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Singular Perturbations of Volterra Equations with Periodic Nonlinearities. Stability and Oscillatory Properties. ⋆

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Abstract: Singularly perturbed integro-differential Volterra equations with MIMO periodic nonlinearities are considered, which describe synchronization circuits (such as phase- and frequency-locked loops) and many other "pendulum-like" systems. Similar to the usual pendulum equation, such systems are typically featured by infinite sequences of equilibria points, and none of which can be globally asymptotically stable. A natural extension of the global asymptotic stability is the *gradient-like* behavior, that is, convergence of any solution to one of the equilibria. In this paper, we offer an efficient frequency-domain criterion for gradient-like behavior. This criterion is not only applicable to a broad class of infinite-dimensional systems with periodic nonlinearities, but in fact ensures the equilibria set stability under *singular perturbation*. In particular, the proposed criterion guarantees the absence of periodic solutions that are considered to be undesirable in synchronization systems. In this paper we also discuss a relaxed version of this criterion, which guarantees the absence of "high-frequency" periodic solutions, whose frequencies lie beyond a certain bounded interval.

Keywords: Singular perturbation, gradient-like behavior, periodic solution, integro-differential equation, phase synchronization systems .

1. INTRODUCTION

Singular perturbation theory is a powerful tool to examine natural and engineered systems, in which slow and fast processes (or two different *time scales*) co-exist (Fridrichs, 1955; Dyke, 1964; Cole, 1968; Imanaliev, 1972, 1974; Kokotovic et al., 1986; O'Malley, 1991; Naidu and Calise, 2001). Mathematically, a "singularly perturbed" system is a family of systems, parameterized by a small scalar parameter $\mu \geq 0$. The system, corresponding to $\mu = 0$, is treated to as an original (*unperturbed*) system, whereas the systems indexed by $\mu > 0$ are considered to be its *perturbations*. Although the system's coefficients are continuous in μ, its structure is changed as the parameter vanishes $\mu = 0$, e.g. the dimension of the state may reduce and a system of ODE may turn into a descriptor system.

In analysis of singular perturbations for various types of dynamical systems two parallel lines can be distinguished. The first of these lines, started by (Tikhonov, 1948), is focused on proving convergence of the perturbed solutions to unperturbed ones as $\mu \to 0$ (Lizama and Prado, 2006b,a; Parand and Rad, 2011). The main concern of the second direction is to find conditions that guarantee global asymptotical stability of the unique equilibrium point under sufficiently small values of the parameter (Klimushev and Krasovskii, 1961; Khalil, 1981; Kokotovic et al., 1986).

Unlike globally asymptotically stable equilibria, well studied in the literature, the behavior of more complicated attractors under singular perturbations remains uncovered by the existing results. In this paper, we consider singular perturbations of a Lur'e-type system, represented as the feedback interconnection of a linear integro-differential equation and a periodic MIMO nonlinearity. Special cases of such a model are pendulum-like systems (Stoker, 1950) and *synchronization systems* (e.g. phase- and frequency-locked loops), arising in electrical and communication engineering (Margaris, 2004; Leonov et al., 2015; Leonov, 2006; Hoppensteadt, 1983). Singular perturbations in such systems may describe the effects of relaxation oscillations (O'Malley, 1991) or "weak" filtering (Hoppensteadt, 1983) (the parameter determines the filter bandwidth).

The presence of periodic nonlinearity in the system usually leads to an infinite sequence of equilibria (which may be stable or unstable), as exemplified by a usual pendulum. One of main questions regarding the dynamics of synchronization system is the convergence of any solution to one

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of the equilibria. This property is sometimes referred to as the *gradient-like* behavior (Leonov, 2006). Efficient sufficient "frequency-algebraic" conditions for the gradient-like behavior, based on the periodic Lyapunov functions and integral quadratic constraints, have been established in (Leonov, 2006; Leonov et al., 1996; Perkin et al., 2012).

If the system is not gradient-like, a natural question arises whether it has periodic solutions. Existence of periodic solutions of some prescribed frequency in phase locked loops (PLL) was studied in (Shakhil'dyan and Lyakhovkin, 1972; Evtyanov and Snedkova, 1968). In (Leonov and Speranskaya, 1985) a general *nonexistence* criterion was obtained, employing the Fourier series method. It was shown that a relaxed version of the condition for the gradient-like behavior guarantees absence of periodic solutions with sufficiently high frequencies. The results of (Leonov and Speranskaya, 1985) were extended to discrete-time systems (Leonov and Fyodorov, 2011) and infinite dimensional systems (Leonov et al., 1996). These latter results were further improved in (Perkin et al., 2015; Smirnova and Proskurnikov, 2016; Smirnova et al., 2016b) with tightening of the frequency-domain conditions.

In this paper we consider singularly perturbed integro-differential equations with periodic nonlinearities. We offer a frequency-domain criterion of gradient-like behavior under small values of the parameter. In the case where the frequency-domain condition is valid only for sufficiently large frequencies, it ensures the absence of the high-frequency periodic oscillations. Unlike the previous works (Smirnova et al., 2016a,b), we consider the case of MIMO nonlinearities and though the methods are the same the technique is much more complicated.

2. PROBLEM SETUP

Consider a system of singularly perturbed integro-differential Volterra equations as follows

$$\begin{aligned} M\ddot{\sigma}_M(t) + \dot{\sigma}_M(t) &= b(t) + R\xi(t-h) - \int_0^t \gamma(t-\tau)\xi(\tau)\,d\tau, \\ \xi(t) &= \varphi(\sigma_M(t)), \quad t \geq 0. \end{aligned} \tag{1}$$

Here $\sigma_M(t), \xi(t) \in \mathbb{R}^l$ and the MIMO nonlinearity φ is input-output decoupled: $\varphi(\sigma_M) = (\varphi_1(\sigma_{M1}), \ldots, \varphi_l(\sigma_{Ml}))$. The map φ_j is C^1–smooth and Δ_j–periodic for any j and has simple and isolated roots on $[0;\Delta_j)$ (being thus non-constant). The matrix $R \in \mathbb{R}^{l\times l}$, delay $h \geq 0$, kernel $\gamma : [0;+\infty) \to \mathbb{R}^{l\times l}$ and $b : [0;+\infty) \to \mathbb{R}^{l\times l}$ are known. The function $b(\cdot)$ is continuous, the function $\gamma(\cdot)$ is summable. The matrix M is diagonal and given by

$$M = \begin{pmatrix} \mathbf{0}_{l_1} & \mathbf{0} \\ \mathbf{0} & \mu I_{l_2} \end{pmatrix}, \quad l_1 + l_2 = l, \quad \mu \geq 0. \tag{2}$$

where I_m, $\mathbf{0}_k$ stand for $m \times m$ identity matrix and for the $k \times k$ null matrix respectively and μ is a small parameter. To determine a solution of (1) uniquely, initial conditions have to be fixed

$$\begin{aligned} \sigma_M(t) \equiv \sigma^0(t)\, \forall t \in [-h;0], \quad \sigma^0 \in C^1[-h;0], \\ \sigma_M(0+0) = \sigma^0(0),\ \dot{\sigma}_M(0+0) = \dot{\sigma}^0(0), \end{aligned} \tag{3}$$

hence the solutions of (1) are continuously differentiable. We impose the additional assumptions that

$$|b(t)| + |\gamma(t)| \leq Ce^{-rt}, \quad C, r > 0. \tag{4}$$

We also introduce the constants m_{1j} and m_{2j} as follows

$$m_{1j} \triangleq \inf_{\zeta\in[0,\Delta_j)} \frac{d\varphi_j(\zeta)}{d\zeta}, m_{2j} \triangleq \sup_{\zeta\in[0,\Delta_j)} \frac{d\varphi_j(\zeta)}{d\zeta}. \tag{5}$$

Since φ_j is periodic and non-constant, $m_{1j} < 0 < m_{2j}\, \forall j$.

The criteria, proposed in this paper, do not require to know the exact values of m_{1j}, m_{2j}. It is assumed, however, that they belong to a certain finite interval $[\alpha_1;\alpha_2]$ (that is, φ_j satisfies a conventional *slope restriction*)

$$-\infty < \alpha_{1j} \leq m_{1j} \leq m_{2j} \leq \alpha_{2j} < \infty \quad \forall j. \tag{6}$$

We introduce the matrices of lower and upper bounds

$$A_i \triangleq diag\,\{\alpha_{i1}, \ldots, \alpha_{il}\} \quad (i = 1, 2).$$

The system (1) arises as a singular perturbation of the following system (corresponding to $\mu = 0$)

$$\begin{aligned} \dot{\sigma}_0(t) &= b(t) + R\xi(t-h) - \int_0^t \gamma(t-\tau)\xi(\tau)\,d\tau, \\ \xi(t) &= \varphi(\sigma_0(t)), \quad t \geq 0. \end{aligned} \tag{7}$$

In papers Perkin et al. (2012), Perkin et al. (2015) the conditions for gradient–like behavior of (7) as well as the conditions for the absence of periodic solutions are obtained. They are formulated in terms of the transfer matrix of its linear part from the input to the output ($-\dot{\sigma}$):

$$K_0(p) = -Re^{-ph} + \int_0^{+\infty} \gamma(t)e^{-pt}\,dt \quad (p \in \mathbf{C}), \tag{8}$$

In this paper we extend the frequency–algebraic criteria for asymptotic behavior of unperturbed system (7) to singular perturbed system (1).

Solving the Cauchy problem for functions $\dot{\sigma}_{Mj}(t)$ with $j = l_1 + 1, \ldots, l$ one can reduce the system (1) to the system

$$\begin{aligned} \dot{\sigma}_M(t) &= b_M(t) + \bar{R}\xi(t-h) - \int_0^t \gamma_M(t-\tau)\xi(\tau)\,d\tau, \\ \xi(t) &= \varphi(\sigma_M(t)) \end{aligned} \tag{9}$$

where $\bar{R}$ is a $l_1 \times l$–matrix, $b_M(t)$ and $\gamma_M(t)$ satisfy the estimate

$$|b_M(t)| + |\gamma_M(t)| < C_1e^{-rt} \quad (C_1 > 0). \tag{10}$$

The transfer matrix for system (1) from the input ξ to the output $-\dot{\sigma}_M(t)$ is as follows

$$K_M(p) = Q(p)K_0(p), \quad Q(p) = \begin{pmatrix} I_{l_1}; & \mathbf{0} \\ \mathbf{0}; & \dfrac{1}{1+\mu p} I_{l_2} \end{pmatrix}. \tag{11}$$

3. THE CONVERGENCE OF SOLUTIONS OF SINGULARLY PERTURBED SYSTEM

The results of this section are based on the following technical lemma, proved in (Smirnova et al., 2014).

Lemma 1. Suppose there exist *diagonal* matrices $\varkappa > 0, \varepsilon > 0, \delta > 0, \tau > 0$ such that for all $\omega \in \mathbb{R}$ one has

$$\Omega(\omega) := Re\Big\{\varkappa K_M(i\omega) - (K_M(i\omega)+ \\ +A_1^{-1}i\omega)^*\tau(K_M(i\omega)+A_2^{-1}i\omega)- \\ -K_M^*(i\omega)\varepsilon K_M(i\omega)\Big\} - \delta > 0 \quad (i^2=-1) \tag{12}$$

Then the quadratic functionals $I_T[\sigma_M(\cdot),\xi(\cdot)]$, defined by

$$I_T \triangleq \int_0^T \{\dot\sigma_M(t)^*\varkappa\xi(t) + \xi(t)^*\delta\xi(t) + \dot\sigma_M(t)^*\varepsilon\dot\sigma_M(t)+ \\ + (\dot\sigma_M(t) - A_1^{-1}\dot\xi(t))^*\tau(\dot\sigma_M(t) - A_2^{-1}\dot\xi(t))\}\,dt \tag{13}$$

are uniformly bounded along any solution of (1):

$$Q := \sup_{T\geq 0} I_T[\sigma_M(\cdot),\xi(\cdot)] < \infty. \tag{14}$$

Henceforth the diagonal entries of matrices $\varkappa, \delta, \varepsilon, \tau$ are denoted by, respectively, $\varkappa_j, \delta_j, \varepsilon_j, \tau_j$. For each $j = 1,\dots,l$ we introduce the following constants and functions

$$\Phi_j(\zeta) \triangleq \sqrt{(1-\alpha_{1j}^{-1}\varphi_j'(\zeta))(1-\alpha_{2j}^{-1}\varphi_j'(\zeta))},$$

$$\nu_j = \frac{\int_0^{\Delta_j}\varphi_j(\zeta)d\zeta}{\int_0^{\Delta_j}|\varphi_j(\zeta)|d\zeta}, \quad \nu_{0j} = \frac{\int_0^{\Delta_j}\varphi_j(\zeta)d\zeta}{\int_0^{\Delta_j}\Phi_j(\zeta)|\varphi_j(\zeta)|d\zeta},$$

$$\nu_{1j}(x,y) = \frac{\int_0^{\Delta_j}\varphi_j(\zeta)d\zeta}{\int_0^{\Delta_j}|\varphi_j(\zeta)|\sqrt{1+\frac{x}{y}\Phi_j^2(\zeta)}d\zeta}.$$

Henceforth we use $ReH = \frac{1}{2}(H+H^*)$ to denote the Hermitian part of a square matrix H.

Our first result gives a condition for the convergence of solutions to equilibria points.

Theorem 2. Suppose that positive definite diagonal matrices $\varkappa$, δ, ε, τ and numbers $a_k \in [0,1]$ $(k = 1,\dots,l)$ exist, satisfying the following conditions:
1) for all $\omega \geq 0$ the inequality (12) is true;
2)the quadratic forms

$$W_j(\xi,\eta,\zeta) := \varepsilon_j\xi^2 + \delta_j\eta^2 + \tau_j\zeta^2 + \varkappa_j a_j\nu_j\xi\eta+ \\ +\varkappa_j(1-a_j)\nu_{0j}\eta\zeta \quad (j=1,\dots,l) \tag{15}$$

are positive definite.
Then the solutions of (1) converges

$$\dot\sigma_M(t) \xrightarrow[t\to\infty]{} 0, \quad \sigma_M(t) \xrightarrow[t\to\infty]{} q, \tag{16}$$

to some equilibrium point q, where $\varphi_k(q_k) = 0\,\forall k$.

Proof. Recalling the definition of Φ_j, the functional (13) can be represented in the following way

$$I_T = \sum_{j=1}^{l}\int_0^T \big\{\varkappa_j\xi_j(t)\dot\sigma_{Mj}(t) + \delta_j\xi_j^2(t)+ \\ +\varepsilon_j\dot\sigma_{Mj}^2(t) + \tau_j\Phi_j^2(\sigma_{Mj}(t))\dot\sigma_{Mj}^2(t)\big\}\,dt. \tag{17}$$

Introducing the functions $F_j(\zeta) \triangleq \varphi_j(\zeta) - \nu_j\,|\varphi_j(\zeta)|$, $\Psi_j(\zeta) \triangleq \varphi_j(\zeta) - \nu_{0j}\Phi_j(\zeta)\,|\varphi_j(\zeta)|$, it can be shown that

$$I_T = \sum_{j=1}^{l}\int_0^T \big\{\varkappa_j a_j\nu_j|\xi_j(t)|\dot\sigma_{Mj}(t) + \varepsilon_j\dot\sigma_{Mj}^2(t)+ \\ \varkappa_j a_{0j}\nu_{0j}|\xi_j(t)|\Phi_j(\sigma_{Mj}(t))\dot\sigma_{Mj}(t) + \tau_j\Phi_j^2(\sigma_{Mj}(t))\dot\sigma_{Mj}^2(t)+ \\ +\delta_j\xi_j^2(t)\big\}\,dt + \sum_{j=1}^{l}\left[\int_0^T \varkappa_j a_j F_j(\sigma_{Mj}(t))\dot\sigma_{Mj}(t)dt+ \right. \\ \left. +\int_0^T \varkappa_j a_{0j}\nu_{0j}\Psi_{0j}(\sigma_{Mj}(t))\dot\sigma_{Mj}(t)dt\right], \tag{18}$$

where $a_{0j} \triangleq 1 - a_j$. By noticing that

$$\int_0^{\Delta_j} F_j(\zeta)d\zeta = \int_0^{\Delta_j}\Psi_j(\zeta)d\zeta = 0\,\forall j, \tag{19}$$

it can be shown that the integrals $\int_0^T F_j(\sigma_{Mj}(t))\dot\sigma_{Mj}(t)dt$ and $\int_0^T \Psi_j(\sigma_{Mj}(t))\dot\sigma_{Mj}(t)dt$ are uniformly bounded. Using (14), one arrives at the following inequality

$$\sum_{j=1}^{l}\int_0^T \{\varkappa_j a_j\nu_j|\xi_j(t)|\dot\sigma_{Mj}(t) + \varkappa_j a_{0j}\nu_{0j}|\xi_j(t)|\cdot \\ \cdot\Phi_j(\sigma_{Mj}(t))\dot\sigma_{Mj}(t) + \varepsilon\dot\sigma_{Mj}^2(t) + \delta_j\xi_j^2(t)+ \\ +\tau_j\Phi_j^2(\sigma_{Mj}(t))\dot\sigma_{Mj}^2(t)\big\}\,dt < C_0 \quad \forall T \geq 0, \tag{20}$$

where C_0 is a constant. Condition 2) and (20) entail that

$$\int_0^{+\infty}\xi_j^2(t)dt < +\infty, \int_0^{+\infty}\dot\sigma_{Mj}^2(t)dt < +\infty. \tag{21}$$

Using the Barbalat lemma (Leonov et al., 1996) it can be shown that $\varphi_j(\sigma_{Mj}(t)) \to 0$ and, due to (1), $\dot\sigma_{Mj} \to 0$, which implies convergence of $\sigma_M(t)$ as $t \to \infty$ to one of the isolated equilibria points.■

Theorem 3. Suppose there exist diagonal matrices $\varkappa > 0, \varepsilon > 0, \delta > 0, \tau > 0$ such that (12) holds for all $\omega \in \mathbb{R}$ and $\nu_{1j} = \nu_{1j}(\tau_j,\varepsilon_j)$ satisfy inequalities

$$2\sqrt{\varepsilon_j\delta_j} > |\nu_{1j}|\varkappa_j \quad (\forall j = 1,\dots,l). \tag{22}$$

Then the solutions converge in the sense of (16).

Proof. Introducing the following diagonal matrix $\Phi(\sigma_M) \triangleq diag\{\Phi_1(\sigma_{M1}),\dots,\Phi_l(\sigma_{Ml})\}$, one may notice that

$$I_T = \int_0^T [\dot\sigma_M(t)^*\varkappa\xi(t) + \xi(t)^*\delta\xi(t)+ \\ +\dot\sigma_M(t)^*(\varepsilon + \tau\Phi^2(\sigma_M))\dot\sigma_M(t)]dt.$$

Additionally, we introduce the functions

$$Y_j(\zeta) \triangleq \varphi_j(\zeta) - \nu_{1j}|\varphi_j(\zeta)|P_j(\zeta),\ P_j(\zeta) \triangleq \sqrt{1+\frac{\tau_j}{\varepsilon_j}\Phi_j^2(\zeta)}$$

and notice that $\xi_j = Y_j(\sigma_{Mj}) + \nu_{1j}|\xi_j|P_j(\sigma_{Mj})$ and the integrals $\int_0^T Y_j(\sigma_{Mj})\dot\sigma_{Mj}(t)dt$ are uniformly bounded. Lemma 1 now implies the uniform boundedness of integrals

$$\int_0^T [\varkappa_j\nu_{1j}|\xi_j|\dot\sigma_{Mj}P_j(\sigma_{Mj}) + \delta_j|\xi_j|^2 + \varepsilon_j\dot\sigma_{Mj}^2P_j^2(\sigma_{Mj})]dt,$$

entailing that $\dot\sigma_{Mj} \in L_2[0;\infty]$, $\xi_j \in L_2[0;\infty]$ due to (22) . The proof is ended by applying the Barbalat lemma.■

In the following two theorems we establish frequency–domain conditions, providing convergence of the solutions of (1) under small parameter. They employ the transfer function $K_0(p)$ of unperturbed system (7).

Theorem 4. Suppose there exist positive definite diagonal matrices $\varepsilon, \delta, \tau, \varkappa$ and numbers $a_j \in [0,1]$ $(j = 1,\dots,l)$ such that the following conditions are satisfied :
1) the frequency inequality

$$\Omega_0(\omega) := Re\Big\{\varkappa K_0(i\omega) - (K_0^*(i\omega)\varepsilon K_0(i\omega) - \\ -(K_0(i\omega) + A_1^{-1}i\omega)^*\tau(K_0(i\omega) + A_2^{-1}i\omega)\Big\} - \delta > 0, \tag{23}$$

is true for all $\omega \geq 0$.
2) the quadratic forms (15) are positive definite.
Then there exists $\mu_0 > 0$ such that for all $\mu \in (0, \mu_0)$ the solutions of (1) converge in the sense of (16).

Proof. It follows from Theorem 2 that (16) holds if the frequency-domain inequality (12) holds for all $\omega \geq 0$. Recalling (11) the function $\Omega(\omega)$ may be decomposed as

$$\Omega(\omega) = \Omega_0(\omega) - \bar\Omega(\omega), \tag{24}$$

where, by definition

$$\bar\Omega(\omega) \triangleq Re\Big\{\varkappa Q_1(i\omega)K_0(i\omega) - K_0^*Q_1^*(i\omega)\tau(K_0(i\omega) + \\ +i\omega A_2^{-1}) - (K_0^*(i\omega) + i\omega A_1^{-1})\tau Q_1(i\omega)K_0(i\omega) + \\ +K_0^*(i\omega)Q_1^*(i\omega)(\tau+\varepsilon)Q_1(i\omega)K_0(i\omega) - \\ -2(K^*(i\omega)\varepsilon Q_1(i\omega)K_0(i\omega)\Big\}, \tag{25}$$

and the diagonal matrix $Q_1(p)$ stands for

$$Q_1(p) \triangleq \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \dfrac{\mu p}{1+\mu p} I_{l_2} \end{pmatrix}.$$

A straightforward computation shows that

$$\bar\Omega(\omega) = O(\mu)T_1(\omega) + O(\mu^2)T_2(\omega),$$

where $T_i(\omega)$ $(i = 1,2)$ are continuous functions. Since $|K_0(i\omega)|$ is bounded for $\omega \in \mathbb{R}$, the inequality (12) is equivalent to

$$-A_1^{-1}\tau A_2^{-1}\omega^2 + P_1(\omega)\omega + P_2(\omega) + \\ +\mu(P_3(\omega) + P_4(\omega)\omega^2) + \mu^2(-A_1^{-1}\tau A_2^{-1}\omega^4 - \\ -\delta\omega^2) > 0 \quad (A_1^{-1}\tau A_2^{-1} < 0), \tag{26}$$

where P_i $(i = 1,2,3,4)$ are bounded for $\omega \in [0, +\infty)$. So we can fix $\bar\mu > 0$ and choose the positive value ω_0 such that the inequality (26) is valid for all $\omega \in [\omega_0, +\infty)$ and all $\mu < \bar\mu$. Next step we choose $\mu_0 < \bar\mu$ so small that $\Omega(\omega)$ is positive for $\omega \in [0, \omega_0]$ and all $\mu < \mu_0$ (this is possible due to (25)). Theorem 4 is proved. ■

Theorem 5. Suppose where exist positive definite diagonal matrices $\varepsilon, \delta, \tau, \varkappa$ such that for all $\omega \geq 0$ the frequency inequality (23) is satisfied. Suppose also that for varying parameters $\varepsilon_j, \delta_j, \tau_j, \varkappa_j$ the inequalities

$$2\sqrt{\varepsilon_j\delta_j} > |\nu_{1j}(\tau_j;\varepsilon_j)|\varkappa_j \quad (j = 1,\dots,l) \tag{27}$$

are valid. Then the conclusion of Theorem 4 is true.

Proof. The proof is similar to the proof of Theorem 4, using Theorem 3 instead of Theorem 2. ■

4. FREQUENCIES OF PERIODIC SOLUTIONS

Definition 6. We say that a solution $\sigma_M(t)$ of (1) has the *period* $T_M > 0$ or the *frequency* $\omega_M = 2\pi/T_M$ if there exists a set of integers I_j $(j = 1,\dots,l)$ such that

$$\sigma_{Mj}(t + T_M) = \sigma_{Mj}(t) + I_j\Delta_j \quad (j = 1,2,\dots,l). \tag{28}$$

In this section we establish the conditions for the absence of periodic solutions with certain frequencies under sufficiently small values of the parameters $\mu < \mu_0$.

We shall need some preliminaries Leonov and Speranskaya (1985); Leonov et al. (1996). Suppose $\sigma_M(t)$ is a T_M–periodic solution of (1). Then $\varphi(\sigma_M(t))$ is a T_M–periodic function. Indeed it follows from (28) that

$$\varphi_j(\sigma_{Mj}(t + T_M)) = \varphi_j(\sigma_{Mj}(t) + I_j\Delta_j) = \varphi_j(\sigma_{Mj}(t)). \tag{29}$$

Consider the Fourier series of this periodic function

$$\varphi(\sigma_M(t)) = \sum_{k=-\infty}^{+\infty} B_k e^{i\omega k t},\ B_k \in \mathbb{R}^l \quad (i^2 = -1), \tag{30}$$

By substituting (30) in (9) we have

$$\dot\sigma_M(t) = b_M(t) + \beta(t) - \sum_{k=-\infty}^{+\infty} K_M(i\omega k)B_k e^{i\omega k t}, \tag{31}$$

where the function $\beta(t)$ is defined as follows

$$\beta(t) \triangleq \int_t^{+\infty} \gamma_M(\tau)\varphi(\sigma_M(t-\tau))\,d\tau. \tag{32}$$

The restrictions (10) imply that that $b_M(t) + \beta(t) \xrightarrow[t\to\infty]{} 0$. Since $\dot\sigma_M(t)$ is T_M–periodic, $b_M(t) + \beta(t) \equiv 0$ and thus

$$\dot\sigma_M(t) = -\sum_{k=-\infty}^{+\infty} K_M(i\omega k)B_k e^{i\omega k t}. \tag{33}$$

Theorem 7. Suppose there exist $\bar\omega > 0$, positive definite matrices $\varkappa$, τ, ε, δ and numbers $a_j \in [0;1]$ $(j = 1,...,l)$, such that the following conditions are valid:
1) for $\omega = 0$ and all $\omega \geq \bar\omega$ the inequality (12) is true;
2) the quadratic forms (15) are positive definite.
Then (1) has no periodic solution with frequency $\omega \geq \bar\omega$.

Proof. Using the functions $F_j(\zeta), \Psi_j(\zeta)$ from the proof of Theorem 2, denote $F(\sigma) \triangleq (F_1(\sigma_1),...,F_l(\sigma_l))^T$ and $\Psi(\sigma) \triangleq (\Psi_1(\sigma_1),...,\Psi_l(\sigma_l))^T$. Introduce the diagonal matrices $A = diag\{a_1,...,a_l\}$, $A_0 = diag\{1-a_1,...,1-a_l\}$. Let $\sigma_M(t)$ be a T_M–periodic solution of (1) and

$$G(t) \triangleq \dot\sigma_M^*(t)\varepsilon\dot\sigma_M(t) + \dot\sigma_M^*(t)\varkappa\varphi(\sigma_M(t)) + \\ +\varphi^*(\sigma_M(t))\delta\varphi(\sigma_M(t)) - F^*(\sigma_M(t))A\varkappa\dot\sigma_M(t) - \\ -\Psi^*(\sigma_M(t))A_0\varkappa\dot\sigma_M(t) + \\ +(\dot\sigma_M(t) - A_1^{-1}\dot\varphi(\sigma_M(t)))^*\tau(\dot\sigma_M(t) - A_2^{-1}\dot\varphi(\sigma_M(t))) \tag{34}$$

and consider the function

$$J(\Theta) = \int_0^\Theta G(t)\,dt \quad (\Theta > 0). \tag{35}$$

Substituting $\Theta = T_M$ and using (19), we have

$$J(T_M)=\int_0^{T_M}\sum_{j=1}^{l}\Big\{\varepsilon_j\dot\sigma_{Mj}^2(t)+\varkappa_j\varphi_j(\sigma_{Mj}(t))\dot\sigma_{Mj}(t)-$$
$$-a_j\varkappa_jF_j(\sigma_{Mj}(t))\dot\sigma_{Mj}(t)+\delta_j\varphi_j^2(\sigma_{Mj}(t))-$$
$$-(1-a_j)\varkappa_j\Psi_j(\sigma_{Mj}(t))\dot\sigma_{Mj}(t)+$$
$$+\tau_j(\dot\sigma_{Mj}(t)-\alpha_{1j}^{-1}\dot\varphi_j(\sigma_{Mj}(t)))\cdot$$
$$\cdot(\dot\sigma_{Mj}(t)-\alpha_{2j}^{-1}\dot\varphi_j(\sigma_{Mj}(t)))\Big\}\,dt=$$
$$=\int_0^{T_M}\sum_{j=1}^{l}\Big\{\varepsilon_j\dot\sigma_j^2(t)+\delta_j\varphi_j^2(\sigma_{Mj}(t))+$$
$$+\tau_j\dot\sigma_{Mj}^2(t)\Phi_j^2(\sigma_{Mj}(t))+\varkappa_ja_j\nu_j|\varphi(\sigma_{Mj}(t))|\dot\sigma_{Mj}(t)+$$
$$+\varkappa_j(1-a_j)\nu_{0j}|\varphi(\sigma_{Mj}(t))|\dot\sigma_{Mj}(t)\Phi_j(\sigma_{Mj}(t))\Big\}\,dt=$$
$$=\int_0^{T_M}\sum_{j=1}^{l}W_j(\dot\sigma_{Mj}(t),|\varphi(\sigma_{Mj}(t))|,\dot\sigma_{Mj}(t)\Phi_j(\sigma_{Mj}(t))). \tag{36}$$

Condition 2) implies that

$$J(T_M)>0. \tag{37}$$

Suppose now that $\sigma_M(t)$ has the frequency $\omega\ge\bar\omega$. Let us transform the functional $J(T_M)$ using expansions (31) and (33) under the following obvious equalities:

$$B_{-k}=\bar B_k\quad(k\in\mathbf{Z}), \tag{38}$$

where the symbol $-$ is used for complex conjugation;

$$\int_0^{T_M}e^{i\omega_Mkt}e^{i\omega_Mmt}\,dt=\begin{cases}0,\ \text{if } k\ne -m,\\ T_M,\ \text{if } k=-m,\end{cases}\quad(k,m\in\mathbf{Z}). \tag{39}$$

Accordingly to Definition 6 the equalities are valid

$$\int_0^{T_M}F_j(\sigma_{Mj}(t))\dot\sigma_{Mj}(t)\,dt=\int_{\sigma_{Mj}(0)}^{\sigma_{Mj}(T_M)}F_j(\zeta)\,d\zeta=0, \tag{40}$$

$$\int_0^{T_M}\Psi_j(\sigma_{Mj}(t))\dot\sigma_{Mj}(t)\,dt=0. \tag{41}$$

We may decompose the integral $J(\Theta)$ from (35) as follows

$$J(\Theta)=\sum_{k=1}^{4}J_k(\Theta), \tag{42}$$

$$J_1(\Theta)=\int_0^\Theta\dot\sigma_M^*(t)\varkappa\varphi(\sigma_M(t))\,dt,$$

$$J_2(\Theta)=\int_0^\Theta\varphi^*(\sigma_M(t))\delta\varphi(\sigma_M(t))\,dt,$$

$$J_3(\Theta)=\int_0^\Theta\dot\sigma_M^*(t)\varepsilon\dot\sigma_M(t)\,dt, \tag{43}$$

$$J_4(\Theta)=\int_0^\Theta(\dot\sigma_M(t)-A_1^{-1}\dot\varphi(\sigma_M(t)))^*\tau\cdot$$

$$\cdot(\dot\sigma_M(t)-A_2^{-1}\dot\varphi(\sigma_M(t)))\,dt.$$

Now we are going to calculate $J_j(T_M)$ using the formulas (31) and (33). We obtain

$$J_1(T_M)=-T_M\Big\{B_0^*K_M^*(0)\varkappa B_0+$$

$$+2\sum_{k=1}^{+\infty}\Big(B_k^*Re(\varkappa K_M(i\omega k))B_k\Big)\Big\}, \tag{44}$$

$$J_2(T_M)=T_M\{B_0^*\delta B_0+2\sum_{k=1}^{+\infty}B_k^*\delta B_k\}, \tag{45}$$

$$J_3(T)=T_M\{B_0^*K_M^*(0)\varepsilon K_M(0)B_0+$$

$$+2\sum_{k=1}^{+\infty}B_k^*K_M^*(i\omega k)\varepsilon K_M(i\omega kB_k\}. \tag{46}$$

For integral $J_4(T_M)$ we use the formula

$$\dot\varphi(\sigma_M(t))=\sum_{k=-\infty}^{+\infty}i\omega kB_ke^{i\omega kt}. \tag{47}$$

From (31), (33), (47) it follows that

$$J_4(T_M)=T_MB_0^*K_M^*(0)\tau K_M(0)B_0+$$

$$+2T_M\sum_{k=1}^{+\infty}B_k^*Re\{(K_M(i\omega k)+A_1^{-1}i\omega k)^*\cdot \tag{48}$$

$$\cdot\tau(K_M(i\omega k)+A_2^{-1}i\omega k)\}B_k.$$

From (44), (45), (46), (48) we get that

$$J(T_M)=-T_MB_0^*\{\varkappa K_M(0)-K_M^*(0)(\varepsilon+\tau)K_M(0)-$$

$$-\delta\}B_0-2T_M\sum_{k=1}^{+\infty}B_k^*\{Re(\varkappa K_M(i\omega k)-$$

$$-(K_M(i\omega k)+i\omega kA_1^{-1})^*\tau(K_M(i\omega k)+i\omega kA_2^{-1}))-$$

$$-\delta-K_M^*(i\omega k)\varepsilon K_M(i\omega k)\}B_k. \tag{49}$$

Condition 1) of the Theorem guarantees that all the terms $B_k^*\Omega(\omega k)B_k\ge 0$ $(k=0,1,2,...)$ and hence

$$J(T)\le 0. \tag{50}$$

The contradiction with (37) implies that (1) has no periodic solution with frequency $\omega\ge\bar\omega$. Theorem 7 is proved.

Theorem 8. Suppose there exist $\bar\omega>0$, positive definite matrices ε, δ, τ, $\varkappa$ and numbers $a_j\in[0,1]$ $(j=1,\dots,l)$ such that the following conditions are valid :
1) for $\omega=0$ and all $\omega\ge\bar\omega$ the inequality (23) is true
2) the quadratic forms (15) are positive definite.
Then there exists $\mu_0>0$ such that system (1) has no periodic solutions with frequencies $\omega\ge\bar\omega$ for $\mu\in(0,\mu_0)$.

Proof. Following the arguments of Theorem 4, under condition 1) there exists $\mu_0>0$ such that inequality (12) is true for $\omega=0$ and $\omega\ge\bar\omega$ for $\mu\in(0,\mu_0)$. Theorem 7 can now be applied to system (1) with $\mu\in(0,\mu_0)$.

Counterparts of Theorem 3 and Theorem 5, ensuring the absence of high-frequency oscillations, can be derived in the same way. We omit these extensions due to the page limit.

5. CONCLUSION

The paper is devoted to asymptotic behavior of singularly perturbed infinite dimensional phase synchronization systems (PSS), described by integro–differential Volterra equations with periodic nonlinear functions and a small parameter at the higher derivative. First of all the problem of gradient–like behavior is considered. It is shown that

for sufficiently small value of the parameter frequency–algebraic stability criteria can by extended from unperturbed PSS to singularly perturbed ones. In this paper we demonstrate also that the relaxation of stability frequency–algebraic conditions guarantees nonexistence of periodic solutions of high frequency. The upper bound for the frequency of periodic solutions is uniform with respect to the small parameter.

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