General relativity and the peeling-off behaviour of the gravitational field in an asymptotically flat space-time

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Abstract

This thesis investigates the peeling-off property of zero rest-mass fields in asymptotically flat space-times, as described by Penrose. Unlike other literature, this work is designed to be accessible to undergraduate physics or mathematics students. It provides a more detailed derivation including numerous explicit calculations, which is appropriate for the assumed background knowledge. The thesis is structured to build from fundamental mathematical concepts to advanced applications in general relativity. The mathematical concepts introduced are topology, compactifications and differential geometry. The main body of the work applies these mathematical tools to the Minkowski space-time and extends the analysis to more general asymptotically flat space-times.

A zero rest-mass field of spin s determines at each event in space-time a set of $2s$ principal null directions. These are related to the radiative behaviour of the field. These directions exhibit the 'peeling-off' behaviour: to order r^{-k-1} $(k = 0, \ldots, 2s)$, $2s - k$ directions coincide radially, where r is an affine parameter on a null geodesic. Criteria for asymptotically simple and asymptotically flat space-times are given, and this peeling-off behaviour is studied in these settings. This involves the introduction of points 'at infinity' through a conformal completion. These points at infinity then become an ordinary hypersurface $\mathscr{I}$ to the conformally completed manifold. The conformal transformations of zero rest-mass fields are investigated so that their behaviour at infinity can be studied at this hypersurface. If the transformed field is continuous at $\mathscr{I}$, we find that the peeling-off property holds. If the Einstein empty-space equations without cosmological constant hold near the boundary, the transformed gravitational field is found to be continuous at the boundary, so that the peeling-off property holds.

Introduction

The main purpose of this thesis is to come to understand a paper by Penrose [1], describing the so-called peeling-off property of zero rest-mass fields in asymptotically flat space-times. This thesis, though mostly a literature study, is still unique in the following sense: Introductory texts to general relativity, like [2] and [3] introduce a lot of the concepts needed to do general relativity, starting from differential geometry. They briefly touch upon asymptotic behaviour, but do not relate it explicitly to the mathematical concept of compactification. They also do not go in the direction of the peeling-off behaviour, choosing to focus on other topics. Of course, the peeling-off behaviour is discussed in papers by Penrose like [1], [4] and by other authors, like Sachs, who first identified this behaviour [5]. These papers, however, assume a lot of background knowledge about differential geometry, spinor calculus and conformal geometry. This thesis is like a bridge between the books and the papers and gathers everything in one place. It starts from a point that an undergraduate student towards the end of their degree in physics or mathematics should be able to understand. However, the foundations are then used to explore the peeling-off behaviour. Unlike the papers on which this thesis is based, the derivations presented here are more in-depth and contain many explicit calculations.

To understand this peeling-off behaviour, we first need to understand how space and time work. This leads us to the theory of general relativity, formulated by Albert Einstein in 1915. This theory revolutionized our understanding of gravitation, replacing Newton's law of universal gravitation with a new framework. At its core, general relativity states that gravity is not a force acting at a distance, but rather a manifestation of the curvature of space-time caused by mass and energy. This theory elegantly describes how massive objects warp the fabric of space-time, creating what we perceive as gravitational attraction. The profound implications of general relativity have led to numerous groundbreaking discoveries and solutions to previously intractable problems in physics and astronomy: it was able to accurately predict the precession of Mercury's orbit, while Newton's theory was not [6]; it predicted gravitational lensing [7], where light from a distant star is bent by the gravitational field of a massive object, such as a galaxy, lying between the star and the observer; it predicts black holes, enigmatic objects that often appear in pop culture, with gravitational fields so intense that not even light can escape and of which we've recently seen the first picture [8]; and the theory did so much more, from shaping our current understanding of cosmology to making the GPS systems in our phones work.

The theory is built up starting in section 1 from topology, the part of mathematics concerned with the properties of a geometric object that are preserved under continuous deformations. Examples of these deformations are stretching, twisting, crumpling, and bending. These deformations do not close holes, open holes, tear or glue the object, or make it pass through itself. Using this foundation of topology, we move on to section 2 to discuss topological compactifications. Compactness is a topological property that (in $\mathbb{R}^n$) states that the set is bounded and that the boundary of the set is included in the set. Compactifying is then a procedure to add one or more points to a set in such a way that the new set doesn't get 'too large' compared to the old one and that the new is compact. This compactification is later used to make space-time compact to make it easier to study. Then we move onto section 3, where the basics of differential geometry are discussed. Differential geometry is the mathematical discipline that studies the geometry of smooth shapes and smooth spaces. We define vectors, one-forms and tensors in these curved spaces. We also quickly introduce spinors since this framework is used in Penrose's paper. In general relativity, we assume space-time to be a smooth space, and so differential geometry is the mathematical backbone of all of the physics of general relativity that we introduce in section 4. In this chapter, we build the bridge between the mathematics developed in the previous chapter and the physics of general relativity. This is a big chapter, since we discuss many fundamentals of general relativity.

The most important concept is the covariant derivative, the analogue of the gradient function that we are used to working with in the flat space $\mathbb{R}^n$. In section 5, we apply our theory of compactifications from section 2 to the (flat) Minkowski space-time that was introduced in section 4. We follow the same method as in [2], but a to my knowledge new proof that this procedure is a compactification is given. We also introduce the concept of a conformal (angle-preserving) map. The concept of compactifying by using a conformal map is then expanded to more general space-times that allow a conformal compactification with similar properties to the Minkowski case in section 6. We call space-times that allow this procedure and satisfy some other properties asymptotically flat. The cosmological constant Λ also plays a fundamental role in this characterisation. We also derive the transformation of some objects under conformal transformations. Finally, in section 7, we reach the end goal where we prove the peeling-off property that Penrose writes about. Specifically, the peeling-off property of zero rest-mass fields in asymptotically flat space-times. Though the proof follows the same arguments as Penrose's proof, it is way more explicit in its execution.

Zero rest-mass fields are, like the name suggests, fields that themselves do not carry any mass. The electromagnetic and gravitational fields are physical examples of such fields and they satisfy the zero rest-mass equation. These fields exhibit a certain asymptotic behaviour, which Penrose calls the peeling-off property. The peeling-off property is best described in terms of principal null directions associated with the field. These are a special set of $2s$ directions along the light cone (null directions), defined at each event in space-time, for any spin s zero rest-mass field along which the field is zero. These principal null directions are only undefined at events at which the field vanishes (in which case, in a sense, every null direction at the event is a principal null direction). However, under certain circumstances, some of them may coincide and if all $2s$ of them coincide, the field is called a null field. If r is an affine parameter on a null geodesic, then it turns out that the part of the field that falls off as r^{-1}, the far zone or radiation field, is effectively null. If we proceed inwards from infinity, or (equivalently) examine the field to increasing orders of accuracy, we encounter successive zones where $r^{-2}, r^{-3}, \cdots$ terms become important in turn. The peeling-off property that we then discover, is that the field in the r^{-k-1} zone ($k = 0, 1, 2, ..., 2s$) has, to this degree of approximation at least $2s - k$ coincident principal null directions pointing along the geodesic (so in the direction of the radiation).

The argument we will use, uses the fact that the zero rest-mass field equations can, with suitable interpretations, be regarded as conformally invariant. This conformal invariance is discussed in section 6. Because of this property, the space-time may be considered from the point of view of its conformal structure, rather than its metric structure, for the study of zero rest-mass fields. Under the conformal structure, we can treat infinity as an ordinary hypersurface boundary to the space-time. Behaviour in the neighbourhood of this hypersurface can then be translated to asymptotic behaviour in the real, physical space. We choose a fictitious conformal metric so that physical infinity becomes a finite hypersurface. Calculations are then carried out using this new metric. This then means that the asymptotic behaviour for zero rest-mass fields can be studied by local techniques, instead of global ones. The peeling-off property of the principal null direction is shown to be equivalent to a requirement that the field is both finite and continuous on the hypersurface representing infinity. It is also shown that this finiteness requirement can, in the case of the gravitational field, be deduced from the relevant field equations and more primitive requirements concerning the asymptotic nature of the space-time itself. We show that these requirements are satisfied if the space-time is asymptotically flat.

Contents

1 Topology

General relativity works by letting gravity be the consequence of the curvature of space-time, rather than a force field acting upon a body. Therefore we will need to develop mathematics that are able to describe such curved spaces. We assume, because it accords with our normal experience, that the curvature of our space-time is smooth. A model that works very well to describe smoothly curved spaces is the so-called smooth manifold, which has its roots in topology. This is why the first chapter is dedicated to discussing some basic topology. We start with the basics, by defining a topological space and a topology. Then we look into how these concepts carry over to other spaces like subspaces or product spaces. We also discuss some common concepts in topological spaces and some topological properties that a model for space-time must have. We finish with a discussion on how to relate one topological space to a different (but possibly equivalent) topological space.

A topological space provides a very general framework to discuss concepts like continuity and convergence without relying on a specific notion of distance. This generality makes topology a powerful tool in many areas of mathematics, including the study of manifolds in space-time models.

Definition 1.1. (Topological space). A topological space is a set M, together with a collection T of subsets $X \subseteq M$. We call a subset $X \subseteq M$ open if it is in T. The open sets are required to satisfy the following rules:

1. The empty set and M itself are open.
2. Unions of open sets are open.
3. Finite intersections of open sets are open.

We call this collection T the topology, and denote the topological space by (M, T), or just by M if the topology is understood from context.

Subspaces of topological spaces are also of great interest. When we have a topological space and consider a subset of it, we can naturally induce a topology on this subset from the parent space. This allows us to treat the subset as a topological space in its own right, equipped with a structure that is consistent with the larger space.

Definition 1.2. (Subspace topology). Given a topological space (M, T) and a subset $X \subseteq M$, the subspace topology T_X on X is defined by $T_X = \{X \cap U : U \in T\}$. So a subset of X is open in the subspace topology if and only if it is the intersection of X with an open set in (M, T). A subspace equipped with the subspace topology are topological spaces on their own, and are called subspaces. Subsets of topological spaces are usually assumed to be equipped with the subspace topology unless otherwise stated.

The concept of subspace topology is particularly useful when we want to focus on a specific part of a topological space while preserving the topological properties. This construction ensures that the subset inherits the same notions of open sets and other topological features from the parent space.

When dealing with multiple topological spaces, we often need to consider their Cartesian product. The product topology allows us to define a topology on this product in a way that is consistent with the topologies of the individual spaces. This is essential for studying multi-dimensional spaces and understanding how topological properties behave in higher dimensions.

Definition 1.3. (Product topology). Let I be a nonempty index set and let M_i be a topological space. Let the Cartesian product of sets M_i be denoted $\prod_{i \in I} M_i$. The open sets in the product topology are arbitrary unions of sets of the form $\prod_{i \in I} U_i$, where each U_i is open in M_i and $U_i \neq M_i$ for only finitely many i.

If the product is finite, this coincides with the box topology, where a set $U \times V$ in $M \times N$ is open if and only if U is open in M and V is open in N. Since we will only be concerned with finite spaces in this work, this is a very intuitive definition to work with. Understanding open and closed sets is fundamental in topology. Closed sets, defined as the complements of open sets, play a crucial role in many topological concepts, such as closure and compactness (to which we will dedicate the entirety of section 2). The closure of a set, for instance, is the smallest closed set that contains the given set, providing a way to extend the set to include its boundary points.

Definition 1.4. (Closed set). A set $X \subseteq M$ is called closed if its complement $X^c = M \backslash X$ is open.

Definition 1.5. (Closure). The closure of a set X, denoted by $\overline{X}$, is the intersection of all closed sets containing X.

The notion of neighbourhoods helps us formalize the idea of points being 'near' each other. This is especially important when discussing continuity, as it allows us to define continuity in terms of open sets rather than relying on a specific distance function.

Definition 1.6. ((Open) neighbourhood). An open neighbourhood of x is an open set $U \subseteq M$ that contains x. A neighbourhood $N \subseteq M$ of x is a set that contains an open neighbourhood of x. In other words, there exists an open subset $U \subseteq N$ with $x \in U \subseteq N$.

Now we introduce a fairly abstract property, which we will equip our model of space-time with. This property is often (implicitly) used in physics because it seems to accord with normal experience.

Definition 1.7. (Hausdorff space). A topological space M is called Hausdorff if for any two distinct points $x, y \in M$, there exist open neighbourhoods U_x of x and U_y of y which do not intersect, $U_x \cap U_y = \emptyset$.

This condition is illustrated by the pun that in Hausdorff spaces any two points can be "housed off" from each other by open sets. One of the most important consequences of this condition is that limits are unique when they exist [9]. The uniqueness of limits makes the whole of calculus work. For example, if limits are not unique, a function couldn't have a unique derivative at a point. Now we introduce another property which our space-time model will have.

Definition 1.8. (Connectedness). A set (or manifold) is called connected if it is not the union of two disjoint nonempty open sets.

Our models will have this property because if our space weren't connected, we wouldn't have any way to find out about any disconnected part.

The next topic we will talk about is relating one topological space to another. When talking about continuity in terms of metric spaces, one of the definitions that turns out to be equivalent is the open set definition. However, in a topological space, we do not have a metric so we cannot define continuity inherently through distance like we do for metric spaces. We do, however, have a notion of open sets. In topological spaces, continuity of a function is defined by the preimage of every open set being open, which generalizes the familiar concept from metric spaces.

Definition 1.9. (Continuity). Let M and N be topological spaces. The map $\phi : M \to N$ is continuous if the preimage $\phi^{-1}(U) \subseteq M$ of any open set $U \subseteq N$ is open in M.

Just like numbers can be equal, we can also talk about 'equal' topological spaces. In this case, these spaces usually aren't exactly equal, meaning their elements are not represented in exactly the same way. But the spaces have the same topological properties and so are topologically indistinguishable. We call such topological spaces homeomorphic, and the map identifying their properties a homeomorphism.

Definition 1.10. (Homeomorphism). Let M and N be topological spaces. A function $f : M \to N$ is a homeomorphism if it satisfies the following 3 properties:

1. f is a bijection.
2. f is continuous.
3. The inverse function f^{-1} is continuous.

We see that homeomorphisms identify points from M with points in N, but also open sets in M with open sets in N. Since the sets M and N then have 'the same open sets', they have 'the same topology' and are therefore topologically indistinguishable. An example of this concept is the homeomorphism between $\mathbb{R}^4$ minus a line and $\mathbb{R}^2 \times \mathbb{S}^2$. This example demonstrates how spaces that appear different at first glance can actually be topologically equivalent.

Example 1.11. ($\mathbb{R}^4$ minus a line is homeomorphic to $\mathbb{R}^2 \times \mathbb{S}^2$).

Proof. Let $(r, \mathbf{s})$ denote an element in $\mathbb{R} \times \mathbb{S}^2$. Using the fact that $\mathbb{S}^2 \subseteq \mathbb{R}^3$, let $g : \mathbb{R} \times \mathbb{S}^2 \to \mathbb{R}^3 \backslash \{(0, 0, 0)\}$ be given by $g(r, \mathbf{s}) = e^r \mathbf{s}$. Since e^x bijective on $(0, \infty)$, continuous and has a continuous inverse $\ln(x)$, as does nonzero scalar multiplication of a nonzero vector, g is a homeomorphism. Now let $f : \mathbb{R} \times \left(\mathbb{R}^3 \backslash \{(0, 0, 0)\}\right) \to \mathbb{R}^4 \backslash \{(t, x, y, z) : x^2 + y^2 + z^2 = 0\}$ be given by $f(t, x, y, z) = (t, x, y, z)$. This is also a homeomorphism. Then $h(t, r, \mathbf{s}) = f(t, g(r, \mathbf{s})) = (t, e^r \mathbf{s})$ is clearly a homeomorphism from $\mathbb{R} \times \left(\mathbb{R} \times \mathbb{S}^2\right) \to \mathbb{R}^4 \backslash \{(t, x, y, z) : x^2 + y^2 + z^2 = 0\}$. The inverse function $h^{-1} : \mathbb{R}^4 \backslash \{(t, x, y, z) : x^2 + y^2 + z^2 = 0\} \to \mathbb{R}^2 \times \mathbb{S}^2$ is given by $h^{-1}(t, x, y, z) = (t, \ln \sqrt{x^2 + y^2 + z^2} \mathbf{s})$. □

The last concept we will discuss here is the embedding. Embeddings allow us to view a topological space as a subspace of another, possibly more familiar, topological space. This can simplify the study of the original space by leveraging the properties of the larger space. An embedding is a function that is injective and homeomorphic onto its image, again ensuring that the topological structure is preserved.

Definition 1.12. (Embedding). Let M and N be topological spaces. A function $f : M \to N$ is called an embedding if the following properties hold:

1. $f : M \to N$ is injective.
2. $f : M \to f(M)$ is a homeomorphism if $f(M)$ carries the subspace topology inherited from N.

2 Compactifications

In this section, we treat the notion of a compact topological space. Compactness seeks to generalise the notion of a closed and bounded subset of Euclidean space. The idea is that a compact space has no holes or missing endpoints, so it must include all limiting values of points. After this concept is introduced, we will prove some theorems about compact sets that will make working with them easier in the later chapters. Lastly, we develop the central concept of this chapter, compactification: a process to make compact sets out of non-compact ones. Some concrete examples as well as the more general one-point compactification will be worked out to illustrate this concept.

Because boundedness is hard to define in general topological spaces (since we don't have a concept of distance to work with), the definition of a compact set is given in terms of covers. Intuitively, a cover of a set is a collection of sets whose union contains that set in question.

Definition 2.1. (Cover). Let K be a subset of M, and let $\{U_i; i \in I\}$ be a (not necessarily finite) collection of subsets of M. We say that the sets U_i cover K if $K \subseteq \bigcup_{i\in I} U_i$.

As a trivial example, the set K covers itself. The concept of a cover is central to understanding compactness. Now that we know what a cover is, we can define a compact set for a general topological space. We first state the definition:

Definition 2.2. (Compact set). A subset $K \subseteq M$ is called compact if every open cover of K has a finite subcover. In other words, if $\bigcup_{i\in I} U_i$ is an open cover of K (all U_i are open), there exists a finite subset $I_f \subseteq I$ such that $\bigcup_{i\in I_f} U_i$ is also a cover of K.

This is a natural definition because it is equivalent to being closed and bounded in $\mathbb{R}^n$ by the Heine-Borel theorem. It is not the most intuitive definition however. We will only be working with Hausdorff spaces, where any compact set is also closed [10]. So at least, the closedness intuitively matches up with our intuition. I try to make make the boundedness intuitive for myself by letting it roughly be captured by the fact that we must be able to find a finite (and so not infinite) subcover.

We now discuss some useful properties of compact spaces, starting with subsets of compact sets. Understanding these properties helps us see how compactness behaves under various operations and transformations, which will be helpful for proving more complex results in section 5.

Theorem 2.3. (A closed subset of a compact set is compact). Let M be a topological space, let $X \subseteq Y \subseteq M$ with Y compact and X closed. Then X is compact.

Proof. Let U be an open cover of X. Since X is closed, X^c is open. Then $U \cup X^c$ is an open cover of M and therefore of Y. Since Y is compact, we can extract a finite subcover $\bigcup_{i\in I_f} U_i \cup X^c$ from this open cover, so I_f is a finite set. Since $X \cap X^c = \emptyset$, $X \subseteq \bigcup_{i\in I_f} U_i$ and so we have found a finite subcover of X. Hence X is compact. □

We find out that compactness is really something that is in a sense 'inherent' to the topological space. Specifically, continuous functions on topological spaces do not affect this property. This will be helpful when we want to leverage results from a topological space that we already know in studying a space that we don't yet know.

Theorem 2.4. (Compactness is a topological property). Let M and N be topological spaces, with M compact. Let $f : M \to N$ be a continuous function. Then $f(M)$ is compact.

Proof. Let $U = \bigcup i \in IU_i$ be a (possibly infinite) open cover of $f(M)$. Since f is continuous, we have that $f^{-1}(U_i)$ is open for all U_i. Now, for any $m \in M$, we have that $f(m) \in U_i$ for some $i \in I$. Therefore, the set $\{f^{-1}(U_i) : i \in I\}$ is an open cover of M. Because M is compact, it has a finite subcover, so that $M \subseteq \bigcup_{i \in I_f} f^{-1}(U_i)$ for some finite subset $I_f \subseteq I$. Therefore, $\bigcup i \in I_f U_i$ is a finite subcover of $f(M)$ and so $f(M)$ is compact. □

Often we want to take the Cartesian product of sets to create new sets, a common example of such a product being $\mathbb{R}^n$. We could prove the compactness of these product spaces separately every time, but it turns out that this is not necessary if we already know that the sets we are taking the product of are already compact.

Theorem 2.5. (A product of compact spaces is compact). Let $\prod_{i \in I} M_i$ be a finite Cartesian product equipped with the product topology. Let $X_i \subseteq M_i$. Then $\prod_{i \in I} X_i$ is compact if and only if all X_i are compact.

To prove the compactness of product spaces, we first need to establish a helpful lemma known as the tube lemma. It has this name because it allows one to conclude that any open subset containing a slice contains an open cylinder that contains that slice.

Lemma 2.6. (Tube lemma). Let M and N be topological spaces. Let X and Y be compact subsets of M and N respectively. If A is an open set containing $X \times Y$, there exists a U open in M and a V open in N such that $X \times Y \subseteq U \times V \subseteq A$.

Proof. The proof is based on [11]. Consider $(x, y) \in X \times Y \subseteq A$. Because A is open, there are open sets $U_{x,y} \subseteq X$ and $V_{x,y} \subseteq Y$ such that $(x, y) \in U_{x,y} \times V_{x,y} \subseteq A$. For any $x \in X$, the set $\{V_{x,y} : y \in Y\}$ is an open cover of Y and hence this cover has a finite subcover. That means there is a finite set $Y_0(x) \subseteq Y$ such that $V_x = \bigcup_{y \in Y_0(x)} V_{x,y}$ contains Y. Now V_x is open in Y because it is a union of open sets. For every $x \in X$, let $U_x = \bigcap_{y \in Y_0(x)} U_{x,y}$, which is also open since the intersection is only over a finite amount of open sets. We now see that $\{x\} \times Y \subseteq U_x \times V_x \subseteq A$. Now let $X_0 \subseteq X$ be a finite subset such that $X \subseteq \bigcup_{x \in X_0} U_x = U$ and $V = \bigcap_{x \in X_0} V_x$. Then by the reasoning from before, it follow that $X \times Y \subseteq U \times V \subset A$ with U and V open. □

With this lemma in hand we are ready to prove theorem 2.5.

Proof. The first part of the proof proof is based on [12], the second on [13]. Let $M \times N$ be a Cartesian product. Let $X \subseteq M$ and $Y \subseteq N$ be such that $X \times Y$ is compact. Let the projection function $p_x : X \times Y \to X$ be defined by $p_x(x, y) = x$. Now let $U \subseteq X$ be open in X. Then $p_x^{-1}(U) = U \times Y$, which is an open set in the product topology. Hence p_x is continuous. Since compactness is conserved under continuous images by theorem 2.4, $X = p_x(X \times Y)$ is compact. Similar reasoning yields that Y is compact.

Now, let $X \subseteq M$ and $Y \subseteq N$ be compact spaces. Now let U be an open cover of $X \times Y$ and let $x \in X$. Now $\{x\} \times Y$ is homeomorphic to Y and is therefore compact. Because U is an open cover of $X \times Y$, it is also an open cover of $\{x\} \times Y$. Hence, there is a finite subcover $\bigcup_{i \in I_f} U_i$ that still covers $\{x\} \times Y$. If we now apply the tube lemma (lemma 2.6) with $X = \{x\}, Y = Y$ and $A = \bigcup_{i \in I_f} U_i$, we get an open neighbourhood W_x of x so that $\{x\} \times Y \subseteq W_x \times Y \subseteq A$. Now let $W = \{W_x : x \in X\}$. This is an open cover of X and hence there exists a finite subcover such that $X \subseteq W_f = \bigcup_{x \in X_f} W_x$, where $X_f \subseteq X$ is a finite set. Now $X \times Y = \bigcup_{x \in X_f} W_x \times Y$. But now, since Y was also compact, each $W_x \times Y$ can also be covered by a finite number of open sets. Hence, the whole of $X \times Y$ admits a finite subcover. By induction, we can extend this property to a finite Cartesian product of spaces. □

This property also holds in the case of infinite Cartesian products by Tychonoff's theorem, but the proof is a lot more involved and relies on the Axiom of Choice. A proof can be found in [14]. However, for our purposes the case of a finite product is sufficient. Similarly to how compactness is carried over in a product space, the closure is also carried over from the spaces the product is formed out of. We study the closure because we often use it in compactifications (of $\mathbb{R}$) to close sets that are not closed while adding the least amount of points.

Theorem 2.7. (The closure of a product space is the product of closures). Let $\prod_{i\in I} M_i$ be a (possibly infinite) Cartesian product with the product topology (definition 1.3). Let $X_i \subseteq M_i$. Then $\overline{\prod_{i\in I} X_i} = \prod_{i\in I} \overline{X_i}$.

Proof. We prove this with two inclusions.

1. Let p_i denote the canonical projection to the i'th space in the cartesian product. Then if $x \in \overline{\prod_{i\in I} X_i}$, then $x \in \overline{p_i^{-1}(X_i)} \subseteq p_i^{-1}(\overline{X_i})$ for all $i \in I$. This is because $p_i^{-1}(X_i) \subseteq p_i^{-1}(\overline{X_i})$ and this set is closed because the projections are continuous functions. Since $\overline{p_i^{-1}(X_i)}$ is the smallest closed set containing $p_i^{-1}(X_i)$ it must be contained in any closed set containing $p_i^{-1}(X_i)$. Therefore, $x \in \prod_{i\in I} \overline{X_i}$ and so $\overline{\prod_{i\in I} X_i} \subseteq \prod_{i\in I} \overline{X_i}$.
2. Now, for the other inclusion, let $x = (x_i)_{i\in I} \in \prod_{i\in I} \overline{X_i}$. Consider an open neighbourhood of x, $U = \prod_{i\in I} U_i$. Then there exist open neighborhoods V_i, such that $x_i \in V_i$ for all $i \in I$ and $\prod_{i\in I} V_i \subseteq U$. Since $x_i \in \overline{X_i}$ for all $i \in I$, we have that there is some $v_i \in V_i \cap X_i$ for all $i \in I$. But then $(v_i)_{i\in I} \in \prod_{i\in I} V_i \cap X_i \subseteq U \cap \prod_{i\in I} X_i$. Since this intersection is therefore nonempty, we must have that $x \in \overline{\prod_{i\in I} X_i}$ and so $\prod_{i\in I} \overline{X_i} \subseteq \overline{\prod_{i\in I} X_i}$.

□

Now we will get to defining a compactification. Our first criterion is of course that it is compact. The second criterion is that the compactification is topologically related to our starting set. Otherwise, we could just pick any compact set and call it a compactification of any non-compact set. Lastly, we want to keep our compactification 'small' because we want the points of our starting set to be in some sense 'close' to the added points of our compactification. Therefore, we introduce the concept of dense sets.

Definition 2.8. (Dense set). A set $X \subseteq Y$ is dense in Y if $\overline{X} = Y$.

With these three criteria in hand, we are ready to formally define a compactification:

Definition 2.9. (Compactification). A compactification of a topological space (X, T) is another topological space (Y, U) and a map $f : X \to Y$ with the following properties:

1. Y is compact.
2. f is an embedding.
3. $f(X)$ is dense in Y.

We will only care about compactifications up to homeomorphism. Two compactifications $f_1 : X \to Y_1$ and $f_2 : X \to Y_2$ are equivalent if there exists a homeomorphism $h : Y_1 \to Y_2$ that fixes the embedded elements of X, so that $h(f_1(x)) = f_2(x)$ for all $x \in X$. Now we show a simple example of a compactification: compactifying the real number line into the circle.

Example 2.10. (Compactification of $\mathbb{R}$ into $\mathbb{S}^1$). Since $\mathbb{S}^1$ is a unit circle in $\mathbb{R}^2$, it is clearly closed and bounded and hence compact. Now, as our embedding, we use the stereographic projection: let $f : \mathbb{R} \to \mathbb{S}^1$ be given by

$$f(x) = \left(\frac{2x}{x^2+1}, \frac{x^2-1}{x^2+1} \right)$$

then $f^{-1} : \mathbb{S}^1 \backslash \{(0,1)\} \to \mathbb{R}$ is given by

$$f^{-1}(x,y) = \frac{x}{1-y}$$

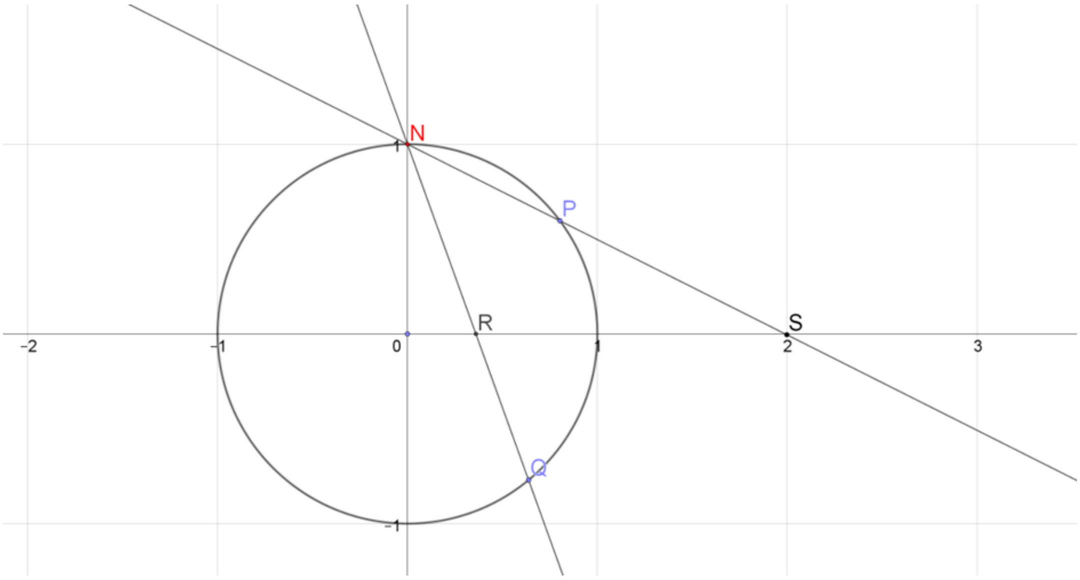

Figure 1: The stereographic projection, where every point on the real number line is mapped to a point on the circle by creating a line through the point on the line (in this figure S and R), and the north pole (point $N = (1,0)$) and intersecting it with the circle (in this case yielding points P and Q).

Now, f is clearly injective, and f and f^{-1} can also be seen to be continuous from the formulae. Then f seen as a function to $f(\mathbb{R})$ is a homeomorphism and therefore an embedding. Lastly, since $f(\mathbb{R}) = \mathbb{S}^1 \backslash \{(0,1)\} \subseteq \overline{f(\mathbb{R})} \subseteq \mathbb{S}^1$, we must have that $\overline{f(\mathbb{R})} = \mathbb{S}^1 \backslash \{(0,1)\}$ or $\overline{f(\mathbb{R})} = \mathbb{S}^1$. But $\mathbb{S}^1 \backslash \{(0,1)\}$ is not a closed set, since the complement, $\{(0,1)\}$, is clearly not an open set. Hence $\overline{f(\mathbb{R})} = \mathbb{S}^1$ and so $f(\mathbb{R})$ is dense in $\mathbb{S}^1$. We conclude that $\mathbb{S}^1$ with f the stereographic projection is a compactification of $\mathbb{R}$.

It turns out that this construction is part of a more general way to compactify non-compact sets by adding one point 'at infinity'. This method of compactifying is called the 1-point compactification, and we will be using it to compactify $\mathbb{R}^n$ into $\mathbb{S}^n$.

Theorem 2.11. (One-point compactification). Let (X, T) be a topological space and let ∞ be a symbol denoting an element not in X. Define the set $X^* = X \cup \{\infty\}$. We define the topology T^* on X^* as $T \cup \{V \subseteq X^* : \infty \in V \text{ and } X \backslash V \text{ is closed and compact in } X\}$. If (X, T) is not compact (X^*, T^*) together with the natural inclusion form a compactification.

Proof. (adapted from [15]). First we should check that T^* is indeed a topology:

1. Since $\emptyset \in T$, $\emptyset \in T^*$. Since X^* contains ∞ and the complement (the empty set) is closed and compact, $X^* \in T^*$ as well.

2. Let $\bigcup U_i$ be an arbitrary union of $U_i \in T^*$. Then for all U_i, we have that either $\infty \in U_i$ or $\infty \notin U_i$. Hence we can split the union as follows: $\bigcup U_i = (\bigcup_{\infty \in U_i} U_i) \cup (\bigcup_{\infty \notin U_i} U_i)$. The second union can only have sets from T and since T is a topology, this union is in T and therefore in T^*. For all U_i in the first union, we have that $X \backslash U_i$ is closed and compact in X. We need to show that $X \backslash \bigcup_{\infty \in U_i} U_i$ is also closed and compact. Since each $X \backslash U_i$ is closed and compact in X, each $U_i \backslash \{\infty\}$ is open in X. Therefore $\bigcup_{\infty \in U_i} U_i \backslash \{\infty\}$ is also open in X. Therefore, $X \backslash \bigcup_{\infty \in U_i} U_i$ is closed in X. Furthermore, $X \backslash \bigcup_{\infty \in U_i} U_i \subseteq X \backslash U_i$ for all U_i. Since a closed subset of a compact set is itself compact by theorem 2.3, $X \backslash \bigcup_{\infty \in U_i} U_i$ is compact and hence in T^*. Now, we need to show that the union of the first and second union is also in T^*. Since we assume these unions are both nonempty, we have that $\infty \in \bigcup U_i$. Then for it to be in T^*, we need that $X \backslash \bigcup U_i$ is closed and compact in X. Since all $U_i \backslash \{\infty\}$ are open in X, their union is open in X and hence the complements are closed. This is again a closed subset of a compact set, which is therefore compact. Hence $\bigcup U_i \in T^*$.

3. For finite intersections, it suffices to check intersections of 2 sets, since bigger intersections follow from induction. Let $U_1, U_2 \in T^*$. Then either both are in T, both are in $\{V \subseteq X^* : \infty \in V$ and $X \backslash V$ is closed and compact in $X\}$, or one of them is in each. In the case they are both in T, their intersection is also in T since T is a topology. If they are both in $\{V \subseteq X^* : \infty \in V$ and $X \backslash V$ is closed and compact in $X\}$, $X \backslash (U_1 \cap U_2) = (X \backslash U_1) \cup (X \backslash U_2)$. Since these are both closed in T, their union is closed in T. Furthermore, the union of compact sets is compact. Therefore $X \backslash (U_1 \cap U_2)$ is closed and compact in X and hence $(U_1 \cap U_2) \in T^*$. Now we have the case where one set is in each of the possibilities. WLOG let $U_1 \in T$ and let $U_2 \in \{V \subseteq X^* : \infty \in V$ and $X \backslash V$ is closed and compact in $X\}$. Since $X \backslash U_2$ is closed, $U_2 \backslash \{\infty\}$ is open in X and hence in T. U_1 is also in T. Since the intersection of U_1 and U_2 doesn't include ∞, it is in also in T and therefore in T^*.

Now, we check the three properties from definition 2.9 to see that it is indeed a compactification:

1. (X^*, T^*) is compact: Let U be an open cover of X^*. Then there is some $V \in U$ for which $\infty \in V$. Then $X^* \backslash V = X \backslash V$, and is therefore compact by the definition of the topology. Since $X^* \backslash V$ is compact and covered by U, $X^* \backslash V$ is covered by a finite subcover $S \subseteq U$. But then $S \cup V \subseteq U$ is still finite, a subset of U and a cover of X^*. Hence U has a finite open subcover.

2. The natural inclusion $i : X \to X^*$ given by $i(x) = x$ is an embedding: clearly i is injective. i viewed as a function from $X \to i(X) = X$ is clearly a homeomorphism, since it is a bijection and since the subspace topology inherited from X^* on X is just T, which is the same topology as that of the domain. Hence i and i^{-1} directly identify open sets.

3. That $\overline{X} = X^*$ we can see as follows: $\overline{X} = \bigcap V$ with $X \subseteq V \subseteq X^*$ and V closed. There exist 2 sets the V could be: X and X^*. Since X^* is clearly closed, we need to show that X is not closed in X^*. Since we assumed that (X, T) was not compact, $\{\infty\} \notin T^*$. Hence, $\{\infty\}$ is not an open set. Therefore, the complement X is not closed. That means that the only closed set containing X, is X^* and so $\overline{X} = X^*$.

□

To make the fact that our compactification of $\mathbb{R}$ to $\mathbb{S}^1$ is a one-point compactification explicit, we can add the point infinity to the number line and extend f to map the north pole N to this new point ∞. Then f is a homeomorphism. The last example that we will provide here will be used in a later proof in section 5. It is the embedding (not compactification) of $\mathbb{R}^4$ into $\mathbb{R} \times \mathbb{S}^3$.

Example 2.12. (Embedding of $\mathbb{R}^4$ into $\mathbb{R} \times \mathbb{S}^3$). Using the stereographic projection (denoted by f), we can show that $\mathbb{S}^3$ is the 1-point compactification of $\mathbb{R}^3$. Then $f(\mathbb{R}^3) \subseteq \mathbb{S}^3$, with f injective in $\mathbb{S}^3$ and a homeomorphism onto $f(\mathbb{R}^3) = \mathbb{S}^3 \backslash \{(0,0,0,1)\}$. If we extend the stereographic projection to be identity on the extra $\mathbb{R}$, we have found an embedding of $\mathbb{R}^4$ into $\mathbb{R} \times \mathbb{S}^3$.

3 Differential geometry

Differential geometry is the area of mathematics that studies the geometry of smooth shapes and smooth spaces, also known as smooth manifolds. It uses techniques of calculus and (multi)linear algebra. The simplest examples of smooth spaces are smooth curves, planes and other smooth surfaces in the three-dimensional Euclidean space. Most of our intuition can be related back to these examples. In both the introduction and in section 1 we alluded to what kind of space-time model we will use: a smooth manifold. Now that we have discussed the topological preliminaries in the earlier chapters, we devote this chapter to developing these smooth manifolds. The main issues that the study of differential geometry can help us with are the facts that parallel lines do not stay parallel in curved spaces and that a smooth space in general is not a vector space, so that we can't just add points to each other. First, we discuss the general definition of manifolds and how to define vectors, one-forms and tensors on them. Most of this material is pulled from [16]. Then we do a lightning introduction to spinors, which is mostly adapted from [3]. Lastly, we discuss how the vectors, one-forms, tensors and spinors transform under coordinate transformations, which as main source had [17].

3.1 Manifolds

Whereas the topological spaces that we defined before only allow us to talk about continuity, a manifold comes equipped with extra structure so that we can also talk about smooth or differentiable functions. This allows to transport techniques of calculus to manifolds, which are more general than $\mathbb{R}^n$. The general requirement of a manifold is that it 'locally looks like $\mathbb{R}^n$'. Since the techniques of calculus are mostly defined in terms of limits, the fact that the space locally looks like $\mathbb{R}^n$ is then enough to make this machinery work. Using what we have developed about topology, we can make the requirement that something locally looks like $\mathbb{R}^n$ rigorous: we require that every point p of a manifold M has a neighbourhood $U \subseteq M$ that is homeomorphic to an open subset of $\mathbb{R}^n$. We call a function that achieves this a chart:

Definition 3.1. (Chart). Let $U \subseteq M$ be open. A chart on M is a homeomorphism

$$\phi : U \to \mathbb{R}^n$$

A chart is defined on open sets.

These open subsets $U \subseteq M$ are called coordinate patches. We use charts to connect patches on a manifold with certain coordinates in $\mathbb{R}^n$. We can therefore write the coordinates of a point $p \in M$ as $\phi(p) = (x^1(p), ..., x^n(p)) = x^\mu(p)$ where μ ranges from 1 to n. Often, we cannot cover a whole manifold using only one chart: if that were possible, the space would be homeomorphic to $\mathbb{R}^n$, and we would not need differential geometry. To handle multiple charts, we define an atlas:

Definition 3.2. (Atlas). Let $A = \{(U_i, \phi_i); i \in I\}$ be a collection of charts, labelled by an index set I. We say that A is a topological atlas for M if M is the union of the coordinate patches, so $M = \bigcup_{i \in I} U_i$.

Now, some of the charts may overlap and they will not always map points on the manifold to the same coordinates: if that were the case, we wouldn't have needed multiple charts in the first place and again the whole space would be homeomorphic to $\mathbb{R}^n$ so that we wouldn't need differential geometry. This overlapping is okay, as long a the charts are connected by smooth function, the transition function.

Definition 3.3. (Transition function). Let M be a topological space. Let (U_i, ϕ_i) and (U_j, ϕ_j) be two charts with a nonempty intersection $U_i \cap U_j$. Let $p \in U_i \cap U_j$. The function that maps the coordinates $\phi_i(p) = x^\mu$ to $\phi_j(p) = \tilde{x}^\mu$ is called the transition function κ_{ij}. In this nonempty part of the intersection, the transition function $\kappa_{ij} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ is defined by $\kappa_{ij} = \phi_j \circ \phi_i^{-1}$.

Because the charts are homeomorphisms, it follows that the transition functions are also homeomorphisms. If a collection of charts covers the entire manifold, we call it a topological atlas. I find this name quite aptly chosen since an atlas is also a collection of maps/charts in the everyday sense of the word. If all charts in an atlas are compatible (so that the transition functions are smooth) we call it a smooth atlas.

Definition 3.4. (Smooth atlas). Two charts (U_i, ϕ_i) and (U_j, ϕ_j) are called compatible if both κ_{ij} and κ_{ji} are smooth. An atlas is called smooth if all of its charts are compatible.

The concept of the smooth atlas can be extended to functions that are compatible in different ways, e.g. we can have an atlas where the transition functions are C^k, or where they are holomorphic.

Now we are ready to define the underlying mathematical object defining our space-time, the smooth manifold:

Definition 3.5. (Smooth manifold). A smooth manifold is a Hausdorff topological space M, together with a smooth atlas A. M is of dimension n if its charts take values in $\mathbb{R}^n$.

Just like there is a natural notion of continuous functions for a topological space, there is also a natural notion of smooth functions for manifolds. We do this by using the coordinate representations of the functions, starting with a smooth map from a manifold M to $\mathbb{R}^n$

Definition 3.6. (Smooth and differentiable function to $\mathbb{R}$). Let M be a smooth manifold. A function $f : M \to \mathbb{R}$ is smooth/differentiable at $p \in M$ if for some chart (U_i, ϕ_i) containing p, the coordinate representation $f_i : \phi_i(U_i) \to \mathbb{R}$ is smooth/differentiable at $\phi_i(p) \in \mathbb{R}^n$.

To extend this definition to a smooth/differentiable map $F : M \to N$ where M and N are manifolds, we need two charts. One chart (U_i, ϕ_i) in M around some point $p \in M$ and a chart (V_j, ψ_j) in N around $F(p) \in N$. If $F(U_i) \subseteq V_j$, the coordinate representation around p of F, $F_{ij} : \phi_i(U_i) \to \psi_j(V_j)$ is given by $F_{ij} = \psi_j \circ F \circ \phi_i^{-1}$.

Definition 3.7. (Smooth map). A continuous map $F : M \to N$ is differentiable/smooth at $p \in M$ if for some chart (U_i, ϕ_i) in M containing p and a chart (V_j, ψ_j) in N containing $F(p)$, the coordinate representation $F_{ij} : \phi_i(U_i) \to \psi_j(V_j)$ is smooth/differentiable at $\phi_i(p) \in \mathbb{R}^n$.

It turns out that this definition does not depend on the choice of coordinates, so a smooth map is well-defined. Now that we have defined the notion of a smooth map between manifolds, we can formulate what it means for two smooth manifolds to be similar. Just like we have homeomorphisms for topological spaces that are indistinguishable by their topological properties, we have manifolds that are 'the same' when it comes to their manifold properties. The functions that replace homeomorphisms are called diffeomorphisms, which roughly correspond to differentiable homeomorphisms.

Definition 3.8. (Diffeomorphism). Let M and N be two smooth manifolds. A diffeomorphism $\phi : M \to N$ is a smooth bijection for which the inverse $\phi^{-1} : N \to M$ is also smooth. We see that a diffeomorphism is then a homeomorphism between manifolds M and N which is also smooth.

We call M and N diffeomorphic if there is a diffeomorphism between them. Now, we will give two small examples of spaces that are diffeomorphic to get slightly more acquainted with this concept.

Example 3.9. The function f in example 1.11 is smooth, so this function is also a diffeomorphism.

Example 3.10. ($\mathbb{R}$ is diffeomorphic to $\mathbb{R}^+$). Let $f : \mathbb{R} \to \mathbb{R}^+$ be given by $f(x) = e^x$. This function is clearly smooth and bijective on the given intervals. Its inverse $f : \mathbb{R}^+ \to \mathbb{R}$ is given by $f^{-1}(x) = \ln(x)$ is also clearly smooth.

We end this subsection by discussing a way to translate functions from one manifold to another. If we have a function f on N mapping to $\mathbb{R}$ and a smooth map $F : M \to N$, we can pull the function on N to a function on M. We call this function the pullback of f by F.

Definition 3.11. (Pullback). Let $F : M \to N$ be a smooth map between manifolds M and N and suppose $f : N \to \mathbb{R}$ is a smooth function on N. Then the pullback of f by F is the smooth function F^*f on M defined by

$$(F^*f)(x) = f(F(x))$$

We will mainly be using the pullback to pull a function from a manifold to its compactification.

3.2 Vectors and the tangent space

We have defined what it means for a map between two smooth manifolds M and N to be differentiable, but we haven't yet defined its derivative. For this, we introduce the notion of a smooth curve:

Definition 3.12. (Smooth curve). Let $U \subseteq \mathbb{R}$ be an open interval containing 0 and let M be a smooth manifold. A smooth curve in M through $p \in M$ is a smooth function $\gamma : U \to M$ such that $\gamma(0) = p$.

In a general manifold, we don't have the surrounding structure of a vector space, so we cannot simply add two points along a smooth curve on the manifold and take the limit $\lim_{h\to 0} \frac{\gamma(h)-\gamma(0)}{h}$ to calculate the tangent vector. This procedure of adding points is only possible if the manifold is embedded in a vector space like $\mathbb{R}^n$. An example of such a space is $\mathbb{S}^n$, which can be embedded in $\mathbb{R}^{n+1}$. However, we can use the fact that points on a manifold can locally be embedded into $\mathbb{R}^n$ and so we will use the coordinate representation of the smooth curve $\gamma_i : I \to \mathbb{R}^n$ given by $\gamma_i = \phi_i \circ \gamma$. Now we can define $v_i^\mu = \dot{\gamma}_i^\mu(0) = \lim_{h\to 0} \frac{\gamma_i^\mu(h)-\gamma_i^\mu(0)}{h}$. If we choose a different chart ϕ_j, we get $\tilde{v}_j^\alpha = \dot{\gamma}_j^\alpha = \lim_{h\to 0} \frac{\gamma_j^\mu(h)-\gamma_j^\mu(0)}{h}$. Since $\gamma_j = \kappa_{ij} \circ \gamma_i$, we get from the chain rule that $\tilde{v}_j^\alpha$ depends linearly on v_i^μ. If we let J_μ^α denote the Jacobian matrix of κ_{ij} at $\phi_i(p)$, we get that

$$\tilde{v}_j^\alpha = \sum_{\mu=1}^{n} J_\mu^\alpha v_i^\mu$$

We will from now on use Einstein's summation convention, where the sum over repeated indices is implied. Then, the above equation turns into

$$\tilde{v}_j^\alpha = J_\mu^\alpha v_i^\mu \tag{1}$$

We call this a contraction over the index μ. Do note that the subscripts i and j are not indices which we contract over, they only indicate the chart with which we express the coordinates. We only contract over an upper and a lower index, so something like $a^\mu b^\mu$ is not possible.

We can define an equivalence relation on curves through p. We call two curves γ and l equivalent, $\gamma \sim_p l$, if there exists a chart around p such that the respective derivatives v_i^μ and w_i^μ are equal. It turns out that we do not need to indicate which chart was used, since two curves that have the same derivative in one chart, will also have the same derivative in another chart [16].

Definition 3.13. (Tangent space). A tangent vector at $p \in M$ is the equivalence class $[\gamma]$ of curves through p with respect to the relation $\sim_p$. The set T_pM of tangent vectors is called the tangent space of M at p.

We call $\dot{\gamma}_i^\mu(0) = (v_i^1, ..., v_i^n) = v_i^\mu$ the coordinate expression of v_p with respect to the chart (U_i, ϕ_i). Then function mapping the tangent vector to its coordinate expression

$$\phi_{i*} : T_pM \to \mathbb{R}^n \text{ given by } \phi_{i*}(v_p) = \dot{\gamma}_i^\mu(0)$$

turns out to be a bijection. Now we can define addition and scalar multiplication on T_pM. If we add vectors in T_pM, we simply add the coordinates in $\mathbb{R}^n$ and map that back to the tangent space. For scalar multiplication, we similarly multiply the coordinates of the vector in $\mathbb{R}^n$ and map them back to the tangent space. This completes the vector space structure of the tangent space.

Since we have now defined vectors, we can also define vector fields. This makes use of the tangent bundle:

Definition 3.14. (Tangent bundle). The tangent bundle TM is the set of all tangent vectors v at every point p. Hence TM is the disjoint union of a tangent spaces T_pM, for all $p \in M$.

$$TM = \bigsqcup_{p \in M} T_pM \tag{2}$$

It turns out that TM is also a smooth manifold, of dimension $2n$. Now we can define vector fields:

Definition 3.15. (Vector field). A vector field is a map $F : M \to TM$ such that $F(p) \in T_pM$ for all $p \in M$.

This definition corresponds with the intuitive idea of defining a vector at every point of the manifold.

3.3 One-forms and general tensors

In the previous part we defined vectors on a manifold. We can easily generalise this procedure to other objects. We define here one-forms (also linear functions or homomorphisms) and then general tensors.

Definition 3.16. (Homomorphism). A homomorphism (of vector spaces) is a map A between two vector spaces U and V over the same field F, that is also linear. So $L : U \to V$ and we have $A(ax+by) = aA(x)+bA(y)$. The set of linear maps from U to V is denoted $\text{Hom}(U, V)$.

It turns out that $\text{Hom}(U, V)$ is also vector space over F if we equip it with the standard operations of addition and scalar multiplications we normally use for functions since it is closed under these operations. The zero map $x \to 0$ is the zero vector. We define the dual space of a vector space V:

Definition 3.17. (Dual space). The dual space of a vector space V over F is the vector space

$$V^* = \text{Hom}(V, F) = \{a : V \to F : a \text{ is linear}\} \tag{3}$$

Its elements a are called linear functionals, linear forms, or one-forms.

If we now apply this concept to the tangent space, we get the so-called cotangent space $T_p^*M = (T_p^M)*$. Its elements thus map vectors in the tangent space T_pM to $\mathbb{R}$ in a linear way. We can define the cotangent bundle in a similar way to the tangent bundle, and define one-form fields in a similar way to vector fields. We can now use these two types of vector spaces to define general tensors on the manifold.

Definition 3.18. (Tensor) A tensor of rank (k, l) is a multilinear map

$$\underbrace{T_pM \times \cdots \times T_pM}_{k \text{ times}} \times \underbrace{T_p^*M \times \cdots \times T_p^*M}_{l \text{ times}} \to \mathbb{R}$$

We denote such a tensor by $T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l}$.

We can define tensor fields on the manifold in a similar way to vector fields and one-form fields. Note that the ordering of the indices within the tensor matters. The tensor $T^{\mu\nu}$ does not in general equal $T^{\nu\mu}$, unless T happens to be symmetric. We use round brackets to denote the symmetrization of a tensor and square brackets to denote anti-symmetrization:

$$T_{(\mu\nu)} = \frac{1}{2!}(T_{\mu\nu} + T_{\nu\mu}) = \frac{1}{2}(T_{\mu\nu} + T_{\nu\mu}) \tag{4}$$

and

$$T_{[\mu\nu]} = \frac{1}{2!}(T_{\mu\nu} - T_{\nu\mu}) = \frac{1}{2}(T_{\mu\nu} - T_{\nu\mu}) \tag{5}$$

This process can also be applied to more than two indices and over multiple tensors in a tensor product. For example, we can have $x^\mu y^{(\nu} z^{\rho\sigma)}$. When there are more than 2 indices involved, the symmetric case generalises easily, in the anti-symmetric case we have to take into account the sign of the permutation of the indices. For 3 indices we then get

$$T_{[\mu\nu\rho]} = \frac{1}{3!}\left(T_{\mu\nu\rho} - T_{\mu\rho\nu} + T_{\nu\rho\mu} - T_{\nu\mu\rho} + T_{\rho\mu\nu} - T_{\rho\nu\mu}\right) \tag{6}$$

If a tensor has 2 indices, we have that the tensor can be split up into its symmetric and anti-symmetric parts as follows: $T_{\mu\nu} = T_{(\mu\nu)} + T_{[\mu\nu]}$. In general however, this does not hold. If we want to (anti-)symmetrize over indices with other indices in between them, we can exclude those by putting them in horizontal bars: $x^\mu y^{(\nu} z^{|\rho|\sigma)}$ denotes symmetrization over ν and σ.

3.4 Spinors

Lastly, we define spinors on the manifold. For this, we define W to be a two-dimensional vector space over $\mathbb{C}$. Then an element of W is denoted by ξ^A, where $A = 0, 1$. We denote by W^* its dual space and denote its elements by ξ_A. We can then form the complex conjugate dual space $\overline{W}^*$, composed of the antilinear maps from $W \to \mathbb{C}$. A map f is antilinear if $f(\xi_1^A + \xi_2^A) = f(\xi_1^A) + f(\xi_2^A)$ and $f(c\xi_A) = \overline{c} f(\xi^A)$ for all ξ_1^A, ξ_2^A and ξ^A in W. We denote an element of this space using a primed lower index, so an element would be denoted by $\xi_{A'}$. Lastly, we define the complex conjugate space $\overline{W}$ to be the dual space of $\overline{W}^*$. Here we use a primed upper index so that elements are denoted $\xi^{A'}$. We denote the image of $\xi^A \in W$ under complex conjugation, defined such that for all $\psi \in \overline{W}^*$, we have $\psi(\xi) = \overline{\xi}(\psi)$, where indices were omitted for clarity. The image of $\xi^A \in W$ under complex conjugation is then denoted by $\overline{\xi}^{A'}$ and similarly the image of $\xi^{A'} \in \overline{W}$ is denoted by $\overline{\xi}^A$. Now we define spinorial tensors in a similar way to normal tensors:

Definition 3.19. (Spinorial tensor). A spinorial tensor of type (k, l, k', l') over W is defined as a multilinear map

$$\underbrace{W \times \cdots \times W}_{k \text{ times}} \times \underbrace{W^* \times \cdots \times W^*}_{l \text{ times}} \times \underbrace{\overline{W} \times \cdots \times \overline{W}}_{k' \text{ times}} \times \underbrace{\overline{W}^* \times \cdots \times \overline{W}^*}_{l' \text{ times}} \to \mathbb{C}$$

We denote such a tensor by $\phi^{A_1 \cdots A_k}{}_{B_1 \cdots B_l}{}^{A_1' \cdots A_{k'}'}{}_{B_1' \cdots B_{l'}'}$.

The relevant ordering of the primed and unprimed indices does not matter, so $T^{AD'B}{}_C$ denotes the same tensor as $T^{AB}{}_C{}^{D'}$. However, the order within the primed and within the unprimed indices does matter, just as it does for normal tensors. Complex conjugation sends spinors of type (k, l, k', l') to (k', l', k, l). We can contract over two primed indices and over two unprimed indices, but a contraction between one primed and one unprimed index is not defined.

3.5 Transforming various fields on the manifold

If we want to have any notion of universally applicable laws of physics, these laws have to be valid irrespective of the choice of coordinates. This is described by the principle of general covariance which states that the general laws of nature are to be expressed by equations that are covariant with respect to coordinate transformations. We will see that we can achieve this by using tensors to express physical laws. We assume that we are on a manifold and we denote the coordinates for an event $p \in M$ by x^μ. We now explore how scalar, vector, one-form and general tensor or spinor fields should transform under coordinate transformations $x^\mu \to \tilde{x}^\mu$. Note that these transformations are passive: by performing these transformations (which can be general diffeomorphisms), we don't actually change the physical space and therefore all physics remain the same. We just parameterise the space using a different set of coordinates. These passive coordinate changes are in general local transformations and can only be extended to global transformations in very special cases. Loosely speaking (in terms of vector spaces, even though the manifold may not be one), we can see this as a 'change of basis'.

3.5.1 Scalar fields

The simplest type of tensor we can have on a manifold on a manifold is a scalar field. A scalar field Φ assigns a number to each point on the manifold. Thus, for a given coordinate system x^μ, it is a function $\Phi(x)$ of x^μ. If we now make a transformation $x^\mu \to \tilde{x}^\mu(x)$. This means that the point that before was now labelled x^μ is now labelled $\tilde{x}^\mu(x)$. Hence, a scalar field $\Phi(x)$ transforms as

$$\tilde{\Phi}(\tilde{x}) = \Phi(x) \tag{7}$$

under a coordinate transformation $x^\mu \to \tilde{x}^\mu(x)$.

3.5.2 Vector fields

Since we found that the tangent space is bijective to $\mathbb{R}^n$ and that the used chart is irrelevant for the operations, we can just denote a vector by its coordinate representation. We let $v^\mu(x)$ denote a vector field with respect to some coordinates x^μ that are clear from context. We already found how vectors transform in eq. (1). However, we can slightly edit this notation to make it slightly more suggestive: A vector field $v^\mu(x)$ transforms as

$$v^\alpha(\tilde{x}) = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} v^\mu(x) \tag{8}$$

under a coordinate transformation $x^\mu \to \tilde{x}^\mu$. This is a contravariant transformation.

3.5.3 One-form fields

We have seen that one-forms at an event p are linear maps that take vectors at p to numbers at p. This means that a one-form needs to have the same number of components as a vector. Therefore we denote a one-form by a_μ, where the index is subscripted, instead of superscripted as with the coordinates and vectors. Then the linear map that this one-form defines at p is $a_\mu v^\mu$. If we now consider this map, we see that it assigns a number to every event p, which means that it is a scalar field. We already saw that we then must have $\tilde{a}_\alpha \tilde{v}^\alpha = a_\mu v^\mu$. From this, we deduce that a one-form must transform oppositely to a vector. Because the coordinate transformation is also smooth, we conclude that a one form field $a_\mu(x)$ transforms as

$$\tilde{a}_\alpha(\tilde{x}) = \frac{\partial x^\mu}{\partial \tilde{x}^\alpha} a_\mu(x) \tag{9}$$

under a coordinate transformation $x^\mu \to \tilde{x}^\mu$. This is a covariant transformation.

3.5.4 Tensors

The scalar, vector and one-form fields are the simplest types of tensors that one can have on a manifold. In general a tensor can have many indices:

$$T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l}(x)$$

denotes a tensor of rank (k,l). Since this tensor is created by taking the tensor product between k vectors and l one-forms, we can just apply the respective rules for each of the upper and lower indices. Hence, a tensor field $T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l}(x)$ transforms as

$$\tilde{T}^{\alpha_1\cdots\alpha_k}{}_{\beta_1\cdots\beta_l}(\tilde{x}) = \frac{\partial\tilde{x}^{\alpha_1}}{\partial x^{\mu_1}}\cdots\frac{\partial\tilde{x}^{\alpha_k}}{\partial x^{\mu_k}}\frac{\partial x^{\nu_1}}{\partial\tilde{x}^{\beta_1}}\cdots\frac{\partial x^{\nu_l}}{\partial\tilde{x}^{\beta_l}}T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l}(x) \tag{10}$$

In general a tensor does not need to have all of the upper indices on the right of the lower indices. Hence, we can have tensors like $T_\mu{}^\nu{}_\rho$ or $T_\mu{}^\nu$. We get these done by changing the order in the tensor product. Because the upper indices come from vectors and the lower indices from one-forms, the upper indices always transform like vectors and the lower indices like one-forms.

The most important feature of tensors is that they can be used to write down equations that hold irrespective of the coordinate system used on the manifold. Suppose that $T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l}(x) = 0$ is satisfied somewhere on the manifold in a particular coordinate system x^μ. If we now make an arbitrary coordinate transformation $x^\mu \to \tilde{x}^\mu(x)$, it follows from eq. (10) that $\tilde{T}^{\alpha_1\cdots\alpha_k}{}_{\beta_1\cdots\beta_l}(\tilde{x}) = 0$ also holds at that point on the manifold. Since this equation keeps its form in all coordinate systems, it is covariant.

We also see that the sum of two tensors of the same type, i.e. the same index structure, transforms in the same way and is therefore also a tensor of that type. We will not explicitly denote the tensor product and also just call it the product. So we will denote the product $L^{\mu\nu} = v^\mu \otimes v^\nu$ as $L^{\mu\nu} = v^\mu v^\nu$. This could cause some confusion when vectors and one forms are multiplied, because $v^\mu a_\mu$ just denotes an inner product between vectors $\mathbf{a}^T\mathbf{v}$ trough the Einstein summation convention, whereas $v^\mu a_\nu$ denotes their tensor product and therefore creates a new tensor $M^\mu{}_\nu$ with an upper and a lower index. We define in general a contraction of two tensors to be the product of two tensors with sums over one or more of the indices. Each contraction is over an upper index and a lower index, so that the resulting object is also a tensor. A tensor can also be contracted with itself. An example would be $R^\mu{}_\mu$.

3.5.5 Spinors

Spinors do not transform under these types of transformations, so that

$$\tilde{\xi}^A(\tilde{x}) = \xi^A(x) \tag{11}$$

The reason for this is that we can absorb the transformation into the quantity $\sigma_\mu^{AB'}$ which relates spinors to tensors. This quantity will be introduced in section 4.2.

4 Space-time

In this section, we will build the bridge between the mathematics developed in the previous sections and the physics of general relativity. First we will talk about some foundations and basic concepts in general relativity. We discuss some postulates of the theory and the mathematical model used for general relativity: a manifold with a metric (tensor) defined at every point, which determines the distance between that point and one that is infinitesimally removed from it. Then we turn to discuss spinors in space-time, give some examples of how they relate to normal tensors and explain why they may be more fundamental objects than tensors. Then we discuss how we quantify the change of various fields on the manifold. Where in normal Euclidean space the gradient derivative is often used, in general relativity this is done by the covariant derivative. Furthermore, we touch upon the concept of causality, related to the fact that information cannot travel faster than the speed of light. Lastly, we quickly present some objects that are often used to do actual calculations in general relativity, and quickly present how mass causes gravitational phenomena in this theory.

4.1 Basic concepts in general relativity

In n-dimensional Euclidian space, the distance between two points is given using the Euclidian metric, where

$$ds^2 = \sum_{i=1}^{n} (dx^i)^2$$

However, we can denote this in a different way, by defining a tensor $g_{\mu\nu}$ to be equal to 1 if $\mu = \nu$ and 0 otherwise. Then we can denote the distance by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

This tensor $g_{\mu\nu}$ is called the metric and can be used to give the line element for different coordinate systems, but also for curved spaces in general. This metric is a symmetric, non-degenerate tensor of type $(0, 2)$. The mathematical model we use for space-time is a pair $(M, g_{\mu\nu})$ where M is a connected four-dimensional smooth manifold and $g_{\mu\nu}$ is a Lorentz metric: if we view the metric tensor as a matrix, the first eigenvalue is negative, and the other three are positive (or the first eigenvalue is positive and the other three are negative). The metric is not a metric in the sense of metric spaces, since it is not positive definite. It does have similar properties however. The metric can change from point to point in space-time and thus defines a tensor field on the manifold. We take the manifold to be connected since we physically have no way to find out anything about the existence of potential disconnected parts.

To simplify notation in general relativity, we use natural units which set the speed of light c equal to 1. We denote an event in space-time by giving the four-vector x^μ of the coordinates: $x^0 = ct = t, x^1 = x, x^2 = y, x^3 = z$. Whenever a Greek letter like μ or ν is used in the index it can go from 0 to 3, whereas if a lowercase Roman letter like i or j is used, it goes from 1 to 3.

Minkowski space-time, also called empty space-time, is the space-time where the laws of special relativity hold. Mathematically it is the manifold $\mathbb{R}^4$ with the metric (denoted by $\eta_{\mu\nu}$

instead of $g_{\mu\nu}$) given everywhere by

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

With this notation, the line element of the space-time for two infinitesimally separated events x^μ and $x^\mu + dx^\mu$ in Minkowski space is given by

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -dt^2 + dx^2 + dy^2 + dz^2 \tag{12}$$

The reason why the Minkowski metric is defined with this minus sign, is due to one of special relativity's central postulates: the speed of light is the same in all inertial frames of reference. To keep the speed of light the same in every inertial frame, we have to use a Lorentz transformation instead of a Galilean transformation when moving from one coordinate system to another that is relatively moving with some speed v. When a Lorentz transformation increases distances between events, we find that the time between the events should be increased by the same amount to keep the speed of light unchanged. This is why the Minkowski line element is conserved under any transformation between inertial frames of reference. It is also why all Lorentz metrics are required to have 1 negative and 3 positive eigenvalues, or 1 positive and 3 negative eigenvalues if the opposite sign convention is used.

The group of 'valid' transformations between inertial frames of references is therefore the group of all bijections $\mathbb{R}^4 \to \mathbb{R}^4$ such that the Minkowski line element is conserved. This group is called the Poincaré group $ISO(3,1)$, where I stands for inhomogeneous. It turns out that every element of the Poincaré group can be decomposed in a translation and a Lorentz transformation [16]. If we remove from the Poincaré group the translations, we are left with the Lorentz group, which is $SO(3,1)$ (technically the Lorentz group is $O(3,1)$, but by setting the determinant equal to $+1$ we disallow spatial reflections and thereby preserve orientation). Since the inner product given by $\eta_{\mu\nu}$ is conserved under $SO(3,1)$ transformations, we must have that (in matrix notation) for any arbitrary Lorentz transformation $\Lambda \in SO(3,1)$ we have $\Lambda^T \eta \Lambda = \eta$. Note that the Poincaré transformations we are talking about here are active transformations. By moving from one inertial system to another, we in some sense 'change physics'. This change is a global change of the space-time. In contrast, we call a pure coordinate transformation a passive transformation: due to the principle of general covariance (we discussed this in section 3.5) this shouldn't change space or physics. It only uses different coordinates to describe the same phenomena. Passive transformations are usually only local transformations. In the case of Poincaré transformations, the difference is very subtle. Loosely speaking in terms of vector spaces, an active transformation keeps the basis the same but changes the vectors, while a passive transformation changes the basis, but keeps the vectors the same. Because the Poincaré group is a symmetry of space-time itself, an equivalent-looking result is obtained whether the transformation is seen as an active or passive transformation. However, physically they are different things. See fig. 2 for an illustration of the difference between active and passive transformations.

It turns out that the metric signature is conserved under a change of basis and therefore under coordinate (passive) transformations by Sylvester's law of inertia [19]. Since the metric is symmetric and real, we can always diagonalise it by the spectral theorem (see [20]). This means that it is invertible and we denote the inverse metric as $g^{\mu\nu}$. It has the property that $g^{\mu\nu}(x) g_{\nu\rho}(x) = g_{\rho\nu}(x) g^{\nu\mu} = \delta^\mu_\rho$ everywhere in space-time. We used the Kronecker delta here, which we define as

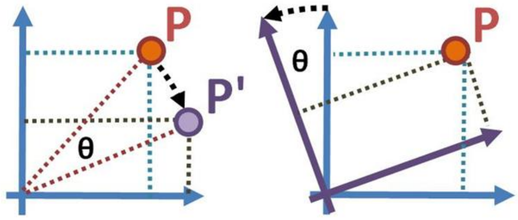

Figure 2: An active (left) and passive (right) transformation of a vector P. In the active transformation the vector P gets rotated by θ degrees clockwise into a different vector P', whereas in the passive transformation the basis gets rotated by θ degrees counterclockwise. These two transformations lead to the same new coordinate representation of the vector, but are actually physically different processes. Image courtesy goes to [18].

$$\delta^{\mu}_{\nu} = \begin{cases} 1 \text{ if } \mu = \nu \\ 0 \text{ otherwise} \end{cases} \tag{13}$$

The inverse metric does not have any direct physical interpretation, but it is a very useful object for performing calculations. We use the metric and inverse metric to raise or lower indices of tensors. We raise a lower index as follows:

$$v^{\mu} = g^{\mu\nu} v_{\nu} \tag{14}$$

A similar process applies to lowering an index. Why do we use the metric to raise and lower indices? We defined the 'norm' of a vector using the metric, so it would make sense to also define a 'dot product'. In normal vector notation we let a dot product be defined by a bilinear form B, so that $\mathbf{x} \cdot \mathbf{x} = \mathbf{x}^T B \mathbf{x}$. We could have also defined the covector $\mathbf{x}^T$ to be $\mathbf{x}^T B$, so that we could denote the dot product by $\mathbf{x}^T \mathbf{x}$. We do a similar thing in this construction, where the 'bilinear form' is denoted by $g_{\mu\nu}$. Then $x^{\mu} \cdot x^{\mu} = x_{\mu} x^{\mu} = g_{\mu\nu} x^{\mu} x^{\nu}$, though the notation with $\cdot$ is never used. We use the following terminology to classify vectors in general relativity:

- If $x_{\mu} x^{\mu} < 0$, $r^2 (= x^2 + y^2 + z^2) < t^2$, so the spatial part is smaller than the distance light travels in the time part and x^{μ} is called timelike.
- If $x_{\mu} x^{\mu} = 0$, $r^2 = t^2$, so the spatial part is equal to the distance light travels in the time part and x^{μ} is called or null (or more infrequently, lightlike)
- If $x_{\mu} x^{\mu} > 0$, $r^2 > t^2$, so the spatial part is greater than the distance light travels in the time part and x^{μ} is called spacelike.

Vectors that are non-spacelike are sometimes called causal because the spatial part is smaller than or equal to the time part. This means that light could travel the spatial part within the time part and thus have a causal effect.

A hypersurface, a 3-dimensional subspace of the manifold, is null if the normal vector at every point is null, This is conceptually strange since then the normal vector is also in the hypersurface which is not possible in normal Euclidean geometry. We call a hypersurface spacelike if the normal vector at every point is timelike and a hypersurface is timelike if the normal vector at every point is spacelike. An alternative characterisation of a null hypersurface is that the pullback of the metric onto the tangent space is degenerate. For surfaces, 2-dimensional subspaces of the manifold, we do the characterisation in terms of the pullback of the metric onto the tangent space: it is null if the pullback of the metric onto the tangent space is degenerate, spacelike if it has signature $(+,+)$ and timelike if it has signature $(+,-)$.

4.2 Spinors in space-time

Spinors in space-time have some properties that are very similar to those of tensors in space-time. Recall that we defined spinors as members of a two-dimensional vector space W over $\mathbb{C}$. The vector space of antisymmetric tensors of type $(0,2,0,0)$ is one-dimensional. If such a tensor $\epsilon_{AB} = -\epsilon_{BA}$ is chosen, the pair (W, ϵ_{AB}) is called a spinor space. The easiest form that ϵ_{AB} can then take is

$$\epsilon_{AB} = \begin{pmatrix} \epsilon_{00} & \epsilon_{01} \\ \epsilon_{10} & \epsilon_{11} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{15}$$

This ϵ symbol is also called the Levi-Civita symbol. It will take this easiest form when an orthonormal basis is chosen for the spinor space. If we want to equip a space-time with spinors, we require the spinors to have some very similar properties as tensors did for a space-time $(M, g_{\mu\nu})$. As a note: it is not trivial that a space-time admits spinors. Their precise definition is quite technical and not within the scope of this work. For the present purposes, we will just assume that a spin structure exists and present and/or derive the properties that we will need. If a reader is interested in learning more about spinors, a place to start could be the books by Penrose and Rindler [21] or for something not quite as in-depth chapter 13 of [3].

We can identify normal tensors with spinorial tensors, where each tensor index gets mapped to a normal spinor index and a complex conjugate index. We translate from tensors to spinors and vice-versa utilizing a quantity $\sigma^{\mu}_{AB'}$ which is Hermitian for each value of μ and satisfies

$$g_{\mu\nu}\sigma^{\mu}_{AB'}\sigma^{\nu}_{CD'} = \epsilon_{AC}\bar{\epsilon}_{B'D'} \tag{16}$$

These σ's are usually called the Infeld-van der Waerden symbols. We define an inner product on the spinors that it is conserved under $SL(2,\mathbb{C})$ transformations to stay consistent with the invariance of the inner product on vectors under Lorentz transformations. This means that we can use the Levi-Civita symbols to raise or lower spinors indices (instead of the space-time metric for tensor indices). When working with spinors, we usually adopt a metric of signature $(+---)$ instead of $(-,+,+,+)$ so that the notion of index rising/lowering is independent of whether the space-time metric is used or the Levi-Civita symbols are used.

Because of the identification of spinors and tensors, we should expect the spinors to active transformations related to the Lorentz transformations which leave the physics unchanged. It turns out that each map $L^{A}{}_{B} \in SL(2,\mathbb{C})$ gives rise to a Lorentz transformation [3]. However, each transformation in $SL(2,\mathbb{C})$ has a negative counterpart which yields the same Lorentz transformation of the physical space. That is to say, $SL(2,\mathbb{C})$ is a double cover of $SO(3,1)$. Just like vectors in space-time transform under actions of the Poincaré group $ISO(3,1)$ (the I stands for inhomogeneous), the spinors transform under actions of $ISL(2,\mathbb{C})$, which turns out to be a double cover of $ISO(3,1)$. This representation of the Poincaré group is therefore only up to sign.

Because ϵ is nondegenerate and used to raise and lower indices just like the space-time metric, it functions in much the same way. However ϵ_{AB} is antisymmetric, unlike the symmetric metric. This means that it now matters which index we use to lower the index. We use the convention of using the first index of ϵ_{AB} to lower indices. Hence we have $\xi_B = \epsilon_{AB}\xi^A = -\epsilon_{BA}\xi^A$. We define ϵ^{AB} to be minus the inverse of ϵ_{AB}. To compensate for the minus sign, we use contraction over the second index to raise an index. Hence, we have $\zeta^A = \epsilon^{AB}\zeta_B = -\epsilon^{BA}\zeta_B$. We thus get

$$\epsilon_{AB}\epsilon^{CB} = \delta_A^C, \quad \epsilon^{AB}\epsilon_{AC} = \delta_C^B \tag{17}$$

The use of the delta symbol here is confusing however, since we now have

$$\delta_A^B = \epsilon_A{}^B = -\epsilon^B{}_A \tag{18}$$

and it is hard to remember what exactly is the right configuration of indices. This is why we shall try not use the Kronecker delta in the context of spinors, preferring to use $\epsilon_A{}^B$ in its place. We regard the first term of eq. (17) as ϵ^{CB} acting on ϵ_{AB} to raise its second index, and similarly for the third term. We can view $\epsilon_A{}^B$ either as ϵ_{AB} with its second index raised or as ϵ^{AB} with its first index lowered. Combining these interpretations, we see that ϵ^{AB} is ϵ_{AB} with both indices raised. Then we find that the Levi-Civita symbol with its indices up takes the same values, e.g. $\epsilon_{00} = \epsilon^{00}$ etc. We denote the spinors obtained from ϵ_{AB} and ϵ^{AB} by complex conjugation as $\epsilon_{A'B'}$ and $\epsilon^{A'B'}$, where the bar that is normally present when working with complex conjugate spinors is now omitted. From these conventions of raising and lowering indices, we can derive a very important result concerning the inner product of two spinors:

Theorem 4.1.

$$\xi_A\zeta^A = -\xi^A\zeta_A \tag{19}$$

Proof.

$$\begin{aligned} \xi_A\zeta^A &= (\epsilon_{BA}\xi^B)\zeta^A \\ &= -\epsilon_{AB}\xi^B\zeta^A \\ &= -\xi^B\zeta_B = -\xi^A\zeta_A \end{aligned} \tag{20}$$

□

In particular, we then have that for any spinor ξ^A, $\xi^A\xi_A = 0$. This important result is reflected in the fact that a spinor vector $m^{AB'}$ can be decomposed in a product of two spinors with one index if and only if the corresponding vector m^μ that it was derived from is null. We will explicitly see this in example 4.5. Another property of spinors that we will be using is that we can extract an ϵ from any spinor that is antisymmetric in 2 of its indices. To prove this, we first need two lemmas.

Lemma 4.2. For any three arbitrary spinors κ_A, ω_A and τ_A, we have

$$\kappa_A\omega^A\tau^B + \omega_A\tau^A\kappa^B + \tau_A\kappa^A\omega^B = 0 \tag{21}$$

Proof. Since spinors are elements of a 2D vector space, we find that at least one of these spinors must be a linear combination of the others. WLOG we can assume that $\tau_A = a\kappa_A + b\omega_A$, where a and b are just arbitrary constants. Then we get (using theorem 4.1)

$$\begin{aligned}
&\kappa_A\omega^A(a\kappa^B + b\omega^B) + \omega_A(a\kappa^A + b\omega^A)\kappa^B + (a\kappa_A + b\omega_A)\kappa^A\omega^B \\
&= a\kappa_A\omega^A\kappa^B + b\kappa_A\omega^A\omega^B + a\omega_A\kappa^A\kappa^B + b\omega_A\kappa^A\omega^B \\
&= a\kappa_A\omega^A\kappa^B + b\kappa_A\omega^A\omega^B - a\omega^A\kappa_A\kappa^B - b\omega^A\kappa_A\omega^B \\
&= 0
\end{aligned} \tag{22}$$

□

From this identity, we get the following identity for ϵ_{AB}:

Lemma 4.3.

$$\epsilon_{AB}\epsilon_{CD} + \epsilon_{BC}\epsilon_{AD} + \epsilon_{CA}\epsilon_{BD} = 0 \tag{23}$$

Proof. Another way of expressing eq. (21) is

$$(\epsilon_{AB}\epsilon_C{}^D + \epsilon_{BC}\epsilon_A{}^D + \epsilon_{CA}\epsilon_B{}^D)\kappa^A\omega^B\tau^C = 0 \tag{24}$$

Since κ^A, ω^B and τ^C were arbitrary, we must then have that

$$\epsilon_{AB}\epsilon_C{}^D + \epsilon_{BC}\epsilon_A{}^D + \epsilon_{CA}\epsilon_B{}^D = 0$$

When we lower D, we get the required identity. □

Equipped with these lemmas, we are ready to prove that we can 'extract' any antisymmetric part of a spinor.

Theorem 4.4. Let $\psi_{\ldots A \ldots B \ldots}$ be antisymmetric in A and B. Then

$$\psi_{\ldots A \ldots B \ldots} = \frac{1}{2}\epsilon_{AB}\psi_{\ldots C \ldots}{}^C{}_{\ldots} \tag{25}$$

Proof. Define $\phi_{\ldots AB} = \psi_{\ldots A \ldots B \ldots}$. Then $\phi_{\ldots AB}$ is still skew in A and B. If we raise C and D in eq. (23), we get the equation

$$\epsilon_A{}^C\epsilon_B{}^D - \epsilon_B{}^C\epsilon_A{}^D = \epsilon_{AB}\epsilon^{CD}$$

This implies that

$$\begin{aligned}
({\epsilon_A}^C {\epsilon_B}^D - {\epsilon_B}^C {\epsilon_A}^D)\phi_{\dots CD} &= \epsilon_{AB}\epsilon^{CD}\phi_{\dots CD} \\
\iff \phi_{\dots AB} - \phi_{\dots BA} &= \epsilon_{AB}{\phi_{\dots C}}^C \\
\iff \phi_{\dots AB} &= \frac{1}{2}\epsilon_{AB}{\phi_{\dots C}}^C \\
\iff \psi_{\dots A \dots B \dots} &= \frac{1}{2}\epsilon_{AB}{\psi_{\dots C \dots}}^C{}_{\dots}
\end{aligned}$$

□

Now that we've discussed some properties of spinors in a space-time, it's time to present some examples so that these processes become a bit more tangible. In Minkowski space-time, we can identify vectors with spinors and vice-versa by letting the Infeld-Van der Waerden symbols be the Pauli matrices.

Example 4.5. (Identification of vectors in Minkowski-space-time with spinors). Let $x^0 = t, x^1 = x, x^2 = y, x^3 = z$ be a vector in Minkowski space-time. Let $\sigma_0^{AA'}$ be the 2×2 identity matrix and let $\sigma_i^{AA'}$ with $i = 1, 2, 3$ be the Pauli spin-matrices. Then

$$x^\mu \sigma_\mu^{AB'} = \begin{pmatrix} t+z & x+iy \\ x-iy & t-z \end{pmatrix} = \kappa^{AB'} \tag{26}$$

denotes a spinorial tensor of type $(1, 0, 1, 0)$. Note that the determinant of this matrix is $t^2 - x^2 - y^2 - z^2$, the original space-time norm of the vector. If the vector was spacelike or timelike, there is not much more we can do now. If however the vector was null, the matrix has zero determinant. That means that the columns are proportional to each other and we can decompose it further into the product of a column vector (spinor ξ^A) and its Hermitian conjugate (conjugate spinor $\overline{\xi}^{A'}$): Let $\xi^0 = \sqrt{t+z}$ and let $\xi^1 = \sqrt{t-z}e^{-i \arctan(y/x)}$ [22]. Then $\kappa^{AB'} = \xi^A \overline{\xi}^{B'}$. We see that this decomposition is determined up to an arbitrary phase factor $e^{i\theta}$ because it gets cancelled by multiplication with the complex conjugate. Conversely, given a spinor ξ^A we can always create a null vector x^μ by multiplying with its complex conjugate spinor. If $\xi^0 = A, \xi^1 = B$ with $A, B \in \mathbb{C}$, we get that $t = \frac{1}{2}(A\overline{A} + B\overline{B}), x = \frac{1}{2}(B\overline{A} + A\overline{B}), y = \frac{1}{2i}(B\overline{A} - A\overline{B}), z = \frac{1}{2}(A\overline{A} - B\overline{B})$.

Now that we've seen how we can relate a normal vector to a spin vector, it might also be instructive to see that the fundamental symmetries of the tensors $SO(3,1)$ and spinors $SL(2,\mathbb{C})$ indeed correspond in a 2:1 manner. As an example here, we will choose spatial x-rotations. Of course, we could do a similar identification for rotations around the other axes and for the 3 different types of boosts.

Example 4.6. ($SL(2,\mathbb{C})$ and $SO(3,1)$ equivalence for spatial x-rotations). Let the following denote an arbitrary rotation around the x-axis in space, rotating the vector x^μ by angle θ. Then we have

$$\begin{pmatrix} \tilde{t} \\ \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\theta) & -\sin(\theta) \\ 0 & 0 & \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} t \\ x \\ \cos(\theta)y - \sin(\theta)z \\ \sin(\theta)y + \cos(\theta)z \end{pmatrix} \tag{27}$$

Now we write this into spinor form using eq. (26), so that we obtain

$$\begin{pmatrix} \tilde{t} + \tilde{z} & \tilde{x} + i\tilde{y} \\ \tilde{x} - i\tilde{y} & \tilde{t} + \tilde{z} \end{pmatrix} = \begin{pmatrix} t + (\sin(\theta)y + \cos(\theta)z) & x + i(\cos(\theta)y - \sin(\theta)z) \\ x - i((\cos(\theta)y - \sin(\theta)z)) & t - (\sin(\theta)y + \cos(\theta)z) \end{pmatrix} \tag{28}$$

We can now use a corresponding $SL(2,\mathbb{C})$ matrix to generate this same transformation when starting in spinor form, by multiplying on the left and right with conjugate transpose matrices:

$$
\begin{aligned}
&\begin{pmatrix} \tilde{t}+\tilde{z} & \tilde{x}+i\tilde{y} \\ \tilde{x}-i\tilde{y} & \tilde{t}+\tilde{z} \end{pmatrix} = \\
&\begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & i\sin\left(\frac{\theta}{2}\right) \\ i\sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{pmatrix} \begin{pmatrix} t+z & x+iy \\ x-it & t-z \end{pmatrix} \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & i\sin\left(\frac{\theta}{2}\right) \\ i\sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{pmatrix}^{\dagger} \\
&= \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\end{aligned} \tag{29}
$$

with

$$
\begin{aligned}
a = \cos\left(\frac{\theta}{2}\right)&\left(\cos\left(\frac{\theta}{2}\right)(t+z) + i\sin\left(\frac{\theta}{2}\right)(x-iy)\right) \\
&- i\sin\left(\frac{\theta}{2}\right)\left(i\sin\left(\frac{\theta}{2}\right)(t-z) + \cos\left(\frac{\theta}{2}\right)(x+iy)\right) \\
b = \cos\left(\frac{\theta}{2}\right)&\left(i\sin\left(\frac{\theta}{2}\right)(t-z) + \cos\left(\frac{\theta}{2}\right)(x+iy)\right) \\
&- i\sin\left(\frac{\theta}{2}\right)\left(\cos\left(\frac{\theta}{2}\right)(t+z) + i\sin\left(\frac{\theta}{2}\right)(x-iy)\right) \\
c = \cos\left(\frac{\theta}{2}\right)&\left(i\sin\left(\frac{\theta}{2}\right)(t+z) + \cos\left(\frac{\theta}{2}\right)(x-iy)\right) \\
&- i\sin\left(\frac{\theta}{2}\right)\left(\cos\left(\frac{\theta}{2}\right)(t-z) + i\sin\left(\frac{\theta}{2}\right)(x+iy)\right) \\
d = \cos\left(\frac{\theta}{2}\right)&\left(\cos\left(\frac{\theta}{2}\right)(t-z) + i\sin\left(\frac{\theta}{2}\right)(x+iy)\right) \\
&- i\sin\left(\frac{\theta}{2}\right)\left(i\sin\left(\frac{\theta}{2}\right)(t+z) + \cos\left(\frac{\theta}{2}\right)(x-iy)\right)
\end{aligned}
$$

This then simplifies into

$$
\begin{pmatrix} t+(\sin(\theta)y+\cos(\theta)z) & x+i(\cos(\theta)y-\sin(\theta)z) \\ x-i((\cos(\theta)y-\sin(\theta)z)) & t-(\sin(\theta)y+\cos(\theta)z) \end{pmatrix} \tag{30}
$$

just like we had before. We clearly see that the same $SL(2,\mathbb{C})$ matrix with a minus sign would have yielded the same Lorentz transformation because the minus would have been present also in the complex conjugate and cancelled out.

The question that now may still remain is: why are we even concerned with spinors? Are tensors not enough? To answer this question, we introduce the null flag, which every spinor has associated with it.

Definition 4.7. (Null flag of a spinor). Given a spinor ψ^A, we define the null flag (real tensor)

$$
F^{AA'BB'} = \psi^A\psi^B\overline{\epsilon}^{A'B'} + \overline{\psi}^{A'}\overline{\psi}^{B'}\epsilon^{AB} \tag{31}
$$

We can view this null flag as a tensor field of type (2, 0) on space-time. Tensor fields are objects which we know how to measure and interpret. We take the physically measurable properties of a spinor ξ^A to be those determinable from the null flag. We see that ξ^A and $-\xi^A$ have the same null flag so that they are physically indistinguishable. However, the dynamical evolution of physical fields represented by spinors is given by differential equations involving the spinor fields themselves and not just their null flags. Then sign differences can affect the evolution of the field. Therefore, in a sense, spinor fields contain more physically relevant information than present in just their null flag fields. Furthermore, we can use spin fields to describe fields in an elegant way. This is also the purpose we will be using them for in a later chapter. If the field has spin s, we can represent the field by a totally symmetric spinor $\phi_{A_1 \cdots A_{2s}}$ with $2s$ indices [1]. Since we can always exchange a tensor index for 2 spinor indices, we could just as well use normal tensors to describe a field of integer spin. However, if the spin is not integer a representation using tensors is not possible and we can only use spinors.

4.3 Covariant derivative

In general relativity, there are some more assumptions added to those of special relativity. Those are the weak equivalence principle [23] and Einstein's equivalence principle (EEP), which states that locally (meaning in a small enough area of space and small enough region of time) the laws of physics should reduce to those of special relativity. It turns out that our mathematical model adheres to these principles (which is why it was chosen as a model in the first place). Since many physical laws involve derivatives, we require a definition of derivatives in our space-time. The difficulty in defining a derivative in a curved space-time is that the (co)tangent spaces aren't directly identifiable like they are in a flat space-time. Hence, we need the derivative/connection to do this for us. If we want to compare quantities at different points on a manifold or quantify the rate of change of a tensor field, we need that the derivative of a tensor is also a tensor. Otherwise, any statements we make about the rate of change of a tensor are dependent on the coordinate system we use. Secondly, we want the derivative to satisfy linearity and the product rule. Thirdly, we want the derivative to commute with contractions. Lastly, it should reduce to the usual partial derivative when applied to a scalar field. We denote a partial derivative by $\partial_\mu = \frac{\partial}{\partial x^\mu}$, and denote our covariant derivative by ∇_μ. We will show that the partial derivative on a scalar field transforms like a tensor (one-form specifically).

$$\tilde{\partial}_\alpha \tilde{\Phi} = \frac{\partial \tilde{\Phi}}{\partial \tilde{x}^\alpha} = \frac{\partial x^\mu}{\partial \tilde{x}^\alpha} \frac{\partial \tilde{\Phi}}{\partial x^\mu} = \frac{\partial x^\mu}{\partial \tilde{x}^\alpha} \partial_\mu \Phi \tag{32}$$

Hence we can define

$$\nabla_\mu \Phi = \partial_\mu \Phi \tag{33}$$

It turns out that for any other type of tensor, the partial derivative does not transform as a tensor and is therefore not suitable as a covariant derivative. If we define the covariant derivative of a vector field

$$\nabla_\mu v^\nu = \partial_\mu v^\nu + \Gamma^\nu_{\mu\rho} v^\rho \tag{34}$$

where

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} (\partial_\rho g_{\nu\sigma} + \partial_\nu g_{\rho\sigma} - \partial_\sigma g_{\nu\rho}) \tag{35}$$

is called the Christoffel symbol, we do get an entity that transforms like a tensor. The Christoffel symbol object does have indices just like a tensor but is not a tensor (hence why it is called a symbol and not a tensor). It is symmetric in the bottom indices. For one-form fields, we define the covariant derivative as

$$\nabla_\mu a_\nu = \partial_\mu v_\nu - \Gamma^\rho_{\mu\nu} a_\rho \tag{36}$$

and for general tensor fields, we can define

$$\begin{aligned}
&\nabla_\rho T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} = \partial_\rho T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} \\
&+ \Gamma^{\mu_1}_{\rho\sigma} T^{\sigma\mu_2 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} + \Gamma^{\mu_2}_{\rho\sigma} T^{\mu_1 \sigma \mu_3 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} + \cdots + \Gamma^{\mu_k}_{\rho\sigma} T^{\mu_1 \cdots \mu_{k-1}\sigma}{}_{\nu_1 \cdots \nu_l} \\
&- \Gamma^{\sigma}_{\rho\nu_1} T^{\mu_1 \cdots \mu_k}{}_{\sigma \nu_2 \cdots \nu_l} - \Gamma^{\sigma}_{\rho\nu_2} T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \sigma \nu_3 \cdots \nu_l} - \cdots - \Gamma^{\sigma}_{\rho\nu_l} T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_{l-1}\sigma}
\end{aligned} \tag{37}$$

If we define the covariant derivative in this way, it satisfies all of the properties that we wanted before. Furthermore, it turns out that this covariant derivative is metric preserving (so $\nabla_\rho g_{\mu\nu} = 0$) and torsion-free (the Christoffel symbol is symmetric), making it the Levi-Civita connection [24]. It turns out that at a point p we can always change coordinates so that the metric evaluated at that point is the Minkowski metric, and that the first derivative of the metric is 0 at that point. Then all Christoffel symbols vanish. This can be seen as the mathematical realisation of Einstein's equivalence principle. We call such a coordinate system a local inertial system.

We can also use the covariant derivative to calculate the derivative along a curve. Let $x^\mu(r)$ denote a curve parameterised by s. We define the covariant derivative along this curve as

$$\frac{D}{ds} = \frac{dx^\mu}{ds} \nabla_\mu \tag{38}$$

$\frac{dx^\mu}{ds}$ defines a vector tangent to the curve, so we see that this notion of a derivative along a curve coincides with the normal definition of the directional derivative given by the inner product of the gradient and a vector specifying the direction in which the derivative is to be taken. The covariant derivative for spinors follows from the covariant derivative for tensors. We define

$$\nabla_{AB'} = \sigma^\mu_{AB'} \nabla_\mu \tag{39}$$

This defines the covariant derivative for spinors uniquely [21] if we additionally require that $\sigma^\mu_{AB'}$ and ϵ_{AB} are covariantly constant:

$$\nabla_\mu \epsilon_{AB} = 0, \quad \nabla_\nu \sigma^\mu_{AB'} = 0 \tag{40}$$

A derivative along a curve is defined analogously for spinors as it was for vectors, one-forms and tensors, where we just transfer the tangent vector and covariant derivative to the spinor world using the $\sigma^\mu_{AB'}$ quantities.

4.3.1 Parallel transport

Now that we have defined the covariant derivative, we can compare vectors, one-forms, tensors and spinors at different points in space-time. However, one might question what it means for vectors to be parallel at different points in a curved space-time. To answer this question as best we can, we develop the notion of parallel transport. Consider a curve $x^\mu(r)$ that goes from p to q as r goes from 0 to 1, where a vector v^μ is defined at p. We then extend this vector to be a vector field on the curve by demanding that the covariant derivative along the curve is zero.

$$\frac{D}{dr}v^\mu = \frac{dx^\nu}{dr}\nabla_\nu v^\mu = \frac{dx^\nu}{dr}\left(\partial_\nu v^\mu + \Gamma^\mu_{\nu\rho}v^\nu\right) = \frac{dv^\mu}{dr} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{dr}v^\rho = 0 \tag{41}$$

From this we get the parallel transported vector $v^\mu(q)$. In Minkowski space, the Christoffel symbol is 0 and so the parallel transport of a vector v^μ along a curve $x^\mu(r)$ means that $\frac{dv^\mu}{dr} = 0$. This just means that the components of v^μ stay the same along the curve. The notion of parallel transported vectors is path-independent in the Minkowski case. However, in a general curved space time, the parallel transport does depend on the path taken. One can use this fact to give an alternate characterisation of the curvature of space-time [17]! Of course, we can also use parallel transport on one-forms, general tensors and spinors, by also requiring that their derivative along the curve is 0.

4.4 Causality

In our daily lives, we often refer to the future and the past, but what do these things mean on our manifold? We can define them as follow: At each point in M the timelike vectors in the point's tangent space can be divided into two classes. We define the following equivalence relation on pairs of timelike tangent vectors. We say that $v^\mu \sim w^\mu$ if $g_{\mu\nu}v^\mu w^\nu < 0$. Then there are two equivalence classes which contain all of the timelike vectors at that point. We can arbitrarily call one of the classes future-directed and the other one past-directed. Physically this corresponds to a choice of time arrow at that point. These timelike directions can be extended to null vectors at a point by continuity. We call a space-time time-orientable if a continuous choice of future-directed and past-directed non-spacelike vectors can be made over the entire manifold. Now we make a classification of some types of curves:

- We call a curve timelike if the tangent vector is timelike at all points in the curve.
- We call a curve null if the tangent vector is null at all points in the curve.
- We call a curve spacelike if the tangent vector is spacelike at all points in the curve.
- We call a curve causal (or non-spacelike) if the tangent vector is timelike or null at all points in the curve.

If the space-time is time orientable, the non-spacelike curves can be classified further depending on their orientation with respect to time. A timelike, null or causal curve in M is

- future-directed if the tangent vector is future-directed for every point in the curve.

- past-directed if the tangent vector is past-directed for every point in the curve.

On time-orientable space-times, we can then define a causal relationship between points. For example, we say that x strictly causally precedes y if there exists a future-directed causal curve from x to y. We say that x causally precedes y if x strictly causally precedes y or if $x = y$. We call a space-time strongly causal if the following condition holds:

Definition 4.8. (Strongly causal space-time). A space-time $(M, g_{\mu\nu})$ is strongly causal if, for all $p \in M$ and every neighbourhood U of p, there exists a neighbourhood V of p contained in U such that no causal curve intersects V more than once.

If a space-time violates strong causality at p, there exist causal curves near p that come arbitrarily close to intersecting themselves. In such a spacetime one could produce closed causal curves by a small modification of the metric in an arbitrarily small neighbourhood of p [3]. Closed causal curves present some interpretational difficulties and are seen as unphysical. Our own universe is believed to be strongly causal (but most likely satisfies even stronger conditions).

4.5 Geodesics

In flat Euclidean space, the shortest route between two points is always a straight line. However, on a curved space, we have to adapt our notion of a straight line. For example, we cannot travel in a Euclidean straight line on a sphere, because if we were to follow the straight line, we would fall off of the sphere. However if on Earth we travel in a line that looks straight to us, we still do take the shortest route. Hence, we can adapt the notion of a straight line to be a path of minimal distance, a so-called geodesic. In space-time, distance can be negative or positive. Therefore a spacelike geodesic is a curve that minimizes the proper distance between two events and a timelike geodesic is a curve that maximizes the proper time between the events. The geodesic equation can be derived through the principle of least action [17]. A curve $x^\mu(r)$, where r is an affine parameter of the curve (in the case of space-, or timelike curves usually proper distance s or proper time τ), is then a geodesic if

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{ds}\frac{dx^\rho}{ds} = 0 \tag{42}$$

Now, suppose a particle follows a timelike curve $x^\mu(\tau)$ parameterised by its proper time τ. We define its velocity as

$$u^\mu = \frac{dx^\mu}{d\tau} \tag{43}$$

This is a vector for each point on the curve. Therefore, we can take the covariant derivative along the curve using eq. (38)

$$\begin{aligned} a^\mu &= \frac{D}{d\tau}u^\mu = \frac{D}{d\tau}\frac{dx^\mu}{d\tau} = \frac{dx^\nu}{d\tau}\nabla_\nu\left(\frac{dx^\mu}{d\tau}\right) \\ &= \frac{dx^\nu}{d\tau}\left(\frac{\partial}{\partial x^\nu}\left(\frac{dx^\mu}{d\tau}\right) + \Gamma^\mu_{\nu\rho}\frac{dx^\rho}{d\tau}\right) = \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} \end{aligned} \tag{44}$$

If we have $a^\mu = 0$, we recognise this as the geodesic equation! Therefore, the geodesic equation is also the equation of motion for freely falling particles. Hence, we could rewrite the geodesic equation into

$$\frac{D}{d\tau}\frac{dx^\mu}{d\tau} = 0 \tag{45}$$

We now also recognise this as the equation for parallel transport (eq. (41)), where the vector that is transported is the tangent vector of the curve! We cannot parameterise null geodesics by either s or τ since they're both 0 along the entire curve. Hence, we define a curve $x^\mu(r)$ to be a null geodesic if the following holds:

$$\frac{D}{dr}\frac{dx^\mu}{dr} = 0, \quad g_{\mu\nu}\frac{dx^\mu}{dr}\frac{dx^\nu}{dr} = 0 \tag{46}$$

where the second condition ensures that the curve is actually a null curve, i.e. that any infinitesimally separated points on the curve are null separated. We call r an affine parameter of the null geodesic.

4.6 Einstein's field equations

In classical physics, Newton's equation of gravity is used to perform calculations where the masses of the bodies involved are significant enough to exhibit a noticeable gravitational force. In general relativity, we no longer see gravitation as a result of a force acting on a body, but as the result of space-time curving. This then means that the geodesics no longer look like straight lines to us, and since we 'see in flat space' the trajectories seem to curve. Before we introduce the field equations, we introduce some quantities that we will be using. We first define the Riemann curvature tensor as

$$R^\mu{}_{\nu\rho\sigma} = -\partial_\rho\Gamma^\mu_{\nu\sigma} + \partial_\sigma\Gamma^\mu_{\nu\rho} - \Gamma^\mu_{\alpha\rho}\Gamma^\alpha_{\nu\sigma} + \Gamma^\mu_{\alpha\sigma}\Gamma^\alpha_{\nu\rho} \tag{47}$$

although other authors use different sign conventions, see [25]. This does not affect the physics of course, but does lead to some sign differences in certain formulae. This tensor is highly significant in general relativity as it gives a measure of whether there is curvature or not. It comes up in the geodesic deviation equation, which describes (as the name suggests) the deviation between two nearby geodesics. This deviation is caused by the tidal gravitational force. It also contains information about how volume changes due to the curvature. If we can find a point p in space-time where the Riemann curvature tensor is 0, we can always find a coordinate system where [17]

$$g_{\mu\nu}(p) = \eta_{\mu\nu}, \quad (\partial_\rho g_{\mu\nu})(p) = 0, \quad (\partial_\rho\partial_\sigma g_{\mu\nu})(p) = 0 \tag{48}$$

This shows that the Riemann curvature tensor contains all coordinate-independent information about the derivative of the metric at that event. Because the Minkowski metric is constant, the derivatives of the metric vanish which means that all of the Christoffel symbols are zero everywhere (see eq. (35)). Therefore, in Minkowski space, $R^\mu{}_{\nu\rho\sigma} = 0$ everywhere. Because $R^\mu{}_{\nu\rho\sigma}$ is a tensor, this must then hold in any coordinate system. The converse also turns out to be true. If $R^\mu{}_{\nu\rho\sigma} = 0$ at all events in a spacetime. Then one can always find a coordinate system such that the metric is that of Minkowski space-time so that $g_{\mu\nu} = \eta_{\mu\nu}$ [17]. In principle, the global structure could still be different, but these scenarios are not physically viable. The Riemann curvature tensor has some interesting properties. We have the following symmetries: upon permutation of the first two or last two indices

$$R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho} = -R_{\nu\mu\rho\sigma} \tag{49}$$

and for exchanging the first two and last two indices we get

$$R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu} \tag{50}$$

Upon permutation of the last three indices we get

$$R_{\mu\nu\rho\sigma} + R_{\mu\sigma\nu\rho} + R_{\mu\rho\sigma\nu} \tag{51}$$

Lastly, we have the Bianchi identity

$$\nabla_\alpha R_{\mu\nu\rho\sigma} + \nabla_\nu R_{\alpha\mu\rho\sigma} + \nabla_\mu R_{\nu\alpha\rho\sigma} = 0, \quad \text{or} \quad \nabla_{[\alpha} R_{\mu\nu]\rho\sigma} = 0 \tag{52}$$

We can also see that the curvature tensor is the commutator of the covariant derivative of a one-form:

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) a_\rho = 2\nabla_{[\mu} \nabla_{\nu]} a_\rho = R^\rho{}_{\sigma\mu\nu} a_\rho \tag{53}$$

Then Riemann tensor also comes up when commuting the covariant derivative of other quantities, see for example [26]. This property can be seen as an alternative definition for the Riemann tensor. We define the Ricci tensor as a contraction of the Riemann tensor with itself:

$$R_{\mu\nu} = R^\rho{}_{\mu\rho\nu} \tag{54}$$

The Ricci tensor is symmetric

$$R_{\mu\nu} = R_{\nu\mu} \tag{55}$$

We further define the Ricci scalar as the trace of the Ricci tensor

$$R = g^{\mu\nu} R_{\mu\nu} = R^\mu{}_\mu \tag{56}$$

Lastly, we define the stress-energy tensor as the tensor $T^{\mu\nu}$ that gives the flux of the μ'th component of the momentum 4-vector across a surface with a constant x^ν coordinate. This tensor is also symmetric and is conserved

$$T^{\mu\nu} = T^{\nu\mu}, \quad \nabla_\mu T^{\mu\nu} = 0 \tag{57}$$

This definition is quite abstract, but its components have the following interpretations:

- T^{00}: energy density.
- T^{i0}: density of the x^i component of the momentum.
- T^{0j}: energy flux through the surface perpendicular to x^j.
- T^{ij}: internal forces per unit area, such as the pressure.

here, as usual $i, j = 1, 2, 3$. The equations that relate the curvature of space-time to the energy and mass that are present are called Einstein's field equations and are given by

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = -8\pi G T_{\mu\nu} \tag{58}$$

where G denotes the Newtonian gravitational constant and Λ denotes the cosmological constant, related to the expansion of the universe. Notice the difference with the now more standard form where the RHS is positive. This is due to our sign conventions. Note that now we have set the signature of the metric to $(+,-,-,-)$ because that's the signature we will be using this with later, while in the next chapter, the signature will be $(-,+,+,+)$ again. This equation can be derived from the Lagrangian principle of least action [17]. We can simplify this equation if we assume that we are in empty space, meaning that $T_{\mu\nu} = 0$. We then get

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0$$

Upon contracting this equation with $g^{\mu\nu}$, we obtain that $R = 4\Lambda$, which we can then plug back into the equation. Then we get

$$R_{\mu\nu} = \Lambda g_{\mu\nu} \tag{59}$$

This equation is called the vacuum Einstein equation. We lastly define two more objects characterising curvature, the Schouten tensor and the Weyl curvature tensor. These are both useful in the setting of conformal transformations, which is also where we will use them in a later chapter. We define the Schouten tensor as

$$P_{\mu\nu} = \frac{1}{2}R_{\mu\nu} - \frac{1}{12}Rg_{\mu\nu} \tag{60}$$

The Weyl tensor is the trace-free part of the curvature tensor and is linked to the Riemann curvature tensor and Schouten tensors by

$$C^{\mu\nu}{}_{\rho\sigma} = R^{\mu\nu}{}_{\rho\sigma} - 4P^{[\mu}{}_{[\rho}\,\delta^{\nu]}{}_{\sigma]} \tag{61}$$

This tensor satisfies the Riemann tensor symmetries and $C^{\mu}{}_{\nu\mu\sigma} = 0$ (expressing part of the tracelessness). The Weyl tensor, just like the Riemann curvature tensor, expresses the tidal force that a body feels when moving along a geodesic. It differs from the Riemann curvature tensor in that it does not contain information on how the volume of the body changes, but only how the shape of the body changes due to the tidal force.

5 Compactification of Minkowski space-time

Now that we have gone through the mathematical preliminaries and the basics of general relativity, it is time to start working towards connecting these fundamentals with the peeling-off behaviour as described by Penrose. We dedicate this section to looking at the compactification of the simplest space-time, Minkowski space-time, and try to get some idea of how this space behaves at infinity. Then we extend this concept and conformally embed this compactification in the Einstein static universe. We follow the same approach as in [2], although here a (to my knowledge unique) proof that this construction is actually a mathematical compactification is provided. Other approaches, with perhaps slightly different features, are possible, see for example [27].

We start by transforming the Minkowski metric to spherical coordinates, after which we obtain

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2) = -dt^2 + dr^2 + r^2 d\Omega^2 \tag{62}$$

if we let $d\Omega^2 = d\theta^2 + \sin^2(\theta)d\phi^2$. This metric looks to be singular for $r = 0$ and $\sin(\theta) = 0$, but this is because these coordinates are not admissable. To obtain regular coordinate neighborhoods the coordinates have to be restricted to for example $0 < r < \infty, 0 < \theta < \pi, 0 < \phi < 2\pi$. We would then actually need two charts to cover the whole space.

Now choose the following coordinate transformation of advanced and retarded null-coordinates:

$$v = t + r, \quad w = t - r$$

Then

$$r = \frac{v-w}{2}, dt = \frac{dv+dw}{2}, dr = \frac{dv-dw}{2}$$

Upon substitution in eq. (62), we obtain the following:

$$\begin{aligned} ds^2 &= -\left(\frac{dv+dw}{2}\right)^2 + \left(\frac{dv-dw}{2}\right)^2 + \left(\frac{v-w}{2}\right)^2 d\Omega^2 \\ &= -dvdw + \frac{1}{4}(v-w)^2 d\Omega^2 \end{aligned} \tag{63}$$

Here $-\infty < v < \infty, -\infty < w < \infty$ (and $v \geq w$). Notice the absence of dv^2 and dw^2, because those surfaces are lightlike. We now introduce another coordinate transformation, which will make the coordinate ranges finite. We define p and q as follows:

$$\tan(p) = v, \tan(q) = w$$

Then we get for the differentials

$$dv = \frac{1}{\cos^2(p)}dp, dw = \frac{1}{\cos^2(q)}dq$$

Upon substitution of this into eq. (63), we obtain:

$$\begin{aligned} ds^2 &= -\frac{dp}{\cos^2(p)}\frac{dq}{\cos^2(q)} + \frac{1}{4}(\tan(p) - \tan(q))^2 d\Omega^2 \\ &= -\frac{dp}{\cos^2(p)}\frac{dq}{\cos^2(q)} + \frac{1}{4}\left(\frac{\sin^2(p)}{\cos^2(p)} - 2\frac{\sin(p)}{\cos(p)}\frac{\sin(q)}{\cos(q)} + \frac{\sin^2(q)}{\cos^2(q)}\right) d\Omega^2 \\ &= \frac{1}{\cos^2(p)\cos^2(q)}(-dpdq + \frac{1}{4}[\sin^2(p)\cos^2(q) \\ &\quad - 2\sin(p)\cos(p)\sin(q)\cos(q) + \sin^2(q)\cos^2(p)]d\Omega^2) \end{aligned}$$

So that in the end

$$ds^2 = \frac{1}{\cos^2(p)\cos^2(q)}\left(-dpdq + \frac{1}{4}\sin^2(p-q)d\Omega^2\right) \tag{64}$$

where $-\frac{1}{2}\pi < p < \frac{1}{2}\pi$ and $-\frac{1}{2}\pi < q < \frac{1}{2}\pi$ (and $p \geq q$). The way that I've presented this equation already hints at what is to come in a bit, the conformal embedding. But first, we define what it means for two metrics to be conformal and finish the compactification.

Definition 5.1. Two metrics $g_{\mu\nu}$ and $h_{\mu\nu}$ are conformal if $g_{\mu\nu} = \lambda^2 h_{\mu\nu}$, with λ a smooth, real-valued and non-zero function.

For two conformal metrics, we have that for any four vectors $x^\mu, y^\mu, v^\mu, w^\mu$ at a point p, we have

$$\frac{g_{\mu\nu}x^\mu y^\nu}{g_{\alpha\beta}v^\alpha w^\beta} = \frac{h_{\mu\nu}x^\mu y^\nu}{h_{\alpha\beta}v^\alpha w^\beta}$$

and

$$\frac{(g_{\mu\nu}x^\mu y^\nu)^2}{g_{\alpha\beta}x^\alpha x^\beta g_{\rho\sigma}y^\rho y^\sigma} = \frac{(h_{\mu\nu}x^\mu y^\nu)^2}{h_{\alpha\beta}x^\alpha x^\beta h_{\rho\sigma}y^\rho y^\sigma}$$

so that ratios of magnitudes and angles are preserved under conformal transformations. Furthermore, the null cone structure is preserved by conformal transformations, since

$$g_{\mu\nu}x^\mu x^\nu > 0, = 0, < 0 \implies h_{\mu\nu}x^\mu x^\nu > 0, = 0, < 0$$

respectively. We see that the metric we found for Minkowski space in eq. (64) has the form $g_{\mu\nu} = \lambda^2 h_{\mu\nu}$, where $\lambda = \frac{1}{2\cos(p)\cos(q)}$, which is a smooth, real valued and nonzero function. Hence the metric is conformal to the metric $h_{\mu\nu}$ given by $d\tilde{s}^2 = -4dpdq + \sin^2(p-q)d\Omega^2$.

We can transform $g_{\mu\nu}$ back to a more usual form by letting $t' = p + q, r' = p - q$. We then get $2dp = dt' + dr'$ and $2dq = dt' - dr'$, so that

$$d\tilde{s}^2 = -4\frac{dt' + dr'}{2}\frac{dt' - dr'}{2} + \sin^2(\frac{t' + r'}{2} - \frac{t' - r'}{2})d\Omega^2$$

Simplifying this, we have

$$d\tilde{s}^2 = -(dt')^2 + (dr')^2 + \sin^2(r')d\Omega^2 \tag{65}$$

where

$$t' + r' < \pi, \quad t' - r' > -\pi, \quad r' > 0 \tag{66}$$

The whole of Minkowski space-time is then given by

$$ds^2 = \frac{1}{4}\frac{1}{\cos^2(\frac{1}{2}(t' + r'))\cos^2(\frac{1}{2}(t' - r'))}d\tilde{s}^2 \tag{67}$$

with the coordinate region given by eq. (66). The total transformation we have now done, starting from the spherical metric given by eq. (62), is then:

$$\begin{aligned} 2t &= \tan\left(\frac{1}{2}(t' + r')\right) + \tan\left(\frac{1}{2}(t' - r')\right) \\ 2r &= \tan\left(\frac{1}{2}(t' + r')\right) - \tan\left(\frac{1}{2}(t' - r')\right) \end{aligned} \tag{68}$$

This transformation yields a homeomorphism from $\mathbb{R} \times \mathbb{R}^+ \to \{t', r' : t' + r' < \pi, \quad t' - r' > -\pi, \quad r' > 0\} = X$. Now, let $Y = \{t', r' : t' + r' \leq \pi, \quad t' - r' \geq -\pi, \quad r' \geq 0\} \subseteq \mathbb{R}^2$. Then Y is closed and bounded and therefore compact. $\mathbb{S}^2$ is also closed and bounded when seen as a subset of $\mathbb{R}^3$ and therefore compact since compactness is a topological property by theorem 2.4. By theorem 2.5, the product $Y \times \mathbb{S}^2$ is also compact. We know from example 1.11 and example 3.10 that $\mathbb{R}^4 \backslash \{t\text{-axis}\} \simeq \mathbb{R}^2 \times \mathbb{S}^2 \simeq \mathbb{R} \times \mathbb{R}^+ \times \mathbb{S}^2$. Since eq. (68) is injective from $\mathbb{R} \times \mathbb{R}^+$ onto Y, by extending it to be the identity map on $\mathbb{S}^2$ gives a diffeomorphism $\mathbb{R}^4 \backslash \{t\text{-axis}\} \simeq \mathbb{R} \times \mathbb{R}^+ \times \mathbb{S}^2 \to X \times \mathbb{S}^2 \subseteq Y \times \mathbb{S}^2$ and is, therefore, an embedding. Furthermore, $Y \times \mathbb{S}^2 = \overline{X} \times \overline{\mathbb{S}^2} = \overline{X \times \mathbb{S}^2}$ due to theorem 2.7 and hence $X \times \mathbb{S}^2$ is dense in $Y \times \mathbb{S}^2$. Therefore we conclude that $Y \times \mathbb{S}^2$ together with the extension of the transformation given by eq. (68) is a compactification of Minkowski space-time minus the spatial origin.

It is instructive to study the boundary of the compactification since it represents the conformal structure of infinity of Minkowski space-time: The boundary corresponding to infinity consists of the null surfaces $t' + r' = \pi$ (labelled $\mathscr{I}^+$) and $t' - r' = -\pi$ (labelled $\mathscr{I}^-$), together with the points $t' = \pi, r' = 0$ (labelled i^+), $t' = 0, r' = \pi$ (labelled i^0) and $t' = -\pi, r' = 0$ (labelled i^-). See fig. 3 for an illustration of this region. We also see that the origin is reincluded in the compactification. At first glance, it looks like the line segment corresponding to $r' = 0$ is part of the boundary, but the diagram is misleading in this case since it is actually part of the interior. When you visualise one of the angular dimensions in 3-D by rotating the triangle around this line segment, this becomes clear. Every point in this diagram corresponds to a 2-sphere, except for i^0 and points with $r' = 0$ (including i^+ and i^-). The fact that i^0 is only a point can be seen from the 1-point compactification of R^3, which only introduces one extra point and not a 2-sphere. We can see that every timelike geodesic originates at i^- and ends at i^+. Similarly, every spacelike geodesic originates at i^0 and ends at i^0, although that is harder to see from the figure since the spherical symmetry is not shown. We can again try to visualise one of the angular dimensions, but it is easier to regard the line $r' = 0$ as a mirror to help visualise this process. Lastly one can regard light rays as originating at $\mathscr{I}^-$ and ending at $\mathscr{I}^+$, which can again be seen from the visual by imagining that the line $r' = 0$ acts like a mirror. Hence, we can see that i^+ and i^- represent future and past timelike infinity, $\mathscr{I}^+$ and $\mathscr{I}^-$ represent future and past null infinity and lastly, i^0 represents spacelike infinity. Non-geodesics do not obey these rules as they could be unphysical and hence any shape you wish for, like a circle, which doesn't originate or end anywhere. We have already introduced the concept of a conformal metric, but we haven't used it yet in the compactification (except maybe to simplify our expressions). We do this in the next subsection.

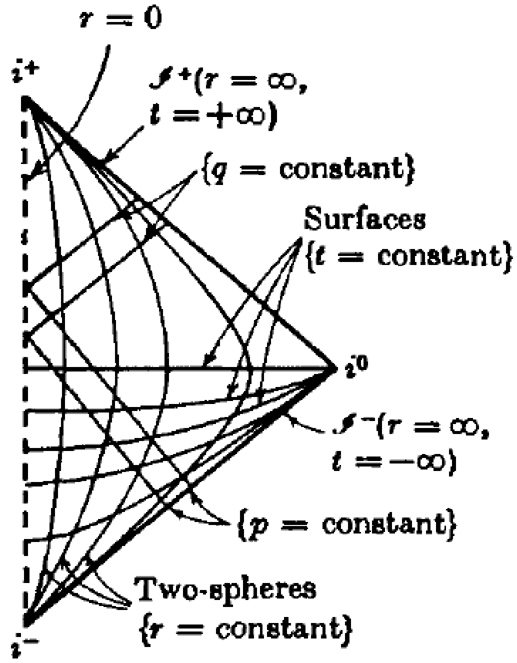

Figure 3: This figure shows the region Y, together with its boundary. This diagram is also called the Penrose diagram of Minkowski space-time. Each point represents a two-sphere of radius 1, except for i^+, i^-, i^0 and points with $r' = 0$ (represented by the dotted line). Radial null geodesics are represented by straight lines at angles of $\pm 45°$. Image from [2].

5.1 Einstein static universe

In this subsection, we will be conformally embedding this space into the Einstein static universe, the cylinder $\mathbb{R} \times \mathbb{S}^3$. We first discuss the Einstein static universe, which has a metric derived from the Friedmann-Lemaître-Robertson-Walker (FLRW) metric that describes the space-time geometry at the largest scale. Then we derive the natural metric on the three-sphere and finally relate it to the Einstein static universe. The FLRW metric is given by:

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega\right) = -dt^2 + a^2(t) d\sigma^2 \tag{69}$$

This metric is determined by $k \in \{-1, 0, 1\}$ and the scale factor $a(t)$. $k = 0$ corresponds to zero curvature, $k = -1$ corresponds to negative curvature and $k = 1$ corresponds to positive curvature.

We can introduce a new radial coordinate χ by demanding that $d\chi = \frac{dr}{\sqrt{1-kr^2}}$ and that $r = 0$ corresponds to $\chi = 0$. There are then three options:

$$r(\chi) = \begin{cases} \sinh(\chi) & \text{if } k = -1 \\ \chi & \text{if } k = 0 \\ \sin(\chi) & \text{if } k = 1 \end{cases}$$

For $k = 1$ the $d\sigma^2$ part of the metric becomes

$$d\sigma^2 = a^2 \left(d\chi^2 + \sin^2(\chi) d\Omega^2\right) \tag{70}$$

It turns out that this is exactly the same as the metric on $\mathbb{S}^3$!

Theorem 5.2. (Line-element on $\mathbb{S}^3$). The line element on $\mathbb{S}^3$ is given by

$$ds^2 = da^2 + a^2 d\chi^2 + a^2 \sin^2(\chi) d\phi^2 \sin^2(\theta) + a^2 \sin^2(\chi) d\theta^2$$

Proof. We can parameterise the 3-sphere of radius a using the following coordinates:

$$\begin{aligned} y^1 &= a\cos(\chi) \\ y^2 &= a\sin(\chi)\cos(\theta) \\ y^3 &= a\sin(\chi)\sin(\theta)\cos(\phi) \\ y^4 &= a\sin(\chi)\sin(\theta)\sin(\phi) \end{aligned}$$

Where $0 < a < \infty$, $0 < \chi, \theta < \pi$ and $0 < \phi < 2\pi$. Then their infinitesimal differences are given by:

$$\begin{aligned} dy^1 &= da\cos(\chi) - a\sin(\chi)d\chi \\ dy^2 &= da\sin(\chi)\cos(\theta) + a\cos(\chi)d\chi\cos(\theta) - a\sin(\chi)\sin(\theta)d\theta \\ dy^3 &= da\sin(\chi)\sin(\theta)\cos(\phi) + a\cos(\chi)d\chi\sin(\theta)\cos(\phi) + \\ &\quad a\sin(\chi)\cos(\theta)d\theta\cos(\phi) - a\sin(\chi)\sin(\theta)\sin(\phi)d\phi \\ dy^4 &= da\sin(\chi)\sin(\theta)\sin(\phi) + a\cos(\chi)d\chi\sin(\theta)\sin(\phi) + \\ &\quad a\sin(\chi)\cos(\theta)d\theta\sin(\phi) + a\sin(\chi)\sin(\theta)\cos(\phi)d\phi \end{aligned}$$

This still parameterises $\mathbb{R}^4$, so we can plug this into the Euclidean metric to obtain a very long equation:

$$
\begin{aligned}
ds^2 = \; & a^2 d\chi^2 \sin^2(\chi) + \\
& a^2 d\chi^2 \sin^2(\phi)\sin^2(\theta)\cos^2(\chi) + \\
& a^2 d\chi^2 \sin^2(\theta)\cos^2(\chi)\cos^2(\phi) + \\
& a^2 d\chi^2 \cos^2(\chi)\cos^2(\theta) + \\
& 2a^2 d\chi d\theta \sin(\chi)\sin^2(\phi)\sin(\theta)\cos(\chi)\cos(\theta) + \\
& 2a^2 d\chi d\theta \sin(\chi)\sin(\theta)\cos(\chi)\cos^2(\phi)\cos(\theta) - \\
& 2a^2 d\chi d\theta \sin(\chi)\sin(\theta)\cos(\chi)\cos(\theta) + \\
& a^2 d\phi^2 \sin^2(\chi)\sin^2(\phi)\sin^2(\theta) + \\
& a^2 d\phi^2 \sin^2(\chi)\sin^2(\theta)\cos^2(\phi) + \\
& a^2 d\theta^2 \sin^2(\chi)\sin^2(\phi)\cos^2(\theta) + \\
& a^2 d\theta^2 \sin^2(\chi)\sin^2(\theta) + \\
& a^2 d\theta^2 \sin^2(\chi)\cos^2(\phi)\cos^2(\theta) + \\
& 2adad\chi \sin(\chi)\sin^2(\phi)\sin^2(\theta)\cos(\chi) + \\
& 2adad\chi \sin(\chi)\sin^2(\theta)\cos(\chi)\cos^2(\phi) + \\
& 2adad\chi \sin(\chi)\cos(\chi)\cos^2(\theta) - \\
& 2adad\chi \sin(\chi)\cos(\chi) + \\
& 2adad\theta \sin^2(\chi)\sin^2(\phi)\sin(\theta)\cos(\theta) + \\
& 2adad\theta \sin^2(\chi)\sin(\theta)\cos^2(\phi)\cos(\theta) - \\
& 2adad\theta \sin^2(\chi)\sin(\theta)\cos(\theta) + \\
& da^2 \sin^2(\chi)\sin^2(\phi)\sin^2(\theta) + \\
& da^2 \sin^2(\chi)\sin^2(\theta)\cos^2(\phi) + \\
& da^2 \sin^2(\chi)\cos^2(\theta) + da^2\cos^2(\chi)
\end{aligned}
$$

Upon repeated use of the identity $\sin^2(x)+\cos^2(x) = 1$ this reduces greatly, so that we obtain:

$$
ds^2 = da^2 + a^2 d\chi^2 + a^2 \sin^2(\chi) d\phi^2 \sin^2(\theta) + a^2 \sin^2(\chi) d\theta^2
$$

If we are moving on the sphere, we cannot move radially, which means that $da = 0$ and we get

$$
\begin{aligned}
ds^2 &= a^2 d\chi^2 + a^2 \sin^2(\chi) d\phi^2 \sin^2(\theta) + a^2 \sin^2(\chi) d\theta^2 \\
&= a^2 \left(d\chi^2 + \sin^2(\chi) d\Omega\right) = d\sigma^2
\end{aligned}
\tag{71}
$$

This is the line element on $\mathbb{S}^3$. □

With the line element on the three-sphere in hand, we are now ready to conformally embed Minkowski space into the Einstein static universe.

Theorem 5.3. (Embedding of Minkowski space in the Einstein static universe). The Minkowski space can be conformally embedded into the cylinder $x^2+y^2+z^2+w^2 = 1$ in a five-dimensional Minkowski space with metric $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 + dw^2$.

Proof. We know that we can use the one-point compactification of a hyperplane to get a sphere using the stereographical projection: $\mathbb{R}^3 \cup \{\infty\} \simeq \mathbb{S}^3$. If we now add another $\mathbb{R}$ and extend the projection to identity on this extra dimension, we can view $\mathbb{R}^4$ as being embedded in the cylinder $\mathbb{R} \times \mathbb{S}^3$. We will now show that the cylinder has a metric conformal to the one given in *eq.* (65). The equation $x^2 + y^2 + z^2 + w^2 = 1$ parameterises a 3-sphere with radius 1, which has the metric: $d\sigma^2 = \left(d\chi^2 + \sin^2(\chi)d\Omega\right)$ as shown in eq. (71). This leaves the t-coordinate untouched so that the final metric indeed becomes $ds^2 = -dt^2 + d\sigma^2 = -dt^2 + \left(d\chi^2 + \sin^2(\chi)d\Omega\right)$, which is conformal to the metric of the Minkowski-space as given by eq. (65). Hence, if we regard r', θ and ϕ as coordinates on $\mathbb{S}^3$, we see that we have covered a subspace of the cylinder $\mathbb{R} \times \mathbb{S}^3$. To make this explicit, we can embed $X \times \mathbb{S}^2$, using $f : X \times \mathbb{S}^2 \to \mathbb{R} \times \mathbb{S}^3$ given by $f(t', r', \theta, \phi) = (t', r', \theta, \phi)$, indeed leaving the metric unchanged. If we now again take the closure, we get a similar picture as before, where we reinclude the origin and also include the infinity boundary consisting of $\mathscr{I}^+, \mathscr{I}^-$ and i^+, i^0 and i^-. We again have coordinate singularities, this time at $\sin(r') = 0$ and $\sin(\theta) = 0$, which can be worked around by moving to a different chart. ☐

If we visualise this embedding, we can do the following: in the description of the compactification from before, we stated that the line $r' = 0$ acted as a mirror of sorts. Now that we are in a cylinder, we can make this behaviour slightly more explicit by making one of the previously suppressed angular dimensions visible again. We can do this by mapping points (t', r', θ, ϕ) with $0 < \phi < \pi$ to the right half of the cylinder and (t', r', θ, ϕ) with $-\pi < \phi < 0$ to the left half. The picture we then get can be seen in fig. 4. Now every point in this image represents half of a two-sphere with area $4\pi \sin^2(r')$. Points with $r' = 0$ (again including i^+ and i^-) and i^0 again represent points instead of two-spheres.

At every point of $\mathscr{I}^+$, we can span the tangent space using the vectors (using the basis t', r', θ, ϕ)

$$x^\mu = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad y^\mu = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad z^\mu = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{72}$$

We see that the tangent space is spanned by one lightlike vector (x^μ) and two spacelike vectors (y^μ and z^μ). Furthermore, the vector x^μ is orthogonal to x^μ, y^μ and z^μ. This means that a null vector is normal to the hypersurface and that the hypersurface is therefore null. We also see that on $\mathscr{I}^+$, we have $t' = \pi - r'$, so that $dt' = -dr'$. Then the pullback of the metric onto the tangent space becomes $d\tilde{s}^2 = \sin^2(r')d\Omega^2$ which is clearly degenerate. By taking a cross-section of $\mathscr{I}^+$ at a certain t', we reduce the basis of the tangent space to y^μ, z^μ and we get that $dt' = 0$ on this cross-section so that the pullback of the metric onto the tangent space is still $d\tilde{s}^2 = \sin^2(r')d\Omega^2$. In this 2-dimensional case this metric is non-degenerate and has signature $(+, +)$. This cross-section is therefore space-like. Futhermore, the metric is conformal to the metric on $\mathbb{S}^2$ with conformal factor $\lambda = \sin(r')$. We will use a similar construction to get a spacelike cross-section of the boundary $\mathscr{I}$ (which turns out to also be topologically $\mathbb{S}^2$ more or less by definition) when proving the peeling-off behaviour of the gravitational field.

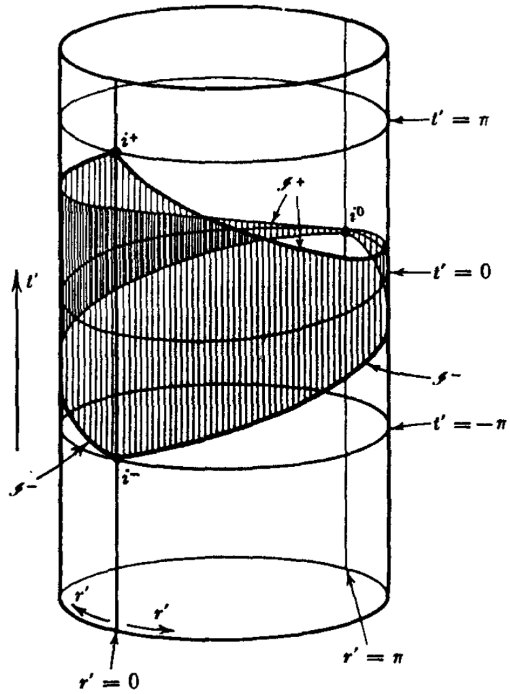

Figure 4: The Einstein static universe represented by an embedded cylinder, where the coordinates θ and ϕ have been suppressed. The shaded region is conformal to the whole of Minkowski space-time. The boundary can be regarded as the conformal infinity of the Minkowski space-time. Each point represents half of a two-sphere of area $4\pi \sin^2(r')$, meaning that i^0 and points with $r' = 0$ (including i^+ and i^-) are points instead of two-spheres. Image taken from [2].

6 Asymptotically flat space-time

As one goes very far away from any matter or energy, the space-time should start to look more and more flat, and so more and more like Minkowski space. Since this behaviour is mainly concerned with what happens very far away, this suggests that we define space-time to be asymptotically flat if we can perform a similar construction to the compactification of Minkowski spacetime: conformally mapping the physical space-time into a new unphysical space-time with similar properties to the Minkowski case. There are two important properties of the construction of conformal infinity for Minkowski space-time which do not carry over to general curved space-times. We want to consider space-times that become flat as we go to 'large distances' in spacelike or null directions, but we do not want to require that space-time also becomes flat at a fixed position at 'early or late times', since we may want to describe a spacetime representing isolated bodies which may remain present at those 'early or late times'. Hence, we can't expect the conformal infinity of a curved space-time to have similar properties to the Minkowski case at past and future timeline infinity, i^+ and i^-. Hence, we do not expect or require these to be present in the general case. In general, a point 'at spatial infinity' will be present but since we are only interested in the behaviour at null infinity, we will exclude this point from our definition. We now extract some properties from the Minkowski case to obtain a sort of preliminary asymptotic flatness called asymptotic simplicity. Note that from now on, we will be using a metric signature of $(+,-,-,-)$ again.

Definition 6.1. (Asymptotically simple space-time). A strongly causal space-time $(\tilde{M}, \tilde{g}_{\mu\nu})$ is called asymptotically simple if some manifold M with boundary $\mathscr{I}(\subseteq M)$ and metric $g_{\mu\nu}$ (called the conformal completion of $\tilde{M}$) exist, whose interior $M \backslash \mathscr{I}$ is conformal to $\tilde{M}$ with $g_{\mu\nu} = \lambda^2 \tilde{g}_{\mu\nu}$ and which satisfies the following properties:

1. M and $g_{\mu\nu}$ are sufficiently differentiable (say C^4 and C^3) everywhere.
2. Defining $\lambda = 0$ on $\mathscr{I}$, λ is sufficiently differentiable (say C^3) everywhere. Furthermore $\nabla_\mu \lambda \neq 0$ on $\mathscr{I}$.
3. Every null geodesic in the interior of M contains, if maximally extended, two distinct points on $\mathscr{I}$.

Remark 6.2. Note that we do not require M to be compact, we only require it to have a boundary (which can have holes, so M does not have to be closed). This is the reason why we call M the conformal completion and not the conformal compactification. However, since we are working with manifolds (which are Hausdorff by assumption), their compactifications are closed. In many physical cases we are working with (products of) embedded submanifolds of $\mathbb{R}^n$, for which the notion of compactness even coincides with being closed and bounded. In those cases, we can often take M to be a compactification of $\tilde{M}$ and then possibly exclude points like i^0, i^+ and i^- in the Minkowski case where the differentiability requirements are too strong. If the map of the compactification is denoted by f, we can take the conformal metric to be $g_{\mu\nu} = \lambda^2 f^*(\tilde{g}_{\mu\nu})$.

The reason for condition 3 is to ensure that $\mathscr{I}$ contains the whole of null infinity for $\tilde{M}$. This condition is a very strong global condition on the physical spacetime. Since null geodesics are mapped to null geodesics under conformal transformation, see section 7, it requires every null geodesic to go off to infinity. Therefore it involves much more than only the asymptotic behaviour. As an example, there are circular null orbits in Schwarzschild's solution which never reach infinity. Thus, this spacetime is not asymptotically simple, even though it would intuitively qualify as being asymptotically flat. The condition on the geodesics can be modified to become an asymptotic condition, so that the space-time becomes weakly asymptotically simple, although we won't be discussing that here.

To gain some familiarity with how some quantities change when going from M to $\tilde{M}$, we first derive and state some transformation formulae. Then we will investigate how the zero rest-mass equation behaves under conformal transformations. Lastly, we discuss the influence of the cosmological constant on the behaviour at infinity and add a condition to asymptotic simplicity to make it asymptotically flat.

6.1 Conformal transformation formulae

Suppose we have the following conformal transformation:

$$\tilde{g}_{\mu\nu} = \lambda^{-2} g_{\mu\nu}, \quad \tilde{g}^{\mu\nu} = \lambda^2 g^{\mu\nu} \tag{73}$$

Note that this is not a coordinate transformation. The coordinates we use are still the same, we have just mapped the metric into a different metric in a manner depending on space and time. In general, we will use a tilde to indicate that we are using the conformal metric. We denote the covariant derivative determined by the $\tilde{g}_{\mu\nu}$ metric as $\tilde{\nabla}$. We get the following for scalar fields:

$$\tilde{\nabla}_\mu \theta = \tilde{\partial}_\mu \theta = \partial_\mu \theta = \nabla_\mu \theta \tag{74}$$

We have $\tilde{\partial}_\mu = \partial_\mu$ because the spaces use the same coordinates. In particular, this also holds for λ itself since it can be seen as a scalar field. Recall that the Christoffel symbol is defined by eq. (35), so that

$$\begin{aligned}
\tilde{\Gamma}^\mu_{\nu\rho} &= \frac{1}{2}\tilde{g}^{\mu\sigma}\left(\tilde{\partial}_\rho \tilde{g}_{\nu\sigma} + \tilde{\partial}_\nu \tilde{g}_{\rho\sigma} - \tilde{\partial}_\sigma \tilde{g}_{\nu\rho}\right) \\
&= \frac{1}{2}\lambda^2 g^{\mu\sigma}\left(\partial_\rho(\lambda^{-2} g_{\nu\sigma}) + \partial_\nu(\lambda^{-2} g_{\rho\sigma}) - \partial_\sigma(\lambda^{-2} g_{\nu\rho})\right) \\
&= \frac{1}{2}\lambda^2 g^{\mu\sigma}(g_{\nu\sigma}\partial_\rho \lambda^{-2} + \lambda^{-2}\partial_\rho g_{\nu\sigma} + g_{\rho\sigma}\partial_\nu \lambda^{-2} + \lambda^{-2}\partial_\nu g_{\rho\sigma} \\
&\quad - g_{\nu\rho}\partial_\sigma \lambda^{-2} - \lambda^{-2}\partial_\sigma g_{\nu\rho}) \\
&= \frac{1}{2} g^{\mu\sigma}(\partial_\rho g_{\nu\sigma} + \partial_\nu g_{\rho\sigma} - \partial_\sigma g_{\nu\rho}) - \frac{g^{\mu\sigma}}{\lambda}(g_{\nu\sigma}\partial_\rho\lambda + g_{\rho\sigma}\partial_\nu\lambda - g_{\nu\rho}\partial_\sigma\lambda) \\
&= \Gamma^\mu_{\nu\rho} - \frac{1}{\lambda}(\delta^\mu_\nu \partial_\rho\lambda + \delta^\mu_\rho \partial_\nu\lambda - g_{\nu\rho}\partial^\mu\lambda) \\
&= \Gamma^\mu_{\nu\rho} - \frac{1}{\lambda}(\delta^\mu_\nu \nabla_\rho\lambda + \delta^\mu_\rho \nabla_\nu\lambda - g_{\nu\rho}\nabla^\mu\lambda)
\end{aligned} \tag{75}$$

Using this result for the Christoffel symbol, we find the transformed derivative of a one-form as follows:

$$
\begin{aligned}
\tilde{\nabla}_\mu k_\nu &= \tilde{\partial}_\mu k_\nu - \tilde{\Gamma}^\rho_{\mu\nu} k_\rho \\
&= \partial_\mu k_\nu - \left(\Gamma^\rho_{\mu\nu} - \frac{1}{\lambda}(\delta^\rho_\mu \nabla_\nu \lambda + \delta^\rho_\nu \nabla_\mu \lambda - g_{\mu\nu}\nabla^\rho \lambda\right) k_\rho \\
&= \partial_\mu k_\nu - \Gamma^\rho_{\mu\nu} k_\rho + \frac{1}{\lambda}(\delta^\rho_\mu \nabla_\nu \lambda + \delta^\rho_\nu \nabla_\mu \lambda - g_{\mu\nu}\nabla^\rho \lambda) k_\rho \\
&= \partial_\mu k_\nu - \Gamma^\rho_{\mu\nu} k_\rho + \frac{1}{\lambda} k_\mu \nabla_\nu \lambda + \frac{1}{\lambda} k_\nu \nabla_\mu \lambda - \frac{1}{\lambda} g_{\mu\nu} k_\rho \nabla^\rho \lambda \\
&= \nabla_\mu k_\nu + 2\frac{1}{\lambda} k_{(\mu} \nabla_{\nu)} \lambda - \frac{1}{\lambda} g_{\mu\nu} k_\rho \nabla^\rho \lambda
\end{aligned} \tag{76}
$$

If we want to raise or lower indices, all quantities with a tilde use $\tilde{g}^{\mu\nu}$ and $\tilde{g}_{\mu\nu}$, and quantities without a tilde use $g^{\mu\nu}$ and $g_{\mu\nu}$. We can treat contravariant indices by lowering them and treat general tensors by treating them as products as we did before. As we can see, the process of deriving all of these transformation formulae is quite arduous, and will not be done here for every quantity that we need. Therefore, I will try to point to other sources where derivations of these quantities can be found (by mentioning specific sections if possible). We will need the transformation of the Schouten tensor, which transforms as (see [21] section 6.8):

$$
\tilde{P}_{\mu\nu} - \lambda^{-1}\nabla_\mu \nabla_\nu \lambda + \frac{1}{2}\lambda^{-2} g_{\mu\nu}(\nabla_\rho \lambda)(\nabla^\rho \lambda) \tag{77}
$$

The Weyl tensor transforms as (this follows from the transformation of the Weyl spinor, which we will introduce in a bit):

$$
\tilde{C}^\mu{}_{\nu\rho\sigma} = C^\mu{}_{\nu\rho\sigma} \tag{78}
$$

Because the metric carries a conformal factor, we obtain that

$$
\tilde{C}_{\mu\nu\rho\sigma} = \lambda^{-2} C_{\mu\nu\rho\sigma} \tag{79}
$$

This is an example of the conformal transformation being dependent on the positioning of the indices. The Weyl tensor is conserved under conformal transformations only if the indices are positioned right. Before we throw spinors into the mix, we introduce the concept of a conformal density. A quantity $\xi^{\cdots}_{\cdots}$ is called a conformal density of weight m if

$$
\tilde{\xi}^{\cdots}_{\cdots} = \lambda^m \xi^{\cdots}_{\cdots}
$$

To treat spinors, we define

$$
\tilde{\sigma}^{AB'}_\mu = \lambda^{-1}\sigma^{AB'}_\mu, \quad \tilde{\sigma}^\mu_{AB'} = \lambda \sigma^\mu_{AB'}, \quad \tilde{\epsilon}_{AB} = \epsilon_{AB} \tag{80}
$$

This makes sure that eq. (16) and eq. (73) stay consistent. Then raising or lowering of spinor indices does not affect the weight of a conformal density. A general spinor conformal density with weight m transforms as

$$\tilde{\xi}_{A_1\cdots A_l B_1'\cdots B_{l'}} = \lambda^m \xi_{A_1\cdots A_l B_1'\cdots B_{l'}} \tag{81}$$

If we denote the total number of indices as $r = l + l'$, the covariant derivative of this spinor conformal density transforms as (see section 5.6 of [21] for a derivation)

$$\begin{aligned}
&\tilde{\nabla}_{XY'}\tilde{\xi}_{A_1\cdots A_l B_1'\cdots B_{l'}'} = \lambda^{m+1}\nabla_{XY'}\xi_{A_1\cdots A_l B_1'\cdots B_{l'}'} \\
&+ \lambda^m\{(m - \frac{1}{2}r)\xi_{A_1\cdots A_l B_1'\cdots B_{l'}'}\nabla_{XY'}\lambda \\
&+ \xi_{XA_2\cdots A_l B_1'\cdots B_{l'}'}\nabla_{A_1Y'}\lambda + ... + \xi_{A_1\cdots A_{l-1}XB_1'\cdots B_{l'}'}\nabla_{A_lY'}\lambda \\
&+ \xi_{A_1\cdots A_l Y'\cdots B_{l'}'}\nabla_{XB_1'}\lambda + ... + \xi_{A_1\cdots A_l B_1'\cdots B_{l'-1}'Y'}\nabla_{XB_{l'}'}\lambda\}
\end{aligned} \tag{82}$$

Now we can decompose the Riemann tensor and the Weyl tensor into spinor form. For a derivation, see section 4.6 of [21]. The Riemann tensor decomposes as:

$$\begin{aligned}
R_{\mu\nu\rho\tau}\sigma^\mu_{AA'}\sigma^\nu_{BB'}\sigma^\rho_{CC'}\sigma^\tau_{DD'} = {} & \Psi_{ABCD}\epsilon_{E'F'}\epsilon_{G'H'} + \epsilon_{AB}\epsilon_{CD}\overline{\Psi}_{E'F'G'H'} \\
& + 2k(\epsilon_{AC}\epsilon_{BD}\epsilon_{E'F'}\epsilon_{G'H'} + \epsilon_{AB}\epsilon_{CD}\epsilon_{E'H'}\epsilon_{F'G'}) \\
& + \epsilon_{AB}\Phi_{CDEF'}\epsilon_{G'H'} \\
& + \epsilon_{CD}\Phi_{ABGH'}\epsilon_{E'F'},
\end{aligned} \tag{83}$$

Where $k = \frac{1}{24}R$. The Weyl tensor decomposes as

$$C_{\mu\nu\rho\sigma}\sigma^\mu_{AA'}\sigma^\nu_{BB'}\sigma^\rho_{CC'}\sigma^\tau_{DD'} = \Psi_{ABCD}\epsilon_{E'F'}\epsilon_{G'H'} + \epsilon_{AB}\epsilon_{CD}\overline{\Psi}_{E'F'G'H'} \tag{84}$$

This is why Ψ_{ABCD} is called the Weyl spinor. The trace-free Ricci tensor decomposes as

$$\sigma^\mu_{AB'}\sigma^\nu_{CD'}(R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu}) = -2\Phi_{ABC'D'} \tag{85}$$

which is why $\Phi_{ABC'D'}$ is called the Ricci spinor. These spinors have the properties

$$\Psi_{ABCD} = \Psi_{(ABCD)}, \quad \Phi_{ABC'D'} = \Phi_{(AB)(C'D')} = \overline{\Phi}_{C'D'AB} \tag{86}$$

The Weyl spinor transforms as

$$\tilde{\Psi}_{ABCD} = \lambda^2\Psi_{ABCD} \tag{87}$$

and the Ricci spinor transforms as

$$\tilde{\Phi}^{C'D'}_{AB} = \lambda^2\Phi^{C'D'}_{AB} + \lambda\nabla^{C'}_{(A}\nabla^{D'}_{B)}\lambda \tag{88}$$

See again section 6.8 of [21] for these transformations. Do note the important difference in raising/lowering indices with the book. Here we have defined the Levi-Civita symbols to be conformally invariant, whereas the Infeld-van der Waerden symbols carry a conformal constant. The book does this in the opposite way. Furthermore, we are transforming in the opposite direction (λ^{-2} instead of λ^2) from the book. We make eq. (78) align with the book (since this is independent of both the Levi-Civita/Infeld-van der Waerden convention and the direction of transformation) and modify the other quantities to align with our conventions. It follows from the Bianchi identity eq. (52) that (see section 4.10 of [21])

$$\nabla^{AE'}\Psi_{ABCD} = 0 \tag{89}$$

Furthermore, when the vacuum field equations hold we have

$$\Phi_{ABC'D'} = 0 \tag{90}$$

Now that we have the transformations of various tensor and spinor quantities, we will investigate the behaviour of the zero rest-mass equation under a conformal transformation in the next subsection.

6.2 The zero rest-mass equation

We can denote a spin s zero-rest mass field by as $\phi_{A_1\cdots A_{2s}}$ which is totally symmetric and satisfies the spin $s(>0)$ zero rest-mass equation

$$\nabla^{A_1B'}\phi_{A_1\cdots A_{2s}} = 0 \tag{91}$$

If we still want this equation to hold after a conformal transformation, we must have that ϕ is a conformal density of weight $s+1$, so that

$$\tilde{\phi}_{A_1\cdots A_{2s}} = \lambda^{s+1}\phi_{A_1\cdots A_{2s}} \tag{92}$$

Then from eq. (82) follows that

$$\begin{aligned}
\tilde{\nabla}^{A_1B'}\tilde{\phi}_{A_1\cdots A_{2s}} =& \tilde{\epsilon}^{A_1X}\tilde{\epsilon}^{B'Y'}\tilde{\nabla}_{XY'}\tilde{\phi}_{A_1\cdots A_{2s}} = \epsilon^{A_1X}\epsilon^{B'Y'}\tilde{\nabla}_{XY'}\tilde{\phi}_{A_1\cdots A_{2s}} \\
=& \epsilon^{A_1X}\epsilon^{B'Y'}(\lambda^{s+2}\nabla_{XY'}\phi_{A_1\cdots A_{2s}} + \\
& \lambda^{s+1}\{(s+1-\frac{1}{2}2s)\phi_{A_1\cdots A_{2s}}\nabla_{XY'}\lambda + \phi_{XA_2\cdots A_{2s}}\nabla_{A_1Y'}\lambda + \\
& \cdots + \phi_{A_1\cdots A_{2s-1}X}\nabla_{A_{2s}Y'}\lambda\}) \\
=& \lambda^{s+2}\nabla^{A_1B'}\phi_{A_1\cdots A_{2s}} + \lambda^{s+1}\epsilon^{A_1X}\epsilon^{B'Y'}(\phi_{A_1\cdots A_{2s}}\nabla_{XY'}\lambda + \\
& \phi_{XA_2\cdots A_{2s}}\nabla_{A_1Y'}\lambda + \cdots + \phi_{A_1\cdots A_{2s-1}X}\nabla_{A_{2s}Y'}\lambda) \\
=& \lambda^{s+1}\epsilon^{A_1X}\epsilon^{B'Y'}(\phi_{A_1\cdots A_{2s}}\nabla_{XY'}\lambda + \phi_{XA_2\cdots A_{2s}}\nabla_{A_1Y'}\lambda + \cdots + \\
& \phi_{A_1\cdots A_{2s-1}X}\nabla_{A_{2s}Y'}\lambda)
\end{aligned} \tag{93}$$

Now we can use the ϵ's to raise either X and Y' to obtain

$$\lambda^{s+1}\left(\phi_{A_1\cdots A_{2s}}\nabla^{XY'}\lambda + \phi^{X}{}_{A_2\cdots A_{2s}}\nabla^{Y'}_{A_1}\lambda + \cdots + \phi_{A_1\cdots A_{2s-1}}{}^{X}\nabla^{Y'}_{A_{2s}}\lambda\right) \tag{94}$$

or we can raise A_1 and Y' to obtain

$$\lambda^{s+1}\left(-\phi^{A_1}{}_{A_2\cdots A_{2s}}\nabla^{Y'}_{X}\lambda - \phi_{XA_2\cdots A_{2s}}\nabla^{A_1Y'}\lambda - \cdots - \phi^{A_1}{}_{A_2\cdots A_{2s-1}X}\nabla^{Y'}_{A_{2s}}\lambda\right) \tag{95}$$

Since the labels themselves are irrelevant, we can exchange the labels A_1 and X and factor the minus to obtain

$$-\lambda^{s+1}\left(\phi^{X}{}_{A_2\cdots A_{2s}}\nabla^{Y'}_{A_1}\lambda + \phi_{A_1A_2\cdots A_{2s}}\nabla^{XY'}\lambda + \cdots + \phi^{X}{}_{A_2\cdots A_{2s-1}A_1}\nabla^{Y'}_{A_{2s}}\lambda\right) \tag{96}$$

Due to the total symmetry of ϕ, this is equal to what we had in eq. (94) with an extra minus sign. Therefore, it must be equal to 0 and hence we have obtained that $\tilde{\nabla}^{A_1B'}\tilde{\phi}_{A_1\cdots A_{2s}} = 0$ under this transformation.

6.3 The cosmological constant

We have defined what it means for a space-time to be asymptotically simple and discussed conformal transformations. What we have not yet done, is discuss how Einstein's field equations, which are assumed to hold in space-time, influence the conformal completion. From the conformal transformation of the Schouten tensor, we find that

$$\begin{aligned}\tilde{g}^{\mu\nu}\tilde{P}_{\mu\nu} &= \lambda^2 g^{\mu\nu}(P_{\mu\nu} - \lambda^{-1}\nabla_\mu\nabla_\nu\lambda + \frac{1}{2}\lambda^{-2}g_{\mu\nu}(\nabla_\rho\lambda)(\nabla^\rho\lambda)) \\ &= \frac{1}{6}\lambda^2 R - \lambda\nabla_\mu\nabla^\mu\lambda + 2(\nabla_\rho\lambda)(\nabla^\rho\lambda)\end{aligned} \tag{97}$$

Since we required that $\lambda = 0$ on $\mathscr{I}$ and that λ was sufficiently well behaved, we have that $\nabla_\mu\nabla^\mu\lambda$ and R are well behaved as well. Making use of this fact, we find that

$$\tilde{g}^{\mu\nu}\tilde{P}_{\mu\nu} = 2(\nabla_\rho\lambda)(\nabla^\rho\lambda) \text{ at } \mathscr{I} \tag{98}$$

Since we also have that $\tilde{g}^{\mu\nu}\tilde{P}_{\mu\nu} = \frac{1}{2}\tilde{R} - \frac{1}{3}\tilde{R} = \frac{1}{6}\tilde{R}$, we obtain

$$\frac{1}{6}\tilde{R} = 2(\nabla_\alpha\lambda)(\nabla^\alpha\lambda) \implies \tilde{R} = 12(\nabla_\alpha\lambda)(\nabla^\alpha\lambda) \tag{99}$$

We now assume that in an asymptotically simple space-time, Einstein's field equations hold. Then we have

$$\tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{R}\tilde{g}_{\mu\nu} + \Lambda\tilde{g}_{\mu\nu} = -8\pi G\tilde{T}_{\mu\nu} \tag{100}$$

In an asymptotically flat space-time, we expect no matter near infinity: the matter would make space-time curve, which means the space-time would not be asymptotically flat. Therefore, we can expect (the trace of) the stress-energy tensor to vanish at large distances in an asymptotically flat space-time (the trace is $\tilde{T}_{\mu}{}^{\mu} = \tilde{T}_{\mu\nu}\tilde{g}^{\mu\nu}$). Then we obtain from the Einstein equations that

$$\begin{aligned} \tilde{R}_{\mu\nu}\tilde{g}^{\mu\nu} - \frac{1}{2}\tilde{R}\tilde{g}_{\mu\nu}\tilde{g}^{\mu\nu} + \Lambda\tilde{g}_{\mu\nu}\tilde{g}^{\mu\nu} &= -8\pi G\tilde{T}_{\mu\nu}\tilde{g}^{\mu\nu} \\ \iff \tilde{R} - 2\tilde{R} + 4\Lambda &= 0 \\ \iff \tilde{R} &= 4\Lambda \\ \iff \Lambda &= 3(\nabla_\alpha\lambda)(\nabla^\alpha\lambda) \end{aligned} \tag{101}$$

From the conditions of an asymptotically simple space-time, the vector $\nabla^\alpha\lambda$ is non-zero. The vector $\nabla_\alpha\lambda$ is normal to $\mathscr{I}$: $\mathscr{I}$ can be seen as the hypersurface (or level set) generated by the equation $\lambda = 0$. Let $\gamma^\mu(a) = x^\mu(a)$ be any curve in $\mathscr{I}$ with $\gamma^\mu(0) = p^\mu$ being (the coordinate representation of) any arbitrary point $p \in \mathscr{I}$. Now let $g^\mu(a) = \lambda(\gamma^\mu(a))$. Since $\gamma^\mu(a)$ lies in $\mathscr{I}$ for all a, we have $g^\mu(a) = 0$ for all a. Then the covariant derivative of g along the curve is zero, so that

$$\frac{Dg}{da} = 0 \implies \frac{d\gamma^\mu}{da}\nabla_\mu\lambda = 0 \tag{102}$$

Since we started off with an arbitrary curve through an arbitrary point, this must hold for any arbitrary tangent vector $\frac{d\gamma^\mu}{da}$. We thus find that $\nabla_\mu\lambda$ is indeed normal to $\mathscr{I}$. This means that the boundary hypersurface $\mathscr{I}$ for an asymptotically simple space-time is timelike, spacelike or null depending on whether Λ is negative, positive or zero.

In an asymptotically flat space-time, we want the curvature to eventually go to zero. In other words, we want the metric to become the Minkowski metric. Since we also want the stress-energy tensor to vanish, the cosmological constant must be equal to zero to have the vacuum field equations generate the Minkowski metric (otherwise we get the (anti)-de Sitter solutions [2]). We then get the following definition for an asymptotically flat space-time:

Definition 6.3. (Asymptotically flat space-time). An asymptotically flat space-time is an asymptotically simple spacetime, has cosmological constant $\Lambda = 0$ and has a vanishing stress-energy tensor at $\mathscr{I}$.

With this definition, we find that for an asymptotically flat space-time, $\mathscr{I}$ is null. Therefore, M lies locally to the past or future of it. Hence, $\mathscr{I}$ must consist of two disconnected components. Just like we saw for Minkowski space we have a $\mathscr{I}^+$ on which null geodesics in M have their future endpoints and a $\mathscr{I}^-$ on which they have their past endpoints. There cannot be more than two components of $\mathscr{I}$ [2] and it turns out that in any asymptotically simple space with $\mathscr{I}$ null, both $\mathscr{I}^+$ and $\mathscr{I}^-$ are topologically $\mathbb{R} \times \mathbb{S}^2$ [2] [1]. $\mathbb{R} \times \mathbb{S}^2$ is a three-dimensional cylinder having cross-sections which are topological spheres ($\mathbb{S}^2$'s). The generators (topologically $\mathbb{R}$) of the cylinders are the null geodesics on $\mathscr{I}$. In particular this property then also holds for asymptotically flat space-times. We will use this fact in the following chapter when proving the peeling-off behaviour of the gravitational field.

7 The peeling-off property

In this final section, we follow the approach of Penrose [1] to derive the so-called peeling-off property of zero rest-mass fields. We have seen some properties of asymptotically simple and flat spacetimes in the previous chapter, including the transformations of various objects under conformal transformations. We will use these in this chapter to derive the peeling-off behaviour. First, we discuss the concept of the principal null directions, because the peeling-off behaviour can conveniently be understood in terms of these directions. Then we discuss the peeling-off property and prove it for the general class of so-called asymptotically regular fields. Then we will derive the property also for the Weyl spinor, where asymptotic regularity is not assumed. We discuss some consequences and then finish off the thesis with some concluding words.

7.1 The principal null directions

To derive the peeling-off property, we consider the principal null directions of our fields. These are directions in which 'the field vanishes'. We let ξ^A denote an arbitrary direction. We can then, for simplicity set $\xi^0 \to \xi^0/\xi^0 = 1, \xi^1 \to \xi^1/\xi^0 = \xi$ (unless $\xi^0 = 0$, but we can always choose our $\sigma_\mu^{AB'}$ so that this does not happen) since the direction is determined by the ratio of ξ^0 and ξ^1 and this is unchanged by our redefinition. We can denote the spin s field by $\phi_{A_1 \cdots A_{2s}}$. We consider the expression

$$\phi_{A_1 \cdots A_{2s}} \xi^{A_1} \cdots \xi^{A_{2s}} \tag{103}$$

This is just a complex polynomial in ξ and so can be factorised:

$$\begin{aligned} \phi_{A_1 \cdots A_{2s}} \xi^{A_1} \cdots \xi^{A_{2s}} &= (\alpha(1)_0 + \alpha(1)_1 \xi)(\alpha(2)_0 + \alpha(2)_1 \xi) \cdots (\alpha(2s)_0 + \alpha(2s)_1 \xi) \\ &= \alpha(1)_{A_1} \xi^{A_1} \alpha(2)_{A_2} \xi^{A_2} \cdots \alpha(2s)_{A_{2s}} \xi^{A_{2s}} \\ &= \alpha(1)_{A_1} \cdots \alpha(2s)_{A_{2s}} \xi^{A_1} \cdots \xi^{A_{2s}} \end{aligned} \tag{104}$$

where function notation is used on the α's instead of subscripts to avoid confusion with the spinor indices. Because the factorisation of a complex polynomial is unique (if $\phi \neq 0$), the coefficients $\alpha(i)_{A_i}$ are unique except for their ordering and multiplication by a nonzero complex number. For example $(x+2)(3x-1) = (-\frac{x}{3}i - \frac{2}{3}i)(9ix - 3i)$. If the degree of eq. (103) is lower than $2s$, say $2s - q$, we must have that q of the $\alpha(i)_{A_i}$ have their second component equal to 0 and hence the factors will be constants. Each $\alpha(i)_{A_i}$ gives rise to a null vector through

$$g^{\mu\nu} \sigma_\nu^{A_i A_i'} \alpha(i)_{A_i} \overline{\alpha(i)}_{A_i'} \tag{105}$$

These null vectors are determined up to proportionality by $\phi_{A_1 \cdots A_{2s}}$ and so the corresponding directions, called the principal null directions, are uniquely determined at each event in spacetime. We see that eq. (104) vanishes if one of the $\alpha(i)_{A_i}$ is a multiple of ξ_{A_i}, since $\xi_A \xi^A = 0$. Hence ξ^A corresponds to a principal null direction if $\phi_{A_1 \cdots A_{2s}} \xi^{A_1} \cdots \xi^{A_{2s}} = 0$. Now, we have from eq. (104)

$$\phi_{A_1\cdots A_{2s}} = \alpha(1)_{A_1}\cdots\alpha(2s)_{A_{2s}} = \alpha(1)_{(A_1}\cdots\alpha(2s)_{A_{2s})} \tag{106}$$

because ϕ is symmetric. Therefore we see that j principal null directions coincide with the direction corresponding to ξ^A if and only if

$$\phi_{A_1\cdots A_{2s}}\xi^{A_{j-1}}\cdots\xi^{A_{2s}} = \alpha(1)_{(A_1}\cdots\alpha(2s)_{A_{2s})}\xi^{A_{j-1}}\cdots\xi^{A_{2s}} = 0 \tag{107}$$

If we have j coincident principal null directions, we have j $\alpha(i)$'s such that the LHS vanishes when matched with ξ^A, while ξ^A appears $2s-j+1$ times. This ensures that then one of them is always matched up in every piece of the symmetrization, and so eq. (107) always holds. If conversely eq. (107) holds, we must have a minimum of j $\alpha(i)$'s proportional to ξ^{A_1} and we can create the j coincident principal null directions with those. If all $2s$ principal null directions coincide we call the field a null field.

7.2 The peeling-off property

The peeling-off property is the following. If r is an affine parameter of a null geodesic, the part of the radiation field that falls off as r^{-1} is effectively null. If we proceed inwards from infinity, we encounter zones where the higher order terms r^{-2}, r^{-3}... start to become important. The behaviour of the principal null direction is then that the field in the r^{k-1} zone (k ranges from $0, 1...2s$) has at least $2s-k$ coincident principal null directions pointing radially. Before we actually get started on deriving the peeling off property however, we will first need to prove that certain properties of our fields are conserved under conformal transformations.

7.2.1 Conformal invariance for the peeling off property

We look at a spin s field $\phi_{A_1\cdots A_{2s}}$ along a null geodesic γ affinely parameterised by r. To each point of γ, we assign a spinor ξ^A corresponding to the tangent (and therefore null) direction to γ at that point. Since $\xi^B\overline{\xi}^{C'}$ then denotes a tangent vector along the curve, $\sigma^{\mu}_{BC'}\xi^B\overline{\xi}^{C'}\nabla_\mu = \xi^B\overline{\xi}^{C'}\nabla_{BC'}$ is a parallel transport operator along γ (compare with eq. (41)). To make sure that ξ has 'the same norm' everywhere on the curve, we normalise ξ^A by setting

$$\xi^B\overline{\xi}^{C'}\nabla_{BC'}\xi_A = 0 \tag{108}$$

so that ξ^A is parelly propagated along g. Note that there is still the freedom to multiply ξ^A with an arbitrary complex constant, as long as this is done for all ξ^A on the curve. This condition only ensures that it is impossible to multiply ξ^A at different points on the curve by different complex constants. To preserve this under conformal transformations, we set

$$\tilde{\xi}_A = \lambda^{1/2}\xi_A \tag{109}$$

Then we get from eq. (82) that

$$
\begin{aligned}
\tilde{\xi}^B\tilde{\bar{\xi}}^{C'}\tilde{\nabla}_{BC'}\tilde{\xi}_A &= \lambda\xi^B\bar{\xi}^{C'}\left(\lambda^{\frac{3}{2}}\nabla_{BC'}\xi_A + \lambda^{\frac{1}{2}}\{(\frac{1}{2}-\frac{1}{2}\cdot 1)\xi_A\nabla_{BC'}\lambda + \xi_B\nabla_{AC'}\lambda\}\right) \\
&= \lambda^{\frac{5}{2}}\xi^B\bar{\xi}^{C'}\nabla_{BC'}\xi_A + \lambda^{\frac{3}{2}}\xi^B\bar{\xi}^{C'}\xi_B\nabla_{AC'}\lambda \\
&= 0
\end{aligned} \tag{110}
$$

where the first part is zero due to the assumed eq. (108) and the second part is zero due to the contraction $\xi^B\xi_B$. Note that setting the conformal tangent spinors in this way does not change the fact that they are tangent to the curve since they are still proportional to ξ^A and the conformal mapping doesn't change the curve ($\gamma = \tilde{\gamma}$). It only affects the normalisation of the tangent spinors. Hence, these null geodesics still parallel transport their tangent null vectors. The conformal invariance of eq. (108) then expresses that null geodesics transform to null geodesics under conformal transformations (we also saw this in our compactification of Minkowski space). Since we assume that γ is parameterised by r (e.g. $\gamma^\mu(r)$ denotes the event in space-time at r), we have

$$
\begin{aligned}
\xi^B\bar{\xi}^{C'}\nabla_{BC'}r &= \sigma_\mu^{BC'}\frac{d\gamma^\mu(r)}{dr}\sigma^\nu_{BC'}\nabla_\nu r = \delta^\nu_\mu\frac{d\gamma^\mu(r)}{dr}\nabla_\nu r \\
&= \frac{d\gamma^\mu(r)}{dr}\nabla_\mu r = \frac{D}{dr}r = 1
\end{aligned}
$$

To also preserve this under conformal transformations, we parameterise $\tilde{\gamma}$ by setting $d\tilde{r} = \lambda^{-2}dr$. Then we have

$$
\tilde{r} = \int d\tilde{r} = \int \lambda^{-2}dr
$$

so that ($\tilde{r}$ is a scalar field, so we use eq. (74))

$$
\begin{aligned}
\tilde{\xi}^B\tilde{\bar{\xi}}^{C'}\tilde{\nabla}_{BC'}\tilde{r} &= \lambda\xi^B\bar{\xi}^{C'}\tilde{\sigma}^\mu_{BC'}\tilde{\nabla}_\mu\int\lambda^{-2}dr \\
&= \lambda^2\xi^B\bar{\xi}^{C'}\sigma^\mu_{BC'}\nabla_\mu\int\lambda^{-2}dr \\
&= \lambda^2\xi^B\bar{\xi}^{C'}\nabla_{BC'}\int\lambda^{-2}dr \\
&= \lambda^2\frac{D}{dr}\int\lambda^{-2}dr \\
&= 1
\end{aligned}
$$

It is now useful to additionally introduce an auxiliary spinor η_A at each point $\gamma^\mu(r)$ which is also parallelly propagated along γ, so that

$$
\xi^B\bar{\xi}^{C'}\nabla_{BC'}\eta_A = 0 \tag{111}
$$

Furthermore, we choose η_A such that it is not a multiple of ξ_A. Then we can choose it so that

$$\xi_A \eta^A = 1$$

We again want these relations to be conserved under conformal maps, so we set

$$\tilde{\eta}_A = \lambda^{-1/2}\eta_A + b\lambda^{1/2}\xi_A$$

where

$$b = \int \lambda^{-2}\eta^B \bar{\xi}^{C'} \nabla_{BC'}\lambda dr$$

so that

$$\xi^B \bar{\xi}^{C'} \nabla_{BC'} b = \lambda^{-2}\eta^B \bar{\xi}^{C'} \nabla_{BC'}\lambda \tag{112}$$

Then we indeed find that

$$\tilde{\xi}_A \tilde{\eta}^A = \lambda^{\frac{1}{2}}\xi_A(\lambda^{-\frac{1}{2}}\eta^A + b\lambda^{\frac{1}{2}}\xi^A) = \xi_A\eta^A = 1$$

and

$$\begin{aligned}
\tilde{\xi}^B \tilde{\bar{\xi}}^{C'} \tilde{\nabla}_{BC'}\tilde{\eta}_A &= \lambda \xi^B \bar{\xi}^{C'} \tilde{\nabla}_{BC'}(\lambda^{-\frac{1}{2}}\eta_A + b\lambda^{\frac{1}{2}}\xi_A) \\
&= \lambda \xi^B \bar{\xi}^{C'} \tilde{\nabla}_{BC'}(\lambda^{-\frac{1}{2}}\eta_A) + \lambda^{\frac{3}{2}}\xi^B\bar{\xi}^{C'}\xi_A \tilde{\nabla}_{BC'} b + b \underbrace{\tilde{\xi}^B \tilde{\bar{\xi}}^{C'} \tilde{\nabla}_{BC'}\tilde{\xi}^A}_{\text{zero, see eq. (110)}} \\
&= \lambda^{\frac{1}{2}} \underbrace{\xi^B \bar{\xi}^{C'} (\lambda \nabla_{BC'}\eta_A}_{\text{zero, see eq. (111)}} - \eta_A \nabla_{BC'}\lambda + \underbrace{\eta_B}_{\to\text{-1}} \nabla_{AC'}\lambda + \underbrace{\lambda^2 \xi_A \nabla_{BC'} b)}_{\text{use eq. (112)}} \\
&= -\lambda^{\frac{1}{2}}\xi^B\bar{\xi}^{C'}\eta_A\nabla_{BC'}\lambda - \lambda^{\frac{1}{2}}\bar{\xi}^{C'}\nabla_{AC'}\lambda + \lambda^{\frac{1}{2}}\xi_A\eta^B\bar{\xi}^{C'}\nabla_{BC'}\lambda
\end{aligned}$$

Now we multiply by ϵ^{DA} (this will not affect whether or not a quantity is equal to 0 or not) and reorder some terms to obtain

$$\lambda^{\frac{1}{2}}\bar{\xi}^{C'}(\xi^D\eta^B\nabla_{BC'}\lambda - \xi^B\eta^D\nabla_{BC'}\lambda - \nabla^D_{C'}\lambda) = \lambda^{\frac{1}{2}}\bar{\xi}^{C'}(2\xi^{[D}\eta^{B]}\nabla_{BC'}\lambda - \nabla^D_{C'}\lambda)$$

Now we can use theorem 4.4 to finally obtain

$$\lambda^{\frac{1}{2}}\bar{\xi}^{C'}(\epsilon^{DB}\underbrace{\xi_E\eta^E}_{1}\nabla_{BC'}\lambda - \nabla^D_{C'}\lambda) = \lambda^{\frac{1}{2}}\bar{\xi}^{C'}(\nabla^D_{C'}\lambda - \nabla^D_{C'}\lambda) = 0$$

Note that there is a freedom of choice in all of this. As we discussed before, the condition that ξ^A is parallel transported only determined ξ^A up to a constant. To compensate in the inner product, we then need to divide out this constant in η^A. An arbitrary multiple of $\tilde{\xi}^A$ can however be added to η^A and its normalisation set by the inner product with $\tilde{\xi}^A$ will remain. Lastly, $\tilde{r}$ was also normalised to be parallel transported along the geodesic. Of course, if we multiply $\tilde{\xi}^A$ by a constant, we need to divide it out here again. However, since it is a scalar and we are only concerned with the derivative, we can add an arbitrary real constant to it (the affine parameter is real). To summarise: if we set

$$\tilde{\xi}^A \to \alpha^{-1}\tilde{\xi}^A, \quad \eta_A \to \alpha\tilde{\eta}^A + \beta\tilde{\eta}^A \quad \tilde{r} \to |\alpha|^2\tilde{r} + \gamma \tag{113}$$

where $\alpha(\neq 0)$ and β can be complex and γ is real, all of the conditions set here will still be fulfilled.

7.2.2 Deriving the peeling-off property

Before we go right into the general peeling-off property, it might be instructive to see what the property that we will derive looks like in the simpler case of Minkowski space-time. In flat space-time, the peeling-off property is that

$$\phi_{A_1 \cdots A_{2s-k} A_{2s-k+1} \cdots A_{2s}} \xi^{A_{2s-k+1}} \cdots \xi^{A_{2s}} = \mathcal{O}(r^{-k-1}) \tag{114}$$

along all null geodesics, where there are k ξ's on the left with $k = 0, 1, \ldots, 2s$. This implies that to order r^{-k} the left-hand side vanishes so that by eq. (107) at least $2s - k + 1$ principal null directions coincide to that order in the radial null direction to which ξ^A corresponds.

Now we will adapt this statement to curved space-times. To give meaning to $\mathcal{O}(\tilde{r}^{-k-1})$, we need to be able to compare expressions at different points of γ, which we cannot immediately do since tensors on a general curved space belong to different tangent spaces. We will overcome this difficulty by making the left-hand side of eq. (114) a scalar by using the auxiliary spinor η_A. The peeling-off property in a curved space-time is then

$$\tilde{\phi}_{A_1 \cdots A_{2s-k} A_{2s-k+1} \cdots A_{2s}} \tilde{\eta}^{A_1} \cdots \tilde{\eta}^{A_{2s-k}} \tilde{\xi}^{A_{2s-k+1}} \cdots \tilde{\xi}^{A_{2s}} = \mathcal{O}(\tilde{r}^{-k-1}) \tag{115}$$

along any null geodesic γ, for $k = 0, 1, \ldots, 2s$, where $\tilde{\eta}$ appears $2s - k$ times and ξ appears k times. Note that if $k = 0$, then no ξ^A is present. Of course, this property does not hold for an arbitrary field. We now introduce a class of fields for which we can prove that the peeling-off property holds, the asymptotically regular fields.

Definition 7.1. (Asymptotically regular field). Let $\tilde{M}$ be an asymptotically simple space-time related to M as in definition 6.1. We call a zero-rest mass field $\tilde{\phi}_{A_1 \cdots A_{2s}}$ asymptotically regular if $\phi_{A_1 \cdots A_{2s}}$ exists throughout M which is related to $\tilde{\phi}_{A_1 \cdots A_{2s}}$ in the interior regions by eq. (92) and which is continuous at the boundary $\mathscr{I}$.

Now we will show that the peeling-off property holds for asymptotically regular fields in asymptotically simple spacetimes. Note that it is not yet required to assume that the space-time is asymptotically flat.

Theorem 7.2. (Peeling-off property). Let $\tilde{M}$ be an asymptotically simple space-time and let $\tilde{\phi}_{A_1 \cdots A_{2s}}$ be an asymptotically regular zero-rest mass field. Then eq. (115) holds. Furthermore, we have that

$$\lim_{\tilde{r} \to \pm\infty} \tilde{r}^{k+1} \tilde{\phi}_{A_1 \cdots A_{2s-k} A_{2s-k+1} \cdots A_{2s}} \tilde{\eta}^{A_1} \cdots \tilde{\eta}^{A_{2s-k}} \tilde{\xi}^{A_{2s-k+1}} \cdots \tilde{\xi}^{A_{2s}} \text{ exists} \tag{116}$$

Proof. Let the null geodesic γ meet $\mathscr{I}$ at a point $G = \gamma(r_0)$ and let ξ^A denote the tangent spinor to γ. From item 2 of the asymptotic flatness conditions we have $\lambda(r_0) = 0$ and $\tilde{\nabla}_{BC'} \lambda \neq 0$. Then $\frac{d\lambda}{dr}(r_0) \neq 0$. Therefore

$$\lim_{r \to r_0} \frac{\lambda(r)}{r - r_0} = \lim_{r \to r_0} \frac{\lambda(r) - \lambda(r_0)}{r - r_0} = \frac{d\lambda}{dr}(r_0) = c \neq 0 \tag{117}$$

This means that λ and $r - r_0$ are of the same order. Furthermore, we have

$$\begin{aligned} \lim_{r \to r_0} \lambda \tilde{r} &= \lim_{r \to r_0} \lambda \int d\tilde{r} = \lim_{r \to r_0} \lambda \int \lambda^{-2} dr \\ &= \lim_{r \to r_0} \frac{\int \lambda^{-2} dr}{\frac{1}{\lambda}} = \lim_{r \to r_0} \frac{\lambda^{-2}}{-\frac{\frac{d\lambda}{dr}}{\lambda^2}} = \lim_{r \to r_0} -\frac{1}{\frac{d\lambda}{dr}} = -c^{-1} \neq 0 \end{aligned} \tag{118}$$

so that also λ and $\tilde{r}^{-1}$ are of the same order. The normal to $\mathscr{I}$ at G and the tangent to γ at G span a plane that contains one other null direction distinct from that of (the tangent vector of) γ: since the tangent to γ is null, we can parameterise the plane by a time coordinate and some spatial coordinate. Any such plane has two null directions. We set η_A to correspond to this other null direction at G. Then $\nabla_{AB'} \lambda$ will be a linear combination of the vectors $\xi_A \overline{\xi}_{B'}$ and $\eta_A \overline{\eta}_{B'}$ at G. This means that at G for some p and q we have

$$\eta^A \overline{\xi}^{B'} \nabla_{AB'} \lambda = \eta^A \overline{\xi}^{B'} (p \xi_A \overline{\xi}_{B'} + q \eta_A \overline{\eta}_{B'}) = 0$$

because $\overline{\xi}^{B'} \overline{\xi}_{B'} = 0$ and $\eta^A \eta_A = 0$. Because in an asymptotically simple space-time λ is assumed to be C^3 everywhere, we can express it as a Taylor polynomial with remainder, so that

$$\begin{aligned} \lambda(r) &= \lambda(r_0) + \frac{d\lambda}{dr}(r_0)(r - r_0) + \frac{1}{2} \frac{d^2\lambda}{dr^2}(r_L)(r - r_0)^2 \\ &= \frac{d\lambda}{dr}(r_0)(r - r_0) + \frac{1}{2} \frac{d^2\lambda}{dr^2}(r_L)(r - r_0)^2 \end{aligned}$$

where $r < r_L < r_0$. This then means that

$$\eta^A \overline{\xi}^{B'} \nabla_{AB'} \lambda = \mathcal{O}(r - r_0)$$

since this is a (directional) derivative. This yields, in combination with eq. (117) that

$$b = \int \lambda^{-2} \eta^B \bar{\xi}^{C'} \nabla_{BC'} \lambda dr = \int \lambda^{-2} \mathcal{O}(r - r_0) dr = \int \mathcal{O}(\lambda^{-1}) = \mathcal{O}(\ln(\lambda))$$

This then means that

$$\tilde{\eta}_A = \lambda^{-\frac{1}{2}} \eta_A + \xi_A \mathcal{O}\left(\lambda^{\frac{1}{2}} \ln(\lambda)\right) \tag{119}$$

If we now substitute eq. (119), eq. (109) and eq. (92) into the left-hand side of eq. (115), we obtain that

$$\begin{aligned} \tilde{\phi}_{A_1 \cdots A_{2s}} \tilde{\eta}^{A_1} \cdots \tilde{\eta}^{A_{2s-k}} \tilde{\xi}^{A_{2s-k+1}} \cdots \tilde{\xi}^{A_{2s}} &= \lambda^{s+1} \phi_{A_1 \cdots A_{2s}} \left(\lambda^{-\frac{1}{2}} \eta^{A_1} + \xi^{A_1} \mathcal{O}(\lambda^{\frac{1}{2}} \ln(\lambda))\right) \\ \cdots \left(\lambda^{-\frac{1}{2}} \eta^{A_{2s-k}} + \xi^{A_{2s-k}} \mathcal{O}(\lambda^{\frac{1}{2}} \ln(\lambda))\right) &(\lambda^{\frac{1}{2}} \xi^{A_{2s-k+1}}) \cdots (\lambda^{\frac{1}{2}} \xi^{A_{2s}}) \end{aligned} \tag{120}$$

we get the first order term by selecting the $\lambda^{-\frac{1}{2}} \eta^A$ part every time in the product of all of the $\tilde{\eta}^A$'s. We get the second order by selecting a $\xi^A \mathcal{O}(\lambda^{\frac{1}{2}} \ln(\lambda))$ term once. We can then gather all of the even higher-order terms under the remainder term given by the second order since those are of even higher order. We then get

$$\begin{aligned} &\lambda^{s+1} \lambda^{-\frac{1}{2}(2s-k)} \lambda^{\frac{1}{2}k} \phi_{A_1 \cdots A_{2s}} \eta^{A_1} \cdots \eta^{A_{2s-k}} \xi^{A_{2s-k+1}} \cdots \xi^{A_{2s}} + \mathcal{O}(\lambda^{s+1} \lambda^{\frac{1}{2}} \ln(\lambda) \lambda^{-\frac{1}{2}(2s-k-1)} \lambda^{\frac{1}{2}k}) \\ &= \lambda^{k+1} \phi_{A_1 \cdots A_{2s}} \eta^{A_1} \cdots \eta^{A_{2s-k}} \xi^{A_{2s-k+1}} \cdots \xi^{A_{2s}} + \mathcal{O}(\lambda^{k+2} \ln(\lambda)) \end{aligned} \tag{121}$$

Now we can use eq. (118) to obtain that $\lambda^{k+1} = \mathcal{O}(\tilde{r}^{-k-1})$. Since we assumed that $\phi_{A_1 \cdots A_{2s}}$ was bounded and ξ^A and η^A stay bounded in parallel transport, we obtain that indeed

$$\tilde{\phi}_{A_1 \cdots A_{2s-k} A_{2s-k+1} \cdots A_{2s}} \tilde{\eta}^{A_1} \cdots \tilde{\eta}^{A_{2s-k}} \tilde{\xi}^{A_{2s-k+1}} \cdots \tilde{\xi}^{A_{2s}} = \mathcal{O}(\tilde{r}^{-k-1})$$

If we now also use the assumption that $\phi_{A_1 \cdots A_{2s}}$ is not only bounded but also continuous, we can obtain eq. (116). If we multiply the LHS with $\tilde{r}^{k+1}$, we get something of $\mathcal{O}(1)$. From the continuity of ϕ at $\mathscr{I}$, we then obtain that indeed

$$\lim_{\tilde{r} \to \pm\infty} \tilde{r}^{k+1} \tilde{\phi}_{A_1 \cdots A_{2s-k} A_{2s-k+1} \cdots A_{2s}} \tilde{\eta}^{A_1} \cdots \tilde{\eta}^{A_{2s-k}} \tilde{\xi}^{A_{2s-k+1}} \cdots \tilde{\xi}^{A_{2s}} \text{ exists}$$

Note that the limits to the past and the future need not be the same. The value of this limit gives the relevant component of $\phi_{A_1 \cdots A_{2s}}$. □

7.3 Peeling-off for the gravitational field

In the previous subsection, it was necessary to assume asymptotic regularity for the zero rest-mass fields involved to obtain the peeling-off property. Here we will show that if instead, the appropriate field equations hold in the neighbourhood of $\mathscr{I}$, we don't always need this assumption. The peeling-off behaviour then follows just from the fact that the space-time is asymptotically simple. Here, we will show that if the Einstein equations without cosmological constant hold in the neighbourhood of $\mathscr{I}$, the gravitational field satisfies the peeling-off property. We will first need to prove some preliminary lemma's.

7.3.1 Preliminary lemma's

First, we prove a lemma that makes essential use of the fact that the boundary is topologically $\mathbb{R} \times \mathbb{S}^2$ if it is null. If we know that in a cross-section of $\mathscr{I}$ the field and its derivative are zero when projected on the null direction in $\mathscr{I}$, this lemma will allow us to conclude that the field itself equals zero on that cross-section.

Lemma 7.3. Let $\mathscr{I}$ be null. Let the null direction in $\mathscr{I}$ be represented by a spinor field i^A satisfying

$$i^A \nabla_{A(B'}\{\bar{i}_{D')} i_C\} = 0 \tag{122}$$

on some smooth space-like cross-section X of $\mathscr{I}^{\pm}$. Suppose a symmetric continuous spinor field $\phi_{A_1 \cdots A_s}$ satisfies

$$i^{A_1} \phi_{A_1 \cdots A_s} = 0, \quad i^B \nabla_{BC'} \phi_{A_1 \cdots A_s} = 0 \tag{123}$$

on X. Then $\phi_{A_1 \cdots A_s} = 0$ on X.

Proof. First, note that if we substitute $i^A \to e^{i\theta} i^A$ where θ is real, not necessarily continuous, the conditions of the lemma still hold. A basis for the tangent space of any null hypersurface consists of one null vector and two space-like vectors [28]. This means that the pullback of the metric on the tangent space has signature $(0, -, -)$. Note that now the spacelike parts are represented by $-$ instead of $+$, because we are using a metric of signature $(+, -, -, -)$. If we now take a cross-section cutting along a generator (null geodesic), we are left with a metric of signature $(-, -)$ and so this cross-section is space-like and has the topological structure of $\mathbb{S}^2$. Since this is then a simply connected, compact Riemann surface, by the uniformization theorem [29] we can conformally map it to the natural metric on the unit sphere (with negative signature). We can then unwrap it into the complex plane X^* with one extra point at infinity since the sphere is the plane's one-point compactification by theorem 2.11. We can use stereographic projection (see example 2.10 for the 1-dimensional case and see [30] for the explicit case of mapping to the complex plane instead of $\mathbb{R}^2$) to get a metric conformal to the Euclidean metric (again, with negative signature). We then have $g^*_{\mu\nu} = \Theta^{-2} g_{\mu\nu}$, where N (the north pole) gets $\Theta = 0$ since it is mapped to infinity in the plane. The conditions of the lemma remain true under this conformal map if

$$i^*_A = i_A, \quad \phi^*_{A_1 \cdots A_s} = \Theta^{\frac{s}{2}} \phi_{A_1 \cdots A_s} \tag{124}$$

Because then

$$
\begin{aligned}
i^{*A}\nabla^*_{A(B'}\{\bar{i}^*_{D')}i^*_C\} &= i^{*A}\nabla^*_{AB'}\{\bar{i}^*_{D'}i^*_C\} + (B' \leftrightarrow D') \\
&= \underbrace{i^A(\Theta\nabla_{AB'}(\bar{i}_{D'}i_C)}_{\text{assumed zero with} B'\leftrightarrow D'} \underbrace{-\bar{i}_{D'}i_C\nabla_{AB'}\Theta + \bar{i}_{B'}i_C\nabla_{AD'}\Theta}_{\text{antisymmetric in } B',D'} \\
&\quad + \underbrace{i_{D'}i_A\nabla_{CB'}\Theta}_{i^A i_A}) + (B' \leftrightarrow D') \\
&= 0
\end{aligned}
\tag{125}
$$

$$
i^{*A_1}\phi^*_{A_1\cdots A_s} = i^{A_1}\Theta^{\frac{s}{2}}\phi_{A_1\cdots A_s} = 0 \tag{126}
$$

$$
\begin{aligned}
i^{*B}\nabla^*_{BC'}\phi^*_{A_1\cdots A_s} &= i^B(\underbrace{\Theta^{\frac{s}{2}+1}\nabla_{BC'}\phi_{A_1\cdots A_s}}_{\text{assumed zero}} + \Theta^{\frac{s}{2}}\{\frac{1}{2}(s-s)\phi_{A_1\cdots A_s}\nabla_{BC'}\Theta \\
&\quad + \underbrace{\phi_{BA_2\cdots A_s}\nabla_{A_1C'}\Theta}_{\text{assumed zero}} + \underbrace{\cdots + \phi_{A_1\cdots A_{s-1}B}\nabla_{A_sC'}\Theta}_{\phi_{A_1\cdots A_s} \text{ is symmetric so assumed zero}} \quad \}) \\
&= 0
\end{aligned}
\tag{127}
$$

Now let X^* be the complex plane of a complex parameter $\sqrt{2}z = x + iy$. We define the vector $m^{*\mu}$ to be the gradient of $-z$ in the plane X^* so that, if a is a quantity defined on X^*, we have

$$
\frac{\partial a}{\partial z} = m^{*\mu}\nabla^*_\mu a = \delta a, \quad \frac{\partial a}{\partial \bar{z}} = \overline{m}^{*\mu}\nabla^*_\mu a = \bar{\delta} a \tag{128}
$$

where we defined the shorthand symbol δ. We then have for $m^{*\mu}$

$$
m^{*\mu} = \nabla^{*\mu}(-z) = \begin{pmatrix} -\frac{\partial z}{\partial x} \\ -\frac{\partial z}{\partial y} \end{pmatrix} = -\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \tag{129}
$$

where $m^*_\mu \overline{m}^{*\mu} = -1, m^*_\mu m^{*\mu} = 0$. At first, this might look strange, how can a complex vector contracted with its conjugate give a negative number? But, remember that we are using (the complex bilinear extension of) a metric with negative signature. We verify:

$$
m^*_\mu m^{*\mu} = \frac{1}{2}(1 \quad i)\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2}(-1+1) = 0 \tag{130}
$$

and

$$
m^*_\mu \overline{m}^{*\mu} = \frac{1}{2}(1 \quad i)\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{1}{2}(-1-1) = -1 \tag{131}
$$

as required. Do note that we now have to be careful separating taking the norm of a complex number and taking the norm of a vector in the plane X^* since these use opposite sign conventions now.

Since the null direction in $\mathscr{I}^{\pm}$ is also normal to $\mathscr{I}^{\pm}$, it is orthogonal to X. Since m^*_μ and $\overline{m}_{\mu^*}$ are tangent to X^* and orthogonality is conserved by conformal maps, we must have that they are orthogonal to the null direction represented by i^*_A. Therefore, $m^*_{AB'} = \sigma^{*\mu}_{AB'} m^*_\mu = \Theta \sigma^{\mu}_{AB'} m^*_\mu$ has the property that $m^*_{AB'} i^A \bar{i}^{B'} = 0$. From this, it follows that either $m^*_{AB'} = i_A \overline{\xi}_{B'}$ or $\overline{m}^*_{A'B} = i_B \overline{\xi}_{A'}$, where ξ_B is just some other spinor that makes the equality work. Now we can choose the rotation of the complex plane in such a way that we have

$$m^*_{AB'} = i_A \overline{\xi}_{B'} \tag{132}$$

Because $m^*_\mu \overline{m}^{*\mu} = -1$, we have that $i_A \overline{\xi}_{B'} \bar{i}^{B'} \xi^A = -1$, so that $i_A \xi^A \bar{i}_{B'} \bar{\xi}^{B'} = 1$. From this it follows that $|i_A \xi^A| = 1$. Now we can use the freedom $i_A \to e^{i\theta} i_A$ (which then also leads to $\xi_A \to e^{i\theta} \xi_A$ to preserve eq. (132)) to obtain

$$i_A \xi^A = 1 \tag{133}$$

We have $\overline{m}^{*\mu} \delta m^*_\mu = 0$ because $m^{*\mu}$ is constant for the plane X^*. This means that we have

$$\xi^A \bar{i}^{B'} \delta(i_A \overline{\xi}_{B'}) = 0 \tag{134}$$

But from the asterisk version of *eq.* (122) we get by contracting with $\bar{\xi}^{B'} \bar{\xi}^{D'}$

$$\bar{\xi}^{B'} i^C \bar{\xi}^{D'} \nabla_{CD'}(\bar{i}_{B'} i_A) + \bar{\xi}^{D'} i^C \bar{\xi}^{B'} \nabla_{CB'}(\bar{i}_{D'} i_A) = 0$$

We can rewrite this with δ to get

$$\bar{\xi}^{B'} \delta(\bar{i}_{B'} i_A) + \bar{\xi}^{D'} \delta(\bar{i}_{D'} i_A) = 0$$

Since B' and D' are just dummy indices, we then find that

$$\bar{\xi}^{B'} \delta(\bar{i}_{B'} i_A) = 0$$

so that we then find

$$\xi^A \bar{\xi}^{B'} \delta(i_A \bar{i}_{B'}) = 0 \tag{135}$$

Now, from the derivative of eq. (133) we find that $\delta(i_A \xi^A) = \delta(1) = 0$. Then $\xi^A \delta i_A + i_A \delta \xi^A = 0$, so that $\xi^A \delta i_A - i^A \delta \xi_A = 0$. If we now use this relation and eq. (133) when we subtract eq. (134) from eq. (135), we obtain

$$\begin{aligned}
&\xi^A\bar{\xi}^{B'}\delta(i_A\bar{i}_{B'})-\xi^A\bar{i}^{B'}\delta(i_A\bar{\xi}_{B'})=0\\
\iff &i_A\xi^A\bar{\xi}^{B'}\delta(\bar{i}_{B'})+\bar{i}_{B'}\bar{\xi}^{B'}\xi^A\delta(i_A)-i_A\xi^A\bar{i}^{B'}\delta(\bar{\xi}_{B'})-\bar{i}^{B'}\bar{\xi}_{B'}\xi^A\delta(i_A)=0\\
\iff &\bar{\xi}^{B'}\delta(\bar{i}_{B'})+\xi^A\delta(i_A)-\bar{i}^{B'}\delta(\bar{\xi}_{B'})+\xi^A\delta(i_A)=0\\
\iff &\xi^A\delta(i_A)+\xi^A\delta(i_A)=0\\
\iff &\xi^A\delta(i_A)=0
\end{aligned} \tag{136}$$

Now, from the asterisk version of the first condition of eq. (123) we find that i^A is an s times coincident principal null direction (see eq. (107)) for $\phi^*_{A_1\cdots A_s}$. This then means that we must have that

$$\phi^*_{A_1\cdots A_s}=ai_{A_1}\cdots i_{A_s} \tag{137}$$

on X^*. From the asterisk version of the second condition of eq. (123), we get that $\delta\phi^*_{A_1\cdots A_s}=0$, so that

$$\xi^{A_1}\cdots\xi^{A_s}\delta\phi^*_{A_1\cdots A_s}=0 \tag{138}$$

Substituting eq. (137) into eq. (138) and using eq. (136) and eq. (133) we get

$$\begin{aligned}
&\xi^{A_1}\cdots\xi^{A_s}\delta(ai_{A_1}\cdots i_{A_s})=0\\
\iff &\xi^{A_1}\cdots\xi^{A_s}(i_{A_1}\cdots i_{A_s}\delta a+ai_{A_2}\cdots i_{A_s}\delta i_{A_1}+\cdots+ai_{A_1}\cdots i_{A_{s-1}}\delta i_{A_s})=0\\
\iff &\xi^{A_1}\cdots\xi^{A_s}i_{A_1}\cdots i_{A_s}\delta a=0\\
\iff &\delta a=0\\
\iff &\frac{\partial a}{\partial z}=0
\end{aligned} \tag{139}$$

This means that a is a constant function on the whole of X^*. If $a\neq 0$, $\phi_{A_1\cdots A_s}$ would become unbounded in the neighbourhood of N since $\Theta=0$ there (remember $\phi_{A_1\cdots A_s}=\Theta^{-\frac{s}{2}}\phi^*_{A_1\cdots A_s}$). But this contradicts the continuity of $\phi_{A_1\cdots A_s}$ and so we must have that $a=0$. But then $\phi^*_{A_1\cdots A_s}=0$ on X^* and so $\phi_{A_1\cdots A_s}=0$ on the whole of X. □

Lemma 7.4. Assume $\mathscr{I}$ is null. Let $\phi_{A_1\cdots A_s}$ be symmetric and defined throughout M, with continuous derivative at $\mathscr{I}$. Suppose $\theta^{B'}_{A_2\cdots A_s}$ is continuous at $\mathscr{I}$ where $\theta^{B'}_{A_2\cdots A_s}=\nabla_{A_1B'}(\lambda^{-1}\phi_{A_1\cdots A_s})$ in the interior of M. Further suppose $\nabla^A_{(B'}\nabla^C_{D')}\lambda=0$ on $\mathscr{I}$. Then $\lambda^{-1}\phi_{A_1\cdots A_s}$ is continuous at $\mathscr{I}$.

Proof.

$$\lambda\theta^{B'}_{A_2\cdots A_s}=\nabla^{A_1B'}\phi_{A_1\cdots A_s}+\lambda\phi_{A_1\cdots A_s}\nabla^{A_1B'}\lambda^{-1}=\nabla^{A_1B'}\phi_{A_1\cdots A_s}-\lambda^{-1}\phi_{A_1\cdots A_s}\nabla^{A_1B'}\lambda=0 \tag{140}$$

because of the assumed continuity of $\theta^{B'}_{A_2\cdots A_s}$ at $\mathscr{I}$ and the fact that $\lambda=0$ there. It also remains continuous. But because $\lambda=0$ at $\mathscr{I}$, we must have that $\phi_{A_1\cdots A_s}\nabla^{A_1B'}\lambda=0$ there too, because otherwise this quantity would blow up due to the λ^{-1} term. $\nabla^{A_1B'}\lambda$ is non-zero and normal to $\mathscr{I}$ and null (because $\mathscr{I}$ is null). Then

$$\nabla^{A_1B'}\lambda = \pm i^{A_1}\bar{i}^{B'} \tag{141}$$

at $\mathscr{I}^{\mp}$ for some i^A representing this null direction. Now $\phi_{A_1\cdots A_s}\nabla^{A_1B'}\lambda = 0$ only if $\phi_{A_1\cdots A_s}i^{A_1} = 0$ on $\mathscr{I}$. This means that the first part of eq. (123) is satisfied. Now

$$\lambda^2\theta^{B'}_{A_2\cdots A_s} = \lambda\nabla^{A_1B'}\phi_{A_1\cdots A_s} - \phi_{A_1\cdots A_s}\nabla^{A_1B'}\lambda \tag{142}$$

must have a vanishing derivative at $\mathscr{I}$, since if $a_{A_1\ldots}$ is continuous at $\mathscr{I}$, $\lambda a_{A_1\ldots}$ is differentiable at $\mathscr{I}$ with gradient $a_{A_1\ldots}\nabla_\mu\lambda$. If we differentiate eq. (142), we get

$$\nabla^{A_1B'}\phi_{A_1\cdots A_s}\nabla^{CD'}\lambda - \nabla^{CD'}\phi_{A_1\cdots A_s}\nabla^{A_1B'}\lambda - \phi_{A_1\cdots A_s}\nabla^{CD'}\nabla^{A_1B'}\lambda = 0$$

on $\mathscr{I}$, where the second derivative in $\phi_{A_1\cdots A_s}$ vanishes because it is multiplied by $\lambda(= 0$ on $\mathscr{I})$. We can lower all of the primed indices with ϵ to obtain

$$\nabla^{A_1}_{B'}\phi_{A_1\cdots A_s}\nabla^{C}_{D'}\lambda - \nabla^{C}_{D'}\phi_{A_1\cdots A_s}\nabla^{A_1}_{B'}\lambda - \phi_{A_1\cdots A_s}\nabla^{C}_{D'}\nabla^{A_1}_{B'}\lambda = 0$$

We clearly see that this equation still holds if we exchange the labels B' and D'. This means we can symmetrize over those indices to obtain

$$\nabla^{A_1}_{(B'}\phi_{|A_1\cdots A_s|}\nabla^{C}_{D')}\lambda - \nabla^{C}_{(D'}\phi_{|A_1\cdots A_s|}\nabla^{A_1}_{B')}\lambda - \phi_{A_1\cdots A_s}\nabla^{C}_{(D'}\nabla^{A_1}_{B')}\lambda = 0 \tag{143}$$

With $\nabla^{C}_{(D'}\nabla^{A_1}_{B')}\lambda = 0$ on $\mathscr{I}$, this gives

$$\nabla^{A_1}_{(B'}\phi_{|A_1\cdots A_s|}\nabla^{C}_{D')}\lambda - \nabla^{C}_{(D'}\phi_{|A_1\cdots A_s|}\nabla^{A_1}_{B')}\lambda = 0 \tag{144}$$

Now using $\nabla^{A_1B'}\lambda = \pm i^A\bar{i}^{B'} \implies \nabla^{A_1}_{B'}\lambda = \pm i^{A_1}\bar{i}_{B'}$ we get

$$i^C\bar{i}_{(D'}\nabla^{A_1}_{B')}\phi_{A_1\cdots A_s} - i^{A_1}\bar{i}_{(B'}\nabla^{C}_{D')}\phi_{A_1\cdots A_s} = 2i^{[C}\bar{i}_{(D'}\nabla^{A_1]}_{B')}\phi_{A_1\cdots A_s} = 0 \tag{145}$$

From this, we obtain (using theorem 4.4) that

$$\epsilon^{CA_1}i_E\bar{i}_{(D'}\nabla^{E}_{B')}\phi_{A_1\cdots A_s} = 0 \tag{146}$$

Now we can remove the $\bar{i}_{D'}$ from the equation by multiplying by $x^{D'}x^{B'}$ (where $x^{D'}$ is arbitrary except for the fact that it is nonzero and not a multiple of $\bar{i}^{D'}$) and dividing by $x^{E'}\bar{i}_{E'}$.

$$\begin{aligned}
&\frac{1}{x^{E'}\bar{i}_{E'}} x^{D'} x^{B'} \epsilon^{CA_1} i_E \bar{i}_{(D'} \nabla^E_{B')} \phi_{A_1\cdots A_s} = 0 \\
\iff \frac{1}{x^{E'}\bar{i}_{E'}} &\left(x^{D'} x^{B'} \epsilon^{CA_1} i_E \bar{i}_{D'} \nabla^E_{B'} \phi_{A_1\cdots A_s} + x^{D'} x^{B'} \epsilon^{CA_1} i_E \bar{i}_{B'} \nabla^E_{D'} \phi_{A_1\cdots A_s} \right) = 0 \\
&\iff x^{B'} \epsilon^{CA_1} i_E \nabla^E_{B'} \phi_{A_1\cdots A_s} + x^{D'} \epsilon^{CA_1} i_E \nabla^E_{D'} \phi_{A_1\cdots A_s} = 0 \\
&\iff x^{B'} \epsilon^{CA_1} i_E \nabla^E_{B'} \phi_{A_1\cdots A_s} = 0
\end{aligned} \tag{147}$$

Since this is valid for arbitrary $x^{B'}$ and ϵ^{CA_1} is nondegenerate, we get

$$i_C \nabla^C_{B'} \phi_{A_1\cdots A_s} = 0 \implies i^B \nabla_{BC'} \phi_{A_1\cdots A_s} = 0 \tag{148}$$

on $\mathscr{I}$. This means that the second part of eq. (123) is satisfied. Using the fact that $i^A \nabla_{AB'}$ operates tangentially in $\mathscr{I}$ and the fact that $\bar{i}_{D'} i_C$ is normal to $\mathscr{I}$, we have $i^A \nabla_{A(B'} \{\bar{i}_{D')} i_C\} = 0$. Now using eq. (141), we obtain

$$\pm i^A \nabla_{A(B'} \nabla_{D')C} \lambda = 0 \tag{149}$$

which means that eq. (122) is satisfied. Then the conditions of lemma 7.3 are all satisfied in the whole of $\mathscr{I}$ and therefore $\phi_{A_1\cdots A_s} = 0$ on $\mathscr{I}$. We have that when we approach any point in $\mathscr{I}$, we get

$$\lim_{\mathscr{I}} \frac{\phi_{A_1\cdots A_s}}{\lambda} \to \frac{0}{0} \tag{150}$$

Since $\phi_{A_1\cdots A_s}$ was assumed to have a continuous derivative at $\mathscr{I}$ and we assume that $\nabla_{AB'}\lambda \neq 0$, we find that the limit exists and so $\lambda^{-1}\phi_{A_1\cdots A_s}$ is continuous at $\mathscr{I}$. □

7.4 Peeling-off in empty space-time

With the lemmas we proved in the previous subsection, we are in a position to discuss solutions to Einstein's vacuum equations. It turns out that, even if we assume nothing about asymptotic regularity, we can still prove that the Weyl spinor satisfies the peeling-off property. This only makes use of the asymptotic flatness of the space-time and the fact that the empty-space equation is satisfied.

Theorem 7.5. The Weyl spinor Ψ_{ABCD} satisfies the peeling-off property.

Proof. We define a new field ϕ_{ABCD} as follows:

$$\tilde{\phi}_{ABCD} = \tilde{\Psi}_{ABCD}, \quad \phi_{ABCD} = \lambda^{-1}\Psi_{ABCD} = \lambda^{-3}\tilde{\Psi}_{ABCD} = \lambda^{-3}\tilde{\phi}_{ABCD} \tag{151}$$

We know that the physical Weyl spinor satisfies the zero rest-mass equation when the empty-space conditions hold near $\mathscr{I}$ so that $\tilde{\nabla}^{AE'}\tilde{\Psi}_{ABCD} = 0$ (eq. (89)). From this we find

$\tilde{\nabla}^{AE'}\tilde{\phi}_{ABCD} = 0$. Since $\tilde{\phi}_{ABCD} = \lambda^3\phi_{ABCD}$, we get that also $\nabla^{AE}\phi_{ABCD} = 0$ by eq. (92). But this then means that

$$\nabla^{AE'}(\lambda^{-1}\Psi_{ABCD}) = 0 \tag{152}$$

We now examine $\nabla^A_{(B'}\nabla^C_{D')}\lambda$. Since $\tilde{\Phi}_{ABC'D'}$ vanishes in empty space-time (eq. (90)), we have by eq. (88).

$$\lambda^2\Phi^{C'D'}_{AB} + \lambda\nabla^{C'}_{(A}\nabla^{D'}_{B)}\lambda = 0$$

When we are not on $\mathscr{I}$, we have $\lambda > 0$, so that we get

$$\lambda\Phi^{C'D'}_{AB} + \nabla^{C'}_{(A}\nabla^{D'}_{B)}\lambda = 0$$

throughout spacetime. But that then means that at $\mathscr{I}$

$$\nabla^A_{(B'}\nabla^C_{D')}\lambda = 0 \tag{153}$$

This means that the conditions for lemma 7.4 are satisfied so that ϕ_{ABCD} is continuous on $\mathscr{I}$. This means that ϕ_{ABCD} is an asymptotically regular field meaning that the peeling-off property (theorem 7.2) holds. Since $\tilde{\phi}_{ABCD} = \tilde{\Psi}_{ABCD}$, the peeling-off property thus holds for the Weyl spinor. □

7.5 Consequences of the peeling-off behaviour

We now discuss some of the consequences of the peeling-off behaviour. If peeling off holds, we have

$$\lim_{\tilde{r}\to\pm\infty} \tilde{r}^{k+1}\tilde{\phi}_{A_1\cdots A_{2s-k}A_{2s-k+1}\cdots A_{2s}}\tilde{\eta}^{A_1}\cdots\tilde{\eta}^{A_{2s-k}}\tilde{\xi}^{A_{2s-k+1}}\cdots\tilde{\xi}^{A_{2s}} \text{ exists} \tag{154}$$

The value of this limit gives the relevant component of $\phi_{A_1\cdots A_2s}$. We now look at the radiation fields, which are given by eq. (154) with $k = 0$. We get the incoming radiation field if the past limit is taken and the outgoing radiation fields radiation field if the future limit is taken. From eq. (154) we obtain the following: We can decompose $\tilde{\phi}_{A_1\cdots A_{2s}}$ into a product of one-forms which can be written as a linear combination of $\tilde{\xi}_A$ and $\tilde{\eta}_A$ since they form a basis. If we assume that a radiation field exists, the limit in eq. (154) with $k = 0$ can't be zero. Therefore, there must be some part of the limit that remains. But since $\tilde{\eta}_A\tilde{\eta}^A = 0$, the only part of the limit that can remain is the part completely linearly independent from $\tilde{\eta}_A$, which means that

$$\lim_{\tilde{r}\to\infty} \tilde{r}\tilde{\phi}_{A_1\cdots A_{2s}} = c\tilde{\xi}_{A_1}\cdots\tilde{\xi}_{A_{2s}} \tag{155}$$

where c is a constant. Therefore all of the principal null directions indeed coincide for the radiation part of the field. We also see that if $\mathscr{I}$ is null, the null direction determined by

η^A is the normal (null tangent) direction to $\mathscr{I}$ at G. Therefore η^A essentially does not depend on the particular null geodesic at G. The radiation field is therefore defined in an origin-dependent way. We also obtain (using that $\tilde{\xi}_A \tilde{\eta}^A = 1$)

$$\lim_{\tilde{r}\to\infty} \tilde{r}\tilde{\phi}_{A_1 \cdots A_{2s}} \tilde{\eta}^{A_1} \cdots \tilde{\eta}^{A_{2s}} = c \tag{156}$$

Therefore, we see that for a given geodesic γ, c is not dependent on any particular choice of $\tilde{\eta}^A$ and $\tilde{\xi}^A$.

Now we turn to the Weyl tensor. We have proved that the Weyl spinor exhibits the peeling-off behaviour, from which it follows that also the Weyl tensor has this property. We have

$$\tilde{C}_{\mu\nu\rho\sigma} = \frac{\tilde{C}^{(1)}_{\mu\nu\rho\sigma}}{\tilde{r}} + \frac{\tilde{C}^{(2)}_{\mu\nu\rho\sigma}}{\tilde{r}^2} + \frac{\tilde{C}^{(3)}_{\mu\nu\rho\sigma}}{\tilde{r}^3} + \frac{\tilde{C}^{(4)}_{\mu\nu\rho\sigma}}{\tilde{r}^4} + \mathcal{O}(\tilde{r}^{-5}) \tag{157}$$

where these components can be algebraically (Petrov) classified according to their principal null directions [3].

7.6 Concluding words

The conformal technique is a technique that is very useful in studying asymptotic behaviour in general relativity. Through this thesis, we have traversed the intricate landscape of topology, compactifications, differential geometry, and general relativity to arrive at a basic understanding of the peeling-off property of zero rest-mass fields in asymptotically flat space-times. However, the point at which this thesis ends is really only the starting point in the study of asymptotic behaviour in general relativity. There is a wealth of literature available on the topic, which I hope that a reader of this thesis may now be able to understand, or at least know where to start looking. I will conclude by providing some examples of how the current results can be built upon and a big limitation of the technique as discussed here.

In Penrose's paper [1], the result of the peeling theorem is extended a bit further. Penrose does not require the cosmological constant to equal zero so that $\mathscr{I}$ can be either spacelike, timelike or null. It turns out that when $\mathscr{I}$ is not null, the arguments required to obtain peeling for the gravitational field are much simpler. It is then also possible to derive the condition $\nabla_\mu \lambda \neq 0$ on $\mathscr{I}$ from the other conditions of asymptotic simplicity. However, in this case, the concept of a radiation field can no longer be defined in an origin-independent way. Lastly, Penrose extends the peeling-off behaviour to apply to both the gravitational and the electromagnetic fields. He proves another lemma to achieve this result, of which lemma 7.4 can be seen as a special case.

A related matter for which the conformal technique can be used is the question of asymptotic groups for curved space-times. It is now called the BMS (Bondi-(Van der Burg)-Metzner-Sachs group), since they were the ones to first investigate this group of asymptotic symmetries in 1962 [31]. They found that the asymptotic symmetry transformations indeed form a group and that the structure of this group does not depend on the particular gravitational field that happens to be present. The puzzling surprise was that the group they discovered was not the expected ten-dimensional Poincaré group, but an infinite-dimensional one. The asymptotic symmetries do include the six Lorentz boost/rotations, but also an additional infinity of symmetries that are not Lorentz. These additional non-Lorentz asymptotic symmetries, which constitute an infinite superset of the four spacetime translations, are named supertranslations.

Since the symmetry group at infinity is not the Lorentz group, general relativity does not reduce to special relativity in the case of weak fields at long distances.

Lastly, in this thesis, a definition of asymptotic flatness was given that, although appropriate for the current work, disregards 2 things: The first, already noted next to the definition, is that certain cases that should definitely qualify for being asymptotically flat, like the Schwarzschild solution, are excluded due to the requirement that every maximally extended null geodesic has an endpoint on $\mathscr{I}$. Furthermore, this definition just 'throws away' the point i^0 at spatial infinity even though spatial infinity functions at the 'link' between $\mathscr{I}^+$ and $\mathscr{I}^-$. In 1978, Ashtekar and Hansen formulated conditions for a notion of asymptotic flatness at both spatial and null infinity [32]. A manifold SPI (spatial infinity) is constructed, which has similar properties to $\mathscr{I}$ discussed here. It turns out to have a corresponding symmetry group that is similar in structure to the BMS group.

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