Resolvent Estimates in (weighted) L^p spaces for the Stokes Operator in Lipschitz Domains

by

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Abstract

Recently, by Z. Shen [34], resolvent estimates for the Stokes operator were established in $L^p(\Omega)$ when Ω is a Lipschitz domain in $\mathbb{R}^d$, with $d \geq 3$ and $\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{1}{2d} + \varepsilon$. This result implies that the Stokes operator generates a bounded analytic semigroup in $L^p(\Omega)$ in the case that Ω is a three-dimensional Lipschitz domain and $\frac{3}{2} - \varepsilon < p < 3 + \varepsilon$.

To fully understand the work of Z. Shen [34] a lot of background information is needed. In this thesis the resolvent estimates are studied in detail in the case $d = 3$. In the end the results of Shen are extended to resolvent estimates in $L^p(w, \Omega)$ where Ω is a three-dimensional Lipschitz domain, $\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{1}{6}$, and $w \in A_{\frac{2p}{3}} \cap \mathrm{RH}_{\frac{3}{3-p}}$ is a weight function that belongs to an intersection of a Muckenhoupt weight class and satisfies a reverse Hölder inequality.

Preface

Given my background in physics and mathematics, I was naturally drawn to a project with a physical motivation. Having already used the (simplified) Navier-Stokes equations to solve relevant problems in engineering, I was very interested in the mathematical mystery behind these equations. The effects of the Navier-Stokes equations are everywhere you look. They describe the movement of the air we breathe, the blood that moves through your veins and the water we drink to survive. However our understanding of these equations is very limited. Basic questions such as the existence and smoothness of the Navier-Stokes equations in $\mathbb{R}^3$, the space we live in, remain unsolved untill today. Solving this question could even earn you a million dollars, since it is one of the famous millennium problems [15].

Luckily, lots of people have shown interest in the mystery and significant progress gets made from time to time. During the beginning of my research I was often thinking, how I could use the developed theory to solve the problems around me. The further I got into the project, the more I started to appreciate the mathematical theory on its own. I learned that the freedom to develop mathematical theory, without directly knowing how to apply it in the "real" world is one of the greatest freedoms in modern science.

Apart from the mathematical theory and concepts, I have also learned a great deal about myself. I have learned to plan ahead, but change the planning when unexpected things happen. I also learned to push through the long days without tangible progress, and I am very happy with the result that lies now before me.

However, I have had significant help in putting this thesis together. First of all I want to thank Mark en Dorothee for their supervision on the project. I really like our discussions, and it really gave me good inspiration on how to tackle the problems in the thesis. I would also like to thank you for your patience with me, and the feedback on my ideas and writing.

Further, I would like to thank my parents, brothers and friends for their interest in the project and their mental support when I needed it the most. I would also like to thank my housemates for their support during the thesis. I really liked it when you came over for coffee and the relaxing Friday afternoons. Finally I would like to thank my girlfriend, Esther, for her amazing support during the project. Thank you for taking care of me when I was feeling sick, thank you for cheering me up when I was down and thank you for all your editing work in this thesis.

Tim Dikland
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Contents

Notation

Number sets

$\mathbb{N}$ – The natural numbers.
$\mathbb{R}$ – The field of real numbers.
$\mathbb{C}$ – The field of complex numbers.

Function spaces

$C(\Omega)$ – The space of continuous functions on Ω.
$C^k(\Omega)$ – The space of k-times continuously differentiable functions on Ω.
$C^k(\overline{\Omega})$ – $f \in C^k(\Omega)$ for which $\partial^\alpha f$ is bounded and uniformly continuous on Ω for $0 \leq |\alpha| \leq k$
$C_c^k(\Omega)$ – The space of k-times continuously differentiable functions on Ω with compact support.
$L^1_{\text{loc}}(\Omega)$ – The space of locally integrable functions.
$L^p(\Omega)$ – The space of measurable functions f with $\left(\int_\Omega |f(x)|^p dx\right)^{1/p} < \infty$.
$W^{k,p}(\Omega)$ – The space of locally integrable functions f such that $\partial^\alpha f \in L^p(\Omega)$, $0 \leq |\alpha| \leq k$.
$H^k(\Omega)$ – The Hilbert space $W^{k,2}(\Omega)$.
$H_0^k(\Omega)$ – The space of functions, such that $f \in H^k(\Omega)$ and f has zero trace.

Weight classes

A_p – The class of the Muckenhoupt weights
RH_s – The class of weights that satisfies a reverse Hölder condition

Functions

$\mathcal{M}(f)$ – The Hardy-Littlewood maximal function of f
$\mathcal{M}_{\partial\Omega}(f)$ – The Hardy-Littlewood maximal function, localized at $\partial\Omega$
$(f)^*$ – The nontangential maximal function of f
δ_{ij} – The Kronecker delta
χ_A – The indicator function of the set A

General

$B_d(p, r)$ – The d-dimensional open ball with center p and radius r
ω_d – The volume of the d-dimensional unit ball
$\partial\Omega$ – The boundary of the set Ω

1 Introduction

This thesis is concerned with the analysis of the Stokes operator. The Stokes operator arises in the analysis of the Navier-Stokes equations. The domain in which the Stokes operator is analysed greatly influences its outcomes. For smooth domains the analysis is well-known [20]. In this thesis we consider the case of Lipschitz domains. Recently Z. Shen proved in [34] that, if Ω is a Lipschitz domain, the resolvent of the Stokes operator is bounded in $L^p(\Omega)$ in the range $\frac{2d}{d+1} - \varepsilon < p < \frac{2d}{d-1} + \varepsilon$, where d indicates the dimension of Ω and ε is a (small) constant depending on d and the Lipschitz character of Ω. This opens the door to study mild solutions of the Navier-Stokes equations. In this thesis the result of Z. Shen is studied in three dimensions. Finally we extend the resolvent bounds to the weighted space $L^p(w,\Omega)$, where $\frac{3}{2} - \varepsilon < p < 3 + \varepsilon$ and w is a weight in the intersection of a Muckenhoupt class and a reverse Hölder class.

Motivation and History

The initial value problem for the Navier-Stokes equations with Dirichlet (no-slip) boundary conditions is given by

$$\text{(NSE)} \quad \begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\,\boldsymbol{u} - \nu \Delta \boldsymbol{u} - \nabla \phi = \boldsymbol{f} & \text{in } (0,T) \times \Omega \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } (0,T) \times \Omega \\ \boldsymbol{u} = 0 & \text{on } (0,T) \times \partial\Omega \\ \boldsymbol{u} = \boldsymbol{u}_0 & \text{in } \Omega. \end{cases} \tag{1.1}$$

Here $\boldsymbol{u}(x,t)$ represents the velocity vector of the fluid at position x and time t. ν represents the viscosity of the fluid, $\nabla\phi$ represents the pressure in the fluid and $\boldsymbol{f}$ is the force applied to the fluid. For simplicity $\nu = 1$ is assumed. The initial mathematical study of this problem can be attributed to the seminal paper of J. Leray in 1934 [30]. Later in 1964 H. Fujita and T. Kato wrote the paper [17] in which they use a functional analytic method to study mild solutions of the Navier-Stokes equations. This method roughly consists of the following three steps, as descibed in [32].

(i) Use the Helmholtz projection ($\mathbb{P}$) on the Navier-Stokes equations to recast it in the form of an abstract initial value problem

$$\text{(PNSE)} \quad \begin{cases} \dfrac{d\boldsymbol{u}}{dt} + A\boldsymbol{u}(t) = -\mathbb{P}\left[(\boldsymbol{u} \cdot \nabla)\,\boldsymbol{u}\right] + \mathbb{P}\boldsymbol{f} & \text{in } (0,T) \\ \boldsymbol{u} = u_0. \end{cases} \tag{1.2}$$

(ii) Convert (PNSE) to an integral equation

$$\boldsymbol{u}(t) = e^{-At}\boldsymbol{u}_0 - \int_0^t e^{-(t-s)A}\mathbb{P}\left[(\boldsymbol{u} \cdot \nabla)\,\boldsymbol{u}\right] ds + \int_0^t e^{-(t-s)A}\mathbb{P}\boldsymbol{f} ds \quad 0 < t < T. \tag{1.3}$$

(iii) Solve the integral equation (1.3) using a fixed point argument.

Within this functional analytic approach there are critical steps that will be discussed in this thesis. First of all, for step (i) it is important that the Helmholtz projection is well defined on the function space in which the problem is posed. We know that if Ω is bounded and has smooth boundary or even C^1 boundary, the Helmholtz decomposition of $L^p(\Omega)$ and the Helmholtz projection exist for all $1 < p < \infty$ [13]. Let Ω be a general domain and $1 < p < \infty$. Now define

$$L^p_\sigma(\Omega) = \text{closure of } \{u \in C_0^\infty(\Omega) : \operatorname{div} u = 0 \text{ in } \Omega\} \text{ in } L^p(\Omega) \tag{1.4}$$
$$G^p(\Omega) = \left\{\nabla u : u \in W^{1,p}(\Omega)\right\}. \tag{1.5}$$

The natural question is, what are the conditions of Ω and p such that we have

$$L^p(\Omega) = L^p_\sigma(\Omega) \oplus G^p(\Omega). \tag{1.6}$$

A decomposition as in (1.6) is known as the Helmholtz (or Hodge) decomposition of $L^p(\Omega)$. In [13], E. Fabes, O. Mendes and M. Mitrea showed that when Ω has a C^1 boundary the Helmholtz decomposition exists for all $1 < p < \infty$. In the same paper, the authors showed that when Ω is a Lipschitz domain the Helmholtz decomposition exists only in the range $\frac{3}{2} - \varepsilon < p < 3 + \varepsilon$, where ε is a (small) constant depending on Ω. This range is sharp in the sense that the weak Neumann problem is no longer uniquely solvable outside this range, which is equivalent to the existance of the Helmholtz decomposition and Helmholtz projection. When the Helmholtz decomposition exists, there also exists a bounded operator $\mathbb{P}_p$ (a projection) that maps $L^p(\Omega)$ onto $L^p_\sigma(\Omega)$. This makes it possible to define the Stokes operator as

$$A_p(\boldsymbol{u}) = \mathbb{P}_p(-\Delta)(\boldsymbol{u}). \tag{1.7}$$

As mentioned before, the Helmholtz decomposition of $L^p(\Omega)$ fails to exist when Ω is a Lipschitz domain and p is not in the range $(\frac{3}{2} - \varepsilon, 3 + \varepsilon)$. In that case we cannot define the Stokes operator using the Helmholtz projection. To circumvent this problem Shen ([34]) defines the Stokes operator A_p in $L^p_\sigma(\Omega)$ by

$$A_p(\boldsymbol{u}) = -\Delta\boldsymbol{u} + \nabla\phi,$$

with the domain

$$D(A_p) = \{\boldsymbol{u} \in W_0^{1,p}(\Omega; \mathbb{C}^d) : \operatorname{div}(\boldsymbol{u}) = 0 \text{ in } \Omega \text{ and}$$
$$- \Delta\boldsymbol{u} + \nabla\phi \in L^p_\sigma(\Omega) \text{ from some } \phi \in L^p(\Omega)\}.$$

To proceed to step (ii) of the Fujita-Kato approach it needs to be shown that $-A_p$ generates a bounded analytic semigroup. For this to be the case we have to show that A_p is closed and densely defined in $L^p_\sigma(\Omega)$. Furthermore A_p has to satisfy the resolvent estimate

$$\left\|(A_p + \lambda)^{-1}\boldsymbol{f}\right\|_{L^p(\Omega)} \leq C|\lambda|^{-1}\|\boldsymbol{f}\|_{L^p(\Omega)}, \tag{1.8}$$

where $\lambda \in \Sigma_\theta = \{\lambda \in \mathbb{C} : \lambda \neq 0 \text{ and } |\arg(\lambda)| < \pi - \theta\}$ and $\theta \in (0, \frac{\pi}{2})$. M. Taylor conjectured in [35] that in the case where Ω is a bounded Lipschitz domain in $\mathbb{R}^3$, $-A_p$ generates a bounded analytic semigroup in $L^p_\sigma(\Omega)$ when $\frac{3}{2} - \varepsilon < p < 3 + \varepsilon$. This was later proved in 2012 by Z. Shen in [34] by considering the Dirichlet problem for the Stokes System.

$$\text{(DSS)} \quad \begin{cases} -\Delta\boldsymbol{u} + \nabla\phi + \lambda\boldsymbol{u} = \boldsymbol{f} & \text{in } \Omega \\ \operatorname{div}(\boldsymbol{u}) = 0 & \text{in } \Omega \\ \boldsymbol{u} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.9}$$

where $\lambda \in \Sigma_\theta$. Shen showed that when $\boldsymbol{f} \in L^2(\Omega; \mathbb{C}^d) \cap L^p(\Omega; \mathbb{C}^d)$ the unique solution $\boldsymbol{u} \in H_0^1(\Omega; \mathbb{C}^d)$ that solves (DSS), satisfies an estimate in the form of

$$\|\boldsymbol{u}\|_{L^p(\Omega)} \leq C_p|\lambda|^{-1}\|\boldsymbol{f}\|_{L^p(\Omega)}, \tag{1.10}$$

when $\frac{2d}{d+1} - \varepsilon < p < \frac{2d}{d-1} + \varepsilon$. This result was already found in the case $\Omega = \mathbb{R}^d_+$ by M. McCracken [31], and in the case that Ω has C^2 boundary by W. Varnhorn [37]. The result on Lipschitz domains (1.10) implies the resolvent estimate (1.8). To come to this conclusion, Shen used so-called layer potentials to solve the (homogeneous) Stokes System

$$\text{(SS)} \begin{cases} -\Delta \boldsymbol{u} + \nabla\phi + \lambda\boldsymbol{u} = 0 & \text{in } \Omega \\ \operatorname{div}(\boldsymbol{u}) = 0 & \text{in } \Omega. \end{cases} \tag{1.11}$$

The result of Shen as well as the techniques used in his paper has lead to several follow-up results. For example P. Kunstmann and L. Weis used similar techniques to show that the Stokes operator has a bounded H^∞-calculus [28]. Later on P. Tolksdorf used the results of Shen together with the results of Kunstmann and Weis to show existence of solutions to the Navier–Stokes equations in the critical space $L^\infty(0,\infty; L^3_\sigma(\Omega))$ whenever the initial velocity is small in the L^3-norm [36].

Instead of studying the Navier-Stokes equations on $L^p(\Omega)$, one would also like to obtain similar results in a weighted space $L^p(w,\Omega)$. One would like to follow the same functional analytic approach. In this thesis the resolvent estimates for the weighted Stokes operator are established in $L^p(w,\Omega)$ in the same range for $w \in A_{\frac{3p}{2}} \cap \mathrm{RH}_{\frac{3}{3-p}}$. This extension is similar to [4, Theorem 8.8]. In order to use the functional analytic method, one would also need the Helmholtz decomposition of $L^p(w,\Omega)$ for this class of weights. In the case that Ω is a domain with C^1 boundary the weighted Helmholtz decomposition exists when $1 < p < \infty$ [16]. However in the case that Ω is a Lipschitz domain there is no positive result on the existence of the weighted Helmholtz decomposition to the best of the author's knowledge.

Goal of the thesis

The main goal of this thesis is to follow up on the result of Z. Shen [34].

Theorem 1.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 3$, and $\lambda \in \Sigma_\theta$. There exists $\varepsilon > 0$ depending only on d, θ and the Lipschitz character of Ω, such that if $\boldsymbol{f} \in L^2(\Omega;\mathbb{C}^d) \cap L^p(\Omega;\mathbb{C}^d)$ and*

$$\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{1}{2d} + \varepsilon,$$

then the unique solution $\boldsymbol{u}$ of (1.9) in $H^1_0(\Omega;\mathbb{C}^d)$ satisfies the estimate

$$\|\boldsymbol{u}\|_{L^p(\Omega)} \leq \frac{C_p}{|\lambda| + r_0^{-2}} \|\boldsymbol{f}\|_{L^p(\Omega)},$$

where $r_0 = \operatorname{diam}(\Omega)$ and C_p depends at most on d, p, θ, and the Lipschitz character of Ω.

This is done by studying the paper of Shen and working out the details that are considered standard in this field. At the end of the thesis we introduce a class of weights and show that similar resolvent estimates hold in certain weighted spaces. This leads to the following extension of Theorem 1.1 in the case $d = 3$.

Theorem 1.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$ and $\lambda \in \Sigma_\theta$. There exists $\varepsilon > 0$ depending only on θ and the Lipschitz character of Ω, such that if $\boldsymbol{f} \in L^2(w,\Omega;\mathbb{C}^3) \cap L^p(w,\Omega;\mathbb{C}^3)$ and*

$$\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{1}{6},$$

and $w \in A_{\frac{2p}{3}} \cap RH_{\frac{3}{3-p}}$, *then the unique solution* $\boldsymbol{u}$ *of* (1.9) *in* $H_0^1(\Omega;\mathbb{C}^3)$ *satisfies the estimate*

$$\|\boldsymbol{u}\|_{L^p(w,\Omega)} \leq \frac{C_p}{|\lambda| + r_0^{-2}} \|\boldsymbol{f}\|_{L^p(w,\Omega)},$$

where $r_0 = \operatorname{diam}(\Omega)$ *and* C_p *depends at most on* p, θ, $[w]_{A_{\frac{2p}{3}}}$, $[w]_{\mathrm{RH}_{\frac{3}{3-p}}}$ *and the Lipschitz character of* Ω.

Structure of the thesis

In order to complete the goal of this thesis, it is structured in the following manner. In **Chapter 2** the Lipschitz domain is defined and the most important properties of Lipschitz domains are stated and proved. Further in this chapter the nontangential maximal function and nontangential convergence are introduced. Using these notions, an approximation scheme is presented to approach the boundary of a Lipschitz domain with domains with a smooth boundary. With this approximation scheme, and some extra conditions on the functions, integration by parts can be justified on Lipschitz domains. In **Chapter 3** a fundamental solution for the Stokes System is constructed. This is done by solving the scalar Helmholtz equation and then using this result to get a fundamental solution for the Stokes System. The rest of the chapter is concerned with establishing inequalities for this fundamental solution of the Stokes System, to be used throughout the rest of the thesis. In **Chapter 4** the single and double layer potential are introduced. The single and double layer potential solve the Stokes System in its domain, Ω, but fail to suffice to the boundary condition on $\partial\Omega$. In order to use the single and double layer potential, its nontangential maximal function should be bounded on $L^p(\partial\Omega)$. This is done in the middle part of this chapter. In the final part of the chapter the nontangential limit of the double layer potential is calculated. It turns out that the conormal derivative of the single layer potential is the adjoint of the double layer potential, hence we also find the nontangential limit of the conormal derivative. In **Chapter 5** the invertibility of the layer potentials is studied. The layer potentials can be inverted by a compactness argument, but this loses the control of the parameter λ. Since explicit dependence on λ is required in the resolvent estimate, an alternative for the compactness arguments needs to be found. Rellich estimates are the right way to do this. Using these Rellich estimates we find that the layer potentials are isomorphisms/Fredholm operators on $L^2(\partial\Omega)$. This leads to the existence of a boundary function such that the layer potential is a unique solution to the Stokes System. Since the unique solutions can be represented by layer potentials, we also obtain a weak reverse Hölder estimate on the solution. In **Chapter 6** extrapolation techniques are developed to extend the (well-known) $L^2(\Omega)$ resolvent estimate to $L^p(\Omega)$. To do so, we use the weak reverse Hölder inequalities of the previous chapter and a good-λ type inequality. This shows the result that Shen had already obtained in [34]. The second part of this chapter introduces an intersection of a Muckenhoupt weight class with a reverse Hölder class. Then we show a resolvent estimate on $L^2(w,\Omega)$. Using the famous extrapolation theory of Rubio de Francia we obtain the resolvent estimates for a similar range as the unweighted case. In **Chapter 7** we draw conclusions and discuss the results. We also look at interesting related questions. Finally the **Appendix** contains basic results in Harmonic Analysis, Calderón-Zygmund theory, theorems from real analysis, inequalities and calculations that would distract too much from the main text.

2 Lipschitz domains

In this section Lipschitz continuous functions and Lipschitz domains are introduced. Then we make sense of integration over a Lipschitz domain and its boundary. This is then used to prove certain scaling properties and the doubling property of the measure on the boundary of the Lipschitz domain. In the next section we discuss nontangential limits in Lipschitz domains and the approximation of a Lipschitz domain with a sequence of smooth domains. The chapter is concluded with a version of the divergence theorem that makes sense on Lipschitz domains.

2.1 Definition of a Lipschitz domain

Before the definition of a Lipschitz domain is given, we define Lipschitz continuous functions.

Definition 2.1 (Lipschitz continuous function)**.** *Let (X, d) and (Y, ρ) be metric spaces. A function $\varphi : X \to Y$ is called a Lipschitz continuous function if there exists an $L \geq 0$ such that*

$$\rho(\varphi(x), \varphi(y)) \leq Ld(x, y), \tag{2.1}$$

for all $x, y \in X$. The smallest value of L for which (2.1) holds is referred to as the Lipschitz constant.

Now that we know the definition of a Lipschitz continuous function we can introduce the definition of the special Lipschitz domain.

Definition 2.2 (Special Lipschitz domain)**.** *Let Ω be an open subset of $\mathbb{R}^d$ and denote its boundary by $\partial\Omega$. Then Ω is a special Lipschitz domain if there exists a rectangular coordinate system (obtained by rotation and translation) and a Lipschitz continuous function $\varphi : \mathbb{R}^{d-1} \to \mathbb{R}$ such that*

$$\Omega = \left\{(\tilde{x}, \tilde{y}) \in \mathbb{R}^{d-1} \times \mathbb{R} \,:\, \tilde{y} < \varphi(\tilde{x})\right\}$$
$$\partial\Omega = \left\{(\tilde{x}, \tilde{y}) \in \mathbb{R}^{d-1} \times \mathbb{R} \,:\, \tilde{y} = \varphi(\tilde{x})\right\},$$

where $\tilde{x} \in \mathbb{R}^{d-1}$ and $\tilde{y} \in \mathbb{R}$ represent the coordinates in the rectangular coordinate system.

Figure 2.1 provides intuition to the somewhat cumbersome definition of a special Lipschitz domain. In this figure, the function $\pi : \mathbb{R}^{d-1} \to \mathbb{R}^d$ is also introduced. This is a convenient function mapping $\tilde{p} \mapsto (\tilde{p}, \varphi(\tilde{p})) = p$. Next we introduce the notion of a Lipschitz domain. This

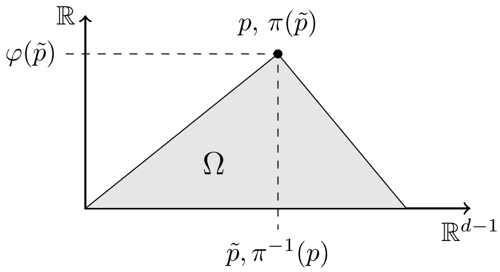

Figure 2.1: A point p of the boundary of a special Lipschitz domain represented in a rectangular coordinate system.

is a domain that is locally represented by a special Lipschitz domain.

Definition 2.3 (Lipschitz domain)**.** *Let Ω be an open subset of $\mathbb{R}^d$ and denote its boundary by $\partial\Omega$. Then Ω is a Lipschitz domain if for all $p \in \partial\Omega$ there exists $r > 0$, a rectangular coordinate system (obtained by rotating and translating) and a Lipschitz continuous function $\varphi : \mathbb{R}^{d-1} \to \mathbb{R}$ such that*

$$\Omega \cap B_d(p,r) = \left\{(\tilde{x},\tilde{y}) \in \mathbb{R}^{d-1} \times \mathbb{R} \,:\, \tilde{x} \in B_d(\tilde{p},r),\, \tilde{y} < \varphi(\tilde{x})\right\}$$
$$(\partial\Omega) \cap B_d(p,r) = \left\{(\tilde{x},\tilde{y}) \in \mathbb{R}^{d-1} \times \mathbb{R} \,:\, \tilde{x} \in B_d(\tilde{p},r),\, \tilde{y} = \varphi(\tilde{x})\right\},$$

where $\tilde{x} \in \mathbb{R}^{d-1}$ and $\tilde{y} \in \mathbb{R}$ denote the coordinates in the rectangular coordinate system and $\tilde{p} \in \mathbb{R}^{d-1}$, $\varphi(\tilde{p}) \in \mathbb{R}$ denote p in the rectangular coordinate system.

From the definitions, it is clear that the boundary of a Lipschitz domain is locally represented by a Lipschitz continuous function. The example of how the coordinate system should be rotated and translated can be found in Figure 2.2 and Figure 2.3. There is an important connection between the space of Lipschitz continuous functions and the Sobolev space $W^{1,\infty}$. This connection is described by Rademacher's Theorem.

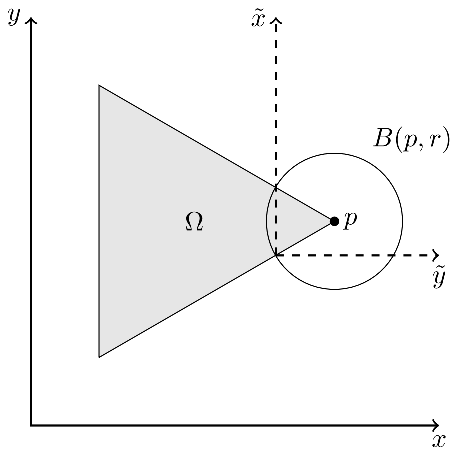

Figure 2.2: Finding a new rectangular coordinate system by rotating and translating.

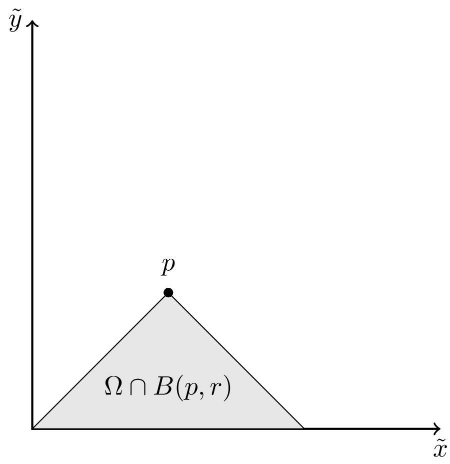

Figure 2.3: The neighborhood of p expressed in the translated and rotated coorinate system.

Theorem 2.4 (Rademacher's Theorem)**.** *Let Ω be an open subset of $\mathbb{R}^d$ and let f be locally Lipschitz continuous in Ω. Then f is differentiable almost everywhere in Ω.*

A proof for this theorem can be found in [11, Section 5.8.3]. From Rademacher's Theorem it is clear that the boundary of a Lipschitz domain is differentiable almost everywhere.

2.2 Integration on the boundary of Lipschitz domains

Rademacher's Theorem ensures that integrals over the boundary of (special) Lipschitz domains are well-defined. To see this we take a look at integrals over the boundary of a special Lipschitz domain. Suppose that Ω is a special Lipschitz domain in $\mathbb{R}^d$ and $f \in L^p(\partial\Omega)$. Then the integral of f over the boundary of Ω is given by

$$\int_{\partial\Omega} f(y)d\sigma(y).$$

Here σ represents the usual surface measure. To evaluate this integral, the surface should be parametrized. This can be done with the help of the definition of a special Lipschitz domain. Let φ be the Lipschitz function from the definition of the special Lipschitz domain. Then we can parametrize the points $p \in \partial\Omega$ as $(\tilde{p}, \varphi(\tilde{p}))$. Therefore we can write the integral over $\partial\Omega$ as

$$\int_{\mathbb{R}^{d-1}} f(\pi(\tilde{y}))\sqrt{1 + |\nabla\varphi(\tilde{y})|^2}d\tilde{y}.$$

Now the importance of Rademacher's Theorem is clear. It ensures that $\nabla\varphi$ is well-defined almost everywhere on $\partial\Omega$ and $\|\nabla\varphi\|_\infty < L$ for some $L > 0$. For integrals over general Lipschitz domains a little more work is needed. For this purpose we introduce the definition of a partition of unity.

Definition 2.5 (Partition of unity)**.** *Let K be a compact subset of $\mathbb{R}^d$ and $(U_i)_{i=1}^n$ a finite open cover of K. A C_c^∞ partition of unity subordinate to the cover $(U_i)_{i=1}^n$ is a collection of functions $(\zeta_i)_{i=1}^n$ such that*

(i) $0 \leq \zeta_i \leq 1$

(ii) $\zeta_i \in C_c^\infty(U_i)$

(iii) $\sum_{i=1}^n \zeta_i(x) = 1$ *for all* $x \in K$.

Such a partition of unity exists for all compact K for every open cover. For details on the construction and existence of a partition of unity one can refer to [10, Theorem 2.15]. The idea now is to evaluate the surface integral on a Lipschitz domain by choosing a cover of the boundary such that in each set the domain behaves like a special Lipschitz domain. We already know how to evaluate the surface integral on these domains. Finally a partition of unity is used to "glue" these results back together.

Now we make this procedure formal. Suppose that Ω is a bounded Lipschitz domain. Then by the definition of a Lipschitz domain we can for each $p \in \partial\Omega$ find a $r_p > 0$ such that $\Omega \cap B_d(p, r_p)$ is a special Lipschitz domain with Lipschitz function φ_p. We can thus calculate

$$\int_{\partial\Omega\cap B(p,r_p)} f(y)d\sigma(y) = \int_{\partial\Omega\cap B(\pi(\tilde{p}),r_p)} f(\pi_p(\tilde{y}))\sqrt{1 + |\nabla\varphi_p(\tilde{y})|^2}d\tilde{y}.$$

Now realize that $\cup_{p\in\partial\Omega} B_d(p, r_p)$ is an open cover of $\partial\Omega$. Also note that $\partial\Omega$ is closed and bounded, hence compact by the Heine-Borel Theorem. We can thus extract a finite subcover. Denote by $\mathcal{C}$ the set of centers that are in the finite subcover. We can now find a partition of unity (ζ_i) subordinate to the cover $\cup_{p\in\mathcal{C}} B_d(p, r_p)$. Now we can write the surface integral as

$$\sum_{p\in\mathcal{C}} \int_{\mathbb{R}^{d-1}} \zeta_p(\pi_p(\tilde{y}))f(\pi_p(\tilde{y}))\sqrt{1 + |\nabla\varphi_p(\tilde{y})|^2}d\tilde{y}.$$

This yields to a well-defined expression for the surface integral over the boundary of a general Lipschitz domain.

2.3 Properties of the surface measure on Lipschitz domains

Now that the surface integral over the boundary of a Lipschitz domain is defined, we will state and prove two lemmas regarding the surface measure. Using these lemmas we show that the surface measure has the doubling property.

Lemma 2.6. *Let Ω be a special Lipschitz domain in $\mathbb{R}^d$. Then there exist constants $C_1, C_2 > 0$, such that for $p \in \partial\Omega$*

$$C_1 r^{d-1} \leq \sigma(\partial\Omega \cap B_d(p,r)) \leq C_2 r^{d-1}, \tag{2.2}$$

where $B_d(p,r)$ denotes the d-dimensional ball with center p and radius r.

Proof. We start the proof by showing the following set of inclusions

$$\pi\left(B_{d-1}\left(\pi^{-1}(p), \frac{r}{\sqrt{1+L^2}}\right)\right) \subset B_d(p,r) \subset \pi\left(B_{d-1}(\pi^{-1}(p), r)\right), \tag{2.3}$$

where L is the essential bound of the gradient of the Lipschitz function describing the $\partial\Omega$ in the ball $B_d(p,r)$. By Theorem 2.4 it must be that $0 \leq |\nabla\varphi(\tilde{p})| \leq L$ almost everywhere. The right inclusion of (2.3) is trivial. For the left inclusion of (2.3) take $\tilde{y} \in B_{d-1}(\pi^{-1}(p), \frac{r}{\sqrt{1+L^2}})$. Then

$$\left|\varphi(\pi^{-1}(p)) - \varphi(\tilde{y})\right|^2 + \left|\pi^{-1}(p) - \tilde{y}\right|^2 \leq (L^2+1)\left|\pi^{-1}(p) - \tilde{y}\right|^2 < r^2$$

and thus $\pi(\tilde{y}) \in \partial\Omega \cap B_d(p,r)$. This shows (2.3). Now notice that

$$\begin{aligned}
\omega_{d-1}\left(\sqrt{1+L^2}\right)^{1-d} r^{d-1} &\leq \int_{\mathbb{R}^{d-1}} \chi_{B_{d-1}(\pi^{-1}(p), \frac{r}{\sqrt{1+L^2}})}(\tilde{y})\sqrt{1+|\nabla\varphi(\tilde{y})|^2}\,d\tilde{y} \\
&\leq \int_{\mathbb{R}^{d-1}} \chi_{B_d(p,r)}(\pi(\tilde{y}))\sqrt{1+|\nabla\varphi(\tilde{y})|^2}\,d\tilde{y} \\
&= \sigma(\partial\Omega \cap B_d(p,r))
\end{aligned}$$

and

$$\begin{aligned}
\sigma(\partial\Omega \cap B_d(p,r)) &= \int_{\mathbb{R}^{d-1}} \chi_{B_d(p,r)}(\pi(\tilde{y}))\sqrt{1+|\nabla\varphi(\tilde{y})|^2}\,d\tilde{y} \\
&\leq \int_{\mathbb{R}^{d-1}} \chi_{B_{d-1}(\pi^{-1}(p),r)}(\tilde{y})\sqrt{1+|\nabla\varphi(\tilde{y})|^2}\,d\tilde{y} \\
&\leq \sqrt{1+L^2}\,\omega_{d-1} r^{d-1},
\end{aligned}$$

where ω_{d-1} is the volume of the $(d-1)$-dimensional unit ball. This completes the proof. ☐

Lemma 2.7. *Let Ω be a special Lipschitz domain in $\mathbb{R}^d$. Then there exist constants $C_1, C_2 > 0$, such that for all $p \in \partial\Omega$*

$$C_1 r^{d-1} \leq \sigma(\partial B_d(p,r) \cap \Omega) \leq C_2 r^{d-1}, \tag{2.4}$$

where $B_d(p,r)$ denotes the d-dimensional ball with center p and radius r.

Proof. It is obvious that

$$\sigma(\partial B(x,r) \cap \Omega) \leq \sigma(\partial B(x,r)) = d\omega_d r^{d-1}.$$

This shows the right-hand side of (2.4) with $C_2 = d\omega_d$. For the left-hand side we define the function $\varphi^* : B_{d-1}(\pi^{-1}(p), \frac{r}{\sqrt{1+L^2}}) \to \partial B_d(p,r)$ by

$$\varphi^*(\tilde{y}) = \varphi(\tilde{p}) - \sqrt{r^2 - |\tilde{p} - \tilde{y}|^2}.$$

We know want to show that φ^* maps $B_{d-1}(\pi^{-1}(p), \frac{r}{\sqrt{1+L^2}})$ into $\partial B_d(p,r) \cap \Omega$. Indeed

$$\begin{aligned}
\varphi^*(\tilde{y}) &= \varphi(\tilde{p}) - \sqrt{r^2 - |\tilde{p} - \tilde{y}|^2} \\
&= \varphi(\tilde{y}) + \varphi(\tilde{p}) - \varphi(\tilde{y}) - \sqrt{r^2 - |\tilde{p} - \tilde{y}|^2} \\
&\leq \varphi(\tilde{y}) + L|\tilde{p} - \tilde{y}| - \sqrt{r^2 - |\tilde{p} - \tilde{y}|^2} \\
&< \varphi(\tilde{y}) + \frac{Lr}{\sqrt{1+L^2}} - \sqrt{\frac{r^2L^2}{1+L^2}} \\
&\leq \varphi(\tilde{y}).
\end{aligned}$$

It is clear from the definition that φ^* is Lipschitz continuous and hence we see that

$$\begin{aligned}
\omega_{d-1} \left(\sqrt{1+L^2}\right)^{1-d} r^{d-1} &= \int_{\mathbb{R}^{d-1}} \chi_{B_{d-1}(\pi^{-1}(p), \frac{r}{\sqrt{1+L^2}})}(\tilde{y}) d\tilde{y} \\
&\leq \int_{\mathbb{R}^{d-1}} \chi_{B_d(p,r)}(\pi(\tilde{y})) \sqrt{1 + |\nabla \varphi^*(\tilde{y})|^2} d\tilde{y}.
\end{aligned}$$

And by Rademacher's Theorem, we find that the left-hand side (2.4) holds with $C_1 = \omega_{d-1} \left(\sqrt{1+L^2}\right)^{1-d}$. This completes the proof. □

Finally we show that the surface measure on a general Lipschitz domain has the doubling property. That way we can use important theorems from harmonic analysis. How exactly this comes into play is explained in Appendix A.

Lemma 2.8 (Doubling Property). *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$. Then for all $p \in \partial\Omega$ and all $r > 0$ there exists a constant $C > 0$ such that*

$$\sigma(B_d(p, 2r) \cap \partial\Omega) \leq C\sigma(B_d(p,r) \cap \partial\Omega). \tag{2.5}$$

Proof. Let Ω' be a special Lipschitz domain. Then by Lemma 2.6 there exist constants C_1, $C_2 > 0$ such that

$$\sigma(B(p, 2r) \cap \partial\Omega') \leq C_1(2r)^{d-1} \quad \text{and} \quad \sigma(B(p,r) \cap \partial\Omega') \geq C_2 r^{d-1}.$$

Combining the above inequalities yields

$$\sigma(B(p, 2r) \cap \partial\Omega') \leq \frac{2^{d-1}C_1}{C_2} \sigma(B(p,r) \cap \partial\Omega').$$

Now the proof is completed by generalising to a Lipschitz domain using a compactness argument. Hereto pick for every $q \in \partial\Omega$ an r_q such that $B(q, r_q) \cap \partial\Omega$ is a special Lipschitz domain. Since Ω is bounded and closed, the boundary $\partial\Omega$ is compact by the Heine-Borel Theorem. Hence we can find a finite set of centers $\mathcal{C}$ such that

$$\partial\Omega \subset \bigcup_{q \in \mathcal{C}} \left(B(q, r_q) \cap \partial\Omega\right).$$

The reverse inclusion trivially holds. Now we calculate

$$\begin{aligned}
\sigma\left(B(p,2r)\cap\partial\Omega\right) &= \sigma\left(B(p,2r)\cap\bigcup_{q\in\mathcal{C}}\left(B(q,r_q)\cap\partial\Omega\right)\right)\\
&= \sigma\left(\bigcup_{q\in\mathcal{C}}\left(B(p,2r)\cap B(q,r_q)\cap\partial\Omega\right)\right)\\
&\leq \sum_{q\in\mathcal{C}}\sigma\left(B(p,2r)\cap\left(B(q,r_q)\cap\partial\Omega\right)\right)\\
&\leq \sum_{q\in\mathcal{C}} C_q\sigma(B(p,r)\cap B(q,r_q)\cap\partial\Omega)\\
&\leq |\mathcal{C}|\max_{q\in\mathcal{C}}\{C_q\}\sigma\left(B(p,r)\cap\partial\Omega\right),
\end{aligned}$$

where $|\mathcal{C}|$ denotes the amount of elements in $\mathcal{C}$ and C_q the doubling constant. We used the subadditivity of the measure and the finiteness of the cover. This completes the proof. ☐

In the last part of this section we want to show that a Lipschitz domain and its boundary equipped with the Euclidean metric and Lebesgue measure is a space of homogeneous type in the sense of Coifman and Weiss [6]. We first give the definition and then show this.

Definition 2.9 (Space of homogeneous type)**.** *(X, d, μ) is called a space of homogeneous type if the quasi-metric d and the measure μ satisfy*

(i) for all $x, y \in X$, $d(x, y) \geq 0$,

(ii) for all $x, y \in X$, $d(x, y) = d(y, x)$,

(iii) $d(x, y) = 0$ if and only if $x = y$,

(iv) there exists a $\kappa > 0$ such that, for all $x, y, z \in X$

$$d(x, y) = \kappa\left(d(x, z) + d(z, y)\right),$$

(v) there exists $C_1, C_2, q > 0$ such that for all $x \in X$ and $r > 0$

$$C_1 r^q \leq \int_{d(x,y)<r} d\mu(y) \leq C_2 r^q.$$

q is called the homogeneous dimension of (X, d, μ).

Lemma 2.10. *Let Ω be a Lipschitz domain in $\mathbb{R}^d$. Then $(\partial\Omega, |\cdot|, \sigma)$, the boundary of a Lipschitz domain with the Euclidean metric and the surface measure, is a space of homogeneous type, with homogeneous dimension $r - 1$.*

Proof. Property $(i) - (iv)$ are trivial. Property (v) is a direct consequence of Lemma 2.6. ☐

2.4 Approximation of Lipschitz domains by smooth domains

The following lemma allows us to approximate Lipschitz domains by a sequence of smooth domains. This is used to justify certain partial integrations. A proof of this lemma can be found in [38, Appendix A].

Lemma 2.11. *Let Ω be a Lipschitz domain in $\mathbb{R}^d$. Then there exists a sequence of C^∞ domains D_j, a sequence of homeomorphisms $\Lambda_j : \partial\Omega \to \partial D_j$, a sequence of functions $\eta_j : \partial\Omega \to \mathbb{R}_+$ and a smooth compactly supported vectorfield $\boldsymbol{h} : \mathbb{R}^d \to \mathbb{R}^d$ such that,*

(i) For each $p \in \partial\Omega$, there exists a $r > 0$ such that $B(p, r) \cap \Omega$ and $B(p, r) \cap D_j$ are special Lipschitz domains with Lipschitz functions φ and φ_k respectively and $\|\nabla\varphi_k\|_\infty \leq \|\nabla\varphi\|_\infty$ and $\nabla\varphi_k \to \nabla\varphi$ pointwise almost everywhere.

(ii) The homeomorphisms $\Lambda_j : \partial\Omega \to \partial D_j$ satisfy

$$\lim_{j\to\infty} \sup_{p\in\partial\Omega} |p - \Lambda_j(p)| = 0,$$

and $\Lambda_j(q)$ approaches q nontangentially, thus there exists a sufficiently large $C > 0$, independent of q, such that

$$|q - \Lambda_j(q)| < (1 + C)\,\mathrm{dist}(\Lambda_j(q), \partial\Omega).$$

(iii) The normals $\boldsymbol{n}_j$ of ∂D_j satisfy

$$\lim_{j\to\infty} \boldsymbol{n}_j(\Lambda_j(p)) = \boldsymbol{n}(p)$$

pointwise almost everywhere.

(iv) The functions η_j satisfy $\delta \leq \eta_j \leq \delta^{-1}$ for some $\delta > 0$, $\eta_j \to 1$ pointwise almost everywhere and

$$\int_A \eta_j(y)d\sigma(y) = \int_{\Lambda_j(A)} d\sigma(y),$$

where A is any measurable subset of $\partial\Omega$.

(v) The vectorfield $\boldsymbol{h}$ satisfies $\langle \boldsymbol{h}, \boldsymbol{n}_j\rangle \geq c > 0$ almost everywhere on each ∂D_j.

(vi) The sequence of domains may be chosen such that $\overline{D}_j \subset \overline{D}_{j+1} \subset \cdots \subset \Omega$. If this is done it is denoted by $D_j \uparrow \Omega$. The sequence of domains may also be chosen such that $\overline{D}_j \supset \overline{D}_{j+1} \supset \cdots \supset \Omega$. If this is done it is denoted by $D_j \downarrow \Omega$.

In order to use this approximation effectively, also convergence needs to be guaranteed. To do so we introduce the nontangential maximal function and the nontangential limit. We define them for Lipschitz domains, which is not the limiting case for which this definition is sensible.

Definition 2.12 (nontangential cone)**.** *Let Ω be a Lipschitz domain in $\mathbb{R}^d$ and $p \in \partial\Omega$. Then the interior nontangential cone of p is given by*

$$E_+(p) = \{x \in \Omega \,:\, |x - p| < C\,\mathrm{dist}(x, \partial\Omega)\}, \tag{2.6}$$

and the exterior nontangential cone of p is given by

$$E_-(p) = \left\{x \in \mathbb{R}^d \setminus \overline{\Omega} \,:\, |x - p| < C\,\mathrm{dist}(x, \partial\Omega)\right\}, \tag{2.7}$$

where $C > 2$, is fixed and sufficiently large depending on the Lipschitz character of Ω.

Before we continue we present a small geometrical lemma regarding this nontangential cone.

Lemma 2.13. *Let $p \in \partial\Omega$ and $C > 2$. Define $E(p) = \{x \in \Omega : |x - p| \leq C\,\mathrm{dist}(x, \partial\Omega)\}$. There exist constants $C_1, C_2 > 0$, such that for all $x \in E(p)$ and $y \in \partial\Omega$ we have*

(i) $|x - y| \geq C_2|p - x|$

(ii) $|x - y| \geq C_1|p - y|$.

Proof. Let $p \in \partial\Omega$ and $x \in E(p)$. For (i) we see that by the definition of $E(p)$ we see that for all $y \in \partial\Omega$ we have

$$|p - x| \leq C\,\mathrm{dist}(x, \partial\Omega) \leq C|x - y|.$$

For (ii) we use the triangle inequality in combination with (i) and find for all $y \in \partial\Omega$ that

$$|p - y| \leq |p - x| + |x - y| \leq (C + 1)|x - y|.$$

This completes the proof. Some visual aid can be found in Figure 2.4. □

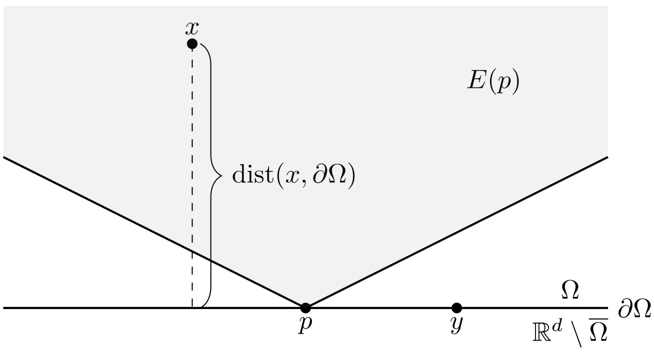

Figure 2.4: Interior nontangential cone of p

Using the definition of the nontangential cone we can define the nontangential maximal function and the nontangential limit.

Definition 2.14 (nontangential maximal function)**.** *Let Ω be a Lipschitz domain in $\mathbb{R}^d$ and $p \in \partial\Omega$. Let $f : \Omega \to \mathbb{C}$ be a continuous function. Then the interior and exterior nontangential maximal function are given by*

$$(f)^*_+(p) = \sup_{x \in E_+(p)} |f(x)|, \tag{2.8}$$

and

$$(f)^*_-(p) = \sup_{x \in E_-(p)} |f(x)| \tag{2.9}$$

respectively.

Definition 2.15 (nontangential limit)**.** *Let Ω be a Lipschitz domain in $\mathbb{R}^d$ and $p \in \partial\Omega$. Let $f : \Omega \to \mathbb{C}$ be a continuous function and (x_n) any sequence such that $x_n \in E_+(p)$ for all n and $x_n \to p$. Then the interior nontangential limit of f is given by*

$$f_+(p) = \lim_{\substack{x_n \in E_+(p) \\ x_n \to p}} f(x_n). \tag{2.10}$$

Analogously the exterior nontangential limit of f is given by

$$f_-(p) = \lim_{\substack{x_n \in E_-(p) \\ x_n \to p}} f(x_n). \tag{2.11}$$

Remark 2.16. *The existence of the nontangential limit* $f(p) = \lim_{\substack{x\in E(p)\\ x\to p}} |f(x)|$ *is equivalent to the statement that for all sequences* (x_n) *such that* $x_n \in E(p)$ *and* $x_n \to p$ *the nontangential limit exists.*

In this thesis we mainly use the interior cones, interior nontangential maximal functions and interior nontangential limits. Therefore we drop the subscript + if it is clear that the interior version is meant and only use the subscripts if both the interior and exterior versions are used at the same time. Now that it is clear what the nontangential maximal function and the nontangential limit are we can prove a version of the divergence theorem tailored to Lipschitz domains. In order to do so, we first state a lemma that will be used in the proof of the theorem.

Lemma 2.17. *Let* Ω *be a Lipschitz domain in* $\mathbb{R}^d$. *Then* $C^m(\overline{\Omega})$ *is dense in* $W^{k,p}(\Omega)$ *for all* k *and all* $1 \le p < \infty$.

The proof of this lemma can be found in [1, Theorem 3.18]. Recall that $C^m(\overline{\Omega})$ is given by the functions $f \in C^m(\Omega)$ for which $\partial^\alpha f$ is bounded and uniformly continuous on Ω for $0 \le |\alpha| \le m$.

Theorem 2.18. *Let* Ω *be a bounded Lipschitz domain in* $\mathbb{R}^d$. *Let* $f, g \in W^{1,2}(\Omega)$ *and* $(f)^*, (g)^* \in L^2(\partial\Omega)$. *Assume that* f *and* g *have nontangential limits almost everywhere on* $\partial\Omega$. *Then*

$$\int_\Omega (f\nabla g - g\nabla f)\, dx = \int_{\partial\Omega} (fg) \cdot \boldsymbol{n} d\sigma. \tag{2.12}$$

Proof. Assume $f, g \in C^1(\overline{\Omega})$. It suffices to study this case, because of the density of $C^1(\overline{\Omega})$ in $W^{1,2}(\Omega)$ (see Lemma 2.17). We now use Lemma 2.11 to obtain a collection of smooth domains, such that $D_j \uparrow \Omega$. Because each of the D_j is smooth, we can apply the "regular" divergence theorem (Theorem B.6) to find

$$\int_{D_j} (f\nabla g - g\nabla f)\, dx = \int_{\partial D_j} (fg) \cdot \boldsymbol{n} d\sigma.$$

The integrals on the left side of the equation converge because of the dominated convergence theorem (Theorem B.5). For the right side of the equation we further use Lemma 2.11 to find

$$\int_{\partial D_j} (f(y)g(y)) \cdot n(y) d\sigma = \int_{\partial\Omega} \eta_j(y) f(\Lambda_j(y)) g(\Lambda_j(y)) \cdot n(\Lambda_j(y)) d\sigma(y).$$

The pointwise boundedness of $\eta_j(y)$ and $n(\Lambda_j(y))$ is obvious. Furthermore, because $\Lambda_j(y) \subset E(y)$ for all $y \in \partial\Omega$, we find

$$f(\Lambda_j(y)) \le \sup_{x\in E(y)} |f(x)| = (f)^*(y).$$

Hence we find

$$|\eta_j(y) f(\Lambda_j(y)) g(\Lambda_j(y)) \cdot n(\Lambda_j(y))| \le \delta^{-1} |(f)^*(y)(g)^*(y)|, \tag{2.13}$$

where δ is the same as in Lemma 2.11. By assumption $(f)^*$ and $(g)^*$ are bounded in $L^2(\partial\Omega)$. Hence the right-hand side of (2.13) is integrable by the Hölder inequality. We can now invoke the dominated convergence theorem and the result follows. □

3 Fundamental solution

In this section we look at a fundamental solution for the Stokes System as given in (1.11) and study its properties. Here again $\lambda \in \Sigma_\theta = \{\lambda \in \mathbb{C} : \lambda \neq 0 \text{ and } |\arg(\lambda)| < \pi - \theta\}$ and $\theta \in (0, \pi/2)$. We can thus write $\lambda = re^{i\tau}$, with $0 < r < \infty$ and $-\pi + \theta < \tau < \pi - \theta$. Now define $k = \sqrt{r}e^{i\frac{(\pi+\tau)}{2}}$ and notice that $k^2 = -\lambda$. A visual representation of the sectors in which λ and k take values can be found below.

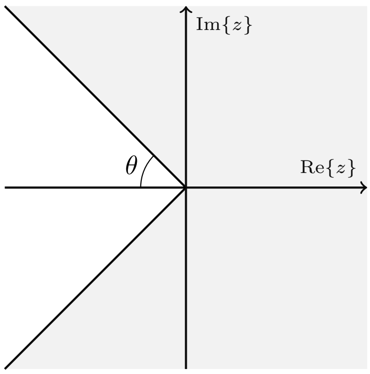

Figure 3.1: The sector (grey) where λ takes possible values.

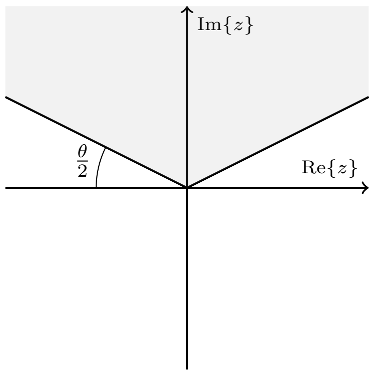

Figure 3.2: The sector (grey) where k takes values, such that $k^2 = -\lambda$.

From now on we will only consider the case $d = 3$, but the results could be extended to $\mathbb{R}^d$ for all $d \geq 3$. Before looking at a fundamental solution of the Stokes System, we derive a candidate fundamental solution to the scalar Helmholtz equation. Next we prove that this is indeed a fundamental solution. Later on this can be used to construct a fundamental solution for the Stokes System. Finally some estimates on the fundamental solutions are proven.

3.1 Fundamental solutions of the Helmholtz equation

First we look at the Helmholtz equation. In this section we are going to construct a family of functions that satisfy the Helmholtz equation. After that we pick such a function and prove that it solves the equation in the distributional sense. Finally we do some estimates on this fundamental solution.

3.1.1 Construction of the fundamental solution

For $\lambda \in \Sigma_\theta$, the scalar Helmholtz equation is given by

$$-\Delta u + \lambda u = -\Delta u - k^2 u = 0. \tag{3.1}$$

The Helmholtz equation is invariant under rotations. Therefore we will look for a solution with radial symmetry. This can be done by looking for solutions in the form of

$$u(x) = v(r),$$

where $r = |x|$. We notice that

$$\frac{\partial r}{\partial x_i} = \frac{1}{2}\left(\sum_{i=1}^{3} x_i^2\right)^{-\frac{1}{2}} 2x_i = \frac{x_i}{r},$$

and thus we find for the derivatives

$$\frac{\partial u}{\partial x_i} = \frac{dv}{dr}\frac{x_i}{r}$$
$$\frac{\partial^2 u}{\partial x_i^2} = \frac{d^2 v}{dr^2}\frac{x_i^2}{r^2} + \frac{dv}{dr}\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right).$$

Using this we find a solution for (3.1) if we find a function $v(r)$ such that

$$\Delta u + k^2 u = \frac{d^2 v}{dr^2}(r) + \frac{3-1}{r}\frac{dv}{dr}(r) + k^2 v(r) = 0. \tag{3.2}$$

The solution $v(r)$ of (3.2) involves Bessel functions when $d > 3$, however if $d = 3$ we can simplify the expression above. To do so, note that

$$\frac{d^2}{dr^2}\left(rv(r)\right) = 2\frac{dv}{dr}(r) + r\frac{d^2 v}{dr^2}(r).$$

Hence (3.2) simplifies to

$$\frac{d^2}{dr^2}\left(rv(r)\right) + k^2 rv(r) = 0.$$

Therefore, if $r > 0$, we find that

$$v(r) = C_1\frac{e^{ikr}}{r} + C_2\frac{e^{-ikr}}{r}.$$

This motivates the following choice for a candidate fundamental solution of the Helmholtz equation,

$$G(x;\lambda) = \frac{e^{ik|x|}}{4\pi|x|}, \tag{3.3}$$

for $x \neq 0$.

3.1.2 Properties of the fundamental solution

First we note a few properties of this candidate fundamental solution. Then we show that this fundamental solution solves the Helmholtz equation in the distributional sense. Before we continue we make a basic remark.

Remark 3.1. *Let $\lambda \in \Sigma_\theta$ and k be such that $k^2 = -\lambda$. Then*

$$\mathrm{Im}(k) > \sin(\theta/2)\sqrt{|\lambda|}. \tag{3.4}$$

Now we can examine the limiting behaviour of (3.3). This is needed to show that (3.3) is a fundamental solution in the distributional sense.

Lemma 3.2. *Let $\lambda \in \Sigma_\theta$ and $G(x;\lambda) = \frac{e^{ik|x|}}{4\pi|x|}$, with $x \in \mathbb{R}^3 \setminus \{0\}$. Let $\varepsilon > 0$. Then*

$$\int_{B(0,\varepsilon)} |G(x;\lambda)| dx \to 0 \tag{3.5}$$

$$\int_{\partial B(0,\varepsilon)} |G(y;\lambda)| d\sigma(y) \to 0, \tag{3.6}$$

when $\varepsilon \to 0$.

Proof. For the first integral we use spherical coordinates and the fact that $\operatorname{Im}(k) > 0$ (Remark 3.1). This yields

$$\begin{aligned}
\int_{B(0,\varepsilon)} |G(x;\lambda)|dx &= \int_{B(0,\varepsilon)} \left|\frac{e^{ik|x|}}{4\pi|x|}\right| dx \\
&= \int_{B(0,\varepsilon)} \frac{e^{-\operatorname{Im}(k)|x|}}{4\pi|x|} dx \\
&= \int_0^\varepsilon \int_{\partial B(0,1)} \frac{e^{-\operatorname{Im}(k)r}}{4\pi r} r^2 d\sigma(y) dr \\
&= \int_{\partial B(0,1)} d\sigma(y) \int_0^\varepsilon \frac{e^{-\operatorname{Im}(k)r}}{4\pi} r dr \\
&\leq \int_0^\varepsilon r dr \to 0 \quad (\varepsilon \to 0).
\end{aligned}$$

For the second integral we again use the fact that $\operatorname{Im}(k) > 0$ (Remark 3.1). Thus

$$\begin{aligned}
\int_{\partial B(0,\varepsilon)} |G(y;\lambda)| d\sigma(y) &= \int_{\partial B(0,\varepsilon)} \left|\frac{e^{ik|y|}}{4\pi|y|}\right| d\sigma(y) \\
&= \int_{\partial B(0,\varepsilon)} \frac{e^{-\operatorname{Im}(k)|y|}}{4\pi|y|} d\sigma(y) \\
&\leq \frac{1}{4\pi\varepsilon} \int_{\partial B(0,\varepsilon)} d\sigma(y) \\
&= \varepsilon \to 0 \quad (\varepsilon \to 0).
\end{aligned}$$

This completes the proof. □

Now it is checked whether $G(x;\lambda)$ really is a fundamental solution in the distributional sense.

Theorem 3.3. *Let* $\lambda \in \Sigma_\theta$ *and* $G(x;\lambda) = \frac{e^{ik|x|}}{4\pi|x|}$ *with* $x \in \mathbb{R}^3 \setminus \{0\}$. *Then* $G(x;\lambda)$ *is a fundamental solution of the Helmholtz equation* (3.1) *in the distributional sense.*

Proof. Since $G(x,\lambda)$ is locally integrable on $\mathbb{R}^3$ we can view $G(x;\lambda)$ as a distribution. Then $G(x;\lambda)$ is a fundamental solution of the scalar Helmholtz equation in the distributional sense if

$$\langle [-\Delta + \lambda]\, G(\cdot\,;\lambda), \psi \rangle = \langle \delta, \psi \rangle, \tag{3.7}$$

holds for all functions $\psi \in \mathcal{D}(\mathbb{R}^3) = C_0^\infty(\mathbb{R}^3)$, where $\delta(x)$ denotes the delta distribution. We can use the definition of the distributional derivative together with the linearity of distributions to write (3.7) as

$$\langle [-\Delta + \lambda]\, G(\cdot\,;\lambda), \psi \rangle = -\langle G(\cdot\,;\lambda), \Delta\psi \rangle + \lambda \langle G(\cdot\,;\lambda), \psi \rangle.$$

We will now evaluate these terms individually. Since $G(x;\lambda)$ blows up at 0, we cut this region out and evaluate it separately. This leads to the following expression.

$$\begin{aligned}
\langle G(\cdot\,;\lambda), \Delta\psi \rangle &= \int_{B(0,\varepsilon)} G(x;\lambda)\Delta\psi(x)dx + \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} G(x;\lambda)\Delta\psi(x)dx \\
&= A_\varepsilon + B_\varepsilon,
\end{aligned}$$

and the expression

$$\begin{aligned}\lambda\langle G(\cdot\,;\lambda),\psi\rangle &= \lambda\int_{B(0,\varepsilon)} G(x;\lambda)\psi(x)dx + \lambda\int_{\mathbb{R}^n\setminus B(0,\varepsilon)} G(x;\lambda)\psi(x)dx\\ &= C_\varepsilon + D_\varepsilon.\end{aligned}$$

Next we can partially integrate the integral B_ε two times. This is possible since the functions are twice continuously differentiable in the domain of integration. This leads to

$$\begin{aligned}B_\varepsilon &= \int_{\partial B(0,\varepsilon)} \frac{\partial\psi}{\partial n}(y)G(y;\lambda)d\sigma(y) - \int_{\partial B(0,\varepsilon)} \frac{\partial G}{\partial n}(y;\lambda)\psi(y)d\sigma(y)\\ &\quad + \int_{\mathbb{R}^n\setminus B(0,\varepsilon)} \Delta G(x;\lambda)\psi(x)dx\\ &= I_\varepsilon + J_\varepsilon + K_\varepsilon,\end{aligned}$$

where $\sigma(y)$ denotes the surface measure and $\boldsymbol{n}$ denotes the inward normal vector (pointing towards the origin). First we will look at the integrals A_ε, C_ε and I_ε. ψ and $\Delta\psi$ are bounded on $B(0,\varepsilon)$ and $\frac{\partial\psi}{\partial n}$ is bounded on $\partial B(0,\varepsilon)$ since they are elements of $\mathcal{D}(\mathbb{R}^3)$. Hence we can pull them out of the integral with the supremum norm. This yields

$$\begin{aligned}|A_\varepsilon| &\le \|\Delta\psi\|_\infty \int_{B(0,\varepsilon)} |G(x;\lambda)|dx \le C\int_{B(0,\varepsilon)} |G(x;\lambda)|dx\\ |C_\varepsilon| &\le \|\psi\|_\infty \int_{B(0,\varepsilon)} |G(x;\lambda)|dx \le C\int_{B(0,\varepsilon)} |G(x;\lambda)|dx\\ |I_\varepsilon| &\le \left\|\frac{\partial\psi}{\partial n}\right\|_\infty \int_{\partial B(0,\varepsilon)} |G(y;\lambda)|d\sigma(y) \le C\int_{\partial B(0,\varepsilon)} |G(y;\lambda)|d\sigma(y).\end{aligned}$$

Furthermore, we know that $K_\varepsilon = D_\varepsilon$, because $\Delta G(x;\lambda) = \lambda G(x;\lambda)$ on $\mathbb{R}^3\setminus\{0\}$. Finally we calculate J_ε. Hereto we notice that

$$\frac{\partial G}{\partial n}(y;\lambda) = \nabla G(y;\lambda)\cdot\boldsymbol{n} = \sum_{i=1}^{3}\left(\frac{iky_i e^{ik|y|}}{4\pi|y|^2} - \frac{y_i e^{ik|y|}}{4\pi|y|^3}\right)\frac{y_i}{|y|} = \frac{ike^{ik|y|}}{4\pi|y|} - \frac{e^{ik|y|}}{4\pi|y|^2}.$$

Using that $\sigma(\partial B(0,\varepsilon)) = 4\pi\varepsilon^2$ we find that

$$\begin{aligned}J_\varepsilon &= \int_{\partial B(0,\varepsilon)} \frac{\partial G}{\partial n}(y,\lambda)\psi(y)d\sigma(y)\\ &= \int_{|y|=\varepsilon}\left(\frac{ike^{ik|y|}}{4\pi|y|} - \frac{e^{ik|y|}}{4\pi|y|^2}\right)\psi(y)d\sigma(y)\\ &= \left(ik\varepsilon e^{ik\varepsilon} - e^{ik\varepsilon}\right)\frac{1}{4\pi\varepsilon^2}\int_{\partial B(0,\varepsilon)} \psi(y)d\sigma(y).\end{aligned}$$

Notice that $\frac{1}{4\pi\varepsilon^2}\int_{\partial B(0,\varepsilon)} \psi(y)d\sigma(y)$ goes to $\psi(0)$ as $\varepsilon\to 0$. Now we put this all together. Since ε was arbitrary we can now let ε go to zero to find

$$\begin{aligned}\langle[-\Delta+\lambda]\,G(\cdot\,;\lambda),\psi\rangle &= -(A_\varepsilon - C_\varepsilon + I_\varepsilon) - (K_\varepsilon - D_\varepsilon) - J_\varepsilon\\ &\to 0 + 0 + \psi(0) = \langle\delta,\psi\rangle.\end{aligned}$$

This shows (3.7) and completes the proof. □

3.1.3 Estimates on the fundamental solutions

Now that it is proven that $G(x;\lambda)$ is a fundamental solution, we can make some estimates on this fundamental solution.

Lemma 3.4. *Let $\lambda \in \Sigma_\theta$. Then for all $x \in \mathbb{R}^3 \setminus \{0\}$ there exist $c, C > 0$ such that*

$$\left|\nabla_x^\ell G(x;\lambda)\right| \leq C\frac{e^{-c\sqrt{|\lambda|}|x|}}{|x|^{\ell+1}}, \tag{3.8}$$

for any integer $\ell \geq 0$, where c depends only on θ, and C depends only on ℓ and θ.

Proof. This will be proved using an induction argument. First of all notice that

$$|G(x;\lambda)| = \left|\frac{e^{ik|x|}}{4\pi|x|}\right| \leq \frac{e^{-\operatorname{Im}(k)|x|}}{4\pi|x|} \leq \frac{e^{-\sin(\theta/2)\sqrt{|\lambda|}|x|}}{4\pi|x|},$$

where in the last inequality Remark 3.1 was used. This shows that the result holds for $\ell = 0$. Notice that on $\mathbb{R}^3 \setminus \{0\}$ the fundamental solution obeys the relation $\Delta G(x;\lambda) = \lambda G(x;\lambda)$. Therefore interior estimates of the Poisson equation can be used to estimate the derivatives of $G(x;\lambda)$ in this domain. Let $B(x,r)$ be an open ball and $w(x)$ such that $\Delta w = f$ on $B(x,r)$. Then we know that

$$\left|\nabla^\ell w(x)\right| \leq Cr^{-\ell} \sup_{B(x,r)} |w(x)| + C \max_{0\leq j\leq \ell-1} \sup_{B(x,r)} r^{j-\ell+2}\left|\nabla^j f\right|. \tag{3.9}$$

A proof of this Caccioppoli-type inequality can be found in the Appendix (Lemma C.4). Now suppose that (3.8) is true if $\ell = k$. If $\ell = k+1$ we can now use the inequality (3.9) together with the induction hypothesis to find

$$\begin{aligned}
\left|\nabla^{k+1} G(x;\lambda)\right| &\leq Cr^{-(k+1)} \sup_{B(x,r)} |G(x;\lambda)| \\
&\quad + C \max_{0\leq j\leq k} \sup_{B(x,r)} r^{j-(k+1)+2}\left|\nabla^j \lambda G(x;\lambda)\right| \\
&\overset{(IH)}{\leq} Cr^{-(k+1)} \sup_{B(x,r)} \frac{e^{-c\sqrt{|\lambda|}|x|}}{|x|} \\
&\quad + \max_{0\leq j\leq l} C_j \sup_{B(x,r)} r^{j-(k+1)+2}|\lambda|\frac{e^{-c\sqrt{|\lambda|}|x|}}{|x|^{j+1}}.
\end{aligned}$$

Now choose $r = \frac{1}{2}|x|$. By setting $y = \sqrt{|\lambda|}|x|$ and using the observation that $y^2 e^{-ay}$ is a bounded function when $c > 0$ and $y > 0$, we find

$$\begin{aligned}
C \max_{0\leq j\leq k} \sup_{B(x,\frac{1}{2}|x|)} |x|^{j-(k+1)+2}|\lambda|\frac{e^{-c\sqrt{|\lambda|}|x|}}{|x|^{j+1}} &\leq C \sup_{B(x,\frac{1}{2}|x|)} |x|^{-(k+1)} y^2 e^{-\frac{c}{2}y}\frac{e^{-\frac{c}{2}y}}{|x|} \\
&\leq C \sup_{B(x,\frac{1}{2}|x|)} |x|^{-(k+1)}\frac{e^{-\frac{c}{2}\sqrt{|\lambda|}|x|}}{|x|}.
\end{aligned}$$

We now use this to find

$$\begin{aligned}\left|\nabla^{k+1}G(x;\lambda)\right| \leq C|x|^{-(k+1)} &\sup_{B(x,\frac{1}{2}|x|)} \frac{e^{-c\sqrt{|\lambda|}|x|}}{|x|} \\ &+ C\max_{0\leq j\leq k} \sup_{B(x,\frac{1}{2}|x|)} |x|^{j-(k+1)+2}|\lambda|\frac{e^{-c\sqrt{|\lambda|}|x|}}{|x|^{j+1}} \\ &\leq C|x|^{-(k+1)} \sup_{B(x,\frac{1}{2}|x|)} \frac{e^{-c\sqrt{|\lambda|}|x|}}{|x|} + C \sup_{B(x,\frac{1}{2}|x|)} |x|^{-(k+1)}\frac{e^{-\frac{c}{2}\sqrt{|\lambda|}|x|}}{|x|}.\end{aligned}$$

We now notice that the functions within the suprema are monotonically decreasing as a function of $|x|$. Therefore their supremum will be their function value at $\frac{|x|}{2}\frac{x}{|x|}$. Therefore we can drop the suprema by changing the constants. We now find

$$\left|\nabla^{k+1}G(x;\lambda)\right| \leq C_\ell \frac{e^{-c\sqrt{\lambda}|x|}}{|x|^{(k+1)+1}}.$$

The proof is now completed by induction. □

We have now considered the case when $\lambda \in \Sigma_\theta$. The case $\lambda = 0$ is just the Laplacian. Let $G(x;0) = \frac{1}{4\pi|x|}$ denote a fundamental solution of the Laplace equation in $\mathbb{R}^3$, with pole at the origin (see e.g. [11]). Then we find the following lemma.

Lemma 3.5. *Let $\lambda \in \Sigma_\theta$. Then for all $\ell \geq 1$ and for all $x \in \mathbb{R}^3 \setminus \{0\}$*

$$\left|\nabla_x^\ell\Big[G(x;\lambda) - G(x;0)\Big]\right| \leq C|\lambda||x|^{1-\ell}, \tag{3.10}$$

where C depends only on ℓ and θ.

Proof. First of all consider the case $|\lambda||x|^2 \geq \frac{1}{2}$. In that case we find using Lemma 3.4

$$\begin{aligned}\left|\nabla_x^\ell\Big[G(x;\lambda) - G(x;0)\Big]\right| &\leq \left|\nabla_x^\ell G(x;\lambda)\right| + \left|\nabla_x^\ell G(x;0)\right| \\ &\leq C_\ell\frac{e^{-c\sqrt{|\lambda|}|x|}}{|x|^{\ell+1}} + \frac{1}{|x|^{\ell+1}}.\end{aligned}$$

Because $e^{-c\sqrt{|\lambda|}|x|} \leq 1$ we can drop this factor and because $2|\lambda||x|^2 \geq 1$ we can multiply by $2|\lambda||x|^2$ to find the result. Now consider the case $|\lambda||x|^2 < \frac{1}{2}$. First of all note that

$$e^{ik|x|} = \sum_{n=0}^{\infty} \frac{(ik|x|)^n}{n!}.$$

This series is uniformly convergent when $|\lambda||x|^2 < \frac{1}{2}$ and thus

$$\frac{\partial}{\partial|x|}\left(\frac{e^{ik|x|} - 1 - ik|x|}{|x|}\right) = \sum_{n=2}^{\infty} \frac{(n-1)\,(ik)^n|x|^{n-2}}{n!}.$$

Hence we find by using this expansion that

$$\left|\nabla_x\Big[G(x;\lambda) - G(x;0)\Big]\right| = \left|\nabla_x\left(\frac{e^{ik|x|} - 1 - ik|x|}{|x|}\right)\right| \leq C\left|-k^2\right| = C|\lambda|,$$

which shows the result in the case $\ell = 1$. We can now again use an interior estimate of the Poisson equation (3.9) for $\ell \geq 2$. Hereto let $w(x) = G(x;\lambda) - G(x;0)$ and notice that $\Delta w(x) = \lambda G(x;\lambda)$ in $\mathbb{R}^3 \setminus \{0\}$. This completes the proof. $\square$

Remark 3.6. *By the same argument as in the $\ell = 1$ case of the previous lemma we find that if $|\lambda||x|^2 \leq \frac{1}{2}$,*

$$|G(x;\lambda) - G(x;0)| \leq C\sqrt{|\lambda|}. \tag{3.11}$$

3.2 Fundamental solutions of the Stokes System

Now after finding a fundamental solution of the Helmholtz equation and proving some of its properties we are ready to introduce a fundamental solution for the Stokes System.

3.2.1 Definition of the fundamental matrix

In this section we give the definition of the fundamental matrix when $\lambda \in \Sigma_\theta$ and $x \in \mathbb{R}^3 \setminus \{0\}$. Since the Stokes System is 3-dimensional we need a fundamental solution in each coordinate. This gives rise to a fundamental matrix, where each row is a fundamental solution with respect to a coordinate. Now define the fundamental matrix as

$$\Gamma_{ij}(x;\lambda) = G(x;\lambda)\delta_{ij} - \frac{1}{\lambda}\frac{\partial^2}{\partial x_i \partial x_j}\Big[G(x;\lambda) - G(x;0)\Big]. \tag{3.12}$$

The corresponding fundamental solution for the pressure term is given by

$$\Phi_j(x) = -\frac{\partial}{\partial x_j}\Big[G(x;0)\Big]. \tag{3.13}$$

When $\lambda = 0$, the fundamental matrix is defined by

$$\Gamma_{ij}(x;0) = \frac{1}{2\omega_3}\left[\frac{\delta_{ij}}{|x|} + \frac{x_i x_j}{|x|^3}\right]. \tag{3.14}$$

In this definition, δ_{ij} denotes the Kronecker delta.

3.2.2 Properties of the fundamental matrix

Now that the fundamental matrix is introduced, we check some properties. First of all we check if the fundamental matrix solves the Stokes System away from zero.

Lemma 3.7. *Let $\lambda \in \Sigma_\theta$. Then for all $x \in \mathbb{R}^3 \setminus \{0\}$*

$$\begin{cases} (-\Delta_x + \lambda)\Gamma_{ij}(x;\lambda) + \dfrac{\partial}{\partial x_i}\Big[\Phi_j(x)\Big] = 0 & i,j = 1,2,3 \\ \displaystyle\sum_{i=1}^{3} \frac{\partial}{\partial x_i}\Big[\Gamma_{ij}(x;\lambda)\Big] = 0 & j = 1,2,3. \end{cases} \tag{3.15}$$

Proof. Let $\lambda \in \Sigma_\theta$ and $x \in \mathbb{R}^3 \setminus \{0\}$. Then we know that $\Delta_x G(x;\lambda) = \lambda G(x;\lambda)$. Hence $(-\Delta_x + \lambda)G(x;\lambda) = 0$ and $\Delta_x G(x;0) = 0$. Now using the definition of the fundamental matrix we find that

$$\begin{aligned}(-\Delta_x + \lambda)\Gamma_{ij}(x;\lambda) &= -\frac{1}{\lambda}\frac{\partial^2}{\partial x_i \partial x_j}\Big[-\lambda G(x;0)\Big]\\ &= -\frac{\partial}{\partial x_i}\Big[-\frac{\partial}{\partial x_j}G(x;0)\Big]\\ &= -\frac{\partial}{\partial x_i}\Big[\Phi_j(x)\Big].\end{aligned}$$

This shows the first equality. Now for the second equality we find

$$\begin{aligned}\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\Big[\Gamma_{ij}(x;\lambda)\Big] &= \sum_{i=1}^{3}\frac{\partial}{\partial x_i}G(x;\lambda)\delta_{ij} - \frac{1}{\lambda}\frac{\partial^3}{\partial x_j \partial x_i^2}\Big[G(x;\lambda) - G(x;0)\Big]\\ &= \frac{\partial}{\partial x_j}G(x;\lambda) - \frac{1}{\lambda}\frac{\partial}{\partial x_j}\Big[\Delta_x G(x;\lambda) - \Delta_x G(x;0)\Big]\\ &= \frac{\partial}{\partial x_j}G(x;\lambda) - \frac{\partial}{\partial x_j}G(x;\lambda) = 0.\end{aligned}$$

This completes the proof. □

We want to check if $(\mathbf{\Gamma}_j, \Phi_j)$ is a fundamental solution in the distributional sense. In order to do so, we first make some preliminary estimates.

Lemma 3.8. *Let $\lambda \in \Sigma_\theta$ and $\Gamma_{ij}(x;\lambda)$ with $x \in \mathbb{R}^3 \setminus \{0\}$ be given by* (3.12)*. Let $\varepsilon > 0$. Then*

$$\int_{B(0,\varepsilon)} |\Gamma_{ij}(x;\lambda)|dx \to 0 \tag{3.16}$$

$$\int_{\partial B(0,\varepsilon)} |\Gamma_{ij}(y;\lambda)|d\sigma(y) \to 0, \tag{3.17}$$

when $\varepsilon \to 0$.

Proof. We use Lemma 3.4 and Lemma 3.5 to find

$$\begin{aligned}\int_{B(0,\varepsilon)} |\Gamma_{ij}(x;\lambda)|dx &\le \int_{B(0,\varepsilon)} C\left(\frac{1}{|x|} + \frac{1}{|\lambda|}\frac{|\lambda|}{|x|}\right)dx\\ &\le C\int_0^\varepsilon \int_{\partial B(0,1)} r\,d\sigma(y)dr\\ &\le C\varepsilon^2 \to 0 \quad (\varepsilon \to 0).\end{aligned}$$

For the second integral we find

$$\begin{aligned}\int_{\partial B(0,\varepsilon)} |\Gamma_{ij}(y;\lambda)|d\sigma(y) &\le \int_{\partial B(0,\varepsilon)} C\left(\frac{1}{|y|} + \frac{1}{|\lambda|}\frac{|\lambda|}{|y|}\right)d\sigma(y)\\ &\le C\frac{1}{\varepsilon}4\pi\varepsilon^2 \to 0 \quad (\varepsilon \to 0).\end{aligned}$$

□

Theorem 3.9. $(\mathbf{\Gamma}_j, \Phi_j)$ *is a fundamental solution of the Stokes System* (1.11) *in the distributional sense.*

Proof. We use a similar proving strategy as in the proof of Lemma 3.3. Let $\boldsymbol{\psi} \in C_c^\infty(\mathbb{R}^3;\mathbb{R}^3)$. Now we want to show that

$$\left\langle (-\Delta_x + \lambda)\Gamma_{ij} + \frac{\partial \Phi_j}{\partial x_i}, \psi_i \right\rangle = \delta_{ij}\psi_i(0) = \langle \delta_{ij}\delta(x), \psi_i \rangle . \tag{3.18}$$

Now using the definition of the distributional derivative together with the linearity of distributions we find

$$\begin{aligned}\left\langle (-\Delta_x + \lambda)\Gamma_{ij} + \frac{\partial \Phi_j}{\partial x_i}, \psi \right\rangle &= -\langle \Gamma_{ij}, \Delta_x \psi \rangle + \left\langle \lambda\Gamma_{ij} + \frac{\partial \Phi_j}{\partial x_i}, \psi \right\rangle \\ &= -\int_{\mathbb{R}^3} \Gamma_{ij}(x;\lambda)\Delta_x\psi(x)dx \\ &\quad + \int_{\mathbb{R}^3} \left(\lambda\Gamma_{ij}(x;\lambda) + \frac{\partial \Phi_j(x)}{\partial x_i} \right) \psi(x)dx.\end{aligned}$$

Because of the singularity at the origin, we will consider these integrals in two regions. The region $B(0,\varepsilon)$ and the region $\mathbb{R}^3 \setminus B(0,\varepsilon)$. Using partial integration we find

$$\begin{aligned}\int_{\mathbb{R}^3\setminus B(0,\varepsilon)} \Gamma_{ij}(x;\lambda)\Delta_x\psi(x)dx &= \int_{\partial B(0,\varepsilon)} \frac{\partial \psi_i(p)}{\partial \nu}\Gamma_{ij}(p;\lambda)d\sigma(p) \\ &\quad + \int_{\partial B(0,\varepsilon)} \frac{\partial \Gamma_{ij}(p;\lambda)}{\partial \nu}\psi_i(p)d\sigma(p) \\ &\quad + \int_{\mathbb{R}^d\setminus B(0,\varepsilon)} \Delta_x\Gamma_{ij}(x;\lambda)\psi_i(x)dx.\end{aligned}$$

By Lemma 3.7 we know that

$$\int_{\mathbb{R}^d\setminus B(0,\varepsilon)} \left(-\Delta_x\Gamma_{ij}(x;\lambda) + \lambda\Gamma_{ij}(x;\lambda) + \frac{\partial \Phi_j(x)}{\partial x_i} \right) \psi(x)dx = 0.$$

Hence we find that

$$\left\langle (-\Delta_x + \lambda)\Gamma_{ij} + \frac{\partial \Phi_j}{\partial x_i}, \psi \right\rangle = \delta_{ij}\psi(0) = \langle \delta_{ij}\delta(x), \psi \rangle .$$

This completes the proof. ∎

We also want to check if the pressure part of the fundamental solution is a harmonic function.

Lemma 3.10. *Let $x \in \mathbb{R}^3 \setminus \{0\}$ and $\Phi_i(x)$ be given by* (3.13)*. Then*

$$\Delta_x\Phi_i(x) = 0.$$

Proof. We start by calculating the second derivative of $\Phi_i(x)$,

$$\begin{aligned}\frac{\partial^2}{\partial x_j^2}\Phi_i(x) &= \frac{\partial}{\partial x_j}\left(\frac{1}{\omega_d|x|^3}\delta_{ij} - 3\frac{x_ix_j}{\omega_d|x|^{3+2}} \right) \\ &= -3\frac{x_i}{\omega_3|x|^{3+2}}\delta_{ij} - 3\frac{x_i}{\omega_3|x|^{3+2}}\delta_{ij} + 3(3+2)\frac{x_ix_j^2}{\omega_3|x|^{3+4}} - 3\frac{x_i}{\omega_3|x|^{3+2}}.\end{aligned}$$

Now we sum all the derivatives to find the Laplacian

$$
\begin{aligned}
\Delta_x \Phi_i(x) &= \sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2} \Phi_i(x) \\
&= -3 \frac{x_i}{\omega_3 |x|^{3+2}} - 3 \frac{x_i}{\omega_3 |x|^{3+2}} + (3^2+6) \frac{x_i |x|^2}{\omega_3 |x|^{3+4}} - 3^2 \frac{x_i}{\omega_3 |x|^{3+2}} \\
&= 0.
\end{aligned}
$$

□

3.2.3 Estimates on the fundamental matrix

In the previous section we have established the important properties of the fundamental matrix. Now we do estimates on the fundamental matrix that will be used throughout the thesis.

Theorem 3.11. *Let $\lambda \in \Sigma_\theta$. Then for any $\ell \geq 0$ and for all $x \in \mathbb{R}^3 \setminus \{0\}$*

$$
\left|\nabla_x^\ell \Gamma(x;\lambda)\right| \leq \frac{C}{(1+|\lambda||x|^2)|x|^{1+\ell}}, \tag{3.19}
$$

where C depends only on ℓ and θ.

Proof. We consider the cases $|\lambda||x|^2 > 1$ and $|\lambda||x|^2 \leq 1$. First let $|\lambda||x|^2 > 1$. By Lemma 3.4 we find that

$$
\begin{aligned}
\left|\nabla_x^\ell \Gamma(x;\lambda)\right| &\leq \left|\nabla_x^\ell G(x;\lambda)\right| + \frac{1}{|\lambda|}\left|\nabla^{\ell+2} G(x;\lambda)\right| + \frac{1}{|\lambda|}\left|\nabla^{\ell+2} G(x;0)\right| \\
&\leq C \frac{e^{-c\sqrt{|\lambda||x|}}}{|x|^{1+\ell}} + C \frac{e^{-c\sqrt{|\lambda||x|}} + 1}{|\lambda||x|^{3+\ell}} \\
&= C \frac{e^{-c\sqrt{|\lambda||x|}} + \frac{1}{|\lambda||x|^2} e^{-c\sqrt{|\lambda||x|}} + \frac{1}{|\lambda||x|^2}}{|x|^{3+\ell}} \\
&\leq C \frac{e^{-c\sqrt{|\lambda||x|}} + e^{-c\sqrt{|\lambda||x|}} + \frac{1}{|\lambda||x|^2}}{|x|^{1+\ell}} \\
&\leq C \frac{1 + \frac{1}{|\lambda||x|^2}}{|x|^{1+\ell}} \\
&\leq \frac{C}{(1+|\lambda||x|^2)|x|^{1+\ell}}.
\end{aligned}
$$

Now let $|\lambda||x|^2 \leq 1$. Using Lemma 3.4 and Lemma 3.5 we find

$$\begin{aligned}
\left|\nabla_x^\ell \Gamma(x;\lambda)\right| &\leq \left|\nabla_x^\ell G(x;\lambda)\right| + \frac{1}{|\lambda|}\left|\nabla^{\ell+2}\left(G(x;\lambda) - G(x;0)\right)\right| \\
&\leq C\frac{e^{-c\sqrt{|\lambda|}|x|}}{|x|^{1+\ell}} + C\frac{1}{|\lambda|}|\lambda||x|^{1-\ell-2} \\
&= C\frac{e^{-c\sqrt{|\lambda|}|x|}+1}{|x|^{1+\ell}} \\
&\leq \frac{C}{(1+|\lambda||x|^2)|x|^{1+\ell}}.
\end{aligned}$$

Where in the last step we used that $1 \leq (1+|\lambda||x|^2) \leq 2$. This completes the proof. □

Theorem 3.12. *Let $\lambda \in \Sigma_\theta$. Suppose that $|\lambda||x|^2 \leq \frac{1}{2}$. Then for all $x \in \mathbb{R}^3 \setminus \{0\}$*

$$\left|\nabla_x\Big[\Gamma(x;\lambda) - \Gamma(x;0)\Big]\right| \leq C\sqrt{|\lambda|}|x|^{-1}, \tag{3.20}$$

where $C > 0$ depends only on θ.

Proof. We know that

$$\begin{aligned}
\Gamma_{\alpha\beta}(x;\lambda) - \Gamma_{\alpha\beta}(x;0) &= \Big[G(x;\lambda) - G(x;0)\Big]\delta_{\alpha\beta} \\
&\quad - \frac{1}{\lambda}\frac{\partial^2}{\partial x_\alpha \partial x_\beta}\left(G(x;\lambda) - G(x;0) - \frac{\lambda|x|}{2\omega_3}\right).
\end{aligned}$$

By Lemma 3.5 we know that

$$|\nabla_x\{G(x;\lambda) - G(x;0)\}| \leq C|\lambda| \leq C\sqrt{|\lambda|}|x|^{-1}. \tag{3.21}$$

Where in the last inequality we used that $|\lambda||x|^2 \leq \frac{1}{2}$. For the second part we notice that

$$e^{ik|x|} = \sum_{n=0}^\infty \frac{(ik|x|)^n}{n!}.$$

Now we use the observation that

$$\begin{aligned}
&\frac{1}{4\pi}\frac{\partial^3}{\partial|x|^3}\left(\frac{e^{ik|x|} - 1 - ik|x| + \frac{k^2|x|^2}{2} - \frac{ik^3|x|^3}{6}}{4\pi|x|}\right) \\
&\qquad = \frac{1}{4\pi}\sum_{n=3}^\infty \frac{(n-1)(n-2)(n-3)}{n!}(ik)^n|x|^{n-4}.
\end{aligned}$$

Hence we can use the series expansion to find that

$$\begin{aligned}
&\left|\nabla_x^3\left(G(x;\lambda) - G(x;0) - \frac{\lambda|x|}{2\omega_3}\right)\right| \\
&\qquad = \left|\nabla_x^3\left(e^{ik|x|} - 1 - ik|x| + \frac{k^2|x|^2}{2} - \frac{ik^3|x|^3}{6}\right)\right| \leq C|\lambda|^2.
\end{aligned}$$

And thus we find

$$\left|\nabla_x \left(\frac{1}{\lambda}\frac{\partial^2}{\partial x_\alpha \partial x_\beta}\left(G(x;\lambda) - G(x;0) - \frac{\lambda|x|}{2\omega_3}\right)\right)\right| \le C|\lambda| \le C\sqrt{|\lambda|}|x|^{-1}.$$

Combining this with (3.21) completes the proof. □

4 Layer potentials for the Stokes System

In this section we study the properties of the single and double layer potential for the Stokes System. In this section we will assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^3$ and $1 < p < \infty$. The proofs in this section rely on techniques from harmonic analysis ([22], [23] for an introduction). Important results from harmonic analysis are also contained in the appendix A. A good introduction to the layer potentials for the Stokes operator can be found in the book of Ladyzhenskaya [29]. The Stokes System was also studied with layer potentials in [31], where the boundary of the domain is the halfspace and in [37], where the boundary of the domain is of class C^2. For convenience we also introduce the Einstein summation convention. This convention means that if an index is on the right-hand side but not on the left-hand side of the equation, it ought to be summed over the dimension.

4.1 The single layer potential

First we study the single layer potential. We start off by giving the definition. Then we show that the single layer potential solves the Stokes System in the domain Ω.

4.1.1 Definition of the single layer potential

Definition 4.1 (Single layer potential)**.** *Let Ω be a bounded and connected Lipschitz domain in $\mathbb{R}^3$. If $\boldsymbol{f} \in L^p(\partial\Omega; \mathbb{C}^3)$ and $\lambda \in \Sigma_\theta$, then the single layer potential with density $\boldsymbol{f}$ is defined by*

$$\mathcal{S}_\lambda(\boldsymbol{f})(x) = (\boldsymbol{u}, \phi) = \begin{cases} \boldsymbol{u}_i(x) = \displaystyle\int_{\partial\Omega} \Gamma_{ik}(x-y;\lambda)\boldsymbol{f}_k(y)d\sigma(y) \\ \phi(x) = \displaystyle\int_{\partial\Omega} \Phi_k(x-y)\boldsymbol{f}_k(y)d\sigma(y). \end{cases} \tag{4.1}$$

We now proceed to show that this definition of the single layer potential yields a class of functions that solves the Stokes System (1.11) in Ω. However, one should keep in mind that the single layer potential does not always satisfy the boundary conditions on $\partial\Omega$.

Lemma 4.2. *Let Ω be a bounded and connected Lipschitz domain in $\mathbb{R}^3$. Let $\lambda \in \Sigma_\theta$ and $\boldsymbol{f} \in L^p(\partial\Omega; \mathbb{C}^3)$. Then the single layer potential is a solution of the Stokes System* (1.11) *in Ω.*

Proof. The idea of the proof is to use Lemma 3.7 in combination with the dominated convergence theorem (Theorem B.5). Now let $x \in \Omega$ and let $\boldsymbol{f} \in L^p(\partial\Omega; \mathbb{C}^3)$. Now let q such that $\frac{1}{p} + \frac{1}{q} = 1$. By Theorem 3.11 we see that the function $y \mapsto \nabla_x^\ell \Gamma_{ik}(x-y;\lambda) \in L^q(\partial\Omega; \mathbb{C})$ for all $\ell \geq 0$. It is also not hard to see that the function $y \mapsto \nabla\Phi_k(x-y) \in L^q(\partial\Omega;\mathbb{C})$. Now we let $(\boldsymbol{u}, \phi) = \mathcal{S}_\lambda(\boldsymbol{f})$. Since Ω is open we can find a radius r such that $B(x,r) \subset \Omega$. Now take a sequence $h_n \to 0$ with $|h_n| < r$. Now consider

$$\frac{\boldsymbol{u}_i(x + h_n e_j) - \boldsymbol{u}_i(x)}{h_n} = \int_{\partial\Omega} \frac{\Gamma_{ik}(x + h_n e_j - y;\lambda) - \Gamma_{ik}(x-y;\lambda)}{h_n} \boldsymbol{f}_k(y)d\sigma(y).$$

By the mean value theorem there exists a $\xi_n \in B(x,r)$ such that

$$\left| \frac{\Gamma_{ik}(x + h_n e_j - y;\lambda) - \Gamma_{ik}(x-y;\lambda)}{h_n} \right| \leq \left| \frac{\partial\Gamma_{ik}(\xi_n - y;\lambda)}{\partial x_j} \right|.$$

Now by the Hölder inequality (Theorem B.2) we find that

$$\int_{\partial\Omega} \left| \frac{\partial\Gamma_{ik}(\xi_n - y;\lambda)}{\partial x_j} \right| |\boldsymbol{f}_k(y)| d\sigma(y) \leq \|\nabla_x \Gamma_{ik}(x - \cdot\,;\lambda)\|_{L^q(\partial\Omega;\mathbb{C})} \|\boldsymbol{f}_k\|_{L^p(\partial\Omega;\mathbb{C})} < \infty.$$

Now we can invoke the dominated convergence theorem to find that

$$\begin{aligned}
\frac{\partial \boldsymbol{u}_i}{\partial x_j}(x) &= \lim_{n\to\infty} \frac{\boldsymbol{u}_i(x+h_n e_j) - \boldsymbol{u}_i(x)}{h_n} \\
&= \lim_{n\to\infty} \int_{\partial\Omega} \frac{\Gamma_{ik}(x+h_n e_j - y;\lambda) - \Gamma_{ik}(x-y;\lambda)}{h_n} \boldsymbol{f}_k(y) d\sigma(y) \\
&= \int_{\partial\Omega} \lim_{n\to\infty} \frac{\Gamma_{ik}(x+h_n e_j - y;\lambda) - \Gamma_{ik}(x-y;\lambda)}{h_n} \boldsymbol{f}_k(y) d\sigma(y) \\
&= \int_{\partial\Omega} \frac{\partial \Gamma_{ik}}{\partial x_j}(x-y;\lambda) \boldsymbol{f}_k(y) d\sigma(y).
\end{aligned}$$

Similar calculations can be done for $\nabla\phi(x)$ and $\Delta_x \boldsymbol{u}_i(x)$. Using this we find

$$\begin{aligned}
-\Delta_x \boldsymbol{u}_i(x) + \frac{\partial}{\partial x_i}\phi(x) + \lambda \boldsymbol{u}_i &= -\Delta_x \int_{\partial\Omega} \Gamma_{ik}(x-y;\lambda) \boldsymbol{f}_k(y) d\sigma(y) + \frac{\partial}{\partial x_i} \int_{\partial\Omega} \Phi_k(x-y) \boldsymbol{f}_k(y) d\sigma(y) \\
&\quad + \lambda \int_{\partial\Omega} \Gamma_{ik}(x-y;\lambda) \boldsymbol{f}_k(y) d\sigma(y) \\
&= \int_{\partial\Omega} \left[-\Delta_x \Gamma_{ik}(x-y;\lambda) + \frac{\partial}{\partial x_i}\Phi_k(x-y) + \lambda \Gamma_{ik}(x-y;\lambda) \right] \boldsymbol{f}_k(y) d\sigma(y) \\
&= 0.
\end{aligned}$$

And similarly we find

$$\begin{aligned}
\operatorname{div}(\boldsymbol{u}) &= \frac{\partial \boldsymbol{u}_i}{\partial x_i} \\
&= \frac{\partial}{\partial x_i} \int_{\partial\Omega} \Gamma_{ik}(x-y;\lambda) \boldsymbol{f}_k(y) d\sigma(y) \\
&= \int_{\partial\Omega} \frac{\partial}{\partial x_i} \Gamma_{ik}(x-y;\lambda) \boldsymbol{f}_k(y) d\sigma(y) \\
&= 0.
\end{aligned}$$

This completes the proof. □

Also note that the pressure part of the single layer potential is still a harmonic function.

Lemma 4.3. *Let Ω be a bounded and connected Lipschitz domain in $\mathbb{R}^3$. Let $\lambda \in \Sigma_\theta$ and $\boldsymbol{f} \in L^p(\partial\Omega;\mathbb{C}^3)$. Let $(\boldsymbol{u},\phi) = \mathcal{S}_\lambda(\boldsymbol{f})$. Then $\Delta_x\phi(x) = 0$ in Ω.*

Proof. By Lemma 3.10 and the same argument as in the proof of Lemma 4.2 we find that $\Delta_x\phi(x) = 0$ in Ω. □

4.2 The double layer potential

4.2.1 Definition of the stress tensor

The internal mechanical stresses in a continuous medium can be modelled using a total stress tensor. This is a combination of forces in the medium and surface forces. In [33], S. Monniaux considers the following class of stress tensors

$$T_{ij}(\boldsymbol{u},\phi)(x) = -\delta_{ij}\phi(x) + \xi \frac{\partial \boldsymbol{u}_i}{\partial x_j}(x) + \frac{\partial \boldsymbol{u}_j}{\partial x_i}(x), \tag{4.2}$$

where $\xi \in (-1, 1]$. In this section we only consider the case $\xi = 0$, similar to Shen [34] and Fabes, Kenig and Verchota [12]. We do this because this yields good Rellich estimates in the next chapter. In order to avoid confusion we introduce a new symbol to denote the stress tensor with $\xi = 0$.

$$S_{ij}(\boldsymbol{u}, \phi)(x) = -\delta_{ij}\phi(x) + \frac{\partial \boldsymbol{u}_i}{\partial x_j}(x). \tag{4.3}$$

What we are interested in is the divergence of this tensor when acting on a vector. Hence we calculate and find that

$$\frac{\partial}{\partial x_j}\Big[S_{ij}(\boldsymbol{u}, \phi)(x)\boldsymbol{w}_i(x)\Big] = -\frac{\partial \phi}{\partial x_i}\boldsymbol{w}_i - \phi\frac{\partial \boldsymbol{w}_i}{\partial x_i} + \boldsymbol{w}_i\Delta\boldsymbol{u}_i + \frac{\partial \boldsymbol{u}_i}{\partial x_j}\frac{\partial \boldsymbol{w}_i}{\partial x_j}.$$

Now if we take a divergence free vector for $\boldsymbol{w}$ we find

$$\frac{\partial}{\partial x_j}\Big[S_{ij}(\boldsymbol{u}, \phi)(x)\boldsymbol{w}_i(x)\Big] = \left(\Delta\boldsymbol{u}_i - \frac{\partial \phi}{\partial x_i}\right)\boldsymbol{w}_i + \frac{\partial \boldsymbol{u}_i}{\partial x_j}\frac{\partial \boldsymbol{w}_i}{\partial x_j}.$$

This observation leads to an important theorem. First define

$$S'_{ij}(\boldsymbol{u}, \phi) = \delta_{ij}\phi + \frac{\partial \boldsymbol{u}_i}{\partial x_j}. \tag{4.4}$$

We conclude this section with the definition of the conormal derivative.

Definition 4.4 (conormal derivative)**.** *Let Ω be a Lipschitz domain in $\mathbb{R}^3$ and $\boldsymbol{u}$ and ϕ be sufficiently nice functions on Ω. Then the conormal derivative is defined as*

$$\frac{\partial \boldsymbol{u}}{\partial \nu} = \boldsymbol{S}_j(\boldsymbol{u}, \phi)\boldsymbol{n}_j = \frac{\partial \boldsymbol{u}}{\partial n} - \phi\boldsymbol{n}. \tag{4.5}$$

4.2.2 Greens type identity for the stress tensor

In this section we prove a Greens type identity for the stress tensor. This helps rewriting the Stokes System inside Ω to an integral involving the conormal derivative on $\partial\Omega$.

Theorem 4.5. *Let Ω be a Lipschitz domain in $\mathbb{R}^3$. Let $\boldsymbol{u}$ and $\boldsymbol{w}$ be sufficiently differentiable and divergence free and let ϕ and ρ be sufficiently differentiable. Then,*

$$\int_\Omega \left[\left(\Delta\boldsymbol{u}_i - \frac{\partial \phi}{\partial x_i} - \lambda\boldsymbol{u}_i\right)\boldsymbol{w}_i - \boldsymbol{u}_i\left(\Delta\boldsymbol{w}_i + \frac{\partial \rho}{\partial x_i} - \lambda\boldsymbol{w}_i\right)\right] dx$$
$$= \int_{\partial\Omega}\Big[S_{ij}(\boldsymbol{w}, \rho)\boldsymbol{u}_i\boldsymbol{n}_j - S'_{ij}(\boldsymbol{u}, \phi)\boldsymbol{w}_i\boldsymbol{n}_j\Big]\, d\sigma. \tag{4.6}$$

Proof. First of all note that

$$\begin{aligned}\int_\Omega &\left[\left(\Delta\boldsymbol{u}_i - \frac{\partial \boldsymbol{u}_i}{\partial x_i} - \lambda\boldsymbol{u}_i\right)\boldsymbol{w}_i - \boldsymbol{u}_i\left(\Delta\boldsymbol{w}_i + \frac{\partial \boldsymbol{w}_i}{\partial x_i} - \lambda\boldsymbol{w}_i\right)\right] dx \\ &= \int_\Omega \left[\left(\Delta\boldsymbol{u}_i - \frac{\partial \boldsymbol{u}_i}{\partial x_i}\right)\boldsymbol{w}_i - \boldsymbol{u}_i\left(\Delta\boldsymbol{w}_i + \frac{\partial \boldsymbol{w}_i}{\partial x_i}\right)\right] dx \\ &= \int_\Omega \operatorname{div}[\boldsymbol{S}_i(\boldsymbol{u}, \phi)]\boldsymbol{u}_i - \operatorname{div}[\boldsymbol{S}'_i(\boldsymbol{u}, \phi)]\boldsymbol{w}_i dx.\end{aligned}$$

Now the result follows directly from the divergence theorem and Lemma 2.11. □

It is natural to call this theorem the Green's identity of the Stokes System. Using the Green's identity it is possible to express the solution to the Stokes System in terms of surface integrals.

Theorem 4.6. *Let Ω be a Lipschitz domain in $\mathbb{R}^3$ and $\lambda \in \Sigma_\theta$. Let $(\mathbf{\Gamma}_k, \Phi_k)$ denote a fundamental solution of the Stokes System* (1.11) *and suppose that $(\boldsymbol{u}, \phi)$ satisfies the Stokes System. Then,*

$$\int_{\partial\Omega} S_{ij}^{'}(\mathbf{\Gamma}_k, \Phi_k)(x-y;\lambda)\boldsymbol{u}_i(y)\boldsymbol{n}_j - S_{ij}(\boldsymbol{u},\phi)(y)\Gamma_{ik}(x-y;\lambda)\boldsymbol{n}_j d\sigma(y) = \begin{cases} \boldsymbol{u}_i(x) & x \in \Omega \\ 0 & x \in \mathbb{R}^3 \setminus \overline{\Omega}. \end{cases} \tag{4.7}$$

Proof. Let $(\mathbf{\Gamma}_k, \Phi_k)$ be a fundamental solution and $(\boldsymbol{u}, \phi)$ be a solution of the homogeneous Stokes System. Now substitute this on the left-hand side of the Green's identity from Theorem 4.5. This yields

$$\begin{aligned} LHS &= \int_\Omega \left[\left(\Delta \boldsymbol{u}_i - \frac{\partial \phi}{\partial x_i} - \lambda \boldsymbol{u}_i \right)(y)\Gamma_{ik}(x-y;\lambda) - \boldsymbol{u}_i(y) \left(\Delta\Gamma_{ik} + \frac{\partial \Phi_i}{\partial x_i} - \lambda\Gamma_{ik} \right)(x-y;\lambda) \right] dy \\ &= \int_\Omega \boldsymbol{u}_i(y)\delta_{ik}\delta(x-y)dy \\ &= \boldsymbol{u}_i(x), \end{aligned}$$

where δ_{ik} denotes the Kronecker delta, and $\delta(x-y)$ denotes the delta function. This completes the proof. □

We want to obtain a similar expression for the pressure term ϕ. Therefore we first establish the following lemma.

Lemma 4.7. *Let $\lambda \in \Sigma_\theta$ and $(\mathbf{\Gamma}_k, \Phi_k)$ denote a fundamental solution of the Stokes System. Then if $x \neq y$,*

$$-\lambda S_{ij}^{'}(\mathbf{\Gamma}_k, \Phi_k)(x-y) + \Delta_x S_{ij}^{'}(\mathbf{\Gamma}_k, \Phi_k)(x-y) = -\frac{\partial \Phi_k}{\partial x_i \partial x_j}(x-y) - \lambda\delta_{ij}\Phi_k(x-y). \tag{4.8}$$

Proof. First of all notice that the differentiation in S is done with respect to y. We can however change to differentiation with respect to x at the cost of a minus sign. We now start by manipulating the Laplacian term.

$$\begin{aligned} \Delta_x S_{ij}^{'}(\mathbf{\Gamma}_k, \Phi_k)(x-y) &= \delta_{ij}\Delta_x\Phi_k(x-y) + \frac{\partial}{\partial y_i}\Delta_x\Gamma_{jk}(x-y) \\ &= \delta_{ij}\Delta_x\Phi_k(x-y) - \frac{\partial}{\partial x_i}\Delta_x\Gamma_{jk}(x-y) \\ &= \delta_{ij}\Delta_x\Phi_k(x-y) - \frac{\partial}{\partial x_i}\left(\lambda\Gamma_{jk}(x-y) + \frac{\partial}{\partial x_j}\Phi_k(x-y) \right) \\ &= -\lambda\frac{\partial}{\partial x_i}\Gamma_{jk}(x-y) - \frac{\partial^2}{\partial x_i \partial x_j}\Phi_k(x-y). \end{aligned}$$

Since $\Phi_k(x-y)$ is a harmonic function. It's Laplacian vanishes. Furthermore we can calculate,

$$\lambda S_{ij}^{'}(\mathbf{\Gamma}_k, \Phi_k)(x-y) = \lambda\delta_{ij}\Phi_k(x-y) + \lambda\frac{\partial}{\partial x_i}\Gamma_{jk}(x-y). \tag{4.9}$$

Now combining the two results completes the proof. □

We are now ready to represent the pressure term of the solution in terms of surface integrals.

Theorem 4.8. *Let Ω be a Lipschitz domain in $\mathbb{R}^3$ and $\lambda \in \Sigma_\theta$. Let $(\boldsymbol{u}, \phi)$ be such that it solves the homogeneous Stokes System. Then for all $x \in \Omega$,*

$$\phi(x) = \int_{\partial\Omega} \frac{\partial}{\partial x_j}\Phi_k(x-y)\boldsymbol{u}_i(y)\boldsymbol{n}_j(y)d\sigma(y) + \int_{\partial\Omega} S_{ij}(\boldsymbol{u},\phi)(y)\Phi_k(x-y)\boldsymbol{n}_j(y)d\sigma(y) - \frac{\lambda}{4\pi}\int_{\partial\Omega}\frac{1}{|x-y|}\boldsymbol{u}_i(y)\boldsymbol{n}_j(y)d\sigma(y) + C, \quad (4.10)$$

where $(\boldsymbol{\Gamma}_k, \Phi_k)$ denotes a fundamental solution of the Stokes System and C is a constant.

Proof. Let $(\boldsymbol{u}, \phi)$ be a solution of the Stokes System. By Theorem 4.6 we can write for all $x \in \Omega$

$$\begin{aligned}\frac{\partial\phi}{\partial x_i} &= \Delta\boldsymbol{u}_i(x) - \lambda\boldsymbol{u}_i(x)\\ &= \int_{\partial\Omega}\Delta_x S'_{ij}(\boldsymbol{\Gamma}_k,\Phi_k)(x-y)\boldsymbol{u}_i(y)\boldsymbol{n}_j - S_{ij}(\boldsymbol{u},\phi)(y)\Delta_x\Gamma_{ik}(x-y;\lambda)\boldsymbol{n}_j d\sigma(y)\\ &\quad - \lambda\left[\int_{\partial\Omega} S'_{ij}(\boldsymbol{\Gamma}_k,\Phi_k)(x-y)\boldsymbol{u}_i(y)\boldsymbol{n}_j - S_{ij}(\boldsymbol{u},\phi)(y)\Gamma_{ik}(x-y;\lambda)\boldsymbol{n}_j d\sigma(y)\right].\end{aligned}$$

We can now use Lemma 4.7 and Lemma 3.7 to find

$$\begin{aligned}\frac{\partial\phi}{\partial x_i} &= -\int_{\partial\Omega} S_{ij}(\boldsymbol{u},\phi)(y)\frac{\partial\Phi_k}{\partial x_i}(x-y)d\sigma(y)\\ &\quad - \int_{\partial\Omega}\left[\frac{\partial\Phi_k}{\partial x_i\partial x_j}(x-y) + \lambda\delta_{ij}\Phi_k(x-y)\right]\boldsymbol{u}_i(y)d\sigma(y).\end{aligned} \quad (4.11)$$

Now let $r > 0$ be such that $B(x,r) \subset \Omega$. pick $x_0 \in B(x,r)$ and let γ be a path from x_0 to x in $B(x,r)$. By the gradient theorem we find

$$\phi(x) - \phi(x_0) = \int_\gamma \nabla\phi(t)\cdot dt.$$

After integrating the right-hand side of (4.11) and grouping all constant terms in one constant C, the proof is completed. □

4.2.3 Definition of the double layer potential

Theorem 4.6 and Theorem 4.8 suggest that it is convenient to introduce the double layer potential.

Definition 4.9 (Double layer potential)**.** *Let Ω be a bounded and connected Lipschitz domain in $\mathbb{R}^3$. If $\boldsymbol{g} \in L^p(\partial\Omega;\mathbb{C}^3)$ and $\lambda \in \Sigma_\theta$. Then the double layer potential with density $\boldsymbol{g}$ is defined by*

$$\mathcal{D}_\lambda(\boldsymbol{g})(x) = (\boldsymbol{w},\rho) = \begin{cases}\boldsymbol{w}_i(x) = -\int_{\partial\Omega} S'_{ij}(\boldsymbol{\Gamma}_k,\Phi_k)(x-y;\lambda)\boldsymbol{g}_k(y)\boldsymbol{n}_j(y)d\sigma(y)\\ \rho(x) = \frac{\partial}{\partial x_j}\int_{\partial\Omega}\Phi_k(x-y)\boldsymbol{n}_j(y)\boldsymbol{g}_k(y)d\sigma(y)\\ \qquad + \frac{\lambda}{4\pi}\int_{\partial\Omega}\frac{1}{|x-y|}\boldsymbol{g}_k(y)\boldsymbol{n}_k(y)d\sigma(y)\end{cases} \quad (4.12)$$

We shall again show that that the double layer potential is a class of functions that solves the Stokes System in Ω.

Lemma 4.10. *Let Ω be a bounded and connected Lipschitz domain in $\mathbb{R}^3$. Let $\lambda \in \Sigma_\theta$ and $\boldsymbol{g} \in L^p(\partial\Omega; \mathbb{C}^3)$. Then the double layer potential is a solution of the Stokes System* (1.11) *in Ω.*

Proof. Let $(\boldsymbol{w}, \rho) = \mathcal{D}_\lambda(\boldsymbol{g})$. For the same reason as in the proof of Lemma 4.2 we can interchange differentiation and integration. Now use Lemma 4.7 to conclude that $-\Delta \boldsymbol{w} + \nabla\rho + \lambda \boldsymbol{w} = 0$ in Ω. To see that $\operatorname{div}(\boldsymbol{u}) = 0$, notice that

$$\sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left(\delta_{ik} \Phi_k(x-y) + \frac{\partial \Gamma_{ik}(x-y;\lambda)}{\partial x_j} \right) = \Delta \left(\frac{1}{4\pi |x-y|} \right) + \frac{\partial}{\partial x_j} \operatorname{div}(\boldsymbol{\Gamma_k}) = 0.$$

□

And again the pressure part of the double layer potential is a harmonic function.

Lemma 4.11. *Let Ω be a bounded and connected Lipschitz domain in $\mathbb{R}^3$. Let $\lambda \in \Sigma_\theta$ and $\boldsymbol{g} \in L^p(\partial\Omega; \mathbb{C}^3)$. Let $(\boldsymbol{w}, \rho) = \mathcal{D}_\lambda(\boldsymbol{g})$. Then $\Delta\rho = 0$ in Ω.*

Proof. Notice that $\Delta\Phi_k = 0$ by Lemma 3.10 and a short calculation shows that $\Delta\left(\frac{1}{|x-y|}\right) = 0$, when $x \neq y$. □

It is not yet clear why we need both the single and double layer potential. We have seen that the single layer potential as well as the double layer potential solve the Stokes System in Ω. However they fulfil different roles in the solution to the boundary value problem for the Stokes System. We shall show in the next sections that the single layer potential is used to solve the Stokes System with Neumann type boundary conditions and the double layer potential is used for Dirichlet type boundary conditions.

4.3 Nontangential maximal functions of layer potentials

In this section we show that the layer potentials defined in the previous section have bounded nontangential maximal functions. This is important, because we want to use smooth approximations of Lipschitz domains. As we know from the section about Lipschitz domains, we need boundedness of the nontangential maximal function in order to use something like the dominated convergence theorem.

4.3.1 Convolutions on Lipschitz domains

We start by stating a well known result that bounds the convolution with a decreasing and radially symmetric kernel.

Lemma 4.12. *Let $\eta : \mathbb{R}^d \to \mathbb{R}$ be a function such that $\eta(x) = \eta(|x|)$ and $\eta(x) \geq \eta(y)$ when $|x| \leq |y|$. Then $\eta_t(x) = \frac{1}{t^d}\eta\left(\frac{x}{t}\right)$ and,*

$$\sup_{t>0} |(f * \eta_t)(x)| \leq \|\eta\|_{L^1(\mathbb{R}^d)} \mathcal{M}(f)(x) \tag{4.13}$$

for all $f \in L^1_{loc}(\mathbb{R}^d)$.

Proof. See for example [22, Theorem 2.1.10] or [24, Proposition 2.3.9]. □

We would like to use a similar lemma to show boundedness of nontangential maximal functions arising from the layer potentials on Lipschitz domains. We cannot directly use this lemma, since f is only defined on $\partial\Omega$. To fix this problem we have to use a parametrization of the boundary. Define $\eta : \mathbb{R}^2 \to \mathbb{R}$ by

$$\eta(y) = \chi_{|y|>1}|y|^{-3}. \tag{4.14}$$

Then the dilation is given by

$$\eta_t(y) = \chi_{|y|>t}\frac{t}{|y|^3}. \tag{4.15}$$

A rather straightforward calculation shows that the L^1 norm of this function equals 2π. However, η is not radially decreasing. Therefore we want to define $\tilde{\eta}$ in such a way that it is a radially decreasing function and that it is a majorant of η. This can be done by defining

$$\tilde{\eta}(y) = \chi_{|y|\leq 1} + \chi_{|y|>1}|y|^{-3}. \tag{4.16}$$

One can show that the L^1 norm again is finite (in fact it equals 3π). Now we are ready to prove the following lemma.

Lemma 4.13. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$ and let $p \in \partial\Omega$. Then there exists $C > 0$ such that*

$$\int_{\substack{y\in\partial\Omega \\ |p-y|>t}} \frac{t}{|p-y|^3} f(y)d\sigma(y) \leq C\mathcal{M}_{\partial\Omega}(f)(p), \tag{4.17}$$

for all $f \in L^1_{loc}(\partial\Omega)$.

Proof. Assume that D is a special Lipschitz domain. Now parametrize the boundary to find

$$\int_{\partial D} \frac{t\chi_{|p-y|>t}(y)}{|p-y|^3} f(y)d\sigma(y) = \int_{\mathbb{R}^2} \frac{t\chi_{|\pi(\tilde{p})-\pi(\tilde{y})|)>t}}{|\pi(\tilde{p})-\pi(\tilde{y})|^3} f(\pi(\tilde{y}))\sqrt{1+|\nabla\varphi(\tilde{y})|^2}d\tilde{y}.$$

We now manipulate the integrand in order to use Lemma 4.12. Notice that

$$|\pi(\tilde{p})-\pi(\tilde{y})|^2 = |\tilde{p}-\tilde{y}|^2 + |\varphi(\tilde{p})-\varphi(\tilde{y})|^2 \leq (1+L)|\tilde{p}-\tilde{y}|^2$$
$$|\pi(\tilde{p})-\pi(\tilde{y})|^2 = |\tilde{p}-\tilde{y}|^2 + |\varphi(\tilde{p})-\varphi(\tilde{y})|^2 \geq |\tilde{p}-\tilde{y}|^2.$$

If we now define

$$\tilde{f}(\tilde{y}) := f(\pi(\tilde{y}))\sqrt{1+|\nabla\varphi(\tilde{y})|^2}.$$

We can easily see that $\tilde{f}$ is locally integrable on $\mathbb{R}^2$ if f is in $L^p(\partial\Omega)$. Also define

$$\eta(x) = \chi_{|x|>1}|x|^{-3}.$$

Then we find that

$$\begin{aligned}\int_{\mathbb{R}^2} \frac{t\chi_{|\pi(\tilde{p})-\pi(\tilde{y})|)>t}}{|\pi(\tilde{p})-\pi(\tilde{y})|^3} f(\pi(\tilde{y}))\sqrt{1+|\nabla\varphi(\tilde{y})|^2}d\tilde{y} &\leq C\int_{\mathbb{R}^2} \frac{t\chi_{|\tilde{p}-\tilde{y}|)>t}}{|\tilde{p}-\tilde{y}|^3}\tilde{f}(\tilde{y})d\tilde{y} \\ &\leq C\int_{\mathbb{R}^2} \tilde{\eta}_t(\tilde{p}-\tilde{y})\tilde{f}(\tilde{y})d\tilde{y} \\ &\leq C\sup_{t>0}\left|(\tilde{\eta}_t * \tilde{f})(\tilde{p})\right| \\ &\leq C\|\tilde{\eta}\|_{L^1(\mathbb{R}^2)}\mathcal{M}(\tilde{f})(\tilde{p}).\end{aligned}$$

On the last line we used Lemma 4.12. We now want to show that there exists a constant such that

$$\mathcal{M}(\tilde{f})(\tilde{p}) \leq C\mathcal{M}_{\partial D}(f)(p).$$

Notice that $\pi(B_2(\tilde{p}, r)) \subset B_3(\pi(\tilde{p}), \sqrt{1+L}r) \cap \partial D$ where L is the Lipschitz constant of φ. To see this we represent points in ∂D by $\pi(\tilde{y})$ and calculate

$$|\pi(\tilde{p}) - \pi(\tilde{y})| = |\tilde{p} - \tilde{y}|^2 + |\varphi(\tilde{p}) - \varphi(\tilde{y})|^2 \leq (1+L)|\tilde{p} - \tilde{y}|^2.$$

and the conclusion follows. Also notice that by Lemma 2.6, the sizes of $\pi(B_2(\tilde{p}, r))$ and $B_3(\pi(\tilde{p}), \sqrt{1+L}r) \cap \partial D$ are comparable. Now

$$\begin{aligned}
\mathcal{M}(\tilde{f})(\tilde{p}) &= \sup_{r>0} \frac{1}{|B_2(\tilde{p}, r)|} \int_{B_2(\tilde{p},r)} \left|\tilde{f}(\tilde{y})\right| d\tilde{y} \\
&\leq \sup_{r>0} \frac{1}{|B_2(\tilde{p}, r)|} \int_{\pi(B_2(\tilde{p},r))} |f(y)| d\sigma(y) \\
&\leq C \frac{1}{\left|B_3(p, \sqrt{1+L}r) \cap \partial D\right|} \int_{B_3(p,\sqrt{1+L}r)\cap\partial\Omega} |f(y)| d\sigma(y) \\
&\leq C\mathcal{M}_{\partial D}(f)(p).
\end{aligned}$$

To summarize, we now showed that for a special Lipschitz domain we have

$$\int_{\partial D} \frac{t\chi_{|p-y|>t}(y)}{|p-y|^3} |f(y)| d\sigma(y) \leq \|\tilde{\eta}\|_{L^1(\mathbb{R}^2)} \mathcal{M}_{\partial\Omega}(f)(p).$$

Now consider a bounded Lipschitz domain Ω in $\mathbb{R}^3$. By compactness there exists a finite collection $(\partial D_j)_{j=1}^n$ of special Lipschitz domains covering $\partial\Omega$ and a partition of unity $(\zeta_j)_{j=1}^n$ subordinate to this covering. Thus we find

$$\begin{aligned}
\int_{\partial\Omega} \frac{t\chi_{|p-y|>t}}{|p-y|^3} f(y) d\sigma(y) &= \sum_{j=1}^n \int_{\partial D_j} \frac{t\chi_{|p-y|>t}}{|p-y|^3} f(y)\zeta_j(y) d\sigma(y) \\
&\leq \sum_{j=1}^n \|\tilde{\eta}\|_{L^1(\mathbb{R}^2)} \mathcal{M}_{\partial D_j}(\zeta_j f)(p) \\
&\leq C \sum_{j=1}^n \|\tilde{\eta}\|_{L^1(\mathbb{R}^2)} \mathcal{M}_{\partial\Omega}(f)(p) \\
&\leq C\mathcal{M}_{\partial\Omega}(f)(p).
\end{aligned}$$

This completes the proof. □

4.3.2 Boundedness of $(\phi)^*$ in $L^p(\partial\Omega)$

We are now able to prove the boundedness of the nontangential maximal functions that arise from the layer potentials. We start by showing this in the case of ϕ, the pressure term. Before we do this, there is also another convolution-type operator that plays an important role. For $\boldsymbol{f}$ locally integrable on $\partial\Omega$ and $p \in \partial\Omega$, define the operators

$$S^t(\boldsymbol{f})(p) := \int_{\substack{y\in\partial\Omega \\ |p-y|>t}} \frac{p_k - y_k}{|p-y|^3} \boldsymbol{f}(y) d\sigma(y), \tag{4.18}$$

and

$$S^*(\boldsymbol{f})(p) := \sup_{t>0} \left| S^t(\boldsymbol{f})(p) \right|. \tag{4.19}$$

We want to use techniques from harmonic analysis, but then we need to integrate over $\mathbb{R}^2$ instead of $\partial\Omega$. This can be overcome by parametrizing the boundary. Assume that Ω is a special Lipschitz domain. Then (4.18) becomes

$$S^t(\boldsymbol{f})(p) = \int_{\mathbb{R}^2} \chi_{|\pi(\tilde{p})-\pi(\tilde{y})|>t}(\pi(\tilde{y})) \frac{\pi(\tilde{p})_k - \pi(\tilde{y})_k}{|\pi(\tilde{p}) - \pi(\tilde{y})|^3} \boldsymbol{f}_k(\pi(\tilde{y})) \sqrt{1 + |\nabla\varphi(\tilde{y})|^2} d\tilde{y}. \tag{4.20}$$

By Rademachers Theorem (Theorem 2.4) we see that $\sqrt{1 + |\nabla\varphi(\tilde{y})|^2}$ is essentially bounded. Hence it is natural to consider the kernel

$$K(x, y) = \frac{(x, \varphi(x)) - (y, \varphi(y))}{|(x, \varphi(x)) - (y, \varphi(y))|^3}. \tag{4.21}$$

It is shown that this is a special kernel, in the sense of Calderón-Zygmund theory.

Lemma 4.14. *There exists a $C > 0$, such that $K(x, y)$ defined by (4.21), is an element of $SK(1, C)$.*

Proof. Notice that by Rademacher's Theorem (Theorem 2.4)

$$|K(x, y)| \leq \frac{C}{|(x, \phi(x)) - (y, \phi(y))|^2} = \frac{1}{|x - y|^2} \frac{C}{\sqrt{1 + \left(\frac{\phi(x)-\phi(y)}{x-y}\right)^2}} \leq \frac{C}{|x - y|^2}.$$

For the gradients of $K(x, y)$ a similar calculation yields that

$$|\nabla_x K_0(x, y)| \leq \frac{C}{|x - y|^3} \quad \text{and} \quad |\nabla_y K_0(x, y)| \leq \frac{C}{|x - y|^3}.$$

Now observe that $|x - x'| \leq \frac{1}{2} \max(|x - y|, |x' - y|)$ implies that $\max(|x - y|, |x' - y|) \leq 2 \min(|x - y|, |x' - y|)$. This follows from the triangle inequality. Now let θ be a point on the line segment between x and x'. Then we find that

$$|\theta - y| + |\theta - x| \geq |x - y|$$
$$|\theta - y| + |\theta - x'| \geq |x' - y|.$$

Adding these inequalities yields together with the previous observations

$$2|\theta - y| \geq -|\theta - x| - |\theta - x'| + |x - y| + |x' - y|$$
$$2|\theta - y| \geq -|x - x'| + 2 \min(|x - y|, |x' - y|)$$
$$2|\theta - y| \geq -\frac{1}{2} \max(|x - y|, |x' - y|) + \max(|x - y|, |x' - y|).$$

And hence we find that

$$|\theta - y| \geq \frac{1}{4} (|x - y| + |x' - y|). \tag{4.22}$$

Now suppose that $|x - x'| \leq \frac{1}{2} \max(|x-y|, |x'-y|)$. We can now use the mean value theorem in combination with inequality 4.22,

$$|K(x,y) - K(x',y)| \leq |\nabla_x K(\theta, y)||x - x'| \leq C \frac{|x-x'|}{|\theta - y|^3} \leq C \frac{|x-x'|}{(|x-y| + |x'-y|)^3}.$$

And a similar calculation yields

$$|K(x,y) - K(x,y')| \leq C \frac{|y-y'|}{(|x-y| + |x-y'|)^3},$$

when $|y - y'| \leq \frac{1}{2} \max(|x-y|, |x-y'|)$. This completes the proof. □

Now, the following Lemma is a consequence of the celebrated Theorem by Coifman, McIntosh and Meyer [5]. They have shown that operators like S are bounded in $L^2(\partial\Omega)$ and hence are Calderón-Zygmund operators. Now, the following Lemma is just a standard result for Calderón-Zygmund operators.

Lemma 4.15. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$ and let $S^*(\boldsymbol{f})$ be defined by (4.19). Then there exists $C > 0$ such that*

$$\|S^*(\boldsymbol{f})\|_{L^p(\partial\Omega)} \leq C\|\boldsymbol{f}\|_{L^p(\partial\Omega)} \tag{4.23}$$

for all $\boldsymbol{f} \in L^p(\partial\Omega; \mathbb{C}^3)$ with $1 < p < \infty$.

Using this important result, we can now show the boundedness of the nontangential maximal function of ϕ by making a decomposition into the case above and a convolution-type integral.

Lemma 4.16. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$. Let $(\boldsymbol{u}, \phi) = \mathcal{S}_\lambda(\boldsymbol{f})$ with $\boldsymbol{f} \in L^p(\partial\Omega; \mathbb{C}^3)$ with $1 < p < \infty$. Then*

$$\|(\phi)^*\|_{L^p(\partial\Omega)} \leq C\|\boldsymbol{f}\|_{L^p(\partial\Omega)}, \tag{4.24}$$

where C depends only on θ, p and the Lipschitz character of Ω.

Proof. Let ϕ be given by the single layer potential. Pick $p \in \partial\Omega$ and define the set

$$E(p) = \{x \in \Omega \,:\, |x - p| \leq C \operatorname{dist}(x, \partial\Omega)\}.$$

Set $t = |x - p|$. By the definition of the non-tangential maximal function

$$\begin{aligned}
(\phi)^*(p) &= \sup_{x \in E(p)} \left| \int_{\partial\Omega} \frac{x_k - y_k}{|x-y|^3} \boldsymbol{f}_k(y) d\sigma(y) \right| \\
&\leq \sup_{x \in E(p)} \left| \int_{\substack{y \in \partial\Omega \\ |p-y| \leq t}} \frac{x_k - y_k}{|x-y|^3} \boldsymbol{f}_k(y) d\sigma(y) \right| + \sup_{x \in E(p)} \left| \int_{\substack{y \in \partial\Omega \\ |p-y| > t}} \frac{x_k - y_k}{|x-y|^3} \boldsymbol{f}_k(y) d\sigma(y) \right| \\
&= A + B.
\end{aligned}$$

We are going to estimate integrals A and B using the estimates from Lemma 2.13

$$\begin{aligned}
A &= \sup_{x\in E(p)} \left| \int_{\substack{y\in\partial\Omega\\|p-y|\le t}} \frac{x_k-y_k}{|x-y|^3}\boldsymbol{f}_k(y)d\sigma(y) \right| \\
&\le \sup_{x\in E(p)} \int_{\substack{y\in\partial\Omega\\|p-y|\le t}} \frac{1}{|p-y|^2}|\boldsymbol{f}_k(y)|d\sigma(y) \\
&\le \sup_{x\in E(p)} \frac{C}{|B_3(p,t)\cap\partial\Omega|} \int_{\substack{y\in\partial\Omega\\|p-y|\le t}} |\boldsymbol{f}_k(y)|d\sigma(y) \\
&\le C\mathcal{M}_{\partial\Omega}(\boldsymbol{f})(p),
\end{aligned}$$

where C only depends on the Lipschitz character of Ω. For part B notice that

$$\left| \frac{x_k-y_k}{|x-y|^3} - \frac{p_k-y_k}{|p-y|^3} \right| \le C\frac{|x-y|}{|p-y|^3} + \frac{|p-y|}{|p-y|^3} \le C\frac{|x-p|}{|p-y|^3}.$$

We now use this to find

$$\begin{aligned}
B &\le \sup_{x\in E(p)} |p-x| \left| \int_{\substack{y\in\partial\Omega\\|p-y|> t}} \frac{1}{|p-y|^3}\boldsymbol{f}_k(y)d\sigma(y) \right| + \sup_{x\in E(p)} \left| \int_{\substack{y\in\partial\Omega\\|p-y|> t}} \frac{p_k-y_k}{|p-y|^3}\boldsymbol{f}_k(y)d\sigma(y) \right| \\
&= \sup_{t>0} \left| \int_{\substack{y\in\partial\Omega\\|p-y|> t}} \frac{t}{|p-y|^3}\boldsymbol{f}_k(y)d\sigma(y) \right| + \sup_{t>0} \left| \int_{\substack{y\in\partial\Omega\\|p-y|> t}} \frac{p_k-y_k}{|p-y|^3}\boldsymbol{f}_k(y)d\sigma(y) \right| \\
&\le C\mathcal{M}(\boldsymbol{f})(p) + S^*(\boldsymbol{f})(p),
\end{aligned}$$

where we used Lemma 4.13 in the last inequality. If we now put it together with the boundedness of the maximal function in L^p (Theorem A.4) and Lemma 4.15 we find

$$\|(\phi)^*\|_{L^p(\partial\Omega)} \le C\|\mathcal{M}_{\partial\Omega}(\boldsymbol{f})\|_{L^p(\partial\Omega)} + \|S^*(\boldsymbol{f})\|_{L^p(\partial\Omega)} \le C\|\boldsymbol{f}\|_{L^p(\partial\Omega)}.$$

This completes the proof. □

4.3.3 Boundedness of $(\nabla \boldsymbol{u})^*$ in $L^p(\partial\Omega)$

Before we can establish the boundedness of the nontangential maximal function of $\nabla\boldsymbol{u}$ we need to do some work. In a similar fashion as the previous section we define

$$T^t_\lambda(\boldsymbol{f})(p) = \int_{\substack{y\in\partial\Omega\\|p-y|>t}} \nabla\boldsymbol{\Gamma}_k(p-y;\lambda)\boldsymbol{f}_k(y)d\sigma(y) \tag{4.25}$$

and

$$T^*_\lambda(\boldsymbol{f})(p) = \sup_{t>0}\left|T^t_\lambda(\boldsymbol{f})(p)\right|. \tag{4.26}$$

Again we would like to show that $T^*_\lambda(\boldsymbol{f})$ is bounded in $L^p(\partial\Omega)$. To use techniques from harmonic analysis, we need to integrate over $\mathbb{R}^2$ ans thus parametrize $\partial\Omega$. Now in the case that Ω is a special Lipschitz domain, we could write (4.25) as

$$T^t_\lambda(f)(p) = \int_{\mathbb{R}^{d-1}} \chi_{|\pi(\tilde{p})-\pi(\tilde{y})|>t}(\pi(\tilde{y}))\nabla_x\boldsymbol{\Gamma}_k((\tilde{p},\varphi(\tilde{p}))-(\tilde{y},\varphi(\tilde{y}));\lambda)\sqrt{1+|\nabla\varphi(\tilde{y})|^2}\boldsymbol{f}_k(\pi(\tilde{y}))d\tilde{y}. \tag{4.27}$$

Since $\sqrt{1+|\nabla\varphi(\cdot)|^2}$ is essentially bounded, this is the motivation to define a kernel given by

$$K_\lambda(x,y) = \nabla_x \mathbf{\Gamma}_k((x,\varphi(x)) - (y,\varphi(y));\lambda). \tag{4.28}$$

In the following we will show that $K_0(x,y)$ is a standard kernel which is associated with a Calderón-Zygmund operator. This will allow for the use of standard results in Calderón-Zygmund theory. First of all, it is showed that $K_0(x,y)$ satisfies the special kernel estimates.

Lemma 4.17. *There exists a $C > 0$, such that $K_0(x,y)$ defined by* (4.28) *is an element of $SK(1,C)$.*

Proof. First of all notice that by Lemma 3.11 we find

$$|K_0(x,y)| = |K(x,y)|$$

and

$$\begin{aligned} |\nabla_x K_0(x,y)| &= |\nabla_x K(x,y)| \\ |\nabla_y K_0(x,y)| &= |\nabla_y K(x,y)|, \end{aligned}$$

with $K(x,y)$ as defined in (4.21). Now the result follows by the proof of Lemma 4.14. □

In order to conclude that T_0 is a Calderón-Zygmund operator, L^2 boundedness of the operator is needed. This is again a consequence of the result of Coifman, McIntosh and Meyer [5], who showed that the Cauchy transform is bounded in L^2. Hence we have the following lemma.

Lemma 4.18. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$ and let $T_0^*(f)$ be defined by* (4.26)*. Then*

$$\|T_0^*(\boldsymbol{f})\|_{L^p(\partial\Omega)} \leq C\|\boldsymbol{f}\|_{L^p(\partial\Omega)}, \tag{4.29}$$

for all $f \in L^p(\partial\Omega)$ with $1 < p < \infty$.

Now the following Theorem extends the result to $\lambda \in \Sigma_\theta$. Hereto it uses the estimates on the fundamental solution derived in Section 3.

Lemma 4.19. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$ and $\lambda \in \Sigma_\theta$. Then for all $\boldsymbol{f} \in L^p(\partial\Omega;\mathbb{C}^3)$ with $1 < p < \infty$ we have*

$$\|T_\lambda^*(\boldsymbol{f})\|_{L^p(\partial\Omega)} \leq C\|\boldsymbol{f}\|_{L^p(\partial\Omega)}, \tag{4.30}$$

where C depends only on θ, p and the Lipschitz character of Ω.

Proof. The case $\lambda = 0$ holds by Lemma 4.18. We want to extend the result to the case $\lambda \in \Sigma_\theta$. Hereto we make estimates on $T_\lambda^*(\boldsymbol{f})$ in order to reduce this problem to the $\lambda = 0$ case. We start by distinguishing two cases. The case $t^2|\lambda| \geq \frac{1}{2}$ and the case $t^2|\lambda| < \frac{1}{2}$. We start with the case

$t^2|\lambda| \geq \frac{1}{2}$. Then by Theorem 3.11

$$\begin{aligned}\left|\int_{\substack{y\in\partial\Omega\\|p-y|>t}} \nabla\boldsymbol{\Gamma}_k(p-y;\lambda)\boldsymbol{f}_k(y)d\sigma(y)\right| &\leq \int_{\substack{y\in\partial\Omega\\|p-y|>t}} |\nabla\boldsymbol{\Gamma}_k(p-y;\lambda)||\boldsymbol{f}_k(y)|d\sigma(y)\\
&\leq C\int_{\substack{y\in\partial\Omega\\|p-y|>t}} \frac{|\boldsymbol{f}_k(y)|}{|\lambda||p-y|^4}d\sigma(y)\\
&\leq C\int_{\substack{y\in\partial\Omega\\|p-y|>t}} \frac{t^2|\boldsymbol{f}_k(y)|}{|p-y|^4}d\sigma(y)\\
&\leq C\int_{\substack{y\in\partial\Omega\\|p-y|>t}} \frac{t}{|p-y|^3}|\boldsymbol{f}_k(y)|d\sigma(y)\\
&\leq C\mathcal{M}_{\partial\Omega}(\boldsymbol{f})(P),\end{aligned}$$

where we used Lemma 4.13 on the last line. Now we consider the case $t^2|\lambda| < \frac{1}{2}$. We now split up the domain of integration in two parts. The first part $t < |p-y| < (2|\lambda|)^{-\frac{1}{2}}$ which we will denote by γ_1 and the second part $|p-y| \geq (2|\lambda|)^{-\frac{1}{2}}$ which we will denote by γ_2. We start by estimating on γ_1. We find

$$\begin{aligned}\left|\int_{\gamma_1} \nabla\boldsymbol{\Gamma}_k(p-y;\lambda)\boldsymbol{f}_k(y)d\sigma(y)\right| &\leq \left|\int_{\gamma_1} \nabla\boldsymbol{\Gamma}_k(p-y;0)\boldsymbol{f}_k(y)d\sigma(y)\right|\\
&+\left|\int_{\gamma_1} \nabla\Big[\boldsymbol{\Gamma}_k(p-y;\lambda)-\boldsymbol{\Gamma}_k(p-y;0)\Big]\boldsymbol{f}_k(y)d\sigma(y)\right|\\
&= A+B.\end{aligned}$$

We now estimate these integrals separately. We notice that $A \leq T_0^*(\boldsymbol{f})(p)$. The second integral is estimated using Theorem 3.12.

$$\begin{aligned}B &\leq \int_{\gamma_1} \left|\nabla\Big[\boldsymbol{\Gamma}_k(p-y;\lambda)-\boldsymbol{\Gamma}_k(p-y;0)\Big]\right||\boldsymbol{f}_k(y)|d\sigma(y)\\
&\leq C\int_{\gamma_1} \frac{\sqrt{|\lambda|}}{|p-y|}|\boldsymbol{f}_k(y)|d\sigma(y)\\
&\leq C\int_{\gamma_1} \frac{|\boldsymbol{f}_k(y)|}{|p-y|^2}d\sigma(y)\\
&\leq C\mathcal{M}_{\partial\Omega}(\boldsymbol{f})(p).\end{aligned}$$

We now estimate the integral on γ_2.

$$\begin{aligned}\left|\int_{\gamma_2} \nabla\boldsymbol{\Gamma}_k(p-y;\lambda)\boldsymbol{f}_k(y)d\sigma(y)\right| &\leq \int_{\gamma_2} |\nabla\boldsymbol{\Gamma}_k(p-y;\lambda)||\boldsymbol{f}_k(y)|d\sigma(y)\\
&\leq \int_{\gamma_2} \frac{|\boldsymbol{f}_k(y)|}{(1+|\lambda||p-y|^2)|p-y|^2}d\sigma(y)\\
&\leq \int_{\gamma_2} \frac{|\boldsymbol{f}_k(y)|}{(1+|\lambda||\lambda|^{-1})|p-y|^2}d\sigma(y)\\
&\leq C\mathcal{M}_{\partial\Omega}(\boldsymbol{f})(p).\end{aligned}$$

Putting these estimates together leads to

$$T_\lambda^*(\boldsymbol{f})(p) \le T_0^*(\boldsymbol{f})(p) + C\mathcal{M}_{\partial\Omega}(\boldsymbol{f})(p). \tag{4.31}$$

Now from the boundedness of the Hardy-Littlewood maximal function (Theorem A.4) and Lemma 4.18 the result follows. □

We can now employ a similar strategy to prove the boundedness of the nontangential maximal function of $\nabla \boldsymbol{u}$.

Lemma 4.20. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$ and $\lambda \in \Sigma_\theta$. Let $(\boldsymbol{u}, \phi) = \mathcal{S}_\lambda(\boldsymbol{f})$ with density $\boldsymbol{f} \in L^p(\partial\Omega)$ with $1 < p < \infty$. Then*

$$\|(\nabla \boldsymbol{u})^*\|_{L^p(\partial\Omega)} \le C\|\boldsymbol{f}\|_{L^p(\partial\Omega)}, \tag{4.32}$$

where $C > 0$ depends only on θ, p and the Lipschitz character of Ω.

Proof. Let $\boldsymbol{u}$ be given by the single layer potential. Pick $p \in \partial\Omega$ and define the set

$$E(p) = \{x \in \Omega : |x - p| \le C \operatorname{dist}(x, \partial\Omega)\}.$$

Set $t = |x - p|$. By the definition of the nontangential maximal function and the fact that we can take the differential inside the integral (cf. proof of Lemma 4.2)

$$\begin{aligned}
(\nabla \boldsymbol{u})^*(p) &= \sup_{x\in E(p)} \left| \int_{\partial\Omega} \nabla \boldsymbol{\Gamma}_k(x-y;\lambda)\boldsymbol{f}_k(y)d\sigma(y) \right| \\
&= \sup_{x\in E(p)} \left| \int_{|p-y|\le t} \nabla \boldsymbol{\Gamma}_k(x-y;\lambda)\boldsymbol{f}_k(y)d\sigma(y) \right| + \sup_{x\in E(p)} \left| \int_{|p-y|>t} \nabla \boldsymbol{\Gamma}_k(x-y;\lambda)\boldsymbol{f}_k(y)d\sigma(y) \right| \\
&= A + B.
\end{aligned}$$

We will look at the two parts separately. Using Theorem 3.11 and Lemma 2.13 we find

$$\begin{aligned}
A &\le \sup_{x\in E(p)} \int_{|p-y|\le t} |\nabla \boldsymbol{\Gamma}_k(x-y;\lambda)||\boldsymbol{f}_k(y)|d\sigma(y) \\
&\le C \int_{|p-y|\le t} \frac{|\boldsymbol{f}_k(y)|}{|x-y|^2} d\sigma(y) \\
&\le C \int_{|p-y|\le t} \frac{|\boldsymbol{f}_k(y)|}{t^2} d\sigma(y) \\
&\le \frac{C}{|B_3(p,t)\cap\partial\Omega|} \int_{|p-y|\le t} |\boldsymbol{f}_k(y)|d\sigma(y) \\
&\le C\mathcal{M}_{\partial\Omega}(\boldsymbol{f})(p).
\end{aligned}$$

For the second part we use the mean value theorem to find

$$\nabla \boldsymbol{\Gamma}_k(x-y;\lambda) \le (x-p)\nabla^2 \boldsymbol{\Gamma}_k(\xi;\lambda) + \nabla \boldsymbol{\Gamma}_k(p-y;\lambda),$$

where $\xi = c(x-y) + (1-c)(x-p)$ with $c \in [0,1]$. If we now use Theorem 3.11 we find

$$\begin{aligned}
B &\le \sup_{x\in E(p)} \int_{|p-y|>t} \frac{|x-p|}{|c(x-y)+(1-c)(p-y)|^3} |\boldsymbol{f}_k(y)|d\sigma(y) + T_\lambda^*(\boldsymbol{f})(p) \\
&\le C \sup_{t>0} \int_{|p-y|>t} \frac{t}{|p-y|^3} |\boldsymbol{f}_k(y)|d\sigma(y) + T_\lambda^*(\boldsymbol{f})(p) \\
&\le C\mathcal{M}_{\partial\Omega}(\boldsymbol{f})(p) + T_\lambda^*(\boldsymbol{f})(p),
\end{aligned}$$

where we used Lemma 4.13 in the last line. Putting it all together leads to

$$\|(\nabla \boldsymbol{u})^*\|_{L^p(\partial\Omega)} \leq C\|\mathcal{M}_{\partial\Omega}(\boldsymbol{f})\|_{L^p(\partial\Omega)} + \|T^*_\lambda(\boldsymbol{f})\|_{L^p(\partial\Omega)} \leq C\|\boldsymbol{f}\|_{L^p(\partial\Omega)}.$$

This completes the proof. □

4.3.4 Boundedness of $(\boldsymbol{u})^*$ in $L^p(\partial\Omega)$

Lemma 4.21. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$. Let $(\boldsymbol{u}, \phi) = \mathcal{S}_\lambda(\boldsymbol{f})$ with density $\boldsymbol{f} \in L^p(\partial\Omega)$ with $1 < p < \infty$. Then*

$$|\lambda|^{\frac{1}{2}}\|(\boldsymbol{u})^*\|_{L^p(\partial\Omega)} \leq C\|\boldsymbol{f}\|_{L^p(\partial\Omega)},$$

where C only depends on θ, p and the Lipschitz character of Ω.

The proof of this lemma is similar to Lemma 4.16 and Lemma 4.20. This section is finished by stating the results in an overarching theorem.

Theorem 4.22. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$ and let $1 < p < \infty$. Let $\boldsymbol{f} \in L^p(\partial\Omega; \mathbb{C}^3)$ and $(\boldsymbol{u}, \phi) = \mathcal{S}_\lambda(\boldsymbol{f})$. Then*

$$\|(\nabla \boldsymbol{u})^*\|_{L^p(\partial\Omega)} + \|(\phi)^*\|_{L^p(\partial\Omega)} + |\lambda|^{\frac{1}{2}}\|(\boldsymbol{u})^*\|_{L^p(\partial\Omega)} \leq C\|\boldsymbol{f}\|_{L^p(\partial\Omega)}, \tag{4.33}$$

where C depends only on θ, p and the Lipschitz character of Ω.

Proof. This is a combination of the previous Lemma 4.16, Lemma 4.20 and Lemma 4.21. □

4.4 Nontangential limits of layer potentials

In this section we derive the limits of the layer potentials when $x \in \Omega$ tends to a point on the boundary. We are especially interested in the nontangential limit of $\boldsymbol{w}$ if $(\boldsymbol{w}, \rho) = \mathcal{D}_\lambda(\boldsymbol{g})$ and the conormal derivative of $\boldsymbol{u}$ if $(\boldsymbol{u}, \phi) = \mathcal{S}_\lambda(\boldsymbol{f})$. Recall that the the double layer potential with density $\boldsymbol{g}$ is given by

$$\boldsymbol{w}_i(x) = \int_{\partial\Omega} \left[\frac{\partial\Gamma_{ik}(x-y;\lambda)}{\partial y_j}\boldsymbol{n}_j(y) - \Phi_i(x-y)\boldsymbol{n}_k(y)\right] \boldsymbol{f}_k(y)d\sigma(y).$$

Further recall that the conormal derivative of the single layer potential is given by

$$\left(\frac{\partial \boldsymbol{u}}{\partial \nu}\right)_i(x) = \frac{\partial \boldsymbol{u}_i}{\partial x_j}\boldsymbol{n}_j - \phi\boldsymbol{n}_i = \int_{\partial\Omega} \left[\frac{\partial\Gamma_{ik}(x-y;\lambda)}{\partial x_j}\boldsymbol{n}_j(y) - \Phi_i(x-y)\boldsymbol{n}_k(y)\right] \boldsymbol{f}_k(y)d\sigma(y).$$

Notice that the kernels of $\boldsymbol{w}_i$ and the conormal derivatie of $\boldsymbol{u}_i$ are very similar. In fact they are adjoint to each other in a certain sense. This motivates the introduction of some extra notation. Define the following kernels

$$K_{ik}(x, y; \lambda) := \frac{\partial\Gamma_{ik}(x-y;\lambda)}{\partial x_j}\boldsymbol{n}_j(y) - \Phi_i(x-y)\boldsymbol{n}_k(y) \tag{4.34}$$

$$K^\star_{ik}(x, y; \lambda) := K^{ad}_{ik}(x, y; \overline{\lambda}) = \frac{\partial\Gamma_{ik}(x-y;\lambda)}{\partial y_j}\boldsymbol{n}_j(y) - \Phi_i(x-y)\boldsymbol{n}_k(y), \tag{4.35}$$

where the superscript ad is used to denote the adjoint of the kernel. These kernels make sense in the integral for almost every $y \in \partial\Omega$ and $x \in \Omega$ or $x \in \mathbb{R}^3 \setminus \overline{\Omega}$. One can see that this kernel is bounded and continuous when $x \neq y$. We can now write the integrals as

$$\boldsymbol{w}_i(x) = \int_{\partial\Omega} K_{ik}(x, y; \lambda)\boldsymbol{f}_k(y)d\sigma(y) \tag{4.36}$$

and

$$\left(\frac{\partial \boldsymbol{u}}{\partial \nu}\right)_i(x) = \int_{\partial\Omega} K^{\star}_{ik}(x, y; \lambda)\boldsymbol{f}_k(y)d\sigma(y). \tag{4.37}$$

It suffices to investigate the limits of (4.36), since (4.37) can then be obtained by using the adjoint kernel. A natural starting point is to investigate if (4.36) exists if $x \in \partial\Omega$. This is done by looking at the gradient of $\boldsymbol{\Gamma}_j$, and Φ_j separately. From the previous section we already know that

$$T^*_\lambda(\boldsymbol{f})(p) = \sup_{t>0}\left|\int_{\substack{y\in\partial\Omega\\|p-y|>t}} \nabla\boldsymbol{\Gamma}_k(p-y;\lambda)\boldsymbol{f}_k(y)d\sigma(y)\right| \tag{4.38}$$

and

$$S^*(\boldsymbol{f})(p) := \sup_{t>0}\left|\int_{\substack{y\in\partial\Omega\\|p-y|>t}} \frac{p_k - y_k}{|p-y|^3}\boldsymbol{f}_k(y)d\sigma(y)\right| = \sup_{t>0}\left|\int_{\substack{y\in\partial\Omega\\|p-y|>t}} \Phi_k(p-y)\boldsymbol{f}_k(y)d\sigma(y)\right| \tag{4.39}$$

are bounded in $L^p(\partial\Omega)$ when $1 < p < \infty$. We can use this in combination with the existence of $\lim_{t\downarrow 0} T^t_\lambda(\boldsymbol{f})(p)$ and $\lim_{t\downarrow 0} S^t(\boldsymbol{f})(p)$ for sufficiently smooth $\boldsymbol{f}$ when $p \in \partial\Omega$. How to combine these things is made clear in the following standard lemma. This lemma can also be found in [22, Theorem 2.1.14]

Lemma 4.23. *Let Ω be a Lipschitz domain and $1 < p < \infty$. Suppose that R^t is a linear operator on $L^p(\partial\Omega)$ and that $R^*(f)(p) = \sup_{t>0}|R^t(f)(p)|$. Further suppose that for all $f \in L^p(\partial\Omega)$ we have that*

$$\|R^* f\|_p \leq C_p \|f\|_p$$

and for all $g \in C^1_c(\partial\Omega)$ we have that $R(g) = \lim_{t\downarrow 0} R^t(g)$ exists pointwise almost everywhere. Then $R(f) = \lim_{t\downarrow 0} R^t(f)$ exists pointwise almost everywhere and

$$\|R(f)\|_{L^p(\partial\Omega)} \leq \|R^*(f)\|_{L^p(\partial\Omega)} \leq C_p\|f\|_{L^p(\partial\Omega)}. \tag{4.40}$$

Proof. Fix $f \in L^p(\partial\Omega)$ and define the oscillation of f as

$$O_f(y) = \limsup_{\varepsilon\to 0}\limsup_{\theta\to 0}\left|R^\varepsilon(f)(y) - R^\theta(f)(y)\right|.$$

We would like to show that $\|O_f\|_{L^p(\partial\Omega)} = 0$, as this would imply that $O_f(y) = 0$ almost everywhere. That would imply that $R^\varepsilon(f)(y)$ is a Cauchy sequence for almost all $y \in \partial\Omega$ and therefore would converge to some $R(f)(y)$ as $\varepsilon \to 0$ for almost all $y \in \partial\Omega$. To show this, we use the density of $C^1_c(\partial\Omega)$ in $L^p(\partial\Omega)$. For any $\eta > 0$ we can find a function $g \in C^1_c(\partial\Omega)$ such that $\|f - g\|_{L^p(\partial\Omega)} < \eta$. Since by assumption $R^\varepsilon(g) \to R(g)$ almost everywhere, we know that $O_g = 0$ almost everywhere. This, in combination with the linearity of R^ε, implies that for almost all $y \in \partial\Omega$

$$O_f(y) \leq O_g(y) + O_{f-g}(y) = O_{f-g}(y).$$

Now we use this to calculate

$$\begin{aligned}\|O_f(y)\|_{L^p(\partial\Omega)} &\leq \|O_{f-g}(y)\|_{L^p(\partial\Omega)} \\ &\leq \|2R^*(f-g)\|_{L^p(\partial\Omega)} \\ &\leq 2C_p\|f-g\|_{L^p(\partial\Omega)} < 2C_p\eta.\end{aligned}$$

We now let $\eta \to 0$ to show that $O_f = 0$ almost everywhere. Hence $R^\varepsilon(f)$ is a Cauchy sequence and converges almost everywhere to some $R(f)$. Since $|R(f)| \leq |R^*(f)|$ inequality (4.40) follows. □

With this lemma in the back of the head, we investigate $\lim_{t\downarrow 0} T^t(\boldsymbol{f})$ and $\lim_{t\downarrow 0} S^t(\boldsymbol{f})$, when $\boldsymbol{f} \in C_c^1(\partial\Omega;\mathbb{C}^3)$. Since $C_c^1(\partial\Omega;\mathbb{C}^3)$ is dense in $L^p(\partial\Omega;\mathbb{C}^3)$ when $1 < p < \infty$, the existence of the limit in L^p is provided by the previous lemma.

Lemma 4.24. *Let Ω be a bounded and connected Lipschitz domain in $\mathbb{R}^3$. Let $\lambda \in \Sigma_\theta$ and suppose that $\boldsymbol{f} \in C_c^1(\partial\Omega;\mathbb{C}^3)$. Then*

$$T_\lambda(\boldsymbol{f}) = \lim_{t\downarrow 0} T_\lambda^t(\boldsymbol{f}) \tag{4.41}$$

exists pointwise almost everywhere.

Proof. Start by taking $\boldsymbol{f} \in C_c^1(\partial\Omega;\mathbb{C}^3)$ and let $p \in \partial\Omega$ be a Lebesgue point of $\boldsymbol{f}$. Let $\eta > 0$ and (t_n) a decreasing sequence such that $t_n \downarrow 0$. We now consider the following difference

$$\left|T_\lambda^{t_n}(\boldsymbol{f})(p) - T_\lambda^{t_m}(\boldsymbol{f})(p)\right|.$$

Without loss of generality we assume that $n > m$. If we now substitute the definition and add and subtract $T_0(\boldsymbol{f})$ we find

$$\begin{aligned}\left|T_\lambda^{t_n}(\boldsymbol{f})(p) - T_\lambda^{t_m}(\boldsymbol{f})(p)\right| &= \int_{\substack{y\in\partial\Omega \\ t_n<|y-p|<t_m}} \Big[\nabla_x\boldsymbol{\Gamma}_k(p-y;\lambda) - \nabla_x\boldsymbol{\Gamma}_k(p-y;0)\Big]\boldsymbol{f}_k(y)d\sigma(y) \\ &\quad + \int_{\substack{y\in\partial\Omega \\ t_n<|y-p|<t_m}} \nabla_x\boldsymbol{\Gamma}_k(p-y;0)\boldsymbol{f}_k(y)d\sigma(y) \\ &= I + J.\end{aligned}$$

Now notice that by Theorem 3.12 and the compactness of the support of $\boldsymbol{f}$ and Lemma 2.6 we find that

$$\begin{aligned}|I| &\leq C\int_{\substack{y\in\partial\Omega \\ t_n<|y-p|<t_m}} \frac{\sqrt{|\lambda|}}{|p-y|}|\boldsymbol{f}_k(y)|d\sigma(y) \\ &\leq C \max_{s\in\operatorname{supp}(\boldsymbol{f})}\big\{|\boldsymbol{f}(s)|\big\}\sqrt{|\lambda|}\int_{\substack{y\in\partial\Omega \\ t_n<|y-p|<t_m}} \frac{1}{|p-y|}d\sigma(y) \\ &\leq C\sqrt{|\lambda|}\int_{t_n}^{t_m} \frac{1}{r}|B(p,r)\cap\partial\Omega|dr \\ &\leq C\sqrt{|\lambda|}\int_{t_n}^{t_m} r dr \\ &\leq C_1\left(t_m^2 - t_n^2\right).\end{aligned}$$

Hence we define $N_1 = \sqrt{\frac{\eta}{3C_1}}$. Now we manipulate J to find

$$\begin{aligned} J &= \int_{\substack{y\in\partial\Omega \\ t_n<|y-p|<t_m}} \nabla_x \mathbf{\Gamma}_k(p-y;0)\Big[\boldsymbol{f}_k(y) - \boldsymbol{f}_k(p)\Big] d\sigma(y) \\ &\quad + \int_{\substack{y\in\partial\Omega \\ t_n<|y-p|<t_m}} \nabla_x \mathbf{\Gamma}_k(p-y;0)\boldsymbol{f}_k(p) d\sigma(y) \\ &= K + \boldsymbol{f}_k(p)L. \end{aligned}$$

Since $\boldsymbol{f} \in C_c^1(\partial\Omega)$ we now use Lemma 3.11 to find that

$$\begin{aligned} |K| &\leq C \int_{\substack{y\in\partial\Omega \\ t_n<|y-p|<t_m}} \frac{|p-y|}{|p-y|^2} \left| \frac{\boldsymbol{f}_k(p) - \boldsymbol{f}_k(y)}{p-y} \right| d\sigma(y) \\ &\leq C \max_{s\in \operatorname{supp}(\boldsymbol{f})} \left\{ |\nabla_x \boldsymbol{f}(s)| \right\} \int_{\substack{y\in\partial\Omega \\ t_n<|y-p|<t_m}} \frac{1}{|p-y|} d\sigma(y). \end{aligned}$$

By a similar calculation as before we find that

$$|K| \leq C_2 \left(t_m^2 - t_n^2 \right),$$

and we define $N_2 = \sqrt{\frac{\eta}{3C_2}}$. For the last bit we again use Lemma 3.11 and Lemma 2.6

$$\begin{aligned} |L| &\leq C \int_{\substack{y\in\partial\Omega \\ t_n<|p-y|<t_m}} \frac{1}{|p-y|^2} d\sigma(y) \\ &\leq C \int_{t_m}^{t_n} \frac{1}{r^2} |B(p,r) \cap \partial\Omega| dr \\ &\leq C_3 \left(t_m - t_n \right), \end{aligned}$$

and hence we define $N_3 = \frac{\eta}{\boldsymbol{f}_k(p)C_3}$. Now we define $N = \max\{N_1, N_2, N_3\}$. Now if $n, m > N$ we find that

$$\left| T_\lambda^{t_n}(\boldsymbol{f})(p) - T_\lambda^{t_m}(\boldsymbol{f})(p) \right| < C_1 N_1^2 + C_2 N_2^2 + C_3 \boldsymbol{f}_k(p) N_3 = \eta$$

This shows that $T_\lambda^{t_n}(\boldsymbol{f})(p)$ is a Cauchy sequence in $\mathbb{C}^3$ for every sequence t_n. Since this is a complete space the limit exists. Thus the limit of $T_\lambda^t(f)$ exists (cf. Remark 2.16) and

$$\lim_{t\downarrow 0} T_\lambda^t(\boldsymbol{f})(p) = T_\lambda(\boldsymbol{f})(p)$$

This completes the proof. □

Now that we have shown existence of $T_\lambda(\boldsymbol{f})$, we now show the same for $S(\boldsymbol{f})$.

Lemma 4.25. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$. Let $\lambda \in \Sigma_\theta$ and suppose that $\boldsymbol{f} \in C_c^1(\partial\Omega; \mathbb{C}^3)$. Then*

$$S(\boldsymbol{f}) = \lim_{t\downarrow 0} S^t(\boldsymbol{f}) \tag{4.42}$$

exists pointwise almost everywhere.

Proof. Start by taking $\boldsymbol{f} \in C_c^1(\partial\Omega; \mathbb{C}^3)$ and let $p \in \partial\Omega$ be a Lebesgue point of $\boldsymbol{f}$. Now let $\eta > 0$ and (t_n) a decreasing sequence such that $t_n \downarrow 0$. We now consider the following difference

$$\left|S^{t_n}(\boldsymbol{f})(p) - S^{t_m}(\boldsymbol{f})(p)\right|.$$

Without loss of generality we assume that $n > m$. If we now substitute the definition we find

$$\left|S^{t_n}(\boldsymbol{f})(p) - S^{t_m}(\boldsymbol{f})(p)\right| \leq \int_{\substack{y\in\partial\Omega \\ t_n<|y-p|<t_m}} \frac{1}{|p-y|^2} \boldsymbol{f}_k(y)d\sigma(y).$$

Now, using the exact same calculations as in the proof of the previous lemma, the result follows. □

We have shown that $\lim_{t\downarrow 0} T^t(\boldsymbol{f})(p)$ and $\lim_{t\downarrow 0} S^t(\boldsymbol{f})(p)$ exist when $\boldsymbol{f} \in C_c^1(\partial\Omega)$. This motivates the following definition

$$\mathcal{K}_\lambda(\boldsymbol{f})(p) = \lim_{t\downarrow 0} \int_{\substack{y\in\partial\Omega \\ |p-y|>t}} \boldsymbol{K}_k(p, y; \lambda)\boldsymbol{f}_k(y)d\sigma(y) \tag{4.43}$$

$$\mathcal{K}^*_{\overline{\lambda}}(\boldsymbol{f})(p) = \lim_{t\downarrow 0} \int_{\substack{y\in\partial\Omega \\ |p-y|>t}} \boldsymbol{K}^\star_k(p, y; \lambda)\boldsymbol{f}_k(y)d\sigma(y). \tag{4.44}$$

Using Lemma 4.23, Lemma 4.24 and Lemma 4.25, we find that (4.43) and (4.44) exist almost everywhere on $\partial\Omega$ if $\boldsymbol{f} \in L^p(\partial\Omega; \mathbb{C}^3)$, $1 < p < \infty$. Notice in the rest of the text that when both arguments in the kernel lie on the boundary of Ω the integral will be taken in this principal value sense. In the following theorem we look at the double layer potential with a constant function as density. This will help in finding the values of $\mathcal{K}_\lambda(\boldsymbol{f})$ and $\mathcal{K}^*_{\overline{\lambda}}(\boldsymbol{f})$ when $\boldsymbol{f}$ is a constant function.

Theorem 4.26. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$ and let $\lambda \in \Sigma_\theta$. Let $\boldsymbol{c} \in \mathbb{C}^3$ be a constant vector and $(\boldsymbol{w}, \rho) = \mathcal{D}_\lambda(\boldsymbol{c})$. Then,*

$$\boldsymbol{w}(x) = \begin{cases} \boldsymbol{c}, & x \in \Omega \\ \frac{1}{2}\boldsymbol{c}, & x \in \partial\Omega \quad (a.e.) \\ 0, & x \in \mathbb{R}^3 \setminus \overline{\Omega}. \end{cases} \tag{4.45}$$

Proof. For $x \notin \partial\Omega$ the Green's identity from Theorem 4.5 can be used with $(\boldsymbol{u}, \phi) = (\boldsymbol{\Gamma}_k, \Phi_k)$ and $(\boldsymbol{w}, \rho) = (\boldsymbol{c}, 0)$ where $\boldsymbol{c}$ is constant. This yields

$$\lambda \int_\Omega \Gamma_{ik}(x-y; \lambda)\boldsymbol{c}_i dy + \boldsymbol{w}_k(x) = \begin{cases} \boldsymbol{c}_k, & x \in \Omega \\ 0, & x \in \mathbb{R}^3 \setminus \overline{\Omega}. \end{cases}$$

Now notice that for $r > 0$, we have by the divergence theorem that

$$\begin{aligned} \int_{\Omega\cap\partial B(x,r)} \Gamma_{ik}(x-y; \lambda)d\sigma(y) &= \int_{\Omega\cap\partial B(0,r)} \Gamma_{ik}(y; \lambda)\boldsymbol{n}_k(y)d\sigma(y) \\ &= \int_{\Omega\cap B(0,r)} \operatorname{div}(\boldsymbol{\Gamma_i})dy = 0. \end{aligned}$$

Hence we have that

$$\lambda\boldsymbol{c}_i \int_\Omega \Gamma_{ik}(x-y; \lambda)dy = \lambda\boldsymbol{c}_i \int_0^\infty \int_{\partial B(x,r)} \Gamma_{ik}(x-y; \lambda)r^2 d\sigma(y)dr = 0.$$

This shows the theorem in the case $x \notin \partial\Omega$. The final part of the theorem is proved by first cutting out a ball with radius ε around $x \in \partial\Omega$. We define $\Omega_\varepsilon = \Omega \setminus B(x,\varepsilon)$. Because $x \notin \Omega_\varepsilon$ we find $\boldsymbol{w}(x) = 0$, by the same arguments as above. If we now define $\partial\tilde{B}(x,\varepsilon) = (\partial B(x,\varepsilon)) \cap \Omega$. Now notice that $\partial\Omega_\varepsilon = (\partial\Omega \cap \partial\Omega_\varepsilon) \cup \partial\tilde{B}(x,\varepsilon)$ Now we can again apply Green's identity.

$$\boldsymbol{w}(x) = \int_{\partial\Omega} S_{ij}(\boldsymbol{\Gamma}_k, \Phi_k)(x-y)\boldsymbol{c}_i d\sigma = \lim_{\varepsilon\to 0}\int_{\partial\tilde{B}(x,\varepsilon)} S_{ij}(\boldsymbol{\Gamma}_k, \Phi_k)(x-y)\boldsymbol{c}_i n_j(y)dy = \frac{\boldsymbol{c}_k}{2}$$

This completes the proof. □

From this theorem we can easily see that

$$\mathcal{K}_\lambda(1)(x) = \begin{cases} 1, & x \in \Omega \\ \dfrac{1}{2}, & x \in \partial\Omega \quad (a.e) \\ 0, & x \in \mathbb{R}^3 \setminus \overline{\Omega}, \end{cases} \tag{4.46}$$

and a similar formula holds for $\mathcal{K}^*_{\overline{\lambda}}$.

Lemma 4.27. *Suppose $\boldsymbol{f} \in C(\partial\Omega)$ and $\boldsymbol{f}(p) = 0$ for some $p \in \partial\Omega$. Then* (4.36) *and* (4.37) *are continuous at p.*

Proof. Let $\varepsilon > 0$. We now show that there exists a $\delta > 0$ such that $|\boldsymbol{w}(x) - \boldsymbol{w}(p)| < \varepsilon$. Define the constant C_1 and C_2 as

$$C_1 = \sup_{x\in\mathbb{R}^3\setminus\partial\Omega}\int_{\partial\Omega} |\boldsymbol{K}_k(x,y;\lambda)|d\sigma(y)$$
$$C_2 = \sup_{p\in\partial\Omega}\lim_{t\downarrow 0}\int_{\substack{y\in\partial\Omega \\ |p-y|>t}} |\boldsymbol{K}_k(p,y;\lambda)|d\sigma(y),$$

which are both finite. Since $\boldsymbol{f}$ is continuous we can find a $\eta > 0$ such that $|\boldsymbol{f}(y)| \leq \frac{\varepsilon}{3(C_1+C_2)}$ when $y \in B(p,\eta) \cap \partial\Omega$. Now we find

$$|\boldsymbol{w}(x) - \boldsymbol{w}(p)| \leq \int_{B(p,\eta)} \left(|\boldsymbol{K}_k(x,y;\lambda)| + |\boldsymbol{K}_k(p,y;\lambda)|\right)|\boldsymbol{f}_k(y)|d\sigma(y)$$
$$+ \int_{\mathbb{R}^3\setminus B(p,\eta)} |\boldsymbol{K}_k(x,y;\lambda) - \boldsymbol{K}_k(p,y;\lambda)||\boldsymbol{f}_k(y)|d\sigma(y).$$

It is clear that the first integral is smaller than $\frac{2\varepsilon}{3}$. If we choose $|x-p| < \frac{1}{2}\eta$ the integrand of the second integral is continuous and bounded since $y \neq x$ and $y \neq p$ in the region of integration. By the continuity of the kernel we can choose $\delta < \frac{1}{2}\eta$ small enough so that second integral is smaller than $\frac{1}{3}\varepsilon$ when $|x-p| < \delta$. This shows the continuity at p. □

Now that we have a continuity property we can find the nontangential limit of the double layer potential.

Theorem 4.28. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$ and let $\lambda \in \Sigma_\theta$. Let $\boldsymbol{f} \in L^p(\partial\Omega;\mathbb{C}^3)$ with $1 < p < \infty$ and $(\boldsymbol{u},\phi) = \mathcal{D}_\lambda(\boldsymbol{f})$ Then,*

$$\boldsymbol{u}_\pm = \left(\mp\frac{1}{2}I + \mathcal{K}^*_{\overline{\lambda}}\right)\boldsymbol{f}. \tag{4.47}$$

Proof. By the definition of the nontangential limit we find

$$\begin{aligned}\boldsymbol{u}_+(p) &= \lim_{\substack{x_n \in E(p)\\ x_n \to p}} \int_{\partial\Omega} \boldsymbol{K}_k(x_n, y; \lambda) \left(\boldsymbol{f}_k(p) + \boldsymbol{f}_k(y) - \boldsymbol{f}_k(p)\right) d\sigma(y)\\ &= \boldsymbol{f}_k(p) \lim_{\substack{x_n \in E(p)\\ x_n \to p}} \int_{\partial\Omega} \boldsymbol{K}_k(x_n, y; \lambda) d\sigma(y) + \lim_{\substack{x_n \in E(p)\\ x_n \to p}} \int_{\partial\Omega} \boldsymbol{K}_k(x_n, y; \lambda) \left(\boldsymbol{f}_k(y) - \boldsymbol{f}_k(p)\right) d\sigma(y).\end{aligned}$$

Using Theorem 4.26 we find that the first integral is 1. By Lemma 4.27 we find that the second integral is continuous is at p. Thus we find

$$\begin{aligned}\boldsymbol{u}_+(p) &= \boldsymbol{f}_k(p) - \boldsymbol{f}_k(p) \int_{\partial\Omega} \boldsymbol{K}_k(p, y; \lambda)) d\sigma(y) + \int_{\partial\Omega} \boldsymbol{K}_k(p, y; \lambda) \boldsymbol{f}_k(y) d\sigma(y)\\ &= -\frac{1}{2} \boldsymbol{f}_k(p) + \mathcal{K}^*_{\overline{\lambda}}(\boldsymbol{f})(p).\end{aligned}$$

The exterior limit can be calculated in a similar fashion. □

Almost the same procedure can be employed to show the limit of the conormal derivative.

Theorem 4.29. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$ and let $\lambda \in \Sigma_\theta$. Let $\boldsymbol{f} \in L^p(\partial\Omega; \mathbb{C}^3)$ with $1 < p < \infty$ and $(\boldsymbol{u}, \phi) = \mathcal{S}_\lambda(\boldsymbol{f})$ Then $\nabla_{tan}\boldsymbol{u}_+ = \nabla_{tan}\boldsymbol{u}_-$,*

$$\left(\frac{\partial \boldsymbol{u}}{\partial \nu}\right)_\pm = \left(\mp\frac{1}{2} I + \mathcal{K}_\lambda\right) \boldsymbol{f}, \tag{4.48}$$

where $\mathcal{K}_\lambda$ is a bounded operator on $L^p(\partial\Omega)$.

Proof. By the definition of the nontangential limit we find

$$\begin{aligned}\left(\frac{\partial \boldsymbol{u}}{\partial \nu}\right)_+ &= \lim_{\substack{x_n \in E(p)\\ x_n \to p}} \int_{\partial\Omega} \boldsymbol{K}_k(x_n, y; \lambda) \left(\boldsymbol{f}_k(p) + \boldsymbol{f}_k(y) - \boldsymbol{f}_k(p)\right) d\sigma(y)\\ &= \boldsymbol{f}_k(p) \lim_{\substack{x_n \in E(p)\\ x_n \to p}} \int_{\partial\Omega} \boldsymbol{K}_k(x_n, y; \lambda) d\sigma(y) + \lim_{\substack{x_n \in E(p)\\ x_n \to p}} \int_{\partial\Omega} \boldsymbol{K}_k(x_n, y; \lambda) \left(\boldsymbol{f}_k(y) - \boldsymbol{f}_k(p)\right) d\sigma(y).\end{aligned}$$

Using Theorem 4.26 we find that the first integral is 1. By Lemma 4.27 we find that the second integral is continuous is at p. Thus we find

$$\begin{aligned}\left(\frac{\partial \boldsymbol{u}}{\partial \nu}\right)_+ &= \boldsymbol{f}_k(p) - \boldsymbol{f}_k(p) \int_{\partial\Omega} \boldsymbol{K}_k(p, y)) d\sigma(y) + \int_{\partial\Omega} \boldsymbol{K}_k(p, y) \boldsymbol{f}_k(y) d\sigma(y)\\ &= -\frac{1}{2} \boldsymbol{f}_k(p) + \mathcal{K}_\lambda(\boldsymbol{f})(p).\end{aligned}$$

The exterior limit can be calculated similarily □

4.5 Layer potentials and boundary conditions

In this section we show how we can use the layer potentials to solve the Dirichlet and the Neumann problem for the homogeneous Stokes System. We show here how one can use the double layer potential to solve the Dirichlet problem for the Stokes System. In exactly the same fashion the Neumann problem can be solved using the single layer potential. Now let us state the Dirichlet problem for the homogeneous Stokes system.

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla\phi + \lambda\boldsymbol{u} = 0 & \text{in } \Omega \\ \operatorname{div}(\boldsymbol{u}) = 0 & \text{in } \Omega \\ \boldsymbol{u} = \boldsymbol{f} & \text{on } \partial\Omega \text{ (nontangential a.e.).} \end{cases} \tag{4.49}$$

First of all note that this problem is not always solvable if $\boldsymbol{f}$ is just in $L^2(\partial\Omega)$. To see this notice that if $\boldsymbol{u}$ solves (4.49) it must be that

$$0 = \int_\Omega \operatorname{div}(\boldsymbol{u})dx = \int_{\partial\Omega} \boldsymbol{u}\cdot\boldsymbol{n}d\sigma(y).$$

We thus define the following space

$$L^2_n(\partial\Omega) = \left\{ \boldsymbol{f} \in L^2(\partial\Omega) \, : \, \int_{\partial\Omega} \boldsymbol{f}\cdot\boldsymbol{n}d\sigma = 0 \right\}. \tag{4.50}$$

Now if $\boldsymbol{f} \in L^2_n(\partial\Omega)$, we can make sense of (4.49). We have already seen in this section that if $(\boldsymbol{u},\phi) = \mathcal{D}_\lambda(\boldsymbol{g})$, then $(\boldsymbol{u},\phi)$ solves the equation in Ω and $\boldsymbol{u}$ is divergence free. We also know from the previous section that $\boldsymbol{u} = \left(-\frac{1}{2}I + \mathcal{K}^*_{\overline{\lambda}}\right)\boldsymbol{g}$. Now notice that if $\left(-\frac{1}{2}I + \mathcal{K}^*_{\overline{\lambda}}\right)$ is an invertible operator we could just choose $\boldsymbol{g} = \left(-\frac{1}{2}I + \mathcal{K}^*_{\overline{\lambda}}\right)^{-1}\boldsymbol{f}$ to solve the boundary value problem. In the case that $\mathcal{K}_\lambda$ is a compact operator, Fredholm theory could be used directly to solve this problem. However the operator $\mathcal{K}_\lambda$ is not a compact operator when Ω is a Lipschitz domain. Therefore Fredholm theory is not directly applicable. However in [12] Fabes, Kenig and Verchota showed that in the case that $\lambda = 0$, $\mathcal{K}_0$ is an invertible operator on $L^2(\partial\Omega)$. It can be shown that $(\mathcal{K}_\lambda - \mathcal{K}_0)$ is a compact operator, and hence $\mathcal{K}_\lambda$ is an invertible operator on $L^2(\partial\Omega)$. The drawback of this method using a compactness argument, is that we lose control of the parameter λ. Therefore we establish so-called Rellich-type estimates in the next section.

5 Invertibility of layer potentials

In this section we study the invertibility of the layer potentials. We have already noticed in the previous section that the invertibility in L^2 is guaranteed by a compactness argument. However, this loses control of the parameter λ. Therefore we establish so-called Rellich estimates for the layer potentials. After this is done, we show invertibility of the layer potentials in L^2 while keeping control of λ. It turns out that the layer potentials solve the Stokes System uniquely. Finally we obtain a weak reverse Hölder inequality.

5.1 Rellich estimates

We start by establishing the Rellich estimates for the layer potentials. For convenience we state a condition that sets the stage for the upcoming lemmas and theorems.

Condition 5.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$ with connected boundary and $\sigma(\partial\Omega) = 1$. Let $\lambda \in \Sigma_\theta$ and $|\lambda| \geq \tau$, where $\tau \in (0,1)$. Let $(\boldsymbol{u}, \phi)$ be a solution of the homogeneous Stokes System* (1.11)*. Suppose $(\nabla \boldsymbol{u})^* \in L^2(\partial\Omega; \mathbb{C}^3)$ and $(\phi)^* \in L^2(\partial\Omega; \mathbb{C})$. Further suppose that $\nabla \boldsymbol{u}$ and ϕ have nontangential limits a.e. on $\partial\Omega$.*

Notice that the layer potentials satisfy this conditions. These conditions ensure that we can use Theorem 2.18 to justify integration by parts. The first Rellich type identity involves the conormal derivative (recall (4.5)).

Lemma 5.2. *Assume Condition 5.1 holds and let $\boldsymbol{h} \in C_0^1(\mathbb{R}^3, \mathbb{R}^3)$ as in Lemma 2.11. Then,*

$$\begin{aligned}\int_{\partial\Omega} \boldsymbol{h}_k \boldsymbol{n}_k |\nabla \boldsymbol{u}| d\sigma = 2\operatorname{Re} \int_{\partial\Omega} \boldsymbol{h}_k \frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_k} \left(\frac{\partial \boldsymbol{u}}{\partial \nu}\right)_i d\sigma + \int_\Omega \operatorname{div}(\boldsymbol{h}) |\nabla \boldsymbol{u}|^2 dx \\ - 2\operatorname{Re} \int_\Omega \frac{\partial \boldsymbol{h}_k}{\partial x_j} \frac{\partial \boldsymbol{u}_i}{\partial x_k} \frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_j} dx + 2\operatorname{Re} \int_\Omega \frac{\partial \boldsymbol{h}_k}{\partial x_i} \frac{\partial \boldsymbol{u}_i}{\partial x_k} \overline{\phi} dx \\ - 2\operatorname{Re} \int_\Omega \boldsymbol{h}_k \frac{\partial \boldsymbol{u}_i}{\partial x_k} \overline{\lambda} \overline{\boldsymbol{u}}_i dx.\end{aligned}$$

Proof. Let $\boldsymbol{h} \in C_0^1(\mathbb{R}^3, \mathbb{R}^3)$. Now the divergence theorem on $\boldsymbol{h}|\nabla \boldsymbol{u}|^2$ yields

$$\int_{\partial\Omega} \boldsymbol{h}_k \boldsymbol{n}_k |\nabla \boldsymbol{u}|^2 d\sigma = \int_\Omega \operatorname{div}(\boldsymbol{h}) |\nabla \boldsymbol{u}|^2 dx + \int_\Omega \boldsymbol{h} \cdot \nabla \left(|\nabla \boldsymbol{u}|^2\right) dx,$$

where we used that $\operatorname{div}(\boldsymbol{h}|\nabla \boldsymbol{u}|^2) = |\nabla \boldsymbol{u}|^2 \operatorname{div}(\boldsymbol{h}) + \boldsymbol{h} \cdot \nabla \left(|\nabla \boldsymbol{u}|^2\right)$. Notice that this operation is justified by Theorem 2.18 and Condition 5.1. It is not guaranteed that $\boldsymbol{u} \in C^2(\overline{\Omega})$. However, this can be overcome by an approximation argument. We omit the this approximation, because it would make a mess out of this already tedious calculation. Now we find the following equalities

$$\begin{aligned}\frac{\partial}{\partial x_j}\left(\frac{\partial \boldsymbol{u}_i}{\partial x_k} \frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_j}\right) = \frac{\partial^2 \boldsymbol{u}_i}{\partial x_j \partial x_k} \frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_j} + \frac{\partial \boldsymbol{u}_i}{\partial x_k} \frac{\partial^2 \overline{\boldsymbol{u}}_i}{\partial x_j^2} \\ \frac{\partial}{\partial x_j}\left(\frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_k} \frac{\partial \boldsymbol{u}_i}{\partial x_j}\right) = \frac{\partial^2 \overline{\boldsymbol{u}}_i}{\partial x_j \partial x_k} \frac{\partial \boldsymbol{u}_i}{\partial x_j} + \frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_k} \frac{\partial^2 \boldsymbol{u}_i}{\partial x_j^2}.\end{aligned}$$

Using these equalities and $\frac{\partial^2 \boldsymbol{u}_i}{\partial x_j \partial x_k} = \frac{\partial^2 \boldsymbol{u}_i}{\partial x_k \partial x_j}$, one finds

$$\begin{aligned}\frac{\partial}{\partial x_k}\left(|\nabla \boldsymbol{u}|^2\right) &= \frac{\partial}{\partial x_k}\left(\frac{\partial \boldsymbol{u}_i}{\partial x_j}\frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_j}\right)\\ &= \frac{\partial^2 \boldsymbol{u}_i}{\partial x_j \partial x_k}\frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_j} + \frac{\partial^2 \overline{\boldsymbol{u}}_i}{\partial x_j \partial x_k}\frac{\partial \boldsymbol{u}_i}{\partial x_j}\\ &= \frac{\partial}{\partial x_j}\left(\frac{\partial \boldsymbol{u}_i}{\partial x_k}\frac{\overline{\boldsymbol{u}}_i}{\partial x_j}\right) + \frac{\partial}{\partial x_j}\left(\frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_k}\frac{\boldsymbol{u}_i}{\partial x_j}\right) - \frac{\partial \boldsymbol{u}_i}{\partial x_k}\Delta \overline{\boldsymbol{u}}_i - \frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_k}\Delta \boldsymbol{u}_i.\end{aligned}$$

We now proceed by calculating the integral involving this term

$$\int_\Omega \boldsymbol{h}_k \frac{\partial}{\partial x_k}|\nabla \boldsymbol{u}|^2 dx = \int_\Omega \boldsymbol{h}_k \left(\frac{\partial}{\partial x_j}\left(\frac{\partial \boldsymbol{u}_i}{\partial x_k}\frac{\overline{\boldsymbol{u}}_i}{\partial x_j}\right) + \frac{\partial}{\partial x_j}\left(\frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_k}\frac{\boldsymbol{u}_i}{\partial x_j}\right) - \frac{\partial \boldsymbol{u}_i}{\partial x_k}\Delta \overline{\boldsymbol{u}}_i - \frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_k}\Delta \boldsymbol{u}_i\right) dx.$$

We only calculate the integral over the first and third term on the right-hand side and obtain the other terms by complex conjugation. First of all

$$\int_\Omega \boldsymbol{h}_k \frac{\partial}{\partial x_j}\left(\frac{\partial \boldsymbol{u}_i}{\partial x_k}\frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_j}\right) dx = \int_{\partial\Omega} \boldsymbol{h}_k \frac{\partial \boldsymbol{u}_i}{\partial x_k}\frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_j}\boldsymbol{n}_j d\sigma - \int_\Omega \frac{\partial \boldsymbol{h}_k}{\partial x_j}\frac{\partial \boldsymbol{u}_i}{\partial x_k}\frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_j} dx,$$

and secondly, using that $\Delta \boldsymbol{u} = \nabla\phi + \lambda \boldsymbol{u}$ and the the fact that $\boldsymbol{u}$ is divergence free,

$$\begin{aligned}\int_\Omega \frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_k}\boldsymbol{h}_k \Delta \boldsymbol{u}_i dx &= \int_\Omega \boldsymbol{h}_k \frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_k}\frac{\partial \phi}{\partial x_i} dx + \int_\Omega \boldsymbol{h}_k \phi \frac{\partial}{\partial x_k}\frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_i} dx + \int_\Omega \boldsymbol{h}_k \frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_k}\lambda \boldsymbol{u}_i dx\\ &= \int_\Omega \boldsymbol{h}_k \frac{\partial}{\partial x_i}\left(\phi \frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_k}\right) dx + \int_\Omega \boldsymbol{h}_k \frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_k}\lambda \boldsymbol{u}_i dx\\ &= \int_{\partial\Omega} \boldsymbol{h}_k \frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_k}\phi \boldsymbol{n}_i d\sigma - \int_\Omega \frac{\partial \boldsymbol{h}_k}{\partial x_i}\frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_k}\phi dx + \int_\Omega \boldsymbol{h}_k \frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_k}\lambda \boldsymbol{u}_i dx.\end{aligned}$$

By the definition of the conormal derivative

$$\int_{\partial\Omega} \boldsymbol{h}_k \frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_k}\left(\frac{\partial \boldsymbol{u}_i}{\partial x_j}\boldsymbol{n}_j\right) d\sigma + \int_{\partial\Omega} \boldsymbol{h}_k \frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_k}(\phi \boldsymbol{n}_i) d\sigma = \int_{\partial\Omega} \boldsymbol{h}_k \frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_k}\left(\frac{\partial \boldsymbol{u}}{\partial \nu}\right)_i d\sigma.$$

Now use the fact that $\boldsymbol{u}_i + \overline{\boldsymbol{u}}_i = 2\operatorname{Re}\boldsymbol{u}_i$ to complete the proof. □

The second Rellich-type identity resembles the first one, except that here the tangential derivative is used, instead of the conormal derivative.

Lemma 5.3. *Assume Condition 5.1 holds and let $\boldsymbol{h} \in C_0^1(\mathbb{R}^3, \mathbb{R}^3)$ as in Lemma 2.11. Then,*

$$\begin{aligned}\int_{\partial\Omega} \boldsymbol{h}_k \boldsymbol{n}_k |\nabla \boldsymbol{u}| d\sigma &= 2\operatorname{Re}\int_{\partial\Omega} \boldsymbol{h}_k \frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_k}\left(\boldsymbol{n}_k \frac{\partial \boldsymbol{u}_i}{\partial x_j} - \boldsymbol{n}_j \frac{\partial \boldsymbol{u}_i}{\partial x_k}\right) d\sigma\\ &\quad + 2\operatorname{Re}\int_{\partial\Omega} \boldsymbol{h}_k \overline{\phi}\left(\boldsymbol{n}_i \frac{\partial \boldsymbol{u}_i}{\partial x_k} - \boldsymbol{n}_k \frac{\partial \boldsymbol{u}_i}{\partial x_i}\right) d\sigma - \int_\Omega \operatorname{div}(\boldsymbol{h})|\nabla \boldsymbol{u}|^2 dx\\ &\quad + 2\operatorname{Re}\int_\Omega \frac{\partial \boldsymbol{h}_k}{\partial x_j}\frac{\partial \boldsymbol{u}_i}{\partial x_k}\frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_j} dx - 2\operatorname{Re}\int_\Omega \frac{\partial \boldsymbol{h}_k}{\partial x_i}\frac{\partial \boldsymbol{u}_i}{\partial x_k}\overline{\phi} dx\\ &\quad + 2\operatorname{Re}\int_\Omega \boldsymbol{h}_k \frac{\partial \boldsymbol{u}_i}{\partial x_k}\overline{\lambda}\overline{\boldsymbol{u}}_i dx.\end{aligned}$$

Proof. Let $\boldsymbol{h} \in C_0^1(\mathbb{R}^3, \mathbb{R}^3)$. Now the divergence theorem on $\boldsymbol{h}|\nabla \boldsymbol{u}|^2$ yields

$$\int_{\partial\Omega} \boldsymbol{h}_k \boldsymbol{n}_k |\nabla \boldsymbol{u}|^2 d\sigma = \int_\Omega \operatorname{div}(\boldsymbol{h}) |\nabla \boldsymbol{u}|^2 dx + \int_\Omega \boldsymbol{h} \cdot \nabla \left(|\nabla \boldsymbol{u}|^2\right) dx,$$

where we used that $\operatorname{div}(\boldsymbol{h}|\nabla \boldsymbol{u}|^2) = |\nabla \boldsymbol{u}|^2 \operatorname{div}(\boldsymbol{h}) + \boldsymbol{h} \cdot \nabla \left(|\nabla \boldsymbol{u}|^2\right)$. Notice that this operation is justified by Theorem 2.18 and Condition 5.1. It is not guaranteed that $\boldsymbol{u} \in C^2(\overline{\Omega})$. However, this can be overcome by an approximation argument. We omit the this approximation, because it would make a mess out of this already tedious calculation. Now again by the divergence theorem

$$\begin{aligned}\int_\Omega \boldsymbol{h} \cdot \nabla \left(|\nabla \boldsymbol{u}|^2\right) dx &= \int_\Omega \boldsymbol{h}_k \frac{\partial}{\partial x_k} \left(\frac{\partial \boldsymbol{u}_i}{\partial x_j} \frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_j}\right) dx \\ &= \int_{\partial\Omega} \boldsymbol{h}_k \frac{\partial \boldsymbol{u}_i}{\partial x_j} \boldsymbol{n}_k \frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_j} d\sigma - \int_\Omega \operatorname{div}(\boldsymbol{h}) |\nabla \boldsymbol{u}|^2 dx.\end{aligned}$$

Now notice the following equalities

$$\begin{aligned}\frac{\partial}{\partial x_j}\left(\frac{\partial \boldsymbol{u}_i}{\partial x_k} \frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_j}\right) &= \frac{\partial^2 \boldsymbol{u}_i}{\partial x_j \partial x_k} \frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_j} + \frac{\partial \boldsymbol{u}_i}{\partial x_k} \frac{\partial^2 \overline{\boldsymbol{u}}_i}{\partial x_j^2} \\ \frac{\partial}{\partial x_j}\left(\frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_k} \frac{\partial \boldsymbol{u}_i}{\partial x_j}\right) &= \frac{\partial^2 \overline{\boldsymbol{u}}_i}{\partial x_j \partial x_k} \frac{\partial \boldsymbol{u}_i}{\partial x_j} + \frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_k} \frac{\partial^2 \boldsymbol{u}_i}{\partial x_j^2}.\end{aligned}$$

Using these equalities and $\frac{\partial^2 \boldsymbol{u}_i}{\partial x_j \partial x_k} = \frac{\partial^2 \boldsymbol{u}_i}{\partial x_k \partial x_j}$, one finds

$$\begin{aligned}\int_\Omega \operatorname{div}(\boldsymbol{h}) |\nabla \boldsymbol{u}|^2 dx &= \int_\Omega \frac{\partial \boldsymbol{h}_k}{\partial x_k} \frac{\partial \boldsymbol{u}_i}{\partial x_j} \frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_j} dx \\ &= \int_{\partial\Omega} \boldsymbol{h}_k \frac{\partial \boldsymbol{u}_i}{\partial x_j} \frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_j} \boldsymbol{n}_k d\sigma - \int_\Omega \boldsymbol{h}_k \frac{\partial^2 \boldsymbol{u}_i}{\partial x_j \partial x_k} \frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_j} dx - \int_\Omega \boldsymbol{h}_k \frac{\partial \boldsymbol{u}_i}{\partial x_j} \frac{\partial^2 \overline{\boldsymbol{u}}_i}{\partial x_j \partial x_k} dx \\ &= \int_{\partial\Omega} \boldsymbol{h}_k \frac{\partial \boldsymbol{u}_i}{\partial x_j} \frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_j} \boldsymbol{n}_k d\sigma - \int_\Omega \boldsymbol{h}_k \frac{\partial}{\partial x_j}\left(\frac{\partial \boldsymbol{u}_i}{\partial x_k} \frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_j}\right) dx - \int_\Omega \boldsymbol{h}_k \frac{\partial}{\partial x_j}\left(\frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_k} \frac{\partial \boldsymbol{u}_i}{\partial x_j}\right) dx \\ &+ \int_\Omega \boldsymbol{h}_k \frac{\partial \boldsymbol{u}_i}{\partial x_k} \Delta \overline{\boldsymbol{u}}_i dx + \int_\Omega \boldsymbol{h}_k \frac{\partial \boldsymbol{u}_i}{\partial x_k} \Delta \boldsymbol{u}_i dx.\end{aligned}$$

Considering that

$$\int_\Omega \boldsymbol{h}_k \frac{\partial}{\partial x_j}\left(\frac{\partial \boldsymbol{u}_i}{\partial x_k} \frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_j}\right) dx = \int_{\partial\Omega} \boldsymbol{h}_k \frac{\partial \boldsymbol{u}_i}{\partial x_k} \frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_j} \boldsymbol{n}_j d\sigma - \int_\Omega \frac{\partial \boldsymbol{h}_k}{\partial x_j} \frac{\partial \boldsymbol{u}_i}{\partial x_k} \frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_j} dx,$$

and using that $\Delta \boldsymbol{u} = \nabla \phi + \lambda \boldsymbol{u}$ and the the fact that $\boldsymbol{u}$ is divergence free,

$$\begin{aligned}\int_\Omega \boldsymbol{h}_k \frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_k} \Delta \boldsymbol{u}_i dx &= \int_\Omega \boldsymbol{h}_k \frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_k} \frac{\partial \phi}{\partial x_i} dx + \int_\Omega \boldsymbol{h}_k \phi \frac{\partial}{\partial x_k} \frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_i} dx + \int_\Omega \boldsymbol{h}_k \frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_k} \lambda \boldsymbol{u}_i dx \\ &= \int_\Omega \boldsymbol{h}_k \frac{\partial}{\partial x_i}\left(\phi \frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_k}\right) dx + \int_\Omega \boldsymbol{h}_k \frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_k} \lambda \boldsymbol{u}_i dx \\ &= \int_{\partial\Omega} \boldsymbol{h}_k \frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_k} \phi \boldsymbol{n}_i d\sigma - \int_\Omega \frac{\partial \boldsymbol{h}_k}{\partial x_i} \frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_k} \phi dx + \int_\Omega \boldsymbol{h}_k \frac{\partial \overline{\boldsymbol{u}}_i}{\partial x_k} \lambda \boldsymbol{u}_i dx.\end{aligned}$$

Now use the fact that $\boldsymbol{u}_i + \overline{\boldsymbol{u}}_i = 2 \operatorname{Re} \boldsymbol{u}_i$ to complete the proof. □

Lemma 5.4. *Assume Condition 5.1 holds. Then there exists a $C > 0$ such that,*

$$\int_\Omega |\nabla \boldsymbol{u}|^2 dx + |\lambda| \int_\Omega |\boldsymbol{u}|^2 dx \leq C \left\| \frac{\partial \boldsymbol{u}}{\partial \nu} \right\|_{L^2(\partial\Omega)} \|\boldsymbol{u}\|_{L^2(\partial\Omega)}, \tag{5.1}$$

where C depends only on θ.

Proof. $(\boldsymbol{u}, \phi)$ is a solution of the homogeneous Stokes System (1.11). Now the homogeneous Stokes System is multiplied by $\overline{\boldsymbol{u}}^T$ (where $\overline{\boldsymbol{u}}$ denotes the complex conjugate of $\boldsymbol{u}$). Using integration by parts one finds for the first term

$$\begin{aligned}\int_\Omega \overline{\boldsymbol{u}} \cdot \Delta \boldsymbol{u} dx &= \int_{\partial\Omega} \overline{\boldsymbol{u}_k} \left(\nabla \boldsymbol{u}_k \cdot \boldsymbol{n}\right) d\sigma - \int_\Omega \nabla \overline{\boldsymbol{u}} \cdot \nabla \boldsymbol{u} dx \\ &= \int_{\partial\Omega} \overline{\boldsymbol{u}} \cdot \frac{\partial \boldsymbol{u}}{\partial n} - \int_\Omega |\nabla \boldsymbol{u}|^2 dx.\end{aligned}$$

For the second term the divergence theorem is used to find

$$\begin{aligned}\int_\Omega \overline{\boldsymbol{u}} \cdot \nabla \phi dx &= \int_\Omega \nabla \cdot (\overline{\boldsymbol{u}} \phi) dx - \int_\Omega \phi (\nabla \cdot \overline{\boldsymbol{u}}) dx \\ &= \int_{\partial\Omega} \overline{\boldsymbol{u}} \cdot (\phi \boldsymbol{n}) d\sigma - \int_\Omega \phi (\nabla \cdot \overline{\boldsymbol{u}}) dx.\end{aligned}$$

Notice that this operation is justified by Theorem 2.18 and Condition 5.1. The last term is rather trivial and yields

$$\int_\Omega \lambda \overline{\boldsymbol{u}} \cdot \boldsymbol{u} dx = \lambda \int_\Omega |\boldsymbol{u}|^2 dx.$$

Since $\boldsymbol{u}$ is divergence free, $\overline{\boldsymbol{u}}$ is also divergence free. Combining the Stokes equation together with the definition of the conormal derivative ($\frac{\partial \boldsymbol{u}}{\partial \nu} = \frac{\partial \boldsymbol{u}}{\partial n} - \phi \boldsymbol{n}$) yields

$$\int_\Omega |\nabla \boldsymbol{u}|^2 dx + \lambda \int_\Omega |\boldsymbol{u}|^2 dx = \int_{\partial\Omega} \frac{\partial \boldsymbol{u}}{\partial \nu} \cdot \overline{\boldsymbol{u}} d\sigma.$$

Now take the real and imaginary part from this equality to find

$$\begin{cases} \displaystyle\int_\Omega |\nabla \boldsymbol{u}|^2 dx + \mathrm{Re}(\lambda) \int_\Omega |\boldsymbol{u}|^2 dx \leq \left| \int_{\partial\Omega} \frac{\partial \boldsymbol{u}}{\partial \nu} \cdot \overline{\boldsymbol{u}} d\sigma \right| \\ \displaystyle|\mathrm{Im}(\lambda)| \int_\Omega |\boldsymbol{u}|^2 dx \leq \left| \int_{\partial\Omega} \frac{\partial \boldsymbol{u}}{\partial \nu} \cdot \overline{\boldsymbol{u}} d\sigma \right|, \end{cases}$$

and hence we find for all $\alpha > 0$

$$\int_\Omega |\nabla \boldsymbol{u}|^2 dx + (\mathrm{Re}(\lambda) + \alpha |\mathrm{Im}(\lambda)|) \int_\Omega |\boldsymbol{u}|^2 dx \leq (1 + \alpha) \left| \int_{\partial\Omega} \frac{\partial \boldsymbol{u}}{\partial \nu} \cdot \overline{\boldsymbol{u}} d\sigma \right|.$$

Now choose $\alpha > 0$ and $c > 0$, both depending only on θ, such that $\mathrm{Re}(\lambda) + \alpha |\mathrm{Im}(\lambda)| \geq c|\lambda|$. Observe that this is possible for all $\lambda \in \Sigma_\theta$. Using this we find that

$$\int_\Omega |\nabla \boldsymbol{u}|^2 dx + |\lambda| \int_\Omega |\boldsymbol{u}|^2 dx \leq C \left| \int_{\partial\Omega} \frac{\partial \boldsymbol{u}}{\partial \nu} \cdot \overline{\boldsymbol{u}} d\sigma \right|. \tag{5.2}$$

Now by the Hölder inequality (B.2) we find that

$$\int_{\partial\Omega} \left| \frac{\partial \boldsymbol{u}}{\partial \nu} \cdot \overline{\boldsymbol{u}} \right| d\sigma \leq \left\| \frac{\partial \boldsymbol{u}}{\partial \nu} \right\|_{L^2(\partial\Omega)} \|\boldsymbol{u}\|_{L^2(\partial\Omega)}. \tag{5.3}$$

combining (5.2) and (5.3) completes the proof. □

We state the following lemma about the pressure part of the solution to the Stokes System. This lemma holds for harmonic functions, and hence is useful in analysing the pressure term of the Stokes System. For the proof we refer to [9].

Lemma 5.5. *Let Condition 5.1 hold. Then there exists* $C > 0$ *such that*

$$\int_\Omega |\phi|^2 dx \leq C\|(\phi)^*\|^2_{L^2(\partial\Omega)} \leq C\|\phi\|^2_{L^2(\partial\Omega)}. \tag{5.4}$$

This lemma helps in the proof of the following lemma, which bounds the gradient of $\boldsymbol{u}$.

Lemma 5.6. *Assume Condition 5.1 holds. Then,*

$$\|\nabla \boldsymbol{u}\|_{L^2(\partial\Omega)} \leq C_\varepsilon \left\|\frac{\partial \boldsymbol{u}}{\partial \nu}\right\|_{L^2(\partial\Omega)} + \varepsilon\left(\|\nabla \boldsymbol{u}\|_{L^2(\partial\Omega)} + \|\phi\|_{L^2(\partial\Omega)} + \left\||\lambda|^{\frac{1}{2}}\boldsymbol{u}\right\|_{L^2(\partial\Omega)}\right)$$

and

$$\|\nabla \boldsymbol{u}\|_{L^2(\partial\Omega)} \leq C_\varepsilon \left(\|\nabla_{tan} \boldsymbol{u}\|_{L^2(\partial\Omega)} + \left\||\lambda|^{1/2}\boldsymbol{u}\right\|_{L^2(\partial\Omega)}\right) + \varepsilon\left(\|\nabla \boldsymbol{u}\|_{L^2(\partial\Omega)} + \|\phi\|_{L^2(\partial\Omega)}\right),$$

for all $\varepsilon \in (0,1)$*, where* $C_\varepsilon > 0$ *depends on* $\theta, \tau, \varepsilon$ *and the Lipschitz character of* Ω*.*

Proof. Start by choosing a vector field $\boldsymbol{h} \in C^1_0(\mathbb{R}^3, \mathbb{R}^3)$ such that $\boldsymbol{h}_k \boldsymbol{n}_k \geq c > 0$ on $\partial\Omega$. Since $\boldsymbol{h} \in C^1_0(\mathbb{R}^3, \mathbb{R}^3)$ there exists a constant $M > 0$ such that $|\boldsymbol{h}| \leq M$ and $|\nabla \boldsymbol{h}| \leq M$. We can now use Lemma 5.2 to find

$$\|\nabla \boldsymbol{u}\|^2_{L^2(\partial\Omega)} \leq C\left(\int_{\partial\Omega} |\nabla \boldsymbol{u}|\left|\frac{\partial \boldsymbol{u}}{\partial \nu}\right| d\sigma + \int_\Omega |\nabla \boldsymbol{u}|^2 dx + \int_\Omega |\nabla \boldsymbol{u}||\phi| dx + |\lambda|\int_\Omega |\nabla \boldsymbol{u}||\boldsymbol{u}| dx\right).$$

We will estimate the integrals on the right-hand side against boundary integrals. Since $\nabla \boldsymbol{u} \in L^2(\partial\Omega)$ and $\frac{\partial \boldsymbol{u}}{\partial \nu} \in L^2(\partial\Omega)$ we can use the Hölder inequality (B.2) to find

$$\int_{\partial\Omega} |\nabla \boldsymbol{u}|\left|\frac{\partial \boldsymbol{u}}{\partial \nu}\right| d\sigma \leq \|\nabla \boldsymbol{u}\|_{L^2(\partial\Omega)}\left\|\frac{\partial \boldsymbol{u}}{\partial \nu}\right\|_{L^2(\partial\Omega)}.$$

The next term can be easily estimated using Lemma 5.4 and we find

$$\int_\Omega |\nabla \boldsymbol{u}|^2 dx \leq C\left\|\frac{\partial \boldsymbol{u}}{\partial \nu}\right\|_{L^2(\partial\Omega)}\|\boldsymbol{u}\|_{L^2(\partial\Omega)}.$$

For the next term we find using Hölder

$$\int_\Omega |\nabla \boldsymbol{u}||\phi| dx \leq \left(\int_\Omega |\nabla \boldsymbol{u}|^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega |\phi|^2 dx\right)^{\frac{1}{2}}.$$

The first factor can again be estimated using Lemma 5.4 and the second factor we use Lemma 5.5. This leads to

$$\int_\Omega |\nabla \boldsymbol{u}||\phi| dx \leq C\|\nabla \boldsymbol{u}\|^{\frac{1}{2}}_{L^2(\partial\Omega)}\left\|\frac{\partial \boldsymbol{u}}{\partial \nu}\right\|^{\frac{1}{2}}_{L^2(\partial\Omega)}\|\phi\|_{L^2(\partial\Omega)}.$$

Now we use the Cauchy inequality (B.1) and Lemma 5.4 to find

$$\begin{aligned}|\lambda|\int_\Omega |\nabla \boldsymbol{u}||\boldsymbol{u}|dx &\leq C|\lambda|^{\frac{1}{2}}\left(\int_\Omega |\nabla \boldsymbol{u}|^2 + |\lambda||\boldsymbol{u}|^2 dx\right)\\ &\leq C|\lambda|^{\frac{1}{2}}\|\boldsymbol{u}\|_{L^2(\partial\Omega)}\left\|\frac{\partial \boldsymbol{u}}{\partial \nu}\right\|_{L^2(\partial\Omega)}\\ &= C\left\||\lambda|^{\frac{1}{2}}\boldsymbol{u}\right\|_{L^2(\partial\Omega)}\left\|\frac{\partial \boldsymbol{u}}{\partial \nu}\right\|_{L^2(\partial\Omega)}.\end{aligned}$$

Now we put it all together

$$\begin{aligned}\|\nabla \boldsymbol{u}\|^2_{L^2(\partial\Omega)} &\leq C\left(\int_{\partial\Omega}|\nabla \boldsymbol{u}|\left|\frac{\partial \boldsymbol{u}}{\partial \nu}\right|d\sigma + \int_\Omega |\nabla \boldsymbol{u}|^2 dx + \int_\Omega |\nabla \boldsymbol{u}||\phi|dx + |\lambda|\int_\Omega |\nabla \boldsymbol{u}||\boldsymbol{u}|dx\right)\\ &\leq C\left\|\frac{\partial \boldsymbol{u}}{\partial \nu}\right\|_{L^2(\partial\Omega)}\left[\|\nabla \boldsymbol{u}\|_{L^2(\partial\Omega)} + \|\phi\|_{L^2(\partial\Omega)} + \left\||\lambda|^{\frac{1}{2}}\boldsymbol{u}\right\|_{L^2(\partial\Omega)}\right].\end{aligned}$$

The result follows from applying the Cauchy inequality with epsilon (Lemma B.1). For the second part of the statement we use Lemma 5.3 to find

$$\begin{aligned}\|\nabla \boldsymbol{u}\|^2_{L^2(\partial\Omega)} \leq C\|\nabla_{\tan}\boldsymbol{u}\|_{L^2(\partial\Omega)}\left[\|\nabla \boldsymbol{u}\|_{L^2(\partial\Omega)} + \|\phi\|_{L^2(\partial\Omega)}\right] + C\int_\Omega |\nabla \boldsymbol{u}|^2 dx\\ + C\int_\Omega |\nabla \boldsymbol{u}||\phi|dx + C|\lambda|\int_\Omega |\nabla \boldsymbol{u}||\boldsymbol{u}|dx.\end{aligned}$$

The result now follows using the same procedure as the first part of the proof. □

Lemma 5.7. *Assume Condition 5.1 holds. Then there exists $C > 0$ such that,*

$$\|\Delta \boldsymbol{u}\cdot \boldsymbol{n}\|_{H^{-1}(\partial\Omega)} \leq C\|\nabla \boldsymbol{u}\|_{L^2(\partial\Omega)}.$$

Proof. First of all we realise that

$$\Delta \boldsymbol{u}\cdot \boldsymbol{n} = \boldsymbol{n}_i\frac{\partial^2 \boldsymbol{u}_i}{\partial x_j^2} = \left(\boldsymbol{n}_i\frac{\partial}{\partial x_j} - \boldsymbol{n}_j\frac{\partial}{\partial x_i}\right)\frac{\partial \boldsymbol{u}_i}{\partial x_j}.$$

Where we use that the divergence of $\boldsymbol{u}$ is zero in $\overline{\Omega}$. Notice that $\left(\boldsymbol{n}_i\frac{\partial}{\partial x_j} - \boldsymbol{n}_j\frac{\partial}{\partial x_i}\right)$ is a tangential derivative and hence well defined on $\partial\Omega$. By the definition of the dual norm we find

$$\begin{aligned}\|\Delta \boldsymbol{u}\cdot \boldsymbol{n}\|_{H^{-1}(\partial\Omega)} &= \sup_{\substack{v\in H^1(\partial\Omega)\\ \|\boldsymbol{v}\|=1}}\langle \Delta \boldsymbol{u}\cdot \boldsymbol{n}, \boldsymbol{v}\rangle\\ &= \sup_{\substack{v\in H^1(\partial\Omega)\\ \|\boldsymbol{v}\|=1}}\left\langle \nabla \boldsymbol{u}, -\left(\boldsymbol{n}_i\frac{\partial}{\partial x_j} - \boldsymbol{n}_j\frac{\partial}{\partial x_i}\right)\boldsymbol{v}_i\right\rangle\\ &\leq \sup_{\substack{v\in H^1(\partial\Omega)\\ \|\boldsymbol{v}\|=1}}\|\nabla \boldsymbol{u}\|_{L^2(\partial\Omega)}\|\boldsymbol{v}\|_{H^1(\partial\Omega)}\\ &\leq C\|\nabla \boldsymbol{u}\|_{L^2(\partial\Omega)}.\end{aligned}$$

This completes the proof. □

Lemma 5.8. *Assume Condition 5.1 holds. Then there exist $C_1, C_2 > 0$ such that,*

$$C_1\left\|\phi - \fint_{\partial\Omega}\phi\right\|_{L^2(\partial\Omega)} \leq \|\nabla\phi\cdot\boldsymbol{n}\|_{H^{-1}(\partial\Omega)} \leq C_2\|\phi\|_{L^2(\partial\Omega)}.$$

Proof. This proof relies on L^2 estimates for the Laplace equation in Lipschitz domains. Pick $g \in L^2(\partial\Omega)$ such that $\fint_{\partial\Omega} g d\sigma = 0$. Now let ψ be such that

$$\begin{cases} \Delta\psi = 0 \quad \text{in } \Omega \\ \dfrac{\partial\psi}{\partial n} = g \quad \text{on } \partial\Omega \\ (\nabla\psi)^* \in L^2(\partial\Omega). \end{cases} \tag{5.5}$$

Since $\Delta\phi = 0$ we can use the Green's identity (Theorem B.7) to find

$$\int_{\partial\Omega}\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n} = \int_\Omega \psi\Delta\phi - \phi\Delta\psi dx = 0. \tag{5.6}$$

Together with the duality of $H^1(\partial\Omega)$ and $H^{-1}(\partial\Omega)$ this yields

$$\left|\int_{\partial\Omega}\phi g d\sigma\right| = \left|\int_{\partial\Omega}\frac{\partial\phi}{\partial n}\psi d\sigma\right| \leq \left\|\frac{\partial\phi}{\partial n}\right\|_{H^{-1}(\partial\Omega)}\|\psi\|_{H^1(\partial\Omega)}.$$

For the Neumann problem (5.5) we know the estimate $\|\psi\|_{H^1(\partial\Omega)} \leq C\|g\|_{L^2(\partial\Omega)}$ [26]. Now we can use the Hahn-Banach Theorem to find that

$$\begin{aligned}
\|\phi\|_{L^2(\partial\Omega)} &= \sup_{\|g\|_{L^2(\partial\Omega)}=1}\left|\int_{\partial\Omega}\phi g d\sigma\right| \\
&\leq \sup_{\|g\|_{L^2(\partial\Omega)}=1} C\left\|\frac{\partial\phi}{\partial n}\right\|_{H^{-1}(\partial\Omega)}\|\psi\|_{H^1(\partial\Omega)} \\
&\leq \sup_{\|g\|_{L^2(\partial\Omega)}=1} C\left\|\frac{\partial\phi}{\partial n}\right\|_{H^{-1}(\partial\Omega)}\|g\|_{L^2(\partial\Omega)} = C\left\|\frac{\partial\phi}{\partial n}\right\|_{H^{-1}(\partial\Omega)}.
\end{aligned}$$

This proves the first part of inequality (5.8). For the second part we let $f \in L^2(\partial\Omega)$ and consider the system

$$\begin{cases} \Delta\psi = 0 \quad \text{in } \Omega \\ \psi = f \quad \text{on } \partial\Omega \\ (\nabla\psi)^* \in L^2(\partial\Omega). \end{cases} \tag{5.7}$$

Now we can again use (5.6) to find

$$\left|\int_{\partial\Omega}\frac{\partial\phi}{\partial n}f d\sigma\right| = \left|\int_{\partial\Omega}\phi\frac{\partial\psi}{\partial n}d\sigma\right| \leq \|\phi\|_{L^2(\partial\Omega)}\|\nabla\psi\|_{L^2(\partial\Omega)}.$$

For the system (5.7) we know the estimate $\|\nabla\psi\|_{L^2(\partial\Omega)} \leq C\|f\|_{H^1(\partial\Omega)}$ [25]. Once again by Hahn-Banach we find

$$\begin{aligned}
\left\|\frac{\partial\phi}{\partial n}\right\|_{H^{-1}(\partial\Omega)} &= \sup_{\|f\|_{H^1(\partial\Omega)}=1}\left|\int_{\partial\Omega}\frac{\partial\phi}{\partial n}f d\sigma\right| \\
&\leq \sup_{\|f\|_{H^1(\partial\Omega)}=1}\|\phi\|_{L^2(\partial\Omega)}\|\nabla\psi\|_{L^2(\partial\Omega)} \\
&\leq \sup_{\|f\|_{H^1(\partial\Omega)}=1} C\|\phi\|_{L^2(\partial\Omega)}\|f\|_{H^1(\partial\Omega)} = C\|\phi\|_{L^2(\partial\Omega)}.
\end{aligned}$$

This completes the proof. □

Lemma 5.9. *Assume Condition 5.1 holds. Then,*

$$\left\| \phi - \fint_{\partial\Omega} \phi d\sigma \right\|_{L^2(\partial\Omega)} \leq C \left(\|\nabla \boldsymbol{u}\|_{L^2(\partial\Omega)} + |\lambda| \|\boldsymbol{u}\cdot\boldsymbol{n}\|_{H^{-1}(\partial\Omega)} \right)$$

and

$$|\lambda| \|\boldsymbol{u}\cdot\boldsymbol{n}\|_{H^1(\partial\Omega)} \leq C \left(\|\phi\|_{L^2(\partial\Omega)} + \|\nabla \boldsymbol{u}\|_{L^2(\partial\Omega)} \right),$$

where $C > 0$ only depends the Lipschitz character of Ω.

Proof. We start by approximating Ω by a sequence of smooth domains. Therefore we may assume that $(\boldsymbol{u}, \phi)$ satisfies the Stokes System (1.11) on some D_k with $\overline{\Omega} \subset D_k$. Hence we can assume $\Delta \boldsymbol{u} = \nabla\phi + \lambda \boldsymbol{u}$ on $\partial\Omega$. Taking the dot product with the normal and using the triangle inequality we find

$$\|\nabla\phi\cdot\boldsymbol{n}\|_{H^{-1}(\partial\Omega)} \leq \|\Delta\boldsymbol{u}\cdot\boldsymbol{n}\|_{H^{-1}(\partial\Omega)} + |\lambda|\|\boldsymbol{u}\cdot\boldsymbol{n}\|_{H^{-1}(\partial\Omega)} \tag{5.8}$$

$$|\lambda|\|\boldsymbol{u}\cdot\boldsymbol{n}\|_{H^{-1}(\partial\Omega)} \leq \|\Delta\boldsymbol{u}\cdot\boldsymbol{n}\|_{H^{-1}(\partial\Omega)} + \|\nabla\phi\cdot\boldsymbol{n}\|_{H^{-1}(\partial\Omega)}. \tag{5.9}$$

Using this together with Lemma 5.8 and Lemma 5.7 we find

$$\begin{aligned} \left\| \phi - \fint_{\partial\Omega} \phi d\sigma \right\|_{L^2(\partial\Omega)} &\leq C \left\| \frac{\partial\phi}{\partial n} \right\|_{H^{-1}(\partial\Omega)} \\ &\leq C \left(\|\Delta\boldsymbol{u}\cdot\boldsymbol{n}\|_{H^{-1}(\partial\Omega)} + |\lambda|\|\boldsymbol{u}\cdot\boldsymbol{n}\|_{H^{-1}(\partial\Omega)} \right) \\ &\leq C \left(\|\nabla\boldsymbol{u}\|_{L^2(\partial\Omega)} + |\lambda|\|\boldsymbol{u}\cdot\boldsymbol{n}\|_{H^{-1}(\partial\Omega)} \right). \end{aligned}$$

Also using (5.8), Lemma 5.8 and Lemma 5.7 we find

$$|\lambda|\|\boldsymbol{u}\cdot\boldsymbol{n}\|_{H^1(\partial\Omega)} \leq C\|\phi\|_{L^2(\partial\Omega)} + C\|\nabla\boldsymbol{u}\|_{L^2(\partial\Omega)}.$$

This completes the proof. ∎

Lemma 5.10. *Assume Condition 5.1 holds. Then there exists $C > 0$ such that,*

$$\left\| |\lambda|^{\frac{1}{2}} \boldsymbol{u} \right\|_{L^2(\partial\Omega)} \leq C \left\| \frac{\partial\boldsymbol{u}}{\partial\nu} \right\|_{L^2(\partial\Omega)}.$$

Proof. First let $\boldsymbol{h} \in C_c^1(\mathbb{R}^3, \mathbb{R}^3)$ a vector field such that $\boldsymbol{h}_k \boldsymbol{n}_k \geq c > 0$ on $\partial\Omega$. Using the divergence theorem we find

$$\begin{aligned} \int_{\partial\Omega} \boldsymbol{h}_k \boldsymbol{n}_k |\boldsymbol{u}|^2 d\sigma &= \int_\Omega \operatorname{div}(\boldsymbol{h})|\boldsymbol{u}|^2 dx + \int_\Omega \boldsymbol{h}_k \frac{\partial\overline{\boldsymbol{u}}_i}{\partial x_k} \boldsymbol{u}_i dx + \int_\Omega \boldsymbol{h}_k \frac{\partial\boldsymbol{u}_i}{\partial x_k} \overline{\boldsymbol{u}}_i dx \\ &= \int_\Omega \operatorname{div}(\boldsymbol{h})|\boldsymbol{u}|^2 dx + 2\operatorname{Re}\int_\Omega \boldsymbol{h}_k \frac{\partial\boldsymbol{u}_i}{\partial x_k} \overline{\boldsymbol{u}}_i dx. \end{aligned}$$

Therefore

$$\|\boldsymbol{u}\|_{L^2(\partial\Omega)}^2 \leq C\int_\Omega |\boldsymbol{u}|^2 dx + C\int_\Omega |\boldsymbol{u}||\nabla\boldsymbol{u}| dx.$$

We now use the Cauchy inequality (Lemma B.1) and the fact that $|\lambda| \geq \tau$ to find

$$\begin{aligned}
|\lambda|\|\boldsymbol{u}\|^2_{L^2(\partial\Omega)} &\leq C|\lambda| \int_\Omega |\boldsymbol{u}|^2 dx + C|\lambda| \int_\Omega |\boldsymbol{u}||\nabla \boldsymbol{u}| dx \\
&\leq C|\lambda|^{\frac{1}{2}} \int_\Omega |\lambda||\boldsymbol{u}|^2 + C|\lambda|^{\frac{1}{2}} \left(\int_\Omega |\nabla \boldsymbol{u}|^2 + |\lambda||\boldsymbol{u}|^2 dx \right) \\
&\leq C|\lambda|^{\frac{1}{2}} \|\boldsymbol{u}\|_{L^2(\partial\Omega)} \left\| \frac{\partial \boldsymbol{u}}{\partial \nu} \right\|_{L^2(\partial\Omega)} \\
&= C \left\| |\lambda|^{\frac{1}{2}} \boldsymbol{u} \right\|_{L^2(\partial\Omega)} \left\| \frac{\partial \boldsymbol{u}}{\partial \nu} \right\|_{L^2(\partial\Omega)}.
\end{aligned}$$

Where in the third inequality we used Lemma 5.4. We can now divide by $\left\| |\lambda|^{\frac{1}{2}} \boldsymbol{u} \right\|_{L^2(\partial\Omega)}$ to complete the proof. □

We are now able to proof the main theorem of this section. This theorem will be important to show the invertibility of the layer potentials.

Theorem 5.11. *Assume Condition 5.1 holds. Then,*

$$\|\nabla \boldsymbol{u}\|_{L^2(\partial\Omega)} + \left\| \phi - \fint_{\partial\Omega} \phi d\sigma \right\|_{L^2(\partial\Omega)} \leq C \left(\|\nabla_{\tan} \boldsymbol{u}\|_{L^2(\partial\Omega)} + |\lambda|^{\frac{1}{2}} \|\boldsymbol{u}\|_{L^2(\partial\Omega)} + |\lambda| \|\boldsymbol{u} \cdot \boldsymbol{n}\|_{H^{-1}(\partial\Omega)} \right)$$

and

$$\|\nabla \boldsymbol{u}\|_{L^2(\partial\Omega)} + |\lambda|^{\frac{1}{2}} \|\boldsymbol{u}\|_{L^2(\partial\Omega)} + |\lambda| \|\boldsymbol{u} \cdot \boldsymbol{n}\|_{H^{-1}(\partial\Omega)} + \|\phi\|_{L^2(\partial\Omega)} \leq C \left\| \frac{\partial \boldsymbol{u}}{\partial \nu} \right\|_{L^2(\partial\Omega)},$$

where $C > 0$ depends only on τ, θ and the Lipschitz character of Ω.

Proof. To prove the first inequality we assume without loss of generality that $\fint_{\partial\Omega} \phi d\sigma = 0$. This can be done by subtracting a constant from ϕ. We now consecutively apply Lemma 5.6 and Lemma 5.9 and find

$$\begin{aligned}
\|\nabla \boldsymbol{u}\|_{L^2(\partial\Omega)} + \|\phi\|_{L^2(\partial\Omega)} &\leq C \left(\|\nabla \boldsymbol{u}\|_{L^2(\partial\Omega)} + |\lambda| \|\boldsymbol{u} \cdot \boldsymbol{n}\|_{H^1(\partial\Omega)} \right) \\
&\leq C_\varepsilon \left(\|\nabla_{\tan} \boldsymbol{u}\|_{L^2(\partial\Omega)} + |\lambda|^{\frac{1}{2}} \|\boldsymbol{u}\|_{L^2(\partial\Omega)} + |\lambda| \|\boldsymbol{u} \cdot \boldsymbol{n}\|_{H^1(\partial\Omega)} \right) \\
&\quad + C\varepsilon \left(\|\nabla \boldsymbol{u}\|_{L^2(\partial\Omega)} + \|\phi\|_{L^2(\partial\Omega)} \right).
\end{aligned}$$

We can now pick ε sufficiently small (e.g. $C\varepsilon = \frac{1}{2}$) and rewrite the inequality to find

$$\|\nabla \boldsymbol{u}\|_{L^2(\partial\Omega)} + \left\| \phi - \fint_{\partial\Omega} \phi d\sigma \right\|_{L^2(\partial\Omega)} \leq C \left(\|\nabla_{\tan} \boldsymbol{u}\|_{L^2(\partial\Omega)} + |\lambda|^{\frac{1}{2}} \|\boldsymbol{u}\|_{L^2(\partial\Omega)} + |\lambda| \|\boldsymbol{u} \cdot \boldsymbol{n}\|_{H^1(\partial\Omega)} \right).$$

For the second inequality we note that by Lemma 5.9

$$\begin{aligned}
\|\nabla \boldsymbol{u}\|_{L^2(\partial\Omega)} + \|\phi\|_{L^2(\partial\Omega)} + |\lambda| \|\boldsymbol{u} \cdot \boldsymbol{n}\|_{H^{-1}(\partial\Omega)} &\leq C \left(\|\nabla \boldsymbol{u}\|_{L^2(\partial\Omega)} + \|\phi\|_{L^2(\partial\Omega)} \right) \\
&\leq C \left(\left\| \frac{\partial \boldsymbol{u}}{\partial \nu} \right\| + \|\nabla \boldsymbol{u}\|_{L^2(\partial\Omega)} \right).
\end{aligned}$$

Now we use Lemma 5.6

$$\|\nabla \boldsymbol{u}\|_{L^2(\partial\Omega)} + \|\phi\|_{L^2(\partial\Omega)} + |\lambda| \|\boldsymbol{u}\cdot \boldsymbol{n}\|_{H^{-1}(\partial\Omega)} \leq CC_\varepsilon \left\| \frac{\partial \boldsymbol{u}}{\partial \nu} \right\|_{L^2(\partial\Omega)} + C\varepsilon \left(\|\nabla \boldsymbol{u}\|_{L^2(\partial\Omega)} + \|\phi\|_{L^2(\partial\Omega)} + \left\| |\lambda|^{\frac{1}{2}} \boldsymbol{u} \right\|_{L^2(\partial\Omega)} \right).$$

For a suitable choice of epsilon (e.g. $C\varepsilon = \frac{1}{2}$) the inequality can be rewritten to

$$\|\nabla \boldsymbol{u}\|_{L^2(\partial\Omega)} + \|\phi\|_{L^2(\partial\Omega)} + |\lambda| \|\boldsymbol{u}\cdot \boldsymbol{n}\|_{H^{-1}(\partial\Omega)} \leq C \left\| \frac{\partial \boldsymbol{u}}{\partial \nu} \right\|_{L^2(\partial\Omega)} + C|\lambda|^{\frac{1}{2}} \|\boldsymbol{u}\|_{L^2(\partial\Omega)}.$$

Now using Lemma 5.10 we find

$$\begin{aligned} \|\nabla \boldsymbol{u}\|_{L^2(\partial\Omega)} + \|\phi\|_{L^2(\partial\Omega)} + |\lambda| \|\boldsymbol{u}\cdot \boldsymbol{n}\|_{H^{-1}(\partial\Omega)} + |\lambda|^{\frac{1}{2}} \|\boldsymbol{u}\|_{L^2(\partial\Omega)} &\leq C \left\| \frac{\partial \boldsymbol{u}}{\partial \nu} \right\|_{L^2(\partial\Omega)} + C|\lambda|^{\frac{1}{2}} \|\boldsymbol{u}\|_{L^2(\partial\Omega)} \\ &\leq C \left\| \frac{\partial \boldsymbol{u}}{\partial \nu} \right\|_{L^2(\partial\Omega)}. \end{aligned}$$

This completes the proof. □

Notice that this kind of estimate also holds for the exterior domain if we assume some decay at infinity. Notice that the layer potentials defined in the previous section indeed satisfy the decay condition.

Theorem 5.12. *Assume Condition 5.1 holds. Further assume that as* $|x| \to \infty$, $|\phi(x)| + |\nabla \boldsymbol{u}(x)| = O(|x|^{-1})$. *Then,*

$$\|\nabla \boldsymbol{u}\|_{L^2(\partial\Omega)} + \|\phi\|_{L^2(\partial\Omega)} \leq C \left(\|\nabla_{\tan} \boldsymbol{u}\|_{L^2(\partial\Omega)} + |\lambda|^{\frac{1}{2}} \|\boldsymbol{u}\|_{L^2(\partial\Omega)} + |\lambda| \|\boldsymbol{u}\cdot \boldsymbol{n}\|_{H^{-1}(\partial\Omega)} \right)$$

and

$$\|\nabla \boldsymbol{u}\|_{L^2(\partial\Omega)} + |\lambda|^{\frac{1}{2}} \|\boldsymbol{u}\|_{L^2(\partial\Omega)} + |\lambda| \|\boldsymbol{u}\cdot \boldsymbol{n}\|_{H^{-1}(\partial\Omega)} + \|\phi\|_{L^2(\partial\Omega)} \leq C \left\| \frac{\partial \boldsymbol{u}}{\partial \nu} \right\|_{L^2(\partial\Omega)},$$

where $C > 0$ *depends only on* τ, θ *and the Lipschitz character of* Ω.

Proof. Similar to Theorem 5.11 □

5.2 Solvability of Neumann and Dirichlet problem

In this section we are going to invert the layer potentials and estimate the norm of the solution in the domain against the norm of the solution on the boundary. It turns out to be the case that the layer potentials are Fredholm operators, for which there exists some nice theory. The following results can be found for example in section 11 of [7].

Definition 5.13 (Fredholm operator)**.** *Let* X, Y *be Banach spaces and* $T : X \to Y$ *a bounded linear operator. Then* T *is a Fredholm operator if*

(i) $\dim \ker T < \infty$

(ii) $\dim \operatorname{Coker} T < \infty$.

The index of a Fredholm operator is given by

$$\operatorname{Ind} T = \dim\ker T - \dim\operatorname{Coker} T.$$

Theorem 5.14. *Let X, Y be Banach spaces, $T : X \to Y$ a Fredholm operator and $K : X \to Y$ a compact operator. Then $T + K$ is a Fredholm operator and* $\operatorname{Ind}(T + K) = \operatorname{Ind}(T)$.

First of all we make the observation that the following operator is compact.

Lemma 5.15. *Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz domain and $\lambda \in \Sigma_\theta$. The operator $\mathcal{K}_\lambda - \mathcal{K}_0$ is compact on $L^2(\partial\Omega; \mathbb{C}^3)$.*

Now we can show that the layer potentials associated to the Neumann and Dirichlet problem are invertible on L^2, independent of λ.

Lemma 5.16. *Let $\lambda \in \Sigma_\theta$ and $|\lambda| \geq \tau$, where $\tau \in (0, 1)$. Suppose that $|\partial\Omega| = 1$. Then $\frac{1}{2}I + \mathcal{K}_\lambda$ is an isomorphism on $L^2(\partial\Omega; \mathbb{C}^3)$ and*

$$\|\boldsymbol{f}\|_{L^2(\partial\Omega)} \leq C \left\| \left(\frac{1}{2}I + \mathcal{K}_\lambda\right) \boldsymbol{f} \right\|_{L^2(\partial\Omega)} \tag{5.10}$$

for all $\boldsymbol{f} \in L^2(\partial\Omega; \mathbb{C}^3)$, where $C > 0$ depends only on θ, τ and the Lipschitz character of Ω.

Proof. Let $\boldsymbol{f} \in L^2(\partial\Omega; \mathbb{C}^3)$ and let $(\boldsymbol{u}, \phi) = \mathcal{S}_\lambda(\boldsymbol{f})$ denote the single layer potential. We know by Lemma 4.2 that the single layer potential satisfies the Stokes System in $\mathbb{R}^3 \setminus \partial\Omega$, $(\nabla \boldsymbol{u})^* \in L^2(\partial\Omega)$ and $(\phi)^* \in \partial\Omega$. Furthermore $\nabla \boldsymbol{u}$ and ϕ have nontangential limits almost everywhere on $\partial\Omega$, and by Theorem 4.29 $\nabla_{\tan}\boldsymbol{u}_+ = \nabla_{\tan}\boldsymbol{u}_-$ and $\left(\frac{\partial \boldsymbol{u}}{\partial \nu}\right)_\pm = (\pm\frac{1}{2}I + \mathcal{K}_\lambda)\boldsymbol{f}$. First of all we prove that

$$\left\| \left(\frac{\partial \boldsymbol{u}}{\partial \nu}\right)_- \right\|_{L^2(\partial\Omega)} \leq C\|\nabla \boldsymbol{u}_-\|_{L^2(\partial\Omega)} + \|\phi_-\|_{L^2(\partial\Omega)} \leq C \left\| \left(\frac{\partial \boldsymbol{u}}{\partial \nu}\right)_+ \right\|_{L^2(\partial\Omega)}. \tag{5.11}$$

The first inequality is obvious by the definition of the conormal derivative. The second inequality is a consequence of Theorem 5.11 and Theorem 5.12. We may apply the last Theorem because the decay of the solution is good enough at infinity. We now show (5.11).

$$\begin{aligned}
\|\nabla \boldsymbol{u}_-\|_{L^2(\partial\Omega)} + \|\phi_-\|_{L^2(\partial\Omega)} &\leq C \left(\|\nabla_{\tan}\boldsymbol{u}_-\| + |\lambda|^{\frac{1}{2}} \|\boldsymbol{u}_-\|_{L^2(\partial\Omega)} + |\lambda| \|\boldsymbol{n} \cdot \boldsymbol{u}_-\|_{H^{-1}(\partial\Omega)} \right) \\
&= C \left(\|\nabla_{\tan}\boldsymbol{u}_+\| + |\lambda|^{\frac{1}{2}} \|\boldsymbol{u}_+\|_{L^2(\partial\Omega)} + |\lambda| \|\boldsymbol{n} \cdot \boldsymbol{u}_+\|_{H^{-1}(\partial\Omega)} \right) \\
&\leq C \left\| \left(\frac{\partial \boldsymbol{u}}{\partial \nu}\right)_+ \right\|_{L^2(\partial\Omega)}.
\end{aligned}$$

We can now use the jump relation to find

$$\|\boldsymbol{f}\|_{L^2(\partial\Omega)} = \left\| \left(\frac{\partial \boldsymbol{u}}{\partial \nu}\right)_+ \right\|_{L^2(\partial\Omega)} + \left\| \left(\frac{\partial \boldsymbol{u}}{\partial \nu}\right)_- \right\|_{L^2(\partial\Omega)} \leq C \left\| \left(\frac{\partial \boldsymbol{u}}{\partial \nu}\right)_+ \right\|_{L^2(\partial\Omega)} = C \left\| \left(\frac{1}{2}I + \mathcal{K}_\lambda\right) \boldsymbol{f} \right\|_{L^2(\partial\Omega)}.$$

This proves the inequality (5.10). It was proven in [12] that in the case $\lambda = 0$, $\frac{1}{2}I + \mathcal{K}_0$ is a Fredholm operator on $L^2(\partial\Omega; \mathbb{C}^3)$ with index zero. Using the fact that $\mathcal{K}_\lambda - \mathcal{K}_0$ is compact on $L^2(\partial\Omega; \mathbb{C}^3)$ we now find in combination with Theorem 5.14 that $\frac{1}{2}I + \mathcal{K}_\lambda$ is a Fredholm operator with index zero. Inequality (5.10) implies that $\|\boldsymbol{f}\|_{L^2(\partial\Omega)} = 0$ when $\boldsymbol{f} \in \ker\left(\frac{1}{2}I + \mathcal{K}_\lambda\right)$. Hence the operator is injective and the dimension of the kernel is zero. Therefore the dimension of the cokernel is zero which implies that the operator is surjective. Thus $\frac{1}{2}I + \mathcal{K}_\lambda$ is an isomorphism on $L^2(\partial\Omega; \mathbb{C}^3)$. □

We want to prove a similar Lemma for the operator $(-\frac{1}{2}I+\mathcal{K}_\lambda)$. It turns out that something weaker is true. Recall the definition of $L^2_n(\partial\Omega) = \left\{\boldsymbol{f} \in L^2(\partial\Omega) : \int_{\partial\Omega} \boldsymbol{f}\cdot\boldsymbol{n} d\sigma(y) = 0\right\}$ as in Section 4.5.

Lemma 5.17. *Let $\lambda \in \Sigma_\theta$. Then $-\frac{1}{2}I+\mathcal{K}_\lambda$ is a Fredholm operator on $L^2(\partial\Omega;\mathbb{C}^3)$ with index zero, and*

$$\|\boldsymbol{f}\|_{L^2(\partial\Omega)} \leq C\left\|\left(-\frac{1}{2}I+\mathcal{K}_\lambda\right)\boldsymbol{f}\right\|_{L^2(\partial\Omega)} \tag{5.12}$$

for all $\boldsymbol{f} \in L^2_n(\partial\Omega)$, where C depends only on θ and the Lipschitz character of Ω.

Proof. By rescaling we may assume that $|\partial\Omega| = 1$. In the case $\lambda = 0$, it was proved in [12] that $-\frac{1}{2}I+\mathcal{K}_0$ is a Fredholm operator on $L^2(\partial\Omega;\mathbb{C}^3)$ with index zero. It was also proven there that the inequality (5.12) holds for $\lambda = 0$. Since by Lemma 5.15, $\mathcal{K}_\lambda - \mathcal{K}_0$ is compact on $L^2(\partial\Omega;\mathbb{C}^3)$ we know by Theorem 5.14 that $-\frac{1}{2}I+\mathcal{K}_\lambda$ is a Fredholm operator with index zero on $L^2(\partial\Omega;\mathbb{C}^3)$ for all $\lambda \in \Sigma_\theta$. We now want to establish the estimate (5.12). Therefore notice that by Theorem 3.12,

$$\begin{aligned}
|(\mathcal{K}_\lambda - \mathcal{K}_0)\,\boldsymbol{f}_k(x)| &\leq C\int_{\partial\Omega}\left|\nabla_x\left[\boldsymbol{\Gamma}_k(x-y;\lambda)-\boldsymbol{\Gamma}_k(x-y;0)\right]\right||\boldsymbol{f}_k(y)|d\sigma(y)\\
&\leq C|\lambda|^{\frac{1}{2}}\int_{\partial\Omega}\frac{|\boldsymbol{f}_k(y)|}{|x-y|}d\sigma(y).
\end{aligned}$$

We conclude that $\|(\mathcal{K}_\lambda - \mathcal{K}_0)\,\boldsymbol{f}\|_{L^2(\partial\Omega)} \leq C|\lambda|^{\frac{1}{2}}\|\boldsymbol{f}\|_{L^2(\partial\Omega)}$. Now it follows that for $\boldsymbol{f} \in L^2_n(\partial\Omega)$

$$\begin{aligned}
\|\boldsymbol{f}\|_{L^2(\partial\Omega)} &\leq C\left\|\left(-\frac{1}{2}I+\mathcal{K}_0\right)\boldsymbol{f}\right\|_{L^2(\partial\Omega)}\\
&\leq C\left\|\left(-\frac{1}{2}I+\mathcal{K}_\lambda\right)\boldsymbol{f}\right\|_{L^2(\partial\Omega)} + C|\lambda|^{\frac{1}{2}}\|\boldsymbol{f}\|_{L^2(\partial\Omega)}.
\end{aligned}$$

Now we can pick $\tau \in (0,1)$ such that $C\tau^{\frac{1}{2}} \leq \frac{1}{2}$. Thus implies that inequality (5.12) holds for $\lambda \in \Sigma_\theta$ with $|\lambda| < \tau$ and τ only depends on θ and the Lipschitz character of Ω. For the case where $|\lambda| \geq \tau$ we use Rellich estimates. To do this let $\boldsymbol{f} \in L^2_n(\partial\Omega)$ and $(\boldsymbol{u},\phi) = \mathcal{S}_\lambda(\boldsymbol{f})$ the single layer potential. By Theorem 5.11

$$\begin{aligned}
\|\nabla\boldsymbol{u}_+\|_{L^2(\partial\Omega)} + \left\|\phi_+ - \fint_{\partial\Omega}\phi_+ d\sigma\right\|_{L^2(\partial\Omega)} &\leq C\left(\|\nabla_{\tan}\boldsymbol{u}\|_{L^2(\partial\Omega)} + |\lambda|^{\frac{1}{2}}\|\boldsymbol{u}\|_{L^2(\partial\Omega)} + |\lambda|\|\boldsymbol{u}\cdot\boldsymbol{n}\|_{H^{-1}(\partial\Omega)}\right)\\
&\leq C\left\|\left(\frac{\partial\boldsymbol{u}}{\partial\nu}\right)_-\right\|_{L^2(\partial\Omega)}.
\end{aligned}$$

Now using the jump relations $\boldsymbol{f} = \left(\frac{\partial\boldsymbol{u}}{\partial\nu}\right)_+ - \left(\frac{\partial\boldsymbol{u}}{\partial\nu}\right)_-$ we find

$$\begin{aligned}
\|\boldsymbol{f}\|_{L^2(\partial\Omega)} &\leq \left\|\left(\frac{\partial\boldsymbol{u}}{\partial\nu}\right)_+\right\|_{L^2(\partial\Omega)} + \left\|\left(\frac{\partial\boldsymbol{u}}{\partial\nu}\right)_-\right\|_{L^2(\partial\Omega)}\\
&\leq C\left\|\left(\frac{\partial\boldsymbol{u}}{\partial\nu}\right)_-\right\|_{L^2(\partial\Omega)} + C\left|\int_{\partial\Omega}\phi_+ d\sigma\right|\\
&= C\left\|\left(-\frac{1}{2}I+\mathcal{K}_\lambda\right)\boldsymbol{f}\right\|_{L^2(\partial\Omega)} + C\left|\int_{\partial\Omega}\phi_+ d\sigma\right|.
\end{aligned}$$

Now we only have to deal with the last term. To do this notice that

$$\left(\frac{\partial \boldsymbol{u}}{\partial \nu}\right)_+ \cdot \boldsymbol{n} = \frac{\partial \boldsymbol{u}_i}{\partial x_j}\boldsymbol{n}_i\boldsymbol{n}_j - \phi_+ = \boldsymbol{n}_j\left(\boldsymbol{n}_i\frac{\partial}{\partial x_j} - \boldsymbol{n}_j\frac{\partial}{\partial x_i}\right)\boldsymbol{u}_i - \phi_+,$$

which leads to

$$\begin{aligned}\left|\int_{\partial\Omega}\phi_+ d\sigma\right| &\leq \left|\int_{\partial\Omega}\left(\frac{\partial \boldsymbol{u}}{\partial \nu}\right)_+ \cdot \boldsymbol{n}\right| + C\int_{\partial\Omega}|\nabla_{\tan}\boldsymbol{u}|d\sigma \\ &\leq C\left|\int_{\partial\Omega}\left(\frac{\partial \boldsymbol{u}}{\partial \nu}\right)_- \cdot \boldsymbol{n}\right| + C\int_{\partial\Omega}|\nabla_{\tan}\boldsymbol{u}|d\sigma \\ &\leq C\left\|\left(\frac{\partial \boldsymbol{u}}{\partial \nu}\right)_-\right\|_{L^2(\partial\Omega)}.\end{aligned}$$

Combining this yields the estimate for $\tau \geq |\lambda|$ which completes the proof. □

5.3 Weak reverse Hölder inequality

In this section we derive a weak reverse Hölder inequality for solutions of the Stokes system. To do so, we first bound the L^2 norm of the solution by the L^2 norm of its non-tangential limit. Then we use the Hardy-Littlewood-Sobolev inequality to obtain an estimate of the L^p norm of the solution for $p = 3 + \varepsilon$. Finally this estimate is integrated to obtain a so-called weak reverse Hölder inequality.

Lemma 5.18. *Let $\lambda \in \Sigma_\theta$ and $(\boldsymbol{u}, \phi)$ be a solution of the Stokes System in Ω. Suppose that $\boldsymbol{u}$ has non-tangential limit a.e. on $\partial\Omega$ and $(\boldsymbol{u})^* \in L^2(\partial\Omega)$. Then,*

$$\int_\Omega |\boldsymbol{u}|^2 dx \leq C\int_{\partial\Omega}|\boldsymbol{u}|^2 d\sigma,$$

where C depends only on θ and the Lipschitz character of Ω.

Proof. By approximating Ω by a sequence of smooth domains with a uniform Lipschitz constant from inside, we may assume that Ω is smooth and $(\boldsymbol{u}, \phi)$ are smooth in $\overline{\Omega}$. Now let $(\boldsymbol{w}, \psi)$ solve the following system

$$\begin{cases} -\Delta \boldsymbol{w} + \lambda \boldsymbol{w} + \nabla\psi = \overline{\boldsymbol{u}} & \text{in } \Omega \\ \operatorname{div} \boldsymbol{w} = 0 & \text{in } \Omega, \end{cases}$$

with $\boldsymbol{w} \in H_0^1(\Omega; \mathbb{C}^3)$ and $\psi \in H^1(\Omega)$. Now using this system and partial integrations we find that

$$\begin{aligned}\int_\Omega |\boldsymbol{u}|^2 dx &= \int_\Omega \boldsymbol{u}\cdot(-\Delta\boldsymbol{w} + \lambda\boldsymbol{w} + \nabla\psi)\,dx \\ &= -\int_{\partial\Omega}\boldsymbol{u}_i\frac{\partial \boldsymbol{w}_i}{\partial x_j}\boldsymbol{n}_j d\sigma + \int_{\partial\Omega}\frac{\partial \boldsymbol{u}_i}{\partial x_j}\boldsymbol{w}_i\boldsymbol{n}_j d\sigma + \int_\Omega -\Delta\boldsymbol{u}_i\boldsymbol{w}_i dx + \int_\Omega \lambda\boldsymbol{u}_i\boldsymbol{w}_i dx \\ &\quad + \int_{\partial\Omega}\boldsymbol{u}_i\psi\boldsymbol{n}_i d\sigma - \int_\Omega \frac{\partial \boldsymbol{u}_i}{\partial x_i}\psi dx.\end{aligned}$$

Since $\boldsymbol{w}$ has zero trace the second term becomes zero. $\boldsymbol{u}$ is a solution to the Stokes System and hence divergence free, so the sixth term is also zero. Now notice that

$$\begin{aligned}\int_{\Omega} (-\Delta \boldsymbol{u}_i + \lambda \boldsymbol{u}_i)\, \boldsymbol{w}_i dx &= \int_{\Omega} -\nabla \phi \boldsymbol{w}_i dx \\ &= -\int_{\partial\Omega} \phi \boldsymbol{w}_i \boldsymbol{n}_i d\sigma + \int_{\Omega} \phi \frac{\partial \boldsymbol{w}_i}{\partial x_i} dx = 0.\end{aligned}$$

Also, because w has zero trace and divergence. This means that

$$\int_{\Omega} |\boldsymbol{u}|^2 dx = -\int_{\partial\Omega} \boldsymbol{u} \cdot \left(\frac{\partial \boldsymbol{w}}{\partial \boldsymbol{n}} - \psi n \right) d\sigma \leq \|\boldsymbol{u}\|_{L^2(\partial\Omega)} \left(\|\nabla \boldsymbol{w}\|_{L^2(\partial\Omega)} + \|\psi\|_{L^2(\partial\Omega)} \right). \tag{5.13}$$

By subtracting a constant from ψ we may assume that $\int_{\partial\Omega} \psi = 0$. Also, note that $\Delta\psi = \operatorname{div}(\overline{\boldsymbol{u}}) = 0$ in Ω. By Lemma 5.8 we now know that

$$\begin{aligned}\|\psi\|_{L^2(\partial\Omega)} &\leq C \|\nabla\psi \cdot \boldsymbol{n}\|_{H^{-1}(\partial\Omega)} \\ &\leq C \left(\|\Delta \boldsymbol{w} \cdot \boldsymbol{n}\|_{H^{-1}(\partial\Omega)} + \|\boldsymbol{u} \cdot \boldsymbol{n}\|_{H^{-1}(\partial\Omega)} \right) \\ &\leq C \left(\|\nabla \boldsymbol{w}\|_{L^2(\partial\Omega)} + \|\boldsymbol{u}\|_{L^2(\partial\Omega)} \right). \end{aligned} \tag{5.14}$$

Combining (5.13) and (5.14) yields

$$\int_{\Omega} |\boldsymbol{u}|^2 dx \leq C \|\boldsymbol{u}\|_{L^2(\partial\Omega)} \|\nabla \boldsymbol{w}\|_{L^2(\partial\Omega)} + C \|\boldsymbol{u}\|^2_{L^2(\partial\Omega)}. \tag{5.15}$$

We now look to bound the norm of $\nabla\boldsymbol{w}$. Hereto we use a Rellich identity like Lemma 5.2. Since $\boldsymbol{w}$ has zero trace we can write

$$\begin{aligned}\int_{\partial\Omega} |\nabla \boldsymbol{w}|^2 d\sigma &\leq C \int_{\Omega} |\nabla \boldsymbol{w}|^2 dx + C \int_{\Omega} |\nabla \boldsymbol{w}||\psi| dx \\ &\quad + C|\lambda| \int_{\Omega} |\nabla \boldsymbol{w}||\boldsymbol{w}| dx + C \int_{\Omega} |\nabla \boldsymbol{w}||\boldsymbol{u}| dx. \end{aligned} \tag{5.16}$$

Using a similar technique as in the proof of Lemma 5.4 we find

$$\int_{\Omega} |\nabla \boldsymbol{w}|^2 dx + |\lambda| \int_{\Omega} |\boldsymbol{w}|^2 dx \leq C \int_{\Omega} |\boldsymbol{w}||\boldsymbol{u}| dx.$$

Using the Poincaré inequality (Theorem B.3) and the Cauchy inequality we find

$$\begin{aligned}\int_{\Omega} |\nabla \boldsymbol{w}|^2 dx + (1+|\lambda|) \int_{\Omega} |\boldsymbol{w}|^2 dx &\leq C \int_{\Omega} |\boldsymbol{w}||\boldsymbol{u}| dx + \int_{\Omega} |\boldsymbol{w}|^2 dx \\ &\leq \varepsilon \int_{\Omega} |\boldsymbol{w}|^2 dx + \frac{C}{\varepsilon} \int_{\Omega} |\boldsymbol{u}|^2 dx + \frac{1}{4} \int_{\Omega} |\boldsymbol{w}|^2 dx + \frac{3}{4} \int_{\Omega} |\nabla \boldsymbol{w}|^2 dx.\end{aligned}$$

If we now choose $\varepsilon = \frac{1+|\lambda|}{2}$ and change the constant C we find

$$\int_{\Omega} |\nabla \boldsymbol{w}|^2 dx + (1+|\lambda|) \int_{\Omega} |\boldsymbol{w}|^2 dx \leq \frac{C}{1+|\lambda|} \int_{\Omega} |\boldsymbol{u}|^2 dx. \tag{5.17}$$

Now, using (5.16) and (5.17) we find

$$\int_{\partial\Omega} |\nabla \boldsymbol{w}|^2 d\sigma \leq C_\varepsilon \int_{\Omega} |\boldsymbol{u}|^2 dx + \varepsilon \int_{\partial\Omega} |\psi|^2 d\sigma,$$

where we used Lemma 5.5 to estimate the norm of ϕ in the domain by the norm of ϕ on the boundary. □

We are now able to state an important theorem that guarantees that the solution of the Stokes System exists, is unique, can be represented by the double layer potential and has a bounded nontangential maximal function.

Theorem 5.19. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$ with connected boundary. Let $\lambda \in \Sigma_\theta$. Given $\boldsymbol{g} \in L^2_n(\partial\Omega)$, there exists a unique $\boldsymbol{u}$ and a harmonic function ϕ, unique up to constants, such that $(\boldsymbol{u}, \phi)$ satisfies the Stokes System* (1.11) *in Ω, $(\boldsymbol{u})^* \in L^2(\partial\Omega)$ and $\boldsymbol{u} = \boldsymbol{g}$ on $\partial\Omega$ in the sense of nontangential convergence. Moreover the solution satisfies the estimate $\|(\boldsymbol{u})^*\|_{L^2(\partial\Omega)} \leq C\|\boldsymbol{g}\|_{L^2(\partial\Omega)}$, and there exists a function $\boldsymbol{f} \in L^2(\partial\Omega)$ such that the solution can be represented by a double layer potential, $(\boldsymbol{u}, \phi) = \mathcal{D}_\lambda(\boldsymbol{f})$ with $\|\boldsymbol{f}\|_{L^2(\partial\Omega)} \leq C\|\boldsymbol{g}\|_{L^2(\partial\Omega)}$, where C depends on θ and the Lipschitz character of Ω.*

Proof. We start by proving uniqueness. Let $(\boldsymbol{u}, \phi)$ and $(\boldsymbol{w}, \rho)$ be solutions given with boundary data $\boldsymbol{g}$. Then $(\boldsymbol{u} - \boldsymbol{w}, \phi - \rho)$ is a solution with boundary data identically zero. We now use Lemma 5.18 and find

$$\int_\Omega |\boldsymbol{u} - \boldsymbol{w}|^2 dx \leq C \int_{\partial\Omega} 0 d\sigma = 0$$

and hence $\boldsymbol{u} = \boldsymbol{w}$. This in turn implies that $\nabla(\phi - \rho) = 0$ and thus ϕ equals ρ up to a constant. We now prove the existance of the solution. By Lemma 5.17 we know that $-\frac{1}{2}I + \mathcal{K}_{\overline{\lambda}}$ is a Fredholm operator on $L^2(\partial\Omega, \mathbb{C}^3)$ with index zero, thus $-\frac{1}{2}I + \mathcal{K}^*_{\overline{\lambda}}$ is a Fredholm operator on $L^2(\partial\Omega, \mathbb{C}^3)$ with index zero. Now let $(\boldsymbol{u}, \phi) = \mathcal{D}_\lambda(\boldsymbol{f})$ be the double layer potential of $\boldsymbol{f} \in L^2(\partial\Omega)$. Then we know that $\text{div}(\boldsymbol{u}) = 0$ and by the divergence theorem we find that $\int_{\partial\Omega} \boldsymbol{u} \cdot \boldsymbol{n} d\sigma = 0$. This shows that $\text{Ran}(-\frac{1}{2}I + \mathcal{K}^*_{\overline{\lambda}}) \subset L^2_{\boldsymbol{n}}(\partial\Omega)$. This implies that the normal vector $\boldsymbol{n}$ is in the kernel of $-\frac{1}{2}I + \mathcal{K}_{\overline{\lambda}}$. By inequality (5.12) we find that $\ker(-\frac{1}{2}I + \mathcal{K}_{\overline{\lambda}}) = \text{Span}\{\boldsymbol{n}\}$. This implies that $\text{Ran}(-\frac{1}{2}I + \mathcal{K}^*_{\overline{\lambda}}) = L^2_{\boldsymbol{n}}(\partial\Omega)$. Hence we find that the operator

$$-\frac{1}{2}I + \mathcal{K}^*_{\overline{\lambda}} : \text{Ran}\left(-\frac{1}{2}I + \mathcal{K}_{\overline{\lambda}}\right) \to L^2_{\boldsymbol{n}}(\partial\Omega)$$

is injective and surjective, hence invertible. Thus given $\boldsymbol{g} \in L^2_{\boldsymbol{n}}(\partial\Omega)$ we choose (by inverting) $\boldsymbol{f} \in \text{Ran}\left(-\frac{1}{2}I + \mathcal{K}_{\overline{\lambda}}\right)$ such that $\left(-\frac{1}{2}I + \mathcal{K}^*_{\overline{\lambda}}\right)\boldsymbol{f} = \boldsymbol{g}$. Thus the double layer potential satisfies $\boldsymbol{u} = \boldsymbol{g}$ on $\partial\Omega$ and $\|(\boldsymbol{u})^*\|_{L^2(\partial\Omega)} \leq \|\boldsymbol{f}\|_{L^2(\partial\Omega)}$. By duality we also know that

$$\|\boldsymbol{f}\|_{L^2(\partial\Omega)} \leq C\left\|\left(-\frac{1}{2}I + \mathcal{K}^*_{\overline{\lambda}}\right)\boldsymbol{f}\right\|_{L^2(\partial\Omega)}$$

and thus $\|\boldsymbol{f}\|_{L^2(\partial\Omega)} \leq C\|\boldsymbol{g}\|_{L^2(\partial\Omega)}$. This completes the proof. □

This theorem leads to the establishment of a reverse Hölder inequality. In the following lemmas we relate the L^p norm of the solution of the Stokes System to the nontangential maximal function of the same solution in L^2.

Lemma 5.20. *Let Ω be a Lipschitz domain in $\mathbb{R}^3$ and let $\boldsymbol{f} \in L^2(\partial\Omega; \mathbb{C}^3)$ and $I_1(\boldsymbol{f})$ as defined in* (A.6). *Then if $p = 3$, there exists a $C > 0$ such that*

$$\|I_1(\boldsymbol{f})\|_{L^p(\Omega)} \leq C\|\boldsymbol{f}\|_{L^2(\partial\Omega)}.$$

Proof. Let $p = 3$ and pick $q = \frac{3}{2}$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Because Ω is a Lipschitz domain we can take $\boldsymbol{h} \in C_0^1(\mathbb{R}^3, \mathbb{R}^3)$ such that $\boldsymbol{h}_k \boldsymbol{n}_k \geq c > 0$. Now using the divergence theorem we find

$$\begin{aligned}\int_{\partial\Omega} |I_1(\boldsymbol{f})|^2 d\sigma &\leq C \int_{\partial\Omega} \left(|I_1(\boldsymbol{f})|^2 \boldsymbol{h}\right) \cdot \boldsymbol{n} d\sigma \\ &= C \int_\Omega \operatorname{div}(\boldsymbol{h}) |I_1(\boldsymbol{f})|^2 dx + C \int_\Omega |I_1(\boldsymbol{f})||\nabla I_1(\boldsymbol{f})| \boldsymbol{h}_k dx \\ &\leq C \int_\Omega |I_1(\boldsymbol{f})|^2 dx + C \int_\Omega |I_1(\boldsymbol{f})||\nabla I_1(\boldsymbol{f})| dx \\ &\leq C \int_\Omega |I_1(\boldsymbol{f})|^2 dx + C \left(\int_\Omega |I_1(\boldsymbol{f})|^p\right)^{\frac{1}{p}} \left(\int_\Omega |\nabla I_1(\boldsymbol{f})|^q\right)^{\frac{1}{q}}.\end{aligned}$$

For the first term we find by Hölder's inequality and the fact that Ω is bounded that

$$\int_\Omega |I_1(\boldsymbol{f})|^2 dx \leq C \|I_1(\boldsymbol{f})\|_{L^p(\Omega)}^2.$$

For the second term we find using the Cauchy inequality (Lemma B.1) that

$$C \left(\int_\Omega |I_1(\boldsymbol{f})|^p\right)^{\frac{1}{p}} \left(\int_\Omega |\nabla I_1(\boldsymbol{f})|^q\right)^{\frac{1}{q}} \leq C \|\nabla I_1(\boldsymbol{f})\|_{L^q(\Omega)} + C \|I_1(\boldsymbol{f})\|_{L^p(\Omega)}.$$

We can easily check the conditions of Theorem A.7, since $\frac{1}{p} = \frac{1}{q} - \frac{1}{3}$. We thus find

$$\|I_1(\boldsymbol{f})\|_{L^p(\Omega)} \leq C \|\boldsymbol{f}\|_{L^q(\Omega)}.$$

Further, by Theorem A.8 we find that

$$\|\nabla I_1(\boldsymbol{f})\|_{L^q(\Omega)} = C \|\boldsymbol{f}\|_{L^q(\Omega)}.$$

Putting this together yields

$$\|I_1(\boldsymbol{f})\|_{L^2(\partial\Omega)} \leq C \|\boldsymbol{f}\|_{L^q(\Omega)}.$$

Since I_1 is a self-adjoint operator we can use duality to find

$$\|I_1(\boldsymbol{f})\|_{L^p(\Omega)} = \|(I_1)^*(\boldsymbol{f})\|_{(L^q(\Omega))^*} \leq C \|\boldsymbol{f}\|_{(L^2(\partial\Omega))^*} = C \|\boldsymbol{f}\|_{L^2(\partial\Omega)},$$

where * denotes the dual. This completes the proof. □

Lemma 5.21. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$ with connected boundary. Let $\boldsymbol{u} \in H^1(\Omega; \mathbb{C}^3)$ and $\phi \in L^2(\Omega)$. Suppose that $(\boldsymbol{u}, \phi)$ satisfies the Stokes System* (1.11) *in Ω for some $\lambda \in \Sigma_\theta$. Then,*

$$\left(\int_\Omega |\boldsymbol{u}|^p dx\right)^{\frac{1}{p}} \leq C \left(\int_{\partial\Omega} |(\boldsymbol{u})^*|^2 d\sigma\right)^{\frac{1}{2}},$$

where $p = 3$ and $C > 0$ depends only on θ and the Lipschitz character of Ω.

Proof. Without loss of generality we can assume that $|\partial\Omega| = 1$. Now define the following set for all $x \in \Omega$

$$\tilde{E}(x) = \{p \in \partial\Omega : |x - p| < C \operatorname{dist}(x, \partial\Omega)\}$$

We can now bound $\boldsymbol{u}(x)$ by the Riesz potential of the nontangential maximal function on the boundary. To do so, observe that when $y \in \tilde{E}(x)$ we have that

$$|\boldsymbol{u}(x)| \leq (\boldsymbol{u})^*(y)$$

and thus

$$\begin{aligned}|\boldsymbol{u}(x)| &\leq \frac{C}{(\operatorname{diam}(\Omega))^2}\int_{\tilde{E}(x)}(\boldsymbol{u})^*(y)d\sigma(y)\\ &\leq C\int_{\tilde{E}(x)}\frac{(\boldsymbol{u})^*(y)}{|x-y|^2}d\sigma(y)\\ &\leq C\int_{\partial\Omega}\frac{(\boldsymbol{u})^*(y)}{|x-y|^2}d\sigma(y) = CI_1((\boldsymbol{u})^*)(x).\end{aligned}$$

Now we use this estimate in combination with Lemma 5.20 to find

$$\left(\int_\Omega |\boldsymbol{u}|^p dx\right)^{\frac{1}{p}} \leq C\left(\int_\Omega |I_1((\boldsymbol{u})^*)|^p dx\right)^{\frac{1}{p}} \leq C\left(\int_{\partial\Omega}|(\boldsymbol{u})^*|^2 d\sigma\right)^{\frac{1}{2}}. \tag{5.18}$$

This completes the proof. □

Theorem 5.22. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$ with connected boundary. Let $\boldsymbol{u} \in H^1(\Omega;\mathbb{C}^3)$ and $\phi \in L^2(\Omega)$. Suppose that $(\boldsymbol{u},\phi)$ satisfies the Stokes System* (1.11) *in Ω for some $\lambda \in \Sigma_\theta$. Then,*

$$\left(\int_\Omega |\boldsymbol{u}|^p dx\right)^{\frac{1}{p}} \leq C\left(\int_{\partial\Omega}|\boldsymbol{u}|^2 d\sigma\right)^{\frac{1}{2}},$$

where $p = 3$ and $C > 0$ depends only on θ and the Lipschitz character of Ω.

Proof. Let $\boldsymbol{f}$ denote the trace of $\boldsymbol{u}$ on $\partial\Omega$ and $\boldsymbol{w}$ the solution of the L^2 Dirichlet problem as given by Theorem 5.19. Now let (Ω_j) be a sequence of smooth domains that approximates Ω from the inside. From Lemma 5.18 it follows that

$$\int_{\Omega_j}|\boldsymbol{u}-\boldsymbol{w}|^2 dx \leq C\int_{\partial\Omega_j}|\boldsymbol{u}-\boldsymbol{w}|^2 d\sigma.$$

If we let $j \to \infty$ we see that $\boldsymbol{u}$ and $\boldsymbol{w}$ are the same on the boundary hence $\boldsymbol{u} = \boldsymbol{w}$ in Ω. As a result we obtain

$$\|(\boldsymbol{u})^*\|_{L^2(\partial\Omega)} = \|(\boldsymbol{w})^*\|_{L^2(\partial\Omega)} \leq C\|\boldsymbol{f}\|_{L^2(\partial\Omega)} \leq C\|\boldsymbol{u}\|_{L^2(\partial\Omega)}.$$

The result now follows by applying Lemma 5.21. □

Let φ be a Lipschitz function such that $\varphi(0) = 0$ and $\|\varphi\|_\infty \leq L$. Define the following sets for $0 < r < \infty$

$$\begin{aligned}D(r) &= \left\{(\tilde{x},\tilde{y}) \in \mathbb{R}^3 \,:\, |\tilde{x}| < r \text{ and } \varphi(\tilde{x}) < \tilde{y} < 30(L+1)r\right\}\\ I(r) &= \left\{(\tilde{x},\tilde{y}) \in \mathbb{R}^3 \,:\, |\tilde{x}| < r \text{ and } \varphi(\tilde{x}) = \tilde{y}\right\}.\end{aligned} \tag{5.19}$$

Lemma 5.23. *Let $\boldsymbol{u} \in H^1(D(2r);\mathbb{C}^3)$ and $\phi \in L^2(D(2r))$. Suppose $(\boldsymbol{u},\phi)$ satisfies* (1.11) *in $D(2r)$ and $\boldsymbol{u} = 0$ on $I(2r)$ for some $0 < r < \infty$ and $\lambda \in \Sigma_\theta$. Let $p = 3$. Then,*

$$\left(\fint_{D(r)}|\boldsymbol{u}|^p dx\right)^{\frac{1}{p}} \leq C\left(\fint_{D(2r)}|\boldsymbol{u}|^2 dx\right)^{\frac{1}{2}},$$

where C depends only on θ and L.

Proof. By a scaling argument it suffices to do the proof for $r = 1$. We are going to proof this lemma, by integrating the result of Theorem 5.22. Let $t \in (1, 2)$. We apply Theorem 5.22 to the Lipschitz domain $D(t)$. This yields

$$\left(\int_{D(t)} |\boldsymbol{u}|^p dx \right)^{\frac{2}{3}} \leq C \int_{\partial D(t)} |u|^2 d\sigma.$$

Notice that $\boldsymbol{u} = 0$ on $I(2)$. Hence we find

$$\left(\int_{D(1)} |\boldsymbol{u}|^p dx \right)^{\frac{2}{3}} \leq C \int_{\partial D(t) \setminus I(2)} |u|^2 d\sigma.$$

Now integrating this equation over the interval $(1, 2)$ yields the desired result. □

Lemma 5.24. *Let $(\boldsymbol{u}, \phi)$ be a solution of the Stokes System* (1.11) *in $B(x_0, r)$. Then,*

$$\left|\nabla^\ell \boldsymbol{u}(x_0)\right| \leq \frac{C_\ell}{r^\ell} \left(\fint_{B(x_0,r)} |\boldsymbol{u}|^2 dx \right)^{\frac{1}{2}} \tag{5.20}$$

for any $\ell \geq 0$, where C_ℓ depends only on ℓ and θ.

Proof. By a scaling argument it suffices to do the proof for $r = 2$. Now let $t \in (1, 2)$ and let $(\boldsymbol{u}, \phi)$ be a solution of 1.11 in $B(x_0, t)$. Now denote the trace of $\boldsymbol{u}$ by $\boldsymbol{g}$. By Theorem 5.19 we can find a $\boldsymbol{f}$ such that $(\boldsymbol{u}, \phi) = \mathcal{D}_\lambda(\boldsymbol{f})$. Using this we find that

$$\nabla^l \boldsymbol{u}(x_0) = \int_{\partial B(x_0,t)} \nabla_x^l S_{ij}(\boldsymbol{\Gamma}_k, \Phi_k) \boldsymbol{f}_k(y) d\sigma(y) \leq C \int_{\partial B(x_0,t)} |\boldsymbol{f}_k(y)|^2 d\sigma(y) \leq C \int_{\partial B(x_0,t)} |\boldsymbol{u}|^2 d\sigma.$$

We can now integrate the result over the interval $(1, 2)$ to find the result. □

The results above allow us to establish a weak reverse Hölder inequality for solutions of the Stokes System.

Theorem 5.25. *Let Ω be a Lipschitz domain in $\mathbb{R}^3$ and let $x_0 \in \overline{\Omega}$ and $0 < r < c\,\mathrm{diam}(\Omega)$. Let $\boldsymbol{u} \in H^1(B(x_0, 2r) \cap \Omega; \mathbb{C}^3)$ and $\phi \in L^2(B(x_0, 2r) \cap \Omega)$. Suppose that $(\boldsymbol{u}, \phi)$ satisfies the Stokes System* (1.11) *in $B(x_0, 2r) \cap \Omega$ and $\boldsymbol{u} = 0$ on $B(x_0, 2r) \cap \partial\Omega$ (if $B(x_0, 2r) \cap \partial\Omega \neq \emptyset$). Then,*

$$\left(\fint_{B(x_0,r)\cap\Omega} |\boldsymbol{u}|^p \right)^{\frac{1}{p}} \leq C \left(\fint_{B(x_0,2r)\cap\Omega} |\boldsymbol{u}|^2 \right)^{\frac{1}{2}}, \tag{5.21}$$

where $p = 3 + \varepsilon$ and $C > 0$, $\varepsilon > 0$ depend only on θ, and the Lipschitz character of Ω.

Proof. We first prove this result in two cases for $p = 3$. In the first case $x_0 \in \Omega$ and there exists an $r > 0$ such that $B(x_0, 3r) \subset \Omega$. In the second case $x_0 \in \partial\Omega$. We start with the first case and use Lemma 5.24 together with the observation that if $x \in B(x_0, r)$, then $B(x, r) \subset B(x_0, 2r)$.

$$\left(\fint_{B(x_0,r)} |\boldsymbol{u}(x)|^p dx \right)^{\frac{1}{p}} \leq \left(\fint_{B(x_0,r)} C \left(\fint_{B(x,r)} |\boldsymbol{u}|^2 dy \right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}} \leq C \left(\fint_{B(x_0,2r)} |\boldsymbol{u}|^2 dy \right)^{\frac{1}{2}}.$$

For the second case we have to do a boundary estimate. We chose r such that $B(x_0, r) \cap \Omega$ is a special Lipschitz domain. We can now use Lemma 5.23 and a covering argument to conclude. To complete the proof we notice that estimate (5.21) is a reverse Hölder inequality. This implies the self-improving behaviour. More details on the self-improvement can be found in old book of Giaquinta [19, Proposition V.1.1] or a more recent paper by Anderson, Hytönen and Tapiola [2, Proposition 6.2]. □

Theorem 5.26. *Let Ω be a Lipschitz domain in $\mathbb{R}^3$ and let $x_0 \in \overline{\Omega}$ and $0 < r < c\,\mathrm{diam}(\Omega)$. Let $\boldsymbol{u} \in H^1(B(x_0, 2r) \cap \Omega; \mathbb{C}^3)$ and $\phi \in L^2(B(x_0, 2r) \cap \Omega)$. Suppose that $(\boldsymbol{u}, \phi)$ satisfies the Stokes System* (1.11) *in $B(x_0, 2r) \cap \Omega$ and $\boldsymbol{u} = 0$ on $B(x_0, 2r) \cap \partial\Omega$ (if $B(x_0, 2r) \cap \partial\Omega \neq \emptyset$). Then,*

$$\left(\fint_{B(x_0,r)\cap\Omega} |\boldsymbol{u}|^2 dx \right)^{\frac{1}{2}} \leq C \left(\fint_{B(x_0,2r)\cap\Omega} |\boldsymbol{u}|^p dx \right)^{\frac{1}{p}}, \tag{5.22}$$

where $p = \frac{3}{2} - \varepsilon$ and $C > 0$, $\varepsilon > 0$ depends only on θ and the Lipschitz character of Ω.

Proof. This is the dual result of Theorem 5.25. □

6 Resolvent estimates

In this section the weak reverse Hölder estimates will be used to prove resolvent estimates for the Stokes operator. First the unweighted case is considered. Using the result of the unweighted case, the weighted resolvent estimates will be proved for a special class of weights.

6.1 Unweighted resolvent estimates

We start off by setting a specific Assumption. Assuming this Assumption an extrapolation theorem is proved. Then we show that the solution of the Stokes problem satisfies the Assumption and as a corollary we obtain resolvent estimates in $L^p(\Omega)$.

Assumption 6.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain, let Q_0 be a cube in $\mathbb{R}^3$ such that $\Omega \subset Q_0$ and let $\boldsymbol{g} \in L^1(2Q_0)$. Let $\boldsymbol{h} \in L^1(2Q_0)$. Let Q be any cube such that $|Q| < |Q_0|$ and define $\boldsymbol{g}_1 = \boldsymbol{g}\chi_{3Q\setminus \operatorname{supp}(\boldsymbol{h})}$ and $\boldsymbol{g}_2 = \boldsymbol{g}\chi_{3Q\cap \operatorname{supp}(\boldsymbol{h})}$. Then,*

(i) there exists a constant $C_1 > 0$ such that the following local reverse Hölder estimate holds

$$\left(\fint_{Q\cap\Omega} |\boldsymbol{g}_1|^p dx\right)^{\frac{1}{p}} \leq C_1 \left(\fint_{2Q\cap\Omega} |\boldsymbol{g}| dx + \sup_{Q'\supset Q} \fint_{Q'} |\boldsymbol{h}| dx\right) \tag{6.1}$$

.

(ii) There exists a constant $C_2 > 0$ such that

$$\fint_{2Q} |\boldsymbol{g}_2| dx \leq C_2 \sup_{Q'\supset Q} \fint_{Q'} |\boldsymbol{h}| dx \tag{6.2}$$

.

Now that we have set the stage for this section we start of with a covering lemma atributed to A. Calderón and A. Zygmund [22, Theorem 4.3.1].

Lemma 6.2. *Let Q be a bounded cube in $\mathbb{R}^3$ and $A \subset Q$ be a measurable set. Then for all $\delta \in (0, 1)$ there exists a sequence of disjoint dyadic cubes $\{Q_j\}$ obtained from Q such that*

(i) $\left|A \setminus \bigcup_j Q_j\right| = 0$

(ii) $|A \cap Q_j| > \delta |Q_j|$

(iii) $\left|A \cap \hat{Q}_j\right| \leq \delta \left|\hat{Q}_j\right|$,

where $\hat{Q}_j$ denotes the dyadic parent cube of Q_j (i.e. Q_j is one of the 2^3 subcubes of $\hat{Q}_j$).

Proof. We construct a collection of disjoint dyadic cubes obtained from Q. We start with $\{Q_j\} = \emptyset$. Divide Q into 2^3 dyadic subcubes and call this collection $\{Q_j^1\}$. Define

$$\begin{aligned} G_1 &:= \left\{Q' \in Q_j^1 \,:\, |Q' \cap A| > \delta |Q'|\right\} \\ B_1 &:= \left\{Q' \in Q_j^1 \,:\, |Q' \cap A| \leq \delta |Q'|\right\}. \end{aligned}$$

Now add G_1 to $\{Q_j\}$ and divide all cubes in B_1 into 2^3 dyadic subcubes and call this collection $\{Q_j^2\}$. The process above is repeated iteratively. We now show that $\{Q_j\}$ has the required properties. By construction, properties *(ii)* and *(iii)* hold. What is left is to show property *(i)*. For this assume $x \neq \cup_j Q_j$ and define $\boldsymbol{f}(x) = \chi_A(x)$. Because of the construction of A there exists a sequence of cubes $\{C_j\}$ with diameter tending to zero such that $|C_j \cap A| \leq \delta |C_j| < |C_j|$. If we now use the Lebesgue differentiation theorem (Theorem A.3), we find for almost every x that

$$\chi_A(x) = \lim_{C_j \to x} \frac{1}{|C_j|} \int_{C_j} \chi_A(y) dy = \lim_{C_j \to x} \frac{1}{|C_j|} \int_{C_j \cap A} dy = \lim_{C_j \to x} \frac{|C_j \cap A|}{|C_j|} < 1$$

Hence we conclude that $\chi_A(x) = 0$ for almost all $x \neq \cup_j Q_j$ and the proof is completed. □

In the following we show an inequality involving level sets. In order to avoid very cumbersome notation we define for $\alpha > 0$ the superlevel set of the (localized) Hardy-Littlewood maximal function as

$$E(\alpha) := \{x \in Q_0 \,:\, \mathcal{M}_{2Q_0}(\boldsymbol{g}\chi_\Omega)(x) > \alpha\}, \tag{6.3}$$

where Ω, Q_0 and $\boldsymbol{g}$ are consistent with Assumption 6.1.

Lemma 6.3. *Assume Assumption 6.1 holds. Then it is possible to choose* $0 < \delta < 1$, $\gamma > 0$ *and* $C_0 > 0$ *depending only on* d, C_1, C_2, p *and* q *such that*

$$|E(A\alpha)| \leq \delta |E(\alpha)| + |\{x \in Q_0 \,:\, \mathcal{M}_{2Q_0}(\boldsymbol{h}\chi_\Omega)(x) > \gamma\alpha\}| \tag{6.4}$$

for all $\alpha > \alpha_0$, *where* $A = (2\delta)^{-\frac{1}{q}}$ *and*

$$\alpha_0 = \frac{C_0}{\delta |Q_0|} \int_\Omega |\boldsymbol{g}| dx. \tag{6.5}$$

Proof. Fix $q \in (2, p)$, let $\delta \in (0,1)$ to be determined later and define $A = (2\delta)^{-\frac{1}{q}}$. By Lemma 6.2 we can find a collection of cubes $\{Q_j\}$ such that

(i) $\left| E(A\alpha) \setminus \bigcup_j Q_j \right| = 0$

(ii) $|E(A\alpha) \cap Q_j| > \delta |Q_j|$

(iii) $\left| E(A\alpha) \cap \hat{Q}_j \right| \leq \delta \left| \hat{Q}_j \right|$.

We start by showing that it is possible to pick $\delta \in (0,1)$ and $\gamma > 0$ such that

$$\left\{ x \in \hat{Q}_j \,:\, \mathcal{M}(\boldsymbol{h}\chi_\Omega)(x) \leq \gamma\alpha \right\} \neq \emptyset \implies \hat{Q}_j \subset E(\alpha). \tag{6.6}$$

We will show this by contradiction. Suppose that there exists $x_0 \in \hat{Q}_j \setminus E(\alpha)$ and $x_1 \in \left\{ x \in \hat{Q}_j \,:\, \mathcal{M}(\boldsymbol{g})(x) \leq \gamma\alpha \right\}$. Notice that when $\hat{Q}_j \subset Q$ we have

$$\fint_Q |\boldsymbol{h}| dx \leq \mathcal{M}(\boldsymbol{h})(x_1) \leq \gamma\alpha \tag{6.7}$$

$$\fint_Q |\boldsymbol{g}| dx \leq \mathcal{M}_{2Q_0}(\boldsymbol{g})(x_0) \leq \alpha. \tag{6.8}$$

Now using the observation that

$$\{x \in \mathbb{R}^3 \,:\, |\boldsymbol{g} + \boldsymbol{h}| > 1\} \subset \left\{ x \in \mathbb{R}^3 \,:\, |\boldsymbol{g}| > \frac{1}{2} \right\} \cup \left\{ x \in \mathbb{R}^3 \,:\, |\boldsymbol{h}| > \frac{1}{2} \right\},$$

we find

$$\begin{aligned}
|Q_k \cap E(A\lambda)| &\le \left|\left\{x \in Q_k \,:\, \mathcal{M}_{2\overline{Q_k}}(\boldsymbol{g})(x) > A\lambda\right\}\right| \\
&\le \left|\left\{x \in Q_k \,:\, \mathcal{M}_{2\overline{Q_k}}(\boldsymbol{g}_1)(x) > \frac{A\lambda}{2}\right\}\right| \\
&\quad + \left|\left\{x \in Q_k \,:\, \mathcal{M}_{2\overline{Q_k}}(\boldsymbol{g}_2)(x) > \frac{A\lambda}{2}\right\}\right| \\
&= F + G.
\end{aligned}$$

We now examine these terms one by one. By the maximal theorem (Theorem A.2), part (ii) of Assumption 6.1 and (6.7) we find

$$G \le \frac{C}{A\alpha}\int_{2\hat{Q}_j} |\boldsymbol{g}_2| dx \le \frac{C}{A\alpha}\int_{2\hat{Q}_j} |\boldsymbol{h}| dx \le \frac{|Q_j| C\gamma}{A}.$$

For the second term we use Chebyshev's inequality, Assumption (6.1) and

$$\begin{aligned}
G &\le \frac{C}{(A\alpha)^p}\int_{2\hat{Q}_j} |\boldsymbol{g}_1|^p dx \\
&\le 2^{p-1}\frac{C|Q_j|N}{(A\alpha)^p}\left[\left(\fint_{2Q_j} |\boldsymbol{g}| dx\right)^p + \left(\sup_{Q' \supset \hat{Q}_j} \fint_{Q'} |\boldsymbol{h}| dx\right)^p\right] \\
&\le |Q_j|\left(\frac{C}{A^p} + \frac{C\gamma^p}{A^p}\right).
\end{aligned}$$

Hence

$$|E(A\alpha) \cap Q_j| \le |Q_j|\left(\frac{C_n C_2 \gamma}{A} + \frac{C_{n,\alpha,p} C_1^p}{A^p}\right) \tag{6.9}$$

$$= \delta |Q_j|\left(C_n C_2 \gamma \delta^{\frac{-1}{q-1}} + C_{n,\alpha,p} C_1^p \delta^{\frac{p}{q-1}}\right). \tag{6.10}$$

We now choose $\delta \in (0,1)$ such that $C_{n,\alpha,p} C_1^p \delta^{\frac{p}{q-1}} \le \frac{1}{2}$, which is possible since $q < p$. Then we choose $\gamma > 0$ such that $C_n C_2 \gamma \delta^{\frac{-1}{q-1}} \le \frac{1}{2}$. It now follows that $|E(A\alpha) \cap Q_j| \le \delta |Q_j|$. This is a contradiction, thus it must be that (6.6) is true. Using this we find that

$$\begin{aligned}
|E(A\alpha) \cap \{x \in Q_0 \,:\, \mathcal{M}(\boldsymbol{h}\chi_\Omega)(x) > \gamma\alpha\}| &\le \sum_{k'} \left|E(A\alpha) \cap \overline{Q_{k'}}\right| \\
&\le \delta \sum_{k'} \left|\overline{Q_{k'}}\right| \\
&\le \delta |E(\alpha)|,
\end{aligned}$$

where $\left\{\overline{Q_{k'}}\right\}$ is a disjoint subcover of $E(A\alpha) \cap \left\{x \in \mathbb{R}^3 \,:\, \mathcal{M}(\boldsymbol{h}\chi_\Omega)(x) > \gamma\alpha\right\}$ with the property that $\overline{Q_{k'}} \cap \left\{x \in \mathbb{R}^3 \,:\, \mathcal{M}(\boldsymbol{h}\chi_\Omega)(x) > \gamma\alpha\right\} \neq \emptyset$. This implies that

$$\begin{aligned}
|E(A\alpha)| &= \left|E(A\alpha) \cap \left\{x \in \mathbb{R}^3 \,:\, \mathcal{M}(\boldsymbol{h}\chi_\Omega)(x) > \gamma\alpha\right\}\right| \\
&\quad + \left|E(A\alpha) \cap \left\{x \in \mathbb{R}^3 \,:\, \mathcal{M}(\boldsymbol{h}\chi_\Omega)(x) \le \gamma\alpha\right\}\right| \\
&\le \delta|E(\alpha)| + \left|E(A\alpha) \cap \left\{x \in \mathbb{R}^3 \,:\, \mathcal{M}(\boldsymbol{h}\chi_\Omega)(x) > \gamma\alpha\right\}\right| \\
&\le \delta|E(\alpha)| + \left|\left\{x \in \mathbb{R}^3 \,:\, \mathcal{M}(\boldsymbol{h}\chi_\Omega)(x) > \gamma\alpha\right\}\right|.
\end{aligned}$$

This completes the proof. □

Now this good-λ type inequality can be integrated to obtain the extrapolation tool that is used to obtain uniform resolvent estimates. This takes a local reverse Hölder property to a global reverse Hölder property, see for instance [4].

Theorem 6.4. *Assume Assumption 6.1 holds. Then for all* $1 < q < p$ *and* $\boldsymbol{h} \in L^q(\Omega)$,

$$\left(\int_\Omega |\boldsymbol{g}|^q dx\right)^{\frac{1}{q}} \leq C \int_\Omega |\boldsymbol{g}| dx + C \left(\int_\Omega |\boldsymbol{h}|^q dx\right)^{\frac{1}{q}}, \tag{6.11}$$

where $C > 0$ *is a constant depending only on* p, q, C_1, C_2, $|Q_0|$ *and the Lipschitz character of* Ω.

Proof. In view of Lemma 6.3 we recall that

$$\begin{aligned}
\alpha_0 &:= \frac{C}{\delta |Q_0|} \int_\Omega |\boldsymbol{g}| dx \\
E(A\alpha) &:= \left\{x \in \mathbb{R}^3 \,:\, \mathcal{M}_{2Q_0}(\boldsymbol{g}\chi_\Omega)(x) > A\alpha\right\} \\
A &:= (2\delta)^{-\frac{1}{q}}.
\end{aligned}$$

Now let $n > \alpha_0$ and define the following integral

$$I_n = \int_0^n q\alpha^{q-1} |E(\alpha)| d\alpha.$$

I_n is bounded. We can now rewrite I_n

$$\begin{aligned}
I_n &= \int_0^{\alpha_0} q\alpha^{q-1} |E(\alpha)| d\alpha + \int_{\alpha_0}^n q\alpha^{q-1} |E(\alpha)| d\alpha \\
&= I + \hat{I}_n.
\end{aligned}$$

The first part of I_n can easily be estimated.

$$I = \int_0^{\alpha_0} q\alpha^{q-1} |E(\alpha)| d\alpha \leq |Q_0| \int_0^{\alpha_0} q\alpha^{q-1} d\alpha = |Q_0| \alpha_0^q.$$

For the second part of I_n we use substitution and Lemma 6.3

$$\begin{aligned}
\hat{I}_n &= A^q \int_{\frac{\alpha_0}{A}}^{\frac{n}{A}} q\alpha^{q-1} |E(A\alpha)| d\alpha \\
&= A^q \int_{\frac{\alpha_0}{A}}^{\frac{n}{A}} q\alpha^{q-1} \delta |E(\alpha)| d\alpha + A^q \int_{\frac{\alpha_0}{A}}^{\frac{n}{A}} q\alpha^{q-1} \left|\left\{x \in \mathbb{R}^3 \,:\, \mathcal{M}(\boldsymbol{h}\chi_\Omega)(x) > \gamma\alpha\right\}\right| d\alpha \\
&\leq \frac{1}{2} \int_{\frac{\alpha_0}{A}}^{\frac{n}{A}} q\alpha^{q-1} |E(\alpha)| d\alpha + C(\delta, \gamma) \int_0^\infty q\alpha^{q-1} \left|\left\{x \in \mathbb{R}^3 \,:\, \mathcal{M}(\boldsymbol{h}\chi_\Omega)(x) > \alpha\right\}\right| d\alpha \\
&\leq \frac{1}{2} \int_0^{\frac{n}{A}} q\alpha^{q-1} |E(\alpha)| d\alpha + C(\delta, \gamma) \int_\Omega |\boldsymbol{h}|^q dx.
\end{aligned}$$

Now if $A \geq 1$ we have that

$$I_n = I + \frac{1}{2} I_n + C(\delta, \gamma) \int_\Omega |\boldsymbol{h}|^q dx$$

and if $0 < A < 1$, then

$$I_{n/A} = I + \frac{1}{2} I_{n/A} + C(\delta,\gamma) \int_\Omega |\boldsymbol{h}|^q dx,$$

which is essentially the same result after simple substitution. Since I_n is finite we can subtract it from both sides. If we now let $n \to \infty$ we find

$$\int_\Omega |\boldsymbol{g}|^q dx \le \frac{C^q}{|Q_0|^{q-1}} \left(\int_\Omega |\boldsymbol{g}| dx \right)^q + C \int_\Omega |\boldsymbol{h}|^q dx,$$

where C only depends on p, q, C_1, C_2, $|Q_0|$ and the Lipschitz character of Ω. The result follows by taking the q^{th} root. This completes the proof. □

We are now able to proof the main result of the paper by Shen [34].

Theorem 6.5. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$ and $\lambda \in \Sigma_\theta$. There exists $\varepsilon > 0$, depending only on θ and the Lipschitz character of Ω, such that if $\boldsymbol{f} \in L^2(\Omega;\mathbb{C}^3) \cap L^p(\Omega;\mathbb{C}^3)$ and*

$$\left| \frac{1}{p} - \frac{1}{2} \right| < \frac{1}{6} + \varepsilon. \tag{6.12}$$

Then the unique solution $\boldsymbol{u}$ of the Dirichlet problem for the Stokes System (1.9) *in $H_0^1(\Omega;\mathbb{C}^3)$ satisfies the estimate*

$$\|\boldsymbol{u}\|_{L^p(\Omega)} \le \frac{C_p}{|\lambda| + r_0^{-2}} \|\boldsymbol{f}\|_{L^p(\Omega)}, \tag{6.13}$$

where $r_0 = \operatorname{diam}(\Omega)$ and C_p depends at most on p, θ and the Lipschitz character of Ω.

Proof. Let $\boldsymbol{f} \in L^2(\Omega;\mathbb{C}^3) \cap L^p(\Omega;\mathbb{C}^3)$. Then there exists a unique $\boldsymbol{u} \in H_0^1(\Omega;\mathbb{C}^3)$ such that

$$\|\boldsymbol{u}\|_{L^2(\Omega)} \le \frac{C}{|\lambda|} \|\boldsymbol{f}\|_{L^2(\Omega)}$$

(see for instance [12]). We start off by going to show that Assumption 6.1 holds for $p = \frac{3}{2} - \varepsilon$, $\boldsymbol{g} = \boldsymbol{u}^2$ and $\boldsymbol{h} = (|\lambda| + 1)^{-2} \boldsymbol{f}^2$. By Theorem 5.25

$$\begin{aligned}\left(⨍_{Q\cap\Omega} \chi_{3Q\setminus \operatorname{supp}(\boldsymbol{f})} \left(|\boldsymbol{u}|^2 \right)^{\frac{3+\varepsilon}{2}} dx \right)^{\frac{2}{3}} &= \left(\left(⨍_{Q} \chi_{3Q\setminus \operatorname{supp}(\boldsymbol{f})} |\boldsymbol{u}|^{3+\varepsilon} dx \right)^{\frac{1}{3+\varepsilon}} \right)^2 \\ &\le \left(C \left(⨍_{2Q\cap\Omega} |\boldsymbol{u}|^2 dx \right)^{\frac{1}{2}} \right)^2.\end{aligned}$$

Thus (i) is fullfilled. For Assumption (ii) we see

$$⨍_{2Q\cap\Omega} \chi_{3Q\cap \operatorname{supp}(\boldsymbol{f})} |\boldsymbol{u}|^2 dx \le \frac{C}{|\lambda| + 1} ⨍_{2Q\cap\Omega} |\boldsymbol{f}|^2 dx \le C \sup_{Q' \supset Q} ⨍_{Q'} \left(\left| \frac{\boldsymbol{f}}{|\lambda| + 1} \right| \right)^2 dx.$$

Now we can use Theorem 6.4 to find

$$\left(\int_\Omega |\boldsymbol{u}|^{2q} dx \right)^{\frac{1}{q}} \le C \int_\Omega |\boldsymbol{u}|^2 dx + \frac{C}{|\lambda| + 1} \left(\int_\Omega |\boldsymbol{f}|^{2q} dx \right)^{\frac{1}{q}}.$$

Now taking the square root of this inequality we find

$$\left(\int_\Omega |\boldsymbol{u}|^{2q} dx \right)^{\frac{1}{2q}} \le \frac{C}{|\lambda| + 1} \left(\int_\Omega |\boldsymbol{f}|^2 dx \right)^{\frac{1}{2}} + \frac{C}{|\lambda| + 1} \left(\int_\Omega |\boldsymbol{f}|^{2q} dx \right)^{\frac{1}{2q}},$$

where $2 < 2q < 3 + \varepsilon$. Since Ω is bounded the result follows by Hölder's inequality. This shows the resolvent bound for $2 \le p < 3 + \varepsilon$. The range $\frac{2}{3} - \varepsilon < p < 2$ follows by duality. □

6.2 Weighted resolvent estimates

In this section we generalize the resolvent estimate (Theorem 6.5) to the weighted case. To do so, we introduce the class of Muckenhoupt weights. We also define the weak reverse Hölder class. We show that for a special class of weights, a weighted L^2 resolvent estimate holds. Using an extrapolation theorem of Rubio de Francia we obtain the weighted L^p resolvent estimate for a range of p.

6.2.1 Basic properties of weights

We now start with the definition of a weight function.

Definition 6.6. *A weight is a nonnegative locally integrable function on $\mathbb{R}^3$ that takes valus in $(0,\infty)$ almost everywhere.*

It is clear that if dx is the Lebesgue measure, $w(x)dx$ also defines a measure. Hence for some measurable set E we can define the weighted volume of E by

$$w(E) := \int_E w(x)dx. \tag{6.14}$$

Similarly, it is natural to introduce weighted L^p spaces. The natural norm on this space is given by

$$\|f\|_{L^p(w)} := \left(\int_{\mathbb{R}^3} |f(x)|^p w(x)dx\right)^{\frac{1}{p}}. \tag{6.15}$$

In the theory of weights, there are certain classes of weights that have special properties. One of them is the class of A_p weights.

Definition 6.7 (A_p weight)**.** *Let $1 < p < \infty$. A weight w is of class A_p if*

$$[w]_{A_p} := \sup_{Q\subset\mathbb{R}^3} \left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-\frac{1}{p-1}}\right)^{p-1} < \infty. \tag{6.16}$$

$[w]_{A_p}$ is called the Muckenhoupt characteristic constant of w (or shorter, the A_p constant of w).

Remark 6.8. *The weights in the Muckenhoupt class A_p are precisely those weights for which the Hardy-Littlewood maximal function is bounded.*

Remark 6.9. *In the definition above we define the Muckenhoupt weights over all cubes in $\mathbb{R}^3$. It is also possible to define the Muckenhoupt weights on cubes that are intersected with a domain (for example a Lipschitz domain, Ω). To be precise, a weights w is in the class $A_{p,\Omega}$ if*

$$\sup_{Q\subset\mathbb{R}^3} \left(\frac{1}{|Q|}\int_{Q\cap\Omega} w(x)dx\right)\left(\frac{1}{|Q|}\int_{Q\cap\Omega} w(x)^{-\frac{1}{p-1}}\right)^{p-1} < \infty.$$

However, it turns out that for all the weights $w \in A_{p,\Omega}$ there exists $\tilde{w} \in A_p$ (defined in $\mathbb{R}^3$) such that $w(x) = \tilde{w}(x)$ for almost all $x \in \Omega$. See for instance [18].

Definition 6.10 (RH_s class)**.** *Let $1 < s < \infty$ and w be a weight. We say $w \in \mathrm{RH}_s$ if it satisfies the reverse Hölder inequality*

$$\left(⨍_Q w^s(x)dx\right)^{\frac{1}{s}} \leq C⨍_Q w(x)dx, \tag{6.17}$$

for all cubes $Q \subset \mathbb{R}^3$. The smallest constant for which this inequality holds is the reverse Hölder characteristic constant of w (or short, the RH_s constant of w) and denoted by $[w]_{\mathrm{RH}_s}$

Now that we have introduced the Muckenhoupt weight class and the reverse Hölder weight class, we are going to proof some properties about these weight classes.

Lemma 6.11. *Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz domain and $w \in \mathrm{RH}_s$. Then*

$$\frac{w(E)}{w(Q)} \leq C \left(\frac{|E|}{|Q|} \right)^{1-\frac{1}{s}},$$

when $E \subset Q \subset \Omega$.

Proof. We will make use of the reverse Hölder property in the denominator together with the "normal" Hölder inequality in the enumerator. Now assume that $E \subset Q \subset \Omega$.

$$\begin{aligned}
\frac{w(E)}{w(Q)} &= \frac{\int_Q \chi_E w(x)dx}{\int_Q w(x)dx} \\
&\leq \frac{\left(\int_Q w^s(x)dx\right)^{\frac{1}{s}} \left(\int_Q \chi_E dx\right)^{1-\frac{1}{s}}}{|Q| \fint_Q w(x)dx} \\
&\leq C \frac{|Q|^{\frac{1}{s}} \left(\fint_Q w^s(x)dx\right)^{\frac{1}{s}} |E|^{1-\frac{1}{s}}}{|Q| \left(\fint_Q w^s(x)dx\right)^{\frac{1}{s}}} \\
&= C \left(\frac{|E|}{|Q|}\right)^{1-\frac{1}{s}}.
\end{aligned}$$

This completes the proof. ☐

Lemma 6.12. *Let $1 < p < 2$. Let $w \in A_{\frac{2}{p}}$. Let Q be a cube. Then*

$$\|f\|_{L^p(Q)} \leq w(Q)^{-\frac{1}{2}} [w]_{A_{\frac{2}{p}}}^{\frac{1}{2}} \|f\|_{L^2(Q,w)}.$$

Proof. Let $f \in L^p(w)$. Now fix a cube Q.

$$\begin{aligned}
\int_Q |f(x)|^p dx &= \int_Q |f(x)|^p w(x)^{\frac{p}{2}} w(x)^{-\frac{p}{2}} dx \\
&\leq \left(\int_Q |f(x)|^2 w(x) dx \right)^{\frac{p}{2}} \left(\int_Q w(x)^{-\frac{1}{\frac{2}{p}-1}} dx \right)^{1-\frac{p}{2}}.
\end{aligned}$$

Hence

$$\begin{aligned}
\|f\|_{L^p} &\leq \|f\|_{L^2(w)} \left(\int_Q w(x)^{-\frac{1}{\frac{2}{p}-1}} dx \right)^{\frac{1}{2}\left(\frac{2}{p}-1\right)} \\
&= \|f\|_{L^2(w)} w(Q)^{-\frac{1}{2}} \left(\int_Q w(x) dx \right)^{\frac{1}{2}} \left(\int_Q w(x)^{-\frac{1}{\frac{2}{p}-1}} dx \right)^{\frac{1}{2}\left(\frac{2}{p}-1\right)} \\
&\leq w(Q)^{-\frac{1}{2}} [w]_{A_{\frac{2}{p}}}^{\frac{1}{2}} \|f\|_{L^2(w)}.
\end{aligned}$$

This completes the proof. ☐

6.2.2 Weighted extrapolation

Now that the necessary definitions and basic lemmas are introduced, it is time to set the stage for this section. Thereafter we again prove an extrapolation theorem and show how this helps in proving the weighted L^2 case.

Assumption 6.13. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain, let Q_0 be a cube in $\mathbb{R}^3$ such that $\Omega \subset Q_0$ and let $\boldsymbol{g} \in L^1(2Q_0)$. Let $1 < q < p$ and $\boldsymbol{h} \in L^q(2Q_0)$. Let Q be such that $|Q| < |Q_0|$ and define $\boldsymbol{g}_1 = \boldsymbol{g}\chi_{3Q\setminus \text{supp}(\boldsymbol{h})}$ and $\boldsymbol{g}_2 = \boldsymbol{g}\chi_{3Q\cap \text{supp}(\boldsymbol{h})}$. Let $w \in A_q$. Assume,*

(i) there exists a constant $C_1 > 0$ such that the following local reverse Hölder estimate holds

$$\left(\fint_{Q\cap\Omega} |\boldsymbol{g}_1|^p dx\right)^{\frac{1}{p}} \leq \left(C_1 \fint_{2Q\cap\Omega} |\boldsymbol{g}| dx + \sup_{Q'\supset Q} \fint_{Q'} |\boldsymbol{h}| dx\right) \tag{6.18}$$

.

(ii) There exists a constant $C_2 > 0$ such that

$$\fint_{2Q} |\boldsymbol{g}_2| dx \leq C_2 \sup_{Q'\supset Q} \fint_{Q'} |\boldsymbol{h}| dx \tag{6.19}$$

.

(iii) There exists a constant $\eta > q/p$ and a $C_3 > 0$ such that for all $E \subset Q \subset Q_0$

$$\frac{w(E)}{w(Q)} \leq C_3 \left(\frac{|E|}{|Q|}\right)^{\eta}. \tag{6.20}$$

I.e. the weights belong to a reverse Hölder class.

Remark 6.14. *Notice that Assumption 6.13 implies Assumption 6.1.*

Recall the definition of $E(\alpha)$ (Equation (6.3)). We now find a weighted version of the good-λ type inequality of the previous section.

Lemma 6.15. *Assume Assumption 6.13 holds. Then,*

$$w(E(A\alpha)) \leq C\delta^{\eta} w(E(\alpha)) + w\left\{x \in Q_0 \,:\, \mathcal{M}_{2Q_0}(\boldsymbol{h}\chi_{\Omega})(x) > \gamma\alpha\right\} \tag{6.21}$$

for all $\alpha > \alpha_0$, where $A = (2\delta)^{\frac{-1}{\tilde{q}}}$ for $\tilde{q} \in (p,q)$ such that $\eta > \frac{q}{\tilde{q}}$ and

$$\alpha_0 = \frac{C_0}{|Q_0|} \int_{\Omega} |\boldsymbol{g}| dx. \tag{6.22}$$

Proof. This follows by slightly modifying the proof of Lemma 6.3. Notice that by (iii) of Assumption 6.13 $|E(A\lambda) \cap Q_j| \leq \delta |Q_j|$ implies $w(E(A\lambda) \cap Q_j) \leq C\delta^{\eta} w(Q_j)$. □

Theorem 6.16. *Assume Assumption 6.13 holds. Then for all $1 < p < q$ and $\boldsymbol{h} \in L^q(2Q_0)$,*

$$\left(\frac{1}{w(Q_0)} \int_{Q_0} |\boldsymbol{g}|^q w dx\right)^{\frac{1}{q}} \leq \frac{C}{|2Q_0|} \int_{2Q_0} |\boldsymbol{g}| dx + C\left(\frac{1}{w(2Q_0)} \int_{2Q_0} |\boldsymbol{h}|^q w(x) dx\right)^{\frac{1}{q}}. \tag{6.23}$$

Proof. In view of Lemma 6.15 we recall that

$$\alpha_0 := \frac{C_0}{\delta|Q_0|}\int_\Omega |\boldsymbol{g}|^2 dx$$
$$E(A\alpha) := \{x \in Q_0 \,:\, \mathcal{M}_{2Q_0}(\boldsymbol{g}\chi_\Omega)(x) > A\alpha\}$$
$$A := (2\delta)^{\frac{-1}{q}}.$$

Now let $n > \alpha_0$ and define the following integral

$$I_n := \int_0^n q\alpha^{q-1} w(E(\alpha))d\alpha. \tag{6.24}$$

Notice that I_n is finite. We now rewrite I_n

$$I_n = \int_0^{\alpha_0} q\alpha^{q-1} w(E(\alpha))d\alpha + \int_{\alpha_0}^n q\alpha^{q-1} w(E(\alpha))d\alpha = I + \tilde{I}_n.$$

We now examine both parts. For the first part of I_n we find

$$I = \int_0^{\alpha_0} q\alpha^{q-1} w(E(\alpha))d\alpha \le w(Q_0)\int_0^{\alpha_0} q\alpha^{q-1} d\alpha \le w(Q_0)\alpha_0^q.$$

For the second part of I_n we use Lemma 6.15

$$\begin{aligned}
\tilde{I}_n &= \int_{\alpha_0}^n q\alpha^{q-1} w(E(\alpha))d\alpha \\
&= A^q \int_{\frac{\alpha_0}{A}}^{\frac{n}{A}} q\alpha^{q-1} w(E(A\alpha))d\alpha \\
&\le A^q C\delta^\eta \int_0^{\frac{n}{A}} q\alpha^{q-1} w(E(\alpha)) + A^q \int_{\alpha_0}^{\frac{n}{A}} q\alpha^{q-1} w\left(\left\{x \in \mathbb{R}^3 \,:\, \mathcal{M}_{2Q_0}(\boldsymbol{h}\chi_\Omega)(x) > \gamma\alpha\right\}\right) d\alpha \\
&\le A^q C\delta^\eta I_n + C_{\delta,\gamma}\int_\Omega |\boldsymbol{h}|^q w(x)dx,
\end{aligned}$$

where we did use the boundedness of the maximal function (Theorem A.5). If we now choose δ small enough, then $A^q C\delta^\eta < \frac{1}{2}$. Since I_n is finite we can subtract $A^q C\delta^\eta I_n$ from both sides to obtain

$$I_n \le I + C(\delta,\gamma)\int_\Omega |\boldsymbol{h}|^q w(x)dx.$$

Now letting $n \to \infty$ we find

$$\int_\Omega |\boldsymbol{g}|^q w(x)dx \le \frac{Cw(Q_0)}{|Q_0|^q}\left(\int_\Omega |\boldsymbol{g}|dx\right)^q + C\int_\Omega |\boldsymbol{h}|^q w(x)dx. \tag{6.25}$$

The result now follows. □

Theorem 6.17. *Let $\Omega \in \mathbb{R}^3$ be a Lipschitz domain and let $w \in A_{\frac{4}{3}} \cap \mathrm{RH}_3$ be a weight. Let $\boldsymbol{f} \in L^2(\Omega, w) \cap L^{\frac{3}{2}}(\Omega)$. Then the solution of (1.9) satisfies*

$$\int_\Omega |\boldsymbol{u}(x)|^2 w(x)dx \le \frac{C}{|\lambda|}\int_\Omega |\boldsymbol{f}(x)|^2 w(x)dx. \tag{6.26}$$

Proof. Assume $\boldsymbol{f} \in L^2(\Omega, w) \cap L^{\frac{3}{2}}(\Omega)$. Then there exists a unique $\boldsymbol{u} \in H_0^1(\Omega)$ such that it solves (1.9). To start we check if Assumption 6.13 holds with $p = 2 + \varepsilon$, $\boldsymbol{g} = \boldsymbol{u}^{\frac{3}{2}}$, $\boldsymbol{h} = \left(\frac{\boldsymbol{f}}{|\lambda|}\right)^{\frac{3}{2}}$. Let $|Q| < |Q_0|$. We start with (i). By Theorem 5.25

$$\begin{aligned}\left(\fint_Q \left(\chi_{5Q\setminus \mathrm{supp}(f)}|\boldsymbol{u}|^{\frac{3}{2}}\right)^2 dx\right)^{\frac{1}{2}} &= \left(\left(\fint_Q \chi_{5Q\setminus \mathrm{supp}(\boldsymbol{f})}|\boldsymbol{u}|^{3+\varepsilon} dx\right)^{\frac{1}{3+\varepsilon}}\right)^{\frac{3}{2}} \\ &\le C\left(\left(\fint_{2Q} \chi_{5Q\setminus \mathrm{supp}(\boldsymbol{f})}|\boldsymbol{u}|^2 dx\right)^{\frac{1}{2}}\right)^{\frac{3}{2}} \\ &\le C\left(\left(\fint_{4Q} |\boldsymbol{u}|^{\frac{3}{2}} dx\right)^{\frac{2}{3}}\right)^{\frac{3}{2}} \\ &= C\fint_{4Q} |\boldsymbol{u}|^{\frac{3}{2}} dx.\end{aligned}$$

This shows (i). By Theorem 6.5 we have

$$\fint_{2Q} \chi_{5Q\cap \mathrm{supp}(\boldsymbol{f})}|\boldsymbol{u}|^{\frac{3}{2}} dx \le C|\lambda|^{-\frac{2}{3}}\fint_{2Q} \chi_{5Q\cap \mathrm{supp}(\boldsymbol{f})}|\boldsymbol{f}|^{\frac{3}{2}} \le C|\lambda|^{-\frac{2}{3}} \sup_{Q'\supset Q} \int_{Q'} |\boldsymbol{f}|^{\frac{3}{2}} dx,$$

which shows (ii). Since we have $w \in A_{\frac{4}{3}} \cap \mathrm{RH}_3$, (iii) follows from Lemma 6.11. Now, using Theorem 6.16, we find

$$\left(\int_\Omega |\boldsymbol{u}(x)|^2 w(x) dx\right) \le Cw(\Omega)^{\frac{1}{2}}\left(\int_\Omega |\boldsymbol{u}(x)|^{\frac{3}{2}} dx\right)^{\frac{2}{3}} + C\int_\Omega |\boldsymbol{f}(x)|^2 w(x) dx. \tag{6.27}$$

Using Lemma 6.12 this leads to

$$\left(\int_\Omega |\boldsymbol{u}(x)|^2 w(x) dx\right) \le C\left(\int_\Omega |\boldsymbol{f}(x)|^2 |\lambda|^{-2} w(x) dx\right)^{\frac{1}{2}}. \tag{6.28}$$

This completes the proof. ☐

A famous result by Rubio de Francia allows for extrapolation of this inequality.

Theorem 6.18 (Rubio de Francia)**.** *Let* $\boldsymbol{f}$ *and* $\boldsymbol{g}$ *be non-negative, measurable functions that are not identically zero. Assume the for all* $w_0 \in A_{\frac{4}{3}} \cap \mathrm{RH}_3$ *and we have that*

$$\int_\Omega |\boldsymbol{g}(x)|^2 w_0(x) dx \le \theta([w]_{A_{\frac{4}{3}}}, [w]_{\mathrm{RH}_3}) \int_\Omega |\boldsymbol{h}(x)|^2 w_0(x) dx, \tag{6.29}$$

where θ *is an increasing function of* $[w]_{A_{\frac{4}{3}}}$ *and* $[w]_{\mathrm{RH}_3}$*. Then, for all* $\frac{3}{2} < p < 3$*, and for all* $w \in A_{\frac{2p}{3}} \cap \mathrm{RH}_{\frac{3}{3-p}}$ *we have*

$$\int_\Omega |\boldsymbol{g}(x)|^p w(x) dx \le C\int_\Omega |\boldsymbol{h}(x)|^p w(x) dx, \tag{6.30}$$

where C *depends on* θ*,* p*,* $[w]_{A_{\frac{2p}{3}}}$ *and* $[w]_{\mathrm{RH}_{\frac{3}{3-p}}}$*.*

The proof of this theorem can be found in [8, Theorem 3.31]. This allows us to state a weighted version of the inequality found by Shen in [34, Theorem 1.1].

Theorem 6.19. *Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz domain. Then for all $\frac{3}{2} < p < 3$, $\boldsymbol{f} \in L^2(\Omega, w) \cap L^p(\Omega, w)$ and all $w \in A_{\frac{2p}{3}} \cap \mathrm{RH}_{\frac{3}{3-p}}$ the unique solution of the Dirichlet problem for the Stokes System (1.9) in $H_0^1(\Omega)$ satisfies the estimate*

$$\|\boldsymbol{u}\|_{L^p(\Omega,w)} \leq \frac{C_p}{|\lambda| + c}\|\boldsymbol{f}\|_{L^p(\Omega,w)}, \tag{6.31}$$

where the constant does not depend on λ.

Proof. Apply Theorem 6.18 to extrapolate the result of Theorem 6.17. □

6.2.3 Examples

In the previous section results were proven for a special class of weights, $A_{\frac{2p}{3}} \cap \mathrm{RH}_{\frac{3}{3-p}}$. In this section we look for examples of weights that belong in this weight class. A candidate for such a weight is the distance to the boundary raised to some power. In this section we show that this is actually the case. In order to start off we state the following lemma.

Lemma 6.20. *Let w be a weight. Then $w \in A_p \cap \mathrm{RH}_s$ if and only if $w^s \in A^q$, where $q = s(p-1)+1$*

For a proof of this lemma we refer to a paper by Johnson and Neugebauer [27]. This already hints in the direction of a weight raised to some power. We now state the following result from Farwig and Sohr [14].

Theorem 6.21. *Let Ω be a Lipschitz domain in $\mathbb{R}^3$ Then,*

$$d(x, \partial\Omega)^\delta \in A_p \quad -1 < \delta < p-1. \tag{6.32}$$

Using Lemma 6.20 and Theorem 6.21 we can show that indeed the distance to the boundary of a Lipschitz domain raised to a certain power is a weight that belongs to the class $A_{\frac{2p}{3}} \cap \mathrm{RH}_{\frac{3}{3-p}}$.

Theorem 6.22. *Let Ω be a Lipschitz domain in $\mathbb{R}^3$ and let $\frac{3}{2} < p < 3$. Then,*

$$\mathrm{dist}(x, \partial\Omega)^\gamma \in A_{\frac{2p}{3}} \cap \mathrm{RH}_{\frac{3}{3-p}} \quad \textit{when} -\frac{3-p}{3} < \gamma < \frac{2p}{3} - 1. \tag{6.33}$$

Proof. Let $w(x) = \mathrm{dist}(x, \partial\Omega)^\gamma$. Now we show for which γ we have that $w \in A_{\frac{2p}{3}} \cap \mathrm{RH}_{\frac{3}{3-p}}$. Then by Lemma 6.20 this is equivalent to finding γ for which $w^{\frac{3\gamma}{3-p}} \in A_{\frac{p}{3-p}}$. By Theorem 6.21 we find that

$$w^{\frac{\gamma}{3-p}} \in A_{\frac{p}{3-p}} \quad \text{when} -1 < \frac{3\gamma}{3-p} < \frac{p}{3-p} - 1.$$

Now the result follows. □

7 Conclusion and Discussion

Following the recent paper by Z. Shen [34], resolvent estimates in L^p for the Stokes operator in three-dimensional Lipschitz domains were studied.

In order to find resolvent estimates, the Dirichlet problem for the Stokes System is considered. The desired inequality between the solution and the inhomogeneous term is known in L^2. In order to extrapolate this inequality to L^p, where $\frac{3}{2} - \varepsilon < p < 3 + \varepsilon$ and ε depends on the Lipschitz domain, we needed to establish a local reverse Hölder property of the solution of the Dirichlet problem of the Stokes System.

Solving the homogeneous Stokes System (1.11) is the stepping stone to the Dirichlet problem. Layer potentials are a good candidate to solve the homogeneous Stokes System, because they solve the Stokes System in the domain, they have bounded nontangential maximal functions and nontangential limits almost everywhere on the boundary of the Lipschitz domain. It can be shown that the layer potential is invertible in L^2 using a compactness argument, hence solving the boundary value problem. However, in using this argument, control is lost over the parameter λ, and thus rendering it unsuitable for finding resolvent estimates in L^p.

To overcome this problem an alternative for compactness is found in Rellich type estimates. These estimates enable us to bound the norm of the operator that maps boundary data to the corresponding density for the layer potential in L^2.

This implies that, given boundary data in L^2_n, a solution to the Stokes System exists, is unique and can be represented by the double layer potential. Because of the nontangential boundedness of the layer potentials a uniform L^p estimate for these solutions is found.

These uniform L^p estimates can be used to establish the before mentioned local reverse Hölder properties of the solution of the Dirichlet problem for the Stokes System and the result follows by extrapolation. This concludes the study of the paper by Z. Shen [34].

The author shows that the local reverse Hölder property of the solution can also be used to proof a weighted resolvent bound in the range $\frac{3}{2} - \varepsilon < p < 3 + \varepsilon$. The class of weights for which these estimates hold, are an intersection of the Muckenhoupt class A_q and the reverse Hölder class RH_s, where q and s depend on p. Examples of weights in this class power weights with power depending on p and the distance to the boundary to a power depending on p.

7.1 Future research

This thesis contains some open ends and also possible directions for further research.

First of all, to the best of the authors knowledge, the literature does not contain a proof of the existence of the Helmholtz decomposition of $L^p(w)$, where $w \in A_q \cap RH_s$ with q and s depending on p. It is conjectured that the Helmholtz decomposition holds for $L^p(w)$, where $w \in A_{\frac{2p}{3}} \cap RH_{\frac{3}{3-p}}$.

Secondly, the result on weighted resolvent estimates can be extended to the higher dimensional case, $d > 3$, and the lower dimensional case, $d = 2$. Then one can show that the weighted Stokes operator generates an analytic semigroup on $L^p_\sigma(w)$. Especially in the higher dimensional case no major obstacles are expected.

Finally, it would be interesting to know if there exists a class of weights for which the resolvent estimates of the weighted Stokes operator go beyond $p = 3$, i.e. more than one ε. The distance to the boundary weights and power weights seem to be good candidates.

A Harmonic analysis

This section contains the results from harmonic analysis that are needed throughout the text of the thesis. The results and proofs are from Grafakos [22], [23] and lecture notes by Auscher [3]. In this literature the results are written down for the space $\mathbb{R}^d$. Most results can be extended to spaces of homogeneous type as introduced by Coifman and Weiss [6], but it needs to be checked. Lipschitz domains with the Euclidean metric and Lebesgue measure are spaces of homogeneous type in this sense. Notice that the surface measure on the boundary of a Lipschitz domain is a doubling measure by Lemma 2.8. We start by introducing the Hardy-Littlewood maximal function.

Definition A.1 (Hardy-Littlewood maximal function)**.** *Let Ω be a bounded and connected Lipschitz domain in $\mathbb{R}^d$ and $f \in L^1_{loc}(\partial\Omega)$. Then the Hardy-Littlewood maximal function of f on the boundary of Ω is given by*

$$\mathcal{M}_{\partial\Omega}(f)(x) = \sup_{B \ni x} \frac{1}{\sigma(B \cap \partial\Omega)} \int_{B \cap \partial\Omega} |f(y)| d\sigma(y). \tag{A.1}$$

One of the key properties of the Hardy-Littlewood maximal function is its weak L^1 boundedness. This is illustrated in the following theorem.

Theorem A.2 (Maximal Theorem)**.** *Let Ω be a bounded and connected Lipschitz domain in $\mathbb{R}^d$ and $f \in L^1_{\text{loc}}(\partial\Omega)$. Then there exists a $C > 0$ such that for all $\alpha > 0$,*

$$\sigma(\{x \in \partial\Omega \,:\, \mathcal{M}_{\partial\Omega}(f)(x) > \alpha\}) \leq \frac{C}{\alpha} \int_{\partial\Omega} |f(y)| d\sigma(y). \tag{A.2}$$

Proof. Define

$$O_\alpha = \{x \in \partial\Omega \,:\, \mathcal{M}_{\partial\Omega}(f)(x) > \alpha\}.$$

Then for all $x \in O_\alpha$, there exists a ball B_x with bounded radius and $x \in B_x$ such that

$$\frac{1}{\sigma(B_x \cap \partial\Omega)} \int_{B_x \cap \partial\Omega} |f(y)| d\sigma(y) > \alpha.$$

The radius is bounded because Ω is bounded. Let $\mathcal{B} = \{B_x\}_{x \in O_\alpha}$ and apply the Vitali covering lemma. So, there exists a countable subset of mutually disjoint balls, denoted by $\mathcal{C}$, such that

$$O_\alpha \subset \bigcup_{B_j \in \mathcal{C}} 5B_j.$$

Therefore we find using Lemma 2.8 that

$$\begin{aligned}
\sigma(O_\alpha) &\leq \sum_{B_j \in \mathcal{C}} \sigma(5B_j) \leq C^3 \sum_{B_j \in \mathcal{C}} \sigma(B_j) \leq \frac{C^3}{\alpha} \sum_{B_j \in \mathcal{C}} \int_{B_j} |f(y)| d\sigma(y) \\
&\leq \frac{C^3}{\alpha} \int_{\bigcup_{B_j \in \mathcal{C}} B_j} |f(y)| d\sigma(y) \leq \frac{C^3}{\alpha} \int_{O_\alpha} |f(y)| d\sigma(y).
\end{aligned}$$

This completes the proof. □

The Maximal Theorem is crucial in the proof of the Lebesgue Differentiation Theorem. This Theorem has applications in in the rest of this section and also in the text of the thesis.

Theorem A.3 (Lebesgue Differentiation Theorem)**.** *Let $f \in L^1(\mathbb{R}^d)$. Then,*

$$\lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)dy = f(x)$$

for almost every $x \in \mathbb{R}^d$.

Proof. The proof can be found in [22, Corollary 2.1.16]. $\square$

Notice that this theorem also holds if the balls are replaced by cubes. Furthermore this is a local Theorem. We can, in the case that Ω is a bounded Lipschitz domain replace f by $f\chi_{\partial\Omega}$. Now we turn to one of the most useful results of this section.

Theorem A.4. *Let Ω be a bounded and connected Lipschitz domain in $\mathbb{R}^d$, let $1 < p \leq \infty$ and let $f \in L^p(\partial\Omega)$. Then $\mathcal{M}_{\partial\Omega}(f) \in L^p(\partial\Omega)$ and there exists a $C > 0$ such that*

$$\|\mathcal{M}_{\partial\Omega}(f)\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p(\partial\Omega)}. \tag{A.3}$$

Proof. Assume that $f \in L^1(\partial\Omega) \cap L^p(\partial\Omega)$. Now let $\alpha > 0$ and define

$$f_\alpha(x) = \chi_{|f(x)|\geq\alpha/2}(x)f(x)$$

and therefore

$$|f| \leq |f_\alpha| + \frac{\alpha}{2}$$

and hence

$$\mathcal{M}_{\partial\Omega}(f) \leq \mathcal{M}_{\partial\Omega}(f_\alpha) + \frac{\alpha}{2}$$

and

$$\{x \in \partial\Omega \, : \, \mathcal{M}_{\partial\Omega}(f) > \alpha\} \subset \left\{x \in \partial\Omega \, : \, \mathcal{M}_{\partial\Omega}(f_\alpha) > \frac{\alpha}{2}\right\}.$$

Therefore by the Maximal Theorem,

$$\sigma(\{x \in \partial\Omega \, : \, \mathcal{M}_{\partial\Omega}(f) > \alpha\}) \leq \sigma(\left\{x \in \partial\Omega \, : \, \mathcal{M}_{\partial\Omega}(f_\alpha) > \frac{\alpha}{2}\right\}) \leq \frac{C}{\alpha}\int_{\partial\Omega} |f_\alpha| d\sigma.$$

Thus, by Fubini,

$$\begin{aligned}
\int_{\partial\Omega} |\mathcal{M}_{\partial\Omega}(f)|^p d\sigma &= p\int_0^\infty \sigma(\{x \in \partial\Omega \, : \, \mathcal{M}_{\partial\Omega}(f) > \alpha\})\alpha^{p-1}d\alpha \\
&\leq p\int_0^\infty \frac{C}{\alpha}\int_{\partial\Omega} |f_\alpha| d\sigma \alpha^{p-1}d\alpha \\
&\leq Cp\int_{\partial\Omega}\int_0^{2f(x)} |f_\alpha|\alpha^{p-2}d\alpha d\sigma \\
&\leq C\frac{2^{p-1}p}{p-1}\int_{\partial\Omega} |f|^p d\sigma.
\end{aligned}$$

Hence $\mathcal{M}$ is bounded in $L^1(\partial\Omega) \cap L^p(\partial\Omega)$. We now want to extend the result to boundedness on $L^p(\partial\Omega)$. By density we can find a sequence $f_k \in L^1(\partial\Omega) \cap L^p(\partial\Omega)$ such that $|f_k| \uparrow |f|$. Then $\mathcal{M}_{\partial\Omega}(f_k) \uparrow \mathcal{M}_{\partial\Omega}(f)$ and now we use the Monotone Convergence Theorem to complete the proof. $\square$

We have thus showed that the Hardy-Littlewood maximal function is bounded on $L^p(\partial\Omega)$ when $1 < p \leq \infty$. This is especially useful if we need to bound something in L^p. With this Theorem in mind a pointwise estimate against the Hardy-Littlewood maximal function is usually enough.

Theorem A.5. *Let Ω be a Lipschitz domain in $\mathbb{R}^3$. Let $1 < p < \infty$ and $w \in A_p$ a weight. Then there exists $C > 0$, such that*

$$\left(\int_\Omega |\mathcal{M}(f)(x)|^p w(x)dx\right)^{\frac{1}{p}} \leq C\left(\int_\Omega |f|^p w(x)dx\right)^{\frac{1}{p}}, \tag{A.4}$$

where C depends at most on p and the Lipschitz character of Ω.

Proof. The proof in the case $\Omega = \mathbb{R}^3$ can be found in [23, Theorem 9.1.9]. This theorem can be extended to spaces of homogenuous type in the natural way. □

Definition A.6 (Riesz potential)**.** *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain, $0 < \alpha < d$ and $f \in L^1_{\text{loc}}(\Omega)$. Then the Riesz potential of f is defined by*

$$I_\alpha(f)(x) = C_{\alpha,d}\int_\Omega \frac{f(y)}{|x-y|^{d-\alpha}}dy.$$

Theorem A.7 (Hardy-Littlewood-Sobolev inequality)**.** *Let $0 < \alpha < d$, $1 < p < q < \infty$ and $\frac{1}{q} \leq \frac{1}{p} - \frac{\alpha}{d}$. Then,*

$$\|I_\alpha(f)\|_{L^q(\Omega)} \leq C\|f\|_{L^p(\Omega)},$$

where C depends on p, q and d.

Theorem A.8. *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain, $1 < p < \infty$ and $f \in L^p(\Omega)$. Then,*

$$\|\nabla I_1(f)\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)}.$$

Definition A.9 (Calderón-Zygmund Kernel)**.** *A function K on $\mathbb{R}^d \times \mathbb{R}^d \setminus \{(x,x) : x \in \mathbb{R}^d\}$ is called a special kernel with constants δ and A if*

(i)

$$|K(x,y)| \leq \frac{A}{|x-y|^d}. \tag{A.5}$$

(ii)

$$|K(x,y) - K(x',y)| \leq \frac{A|x-x'|^\delta}{(|x-y| + |y-x'|)^{d+\delta}} \tag{A.6}$$

whenerver $|x-x'| \leq 2\max(|x-y|, |y-x'|)$.

(iii)

$$|K(x,y) - K(x,y')| \leq \frac{A|y-y'|^\delta}{(|y-x| + |x-y'|)^{d+\delta}} \tag{A.7}$$

whenerver $|y-y'| \leq 2\max(|y-x|, |x-y'|)$.

The class of special kernels with constants δ and A is denoted by $SK(\delta, A)$. If $K \in SK(\delta, A)$, then the adjoint kernel $K^*(x, y) := \overline{K(y, x)}$ is also in $SK(\delta, A)$. In the thesis we only use special kernels with $\delta = 1$. The exact value of the constant A is usually not of our interest. Now that we have defined the Calderón-Zygmund kernel we can define the operators associated with these kernels.

Definition A.10 (Associated kernel). *Let $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ be a bounded operator. A kernel K is associated to T if for all f smooth enough with compact support*

$$Tf(x) = \int_{\mathbb{R}^d} K(x, y)f(y)dy \tag{A.8}$$

for almost every $x \in \mathbb{R}^d \setminus \operatorname{supp}(f)$.

If we now take a special kernel in the previous definition, we find the definition of a Calderón-Zygmund operator.

Definition A.11 (Calderón-Zygmund Operator). *A Calderón-Zygmund Operator of order δ is a bounded operator $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ that is associated to a kernel $K \in SK(\delta, A)$.*

It turns out to be the case that Calderón-Zygmund operators are bounded on L^p, when $1 < p < \infty$. If $K(x, y)$ is a special kernel, we can also define the truncated kernel as

$$K^\varepsilon(x, y) = K(x, y)\chi_{|x-y|>\varepsilon}.$$

Using this truncated kernel we define the following operators.

$$T^\varepsilon(f)(x) = \int_{\mathbb{R}^d} K^\varepsilon(x, y)f(y)d\sigma(y)$$
$$T^\varepsilon(f)(x) = \sup_{\varepsilon>0} |T^\varepsilon(f)(x)|.$$

We call these operators the truncated operator and the maximal singular operator respectively. Regarding these operators we state the following two theorems. These theorems are formulated on $\partial\Omega$ which have homogeneous dimension $d - 1$.

Theorem A.12. *Let Ω be a Lipschitz domain. Let K be a standard kernel and let T be a Calderón-Zygmund operator associated with kernel K. Then T has a bounded extension that maps $L^1(\partial\Omega)$ into $L^{1,\infty}(\partial\Omega)$ with norm*

$$\|Tf\|_{L^{1,\infty}(\partial\Omega)} \leq C\|f\|_{L^1(\partial\Omega)} \tag{A.9}$$

and for $1 < p < \infty$, T has a bounded extension that maps $L^p(\partial\Omega)$ to $L^p(\partial\Omega)$ with norm

$$\|Tf\|_{L^p(\partial\Omega)} \leq C_p\|f\|_{L^p(\partial\Omega)} \tag{A.10}$$

for all $f \in L^p(\partial\Omega)$.

Proof. The proof for the case $\Omega = \mathbb{R}^d$ can be found in [23, Theorem 8.2.1]. It can be extended to spaces of homogeneous type. □

Theorem A.13. *Let Ω be a Lipschitz domain. Let K be a standard kernel and let T be a Calderón-Zygmund operator associated with kernel K. Let $1 < p < \infty$. Then there exists a constant $C > 0$ such that*

$$\|T^*(f)\|_{L^p(\partial\Omega)} \leq C_p\|f\|_{L^p(\partial\Omega)} \tag{A.11}$$

for all $f \in L^p(\partial\Omega)$.

Proof. The proof for the case $\Omega = \mathbb{R}^d$ can be found in [23, Corollary 8.2.4]. It can be extended to spaces of homogeneous type. □

B Standard results in real analysis

Lemma B.1 (Cauchy inequality)**.** *Let $a, b \in \mathbb{R}$ and $\varepsilon > 0$,*

$$ab \leq \varepsilon a^2 + \frac{b^2}{4\varepsilon}. \tag{B.1}$$

Proof. The result follows from a simple calculation,

$$\begin{aligned} 0 &\leq \left(a\sqrt{\varepsilon} - \frac{b}{2\sqrt{\varepsilon}}\right)^2 \\ 0 &\leq \varepsilon a^2 - ab + \frac{b^2}{4\varepsilon} \\ ab &\leq \varepsilon a^2 + \frac{b^2}{4\varepsilon}. \end{aligned}$$

□

Theorem B.2 (Hölder inequality)**.** *Let $p, q \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then for all measurable functions f and g we have*

$$\|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^q}.$$

Theorem B.3 (Poincaré inequality)**.** *Let $1 \leq p < \infty$ and let Ω a bounded set. Then there exists a constant such that for all functions $u \in W_0^{1,p}(\Omega)$ we have the inequality*

$$\|u\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)}. \tag{B.2}$$

Theorem B.4 (Monotone Convergence Theorem)**.** *Let $(f_n)_{n \geq 1}$ be a sequence of measurable functions such that $0 \leq f_n \uparrow f$. Then f is measurable and*

$$\lim_{n \to \infty} \int_{\mathbb{R}^d} f_n d\mu = \int_{\mathbb{R}^d} f d\mu.$$

Theorem B.5 (Dominated Convergence Theorem)**.** *Let $(f_n)_{n \geq 1}$ be a sequence of integrable functions such that $\lim_{n \to \infty} f_n = f$ pointwise. If there exists a integrable function $g : \mathbb{R}^d \to [0, \infty]$ such that $|f_n| \leq g$ for all $n \geq 1$, then f is integrable and*

$$\lim_{n \to \infty} \int_{\mathbb{R}^d} f_n d\mu = \int_{\mathbb{R}^d} f d\mu.$$

Theorem B.6 (Divergence Theorem)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with C^1-boundary $\partial\Omega$ and let $X : \overline{\Omega} \to \mathbb{R}^d$ be a $C^1(\overline{\Omega})$ vector field. Then,*

$$\int_{\Omega} \operatorname{div}(X) dx = \int_{\partial\Omega} X \cdot n d\sigma$$

where n denotes the outward unit normal to $\partial\Omega$

The following theorem is a direct consequence of the divergence theorem.

Theorem B.7 (Green's second identity)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with C^1-boundary $\partial\Omega$. Let $\phi, \psi \in C^2(\overline{\Omega})$. Then,*

$$\int_{\Omega} \psi \Delta\phi - \phi \Delta\psi dx = \int_{\partial\Omega} \psi \frac{\partial \phi}{\partial n} - \phi \frac{\partial \psi}{\partial n} d\sigma.$$

C Interior estimate of the Poisson equation

Lemma C.1. *Let u be harmonic in Ω. Let $\Omega^{'}$ be a compact subset of Ω. Then for all multiincides α the following inequality holds*

$$\sup_{y\in\Omega'} |D^\alpha u(y)| \le Cr^{-\alpha} \sup_{y\in\Omega} |u(y)|, \tag{C.1}$$

where $r = \mathrm{dist}(\Omega', \partial\Omega)$ and where the constant C only depends on the dimension of Ω and the multiindex.

Proof of Lemma C.1 can be found in [21, Theorem 2.10]

Lemma C.2. *Let f be a bounded and integrable function on Ω. Let w be the Newtonian potential of f. Then $w \in C^1(\mathbb{R}^n)$ and for all $x \in \Omega$,*

$$D_i w(x) = \int_\Omega D_i\Gamma(x-y)f(y)dy \quad i = 1, \ldots, n. \tag{C.2}$$

Proof of Lemma C.2 can be found in [21, Lemma 4.1]

Lemma C.3. *Let Ω be a domain in $\mathbb{R}^n$. Let $u \in C^2(\Omega)$ and $f \in C(\Omega)$ and they satisfy the relation $\Delta u = f$ in Ω. Define the concentric balls $B_1 = B(x, R)$ and $B_2 = B(x, 2R) \subset\subset \Omega$. Then for all $x \in \Omega$ one can writes $u(x) = v(x) + w(x)$, where $v(x)$ is harmonic and $w(x)$ is the Newtonian potential of f.*

Lemma C.4 (Poisson derivative in $\mathbb{R}^d$)**.** *Fix $x \in \mathbb{R}^3$ and let $\Delta u = f$ in $B(x, r)$ and $\ell \ge 1$. Then the derivatives of u in x can be estimated by*

$$\left|\nabla^\ell u(x)\right| \le Cr^{-\ell} \sup_{B(x,r)} |u| + C \max_{0\le j\le \ell-1} \sup_{B(x,r)} r^{j-\ell+2} \left|\nabla^j f\right|. \tag{C.3}$$

Proof. Start by writing $u(x) = v(x)+w(x)$, where v is harmonic and w is the Newtonian potential of f, by the virtue of Lemma C.3. Now by Lemma C.1 we know that

$$\sup_{y\in B(x,\frac{r}{2})} |Dv(y)| \le Cr^{-1} \sup_{y\in B(x,r)} |u(y)|.$$

Furthermore we know by Lemma C.2 that

$$\begin{aligned}
\sup_{y\in B(x,\frac{r}{2})} |Dw(y)| &= \sup_{y\in B(x,\frac{r}{2})} \int_{B(x,r)} |D\Gamma(y-\eta)||f(\eta)|d\eta \\
&\le \sup_{y\in B(x,\frac{r}{2})} \sup_{s\in B(x,r)} |f(s)| \int_{B(x,r)} |D\Gamma(y-\eta)|d\eta \\
&\le \sup_{s\in B(x,r)} |f(s)| \sup_{y\in B(x,\frac{r}{2})} \int_0^r \int_{\partial B(x,1)} |D\Gamma(y+\tilde{r}\eta)|r^2 d\sigma(\eta)d\tilde{r} \\
&\le C \sup_{s\in B(x,r)} |f(s)| \int_0^r \int_{\partial B(0,1)} d\sigma(\eta)d\tilde{r} \\
&\le Cr \sup_{s\in B(x,r)} |f(s)|.
\end{aligned}$$

Putting this estimates together yields

$$
\begin{aligned}
|Du(x)| &\leq \sup_{y\in B(x,\frac{r}{2})} |Du(y)| \\
&\leq \sup_{y\in B(x,\frac{r}{2})} |Dv(y)| + \sup_{y\in B(x,\frac{r}{2})} |Dw(y)| \\
&\leq Cr^{-1} \sup_{y\in B(x,r)} |u(y)| + Cr \sup_{y\in B(x,r)} |f(y)|.
\end{aligned}
$$

We now finish the proof using an induction argument. The claim is already shown to hold for $\ell = 1$. Now assume the induction hypothesis and work out the case $\ell = k+1$. To do this first notice that u and f statisfy $\Delta D^k u = D^k f$ on Ω. Now we calculate

$$
\begin{aligned}
\left|D^{k+1}u(x)\right| &= \left|D(D^k u)(x)\right| \\
&\leq Cr^{-1} \sup_{y\in B(x,r)} \left|D^k u(y)\right| + Cr \sup_{y\in B(x,r)} \left|D^k f(y)\right| \\
&\leq Cr^{-1} \left(Cr^{-k} \sup_{y\in B(x,r)} |u(y)| + C \max_{0\leq j\leq k-1} \sup_{y\in B(x,r)} r^{j-k+2} \left|D^j f\right| \right) + Cr \sup_{y\in B(x,r)} \left|D^k f(y)\right| \\
&\leq Cr^{-(k+1)} \sup_{y\in B(x,r)} |u(y)| + C \max_{0\leq j\leq k} r^{j-(k+1)+2} \left|D^j f\right|,
\end{aligned}
$$

where in the second inequality the induction hypthesis was used. This completes the proof. □

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