M$, the expression for $Q_k(\omega, \omega)$ at p reduces to the one of the previous examples. Consequently, $Q_k(\omega, \omega) \geq 0$ for any k -form ω if for any $p \in M$ and any $1 \leq i_1 < \dots < i_k \leq n$ one has $\sum_{r=1}^k \partial_{i_r}^2 (\log \rho)(p) \leq a_k$, where the last expression is in normal coordinates around p .

3.2 The Heat Semigroup Generated by $-L_k$

We return to the general setting described at the beginning of this section. For each $k = 0, 1, \dots, n$ the operator L_k is essentially self-adjoint on $L^2(\Lambda^k TM, m)$ (see [9, 54] for the case $\rho \equiv 1$ and [56]) and satisfies $\langle L_k \omega, \omega \rangle_\rho = |d_k \omega|^2 + |\delta_{k-1} \omega|^2 \geq 0$ for all smooth k -forms ω . Consequently, its closure is a self-adjoint operator on $L^2(\Lambda^k TM, m)$. With slight abuse of notation, we shall denote this closure by L_k again. By the spectral theorem, $-L_k$ generates a strongly continuous contraction semigroup

$$P_t^k := e^{-tL_k}, \quad t \geq 0,$$

on $L^2(\Lambda^k TM, m)$.

From now on, we assume that Hypothesis 3.5 is satisfied. As was shown in [9, 56], under this assumption, the restriction of $(P_t^k)_{t \geq 0}$ to $L^p(\Lambda^k TM, m) \cap L^2(\Lambda^k TM, m)$ extends to a strongly continuous contraction semigroup on $L^p(\Lambda^k TM, m)$ for any $p \in [1, \infty)$. These extensions are consistent, i.e. the semigroups $(P_t^k)_{t \geq 0}$ on $L^{p_i}(\Lambda^k TM, m)$, $i = 1, 2$, agree on the intersection $L^{p_1}(\Lambda^k TM, m) \cap L^{p_2}(\Lambda^k TM, m)$.

The infinitesimal generator of the semigroup $(P_t^k)_{t \geq 0}$ in $L^p(\Lambda^k TM, m)$ will be denoted (with slight abuse of notation) by $-L_k$ and its domain by $D_p(L_k)$.

As an operator acting in $L^2(\Lambda^k TM, m)$, L_k is the closure of an operator defined a priori on $C_c^\infty(\Lambda^k TM)$ and therefore the inclusion $C_c^\infty(\Lambda^k TM) \subseteq D_2(L_k)$ trivially holds. The definition of the domain $D_p(L_k)$ is indirect, however, and based on the fact that L_k generates a strongly continuous semigroup on $L^p(\Lambda^k TM, m)$. Nevertheless we have:

Lemma 3.8 $C_c^\infty(\Lambda^k TM)$ is contained in $D_p(L_k)$ for all $1 < p < \infty$.

Proof We follow the idea of [48, Lemma 4.8]. Pick an arbitrary k -form $\omega \in C_c^\infty(\Lambda^k TM, m)$. Then $\omega \in D_2(L_k)$ (by definition of L_k on $L^2(\Lambda^k TM, m)$) and also $\omega \in L^p(\Lambda^k TM, m)$. Since $L^p(\Lambda^k TM, m)$ is a reflexive Banach space, a standard result in semigroup theory states that in order to show that $\omega \in D_p(L_k)$ it suffices to show that

$$\limsup_{t \downarrow 0} \frac{1}{t} \|P_t^k \omega - \omega\|_p < \infty$$

(see, e.g. [14]). Note that $\frac{1}{t}(P_t^k \omega - \omega) = -\frac{1}{t} \int_0^t P_s^k L_k \omega \, ds$ in $L^2(\Lambda^k TM, m)$. However, since $L_k \omega \in C_c^\infty(\Lambda^k TM)$ (as both d and δ map $C_c^\infty(\Lambda TM)$ to $C_c^\infty(\Lambda TM)$), we can interpret the integral on the right-hand side as a Bochner integral in the Banach space $L^p(\Lambda^k TM, m)$ (see [35, Chapter 1]). Consequently, we may estimate

$$\frac{1}{t} \|P_t^k \omega - \omega\|_p \leq \frac{1}{t} \int_0^t \|P_s^k L_k \omega\|_p \, ds \leq \frac{1}{t} \int_0^t \|L_k \omega\|_p \, ds = \|L_k \omega\|_p.$$

But then $\limsup_{t \downarrow 0} \frac{1}{t} \|P_t^k \omega - \omega\|_p \leq \|L_k \omega\|_p < \infty$. This proves the claim. □

By the Stein interpolation theorem [53, Theorem 1 on p.67], for $p \in (1, \infty)$ and $k = 0, 1, \dots, n$ the mapping $t \mapsto P_t^k$ extends analytically to a strongly continuous $\mathcal{L}(L^p(\Lambda^k TM, m))$ -valued mapping $z \mapsto P_z^k$ defined on the sector Σ_{ω_p} with $\omega_p = \frac{\pi}{2}(1 - |2/p - 1|)$. On this sector, the operators P_z^k are contractive. This implies that L_k is sectorial of angle ω_p .

As explained in [56, p. 625], it follows from the general theory of Dirichlet forms [25] that there exists a Markov process $(X_t)_{t \geq 0}$ such that

$$P_t^0 f(x) = \mathbb{E}^x(f(X_t)) \tag{3.6}$$

for all $f \in C_c^\infty(M)$. Here, \mathbb{E}^x denotes expectation under the law of the process $(X_t)_{t \geq 0}$ starting almost surely in $x \in M$. Using this together with Hypothesis 3.5 (this corresponds to the assumption made in [56, Eq. (1.2)], see the explanation preceding the proof of theorem 3.12), it is then shown in [56, Proposition 2.3] that there exists a Markov process $(V_t)_{t \geq 0}$ such that

$$P_t^k \omega(v) = \mathbb{E}^v(\omega(V_t))$$

for all $\omega \in C_c^\infty(\Lambda^k TM)$. Here, \mathbb{E}^v denotes expectation under the law of the process $(V_t)_{t \geq 0}$ starting almost surely in $v \in M$.

As a consequence of (3.6), the operators P_t^0 are positive, in the sense that they send non-negative functions to non-negative functions. This, together with the following lemma, allows us to show that L_k is in fact R -sectorial of angle $< \frac{1}{2}\pi$.

Lemma 3.9 (*R-sectoriality via pointwise domination*) *Let M be a Riemannian manifold of dimension n equipped with a measure m . Let $k \in \{0, 1, \dots, n\}$ and suppose A and B are sectorial operators of angle $< \frac{1}{2}\pi$ on the space $L^p(M, m)$ and $L^p(\Lambda^k TM, m)$, respectively, with $1 \leq p < \infty$. Suppose the bounded analytic C_0 -semigroups $(S_t)_{t \geq 0}$ and $(T_t)_{t \geq 0}$ generated by $-A$ and $-B$ satisfy the pointwise bound*

$$|T_t \omega| \leq C S_t |\omega|$$

for all $\omega \in L^p(\Lambda^k TM, m)$ and $t \geq 0$, where C is a constant. If the set $\{(I + sA)^{-1} : s > 0\}$ is R -bounded (in particular, if A is R -sectorial), then B is R -sectorial of angle $< \frac{1}{2}\pi$.

For the proof of this lemma, we need the following result.

Lemma 3.10 *Let (M, g) be a Riemannian manifold of dimension n equipped with a measure m . For all $\omega_1, \dots, \omega_N \in L^p(\Lambda^k TM, m)$, we have*

$$\mathbb{E} \left\| \sum_{i=1}^N r_i \omega_i \right\|_{L^p(\Lambda^k TM, m)} \approx_p \left\| \left(\sum_{i=1}^N |\omega_i|^2 \right)^{1/2} \right\|_{L^p(M, m)},$$

where $(r_i)_i$ is a Rademacher sequence; the implicit constant only depends on p .

Proof Step 1—First we assume that $\omega_1, \dots, \omega_N$ are supported in a single coordinate chart (U, x) . With slight abuse of notation, we will identify each $\omega_i|_U$ with the corresponding \mathbb{C}^{d_k} -valued function on U ; here, $d_k = \binom{n}{k}$ is the dimension of $\Lambda^k T U$.

Denote by G_k^{-1} the symmetric, positive definite $d_k \times d_k$ -matrix with elements

$$(G_k^{-1})_{i_1 i_2 \dots i_k, j_1 j_2 \dots j_k} = (dx^{i_1} \wedge \dots \wedge dx^{i_k}) \cdot (dx^{j_1} \wedge \dots \wedge dx^{j_k}),$$

where $1 \leq i_1 < \dots < i_k \leq n$ and $1 \leq j_1 < \dots < j_k \leq n$.

Since G_k^{-1} is orthogonally diagonalisable, we have $G_k^{-1}(p) = Q(p)D(p)Q(p)^T$, where $D(p)$ is diagonal with positive diagonal entries. Now set

$$\eta_i(p) := \sqrt{D(p)}Q(p)^T \omega_i(p)$$

for $p \in U$. By using the Kahane-Khintchine inequality,

$$\begin{aligned} \mathbb{E} \left\| \sum_i r_i \omega_i \right\|_{L^p(\Lambda^k T M, m)}^p &= \mathbb{E} \left\| \sum_i r_i \omega_i \right\|_{L^p(\Lambda^k T U, m|_U)}^p \\ &\approx_p \mathbb{E} \left\| \sum_i r_i \omega_i \right\|_{L^2(\Lambda^k T U, m|_U)}^p \\ &= \left(\mathbb{E} \int_U \left| \sum_i r_i \omega_i \right|^2 dm \right)^{p/2} \\ &= \left(\mathbb{E} \int_U \sum_{i,j} r_i r_j (\omega_i \cdot \bar{\omega}_j)_{G_k^{-1}} dm \right)^{p/2} \\ &= \left(\mathbb{E} \int_U \sum_{i,j} r_i r_j \omega_i^T G_k^{-1} \bar{\omega}_j dm \right)^{p/2} \\ &= \left(\mathbb{E} \int_U \sum_{i,j} r_i r_j \eta_i^T \bar{\eta}_j dm \right)^{p/2} \\ &= \left(\int_U \mathbb{E} \left| \sum_i r_i \eta_i \right|^2 dm \right)^{p/2} \\ &= \mathbb{E} \left\| \sum_i r_i \eta_i \right\|_{L^2(U, m|_U; \mathbb{C}^{d_k})}^p. \end{aligned}$$

Next, by the square function characterisation of Rademacher sums for \mathbb{C}^{d_k} -valued functions,

$$\begin{aligned} \mathbb{E} \left\| \sum_i r_i \eta_i \right\|_{L^2(U, m|_U; \mathbb{C}^{d_k})}^p &\approx_p \left\| \left(\sum_i |\eta_i|^2 \right)^{1/2} \right\|_{L^p(U, m|_U)}^p \\ &= \left\| \left(\sum_i \eta_i^T \bar{\eta}_i \right)^{1/2} \right\|_{L^p(U, m|_U)}^p \\ &= \left\| \left(\sum_i \omega_i^T G_k^{-1} \bar{\omega}_i \right)^{1/2} \right\|_{L^p(U, m|_U)}^p \\ &= \left\| \left(\sum_i \omega_i \cdot \omega_i \right)^{1/2} \right\|_{L^p(U, m|_U)}^p \\ &= \left\| \left(\sum_i |\omega_i|^2 \right)^{1/2} \right\|_{L^p(M, m)}^p . \end{aligned}$$

Step 2—We now turn to the general case. Let $(\phi_U)_{U \in \mathcal{U}}$ be a partition of unity subordinate to a collection of coordinate charts \mathcal{U} covering M . Then, using Fubini’s theorem and the result of Step 1,

$$\begin{aligned} \mathbb{E} \left\| \sum_i r_i \omega_i \right\|_{L^p(\Lambda^k TM, m)}^p &= \mathbb{E} \sum_U \int_M \phi_U \left| \sum_i r_i \omega_i \right|^p dm \\ &= \mathbb{E} \sum_U \left\| \sum_i r_i \phi_U^{1/p} \omega_i \right\|_{L^p(\Lambda^k TM, m)}^p \\ &\approx_p \sum_U \left\| \left(\sum_i |\phi_U^{1/p} \omega_i|^2 \right)^{1/2} \right\|_{L^p(M, m)}^p \\ &= \sum_U \int_M \left(\sum_i |\phi_U^{1/p} \omega_i|^2 \right)^{p/2} dm \\ &= \sum_U \int_M \phi_U \left(\sum_i |\omega_i|^2 \right)^{p/2} dm \\ &= \int_M \left(\sum_i |\omega_i|^2 \right)^{p/2} dm \end{aligned}$$

$$= \left\| \left(\sum_i |\omega_i|^2 \right)^{1/2} \right\|_{L^p(M,m)}^p .$$

□

Proof of Lemma 3.9 Upon taking Laplace transforms, the pointwise assumption implies, for $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > 0$,

$$|(I + \lambda B)^{-1} \omega| \leq C(I + \operatorname{Re} \lambda A)^{-1} |\omega|.$$

Hence if $\operatorname{Re} \lambda_1, \dots, \operatorname{Re} \lambda_N > 0$, then for all $\omega_1, \dots, \omega_N \in L^p(\Lambda^k TM, m)$ we find, by Lemma 3.10,

$$\begin{aligned} \mathbb{E} \left\| \sum_{i=1}^N r_i (I + \lambda_i B)^{-1} \omega_i \right\|_{L^p(\Lambda^k TM, m)} &\approx_p \left\| \left(\sum_{i=1}^N |(I + \lambda_i B)^{-1} \omega_i|^2 \right)^{1/2} \right\|_{L^p(M, m)} \\ &\leq C \left\| \left(\sum_{i=1}^N [(I + \operatorname{Re} \lambda_i A)^{-1} |\omega_i|]^2 \right)^{1/2} \right\|_{L^p(M, m)} \\ &\approx_p C \mathbb{E} \left\| \sum_{i=1}^N r_i (I + \lambda_i A)^{-1} |\omega_i| \right\|_{L^p(M, m)} \\ &\leq CR \mathbb{E} \left\| \sum_{i=1}^N r_i |\omega_i| \right\|_{L^p(M, m)} \\ &\approx_p CR \left\| \sum_{i=1}^N |\omega_i|^2 \right\|_{L^p(M, m)}^{1/2} \\ &\approx_p CR \mathbb{E} \left\| \sum_{i=1}^N r_i \omega_i \right\|_{L^p(\Lambda^k TM, m)} . \end{aligned}$$

Here, R denotes the R -bound of the set $\{(I + sA)^{-1} : s > 0\}$. This gives the R -boundedness of the set $\{(I + \lambda B)^{-1} : \operatorname{Re} \lambda > 0\}$. A standard Taylor expansion argument allows us to extend this to the R -boundedness of the set $\{(I + \lambda B)^{-1} : \lambda \in \Sigma_\nu\}$ for some $\nu > \frac{1}{2}\pi$. □

We now return to the setting considered at the beginning of this section. Combining the preceding lemmas, we arrive at the following result.

Proposition 3.11 (*R*-sectoriality of L_k) *Let Hypothesis 3.5 be satisfied. For all $1 < p < \infty$ and $k = 0, 1, \dots, n$, the operator L_k is R -sectorial on $L^p(\Lambda^k TM, m)$ with angle $\omega_R^+(L_k) < \frac{1}{2}\pi$.*

Proof Fix $1 < p < \infty$. As we have already noted, $-L_k$ generates a strongly continuous analytic contraction semigroup on $L^p(\Lambda^k TM)$. By [9,56], these semigroups satisfy the pointwise bound

$$|P_t^k \omega| \leq P_t^0 |\omega| \tag{3.7}$$

for all $\omega \in L^p(\Lambda^k TM, m)$. Since the semigroup generated by $-L_0$ is positive, L_0 is R -sectorial by [38, Corollary 5.2]. Lemma 3.9 then implies that L_k is R -sectorial, of angle $< \frac{1}{2}\pi$. \square

We are now ready to state our first main result.

Theorem 3.12 [Bounded H^∞ -calculus for L_k] *Let Hypothesis 3.5 be satisfied. For all $1 < p < \infty$ and all $k = 0, 1, \dots, n$, the operator L_k has a bounded H^∞ -calculus on $L^p(\Lambda^k TM, m)$ of angle $< \frac{1}{2}\pi$.*

For $k = 0$ the proposition is an immediate consequence of [38, Corollary 5.2]; see [16] for a more detailed quantitative statement. For $k = 1, \dots, n$ this argument cannot be used and instead we shall apply the square function estimates of [56]. To make the link between the definitions used in that paper and the ones used here, we need to make some preliminary remarks.

In [56], the Hodge Laplacian on k -forms is defined as

$$\tilde{\Delta}_k := -\text{Tr}(\nabla \nabla).$$

This is motivated by the fact that on functions this operator agrees with Δ_k (see [27]). Similarly in [56] one defines

$$\tilde{L}_k := \tilde{\Delta}_k - \text{Tr}(\nabla(\log \rho) \otimes \nabla). \tag{3.8}$$

Actually, the definition in [56] there differs notationally from (3.8) in that $e^{-\rho}$ is written for the strictly positive function that we denote by ρ .

Define

$$V_k := L_k - \tilde{L}_k$$

as a linear operator on $C_c^\infty(\Lambda^k TM)$ (cf. [56, eq. (1.2)], recalling our convention of considering the negative Laplacian). We will show in a moment that

$$\omega \cdot V_k \omega = Q_k(\omega, \omega), \tag{3.9}$$

so that Hypothesis 3.5 can be rephrased as assuming that $\omega \cdot V_k \omega \geq 0$. This corresponds to the assumption made in [56, Eq. (1.4)]. Thus, the results from [56] may be applied in the present situation.

Turning to the proof of (3.9), first observe that $\tilde{\Delta}_k$ satisfies

$$\frac{1}{2} \tilde{\Delta}_0 |\omega|^2 = \omega \cdot \tilde{\Delta}_k \omega - |\nabla \omega|^2,$$

from which it follows that

$$\frac{1}{2}\tilde{L}_0|\omega|^2 = \omega \cdot \tilde{L}_k\omega - |\nabla\omega|^2 - \frac{1}{2}\text{Tr}(\nabla(\log \rho) \otimes \nabla|\omega|^2) + \omega \cdot \text{Tr}(\nabla(\log \rho) \otimes \nabla\omega).$$

This can be simplified to

$$\frac{1}{2}\tilde{L}_0|\omega|^2 = \omega \cdot \tilde{L}_k\omega - |\nabla\omega|^2. \tag{3.10}$$

Indeed, in a coordinate chart one has

$$\begin{aligned} \frac{1}{2}\text{Tr}(\nabla(\log \rho) \otimes \nabla|\omega|^2) &= \frac{1}{2} \sum_{j=1}^n \nabla^j(\log \rho) \nabla_j|\omega|^2 \\ &= \sum_{j=1}^n \nabla^j(\log \rho) \nabla_j\omega \cdot \omega = \text{Tr}(\nabla(\log \rho) \otimes \nabla\omega) \cdot \omega. \end{aligned}$$

Noting that $L_0 = \tilde{L}_0$, combining (3.3) and (3.10) gives $\omega \cdot V_k\omega = Q_k(\omega, \omega)$ as desired.

Proof of Theorem 3.12 Fix $1 < p < \infty$. By Proposition 3.11, L_k is R -sectorial on $L^p(\Lambda^k TM, m)$ and $\omega_R^+(L_k) < \frac{1}{2}\pi$. Pick $\vartheta \in (\omega_R^+(L_k), \frac{1}{2}\pi)$. The function $\psi(z) := \frac{1}{\sqrt{2}}\sqrt{z}e^{-\sqrt{z}}$ belongs to $H^1(\Sigma_\vartheta^+) \cap H^\infty(\Sigma_\vartheta^+)$. Using the substitution $t = s^2$, we see that

$$\int_0^\infty |\psi(tL_k)\omega|^2 \frac{dt}{t} = \int_0^\infty \left| \frac{\partial}{\partial t} \Big|_{t=s} e^{-tL_k^{1/2}} \omega \right|^2 s \, ds.$$

Accordingly, by [56, Theorem 5.3],

$$\|\omega - E_0^k\omega\|_p \lesssim_p \left\| \int_0^\infty |\psi(tL_k)\omega|^2 \frac{dt}{t} \right\|_p \lesssim_p \|\omega\|_p \tag{3.11}$$

for all $\omega \in C_c^\infty(\Lambda^k TM)$, where E_0^k denotes projection onto the kernel of L_k . By a routine density argument (using that convergence in the mixed $L^p(L^2)$ -norm implies almost everywhere convergence along a suitable subsequence), these inequalities extend to arbitrary k -forms $\omega \in L^p(\Lambda^k TM, m)$.

Now it is well known that for an R -sectorial operator, the square function estimate (3.11) implies the operator having a bounded H^∞ -calculus of angle at most equal to its angle of R -sectoriality (see [39] or [36, Chapter 10]). \square

4 The Hodge–Dirac Operator

Throughout this section, we shall assume that Hypothesis 3.5 is in force. Under this assumption one may check, using the Bochner-Lichnérowicz-Weitzenböck formula

(3.3) instead of (3.1), that the results in [9, Sect. 5] proved for the special case $\rho \equiv 1$ carry over to general strictly positive functions $\rho \in C^\infty(M)$. Whenever we refer to results from [9], we bear this in mind.

Definition 4.1 [Hodge–Dirac operator associated with ρ] The *Hodge–Dirac operator* associated with ρ is the linear operator D on $C_c^\infty(\Lambda TM)$ defined by

$$D := d + \delta.$$

As in Remark 3.1 it would be more accurate to denote this operator by D_ρ , but again we prefer to keep the notation simple.

With respect to the decomposition $C_c^\infty(\Lambda TM) = \bigoplus_{k=0}^n C_c^\infty(\Lambda^k TM)$, D can be represented by the $(n + 1) \times (n + 1)$ -matrix

$$D = \begin{pmatrix} 0 & \delta_0 & & & & \\ d_0 & 0 & \delta_1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & d_{n-2} & 0 & \delta_{n-1} & \\ & & & d_{n-1} & 0 & \end{pmatrix},$$

From $d^2 = \delta^2 = 0$, it follows that

$$D^2 = \begin{pmatrix} L_0 & & & \\ & \ddots & & \\ & & & L_n \end{pmatrix} =: L.$$

Lemma 4.2 For all $1 \leq p < \infty$, the operator D is closable as a densely defined operator on $L^p(\Lambda TM, m)$.

Proof For the reader’s convenience, we include the easy proof. Let $(\omega_n)_n$ be a sequence in $C_c^\infty(\Lambda TM)$ and suppose that $\omega_n \rightarrow 0$ and $D\omega_n \rightarrow \eta$ in $L^p(\Lambda TM, m)$. Decomposing along the direct sum, we find that $\omega_n^k \rightarrow \omega^k$ in $L^p(\Lambda^k TM, m)$ for $0 \leq k \leq n$ and $d_{k-1}\omega_n^{k-1} + \delta_k\omega_n^{k+1} \rightarrow \eta^k$ in $L^p(\Lambda^k TM, m)$ for $1 \leq k \leq n - 1$; for $k = 0$ we have $\delta_0\omega_n^1 \rightarrow \eta^0$ in $L^p(\Lambda^0 TM, m)$ and for $k = n$ we have $d_{n-1}\omega_n^{n-1} \rightarrow \eta^n$ in $L^p(\Lambda^n TM, m)$.

First consider $1 \leq k \leq n - 1$, and pick $\phi \in C_c^\infty(\Lambda^k TM, m)$. By Hölder’s inequality,

$$\begin{aligned} \langle \eta^k, \phi \rangle_\rho &= \lim_{n \rightarrow \infty} \langle d_{k-1}\omega_n^{k-1} + \delta_k\omega_n^{k+1}, \phi \rangle_\rho \\ &= \lim_{n \rightarrow \infty} \langle \omega_n^{k-1}, \delta_{k-1}\phi \rangle_\rho + \langle \omega_n^{k+1}, d_k\phi \rangle_\rho \\ &= \langle 0, \delta_{k-1}\phi \rangle_\rho + \langle 0, d_k\phi \rangle_\rho \\ &= 0. \end{aligned}$$

This is justified since both ω_n^{k+1} and ϕ are compactly supported and therefore belong to $D_q(\delta_k)$, respectively, $D_q(\delta_{k-1})$, with $\frac{1}{p} + \frac{1}{q} = 1$. It follows that $\eta^k = 0$ by density.

The cases $k = 0$ and $k = n$ are treated similarly. We conclude that $\eta^k = 0$ for all k , so $\eta = 0$. □

With slight abuse of notation, we will denote the closure again by D and write $D_p(D)$ for its domain in $L^p(\Lambda^k TM, m)$. The main result of this section asserts that, under Hypothesis 3.5, for all $1 < p < \infty$ the operator D is R -bisectorial on $L^p(\Lambda^k TM, m)$ and has a bounded H^∞ -calculus on this space.

Since L_k is sectorial on $L^p(\Lambda^k TM, m)$, $1 < p < \infty$, its square root is well defined and sectorial. Moreover, we have $C_c^\infty(\Lambda^k TM) \subseteq D_p(L_k) \subseteq D_p(L_k^{1/2})$ (cf. Lemma 3.8).

Lemma 4.3 *For all $1 < p < \infty$ and $k = 0, 1, \dots, n$, $C_c^\infty(\Lambda^k TM)$ is dense in $D_p(L_k^{1/2})$.*

Proof Pick an arbitrary $\omega \in D_p(L_k^{1/2})$. By [2, Proposition 3.8.2], we have $\omega \in D_p((I - L_k)^{1/2})$. From the proof of [9, Corollaries 4.3 and 5.3], we see that there exists a sequence $(\omega_n)_n$ in $C_c^\infty(\Lambda^k TM)$ such that $(I + L_k)^{1/2}\omega_n \rightarrow (I + L_k)^{1/2}\omega$ in $L^p(\Lambda^k TM, m)$. By [9, Lemmas 4.2 and 5.2], we then find that

$$\|\omega_n - \omega\|_{D_p(L_k^{1/2})} = \|\omega_n - \omega\|_p + \|L_k^{1/2}(\omega_n - \omega)\|_p \lesssim \|(I + L_k)^{1/2}(\omega_n - \omega)\|_p.$$

By the choice of the sequence ω_n , the latter tends to 0 and consequently we have $\omega_n \rightarrow \omega$ in $D_p(L_k^{1/2})$. □

The following result is essentially a restatement of [9, Theorem 5.1, Corollary 5.3] in the presence of non-negative curvature. The results in [9] are stated only for the case $\rho \equiv 1$ and given in the form of inequalities for smooth compactly supported k -forms.

Theorem 4.4 (Boundedness of the Riesz transform associated with L_k) *Let Hypothesis 3.5 hold. For all $1 < p < \infty$ and $k = 0, 1, \dots, n$, we have*

$$D_p(L_k^{1/2}) = D_p(d_k + \delta_{k-1}),$$

and for all ω in this common domain we have

$$\|L_k^{1/2}\omega\|_p \simeq_{p,k} \|(d_k + \delta_{k-1})\omega\|_p.$$

Here, $D_k := d_k + \delta_{k-1}$ is the restriction of D as a densely defined operator acting from $L^p(\Lambda^k TM, m)$ into $L^p(\Lambda^k TM, m)$.

Proof We start by showing that $D_p(L_k^{1/2}) \subseteq D_p(d_k + \delta_{k-1})$ together with the estimate

$$\|(d_k + \delta_{k-1})\omega\|_p \lesssim_{p,k} \|L_k^{1/2}\omega\|_p.$$

Pick an arbitrary $\omega \in D_p(L_k^{1/2})$. As $C_c^\infty(\Lambda^k TM)$ is dense in $D_p(L_k^{1/2})$ by Lemma 4.3, we can find a sequence $(\omega_i)_i$ of k -forms in this space converging to ω in $D_p(L_k^{1/2})$.

By [9, Theorem 5.1] we then find, for all i, j ,

$$\begin{aligned} & \|\omega_i - \omega_j\|_p + \|(\mathbf{d}_k + \delta_{k-1})(\omega_i - \omega_j)\|_p \\ & \lesssim \|\omega_i - \omega_j\|_p + \|\mathbf{d}_k\omega_i - \mathbf{d}_k\omega_j\|_p + \|\delta_{k-1}\omega_i - \delta_{k-1}\omega_j\|_p \\ & \lesssim \|\omega_i - \omega_j\|_p + \|L_k^{1/2}\omega_i - L_k^{1/2}\omega_j\|_p \end{aligned}$$

which shows that $(\omega_i)_i$ is Cauchy in $\mathbf{D}_p(\mathbf{d}_k + \delta_{k-1})$. By the closedness of $\mathbf{d}_k + \delta_{k-1}$, this sequence converges to some $\eta \in \mathbf{D}_p(\mathbf{d}_k + \delta_{k-1})$. Since both $\mathbf{D}_p(L_k^{1/2})$ and $\mathbf{D}_p(\mathbf{d}_k + \delta_{k-1})$ are continuously embedded into $L^p(\Lambda^k TM, m)$, we have $\omega_i \rightarrow \omega$ and $\omega_i \rightarrow \eta$ in $L^p(\Lambda^k TM, m)$, and therefore $\eta = \omega$. This shows that $\omega \in \mathbf{D}_p(\mathbf{d}_k + \delta_{k-1})$. To prove the estimate, by [9, Theorem 5.1] we obtain, for all i ,

$$\|(\mathbf{d}_k + \delta_{k-1})\omega_i\|_p \leq \|\mathbf{d}_k\omega_i\|_p + \|\delta_{k-1}\omega_i\|_p \leq C_{p,k} \|L_k^{1/2}\omega_i\|_p.$$

Since $\omega_i \rightarrow \omega$ both in $\mathbf{D}_p(L_k^{1/2})$ and $\mathbf{D}_p(\mathbf{d}_k + \delta_{k-1})$, it follows that

$$\|(\mathbf{d}_k + \delta_{k-1})\omega\|_p \leq C_{p,k} \|L_k^{1/2}\omega\|_p.$$

The reverse inclusion and estimate may be proved in a similar manner. Now one uses that $C_c^\infty(\Lambda^k TM)$ is dense in $\mathbf{D}_p(\mathbf{d}_k + \delta_{k-1})$, $\mathbf{d}_k + \delta_{k-1}$ being the closure of its restriction to $C_c^\infty(\Lambda^k TM)$. One furthermore uses the estimate in [9, Corollary 5.3] which holds (with $e = 0$ in the notation of [9]) by Hypothesis 3.5. Finally, by definition of the norm on $L^p(\Lambda TM, m)$, for all $\omega \in C_c^\infty(\Lambda^k TM)$, we have

$$\|\mathbf{d}_k\omega\|_p + \|\delta_{k-1}\omega\|_p \approx_p \|(\mathbf{d}_k + \delta_{k-1})\omega\|_p \tag{4.1}$$

noting that $\mathbf{d}_k\omega \in C_c^\infty(\Lambda^{k+1} TM)$ and $\delta_{k-1}\omega \in C_c^\infty(\Lambda^{k-1} TM)$. □

Our proof of the R -bisectoriality of D will be based on R -gradient bounds to which we turn next. We begin with a lemma.

Lemma 4.5 *For all $1 < p < \infty$ and $k = 0, 1, \dots, n$, we have $\mathbf{D}_p(L_k^{1/2}) \subseteq \mathbf{D}_p(\mathbf{d}_k) \cap \mathbf{D}_p(\delta_{k-1})$.*

Proof Pick $\omega \in \mathbf{D}_p(L_k^{1/2})$ arbitrarily. As $C_c^\infty(\Lambda^k TM)$ is dense in $\mathbf{D}_p(L_k^{1/2})$ by Lemma 4.3, we can find a sequence $(\omega_i)_i$ of k -forms in this space converging to ω in $\mathbf{D}_p(L_k^{1/2})$. By [9, Theorem 5.1] we then find, for all i, j ,

$$\|\omega_i - \omega_j\|_p + \|\mathbf{d}_k\omega_i - \mathbf{d}_k\omega_j\|_p \lesssim \|\omega_i - \omega_j\|_p + \|L_k^{1/2}\omega_i - L_k^{1/2}\omega_j\|_p \tag{4.2}$$

which shows that $(\omega_i)_i$ is Cauchy in $\mathbf{D}_p(\mathbf{d}_k)$. By the closedness of \mathbf{d}_k , we then find that this sequence converges to some $\eta \in \mathbf{D}_p(\mathbf{d}_k)$. As in the proof of Theorem 4.4, we show that $\omega = \eta$. It follows that $\omega \in \mathbf{D}_p(\mathbf{d}_k)$.

This proves the inclusion $\mathbf{D}_p(L_k^{1/2}) \subseteq \mathbf{D}_p(\mathbf{d}_k)$. The inclusion $\mathbf{D}_p(L_k^{1/2}) \subseteq \mathbf{D}_p(\delta_k)$ is proved in the same way. □

Thanks to the lemma, the operators

$$d_k L_k^{-1/2} : R_p(L_k^{1/2}) \rightarrow R_p(d_k), \quad L_k^{1/2} \omega \mapsto d_k \omega$$

and

$$\delta_{k-1} L_k^{-1/2} : R_p(L_k^{1/2}) \rightarrow R_p(\delta_{k-1}), \quad L_k^{1/2} \omega \mapsto \delta_{k-1} \omega$$

are well defined, and by Theorem 4.4 combined with the equivalence of norms (4.1) they are in fact L^p -bounded.

It also follows from the lemma that the operators $d_k(I + t^2 L_k)^{-1}$ and $\delta_{k-1}(I + t^2 L_k)^{-1}$ are well defined and L^p -bounded for all $t \in \mathbb{R}$; indeed, just note that $D_p(L_k) \subseteq D_p(L_k^{1/2}) \subseteq D_p(d_k) \cap D_p(\delta_{k-1})$. The next proposition asserts that these operators form an R -bounded family:

Proposition 4.6 (*R*-gradient bounds) *Let Hypothesis 3.5 hold. For all $1 < p < \infty$ and $k = 0, 1, \dots, n$ the families of operators*

$$\{t d_k (I + t^2 L_k)^{-1} : t > 0\}$$

and

$$\{t \delta_{k-1} (I + t^2 L_k)^{-1} : t > 0\}$$

are both R -bounded.

Proof We will only prove that the first set is R -bounded. The R -boundedness of the other set is proved in exactly the same way.

For $t > 0$, standard functional calculus arguments show that

$$\begin{aligned} t d_k (I + t^2 L_k)^{-1} &= (d_k L_k^{-1/2}) \left((t^2 L_k)^{1/2} (I + t^2 L_k)^{-1} \right) \\ &= (d_k L_k^{-1/2}) \left(\psi (t^2 L_k) \right), \end{aligned}$$

where $\psi(z) = \frac{\sqrt{z}}{1+z}$. Observe that $\psi \in H^1(\Sigma_\vartheta^+) \cap H^\infty(\Sigma_\vartheta^+)$ for any $\vartheta \in (0, \frac{1}{2}\pi)$. By a result of [39] (see also [40, Chapter 12]) the set

$$\{\psi (t^2 L_k) : t > 0\}$$

is R -bounded in $\mathcal{L}(L^p(\Lambda^k T M, m))$. Since $d_k L_k^{-1/2}$ is bounded, it follows that the set

$$\{(d_k L_k^{-1/2}) (\psi (t^2 L_k)) : t > 0\}$$

is R -bounded in $\mathcal{L}(L^p(\Lambda^k TM, m), L^p(\Lambda^{k+1} TM, m))$. This concludes the proof. \square

In order to prove the R -bisectoriality of the Hodge–Dirac operator, we need one more lemma, which concerns commutativity rules used in the computation of the resolvents of the Hodge–Dirac operator.

Lemma 4.7 *For all $1 \leq p < \infty, k = 0, 1, \dots, n$, and $t > 0$, the following identities hold on $D_p(d_k)$ and $D_p(\delta_k)$, respectively:*

$$(I + t^2 L_{k+1})^{-1} d_k = d_k (I + t^2 L_k)^{-1}$$

and

$$(I + t^2 L_k)^{-1} \delta_k = \delta_k (I + t^2 L_{k+1})^{-1}.$$

Similar identities hold with $(I + t^2 L_{k+1})^{-1}$ replaced by $(I + t^2 L_{k+1})^{-1/2}$ or P_t^{k+1} .

Proof We will only prove the first identity; the second is proved in a similar manner. The corresponding results for P_t^{k+1} can be proved along the same lines, or deduced from the results for the resolvent using Laplace inversion, and in turn the identities involving $(I + t^2 L_{k+1})^{-1/2}$ follow from this.

For k -forms $\omega \in C_c^\infty(\Lambda^k TM, m)$, we have $P_t^{k+1} d_k \omega = d_k P_t^k \omega$ (see [9]). Here, the right-hand side is well defined as $P_t^k \omega \in D_p(L_k) \subseteq D_p(d_k)$ (which holds by analyticity of P_t^k). Now pick $\omega \in D_p(d_k)$ and let $\omega_n \in C_c^\infty(\Lambda^k TM)$ be a sequence converging to $\omega \in D_p(d_k)$. Such a sequence exists by the definition of d_k as a closed operator. Thus $\omega_n \rightarrow \omega$ and $d_k \omega_n \rightarrow d_k \omega$ in $L^p(\Lambda^k TM, m)$ respectively $L^p(\Lambda^{k+1} TM, m)$. The boundedness of P_t^k and P_t^{k+1} then implies that $P_t^k \omega_n \rightarrow P_t^k \omega$ and $P_t^{k+1} d_k \omega_n \rightarrow P_t^{k+1} d_k \omega$ in $L^p(\Lambda^k TM, m)$ respectively $L^p(\Lambda^{k+1} TM, m)$. As $P_t^{k+1} d_k \omega_n = d_k P_t^k \omega_n$ for every n , and as the left-hand side converges, we obtain that $d_k P_t^k \omega_n$ converges in $L^p(\Lambda^{k+1} TM, m)$. The closedness of d_k shows that $P_t^k \omega \in D_p(d_k)$ and that $P_t^{k+1} d_k \omega = d_k P_t^k \omega$.

Taking Laplace transforms on both sides, we obtain

$$(t^{-2} + L_{k+1})^{-1} d_k \omega = d_k (t^{-2} + L_k)^{-1} \omega$$

from which one deduces the desired identity. \square

Remark 4.8 Although we will not need it, we point out the following consequence of the preceding results: for all $k = 0, 1, \dots, n$ we have

$$D_p(D_k) = D_p(d_k) \cap D_p(\delta_{k-1})$$

with equivalent norms.

To prove this, we note that Lemma 4.5, combined with the domain equality of Theorem 4.4, gives the inclusion $D_p(D_k) \subseteq D_p(d_k) \cap D_p(\delta_{k-1})$. To prove the reverse inclusion we argue as follows: For $\omega \in C_c^\infty(\Lambda^k TM)$ we observed in (4.1) that

$$\|D_k \omega\|_p \approx_{p,k} \|d_k \omega\|_p + \|\delta_{k-1} \omega\|_p. \tag{4.3}$$

By Theorem 4.4 and the estimate (4.2) used in the proof of Lemma 4.5 and its analogue for δ_{k-1} , this equivalence of norms extends to arbitrary $\omega \in D_p(L_k^{1/2})$.

Now let $\omega \in D_p(d_k) \cap D_p(\delta_{k-1})$ be arbitrary. For $t > 0$ we have $P_t^k \omega \in D_p(L_k) \subseteq D_p(L_k^{1/2})$, so that

$$\|D_k P_t^k \omega\|_p \approx_{p,k} \|d_k P_t^k \omega\|_p + \|\delta_{k-1} P_t^k \omega\|_p. \tag{4.4}$$

By Lemma 4.7 we have $\|d_k P_t^k \omega\|_p = \|P_t^{k+1} d_k \omega\|_p \rightarrow \|d_k \omega\|_p$ as $t \downarrow 0$, and similarly $\|\delta_{k-1} P_t^k \omega\|_p \rightarrow \|\delta_{k-1} \omega\|_p$. As a consequence, $P_t^k \omega \rightarrow \omega$ in $D_p(d_k) \cap D_p(\delta_{k-1})$. By (4.4) and the closedness of D_k , we then also have $\omega \in D_p(D_k)$ and $P_t^k \omega \rightarrow \omega$ in $D_p(D_k)$. We conclude that $D_p(d_k) \cap D_p(\delta_{k-1}) \subseteq D_p(D_k)$ and that (4.3) holds for all $\omega \in D_p(d_k) \cap D_p(\delta_{k-1})$.

We now obtain the following result.

Theorem 4.9 (*R-bisectoriality of D*) *Let Hypothesis 3.5 hold. For all $1 < p < \infty$ the Hodge–Dirac operator D is R-bisectorial on $L^p(\Delta TM, m)$.*

Proof We will start by showing that the set $\{it : t \in \mathbb{R}, t \neq 0\}$ is contained in the resolvent set of D . We will do this by showing that $I - itD$ has a two-sided bounded inverse given by

$$\begin{pmatrix} (I + t^2 L_0)^{-1} & it\delta_0 (I + t^2 L_1)^{-1} & & & \\ itd_0 (I + t^2 L_0)^{-1} & (I + t^2 L_1)^{-1} & it\delta_1 (I + t^2 L_2)^{-1} & & \\ & \ddots & \ddots & \ddots & \\ & & itd_{n-2} (I + t^2 L_{n-2})^{-1} & (I + t^2 L_{n-1})^{-1} & it\delta_{n-1} (I + t^2 L_n)^{-1} \\ & & & itd_{n-1} (I + t^2 L_{n-1})^{-1} & (I + t^2 L_n)^{-1} \end{pmatrix}$$

with zeroes in the remaining entries away from the three main diagonals. By the R -sectoriality of L_k (Proposition 3.11) and the R -gradient bounds (Proposition 4.6) all entries are bounded. It only remains to check that this matrix defines a two-sided inverse of $I - itD$. Let us first multiply with $I - itD$ from the left. It suffices to compute the three diagonals, as the other elements of the product clearly vanish. It is easy to see that the k -th diagonal element becomes

$$\begin{aligned} & t^2 d_{k-2} \delta_{k-2} (I + t^2 L_{k-1})^{-1} + (I + t^2 L_{k-1})^{-1} + t^2 \delta_{k-1} d_{k-1} (I + t^2 L_{k-1})^{-1} \\ & = (I + t^2 L_{k-1}) (I + t^2 L_{k-1})^{-1} = I \end{aligned} \tag{4.5}$$

using that $L_{k-1} = -(\mathfrak{d}_{k-2}\delta_{k-2} + \delta_{k-1}\mathfrak{d}_{k-1})$; obvious adjustments need to be made for $k = 1$ and $k = n$. For the two other diagonals, it is easy to see that one gets two terms which cancel.

To make this argument rigorous, note that both $\mathfrak{d}_{k-2}\delta_{k-2}(I + t^2L_{k-1})^{-1}$ and $\delta_{k-1}\mathfrak{d}_{k-1}(I + t^2L_{k-1})^{-1}$ are well defined as bounded operators, so that it suffices to check the computations for $\omega \in C_c^\infty(\Lambda TM)$. The asserted well-definedness and boundedness of the first of these operators can be seen by noting that

$$\mathfrak{d}_{k-2}\delta_{k-2} \left(I + t^2L_{k-1} \right)^{-1} = \mathfrak{d}_{k-2} \left(I + t^2L_{k-2} \right)^{-1/2} \circ \delta_{k-2} \left(I + t^2L_{k-1} \right)^{-1/2},$$

using Lemma 4.7; the boundedness of the other operator follows similarly.

If we multiply with $I - itD$ from the right and use Lemma 4.7, we easily see that the product is again the identity.

It remains to show that the set $\{it(it - D)^{-1} : t \neq 0\} = \{(it - D)^{-1} : t \neq 0\}$ is R -bounded. For this, observe that the diagonal entries are R -bounded by the R -sectoriality of L_k . The R -boundedness of the other entries follows from the R -gradient bounds (Proposition 4.6). Since a set of operator matrices is R -bounded precisely when each entry is R -bounded, we conclude that D is R -bisectorial. \square

Proposition 4.10 *Let $1 < p < \infty$. Then $D^2 = L$ as densely defined closed operators on $L^p(\Lambda TM, m)$.*

This result may seem obvious by formal computation, but the issue is to rigorously justify the matrix multiplication involving products of unbounded operators.

Proof It suffices to show that $\mathbf{D}_p(L) \subset \mathbf{D}_p(D^2)$ and $D^2(I + t^2L)^{-1} = L(I + t^2L)^{-1}$, or equivalently, $(\mathfrak{d}_{k-1}\delta_{k-1} + \delta_k\mathfrak{d}_k)(I + t^2L_k)^{-1} = L_k(I + t^2L_k)^{-1}$ for all $k = 0, 1, \dots, n$. The rigorous justification of the equivalent identity (4.5) has already been given in the course of the above proof.

If $\omega \in \mathbf{D}_p(D^2)$, then by Lemma 4.7 we find

$$D^2 \left(I + t^2L \right)^{-1} \omega = \left(I + t^2L \right)^{-1} D^2 \omega \rightarrow D^2 \omega, \quad t \rightarrow 0.$$

Here we used that $(I + t^2L)^{-1}$ converges to I strongly as $t \rightarrow 0$ by the general theory of sectorial operators. But then we find that

$$L \left(I + t^2L \right)^{-1} \omega = D^2 \left(I + t^2L \right)^{-1} \omega \rightarrow D^2 \omega, \quad t \rightarrow 0.$$

As $(I + t^2L)^{-1}\omega \rightarrow \omega$ as $t \rightarrow 0$, the closedness of L gives $\omega \in \mathbf{D}(L)$ and $L\omega = D^2\omega$. \square

We are now ready to prove that D has a bounded H^∞ -calculus on $L^p(\Lambda TM, m)$.

Theorem 4.11 (Bounded H^∞ -functional calculus for D) *Let Hypothesis 3.5 hold. For all $1 < p < \infty$ the Hodge–Dirac operator D on $L^p(\Lambda TM, m)$ has a bounded H^∞ -calculus on a bisector.*

Proof With all the preparations done, this now follows by combining Proposition 2.3 with Theorems 3.12 and 4.9 and Proposition 4.10. \square

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