Reframing existence and uniqueness theorems

for the non-homogeneous wave equation

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TUDelft

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by

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Preface

Working on this project has been an interesting experience. This is the first time in my mathematical career, so to speak, that I can do a project involving the more theoretical side of mathematics. At first, working through references that are written for people with a lot more mathematical knowledge and experience than me was discouraging. However, as I read more and more, these same references became easier to grasp and I started to actually understand what the authors were saying. Now, at the end, I feel satisfied with what I have been able to do in the project.

I would like to thank my supervisor, Carolina Urzúa-Torres, for familiarizing me with the process of mathematical research and telling me that my 'failed attempts' at proving results were actually useful conclusions. Finally, I would also thank my friends, parents, grandparents and my brother for letting me vent about my frustrations and my successes.

Cas van Ooijen
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Summary

Layman's summary

This thesis is about the wave equation. The wave equation describes waves that propagate through a certain medium. The *solution* of the wave equation is a mathematical description of what that wave looks like. There are many fields of study where the wave equation is used to model processes that behave like travelling waves. For instance, a vibrating string or membrane can be modelled by the wave equation very well. Furthermore, the wave equation can be used to model light or sound waves and how they reflect and refract when travelling through different materials.

Because the wave equation is such an important tool to model these phenomena, there are many people working on *solving* the wave equation in different contexts, i.e. finding a solution. However, it is difficult (often impossible) to find an exact solution. Therefore, people usually calculate a 'solution' that is approximately correct.

An important question to ask is: 'Does the wave equation always have a solution? And is there only *one* solution?'. Physically, the answer is clear. If a string is vibrating, it clearly cannot vibrate in two ways at the same time and it will always vibrate in some particular way. Even though the answer is clear physically, answering this question mathematically is more difficult. It is good to remember that the wave equation is just a mathematical model of propagating waves, and it could very well be that the wave equation has more than one solution or no solution at all in some particular context.

The answer of this question is important for the people calculating approximate solutions to the wave equation. If there does not exist a unique solution, the process of calculating the approximate solution may break down.

Answering that question is the subject of this thesis. We will first introduce the necessary mathematical tools, after which we will use existing research to find out under which conditions the wave equation has one, and only one solution. Finally, we will extend our results to more general equations than just the wave equation.

Summary

The wave equation is a partial differential equation with many applications, such as vibrating strings and membranes, or reflecting light or sound waves. In this thesis, we investigate what regularity conditions on the boundary conditions, initial conditions and right-hand side term f we have to assume, so that the wave equation, given below, with a Dirichlet boundary condition has a unique solution in the Sobolev space $H^1(Q)$, which is a space in which solutions of the wave equation make natural sense.

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u = f & \text{for } (x,t) \in Q = \Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{for } x \in \Omega, \\ \frac{\partial u}{\partial t}(x,0) = u_1(x) & \text{for } x \in \Omega, \\ u = g & \text{for } (x,t) \in \Sigma := \partial\Omega \times (0,T). \end{cases}$$

Here we assume that $\Omega \subseteq \mathbb{R}^n$ satisfies the Lipschitz condition (page 10).

In order to answer this question, we first introduce the necessary theory, about Sobolev spaces, interpolation between Banach spaces (and in particular, Sobolev spaces) and anisotropic Sobolev spaces in Chapter 2. By reframing existence-uniqueness theorems for the wave equation by Lasiecka, Lions & Triggiani [6], we find the following result. Given that $f \in L^2(Q)$, $u_0 \in H^1(\Omega)$ (Sobolev space of order 1), $u_1 \in L^2(\Omega)$ and $g \in H^1(\Sigma)$ along with a compatibility condition, the wave equation has a unique solution $u \in H^1(Q)$.

Furthermore, using interpolation between the above result and a weaker result, we investigate what we can say about the regularity of the unique solution u if we have boundary data g in $H^{1/2}(\Sigma)$. It turns out that, given that $f \in L^2(0,T;(H_{00}^{1/2}(\Omega))')$, $u_0 \in H^{1/2}(\Omega)$, $u_1 \in (H_{00}^{1/2}(\Omega))'$ and $g \in H^{1/2}(\Sigma)$, the unique solution u is in $H^{1/2}(Q)$, which is not yet the desired $H^1(Q)$-regularity, in which the solution would make natural sense. Here $(H_{00}^{1/2}(\Omega))'$ is the dual space of a space introduced in Subsection 2.2.4.

Finally, we extend our first result to more general wave equations with a wave speed c and to general second order hyperbolic partial differential equations. However, for the general second order hyperbolic equations, we only prove existence, and not uniqueness.

Contents

1

Introduction

The wave equation is a hyperbolic partial differential equation that describes a wide range of physical phenomena involving waves. For instance, the wave equation can be used to model vibrating strings or light and sound waves travelling through the air (see Haberman [4, Chapter 4]). Furthermore, the wave equation shares a lot of similarities with other *hyperbolic* partial differential equations, modelling for example seismics and electro-magnetics.

In this thesis, we will study the wave equation with a non-homogeneous Dirichlet boundary condition, given below by (1.1). Here $\Omega \subseteq \mathbb{R}^n$ is a Lipschitz domain (see Subsection 2.2.2).

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u = f & \text{for } (x,t) \in Q = \Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{for } x \in \Omega, \\ \frac{\partial u}{\partial t}(x,0) = u_1(x) & \text{for } x \in \Omega, \\ u = g & \text{for } (x,t) \in \Sigma := \partial\Omega \times (0,T). \end{cases} \tag{1.1}$$

Equation (1.1) is a strong formulation of the wave equation. Strong formulations of partial differential equations require that solutions are differentiable in the standard sense. Often, this is too strong of an assumption, as it could rule out realistic and experimentally found solutions (see Example 2.3). In the study of partial differential equations, we often consider *weak formulations* of problems. Weak formulations are generally derived by multiplying the equation with a differentiable *test function* and integrating the equation over the domain Q. For example, a weak formulation of (1.1) using test function ϕ would be

$$\left[\int_\Omega \phi \frac{\partial u}{\partial t}\, dx\right]_{t=T} - (\phi, u_1)_\Omega - \left(\frac{\partial \phi}{\partial t}, \frac{\partial u}{\partial t}\right)_Q - \left(\phi, \frac{\partial g}{\partial \mathbf{n}}\right)_\Sigma + \int_0^T \int_\Omega \nabla\phi \cdot \nabla u \, dx\, dt = (f,\phi)_Q, \tag{1.2}$$

where $(\cdot,\cdot)_Z$ denotes the $L^2(Z)$-inner product (to be defined in Section 2.1). All the functions present in (1.2) are contained in some function space, to be introduced in Chapter 2.

Using weak formulations and a weaker sense of derivatives (see Subsection 2.1.2) allows us to consider a wider range of problems, because we do not have to assume as much regularity in the boundary and initial conditions and the function f. Furthermore, using the weaker sense of derivatives, we can also work with solutions that are not differentiable in the classical sense (see Example 2.3 on page 4).

Moreover, certain widely used numerical methods, such as the finite element method or the boundary element method (see [11] and [10] respectively) solve the weak formulations of partial differential equations, instead of the strong formulations. These kinds of numerical methods (that are part of a larger class of methods called *Galerkin methods*) lead to a linear system of algebraic equations, given by

$$A\mathbf{x} = \mathbf{b} \tag{1.3}$$

for some matrix A and vector $\mathbf{b}$. The vector $\mathbf{x}$ is the numerical solution obtained by solving (1.3) with a computer. It is important to be able to guarantee that a solution of (1.3) exists, is unique, and approximates the true solution of (1.2) well. For this, we need existence and uniqueness of the true solution in the corresponding energy spaces, which will be introduced in Chapter 2.

If we are not able to guarantee that the solution of (1.2) exists and is unique, standard boundary element methods (see Steinbach & Urzúa-Torres [10]) fail to be stable, which causes the numerical solutions to be inaccurate. Hence, in order to improve the current numerical methods, it is important to study under which conditions the wave equation has a unique solution in a corresponding energy space. This is the subject of the thesis.

The structure of this thesis is as follows. In Chapter 2, we will introduce the necessary tools to study existence and uniqueness of solutions of the wave equation, such as Sobolev spaces, real interpolation methods and anisotropic Sobolev spaces. Then, Chapter 3 is about the existence and uniqueness results and their proofs. Chapter 3 is essentially a re-interpretation from a paper by Lasiecka, Lions & Triggiani [6], which proves existence and uniqueness in different spaces than Sobolev spaces.

2

Preliminaries

In this first chapter, we will define the important concepts and theorems that we will use for the proof of the main result. This Chapter has the following structure:

Section 2.1: Sobolev spaces and distributions. When looking for (weak) solutions of differential equations, we find the solutions in Sobolev spaces. In Sobolev spaces, a weaker notion of the derivative is used, for which the notion of (Schwartz) distributions is also needed.

Section 2.2: Interpolation between Banach spaces. Interpolation is a tool we will need in order to extend existing results on existence-uniqueness of the wave equation to Sobolev spaces. In particular, Theorem 2.20, applied to solution operators (to be defined in Chapter 3) is an important tool.

Section 2.3: Anisotropic Sobolev spaces. In evolution problems, such as the wave equation or the heat equation, anisotropic Sobolev spaces are typically used as the space we find our solutions in. Here, we may have a different regularity in time than in space.

2.1. Sobolev spaces and distributions

2.1.1. The space $L^p(\Omega)$ and the Sobolev space

In order to give a definition of Sobolev spaces, we first briefly recall the spaces $L^p(\Omega)$ of p-integrable functions. Here, Ω is a subset of $\mathbb{R}^n$. Ω will be the space-domain of (1.1) and eventually, it will have to satisfy the strong local Lipschitz condition (Definition 2.22). However, for now we will only need that Ω is open.

Definition 2.1 (Carothers [2, page 342]). *Consider the measure space $(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n), \lambda)$ (Lebesgue measure on $\mathbb{R}^n$). If $\Omega \subseteq \mathbb{R}^n$ is open and $1 \le p < \infty$, then we say that a function $u : \Omega \to \mathbb{R}$ is in $\mathscr{L}^p(\Omega)$ if it is measurable and*

$$\|u\|_p = \left(\int_\Omega |u|^p \, d\lambda\right)^{\frac{1}{p}} < \infty$$

The space $L^p(\Omega)$ then consists of all equivalence classes of functions $u \in \mathscr{L}^p(\Omega)$ that are equal almost everywhere.

Although $L^p(\Omega)$ is technically a space containing equivalence classes, we will frequently say that a function $u \in L^p(\Omega)$. This means that $\|u\|_p$ is finite. Because $L^p(\Omega)$ has a much nicer structure than $\mathscr{L}^p(\Omega)$, it is convenient to use this notation. This nice structure is given by Theorem 2.2.

Theorem 2.2 (Theorem 19.10 in Carothers [2, page 346]). ***Riesz-Fischer***
Let $1 \le p < \infty$ and let $\Omega \subseteq \mathbb{R}^n$ be open. Then the normed space $(L^p(\Omega), \|\cdot\|_p)$ is a Banach space. Furthermore, if $p = 2$, then $(L^2(\Omega), \|\cdot\|_2)$ is a Hilbert space with inner product

$$(u, v)_2 = \int_\Omega u \cdot v \, d\lambda \tag{2.1}$$

for any $u, v \in L^2(\Omega)$.

Using L^p-spaces, and more specifically L^2-spaces, we can now define Sobolev spaces. First, however, we will motivate why Sobolev spaces are commonly used as a space in which solutions to differential equations live.

Roughly speaking, a Sobolev space contains all solutions of differential equations that use a finite amount of energy. For instance, a solution of the wave equation whose amplitude - which corresponds to potential energy - diverges to infinity is generally not in the relevant Sobolev space.

In the context of solutions of differential equations, using Sobolev spaces makes sense physically, for differential equations that model physical processes, because the entire universe contains only a finite amount of energy. Then it is logical to assume that our models of reality, the differential equations, have solutions that have a finite amount of energy as well.

Sobolev spaces are also less restrictive than other spaces in which solutions may live. For instance, looking for solutions of the wave equation in $C^2(Q)$, the space of twice continuously differentiable functions on Q, where Q is the domain in time *and* space, would rule out solutions that are not differentiable in the standard sense, but have been experimentally shown to exist. An example of this is Example 2.3.

Example 2.3. *Consider the one-dimensional wave equation*

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} & \text{for } (x,t) \in (0,\infty) \times (0,T], \\ u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0 & \text{for } x \in (0,\infty), \\ u(0,t) = \mathbf{1}_{(0,\pi)}(t)\sin(t) & \text{for } t \in (0,T]. \end{cases} \tag{2.2}$$

It can be shown that the function $u(x,t) = \mathbf{1}_{(0,\pi)}(t-x)\sin(t-x)$ solves Problem (2.2). However, this solution is clearly not differentiable, because of the sharp turns in its graph. This can be seen in Figure 2.1.

Because of these sharp turns, this solution cannot be differentiable nor twice continuously differentiable, i.e. $u \notin C^2(\Omega)$. However, because we only consider a finite amount of time, the total potential and kinetic energy of this solution is finite, so $u(x,t)$ ***is*** *in the relevant Sobolev space.*

We can make use of so-called *distributional derivatives* to circumvent this problem with non-differentiabilty. First we will define Sobolev spaces. Then, we discuss how the distributional derivatives are constructed in Subsection 2.1.2. Since energies of solutions can usually be represented using integrals, we define Sobolev spaces as follows (the definition is combined from Lions & Magenes [8, page 1] and Evans [3, page 258]).

Definition 2.4. *Let $\Omega \subseteq \mathbb{R}^n$ be open, let $1 \le p < \infty$ and $m \in \mathbb{N}$. The* ***Sobolev space*** *of order m on Ω is defined by*

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) : \forall |\alpha| \le m : D^\alpha u \in L^p(\Omega)\},$$

where for any multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $D^\alpha u$ is given by

$$D^\alpha u = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots x_n^{\alpha_n}} u \quad \text{with}$$
$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n,$$

where the derivatives are ***distributional derivatives****. Furthermore, if $p = 2$, then we use the notation*

$$H^m(\Omega) = W^{m,2}(\Omega).$$

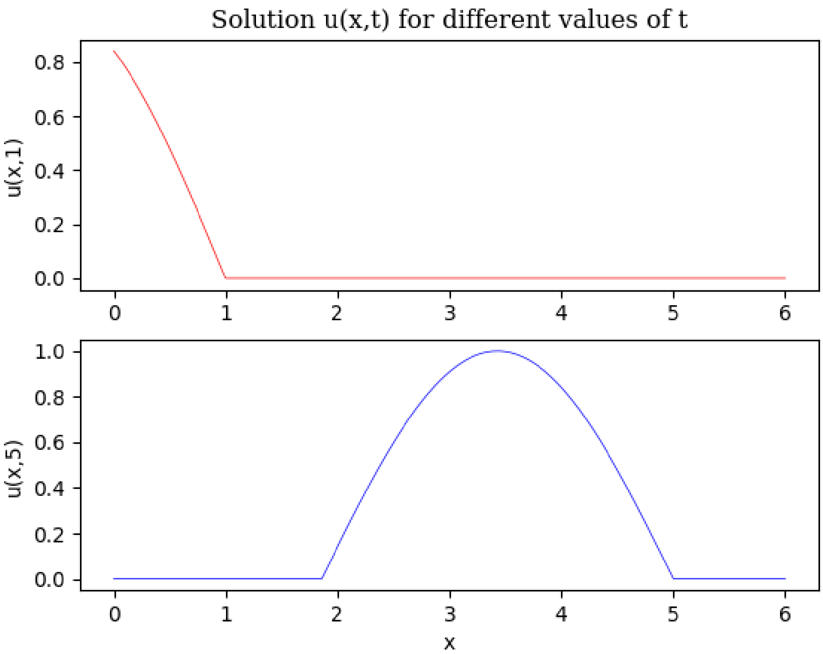

Figure 2.1: Solution $u(x,t)$ of (2.2) for $t=1$ (above) and $t=5$ (below).

In other words, a function $u:\Omega\to\mathbb{R}$ is in the Sobolev space $W^{m,p}$ if it is in $L^p(\Omega)$ and all of its (distributional) derivatives up to and including degree m are in $L^p(\Omega)$ as well.

Studying the wave equation (1.1), we will assume that solutions are contained in the Sobolev space $H^1(Q)$, since the only relevant energies are the potential energy and the kinetic energy. The potential energy is only dependent on the value of the solution itself, and the kinetic energy is a function of the velocity, the *first* derivative of the solution. Note that Q is again the domain in space and time.

Using these new kinds of derivatives not only allows us to work with solutions that have sharp turns, but it also causes $W^{m,p}(\Omega)$ to be a Banach space (see Subsection 2.1.3).

2.1.2. Distributional derivatives

As we have seen in Example 2.3, it is easy to find an initial-boundary value problem that has a non-differentiable solution. In order to solve this apparent paradox, we will need to introduce a weaker concept of derivatives, such that these sharp turns do not cause problems. More precisely, we want to consider a notion of derivatives so that non-differentiability on a set of measure zero is not an issue.

The precise definition of a distribution makes use of so-called *test functions* - which will be defined shortly - and *dual spaces*. Hence, we proceed to introduce these objects.

Definition 2.5 (Lions & Magenes [8, page 2])**.** *Let $\Omega\subseteq\mathbb{R}^n$ be open. The function $\phi:\Omega\to\mathbb{R}$ is called a* ***test function*** *if it is infinitely differentiable on Ω and if it has compact support, i.e.*

$$\operatorname{supp}(\phi)=\{x\in\Omega:\phi(x)\neq 0\}$$

is compact. The space of all test functions ϕ on Ω is denoted by $\mathcal{D}(\Omega)$.

Essentially, a test function ϕ is a smooth function that is zero everywhere except on a closed and bounded set. These test functions are generally used if we want to determine the value of a function not in a single point, but as a kind of average over a compact set. If $f:\Omega\to\mathbb{R}$, we do this by integrating the product $f(x)\phi(x)$ over the support of ϕ. Considering the values of functions on compact sets of non-zero measure instead of single points allows us to disregard issues with differentiability if these issues occur on a set of measure zero, because sets of measure zero do not contribute to the value of integrals. Still, test functions may measure the value of functions quite accurately, as the support may have an arbitrarily small measure.

A distribution then maps a test function - a sort of measuring device, as we have seen previously - to a real number. Thus, a distribution assigns a value to every test function.

To give a precise definition of distributions, we have to introduce the concept of dual spaces.

Definition 2.6 (Lions & Magenes [8], page 2). *Let X be a normed vector space over $\mathbb{R}$. A **functional** is a map from the normed space X to the field $\mathbb{R}$. The **dual space** of X, denoted by X', is the space of all continuous linear functionals $f : X \to \mathbb{R}$.*

Definition 2.7 (Lions & Magenes [8], page 2). *Let $\Omega \subseteq \mathbb{R}^n$ be open. Then the **space of (Schwartz) distributions** on Ω is the dual space of $\mathscr{D}(\Omega)$. A **distribution** is an element $T \in \mathscr{D}'(\Omega)$. If $T \in \mathscr{D}'(\Omega)$ is a distribution and $\phi \in \Omega$ is a test function, then we define the value of T at ϕ, denoted as $\langle T, \phi \rangle$, by*

$$\langle T, \phi \rangle = T(\phi).$$

*The value $\langle T, \phi \rangle$ is commonly called a **duality pairing**.*

Note that the space of distributions, the dual of $\mathscr{D}(\Omega)$, is denoted by $\mathscr{D}'(\Omega)$ instead of $\mathscr{D}(\Omega)'$, as one would expect. The notation $\mathscr{D}'(\Omega)$ is commonly used in the references (see for example all throughout the two volumes of Lions & Magenes [8] and [9]). Therefore, we will use $\mathscr{D}'(\Omega)$ in this thesis as well.

In Example 2.8, two examples of distributions will be given.

Example 2.8. **(a)** *The Dirac delta 'function' δ is a distribution on $\mathbb{R}$. Indeed, for any $\phi \in \mathscr{D}(\mathbb{R})$, we have that*

$$\langle \delta, \phi \rangle = \phi(0).$$

(b) *If $f : \Omega \to \mathbb{R}$ is integrable on all compact subsets $K \subseteq \Omega$, then there is a corresponding distribution $T_f \in \mathscr{D}'(\Omega)$ defined by*

$$\langle T_f, \phi \rangle = \int_\Omega f(x)\phi(x)\, dx$$

In fact, if this is the case, we can interpret f as being a distribution itself, i.e. $f \in \mathscr{D}'(\Omega)$.

Using distributions, we can now define the distributional derivatives that will solve our problems with non-differentiability on sets of measure zero.

Definition 2.9 (Lions & Magenes [8], page 3). *Let $T \in \mathscr{D}'(\Omega)$ be a distribution. Then we define the **distributional derivative** $\frac{\partial T}{\partial x_j}$ of T by*

$$\forall \phi \in \mathscr{D}(\Omega) : \left\langle \frac{\partial T}{\partial x_j}, \phi \right\rangle = -\left\langle T, \frac{\partial \phi}{\partial x_j} \right\rangle.$$

The distributional derivative is itself a distribution. $D^\alpha T$ is defined by iteration.

Example (2.10) shows that we can calculate distributional derivatives even for highly irregular functions.

Example 2.10. *The zero function on $\mathbb{R}$ is the distributional derivative of the indicator function of $\mathbb{Q}$. Note that the indicator function $\mathbf{1}_\mathbb{Q}$ can be interpreted to be a distribution as in Example 2.8 (b). Indeed, for any $\phi \in \mathscr{D}(\mathbb{R})$,*

$$\int_\mathbb{R} \mathbf{1}_\mathbb{Q}(x) \frac{d}{dx}\phi(x)\, dx = 0$$

because $\mathbf{1}_\mathbb{Q} = 0$ almost everywhere. Furthermore, we obviously have

$$-\int_\mathbb{R} 0 \cdot \phi(x)\, dx = 0.$$

We conclude that the zero function on $\mathbb{R}$ is the distributional derivative of $\mathbf{1}_\mathbb{Q}$.

It can be shown that the distributional derivative is unique up to a set of measure zero, i.e. the equivalence class of functions in $L^2(\Omega)$ that is assigned to the distributional derivative is unique. In fact, if v_1 is a distributional derivative of u, then v_2 is also a distributional derivative of u if $v_1 = v_2$ almost everywhere. Applying this notion to Example (2.10), we see that $\mathbf{1}_{\mathbb{Q}}$ is a distributional derivative of itself.

2.1.3. Completeness of Sobolev spaces

The following theorem shows that Sobolev spaces not only make sense physically (using finite energy), but also have a very nice mathematical structure.

Theorem 2.11 (Theorem 2 in Evans [3, page 262]). *Let $\Omega \subseteq \mathbb{R}^n$ be open. If $1 \le p < \infty$ and $m \in \mathbb{N}$, then $W^{m,p}(\Omega)$ equipped with the norm*

$$\|u\|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha| \le m} \|D^\alpha u\|_p^p \right)^{\frac{1}{p}} \tag{2.3}$$

is a Banach space. Moreover, if $p = 2$, $H^m(\Omega)$ is a Hilbert space with inner product

$$\langle u, v \rangle_{H^m(\Omega)} = \sum_{|\alpha| \le m} \langle D^\alpha u, D^\alpha v \rangle_2 \tag{2.4}$$

for any two functions $u, v \in H^m(\Omega)$.

For the proof of this theorem we refer to Evans [3, page 262].

Moving forward, we will not use the general Sobolev space $W^{m,p}(\Omega)$ anymore. Instead, we will state all definitions and results using the notation $H^m(\Omega)$, where $p = 2$, because most of the references state results only for $H^m(\Omega)$ and not for $W^{m,p}(\Omega)$, since $H^m(\Omega)$ is not only a Banach space, but also a Hilbert space.

2.2. Interpolation between Banach spaces

We will now build up some theory about interpolation (primarily based on Chapter 7 of Adams [1]). The general idea of interpolation between two Banach spaces X_0 and X_1 is that we find a new space X that is somehow 'between' X_0 and X_1 based on a parameter $\theta \in [0,1]$. Here θ determines how 'close' the intermediate space is to either X_0 or X_1.

First we will discuss the general theory of interpolation for two Banach spaces X_0 and X_1 in Subsection 2.2.1. In Subsection 2.2.2 we will apply these ideas to Sobolev spaces and Subsection 2.2.3 will discuss the trace theorem; a result that tells us how smooth functions are on the boundary of their domain. Finally, Subsection 2.2.4 will discuss how we can define Sobolev spaces for negative orders as well.

2.2.1. General theory

In order to make sense of interpolation between spaces, we first have to make sense of what we mean by saying that a space is 'between' two other spaces. We will make this notion precise using continuous embeddings.

Definition 2.12. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces. We say that X is **(continuously) embedded** in Y if $X \subseteq Y$ and the identity map $I : X \to Y$ is continuous, i.e.*

$$\|x\|_Y \le C\|x\|_X$$

for some constant $C > 0$. If X is (continuously) embedded in Y, then we write $X \hookrightarrow Y$.

Intuitively, if $X \hookrightarrow Y$, then elements $x_0, x_1 \in X$ being close to each other in the norm $\|\cdot\|_X$ implies that they are also close to each other in the norm $\|\cdot\|_Y$.

Definition 2.13 (Adams [1, page 208]). *Let* $(X_0, \|\cdot\|_{X_0})$ *and* $(X_1, \|\cdot\|_{X_1})$ *be two Banach spaces. A Banach space* X *with norm* $\|\cdot\|_X$ *is an* ***intermediate space*** *between* X_0 *and* X_1 *if*

$$X_0 \cap X_1 \hookrightarrow X \hookrightarrow X_0 + X_1,$$

where

$$X_0 + X_1 := \{u = u_0 + u_1 : u_0 \in X_0 \text{ and } u_1 \in X_1\}.$$

Here we choose the sum $X_0 + X_1$ instead of the union $X_0 \cup X_1$ because the union is generally not even a vector space. In all of the situations we will study, we will have $X_1 \hookrightarrow X_0$, so that $X_0 \cap X_1 = X_1$ and $X_0 + X_1 = X_0$. In that case, a space X is intermediate between X_0 and X_1 if $X_0 \hookrightarrow X \hookrightarrow X_1$.

Interpolation is a way to find such intermediate spaces X. We will discuss two methods of interpolation in this thesis: the K-method and the J-method. It will turn out that both methods are equivalent. In other words, using each method will result in the same space with equivalent norms. The K-method and J-method make use of the aptly named K-functional and J-functional.

Definition 2.14 (Adams [1, page 208]). *Let* $t > 0$. *Then the* ***K-functional*** K *on* X_0+X_1 *and the* ***J-functional*** J *on* $X_0 \cap X_1$ *define norms on* $X_0 + X_1$ *and* $X_0 \cap X_1$ *and are given by*

$$K(t,u) = \inf\{\|u_0\|_{X_0} + t\|u_1\|_{X_1} : u = u_0 + u_1, u_0 \in X_0, u_1 \in X_1\}, \tag{2.5}$$
$$J(t,u) = \max\{\|u\|_{X_0}, t\|u\|_{X_1}\}. \tag{2.6}$$

Using the value $t > 0$ in the definition allows us to skew the value of the norms either toward the X_0-norm or the X_1-norm. For instance, if t is close to zero, the values of the functionals are closer to the X_0-norm than the X_1-norm. If t is large, then the X_1-norm contributes more to the values of the functionals.

Using these functionals, we can introduce interpolation methods.

Theorem 2.15 (7.9 and 7.10 in Adams [1, page 209]). ***The K-method***
Let $0 < \theta < 1$ *and* $1 \le q < \infty$ *or* $0 \le \theta \le 1$ *and* $q = \infty$. *We say that an element* $u \in X_0 + X_1$ *is in the space* $(X_0, X_1)_{\theta,q;K}$ *if*

$$\|u\|_{\theta,q;K} = \begin{cases} \left(\int_0^\infty (t^{-\theta}K(t,u))^q \,\frac{dt}{t}\right)^{1/q} & \text{if } q < \infty \\ \operatorname*{ess\,sup}_{0<t<\infty} \left(t^{-\theta}K(t,u)\right) & \text{if } q = \infty \end{cases} \tag{2.7}$$

is finite. The space $(X_0, X_1)_{\theta,q;K}$ *is a non-trivial Banach space with norm* $\|\cdot\|_{\theta,q;K}$. *Furthermore, it is intermediate between* X_0 *and* X_1.

In the integral we divide by t to make sure that the X_1-norm does not contribute that much more to the integral than the X_0-norm. If we would not do this, there would be no balance between the norms, since the interval where the X_1-norm contributes more has infinite Lebesgue measure and the interval where the X_0-norm contributes more has a Lebesgue measure of one.

Furthermore, if we increase θ, we also increase the contribution of the X_0-norm in the integral. Indeed, if t is small, i.e. the X_0-norm contributes more to the value of the K-functional, then the value of the function $t \mapsto t^{-\theta}K(t,u)$ is larger than the value of the K-functional for larger t.

Note that we use the *essential supremum* $\operatorname{ess\,sup}_{0<t<\infty}\left(t^{-\theta}K(t,u)\right)$ to disregard sets of measure zero in determining the supremum.

Theorem 2.16 (7.12 and 7.13 in Adams [1, page 211]). ***The J-method***
Let $0 < \theta < 1$ *and* $1 < q \leq \infty$ *or* $0 \leq \theta \leq 1$ *and* $q = 1$. *We say that an element* $u \in X_0 + X_1$ *is in the space* $(X_0, X_1)_{\theta,q;J}$ *if it can be written as* $u = \int_0^\infty f(t)\,\frac{dt}{t}$ *for some function* f *that takes values in* $X_0 \cap X_1$, *such that* $f(t)/t$ *is integrable with respect to the norm* $\|\cdot\|_{X_0\cap X_1}$ *and if the quantity*

$$\int_0^\infty (t^{-\theta} J(t, f))^q \,\frac{dt}{t}$$

is finite. Here $\|\cdot\|_{X_0\cap X_1}$ *is given by* $\|\cdot\|_{X_0\cap X_1} = J(1,\cdot)$. *Then* $(X_0, X_1)_{\theta,q;J}$ *is a nontrivial Banach space with norm*

$$\|u\|_{\theta,q;J} = \inf_{f\in S(u)} \left(\int_0^\infty (t^{-\theta} J(t, f))^q \,\frac{dt}{t}\right)^{1/q}, \tag{2.8}$$

with

$$S(u) = \left\{ f : u = \int_0^\infty f(t)\,\frac{dt}{t} \right\},$$

with all $f \in S(u)$ *satisfying the same requirements of* f *as above. The space* $(X_0, X_1)_{\theta,q;J}$ *is intermediate between* X_0 *and* X_1.

For similar reasons as the K-method, increasing θ increases the contribution of the X_0-norm in the integral. However, instead of using u in the integral, we use this function f that is associated with u. The function f is used because u is generally not in the intersection $X_0 \cap X_1$, so taking the J-functional would be ill-defined.

Also note that the integral of $f(t)$ is a *Bochner integral* (see Adams [1, page 206]). The Bochner integral is a generalization of the Lebesgue integral where functions may take values in any Banach space, instead of only in $\mathbb{R}$ or $\mathbb{C}$.

Theorem 2.17 shows that it does not matter which interpolation method we use.

Theorem 2.17 (Theorem 7.16 in Adams [1, page 215]). *If* $0 < \theta < 1$ *and* $1 \leq q \leq \infty$, *then*

$$(X_0, X_1)_{\theta,q;K} \hookrightarrow (X_0, X_1)_{\theta,q;J} \text{ and } (X_0, X_1)_{\theta,q;J} \hookrightarrow (X_0, X_1)_{\theta,q;K},$$

so $(X_0, X_1)_{\theta,q;K} = (X_0, X_1)_{\theta,q;J}$ *with both spaces having equivalent norms.*

Because the K-method and the J-method are equivalent, we can omit the subscript K or J specifying which interpolation method we use. We will write $(X_0, X_1)_{\theta,q}$ for both interpolation methods. If, additionally, $q = 2$ (as it will always be in this thesis), we write $[X_0, X_1]_\theta$ following the notation of Lions & Magenes [8, page 10].

In the remainder of this subsection, we include some useful results - mainly some inequalities - about interpolation between Banach spaces. The proofs of these results can be found in of Adams [1, Chapter 7].

Theorem 2.18 (Theorems 7.18 and 7.20 in Adams [1, page 216,217]). *Let* $0 < \theta < 1$ *and* $1 \leq q \leq \infty$. *Then for any* $u \in (X_0, X_1)_{\theta,q}$ *and* $v \in X_0 \cap X_1$, *we have that*

$$K(t, u) \leq C_1 t^\theta \|u\|_{\theta,q},$$
$$\|v\|_{\theta,q} \leq C_2 t^{-\theta} J(t, v),$$

where C_1 *and* C_2 *are positive constants and the norm* $\|\cdot\|_{\theta,q}$ *can either be the* K*-norm or the* J*-norm.*

Theorem 2.20 relates the X_0- and X_1-norms to the norm of the intermediate space. To state this theorem, we first need another definition.

Definition 2.19 (7.22 in Adams [1, page 220]). *Let $\mathscr{P} = \{X_0, X_1\}$ and $\mathscr{Q} = \{Y_0, Y_1\}$ be two pairs of Banach spaces. Let $T : X_0 + X_1 \to Y_0 + Y_1$ be a continuous linear operator such that T is also continuous from X_i to Y_i with operator norm at most M_i for $i = 0, 1$, i.e. for any $u_i \in X_i$ we have*

$$\|Tu_i\|_{Y_i} \leq M_i \|u_i\|_{X_i}.$$

If X and Y are intermediate spaces for $\mathscr{P}$ and $\mathscr{Q}$ respectively, then we call X and Y ***interpolation spaces of type θ*** *($\theta \in [0,1]$) if every such operator T is also continuous from X to Y with operator norm M satisfying*

$$M \leq C M_0^{1-\theta} M_1^{\theta}. \tag{2.9}$$

Here $C \geq 1$ is independent of the operator T. The interpolation spaces X and Y are ***exact*** *if (2.9) holds with $C = 1$.*

Theorem 2.20 (Theorem 7.23 in Adams [1, page 220]). *Let $\mathscr{P} = \{X_0, X_1\}$ and $\mathscr{Q} = \{Y_0, Y_1\}$ be two pairs of Banach spaces. If $0 < \theta < 1$ and $1 < q < \infty$, then the intermediate spaces $(X_0, X_1)_{\theta,q}$ and $(Y_0, Y_1)_{\theta,q}$ are exact interpolation spaces of type θ for $\mathscr{P}$ and $\mathscr{Q}$.*

One final theorem we will need is the reiteration theorem (Theorem 2.21). This result described how we can apply interpolation methods multiple times.

Theorem 2.21 (7.21 in Adams [1, page 218]). ***The Reiteration Theorem***
Let $0 \leq \theta_0 < \theta_1 \leq 1$ and let X_0 and X_1 be two Banach spaces. Suppose that $X_{\theta_0} = (X_0, X_1)_{\theta_0,q}$ and $X_{\theta_1} = (X_0, X_1)_{\theta_1,q}$ for $1 \leq q \leq \infty$. Finally, let $0 < \lambda < 1$. Then, if $\theta = (1-\lambda)\theta_0 + \lambda\theta_1$, we have that

$$(X_{\theta_0}, X_{\theta_1})_{\lambda,q} = (X_0, X_1)_{\theta,q}.$$

2.2.2. Application to Sobolev spaces

In this subsection, we will apply the above interpolation techniques to Sobolev spaces. First, however, we will define a property of our domain $\Omega \subseteq \mathbb{R}^n$. Although the strong local Lipschitz property is only indirectly connected to interpolation of Sobolev spaces, we will need it as an assumption in many results later on. Furthermore, in our main result we will assume that our domain satisfies this condition.

Definition 2.22 (Definition 4.9 in Adams [1, page 83]). *A subset $\Omega \subseteq \mathbb{R}^n$ satisfies the* ***strong local Lipschitz condition*** *if it is open and there exist $\delta, M > 0$, a locally finite open cover $\{U_j : j \in J\}$ of $\partial\Omega$ for some index set J, and for each $j \in J$ a real-valued function f_j of $n-1$ variables such that the following four conditions hold.*

1. *For some finite $R \in \mathbb{N}$, every collection of $R+1$ subsets U_j is disjoint.*
2. *For every pair of points $x, y \in \Omega_\delta = \{x \in \Omega : d(x, \partial\Omega) < \delta\}$ with $|x-y| < \delta$ there is a $j \in J$ such that*
$$x, y \in \{u \in U_j : d(u, \partial U_j) > \delta\}.$$
3. *Each function f_j is Lipschitz continuous with constant M, i.e. for any $\alpha, \beta \in \mathbb{R}^{n-1}$ we have that*
$$|f_j(\alpha) - f_j(\beta)| \leq M|\alpha - \beta|.$$
4. *For some Cartesian coordinate system $(\zeta_{j,1}, \zeta_{j,2}, \ldots, \zeta_{j,n})$ in U_j, $\Omega \cap U_j$ is represented by the inequality*
$$\zeta_{j,n} < f_j(\zeta_{j,1}, \ldots, \zeta_{j,n-1}).$$

If these conditions hold, we call Ω a ***Lipschitz domain***.

First, if Ω is a bounded set, then Definition 2.22 can be reduced to the following: For each point $x \in \partial\Omega$, there is a neighbourhood U of x such that the intersection $\partial\Omega \cap U$ is the graph of a Lipschitz continuous function.

Intuitively, we can imagine Lipschitz-domains as follows: Suppose that the boundary of Ω is piecewise Lipschitz continuous. Then for every point x on the boundary, we can draw an open ball such that locally (in some rotated or scaled coordinate system), Ω consists of all the points that are below a Lipschitz continuous function (namely the boundary). Figure 2.2 gives an example of a subset $\Omega \subseteq \mathbb{R}^n$ that does not satisfy the strong local Lipschitz condition.

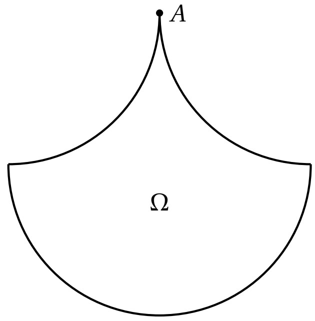

Figure 2.2: A domain that does not satisfy the strong local Lipschitz condition. The point A is a point where condition (4) in Definition 2.22 does not hold.

Having defined Lipschitz domains, we can interpolate between Sobolev spaces. Following the notation of Lions & Magenes [8, page 40], we give a definition of the space $H^s(\Omega)$ for real $s \geq 0$.

Definition 2.23 (Lions & Magenes [8, page 40]). *Let $\Omega \subseteq \mathbb{R}^n$ be a Lipschitz domain, let $0 < \theta < 1$ and let $m \in \mathbb{N}$. Suppose that $s = (1-\theta)m$. Then we set*

$$H^s(\Omega) = [H^m(\Omega), L^2(\Omega)]_\theta.$$

Here we require Ω to be a Lipschitz domain to make sure that $H^s(\Omega)$ is the same space as in Definition 2.4 if s is integer. Furthermore, by setting Ω to be Lipschitz we make $H^s(\Omega)$ independent of the choice of m, as long as $(1-\theta)m = s$. For proofs of these statements, see Lions & Magenes [8, page 40].

The space $H^s(\Omega)$ for non-integer s is also commonly called a *Besov space*. See Adams [1, page 230] for a more general definition of Besov spaces where we interpolate between Sobolev spaces $W^{m,p}(\Omega)$.

Using the Reiteration Theorem (Theorem 2.21), we can also interpolate between the spaces $H^s(\Omega)$.

Theorem 2.24 (Lions & Magenes [8, page 43]). *If $\Omega \subseteq \mathbb{R}^n$ is a Lipschitz domain, then*

$$[H^{s_1}(\Omega), H^{s_2}(\Omega)]_\theta = H^{(1-\theta)s_1+\theta s2}(\Omega)$$

for any $0 < \theta < 1$, $s1 > s2 > 0$ (with equivalent norms).

A useful result concerning these *fractional order Sobolev spaces* $H^s(\Omega)$ states that the space of test functions on the closure of Ω is dense in $H^s(\Omega)$.

Theorem 2.25 (Theorem 9.3 in Lions & Magenes [8, page 41]). *Let Ω be an open Lipschitz domain and let $s \in \mathbb{R}_{\geq 0}$. Then $\mathscr{D}(\bar{\Omega})$ is dense in $H^s(\Omega)$.*

The last result of this subsection relates the norms of fractional order Sobolev spaces $H^s(\Omega)$ with the norms of the neighboring integer order Sobolev spaces. This result follows almost directly from the Rellich-Kondrachov embedding theorem (Theorem 2.26), but we provide an alternative proof using only the tools studied so far.

Theorem 2.26 (Proposition 4.4 in Taylor [12, page 334]). ***Rellich-Kondrachov embedding theorem.*** *Let $\Omega \subseteq \mathbb{R}^n$ be a Lipschitz domain. For $s \geq 0$ and $\sigma > 0$, we have that*

$$H^{s+\sigma}(\Omega) \hookrightarrow H^s(\Omega).$$

Lemma 2.27. *Let $\Omega \subseteq \mathbb{R}^n$ be a Lipschitz domain and let s and $\theta \in (0,1)$ be such that $(1-\theta)\lceil s \rceil + \theta \lfloor s \rfloor = s$. Then for any $u \in H^{\lceil s \rceil}(\Omega)$, we have that*

$$C\|u\|_{H^{\lfloor s \rfloor}(\Omega)} \leq \|u\|_{H^s(\Omega)} \leq \|u\|_{H^{\lceil s \rceil}(\Omega)},$$

for some constant $C > 0$.

Proof. First note that by Theorem 2.24, $H^s(\Omega) = [H^{\lceil s \rceil}(\Omega), H^{\lfloor s \rfloor}(\Omega)]_\theta$. Then it follows from Theorem 2.18 that there is a constant $C > 0$ such that for all $t \in (0,\infty)$

$$\|u\|_{H^s(\Omega)} \geq Ct^{-\theta}K(t,u).$$

In particular,

$$\|u\|_{H^s(\Omega)} \geq CK(1,u).$$

Now, let $u_0 \in H^{\lfloor s \rfloor}(\Omega)$ and $u_1 \in H^{\lceil s \rceil}(\Omega)$ be such that $u = u_0 + u_1$. Then

$$\|u\|_{H^{\lfloor s \rfloor}(\Omega)} = \|u_0 + u_1\|_{H^{\lfloor s \rfloor}(\Omega)} \leq \|u_0\|_{H^{\lfloor s \rfloor}(\Omega)} + \|u_1\|_{H^{\lfloor s \rfloor}(\Omega)} \leq \|u_0\|_{H^{\lfloor s \rfloor}(\Omega)} + \|u_1\|_{H^{\lceil s \rceil}(\Omega)},$$

since the $H^{\lceil s \rceil}(\Omega)$-norm is the sum of the $H^{\lfloor s \rfloor}(\Omega)$ and other non-negative numbers. Then, as

$$K(1,u) = \inf\left\{\|u_0\|_{H^{\lfloor s \rfloor}(\Omega)} + \|u_1\|_{H^{\lceil s \rceil}(\Omega)} \mid u = u_0 + u_1, u_0 \in H^{\lfloor s \rfloor}(\Omega), u_1 \in H^{\lceil s \rceil}(\Omega)\right\},$$

and since we chose an arbitrary pair u_0, u_1 with $u = u_0 + u_1$, we also have that

$$\|u\|_{H^{\lfloor s \rfloor}(\Omega)} \leq K(1,u).$$

Therefore, we have that

$$\|u\|_{H^s(\Omega)} \geq C\|u\|_{H^{\lfloor s \rfloor}(\Omega)}.$$

Now we prove the other inequality. Let T be the identity operator. Then T is continuous from $H^{\lfloor s \rfloor}(\Omega)$ to $H^{\lfloor s \rfloor}(\Omega)$ and it is continuous from $H^{\lceil s \rceil}(\Omega)$ to $H^{\lceil s \rceil}(\Omega)$. Furthermore, the operator norm of T is equal to 1 in both cases. Then by Theorem 2.20, T is also continuous from $H^s(\Omega)$ to $H^s(\Omega)$ with

$$\|u\|_{H^s(\Omega)} \leq \|u\|^{1-\theta}_{H^{\lfloor s \rfloor}(\Omega)}\|u\|^{\theta}_{H^{\lceil s \rceil}(\Omega)} \leq \|u\|^{1-\theta+\theta}_{H^{\lceil s \rceil}(\Omega)} = \|u\|_{H^{\lceil s \rceil}(\Omega)}.$$

□

2.2.3. The trace theorem

In this final subsection, we will discuss the trace theorem for Sobolev spaces. Essentially, the trace theorem states that functions u that are in $H^s(\Omega)$ in the interior of Ω, are in $H^{s-\frac{1}{2}}(\Gamma)$ on the boundary Γ of Ω. The precise statement (from Lions & Magenes [8, page 41]) is given in Theorem 2.28.

Theorem 2.28 (Theorem 9.4 in Lions & Magenes [8, page 41]). ***The trace theorem***
*Let $\Omega \subseteq \mathbb{R}^n$ be an open Lipschitz domain with boundary Γ. Define the **trace operator** γ on $\mathscr{D}(\bar{\Omega})$ (the space of test functions on the closure of Ω) as the restriction of a test function $\phi \in \mathscr{D}(\bar{\Omega})$ to the boundary Γ:*

$$\gamma\phi = \phi|_\Gamma.$$

This mapping from $\mathscr{D}(\bar{\Omega})$ to $\mathscr{D}(\Gamma)$ extends by continuity to a continuous trace operator γ from $H^s(\Omega)$ to $H^{s-1/2}(\Gamma)$. Furthermore, the trace operator γ is surjective.

Note that we can only extend this mapping continuously because $\mathscr{D}(\bar{\Omega})$ is dense in $H^s(\Omega)$ for any $s \geq 0$ (see Theorem 2.25).

Furthermore, this theorem shows us that interpolation and fractional order spaces are not just mathematical curiosities, but also arise naturally in the study of partial differential equations.

2.2.4. Negative order Sobolev spaces

The need for negative order Sobolev spaces is best illustrated using an example. Example 2.29 shows when these kinds of spaces are needed.

Example 2.29. *Consider the Poisson equation on a Lipschitz domain $\Omega \subseteq \mathbb{R}^n$*

$$-\Delta u = f \tag{2.10}$$

with homogeneous Dirichlet boundary conditions $u = 0$ on the boundary Γ of Ω. We are interested in what kind of regularity the function f must have, for a solution to exist. We find a weak formulation of (2.10) by multiplying with a test function and integrating over the domain (assuming the test function vanishes at the boundary):

$$\int_\Omega \nabla u \cdot \nabla \phi \, dx = \int_\Omega f\phi \, dx. \tag{2.11}$$

As we can see, as long as $u \in H_0^1(\Omega)$, we really only need that $\phi \in H_0^1(\Omega)$; we don't need infinite differentiability. Now, we can interpret the function f as a linear functional acting on a test function $\phi \in H_0^1(\Omega)$. Therefore, using the same logic as Example (2.8) (b), we will assume that $f \in \left(H_0^1(\Omega)\right)' = H^{-1}(\Omega)$.

Had we chosen our test function $\phi \in \mathscr{D}(\Omega)$ infinitely differentiable, we would only need to assume that f corresponds to a distribution.

Definitions 2.30 and 2.31 make negative order Sobolev spaces precise.

Definition 2.30 (Lions & Magenes [8, page 54]). *Let $\Omega \subseteq \mathbb{R}^n$ be an open Lipschitz domain. Then for $s \geq 0$ we set $H_0^s(\Omega)$ to be the closure of $\mathscr{D}(\Omega)$ in $H^s(\Omega)$. It can be shown (see Lions & Magenes [8, page 62]) that a function u is in $H_0^s(\Omega)$ if and only if $u \in H^s(\Omega)$ and the normal derivatives vanish on the boundary:*

$$\frac{\partial^j u}{\partial \mathbf{n}^j} = 0$$

for any integer $0 \leq j \leq s - \frac{1}{2}$.

Definition 2.31 (Lions & Magenes [8, page 70]). *Let $\Omega \subseteq \mathbb{R}^n$ be an open Lipschitz domain and let $s > 0$. Then we define the negative order Sobolev space $H^{-s}(\Omega)$ by*

$$H^{-s}(\Omega) = \left(H_0^s(\Omega)\right)'.$$

In Section 3.3, we will have to interpolate between positive and negative order Sobolev spaces. In some specific cases (such as the one that will be discussed in Section 3.3), the resulting intermediate space is not the obvious one, where we use linear interpolation of the numbers s. To this end, we introduce the following space.

Definition 2.32 (See Theorem 11.7 in Lions & Magenes [8, page 66]). *Let Ω be a Lipschitz domain and let $s_1, s_2 \geq 0$ be such that $s_1 > s_2$ and $s_2 \notin \{k + \frac{1}{2} | k \in \mathbb{Z}\}$. Suppose that for $\theta \in (0,1)$, we have that*

$$(1-\theta)s_1 + \theta s_2 = \mu + \frac{1}{2}$$

for some integer $\mu \geq 0$. We define the space

$$H_{00}^{\mu+\frac{1}{2}}(\Omega) = [H_0^{s_1}(\Omega), H_0^{s_2}(\Omega)]_\theta.$$

The space $H_{00}^{\mu+\frac{1}{2}}(\Omega)$ is independent of s_1, s_2 and θ.

We can make the space $H_{00}^{\mu+\frac{1}{2}}$ more explicit by introducing a class of functions $\varrho : \bar{\Omega} \to \mathbb{R}$ with the following properties (see Lions & Magenes [8, page 57]:

- ϱ is infinitely differentiable on $\bar{\Omega}$, positive on Ω, vanishing on Γ of the order of $d(x,\Gamma)$, which is the distance from the point $x \in \Omega$ to the boundary Γ, i.e. such that

$$\lim_{x \to x_0} \frac{\varrho(x)}{d(x,\Gamma)} = d \neq 0,$$

  if $x_0 \in \Gamma$.

The functions ϱ^{-s} are exactly those functions that, when multiplied with an $H_0^s(\Omega)$-function f for $0 \leq s \leq 1$, convert the function f to an $L^2(\Omega)$-function (see Theorem 11.2 in Lions & Magenes [8, page 57]). In fact, the mapping $f \mapsto \varrho^{-s} f$ is continuous and linear from $H_0^s(\Omega)$ to $L^2(\Omega)$. This fact also provides a way to visualize $H_0^s(\Omega)$-functions for non-integer s.

In Figure 2.3, an example function $\varrho : (0,1) \to \mathbb{R}$ is plotted along with an $H^1(0,1)$-function f, given by

$$f(x) = \begin{cases} \frac{3}{10}\sin(2\pi x) + \frac{7}{10} & \text{if } 0 < x < \frac{2}{3}, \\ \left(\frac{18+9\sqrt{3}}{20}\right)x - \frac{9\sqrt{3}-2}{20} & \text{if } \frac{2}{3} \leq x < 1, \end{cases}$$

and along with the functions ϱf and $\varrho^{-\frac{1}{2}} f$ to illustrate what multiplication by ϱ looks like. In this specific case, ϱ is given by

$$\varrho(x) = \begin{cases} x & \text{if } 0 < x < \frac{1}{3}, \\ -3(x - \frac{1}{2})^2 + \frac{5}{12} & \text{if } \frac{1}{3} \leq x < \frac{2}{3}, \\ 1 - x & \text{if } \frac{2}{3} \leq x < 1, \end{cases}$$

which is technically not infinitely differentiable, but it does provide an accurate representation of the shape of a function ϱ that is infinitely differentiable, since it vanishes at the boundary of the order of $d(x,\Gamma)$.

Using this class of functions ϱ, we can characterise the space $H_{00}^{\mu+\frac{1}{2}}(\Omega)$. This characterisation is given by Theorem 2.33

Theorem 2.33 (Theorem 11.7 in Lions & Magenes [8, page 66]). *If Ω is a Lipschitz domain and $\mu \geq 0$ is an integer, the space $H_{00}^{\mu+\frac{1}{2}}(\Omega)$ is given by*

$$H_{00}^{\mu+\frac{1}{2}}(\Omega) = \left\{ u \mid u \in H_0^{\mu+\frac{1}{2}}(\Omega), \forall \alpha : |\alpha| = \mu : \varrho^{-\frac{1}{2}} D^\alpha u \in L^2(\Omega) \right\},$$

where α is a multi-index.

This definition of $H_{00}^{\mu+\frac{1}{2}}(\Omega)$ makes sense, because - very roughly speaking - one can imagine multiplying $D^\alpha u$ by $\varrho^{-\frac{1}{2}}$ as transforming $D^\alpha u$ so that it is in $H^{\mu-|\alpha|}(\Omega)$ instead of $H^{\mu-|\alpha|+\frac{1}{2}}(\Omega)$.

The space $H_{00}^{\frac{1}{2}}(\Omega)$-space is often referred to as a *Lions-Magenes space*, as it was first introduced by Jacques-Louis Lions and Enrico Magenes.

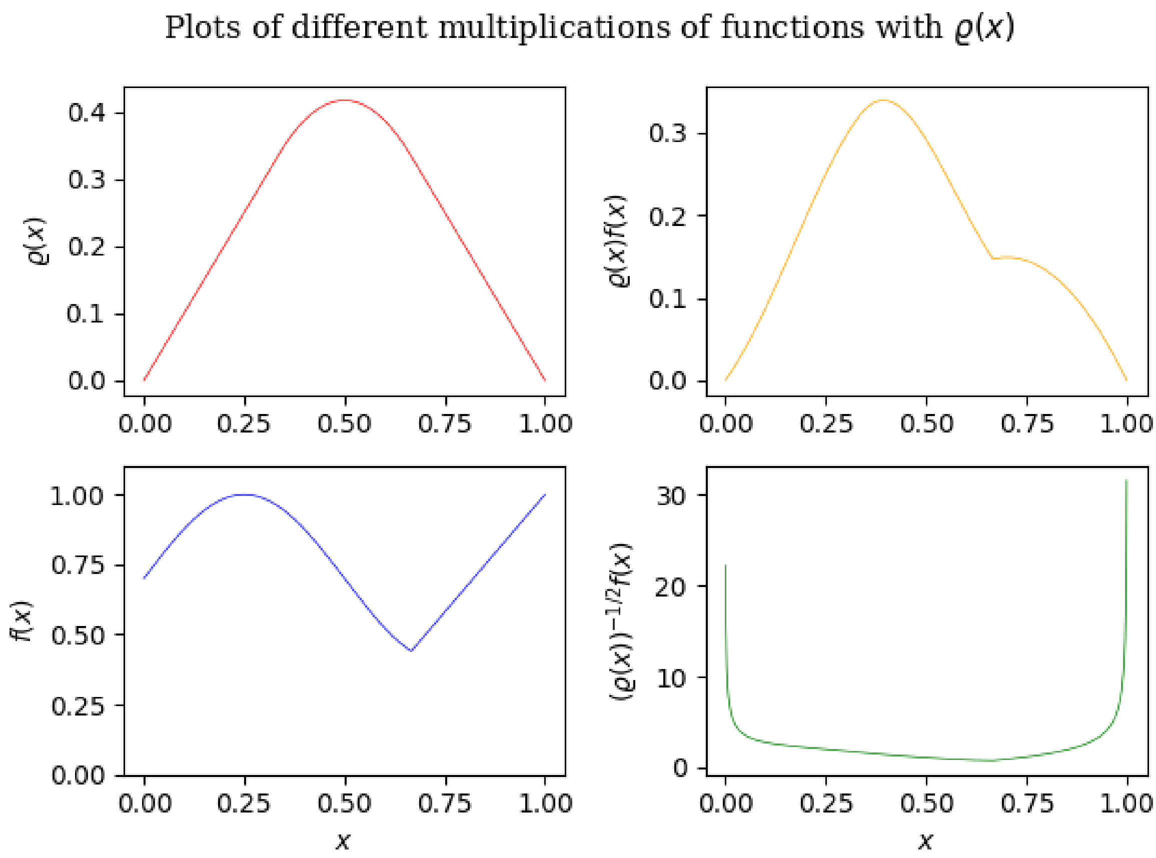

Figure 2.3: The functions $\varrho, f, \varrho f, \varrho^{-\frac{1}{2}} : (0,1) \to \mathbb{R}$.

$H_{00}^{\mu+\frac{1}{2}}(\Omega)$ is used often in the world of boundary integral equations, where it is denoted by $\tilde{H}^{\mu+\frac{1}{2}}(\Omega)$ (Equation (4.3.10) in Hsiao & Wendland [5, page 190]).

The interpolation between positive and negative order Sobolev spaces involves these $H_{00}^{\mu+\frac{1}{2}}$-spaces in some cases.

Theorem 2.34 (Theorem 12.4 in Lions & Magenes [8, page 73]). *Let Ω be a Lipschitz domain. Let $s_1, s_2 \geq 0$ be such that $s_2 \notin \{k + \frac{1}{2} | k \in \mathbb{Z}\}$. Let $\theta \in (0,1)$.*

- *If $(1-\theta)s_1 - \theta s_2 \neq -\frac{1}{2} - \mu$ for some integer $\mu \geq 0$, then*

$$[H^{s_1}(\Omega), H^{-s_2}(\Omega)]_\theta = H^{(1-\theta)s_1 - \theta s_2}(\Omega).$$

- *If $(1-\theta)s_1 - \theta s_2 = -\frac{1}{2} - \mu$ for some integer $\mu \geq 0$, then*

$$[H^{s_1}(\Omega), H^{-s_2}(\Omega)]_\theta = \left(H_{00}^{\mu+\frac{1}{2}}(\Omega)\right)'.$$

2.3. Anisotropic Sobolev spaces

In this final part of the introduction of the theory, we will introduce anisotropic Sobolev spaces. The idea is that the time derivatives can be of a different order of smoothness than the space derivatives. Roughly speaking, we would assume that the time derivatives are in $H^m(0,T)$ and the space derivatives in $H^k(\Omega)$ for possibly different non-negative real numbers m and k.

In evolution problems (like the heat- or wave equation), the time variable is often treated differently than the space variables. For instance, in the method of lines for solving time-dependent partial differential equations numerically (see Van Kan et al. [7, page 100]), one treats the discretisation in time in a different way than the discretisation in space.

The method of separation of variables (see Haberman [4, page 32]) is an analytic example of time and space being treated differently. Here, one assumes that the solution of a partial differential equation can be written as a product of a function of time and a function of space.

Because time and space are used in different ways, it is logical to assume that solutions of differential equations may have different regularity properties in time than in space as well.

In the previous sections we have considered Lipschitz domains $\Omega \subseteq \mathbb{R}^n$. Moving forward, we will consider a domain that involves both time and space, the *space-time cylinder*. This domain and some notation about the domain will be defined in Subsection 2.3.1. Subsection 2.3.2 will define the anisotropic Sobolev spaces on the space-time cylinder. After the definition, we will discuss a related function space that is useful for the results in the next chapter. The last Subsection is about interpolation between these anisotropic Sobolev spaces.

2.3.1. A note about the domain

As has been said before, our main result will be defined on a domain involving both space and time. For a Lipschitz domain Ω, and a time interval $(0,T)$ for some $T<\infty$, we define

$$Q = \Omega \times (0,T). \tag{2.12}$$

Note that Q is a Lipschitz domain, because Ω and $(0,T)$ are as well. The different boundaries of Q are denoted below.

- The boundary of Ω is denoted by Γ.
- The *lateral boundary* of Q is defined as $\Sigma = \Gamma \times (0,T)$.
- The *initial boundary* of Q is defined as $\Sigma_0 = \Omega \times \{0\}$.
- The *final boundary* of Q is defined as $\Sigma_T = \Omega \times \{T\}$.

Figure 2.4 visualizes the domain Q for a circular domain $\Omega \subseteq \mathbb{R}^2$.

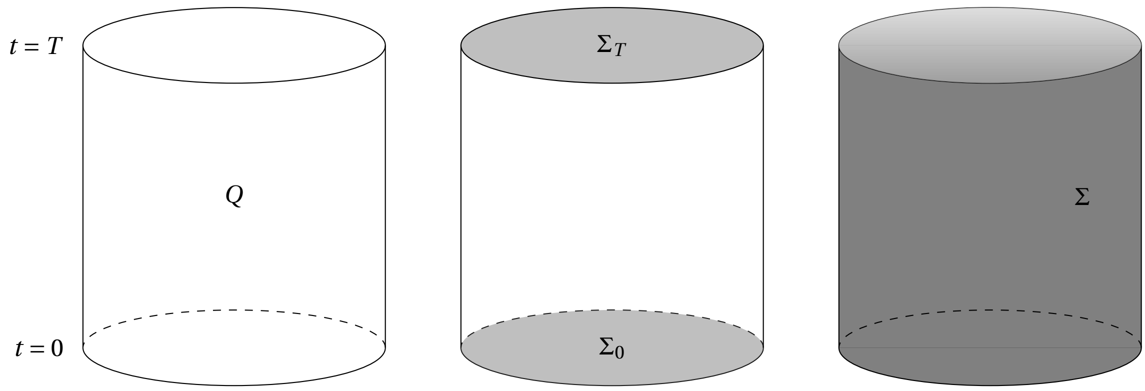

Figure 2.4: The domain $Q = \Omega \times (0,T)$ and its boundaries.

Note that the initial and boundary conditions of the wave equation (1.1) require that the trace γu of the solution u, restricted to the initial boundary Σ_0 and lateral boundary Σ respectively, is equal to the initial and boundary data, i.e.

$$\gamma u|_\Sigma = g \text{ and } \gamma u|_{\Sigma_0} = u_0.$$

The trace theorem (Theorem 2.28) then states that, given that the solution u of (1.1) satisfies $u \in H^1(Q)$, the initial and boundary data u_0 and g satisfy $u_0 \in H^{1/2}(\Sigma_0)$ and $g \in H^{1/2}(\Sigma)$.

2.3.2. Bochner spaces and anisotropic Sobolev spaces

The remainder of this section is primarily based on the second volume of Lions & Magenes [9, page 6-9]. The definition of the *Bochner space* is adapted from Adams [1, page 207].

Anisotropic Sobolev spaces are defined in terms of Bochner spaces. A Bochner space consists of functions $f : (a,b) \to X$, where (a,b) is an interval and X is a Banach space. Solutions of time-dependent partial differential equations are also of this type. Indeed, if $u(x,t)$ is a solution of a partial differential equation on the domain $Q = \Omega \times (0,T)$, then the mapping $f : (0,T) \to L^2(\Omega)$ defined by $f(t) = u(x,t)$ (for each $t \in (0,T)$ there is an $L^2(\Omega)$-function $u(x,t)$) is equivalent to the solution $u : Q \to \mathbb{R}$ defined by $u(x,t)$. As we will shortly see, we then denote $f \in L^2(0,T; L^2(\Omega))$.

Definition 2.35. *Let $(X, \|\cdot\|_X)$ be a Banach space. For $1 \le p \le \infty$ we say that a function f is in the **Bochner space** $L^p(a,b;X)$ if $\|f\|_{L^p(a,b;X)} < \infty$ where*

$$\|f\|_{L^p(a,b;X)} = \begin{cases} \left(\int_a^b \|f(t)\|_X \, dt\right)^{\frac{1}{p}} & \text{if } p < \infty \\ \operatorname*{ess\,sup}_{a<t<b} \|f(t)\|_X & \text{if } p = \infty. \end{cases}$$

First, we shall state some useful results about Bochner spaces themselves, since we will need these results later in Chapter 3. After stating the results, we will proceed with defining anisotropic Sobolev spaces.

Theorem 2.36 is a useful result about interpolation between two Bochner spaces $L^p(a,b;X)$ and $L^p(a,b;Y)$.

Theorem 2.36 (Remark 14.4 in Lions & Magenes [8, page 96])**.** *Let X and Y be two Banach spaces such that $X \subseteq Y$. Then for any $1 \le p \le \infty$ and $\theta \in (0,1)$, we have*

$$[L^p(a,b;X), L^p(a,b;Y)]_\theta = L^p(a,b,[X,Y]_\theta).$$

If we know the regularity of a function and the m-th derivative, we can relate the j-th derivatives for $1 \le j \le m-1$ with the intermediate Sobolev spaces defined in Definition 2.23.

Theorem 2.37 (Theorem 2.3 in Lions & Magenes [8, page 15])**.** ***Intermediate derivatives theorem.*** *Let X and Y be two Hilbert spaces such that $X \subseteq Y$. If $u \in L^2(a,b;X)$, $u^{(m)} \in L^2(a,b;Y)$ for $m \in \mathbb{N}$ and $j \in \mathbb{N}$ is such that $1 \le j \le m-1$, then*

$$u^{(j)} \in L^2(a,b;[X,Y]_{j/m}).$$

Moreover, the mapping $u \mapsto u^{(j)}$ is linear and continuous.

For example, if a function u is in $L^2(0,T;H^2(\Omega))$ and its second time derivative u'' is in $L^2(0,T;L^2(\Omega))$, then Theorem 2.37 implies that the first time derivative u' is in $L^2(0,T;[H^2(\Omega),L^2(\Omega)]_{1/2}) = L^2(0,T;H^1(\Omega))$.

Now, we continue with the definition of anisotropic Sobolev spaces. To this end, we first define the following (see Lions & Magenes [9, page 6]).

Definition 2.38. *Let $(X, \|\cdot\|_X)$ be a Banach space. For $m \in \mathbb{N}$ we define $H^m(a,b;X)$ for an interval (a,b) (possibly infinite) as follows.*

$$H^m(a,b;X) = \left\{ u : \; u, \frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial t^2}, \dots, \frac{\partial^m u}{\partial t^m} \in L^2(a,b;X) \right\}.$$

For any real $s \ge 0$ with $\theta \in [0,1]$ such that $(1-\theta)m = s$, we define

$$H^s(a,b;X) = [H^m(a,b;X), L^2(a,b;X)]_\theta.$$

Using the spaces $L^2(a,b;X)$ and $H^s(a,b;X)$, we can define anisotropic Sobolev spaces.

Definition 2.39 (Lions & Magenes [9, page 6]). *Let $r, s \geq 0$. If $\Omega \subseteq \mathbb{R}^n$ is an open Lipschitz domain, we define the* ***anisotropic Sobolev space*** *$H^{r,s}(\Omega)$ by*

$$H^{r,s}(Q) = L^2(0,T;H^r(\Omega)) \cap H^s(0,T;L^2(\Omega)).$$

It can be shown that $H^{r,s}(Q)$ is a Hilbert space with norm

$$\|u\|_{H^{r,s}(Q)} = \left(\int_0^T \|u(t)\|^2_{H^r(\Omega)}\, dt + \|u\|^2_{H^s(0,T;L^2(\Omega))}\right)^{\frac{1}{2}}.$$

For integer r and s, the definition of the $H^{r,s}(Q)$-norm is easier to grasp. Indeed, if $r, s \in \mathbb{N}$, then

$$\begin{aligned}
\|u\|^2_{H^{r,s}(Q)} &= \int_0^T \sum_{|\alpha|\leq r} \|D_x^\alpha u(t)\|_2^2\, dt + \sum_{k=0}^s \int_0^T \left\|\frac{\partial^k u}{\partial t^k}\right\|_2^2 dt \\
&= \sum_{|\alpha|\leq r} \int_0^T \int_\Omega |D_x^\alpha u|^2\, dx\, dt + \sum_{k=0}^s \int_0^T \int_\Omega \left|\frac{\partial^k u}{\partial t^k}\right|^2 dx\, dt.
\end{aligned}$$

As we can see, a function u is in the anisotropic space $H^{r,s}(Q)$ if its spatial derivatives up to order r and its time derivatives up to order s are square integrable on Q. The norm for any non-negative r and s is derived from a real interpolation method (see for example Theorem 2.15 on page 8).

2.3.3. A related space

The first space of this subsection is used a lot in Lasiecka, Lions & Triggiani [6]. We will reference this paper many times in Chapter 3, as it provides existence-uniqueness results for the wave equation in slightly different situations.

Definition 2.40 (Evans [3, Page 301]). *Let $(X, \|\cdot\|_X)$ be a Banach space. Then $C([a,b];X)$ is the space of all* ***continuous*** *functions $u : [a,b] \to X$. We endow $C([a,b];X)$ with the norm*

$$\|u\|_{C([a,b];X)} = \max_{a\leq t\leq b} \|u(t)\|_X.$$

Note that the norm $\|u\|_{C([a,b];X)}$ is always finite, because continuous functions on compact intervals are bounded and attain their maximum. This also means that $C([a,b];X)$ is a subset of the space $L^\infty(a,b;X)$ as defined in Definition 2.35.

A very useful theorem from Lions & Magenes [8] can relate these spaces with Sobolev spaces. The theorem is given by Theorem 2.41.

Theorem 2.41 (Theorem 3.1 in Lions & Magenes [8, page 19]). *Let $u \in L^2(a,b;X)$ and $u^{(m)} \in L^2(a,b;Y)$ for $m \in \mathbb{N}$. Then, as long as $-\infty < a < b < \infty$ and $j \in \mathbb{N}$ is such that $0 \leq j \leq m-1$, we have that*

$$u^{(j)} \in C([a,b];[X,Y]_{(j+1/2)/m}).$$

2.3.4. Interpolation between anisotropic Sobolev spaces

Since the theory of Section 2.2 was built up for any Banach space, all the theorems and methods in that section also apply to interpolation between anisotropic Sobolev spaces, since $H^{r,s}(Q)$ is a Banach space for any $r, s \geq 0$. In particular, Theorem 2.20 is of importance for the main result.

Theorem 2.42 (Proposition 2.1 in Lions & Magenes [9, page 7]). *Let $r_1, r_2, s_1, s_2 \geq 0$ and let $\theta \in [0,1]$. Then*

$$[H^{r_1,s_1}(Q), H^{r_2,s_2}(Q)]_\theta = H^{(1-\theta)r_1+\theta r_2,(1-\theta)s_1+\theta s_2}(Q).$$

3

Results

This Chapter is about the existence-uniqueness results for the wave equation. We will try to find assumptions on the regularity of the boundary- and initial conditions, and the right-hand side function f of the wave equation (1.1) so that there is a unique solution $u \in H^1(Q)$.

The paper by Lasiecka, Lions and Triggiani [6] contains a few theorems about existence-uniqueness of solutions of the wave equations. However, most of their results imply existence and uniqueness in $C([0,T];H^m(\Omega))$-spaces. Essentially, this Chapter will be an interpretation of the results of Lasiecka, Lions and Triggiani, so that we obtain existence and uniqueness in $H^1(Q)$. Note that the reason we are proving existence and uniqueness in $H^1(Q)$ is that (weak) solutions of the wave equation make natural sense in $H^1(Q)$; we have to have *finite* potential and kinetic energy.

Proving existence and uniqueness of solutions of the wave equation - a hyperbolic partial differential equation - is difficult, compared to proving existence-uniqueness for other types of partial differential equations, such as the Poisson equation (an elliptic equation). This is illustrated by Example 3.1, where we try to prove uniqueness for the wave equation using a standard approach that works for the Poisson equation.

Example 3.1. *Consider the wave equation (1.1). Suppose that (1.1) has two solutions u^0 and u^1. Define $v := u^0 - u^1$. Note that*

$$\begin{aligned}\frac{\partial^2 v}{\partial t^2} - \Delta v &= \frac{\partial^2 u^0}{\partial t^2} - \Delta u^0 - \frac{\partial^2 u^1}{\partial t^2} + \Delta u^1 \\ &= f - f = 0.\end{aligned}$$

Similar calculations show that the difference v satisfies the following partial differential equation.

$$\begin{cases} \frac{\partial^2 v}{\partial t^2} - \Delta v = 0 & \text{for } (x,t) \in Q, \\ v(x,0) = 0 & \text{for } x \in \Omega, \\ \frac{\partial v}{\partial t}(x,0) = 0 & \text{for } x \in \Omega, \\ v = 0 & \text{for } (x,t) \in \Sigma. \end{cases} \tag{3.1}$$

Now, multiply the partial differential equation by v and integrate over the domain Q[1]. We obtain

$$\int_0^T \int_\Omega \left(v \frac{\partial^2 v}{\partial t^2} - v \Delta v \right) dx\, dt = 0 \tag{3.2}$$

[1]This essentially also the process by which we derive weak formulations of initial-boundary value problems. Here we pick the test function v itself. See Example 2.29 for the derivation of the weak formulation of a Poisson equation.

Using the identities (3.3) and (3.4):

$$v\Delta v = \nabla\cdot(v\nabla v) - \nabla v\cdot\nabla v, \tag{3.3}$$

$$\frac{\partial}{\partial t}\left[v\frac{\partial v}{\partial t}\right] = \left(\frac{\partial v}{\partial t}\right)^2 + v\frac{\partial^2 v}{\partial t^2}, \tag{3.4}$$

and Gauss' divergence theorem, we can rewrite the above expression to the following.

$$\left[\int_\Omega v\frac{\partial v}{\partial t}\,dx\right]_{t=T} - \int_0^T\int_\Omega\left(\frac{\partial v}{\partial t}\right)^2\,dx\,dt + \int_0^T\int_\Omega \|\nabla v\|^2\,dx\,dt = 0, \tag{3.5}$$

where $\|\cdot\|$ denotes the Euclidean norm. The goal of this approach is to prove that (3.2) implies that $v = 0$. However, the issue here is that the terms of (3.5) may cancel with each other, giving us no information about v. Therefore, we will need another approach to prove uniqueness of solutions of the wave equation (1.1).

The sequel of this Chapter will be subdivided as follows. In the first section we will prove existence and uniqueness for relatively strong regularity assumptions on the initial- and boundary data. The second section contains a proof for a weaker result using weaker regularity assumptions. Thirdly, we will use interpolation between the results of the first two sections. Finally, we will discuss some extensions and discuss the obtained results.

3.1. Strong regularity assumptions

As has been stated before, this section contains the proof for an existence-uniqueness result using relatively strong regularity assumptions on the data. This stronger existence-theorem corresponds to the first existence-uniqueness result in the paper by Lasiecka, Lions & Triggiani [6, page 151].

In order to translate that result to a standard Sobolev space, we will use two lemmas - stated and proved in Subsection 3.1.1. These two lemmas will also be useful for the weaker regularity assumptions. Since proving existence will require a different approach to uniqueness, we will divide these to proofs into two subsections: Subsection 3.1.2 and Subsection 3.1.3.

3.1.1. Reframing lemmas

The two lemmas that are stated and proved in this subsection will be essential in reframing the results of Lasiecka, Lions and Triggiani [6] to results in terms of standard (isotropic) Sobolev spaces.

Lemma 3.2 is useful for reframing existence-uniqueness results in $C([0,T];X)$-spaces to Sobolev spaces.

Lemma 3.2. *Let $m\in\mathbb{N}$. Suppose that $u\in C([0,T];H^m(\Omega))$ and $u^{(m)} = \frac{d^m u}{dt^m}\in C([0,T];L^2(\Omega))$, with $T<\infty$. Then $u\in H^{m,m}(Q)$.*

Proof. Let u satisfy the requirements of Lemma 3.2. We will first show that for any integer k with $1\le k\le m-1$, $u^{(k)}\in L^2(0,T;H^{m-k}(\Omega))$.

Since $u:[0,T]\to H^m(\Omega)$ and $u^{(m)}:[0,T]\to L^2(\Omega)$ are continuous on a compact interval, there exist constants $0<M_0, M_m<\infty$ such that

$$\forall t\in[0,T]: \|u(t)\|_{H^m(\Omega)}\le M_m \text{ and} \|u^{(m)}(t)\|_{L^2(\Omega)}\le M_0.$$

Hence,

$$\|u\|_{L^2(0,T;H^m(\Omega))} = \int_0^T \|u(t)\|^2_{H^m(\Omega)}\,dt \le M_m^2 < \infty,$$

and

$$\|u^{(m)}\|_{L^2(0,T;L^2(\Omega))} = \int_0^T \|u^{(m)}(t)\|^2_{L^2(\Omega)}\,dt \le TM_0^2 < \infty,$$

so $u\in L^2(0,T;H^m(\Omega))$ and $u^{(m)}\in L^2(0,T;L^2(\Omega))$. Then by the intermediate derivatives theorem (Theorem 2.37), we have that for any integer k such that $1\le k\le m-1$, $u^{(k)}\in L^2(0,T;H^{m-k}(\Omega))$, since

$$[H^m(\Omega), L^2(\Omega)]_{k/m} = H^{(1-k/m)m}(\Omega) = H^{m-k}(\Omega).$$

Define the numbers M_k for $1 \le k \le m-1$ by

$$M_k := \|u^{(k)}\|_{L^2(0,T;H^{m-k}(\Omega))} < \infty.$$

M_m and M_0 were defined earlier. We will now show that $u \in H^{1,1}(\Omega) = L^2(0,T;H^m(\Omega)) \cap H^m(0,T;L^2(\Omega))$. Note that for any integer $k \le m$ and for all $t \in [0,T]$, we have that

$$\|u^{(k)}(t)\|_{H^{m-k}(\Omega)} \ge \|u^{(k)}(t)\|_{L^2(\Omega)}.$$

It suffices to show that $u \in H^m(0,T;L^2(\Omega))$, since we have already shown that $u \in L^2(0,T;H^m(\Omega))$. Now,

$$\begin{aligned}
\|u\|^2_{H^m(0,T;L^2(\Omega))} &= \sum_{k=0}^{m} \int_0^T \|u^{(k)}(t)\|^2_{L^2(\Omega)}\, dt \\
&\le \sum_{k=0}^{m} \int_0^T \|u^{(k)}(t)\|^2_{H^{m-k}(\Omega)}\, dt \\
&= \sum_{k=0}^{m} \|u^{(k)}\|^2_{L^2(0,T;H^{m-k}(\Omega))} \\
&= \sum_{k=0}^{m} M_k^2 < \infty.
\end{aligned}$$

Hence, $u \in H^m(0,T;L^2(\Omega))$. We conclude that $u \in H^{m,m}(\Omega)$. □

An extension of Lemma 3.2 to fractional order Sobolev spaces $H^{s,s}(Q)$ might also be useful in re-interpreting existence-uniqueness theorems. However, we expect that a statement and proof of such an extension would require good definitions of fractional derivatives, which are outside the scope of this thesis.

The second lemma allows us to identify the anisotropic space $H^{s,s}(Q)$ with the isotropic space $H^s(Q)$ for any $0 \le s \le 1$.

Lemma 3.3. *Let $\Omega \subseteq \mathbb{R}^n$ be an open Lipschitz domain and let s be a real number with $0 \le s \le 1$. Then, if $Q = \Omega \times (0,T)$, we have $H^{s,s}(Q) = H^s(Q)$.*

Proof. We first treat the case $s = 1$. If $u \in H^{1,1}(Q)$, then $u \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$. Then $u \in L^2(Q)$. Furthermore, for any space variable x_j, $\frac{du}{dx_j} \in L^2(Q)$ as well. This applies to the time variable t as well. Then, because we only consider first derivatives (so no mixed derivatives between time and space), we conclude $u \in H^1(Q)$. Using the definitions, the converse inclusion also holds, so $H^{1,1}(Q) = H^1(Q)$.

If $s = 0$, then $H^{0,0}(Q) = H^0(Q) = L^2(Q)$, because

$$\|u\|^2_{H^{0,0}(Q)} = \|u\|^2_{L^2(0,T;L^2(\Omega))} = \int_0^T \int_\Omega |u|^2 dx\, dt = \int_Q |u|^2 d\lambda = \|u\|^2_{L^2(Q)}.$$

The case $0 < s < 1$ then follows by interpolation. Suppose that $\theta \in (0,1)$ is such that $1-\theta = s$. Then

$$H^{s,s}(Q) = [H^{1,1}(Q), H^{0,0}(Q)]_\theta = [H^1(Q), L^2(Q)]_\theta = H^s(Q).$$

□

3.1.2. Existence

In all the proofs of Lasiecka, Lions & Triggiani [6], a *solution operator* is used to prove existence and uniqueness. The term *solution operator* refers to an operator S, mapping the data - the initial and boundary conditions and force term f - to the solution of the problem (1.1). If this operator S is continuous, then we know that the wave equation (1.1) has a unique solution[2].

In the case of Theorem 3.4, we would say that the solution operator S is continuous from $L^1(0,T;L^2(\Omega)) \times H^1(\Omega) \times L^2(\Omega) \times H^1(\Sigma)$ to $C([0,T];H^1(\Omega))$.

In the remainder of this thesis, *every* existence-uniqueness theorem that states that - under a certain set of assumptions - there exists a unique solution in some space, will always be seen as the solution operator being continuous.

Using the lemmas from the previous subsection, we can now start with reframing the existence-uniqueness result of Lasiecka, Lions & Triggiani [6, page 151]. The original result we will reframe is given by Theorem 3.4.

Theorem 3.4 (Theorem 2.1 in Lasiecka, Lions & Triggiani [6, page 151])**.** *Consider the wave equation (1.1). Suppose that*

$$\begin{cases} f \in L^1(0,T;L^2(\Omega)), \\ u_0 \in H^1(\Omega), \\ u_1 \in L^2(\Omega), \\ g \in H^1(\Sigma), \end{cases} \tag{3.6}$$

with the compatibility condition

$$g|_{t=0} = u_0|_{\partial\Omega}. \tag{3.7}$$

Then the unique solution of (1.1) satisfies

$$\begin{cases} u \in C([0,T];H^1(\Omega)), \\ u_t \in C([0,T];L^2(\Omega)), \\ \frac{\partial u}{\partial \mathbf{n}} \in L^2(\Sigma), \end{cases} \tag{3.8}$$

where $\frac{\partial u}{\partial \mathbf{n}}$ denotes the inward normal derivative along the boundary Σ.

The compatibility condition (3.7) essentially states that the Dirichlet boundary condition g 'agrees with' the initial condition u_0 on the boundary of Ω, illustrated in Figure 3.1.

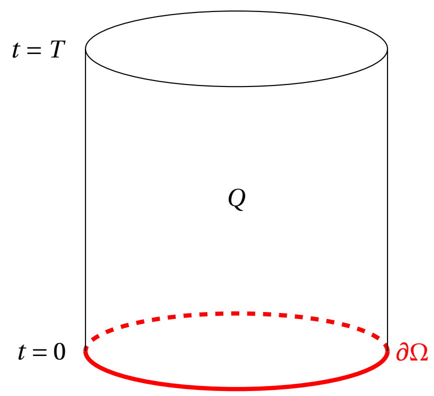

Figure 3.1: The boundary $\partial\Omega$. In the compatibility condition (3.7), the boundary condition g and initial condition u_0 have to be equal here.

Existence of a solution in $H^1(Q)$ now immediately follows by applying Lemmas 3.2 and 3.3 to (3.8).

[2] In fact, existence and uniqueness already follows from the well-posedness of the solution operator. Indeed, every input, i.e. set of data, would then have a unique output.

Corollary 3.5. *Consider the wave equation (1.1). Suppose that assumptions (3.6) and compatibility condition (3.7) are satisfied. Then there exists a solution $u \in H^1(Q)$ that solves (1.1).*

Proof. Suppose that (3.6) and (3.7) hold. Then we know that there exists a unique solution u such that

$$\begin{cases} u \in C([0,T]; H^1(\Omega)), \\ u' \in C([0,T]; L^2(\Omega)). \end{cases}$$

But then by Lemma 3.2, we know that $u \in H^{1,1}(Q)$. Then by Lemma 3.3, we see that $u \in H^1(Q)$. Thus, there exists a solution $u \in H^1(Q)$ of the wave equation (1.1). □

It should be noted that we *cannot* conclude that the solution $u \in H^1(Q)$ is unique yet, since the set

$$\mathscr{H}(Q) := \left\{ u | u \in C([0,T]; H^1(\Omega)), u' \in C([0,T]; L^2(\Omega)) \right\} \subseteq H^1(Q).$$

Indeed, from Lemmas 3.2 and 3.3 it follows that $H^1(Q)$ is a *larger* space than $\mathscr{H}(Q)$. Therefore, uniqueness in $\mathscr{H}(Q)$ does not imply uniqueness in $H^1(Q)$, since there could exist a solution that is in $H^1(Q)$ but not in $\mathscr{H}(Q)$. It is clear that we have to work some more to obtain uniqueness in $H^1(\Omega)$.

3.1.3. Uniqueness

We will use an existence-uniqueness theorem that applies to more partial differential equations than just the wave equation. The following context applies (see Lions & Magenes [8, page 265]).

Let $(V, \|\cdot\|_V)$ and $(H, \|\cdot\|_H)$ be two Hilbert spaces such that $V \subseteq H$ and V is dense in H. Let $a(t; u, v)$ be a continuous bilinear form (linear in both its arguments) on V, such that the mapping $t \mapsto a(t; u, v)$ is once continuously differentiable in $[0,T]$ with $T < \infty$ for all $u, v \in V$ and there is λ and $\alpha > 0$ such that for all $v \in V$ we have

$$a(t; v, v) + \lambda \|v\|_H^2 \geq \alpha \|v\|_V^2. \tag{3.9}$$

Let $A(t)$ be the operator defined by

$$a(t; u, v) = \langle A(t)u, v \rangle, \tag{3.10}$$

with $A(t)u \in V'$.

We consider the equation

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + A(t)u = f & \text{for } (x,t) \in Q, \\ u(x,0) = u_0(x) & \text{for } x \in \Omega, \\ u'(x,0) = u_1(x) & \text{for } x \in \Omega, \\ u = 0 & \text{for } (x,t) \in \Sigma. \end{cases} \tag{3.11}$$

Examples 9.6.1, 9.6.2 and 9.6.3 in Lions & Magenes [8, pages 292-296] show that the wave equation with a homogeneous Dirichlet boundary condition, corresponding to $A(t) = -\Delta$, is a special case of this more general context.

Theorem 3.6 (Theorem 8.1 in Lions & Magenes [8, page 265])**.** *Suppose that*

$$\begin{cases} f \in L^2(0,T; H), \\ u_0 \in V, \\ u_1 \in H. \end{cases} \tag{3.12}$$

Then there exists a unique solution u satisfying (3.11) with

$$\begin{cases} u \in L^2(0,T; V), \\ u' \in L^2(0,T; H). \end{cases} \tag{3.13}$$

If we apply this to the wave equation with homogeneous boundary conditions, we obtain Corollary 3.7.

Corollary 3.7. *Consider the wave equation (1.1) with $g = 0$. Assume that $f \in L^2(Q)$, $u_0 \in H^1(\Omega)$ and $u_1 \in L^2(\Omega)$. Then there exists a unique solution u of (1.1) with $u \in H^1(Q)$.*

Proof. Let $V = H^1(\Omega)$ and $H = L^2(\Omega)$. Since $H^1(\Omega)$ and $L^2(\Omega)$ are both Hilbert spaces and since $H^1(\Omega)$ is dense in $L^2(\Omega)$, it follows by Theorem 3.6 that there is a unique solution u with $u \in L^2(0,T;H^1(\Omega))$ and $u' \in L^2(Q)$. Therefore, $u \in H^{1,1}(Q) = L^2(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$. By Lemma 3.3, we conclude that $u \in H^1(\Omega)$. □

In the remainder of this subsection, we will prove that the wave equation (1.1) with *non-homogeneous* boundary conditions also has a unique solution $u \in H^1(Q)$. This proof follows along exactly the same lines as the proof of Theorem 2.1 in Lasiecka, Lions & Triggiani [6, pages 152-161], but substituting the spaces corresponding to Corollary 3.7 for $C([0,T];H^m(\Omega))$-spaces. That means that the proof of Proposition 3.8 will have the same level of detail as the proof of Theorem 3.4 by Lasiecka, Lions & Triggiani [6, pages 152-161].

Proposition 3.8. *Consider the wave equation (1.1). Suppose that*

$$\begin{cases} f \in L^2(Q), \\ u_0 \in H^1(\Omega), \\ u_1 \in L^2(\Omega), \\ g \in H^1(\Sigma), \end{cases} \tag{3.14}$$

and that compatibility condition (3.7) holds. Then there exists a unique solution $u \in H^1(Q)$.

The proof in Lasiecka, Lions & Triggiani [6] uses a set of four Lemmas to prove the existence-uniqueness result. We will adapt three of those Lemmas in order to prove Proposition 3.8 (the first lemma in Lasiecka, Lions & Triggiani is replaced by Corollary 3.7).

Lemma 3.9 (Adapted from Lemma 2.2 in Lasiecka, Lions & Triggiani [6, page 157])**.** *Suppose that*

$$\begin{cases} f \in L^2(0,T;H^{-1}(\Omega)), \\ u_0 \in L^2(\Omega), \\ u_1 \in H^{-1}(\Omega), \\ g \in L^2(\Sigma). \end{cases} \tag{3.15}$$

Then the unique solution u of (1.1) satisfies $u \in L^2(Q)$ and $u' \in L^2(0,T;H^{-1}(\Omega))$.

Proof. Let $(\cdot,\cdot)_Z$ denote the $L^2(Z)$-inner product. We first assume that $f = 0$.

1. If all data are smooth (we can then later extend the solution operator by continuity), and if u is the solution of the wave equation (1.1) and φ is the solution of

$$\begin{cases} \varphi'' - \Delta\varphi = \psi & \text{in } Q, \\ \varphi(x,T) = \varphi'(x,T) = 0 & \text{in } \Omega, \\ \varphi = 0 & \text{on } \Sigma, \end{cases} \tag{3.16}$$

we compare $(f,\varphi)_Q$ with $(\varphi'',\varphi)_Q$ by integration by parts. We obtain

$$0 = (f,\varphi)_Q + \left(\frac{\partial u}{\partial \mathbf{n}}, \varphi\right)_\Sigma = \int_\Sigma g \frac{\partial \varphi}{\partial \mathbf{n}}\, d\Sigma - (u_1, \varphi(0))_\Omega + (u_0, \varphi'(0))_\Omega + (u,\psi)_Q.$$

2. By virtue of Corollary 3.7, the mapping

$$\psi \mapsto (u_1,\varphi(0))_\Omega - (u_0,\varphi'(0))_\Omega - \int_\Sigma g \frac{\partial\varphi}{\partial\mathbf{n}}\, d\Sigma$$

is continuous on $L^2(Q)$ and according to (3.16) the above right-hand side equals $(u,\psi)_Q$. Therefore, u belongs to the dual space of $L^2(Q)$, i.e. $L^2(Q)$.

3. It follows that
$$u'' = \Delta u \in L^2(0,T;H^{-2}(\Omega))$$
and by the intermediate derivatives theorem (Theorem 2.37) we have that
$$u' \in L^2(0,T;H^{-1}(\Omega))$$
and the case $f = 0$ is proved.

4. The same conclusion holds when $f \neq 0$ and
$$f \in L^2(0,T;H^{-1}(\Omega)).$$
Indeed, one has only to add to the u obtained above the solution of
$$\begin{cases} \Psi'' - \Delta\Psi = f & \text{in } Q, \\ \Psi(x,0) = \Psi'(x,0) = 0 & \text{in } \Omega, \\ \Psi = 0 & \text{on } \Sigma, \end{cases}$$
which exists and is unique (by application of Theorem 3.6 with $H = H^{-1}(\Omega)$ and $V = L^2(\Omega)$; note that $L^2(\Omega)$ is dense in $H^{-1}(\Omega)$).

□

Lemma 3.10 (Adapted from Lemma 2.3 in Lasiecka, Lions & Triggiani [6, page 158])**.** *Consider the wave equation (1.1) with $f = 0$. We assume that*
$$\begin{cases} g, g' \in L^2(\Sigma), \\ u_0 \in H^1(\Omega), u_1 \in L^2(\Omega), \end{cases} \tag{3.17}$$
and that compatibility condition (3.7) holds. Then
$$\begin{cases} u, u' \in L^2(Q), \\ u'' \in L^2(0,T;H^{-1}(\Omega)). \end{cases} \tag{3.18}$$

Proof. Let u be the solution of (1.1) with $f = 0$. We assume all data to be smooth and then extend by continuity. Set $v := u'$; it satisfies
$$\begin{cases} v'' - \Delta v = 0 & \text{in } Q, \\ v(x,0) = u_1(x) & \text{in } \Omega, \\ v'(x,0) = \Delta u_0(x) & \text{in } \Omega, \\ v = g' & \text{on } \Sigma. \end{cases} \tag{3.19}$$
and we can extend by continuity as long as the condition (3.7) holds. By applying Lemma 3.9 to (3.19) we have that
$$v \in L^2(Q), v' \in L^2(0,T;H^{-1}(\Omega)).$$
Therefore, the result follows from $u' = v$. □

Lemma 3.11 (Adapted from Lemma 2.4 in Lasiecka, Lions & Triggiani [6, page 159]). *Let $\Gamma := \partial\Omega$. Consider again the wave equation (1.1) with $f = 0$. Suppose we have all the assumptions of Lemma 3.10 and, moreover*

$$g \in L^2(0,T;H^{1/2}(\Gamma)) \tag{3.20}$$

Then one also has

$$u \in L^2(0,T;H^1(\Omega)).$$

Proof. According to Lemma 3.10 one has

$$\Delta u = u'' \in L^2(0,T;H^{-1}(\Omega)),$$

and by construction and the assumption on g,

$$u|_\Gamma = g \in L^2(0,T;H^{1/2}(\Gamma)).$$

The classical Dirichlet problem with parameter t implies the result. □

Using these three lemmas, we can now prove Proposition 3.8.

Proof of Proposition 3.8. By the same logic as in the proof of Lemma 3.9, it suffices to consider the wave equation (1.1) with $f = 0$. Since $g \in H^1(\Sigma)$, we have that

$$g \in L^2(0,T;H^1(\Gamma)), g' \in L^2(0,T;L^2(\Gamma)).$$

Then by Theorem 2.41, g satisfies (3.20). Therefore, the solution of (1.1) is unique in $H^1(Q)$. This is exactly what we were aiming to prove. □

Proposition 3.8 is not yet as strong as we would like. Because Proposition 3.8 requires $H^1(\Sigma)$-boundary data, the trace theorem (Theorem 2.28) might indicate that the solution u - which has its trace (restricted to Σ) in $H^1(\Sigma)$ - has a higher regularity than $H^1(Q)$, namely $H^{\frac{3}{2}}(Q)$. Unfortunately, the trace operator is only surjective, and not injective; we know that there exists some $\phi \in H^{\frac{3}{2}}(Q)$ that has $g \in H^1(\Sigma)$ as its trace (restricted to Σ), but we cannot conclude that this ϕ then also solves the wave equation (1.1).

Furthermore, it is clear from the proof of our main reframing lemma (Lemma 3.2) that we cannot reach a higher degree of regularity using the translation methodology we developed in Subsection 3.1.1. For instance, if $u \in C([0,T];H^m(\Omega))$ and $u^{(m)} \in C([0,T];L^2(\Omega))$, we cannot conclude that $u \in H^{m+\frac{1}{2}}(Q)$, because $u \in C([0,T];H^m(\Omega))$ does not provide any information about the $H^{m+\frac{1}{2}}(Q)$-norm of u.

There is another way in which Proposition 3.8 may not yet be as strong as possible. Note that in order to obtain uniqueness in $H^1(Q)$, we had to strengthen the assumption on the force term f in (1.1)[3]. Indeed, in Corollary 3.5 we had to assume $f \in L^1(0,T;L^2(\Omega))$ and in Proposition 3.8 we had to assume $f \in L^2(Q) = L^2(0,T;L^2(\Omega))$. As of yet, it is unclear to us why Lasiecka, Lions & Triggiani [6] only have to assume that $f \in L^1(0,T;L^2(\Omega))$ in their existence-uniqueness result. We suspect that it has something to do with $L^1(0,T;L^2(\Omega))$ being the dual space of $L^\infty(0,T;L^2(\Omega))$, which is very closely related to $C([0,T];L^2(\Omega))$.

[3] For bounded domains $Q \subseteq \mathbb{R}^n$, we have that $L^2(Q) \subseteq L^1(Q)$. Indeed, if a function has a finite $L^2(Q)$-norm, the square of its absolute value is integrable. The $L^1(Q)$-norm, the integral over Q of the absolute value itself, is less than the $L^2(Q)$-norm (and thus finite), since squaring a large number makes it even larger. On bounded domains, the only way for a function to not be integrable is by growing too quickly.

3.2. Weak regularity assumptions

This section will be an adaptation of another result by Lasiecka, Lions & Triggiani [6, page 153], with weaker regularity assumptions on the data, but also with lower regularity of the solution.

A large part of the proof of this Section has already been done in Section 3.1. Therefore, we will not subdivide this section into separate subsections for existence and uniqueness. The result we will reinterpret is given by Theorem 3.12

Theorem 3.12 (Theorem 2.3 in Lasiecka, Lions & Triggiani [6, page 153])**.** *Consider the wave equation (1.1). Suppose that*

$$\begin{cases} f \in L^1(0,T;H^{-1}(\Omega)), \\ u_0 \in L^2(\Omega), \\ u_1 \in H^{-1}(\Omega), \\ g \in L^2(\Sigma). \end{cases} \tag{3.21}$$

Then, the unique solution u of (1.1) satisfies

$$\begin{cases} u \in C([0,T];L^2(\Omega)), \\ u_t \in C([0,T];H^{-1}(\Omega)), \\ \frac{\partial u}{\partial \mathbf{n}} \in H^{-1}(\Sigma). \end{cases} \tag{3.22}$$

Note that every regularity assumption and conclusion in Theorem 3.12 has been shifted down by one degree as compared to Theorem 3.4.

One might also note that no compatibility condition like (3.7) is required. This can be explained using a Proposition from Lions & Magenes [9, page 11].

Proposition 3.13 (Proposition 2.2 in Lions & Magenes [9, page 11])**.** *Let $u \in H^{r,s}(Q)$ for some $r, s > 0$ with $1 - \frac{1}{2}\left(\frac{1}{r} + \frac{1}{s}\right) > 0$ and consider the **compatibility relation***

$$\frac{\partial^k}{\partial t^k}\left(\left[\frac{\partial^j u}{\partial \mathbf{n}^j}\right]_\Sigma\right)_{t=0} = \gamma_j\left(\frac{\partial^k u}{\partial t^k}(x,0)\right), \tag{3.23}$$

where γ_j denotes the j-th degree normal derivative along the boundary Γ of Ω (similarly defined as the trace operator in Theorem 2.28). The compatibility relation (3.23) is valid for all integer couples j,k such that

$$\frac{j}{r} + \frac{k}{s} < 1 - \frac{1}{2}\left(\frac{1}{r} + \frac{1}{s}\right). \tag{3.24}$$

Since the initial condition u_0 in Theorem 3.12 is in $L^2(\Omega) = H^{0,0}(\Omega)$, $1 - \frac{1}{2}\left(\frac{1}{r} + \frac{1}{s}\right)$ actually tends to $-\infty$ if we approach 0 from above. Therefore, no compatibility condition is needed, since (3.24) does not make sense in this case. Indeed, if we are working in $L^2(\Omega)$, we are actually working with equivalence classes of functions that are equal almost everywhere. Therefore, pointwise evaluation at the boundary Γ does not really make sense, as Γ has measure zero if it is seen as a subset of Q.

Now, we can reframe Theorem 3.12 to a result in which the solution is contained in $L^2(Q)$.

Proposition 3.14. *Consider the wave equation (1.1). Suppose that*

$$\begin{cases} f \in L^2(0,T;H^{-1}(\Omega)), \\ u_0 \in L^2(\Omega), \\ u_1 \in H^{-1}(\Omega), \\ g \in L^2(\Sigma). \end{cases} \tag{3.25}$$

Then the unique solution u of (1.1) satisfies $u \in L^2(Q)$.

Proof. According to Theorem 3.12, the unique solution u of (1.1) satisfies $u \in C([0,T];L^2(\Omega))$. Then Lemma 3.2 with $m = 0$ implies that $u \in L^2(\Omega)$ as well. Lemma 3.9 shows that this solution u is unique in $L^2(Q)$. □

3.3. Interpolation between the results

When working with boundary integral equations corresponding to the wave equation with zero initial conditions (see for example Steinbach & Urzúa-Torres [10]), one usually has boundary data in $H^{1/2}(\Sigma)$. Therefore, it would be interesting and useful to investigate what can be said about the regularity of solutions of the wave equation (1.1) if one has boundary data in $H^{1/2}(\Sigma)$. This result can be achieved by interpolation between Propositions 3.8 and 3.14. Note that we actually need the *Lions-Magenes space* $H^{1/2}_{00}(\Omega)$ here, as stated in Subsection 2.2.4.

Corollary 3.15. *Consider the wave equation (1.1). Set*

$$M := H^{\frac{1}{2}}_{00}(\Omega).$$

Suppose that

$$\begin{cases} f \in L^2(0,T;M') \\ u_0 \in H^{\frac{1}{2}}(\Omega), \\ u_1 \in M', \\ g \in H^{\frac{1}{2}}(\Sigma). \end{cases} \tag{3.26}$$

Then the unique solution u of (1.1) satisfies $u \in H^{\frac{1}{2}}(Q)$.

Proof. Let $S_1 : L^2(Q) \times H^1(\Omega) \times L^2(\Omega) \times H^1(\Sigma) \to H^1(Q)$ be the solution operator of Proposition 3.8 defined by $(f, u_0, u_1, g) \mapsto u$. By Proposition 3.8, this operator is continuous. Similarly, let $S_2 : L^2(0,T;H^{-1}(\Omega)) \times L^2(\Omega) \times H^{-1}(\Omega) \times L^2(\Sigma) \to L^2(Q)$ be the solution operator of Proposition 3.14 defined by $(f, u_0, u_1, g) \mapsto u$. By Proposition 3.14, S_2 is continuous. We will interpolate this operator using parameter $\theta = 1/2$.

By Theorem 2.36 and Theorem 2.34, we have that

$$[L^2(Q), L^2(0,T;H^{-1}(\Omega))]_\theta = L^2(0,T,M').$$

Note that we introduce the Lions-Magenes space M, since we are in the special case of Theorem 2.34 where $(1-\theta)s_1 - \theta s_2 = -1/2$.

By definition, we have that

$$[H^1(\Omega), L^2(\Omega)]_\theta = H^{\frac{1}{2}}(\Omega) \text{ and } [H^1(\Sigma), L^2(\Sigma)]_\theta = H^{\frac{1}{2}}(\Sigma).$$

Finally, by Theorem 2.34 we have that

$$[L^2(\Omega), H^{-1}(\Omega)]_\theta = M'.$$

Combining all this with Theorem 2.20, the solution operator $S : L^2(0,T,M') \times H^{1/2}(\Omega) \times M' \times H^{1/2}(\Sigma) \to H^{1/2}(Q)$ is also continuous. Therefore, with assumptions (3.26), there exists a unique solution $u \in H^{1/2}(Q)$.

□

Note that we also do not need any compatibility condition here, since

$$1 - \frac{1}{2}\left(\frac{1}{1/2} + \frac{1}{1/2}\right) = -1 < 0,$$

where we used the fact that $H^{1/2}(Q)$ can be identified with $H^{\frac{1}{2},\frac{1}{2}}(Q)$ (see Lemma 3.3).

It would be interesting to take a different approach to interpolating the results with strong and weak regularity assumptions. In this approach, one would first interpolate between Theorems 3.4 and 3.12 and then apply the translation lemmas (Lemmas 3.2 and 3.3). However, we would need a version of Lemma 3.2 that works with fractional order Sobolev spaces as well. As of yet, proving such a result has been unsuccessful. If such a result were found, we may be able to draw stronger regularity conclusions for the solution u. As has been stated before, the wave equation only really makes sense for solutions in $H^1(Q)$, so our conclusion $u \in H^{1/2}(Q)$ is not yet as strong as we would like it to be.

Like in Section 3.1, the boundary data being in $H^{1/2}(\Sigma)$ and the trace theorem (Theorem 2.28) indicate that it may be possible to prove $H^1(Q)$-regularity for our solution u.

3.4. Extensions

Wave equation (1.1) is only a specific case of the wave equations that are used in practice. Indeed, wave equation (1.1) only provides solutions with a wave speed of 1. A more general wave equation with a Dirichlet boundary condition is given by (3.27).

$$\begin{cases} \frac{\partial^2 v}{\partial t^2} - c^2 \Delta v = f & \text{for } (x,t) \in Q, \\ v(x,0) = v_0(x) & \text{for } x \in \Omega, \\ v'(x,0) = v_1(x) & \text{for } x \in \Omega, \\ v = g & \text{for } (x,t) \in \Sigma. \end{cases} \tag{3.27}$$

Proposition 3.8 can be extended to wave equations like (3.27) using a coordinate transformation.

Corollary 3.16. *Let Ω be a Lipschitz domain and consider wave equation (3.27). Suppose that all the assumptions (3.14) and compatibility condition (3.7) hold. Then there exists a unique $v \in H^1(Q)$ that solves (3.27).*

Proof. Let $u \in H^1(Q_c)$ be the solution of (1.1) on $Q_c = \Omega_c \times (0,T)$ with

$$\Omega_c := \{cx \mid x \in \Omega\}.$$

Note that u exists and is unique because we transform the domain continuously (so the Lipschitz condition still holds for Ω_c); the transformation is given by $z = cx$. Then it follows that $v = u(z,t)$ solves (3.27) on Q and $v \in H^1(Q)$. □

Another interesting extension is the extension to more general second order hyperbolic partial differential equations. The paper by Lasiecka, Lions & Triggiani [6, page 184] contains a section about these kinds of partial differential equations. Proposition 3.8 cannot yet be extended to these more general differential equations fully; we can extend existence in $H^1(Q)$, but not uniqueness in $H^1(Q)$.

Consider the following differential equation.

$$\begin{cases} u'' + Au = f & \text{for } (x,t) \in Q, \\ u(x,0) = u_0(x) & \text{for } x \in \Omega, \\ u'(x,0) = u_1(x) & \text{for } x \in \Omega, \\ u = g & \text{for } x \in \Sigma, \end{cases} \tag{3.28}$$

where the differential operator A is given by

$$A = -\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial}{\partial x_j}\right). \tag{3.29}$$

Here we assume that $\Omega \subseteq \mathbb{R}^n$ and

$$\begin{cases} a_{ij} = a_{ji} \in L^\infty(Q) & \forall i, j, \\ a_{ij}(x,t)\xi_i\xi_j > \alpha\xi_i\xi_i & \forall \xi_i \in \mathbb{R}, \alpha > 0 \text{ a.e. in } Q. \end{cases} \tag{3.30}$$

Note that (3.29) can also be written as

$$A = -\operatorname{div}(\boldsymbol{\alpha}(x,t)\nabla), \tag{3.31}$$

where $\boldsymbol{\alpha}(x,t)$ is a matrix with entries $\alpha_{ij}(x,t) = a_{ij}(x,t)$. Assumptions (3.30) then state that $\boldsymbol{\alpha}(x,t)$ is bounded and positive definite. This guarantees that the operator A satisfies assumptions (3.9) and (3.10).

Lasiecka, Lions & Triggiani [6, page 185] conclude the following about existence and uniqueness of solutions of (3.28).

Theorem 3.17 (Theorem 4.1 in Lasiecka, Lions & Triggiani [6, page 185])**.** *Consider (3.28). Assume that*

$$\begin{cases} f \in L^1(0,T;L^2(\Omega)), \\ u_0 \in H^1(\Omega), \\ u_1 \in L^2(\Omega), \\ g \in H^1(\Sigma), \end{cases} \tag{3.32}$$

and assume that compatibility condition (3.7). In addition, assume that for any integer $0 \le k \le n$, *we have*

$$\frac{\partial a_{ij}}{\partial t}, \frac{\partial a_{ij}}{\partial x_k} \in L^\infty(Q).$$

Then there exists a unique solution u satisfying

$$\begin{cases} u \in C([0,T];H^1(\Omega)), \\ u' \in C([0,T];L^2(\Omega)), \end{cases} \tag{3.33}$$

as a continuous map with respect to f, u_0, u_1 *and* g.

Corollary 3.18. *Consider the hyperbolic differential equation (3.28) and suppose that all assumptions of Theorem 3.17 are satisfied. Then there exists a solution* $u \in H^1(Q)$ *of (3.28).*

Proof. The result follows directly from applying Lemma 3.2 to the conclusion of Theorem 3.17. □

It is likely that uniqueness can be obtained using the same process as Subsection 3.1.3. However, one would have to check whether the proof of Theorem 3.17 is compatible with the spaces from Theorem 3.6.

3.5. Discussion

To be able to conclude existence and uniqueness of solutions u of the wave equation (1.1) in $H^1(Q)$ requires $f \in L^2(Q)$, $u_0 \in H^1(\Omega)$, $u_1 \in L^2(\Omega)$ and $g \in H^1(\Sigma)$. There are some indications that the result is not sharp and we may be able to strengthen the conclusion, or weaken the assumptions.

The trace theorem (Theorem 2.28) implies that $H^{3/2}(Q)$-functions have $H^1(\Sigma)$-traces. This indicates that it may be possible that the solution u of (1.1) is actually in $H^{3/2}(Q)$. Likewise, $H^1(Q)$-functions have $H^{1/2}(\Sigma)$-traces, so it is possible that in the case of Corollary 3.15, we actually have $H^1(Q)$-regularity for the solution u, which is the desired regularity for solutions of the wave equation. If this were true, we would be able to weaken the assumptions of Proposition 3.8 to the assumptions of Corollary 3.15 and still be able to conclude existence and uniqueness in $H^1(Q)$.

Secondly, if one were to find out why Lasiecka, Lions & Triggiani [6] are able to only assume $L^1(0,T;L^2(\Omega))$-regularity of f, instead of the $L^2(Q)$-regularity *we* have to assume (in the case of Proposition 3.8), one could be able to slightly strengthen Proposition 3.8 by weakening the assumptions a little.

The above two remarks provide interesting avenues of further research. Three other directions would be the following:

1. Investigate whether the proof of Theorem 3.17 can be adapted to $u \in H^1(Q)$, in order to extend the results of this thesis to arbitrary second order hyperbolic differential equations with a Dirichlet boundary condition.
2. Find a non-integer extension of Lemma 3.2 in order to interchange translation and interpolation in the order of the proof of Corollary 3.15. It is possible that a stronger conclusion can be drawn.
3. As a final avenue of further research, one could try to translate different existence-uniqueness theorems altogether, following a similar approach as in this thesis.

4

Conclusion

The wave equation (1.1) - given below for convenience - is a partial differential equation that is relatively difficult to work with. In this thesis, we have tried to find suitable assumptions on the initial- and boundary conditions and force term f, so that the wave equation has a unique solution u that is in $H^1(Q)$, the energy space in which solutions of the wave equation make sense.

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u = f & \text{for } (x,t) \in Q = \Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{for } x \in \Omega, \\ \frac{\partial u}{\partial t}(x,0) = u_1(x) & \text{for } x \in \Omega, \\ u = g & \text{for } (x,t) \in \Sigma := \partial\Omega \times (0,T). \end{cases}$$

To this end, we have adapted two existence-uniqueness theorems from a paper by Lasiecka, Lions & Triggiani [6], so that we can draw conclusions about existence and uniqueness of solutions in $H^1(Q)$.

In order to conclude that the wave equation has a unique solution in $H^1(Q)$, we have determined that a sufficient set of assumptions is the following:

$$\begin{cases} f \in L^2(Q), \\ u_0 \in H^1(\Omega), \\ u_1 \in L^2(\Omega), \\ g \in H^1(\Sigma), \end{cases}$$

along with a compatibility condition given by

$$g|_{t=0} = u_0|_\Gamma,$$

i.e. the initial condition and the boundary condition agree on the boundary Γ of Ω. Since, in the context of boundary integral equations (see Steinbach & Urzúa-Torres [10]), one usually works with boundary data in $H^{1/2}(\Sigma)$, we have investigated what can be said in this case as well. By interpolating between a weaker existence-uniqueness theorem (see Proposition 3.14) and the stronger existence-uniqueness theorem stated above, we arrived at Corollary 3.15. Corollary 3.15 states that with boundary data in $H^{1/2}(\Sigma)$ (and other corresponding assumptions on initial conditions and force term f), one may conclude that the wave equation has a unique solution in $H^{1/2}(Q)$, which is not yet the desired regularity in which solutions of the wave equation make sense. However, the trace theorem (Theorem 2.28) suggests that it may be possible to prove $H^1(Q)$-regularity in this case as well, since $H^1(Q)$-functions have $H^{1/2}(\Sigma)$-traces.

The stronger existence-uniqueness theorem (Proposition 3.8) can be extended to more general wave equations (3.27) with wave speed c. Furthermore, it may also be partially extended to arbitrary second order hyperbolic partial differential equations with Dirichlet boundary conditions. The existence-part can be extended to the general hyperbolic differential equations, but it is unclear whether the uniqueness may be extended as well.

Bibliography

[1] R.A. Adams and J.J.F. Fournier. *Sobolev Spaces.* Elsevier Science & Technology, 2nd edition, 2003.

[2] N.L. Carothers. *Real Analysis.* Cambridge University Press, 2000.

[3] L.C. Evans. *Partial Differential Equations.* American Mathematics Society, 2nd edition, 1998.

[4] R. Haberman. *Applied Partial Differential Equations.* Pearson, 5th edition, 2013.

[5] G.C. Hsiao and W.L. Wendland. *Boundary Integral Equations.* Springer, 2008.

[6] J.L. Lions I. Lasiecka and R. Triggiani. Non homogeneous boundary value problems for second order hyperbolic operators. *Journale Mathématiques Pures et Appliqués*, 65:149 – 192, 1986.

[7] J. van Kan, A. Segal, F.J. Vermolen and J.F.B.M. Kraaijevanger. *Numerical Methods for Partial Differential Equations.* Delft Academic Press, 2019.

[8] J.L. Lions and E. Magenes. *Non-Homogeneous Boundary Value Problems and Applications. Volume I.* Springer-Verlag, 1972.

[9] J.L. Lions and E. Magenes. *Non-Homogeneous Boundary Value Problems and Applications. Volume II.* Springer-Verlag, 1972.

[10] O. Steinbach and C.A. Urzúa-Torres. A new approach to space-time boundary integral equations for the wave equation. *SIAM J. Math. Anal.*, 54:1370 – 1392, 2022.

[11] O. Steinbach and M. Zank. Coercive space-time finite element methods for initial boundary value problems. *Electron. Trans. Numer. Anal.*, 52:154 – 194, 2020.

[12] M.E. Taylor. *Partial Differential Equations I - Basic Theory.* Springer, 1996.