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**Scaling limits of long-range quantum random walks**

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“Scaling limits of long-range quantum random walks”

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## Abstract

In this thesis we introduce a variation on the quantum random walk to discuss shifts in an arbitrary range. The concept of Hadamard coin was therefore generalised to a higher order. By a Fourier transform method and a tensor product decomposition of the evolution matrix the long-range quantum random walk was found to converge in distribution to a random variable, different for every range. The limiting random variable consists of three parts: one part fast decaying with the range size, a non-convergent part and a convergent part. Lastly, an introduction was made into the topic of trapped quantum random walks. As a starting point, the survival probability of such a walk on a 3-cycle was calculated and found to scale as  $2^{-n}$ , as does the classical trapped random walk on this topology.

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## Introduction

Over the past decades researchers have made efforts towards the realisation of a quantum computer for the implementation of quantum algorithms [1]. A functioning quantum computer could for instance be used for integer factorisation via Shor's quantum algorithm, which is substantially faster than any known classical factorisation algorithm [2]. In order for quantum computation to be successful, quantum algorithms must be developed that supersede their classical counterparts. A powerful tool in the development of quantum algorithms is the quantum random walk [3].

The process of a quantum random walk has gained interest in the past years due to its potential to speed up classical algorithms. The process describes the evolution of the state of a walker in terms of a wave-function with an internal degree of freedom. The position of this walker at time  $n$ , denoted by  $X_n$ , travels ballistically, contrary to a classical random walk. Meaning,  $X_n/n$  converges to a limiting random variable as  $n$  tends to infinity [4]. Like the classical random walk, there exist many variations on its standard version. In this thesis we focus on a variation on the quantum random walk that extends the steps or shifts in the quantum random walk to a longer range. An effort is made to investigate the resulting limiting random variable for arbitrary ranges. In the classical random walk  $X_n/\sqrt{n}$  converges to a normal distribution, regardless of the step-distribution [5]. We examine if the long-range quantum random walk shares this property of universality. Furthermore, we investigate the limit of infinite range. Finally, an introduction is made into the topic of trapped quantum random walks.

The thesis starts with an overview of the theory on quantum random walks in Section 1, where the concept is described and a result of a limiting distribution is presented. Section 2 summarizes a method for physical implementation of the quantum random walk in an ion-trap. An extension of the quantum random walk to a long-range version is formalised in Section 3 where we discuss the concept and investigate a limiting random variable for the process. We introduce the topic of trapped quantum random walks and analyse a starting point for further research in Section 4. Finally, Section 5 treats the conclusions of the obtained results. This thesis is part of the bachelor programs Applied Mathematics and Applied Physics of the EEMCS and AS faculties of the Delft University of Technology.

# 1 Theory

## Concept & limiting distribution

In this section we present an overview of the basics of quantum random walks (QRW) and some important convergence results on the topic based on [4]. Section 1.1 starts with an introduction of the formalism behind the QRW followed by the dynamics of the QRW in Section 1.2. We proceed with a derivation of the limiting distribution of the process based on Fourier transform methods in Section 1.3. The derivation continues with a theorem regarding convergence in distribution based on the method of moments in Section 1.4 after which a more explicit calculation of the limiting distribution is presented in Section 1.5.

### 1.1 State space

In the classical random walk a state is simply an integer value which we interpret as the position of the walker. In order to describe the dynamics of a QRW, a more complicated setting is needed [6]. A QRW is a quantum system that evolves in discrete time. The states that describe the system consist of a position part which can be thought of as modelling the position of a particle, and an internal degree of freedom analogous to quantum mechanical spin. In its most common form, the state of the system is of the form  $\psi = \begin{pmatrix} \psi(x, 1) \\ \psi(x, -1) \end{pmatrix}$  where  $\psi(\cdot, i) \in \ell^2(\mathbb{Z})$ . Here  $\psi(\cdot, i)$  can be thought of as the position part and the internal degree of freedom, or coin state, is an element of  $\mathbb{C}^2$ . With this choice for the position the QRW is a walk on the integers. The total state  $\psi$  is then a tensor product of this position and coin state such that the total state is an element of the space

$$\mathcal{H} = \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2 \simeq (\ell^2(\mathbb{Z}))^2.$$

We will write  $\{e_x\}_{x \in \mathbb{Z}}$  for the usual basis elements of  $\ell^2(\mathbb{Z})$ . i.e.,  $e_x$  is the column with 1 at  $x$  and zero elsewhere and further denote

$$\left\{ |1\rangle = w_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |-1\rangle = w_{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

for the basis of  $\mathbb{C}^2$ . We can now write the total state as

$$\psi = \sum_{x \in \mathbb{Z}} \psi(x, 1) e_x \otimes |1\rangle + \psi(x, -1) e_x \otimes |-1\rangle.$$

Or in shorter notation:

$$\psi = \sum_{i=\pm 1} \sum_{x \in \mathbb{Z}} \psi(x, i) e_x \otimes w_i.$$

With this state we associate a probability to find the particle at position  $x$  equal to

$$\mathbb{P}(x) = \sum_{i=\pm 1} |\langle e_x \otimes w_i, \psi \rangle|^2 = |\psi(x, 1)|^2 + |\psi(x, -1)|^2.$$

In order for this rule to be meaningful, we require that the state is normalised. Meaning that the sum of this probability over all integers equals one:  $\sum_{i=\pm 1} \sum_{x \in \mathbb{Z}} |\psi(x, i)|^2 = 1$ . Essentially, the state of a QRW is a wave-function on the integers with an internal degree of freedom. In contrast to a classical random walk, there is no localisation of position. As such, it is important to note that the QRW does not define a stochastic process like the classical random walk, but rather a sequence of probability measures. Like in quantum mechanics, the position is a probabilistic concept.



## 1.2 Dynamics

In this section we describe the evolution between states that compose a QRW. Given an initial state  $\psi_0 = \begin{pmatrix} \psi_0(x, 1) \\ \psi_0(x, -1) \end{pmatrix}$  the QRW defines a time-evolution of  $\psi_0$  in discrete time by repeated application of a unitary operator  $U$ ;  $\psi_n = U^n \psi_0$ . Indeed, the operator must be unitary to ensure conservation of total probability. We denote by  $X_n^\psi$  a random variable assuming integer values with probability as described in Section 1.1 using the state  $\psi_n$ . The unitary operator  $U$  consists of two parts, one acting on the coin-state and the other on the position. These two parts of the dynamics are explained in Section 1.2.1 and Section 1.2.2 respectively.

### 1.2.1 Coin operation

In the simplest case one can take the Hadamard operator  $H$  as operator on the coin state:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$H$  acting on a state  $\psi$  gives

$$H\psi = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \psi(x, 1) \\ \psi(x, -1) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi(x, 1) + \psi(x, -1) \\ \psi(x, 1) - \psi(x, -1) \end{pmatrix}.$$

In Section 1.2.2 we will see that the  $|1\rangle$ -part of the state,  $\psi(x, 1)$ , undergoes a shift to the right. Similarly the  $|-1\rangle$ -part of the state,  $\psi(x, -1)$ , undergoes a shift to the left. The Hadamard coin is considered a fair coin, in the sense that it does not favor a shift to the right over a shift to the left or vice versa. More precisely: after the operation of the Hadamard coin, the parts of the state that previously corresponded to a right shift and left shift are distributed in superpositions of equal amplitude over both coin-states. But one might also consider a more general unitary operator  $A$ , for example:

$$A = \begin{pmatrix} \sqrt{\rho} & \sqrt{1-\rho} \\ \sqrt{1-\rho} & -\sqrt{\rho} \end{pmatrix}$$

As long as  $A \in U_2(\mathbb{C})$ , that is  $A$  is a unitary operator on  $\mathbb{C}^2$ .  $A$  acting on a state  $\psi$  gives:

$$A\psi = \begin{pmatrix} \sqrt{\rho} & \sqrt{1-\rho} \\ \sqrt{1-\rho} & -\sqrt{\rho} \end{pmatrix} \begin{pmatrix} \psi(x, 1) \\ \psi(x, -1) \end{pmatrix} = \begin{pmatrix} \sqrt{\rho}\psi(x, 1) + \sqrt{1-\rho}\psi(x, -1) \\ \sqrt{1-\rho}\psi(x, 1) - \sqrt{\rho}\psi(x, -1) \end{pmatrix}.$$

Choosing  $\rho$  unequal to  $1/2$  introduces a bias in the walk. If for example  $\rho > 1/2$ , the part of  $H\psi$  that is shifted to the right consist is a superposition of the previous coin-states. But in this superposition, the part that previously corresponded to a right shift has a higher amplitude.

### 1.2.2 Shift operation

As mentioned in Section 1.2.1, the second part of the evolution  $U$  is a shift to the right or left, corresponding to the coin-state. The shift operator is defined as follows:

$$S\psi = S \begin{pmatrix} \psi(x, 1) \\ \psi(x, -1) \end{pmatrix} = \begin{pmatrix} \psi(x-1, 1) \\ \psi(x+1, -1) \end{pmatrix}.$$

We can now define the total evolution operator  $U$  as  $U = S(I \otimes A)$ , with  $I$  the identity on  $\ell^2(\mathbb{Z})$ . The tensor product with the identity indicates that the coin operator  $A$  only acts on the coin-state. In case we take the Hadamard coin as coin operator, we find the evolution of a state to be the following:

$$U\psi = U \begin{pmatrix} \psi(x, 1) \\ \psi(x, -1) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi(x-1, 1) + \psi(x-1, -1) \\ \psi(x+1, 1) - \psi(x+1, -1) \end{pmatrix}$$

Recall that the evolution operator is repeatedly applied to an initial state  $\psi_0$  to constitute a QRW;  $\psi_n = U^n \psi_0$ . In the current setting it is complicated to express  $U^n$  for arbitrary  $n \in \mathbb{N}$ . This makes it hard to investigate the behaviour of the QRW after an arbitrary amount of steps. In Section 1.3 we present a solution to this difficulty.

### 1.3 Fourier transform

In this section we introduce the Fourier transform methods which enable us to show that  $X_n/n$  converges in distribution to a limiting random variable as  $n \rightarrow \infty$ . This result is comparable to the convergence of the classical random walk, where  $X_n/\sqrt{n}$  converges to a Gaussian as  $n \rightarrow \infty$ . However, the division by  $n$  instead of  $\sqrt{n}$  forms a remarkable difference between the classical random walk and the QRW. To this extend we will perform calculations in the Fourier space  $L^2(\mathbb{K}) = \widehat{\ell^2(\mathbb{Z})}$  of  $\ell^2(\mathbb{Z})$ . Here  $\mathbb{K}$  denotes the torus. That is the interval  $[0, 2\pi]$  with opposite ends identified. Alternatively we can think of  $\mathbb{K}$  as  $\mathbb{R}$  with all points that are  $2\pi$  apart identified. For  $\psi(x) \in \ell^2(\mathbb{Z})$  we have  $\hat{\psi}(k) = \sum_{x \in \mathbb{Z}} e^{ikx} \psi(x)$ . This allows us to describe the total state as an element of  $L^2(\mathbb{K}) \otimes \mathbb{C}^2$ . So:

$$\hat{\psi}(k) = \begin{pmatrix} \hat{\psi}(k, 1) \\ \hat{\psi}(k, -1) \end{pmatrix}$$

The evolution of states as described in Section 1.2 is somewhat cumbersome to work with. We will see that the evolution takes a much simpler form in the Fourier space. Let us therefore compute  $\widehat{U\psi} = \widehat{SA\psi}$  (without specifying the coin operator  $A$ ). Note that

$$\begin{aligned} \widehat{(S\psi)}(k) &= \begin{pmatrix} \widehat{\psi(x-1, 1)} \\ \widehat{\psi(x+1, -1)} \end{pmatrix} = \begin{pmatrix} \sum_{x \in \mathbb{Z}} e^{ikx} \psi(x-1, 1) \\ \sum_{x \in \mathbb{Z}} e^{ikx} \psi(x+1, -1) \end{pmatrix} = \begin{pmatrix} \sum_{x \in \mathbb{Z}} e^{ik(x+1)} \psi(x, 1) \\ \sum_{x \in \mathbb{Z}} e^{ik(x-1)} \psi(x, -1) \end{pmatrix} \\ &= \begin{pmatrix} e^{ik} \sum_{x \in \mathbb{Z}} e^{ikx} \psi(x, 1) \\ e^{-ik} \sum_{x \in \mathbb{Z}} e^{ikx} \psi(x, -1) \end{pmatrix} = \begin{pmatrix} e^{ik} \hat{\psi}(k, 1) \\ e^{-ik} \hat{\psi}(k, -1) \end{pmatrix} = \begin{pmatrix} e^{ik} & 0 \\ 0 & e^{-ik} \end{pmatrix} \hat{\psi}(k). \end{aligned}$$

Also,  $A$  acts only on the coin state, and therefore commutes with the Fourier transform:

$$A\psi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \psi(x, 1) \\ \psi(x, -1) \end{pmatrix} = \begin{pmatrix} a\psi(x, 1) + b\psi(x, -1) \\ c\psi(x, 1) + d\psi(x, -1) \end{pmatrix}$$

Such that by linearity of the Fourier transform

$$\widehat{A\psi} = \begin{pmatrix} a\hat{\psi}(k, 1) + b\hat{\psi}(k, -1) \\ c\hat{\psi}(k, 1) + d\hat{\psi}(k, -1) \end{pmatrix} = A\hat{\psi}.$$

So we can describe one step of the QRW in the Fourier space as

$$\widehat{U\psi} = \widehat{SA\psi} = \begin{pmatrix} e^{ik} & 0 \\ 0 & e^{-ik} \end{pmatrix} A\hat{\psi}(k) = \hat{U}(k)\hat{\psi}(k),$$

with

$$\hat{U}(k) = \begin{pmatrix} e^{ik} & 0 \\ 0 & e^{-ik} \end{pmatrix} A.$$

And for the  $n$ -th state we obtain

$$\hat{\psi}_n(k) = \hat{U}^n(k) \hat{\psi}_0(k).$$

The matrix  $\hat{U}$  that characterises evolution in the Fourier space has the advantageous property that it is unitary and therefore normal. So each state has an eigendecomposition of orthonormal eigenvectors  $v_1(k), v_2(k)$  with corresponding eigenvalues  $\lambda_1(k), \lambda_2(k)$ . In the case that  $\lambda_1(k) = \lambda_2(k) = \lambda(k)$  the evolution is somewhat trivial, see Appendix C.1. Let us therefore assume that  $\lambda_1(k) \neq \lambda_2(k)$ . Since the eigenvectors  $v_1(k)$  and  $v_2(k)$  are orthogonal and span  $\mathbb{C}^2$  we can write

$$\hat{\psi}_0 = \langle v_1(k), \hat{\psi}_0 \rangle v_1(k) + \langle v_2(k), \hat{\psi}_0 \rangle v_2(k),$$

such that the state at time  $n$  reads

$$\hat{\psi}_n = U(k)^n \hat{\psi}_0 = \lambda_1(k)^n \langle v_1(k), \hat{\psi}_0 \rangle v_1(k) + \lambda_2(k)^n \langle v_2(k), \hat{\psi}_0 \rangle v_2(k) = \sum_{j=1}^2 \lambda_j(k)^n \langle v_j(k), \hat{\psi}_0 \rangle v_j(k).$$

## 1.4 Method of moments

The moments of a random variable can provide helpful insights for understanding its behaviour. Moreover, a bounded random variable is uniquely determined by its moments [7] (Hausdorff moment problem). We will investigate the moments of  $X_n/n$  as  $n$  tends to infinity and find that each sequence of moments converges to some limiting moment. Theorem 1 then ensures that  $X_n/n$  converges in distribution to a random variable whose moments are these limits. See Definition 1 for the precise meaning of convergence in distribution.

**Definition 1.** *A sequence of random variables  $X_n$  converges in distribution to a random variable  $X$  if for every bounded and continuous  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  and every integer  $n$ :*

$$\mathbb{E}(\phi(X_n)) \rightarrow \mathbb{E}(\phi(X))$$

And we write  $X_n \xrightarrow{d} X$ .

**Theorem 1.** *Let  $X$  be a random variable of bounded support and  $X_n$  a sequence of random variables such that for each  $k \in \mathbb{N}$*

$$\mathbb{E}[X_n^k] \rightarrow \mathbb{E}[X^k].$$

*Then if  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  is bounded and continuous,  $X_n \xrightarrow{d} X$ .*

See Appendix B for a proof of Theorem 1. Note that  $X_n$  must be divided by  $n$ , in contrast to the convergence of  $\frac{X_n}{\sqrt{n}}$  in the classical random walk. This scaling is called ballistic. Intuitively, it means that the QRW spreads faster than its classical counterpart.

Let us now examine the moments of  $X_n$ . They are by definition given by

$$\mathbb{E}(X_n^r) = \langle \psi_n, X^r \psi_n \rangle.$$

Where the inner product is that of  $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$ . We can make use of the Fourier transform results obtained in Section 1.3 to find a more explicit expression. Inner products in this space are related to those in  $L^2(\mathbb{K})$  and  $\mathbb{C}^2$  by

$$\langle \psi, \phi \rangle_{\ell^2(\mathbb{Z}) \otimes \mathbb{C}^2} = \int_0^{2\pi} \langle \hat{\psi}, \hat{\phi} \rangle_{\mathbb{C}^2} \frac{dk}{2\pi}.$$

See Appendix C.2 for a derivation. In order to express the moments of  $X_n$  in terms of the Fourier transform of the state  $\psi_n$  we must calculate  $\widehat{X^r \psi}$  or as a first step  $\widehat{X \psi}$ :

$$\widehat{X \psi} = \sum_{x \in \mathbb{Z}} x e^{ikx} \psi(x) = -i \frac{d}{dk} \sum_{x \in \mathbb{Z}} e^{ikx} \psi(x) = -i \frac{d}{dk} \hat{\psi}(k),$$

hence

$$\widehat{X^r \psi} = \left( -i \frac{d}{dk} \right)^r \hat{\psi}.$$

Finally we can express the moments of  $X_n$  in terms of its Fourier transforms:

$$\mathbb{E}(X_n^r) = \langle \psi_n, X^r \psi_n \rangle_{\ell^2(\mathbb{Z}) \otimes \mathbb{C}^2} = \int_0^{2\pi} \langle \hat{\psi}_n, D^r \hat{\psi}_n \rangle_{\mathbb{C}^2} \frac{dk}{2\pi},$$

where  $D = -i \frac{d}{dk}$ .  $D^r \hat{\psi}_n$  can be expressed more explicitly using the eigendecomposition found in Section 1.3 and Leibniz rule:

$$D^r \hat{\psi}_n = (-i)^r \sum_{j=1}^2 \langle v_j(k), \hat{\psi}_0 \rangle n^r \lambda_j(k)^{n-r} \frac{d^r \lambda_j}{dk^r} v_j(k) + \mathcal{O}(n^{r-1}).$$

See Appendix C.3 for a derivation. Consequently,

$$\frac{D^r \hat{\psi}_n}{n^r} = \sum_{j=1}^2 \lambda_j(k)^{n-r} (D \lambda_j)^r \langle v_j(k), \hat{\psi}_0 \rangle v_j(k) + \mathcal{O}(n^{-1}).$$

And the moments are given by

$$\begin{aligned} \mathbb{E} \left[ \left( \frac{X_n}{n} \right)^r \right] &= \int_0^{2\pi} \left\langle \hat{\psi}_n, \frac{D^r \hat{\psi}_n}{n^r} \right\rangle \frac{dk}{2\pi} \\ &= \int_0^{2\pi} \sum_{j=1}^2 \lambda_j(k)^{n-r} (D \lambda_j)^r \langle v_j(k), \hat{\psi}_0 \rangle \langle \hat{\psi}_n, v_j(k) \rangle \frac{dk}{2\pi} + \mathcal{O}(n^{-1}). \end{aligned}$$

Note that  $\langle \hat{\psi}_n, v_j(k) \rangle = \lambda_j(k)^{-n} \langle \hat{\psi}_0, v_j(k) \rangle$ , leaving us with:

$$\mathbb{E} \left[ \left( \frac{X_n}{n} \right)^r \right] = \int_0^{2\pi} \sum_{j=1}^2 \left( \frac{D \lambda_j}{\lambda_j(k)} \right)^r |\langle v_j(k), \hat{\psi}_0 \rangle|^2 \frac{dk}{2\pi} + \mathcal{O}(n^{-1}).$$

We recognise in this last expression the moments of a random variable. Indeed, take  $\Omega = \mathbb{K} \times \{1, 2\}$  and let  $\mu$  be the probability measure on  $\Omega$  defined by  $|\langle \hat{\psi}_0, v_j(k) \rangle|^2 \frac{dk}{2\pi}$ . Now define  $h(k, j) = \frac{D \lambda_j}{\lambda_j(k)}$ . Then  $h : \mathbb{K} \times \{1, 2\} \rightarrow \mathbb{R}$  is a random variable. To see that  $h$  is indeed real we note that  $|\lambda_j(k)| = 1$  such that we can write  $\lambda_j(k) = e^{ia_j(k)}$  with  $a_j(k) \in \mathbb{R}$ . Now

$$h_j(k) = \frac{-i \frac{d}{dk} e^{ia_j(k)}}{e^{ia_j(k)}} = \frac{\frac{d}{dk} a_j(k) e^{ia_j(k)}}{e^{ia_j(k)}} = \frac{d}{dk} a_j(k) \in \mathbb{R}.$$

With this interpretation we can write

$$\mathbb{E} \left[ \left( \frac{X_n}{n} \right)^r \right] \rightarrow \int_{\Omega} h^r d\mu \quad \text{as } n \rightarrow \infty.$$

And Theorem 1 guarantees that

$$\frac{X_n}{n} \xrightarrow{d} Y = h(Z),$$

with  $Z$  a random element of  $\Omega$  whose distribution is given by  $\mu$ . Here ' $\xrightarrow{d}$ ' denotes convergence in distribution. This result allows us, with a suitable initial state, to compute the asymptotic cumulative distribution of  $\frac{X_n}{n}$  and its asymptotic probability density function by examining the eigenvalues of  $\hat{U}(k)$ .

## 1.5 Limiting distribution

Let us now apply the results of the previous section to the Hadamard case. That is

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Therefore,

$$\hat{U}(k) = \begin{pmatrix} e^{ik} & 0 \\ 0 & e^{-ik} \end{pmatrix} H = \begin{pmatrix} e^{ik} & 0 \\ 0 & e^{-ik} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{ik} & e^{ik} \\ e^{-ik} & -e^{-ik} \end{pmatrix}.$$

For convenience we first compute the eigenvalues  $\tilde{\lambda}$  of  $\sqrt{2}\hat{U}(k)$ :

$$\begin{aligned} \begin{vmatrix} e^{ik} - \tilde{\lambda} & e^{ik} \\ e^{-ik} & -e^{-ik} - \tilde{\lambda} \end{vmatrix} &= (e^{ik} - \tilde{\lambda})(-e^{-ik} - \tilde{\lambda}) - 1 = 0 \\ &= -1 - \tilde{\lambda}e^{ik} + \tilde{\lambda}e^{-ik} + \tilde{\lambda}^2 - 1 = 0 \\ &= \tilde{\lambda}^2 - 2i \sin(k)\tilde{\lambda} - 2 = 0 \\ \implies \tilde{\lambda}_j(k) &= i \sin(k) \pm \sqrt{2 - \sin^2(k)} \end{aligned}$$

So we find for the eigenvalues  $\lambda$  of  $\hat{U}(k)$ :

$$\lambda_j(k) = \frac{\tilde{\lambda}_j(k)}{\sqrt{2}} = \frac{i \sin(k)}{\sqrt{2}} \pm \sqrt{1 - \frac{1}{2} \sin^2(k)}$$

Furthermore,

$$D\lambda_j = -i \frac{d}{dk} \lambda_j(k) = \frac{\cos(k)}{\sqrt{2}} \pm \frac{\sin(k) \cos(k)}{\sqrt{4 - 2 \sin^2(k)}} i.$$

The random variable  $h$  now equals

$$h(j, k) = \frac{D\lambda_j}{\lambda_j(k)} = h_j(k) = \pm \frac{\cos(k)}{\sqrt{2 - \sin^2(k)}}.$$

See Appendix C.4 for a derivation. We can use this expression to gain more insight in the limit distribution. First of all note that  $h$  takes values in  $[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]$ . If we take an initial state with

$X_0 = 0$  and one of  $w_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  and  $w_{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ , then  $\hat{\psi}_0(k) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  or  $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ . In which case

$$\mu = |\langle \psi_0(k), v_j(k) \rangle|^2 \frac{dk}{2\pi} = |v_{ji}(k)|^2 \frac{dk}{2\pi}.$$

If we instead consider a random initial state, that is  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$  or  $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$  with equal probability, then the measure  $\mu$  becomes

$$\mu = \frac{1}{2} \sum_{i=1,2} |v_{ji}(k)|^2 \frac{dk}{2\pi} = |v_j(k)| \frac{dk}{4\pi} = \frac{dk}{4\pi}.$$

We can then calculate explicitly the cumulative distribution function:

$$\begin{aligned} \mathbb{P}(Y \leq y) &= \int_{h^{-1}([-\infty, y])} d\mu = 2 \int_{\cos(k)/\sqrt{1+\cos^2(k)} \leq y} \frac{dk}{4\pi} \\ &= \int_{\cos^{-1}(y/\sqrt{1-y^2})}^{2\pi - \cos^{-1}(y/\sqrt{1-y^2})} \frac{dk}{2\pi} \\ &= 1 - \frac{1}{\pi} \cos^{-1} \left( \frac{y}{\sqrt{1-y^2}} \right) \end{aligned}$$

Finally we can take the derivative with respect to  $y$  to obtain the probability density function:

$$\frac{d}{dy} \left[ 1 - \frac{1}{\pi} \cos^{-1} \left( \frac{y}{\sqrt{1-y^2}} \right) \right] = -\frac{1}{\pi} \frac{d}{du} \cos^{-1}(u) \frac{d}{dy} \left( \frac{y}{\sqrt{1-y^2}} \right)$$

where  $u = \frac{y}{\sqrt{1-y^2}}$ . So

$$f(y) = \frac{1}{\pi} \frac{\sqrt{1-y^2}}{\sqrt{1-2y^2}} \frac{1}{(1-y^2)^{3/2}} dy = \frac{dy}{\pi \sqrt{1-2y^2}(1-y^2)}.$$

This result can be summarised in the following theorem:

**Theorem 2.** *Let  $X_n$  be a random variable defined by a QRW with the Hadamard coin*

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

then  $X_n/n \xrightarrow{d} Y$ . Where  $Y$  is a random variable with probability density function

$$f_Y(y) = \frac{dy}{\pi \sqrt{1-2y^2}(1-y^2)}.$$

See Figure 1a for a plot of this probability density. In Section 1.2 we have seen that a biased coin can be used to form a biased QRW. The derivation for its limiting distribution is analogous to that of the Hadamard coin. It is presented in Appendix C.5 and the result is summarised in the following theorem:

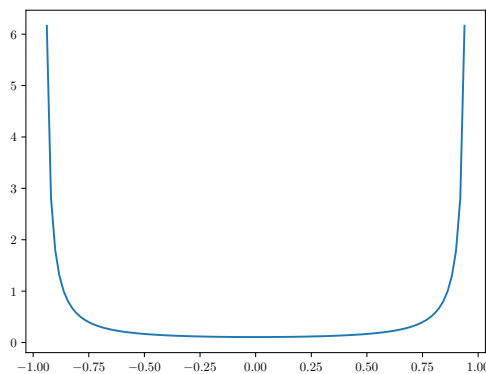
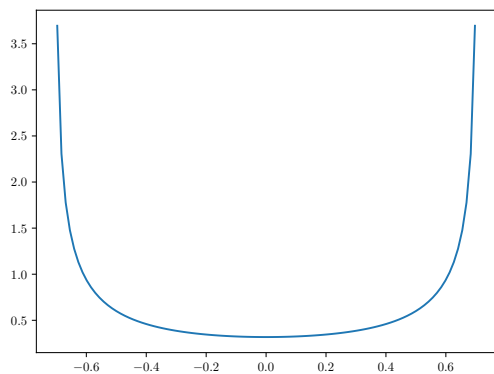
**Theorem 3.** Let  $X_n$  be a random variable defined by a QRW with biased coin

$$A = \begin{pmatrix} \sqrt{\rho} & \sqrt{1-\rho} \\ \sqrt{1-\rho} & -\sqrt{\rho} \end{pmatrix},$$

then  $X_n/n \xrightarrow{d} Y$ . Where  $Y$  is a random variable with probability density function

$$f_Y(y) = \frac{\sqrt{1-\rho} dy}{\pi \sqrt{\rho - y^2(1-y^2)}}.$$

Indeed, if we choose  $\rho = \frac{1}{2}$  we retrieve the result of Theorem 2. This probability density is plotted in Figure 1b.



(a). Probability density of the limiting distribution of  $X_n/n$  with Hadamard coin. (b). Probability density of the limiting distribution of  $X_n/n$  with biased coin ( $\rho = 0.9$ ).

## 2 Theory

### Physical implementation

In this section we present the basic idea behind a physical implementation of a QRW. A variety of systems might be used to physically implement a QRW, see for example [8, 9, 10]. This section focusses on the ion-trap configuration as described in [8, 11] as this is perhaps the simplest concept for physical implementation. We start with an overview of the configuration in Section 2.1. Then, we present an approach for treating a harmonic perturbation in Section 2.2. Finally, we discuss the concept of coherent states and displacement in Section 2.3 based on [12].

#### 2.1 Ion-trap

More specifically, we consider a  ${}^9\text{Be}^+$  ion confined in a coaxial-resonator radio frequency trap. For a detailed overview of the set-up, see [13, 14]. The state of the ion is initially of the form  $|0\rangle|\downarrow\rangle$  where  $|\alpha\rangle$  denotes a coherent state of the ion-trap and the  $|\downarrow\rangle$  indicates the internal state, or spin state of the ion. The meaning of coherent state will be explained later in Section 2.3. The ion then goes through a series of four laser pulses each serving a different purpose. How these pulses affect the state of the ion in the desired way will be explained in Section 2.2. The first pulse acts on internal state. A  $\pi/2$ -pulse creates equal superpositions of  $|0\rangle|\downarrow\rangle$  and  $|0\rangle|\uparrow\rangle$ . This is exactly the function of the Hadamard coin in the dynamics of a QRW. After this first pulse, a displacement beam shifts the coherent state corresponding to  $|\uparrow\rangle$  by one position. This results in a state  $\frac{1}{\sqrt{2}}|\alpha\rangle|\downarrow\rangle + \frac{1}{\sqrt{2}}|0\rangle|\uparrow\rangle$ . So this pulse partly accomplishes the function of the shift operation. The third pulse, a  $\pi$ -pulse, interchanges the internal states, resulting in:  $\frac{1}{\sqrt{2}}|\alpha\rangle|\uparrow\rangle + \frac{1}{\sqrt{2}}|0\rangle|\downarrow\rangle$ . Lastly, a displacement beam is applied again that creates the state  $\frac{1}{\sqrt{2}}|\alpha\rangle|\downarrow\rangle + \frac{1}{\sqrt{2}}|-\alpha\rangle|\uparrow\rangle$ . After this last pulse, the shift operation is completed. With these four pulses, we are left with a state that corresponds to a QRW after one iteration. Repeating the process will then induce a QRW as desired.

#### 2.2 Harmonic perturbation

In order to investigate the effect of a laser pulse on an ion we use the rotating frame approach for time-dependent perturbations of a Hamiltonian, as explained in Appendix C.6. We consider a Hamiltonian acting on a two-level system consisting of a constant part  $H_0$  and a time-dependent part  $V(t)$ :

$$H(t) = H_0 + V(t).$$

In the rotating frame approach we switch to a rotating frame of reference for which we choose a rotation equal to  $R(t) = e^{-iH_0 t/\hbar}$ . The approach then states that in order to find the wave function  $|\psi(t)\rangle$  for our Hamiltonian, we need to solve for the Hamiltonian in the rotated frame

$$H_{rot} := R^\dagger(t)V(t)R(t),$$

and then apply the rotation  $R(t)$ . This approach can be applied to a perturbing oscillatory magnetic field on a two-level system in an otherwise constant magnetic field. We investigate the situation where the constant magnetic field points in the  $z$ -direction and the perturbing field points in the  $x$ -direction, such that the Hamiltonian takes the form

$$H(t) = \frac{\hbar\omega}{2}\sigma_z + \epsilon\sigma_x \cos(\omega_1 t),$$



where  $\hbar\omega$  is the energy gap between  $|\uparrow\rangle$  and  $|\downarrow\rangle$  and  $\sigma_x, \sigma_z$  are Pauli spin matrices. So according to the rotating frame approach we separate this Hamiltonian into  $H_0 = \frac{\hbar\omega}{2}\sigma_z$  and  $V(t) = \epsilon\sigma_x \cos(\omega_1 t)$ . We then take the rotation equal to  $R(t) = e^{-iH_0 t/\hbar} = e^{-i\omega t\sigma_z/2}$ , such that the Hamiltonian in the rotated frame becomes

$$H_{rot} = e^{i\omega t\sigma_z/2}\sigma_x e^{-i\omega t\sigma_z/2}\epsilon \cos(\omega_1 t).$$

In order to proceed with the calculation we will first prove a useful equality:

$$e^{i\alpha\sigma_z} = \cos(\alpha) + i \sin(\alpha)\sigma_z.$$

By definition

$$e^{i\alpha\sigma_z} = \sum_{n=0}^{\infty} \frac{(i\alpha\sigma_z)^n}{n!},$$

and when we separate this sum into even and odd powers we find

$$e^{i\alpha\sigma_z} = \sum_{m=0}^{\infty} \frac{(i\alpha\sigma_z)^{2m}}{(2m)!} + \sum_{m=0}^{\infty} \frac{(i\alpha\sigma_z)^{2m+1}}{(2m+1)!}.$$

One can verify that all even powers of  $\sigma_z$  equal the identity, and as a consequence all odd powers equal  $\sigma_z$ . So we have

$$e^{i\alpha\sigma_z} = \sum_{m=0}^{\infty} \frac{(i\alpha)^{2m}}{(2m)!} + \sum_{m=0}^{\infty} \frac{(i\alpha)^{2m+1}}{(2m+1)!}\sigma_z = \cos(\alpha) + i \sin(\alpha)\sigma_z.$$

By writing  $R(t)$  in this form the Hamiltonian in the rotated frame becomes

$$H_{rot} = [\cos(\omega t/2) + i \sin(\omega t/2)\sigma_z]\sigma_x [\cos(\omega t/2) - i \sin(\omega t/2)\sigma_z]\epsilon \cos(\omega_1 t).$$

And after multiplication:

$$H_{rot} = [\cos^2(\omega t/2)\sigma_x + i \cos(\omega t/2) \sin(\omega t/2)(\sigma_z\sigma_x - \sigma_x\sigma_z) + \sin^2(\omega t/2)\sigma_z\sigma_x\sigma_z]\epsilon \cos(\omega_1 t).$$

Now we simplify further by noting that  $\sigma_z\sigma_x - \sigma_x\sigma_z = 2i\sigma_y$  and  $\sigma_z\sigma_x\sigma_z = -\sigma_x$ , hence

$$H_{rot} = [(\cos^2(\omega t/2) - \sin^2(\omega t/2))\sigma_x - 2 \cos(\omega t/2) \sin(\omega t/2)\sigma_y]\epsilon \cos(\omega_1 t).$$

We recognise the terms between brackets from the doubling formula, so

$$H_{rot} = [\cos(\omega t)\sigma_x - \sin(\omega t)\sigma_y]\epsilon \cos(\omega_1 t).$$

Let us investigate the case where  $\omega_1 = \omega$ :

$$H_{rot} = [\cos^2(\omega t)\sigma_x - \cos(\omega t) \sin(\omega t)\sigma_y]\epsilon.$$

Again the doubling formula can be used for simplification:

$$H_{rot} = \frac{\epsilon}{2}\sigma_x + \frac{\epsilon}{2}\cos(2\omega t)\sigma_x - \frac{\epsilon}{2}\sin(2\omega t)\sigma_y.$$

These last two terms do not contribute over time to the solution. They are an example of non-secular terms, their oscillation is non-resonant. See for example [15]. So finally, we find that the Hamiltonian in the rotated frame is that of a constant magnetic field in the x-direction. So in the rotated frame, the spins undergo Larmor precession about the x-axis with an angular frequency of  $\epsilon/2\hbar$ . The transition from the rotated frame to the original frame  $R(t)$  is a precession about the z-axis with angular frequency  $\omega/2$ . By combining these two rotations about orthogonal axes for appropriate times any rotation of the spin state can be achieved. This is a consequence of the fact that the Pauli spin matrices, after multiplication with the complex number  $i$ , generate the rotation group  $SU(2)$  [16]. So essentially, the strength and duration of the perturbing magnetic field can be tuned to obtain any rotation of the spin.

### 2.3 Coherent state and displacement

As mentioned before, the implementation of a QRW with an ion-trap is not a walk over actual positions. For simplicity, we can regard the ion-trap as a harmonic potential; giving rise to the motional states of the ion. The Hamiltonian of a harmonic potential is given by

$$\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}),$$

where  $\hat{a}^\dagger$  and  $\hat{a}$  denote the creation operator and annihilation operator respectively. The eigenstates of this Hamiltonian are called number states or Fock states, for which we write  $|n\rangle$ . In order to implement the shift operation of the QRW, a laser is applied to the ion. This affects the motional state of the ion as the operator

$$\hat{D}(\alpha) = e^{\alpha\hat{a}^\dagger - \bar{\alpha}\hat{a}},$$

the displacement operator. It has the following effect on the ground state:

$$\hat{D}(\alpha)|0\rangle = |\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

We call this resulting state  $|\alpha\rangle$  a coherent state. Such a state is an eigenstate of the annihilation operator. Its definition shows that the outcome of an energy measurement on  $|\alpha\rangle$  is distributed as a Poisson distribution. Coherent states have various interesting properties. For instance, the expected value of position oscillates according to classical harmonic oscillation. The coherent state also minimises the Heisenberg uncertainty relation. Due to these properties, coherent states resemble the behaviour of classical states. Repeated application of the displacement operator further displaces the state as  $\hat{D}(\alpha)|\alpha\rangle = |2\alpha\rangle$ , enabling a walk over coherent states.

It is however not straightforward to measure the coherent state of an ion. Indeed, the coherent states are eigenstates of the non-hermitian annihilation operator which is not observable. There do exist methods for distinguishing two different coherent states using the method of photon counting. See for instance [17].

### 3 Long-range QRW

In this section we introduce a variation on the QRW that allows for multiple steps, meaning that we will not restrict the QRW to shifts of  $+1$  or  $-1$  but allow shifts of  $-s$  up to  $+s$ . The section aims to investigate how this extension of the dynamics affects the limiting distribution. Furthermore, we attempt to identify the behaviour in the limit of a long range. i.e. when  $s$  tends to infinity. We start by describing the setting of a long range QRW in Section 3.1 whereafter we generalise the results of Section 1 to this extension in Section 3.2. An extension of the Hadamard coin is introduced in Section 3.3. Thereafter, we focus on the most simple extension where we restrict the QRW to shifts between  $+2$  and  $-2$  in Section 3.4 after which we analyse the general case of shifts between  $-s$  and  $s$  in Section 3.5. We conclude by investigating the limit of an infinite range of shifts in Section 3.6.

#### 3.1 Setting & notation

We start by extending the coin-state to an element of  $\mathbb{C}^{2s}$  instead of  $\mathbb{C}^2$ . We will write

$$\left\{ |i\rangle = w_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \right\}_{i \in \sigma(s)}$$

as a basis for  $\mathbb{C}^{2s}$ , with  $\sigma(s) = \{-s, \dots, -1, 1, \dots, s\}$ . The total state now reads

$$\psi = \sum_{i \in \sigma(s)} \sum_{x \in \mathbb{Z}} \psi(x, i) e_x \otimes w_i.$$

With this state we associate a probability to find the particle at position  $x$  equal to

$$\mathbb{P}(x) = \sum_{i \in \sigma(s)} |\langle e_x w_i, \psi \rangle|^2 = \sum_{i \in \sigma(s)} |\psi(x, i)|^2.$$

The coin operator  $A$  must now be a unitary  $2s \times 2s$  matrix. We alter the shift  $S : \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2 \rightarrow \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$  on the total state to:

$$S(p \otimes |i\rangle) = \tau_i p \otimes |i\rangle$$

Here  $\tau_i$  denotes a shift of  $i$  steps to the right. This construction allows for a maximum of  $s$  steps either to the left or right. We can extend the results of Section 1.3 by writing

$$\hat{\psi} = \begin{pmatrix} \hat{\psi}(k, s) \\ \vdots \\ \hat{\psi}(k, -s) \end{pmatrix}$$

for the Fourier transform of  $\psi$ . The shift operator in the transform space generalises to:

$$\begin{aligned} \widehat{(S\psi)}(k) &= \begin{pmatrix} \widehat{\psi(x-s, s)} \\ \vdots \\ \widehat{\psi(x+s, -s)} \end{pmatrix} = \begin{pmatrix} \sum_{x \in \mathbb{Z}} e^{ikx} \psi(x-s, s) \\ \vdots \\ \sum_{x \in \mathbb{Z}} e^{ikx} \psi(x+s, -s) \end{pmatrix} = \begin{pmatrix} \sum_{x \in \mathbb{Z}} e^{ik(x+s)} \psi(x, s) \\ \vdots \\ \sum_{x \in \mathbb{Z}} e^{ik(x-s)} \psi(x, -s) \end{pmatrix} \\ &= \begin{pmatrix} e^{iks} \sum_{x \in \mathbb{Z}} e^{ikx} \psi(x, s) \\ \vdots \\ e^{-iks} \sum_{x \in \mathbb{Z}} e^{ikx} \psi(x, -s) \end{pmatrix} = \begin{pmatrix} e^{iks} \hat{\psi}(k, s) \\ \vdots \\ e^{-iks} \hat{\psi}(k, -s) \end{pmatrix} = \begin{pmatrix} e^{iks} & & \emptyset \\ & \ddots & \\ \emptyset & & e^{-iks} \end{pmatrix} \hat{\psi}(k). \end{aligned}$$

Such that a step is described by

$$\widehat{U\psi} = \widehat{SA\psi} = \begin{pmatrix} e^{iks} & & \emptyset \\ & \ddots & \\ \emptyset & & e^{-iks} \end{pmatrix} A \hat{\psi}(k) = \hat{U}(k) \hat{\psi}(k),$$

with

$$\hat{U}(k) = \begin{pmatrix} e^{iks} & & \emptyset \\ & \ddots & \\ \emptyset & & e^{-iks} \end{pmatrix} A.$$

We denote by  $v_1(k), \dots, v_{2s}(k)$  and  $\lambda_1(k), \dots, \lambda_{2s}(k)$  the eigenvectors and corresponding eigenvalues of  $\hat{U}$ . Again we have

$$\hat{\psi}_n(k) = \hat{U}^n(k) \hat{\psi}_0(k),$$

and a result for the total evolution similar to that in Section 1.3:

$$\hat{\psi}_n = U(k)^n \hat{\psi}_0 = \sum_{j=1}^{2s} \lambda_j(k)^n \langle \hat{\psi}_0, v_j(k) \rangle v_j(k).$$

### 3.2 Convergence

The method of moments can also be extended to the multiple step walk. As before

$$\mathbb{E}(X_n^r) = \langle \psi_n, X^r \psi_n \rangle.$$

The appropriate relation of inner products of the state-space and transform-space is now

$$\langle \psi, \phi \rangle_{\ell^2(\mathbb{Z}) \otimes \mathbb{C}^{2s}} = \int_0^{2\pi} \langle \hat{\psi}, \hat{\phi} \rangle_{\mathbb{C}^{2s}} \frac{dk}{2\pi}.$$

The remainder of the derivation leading to the limiting distribution can simply be extended to the multiple step walk by replacing all sums over  $j = 1, 2$  to sums over  $j = 1, \dots, 2s$ . This results in

$$\mathbb{E} \left[ \left( \frac{X_n}{n} \right)^r \right] = \int_0^{2\pi} \sum_{j=1}^{2s} \left( \frac{D\lambda_j}{\lambda_j(k)} \right)^r |\langle \hat{\psi}_0, v_j(k) \rangle|^2 \frac{dk}{2\pi} + \mathcal{O}(n^{-1}).$$

It now follows that if we take  $\Omega = \mathbb{K} \times \{1, \dots, 2s\}$  and let  $\mu$  be the probability measure on  $\Omega$  defined by  $|\langle \hat{\psi}_0, v_j(k) \rangle|^2 \frac{dk}{2\pi}$ . Now define  $h(k, j) = \frac{D\lambda_j}{\lambda_j(k)}$ . Then  $h : \mathbb{K} \times \{1, \dots, 2s\} \rightarrow \mathbb{R}$  is a random variable. With this interpretation we can write

$$\mathbb{E} \left[ \left( \frac{X_n}{n} \right)^r \right] \rightarrow \int_{\Omega} h^r d\mu \quad \text{as } n \rightarrow \infty.$$

And Theorem 1 guarantees that

$$\frac{X_n}{n} \xrightarrow{d} Y = h(Z),$$

with  $Z$  a random element of  $\Omega$  whose distribution is given by  $\mu$ .

### 3.3 Multiple Hadamard coins

In Section 1.2 we introduced the Hadamard coin:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

This coin was chosen because it is fair in the sense that it creates equal superpositions from single coin states, yet different for every coin state. Namely  $H$  on  $|1\rangle$  gives  $\frac{|1\rangle+|-1\rangle}{\sqrt{2}}$  and  $H$  on  $|-1\rangle$  gives  $\frac{|1\rangle-|-1\rangle}{\sqrt{2}}$ . The key property here is that all entries of the matrix have the same absolute value, meaning that it will create superpositions of the coin states with equal probability. In this sense the coin is fair. This property can easily be extended to higher order matrices by defining the  $2^m$ -th order Hadamard coin as a tensor product

$$H_{2^m} = \underbrace{H \otimes H \otimes \dots \otimes H}_m = H^{\otimes m}.$$

We will start with the case of  $s = m = 2$ , after which we can continue to the more difficult general case.

### 3.4 Example: range two

In this case we have

$$H_4 = H \otimes H$$

and

$$\hat{U}_4(k) = \begin{pmatrix} e^{2ik} & 0 & 0 & 0 \\ 0 & e^{ik} & 0 & 0 \\ 0 & 0 & e^{-ik} & 0 \\ 0 & 0 & 0 & e^{-2ik} \end{pmatrix} (H \otimes H).$$

Or by decomposing the first matrix into a Kronecker product

$$\hat{U}_4(k) = \left( \begin{pmatrix} e^{\frac{3}{2}ik} & 0 \\ 0 & e^{-\frac{3}{2}ik} \end{pmatrix} \otimes \begin{pmatrix} e^{\frac{1}{2}ik} & 0 \\ 0 & e^{-\frac{1}{2}ik} \end{pmatrix} \right) (H \otimes H).$$

We have a degree of freedom in choosing the decomposition, due to the fact that we can multiply one of the two matrices by a constant as long as we divide the other by the same constant. This particular decomposition was chosen because the elements on the diagonal are now conjugates. From the mixed-product property it follows that

$$\hat{U}_4(k) = \left( \begin{pmatrix} e^{\frac{3}{2}ik} & 0 \\ 0 & e^{-\frac{3}{2}ik} \end{pmatrix} H \right) \otimes \left( \begin{pmatrix} e^{\frac{1}{2}ik} & 0 \\ 0 & e^{-\frac{1}{2}ik} \end{pmatrix} H \right).$$

The eigenvalues of a Kronecker product  $A \otimes B$  are products of the eigenvalues of  $A$  and  $B$ . Therefore we would like to calculate the eigenvalues of the two matrices above. These are both of the form

$$\begin{pmatrix} e^{aik} & 0 \\ 0 & e^{-aik} \end{pmatrix} H = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{aik} & e^{aik} \\ e^{aik} & -e^{-aik} \end{pmatrix},$$

which is a product of unitary matrices and is therefore unitary. Note that its eigenvalues lie on the unit circle in  $\mathbb{C}^2$ . They are determined in the usual fashion:

$$\begin{aligned} \left| \begin{array}{cc} e^{aik} - \lambda & e^{aik} \\ e^{-aik} & -e^{-aik} - \lambda \end{array} \right| &= (e^{aik} - \lambda)(-e^{-aik} - \lambda) - 1 \\ &= \lambda^2 + (e^{-aik} - e^{aik})\lambda - 2 \\ &= \lambda^2 - 2i \sin(ak)\lambda - 2 = 0 \\ \implies \lambda_{1,2} &= i \sin(ak) \pm \sqrt{2 - \sin^2(ak)} \end{aligned}$$

Where we have excluded the factor  $\frac{1}{\sqrt{2}}$ . With this factor the eigenvalues read

$$\lambda_{1,2} = \frac{i \sin(ak)}{\sqrt{2}} \pm \sqrt{1 - \sin^2(ak)/2},$$

and

$$D\lambda_j(k) = a \frac{\cos(ak)}{\sqrt{2}} \pm a \frac{\sin(ak) \cos(ak)}{\sqrt{4 - 2 \sin^2(ak)}} i.$$

This yields for the random variable  $h_j(k)$  related to these eigenvalues:

$$h_j(k) = \pm \frac{a \cos(ak)}{\sqrt{2 - \sin^2(ak)}}$$

See Appendix C.7. Note that we have found eigenvalues for  $\hat{U}_4(k)$  of the form  $\nu = \lambda\mu$ , where  $\lambda, \mu$  are eigenvalues of matrices of the discussed form. Now, since  $\lambda$  and  $\mu$  lie on the unit circle in  $\mathbb{C}^2$  we can write  $\lambda = e^{ia(k)}$  and  $\mu = e^{ib(k)}$ . Therefore  $\nu = e^{i(a(k)+b(k))}$ . Recall that the random variable  $h_j(k)$  equals the derivative of the phase of the eigenvalues. i.e.  $h_j(k) = \frac{d}{dk} \phi_j(k)$  for the eigenvalue  $\mu_j = e^{i\phi_j(k)}$ . We conclude that

$$h_\nu = \frac{d}{dk} [a(k) + b(k)] = \frac{d}{dk} a(k) + \frac{d}{dk} b(k) = h_\lambda + h_\mu.$$

So the random variables associated with the total evolution operator is the sum of the random variables associated with the matrices in its tensor-product decomposition. Therefore we can substitute  $a = 3/2$  and  $a = 1/2$  in the result for  $h_j(k)$  and sum the results which gives four values for the total  $h_j(k)$  that is associated with  $\hat{U}_4(k)$ :

$$h_{1,2}(k) = \pm \frac{\frac{3}{2} \cos(\frac{3}{2}k)}{\sqrt{2 - \sin^2(\frac{3}{2}k)}} \pm \frac{\frac{1}{2} \cos(\frac{1}{2}k)}{\sqrt{2 - \sin^2(\frac{1}{2}k)}}$$

and

$$h_{3,4}(k) = \pm \frac{\frac{3}{2} \cos(\frac{3}{2}k)}{\sqrt{2 - \sin^2(\frac{3}{2}k)}} \mp \frac{\frac{1}{2} \cos(\frac{1}{2}k)}{\sqrt{2 - \sin^2(\frac{1}{2}k)}}$$

To see how this results compares to the 1-step QRW we will the moments of the 1-step QRW with the moments of the 2-step QRW. If we take an initial state with  $X_0 = 0$  and one of  $w_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ ,  $w_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ ,  $w_{-1} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$  and  $w_{-2} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ , then  $\hat{\psi}_0(k) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$  or  $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ . In which case

$$\mu = |\langle \psi_0(k), v_j(k) \rangle|^2 \frac{dk}{2\pi} = |v_{ji}(k)|^2 \frac{dk}{2\pi}.$$

If we instead consider a random initial state, that is  $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$  or  $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$  with equal probability, then the measure  $\mu$  becomes

$$\mu = \frac{1}{4} \sum_{i=1}^4 |v_{ji}(k)|^2 \frac{dk}{2\pi} = |v_j(k)|^2 \frac{dk}{8\pi} = \frac{dk}{8\pi}.$$

And the moments of the 2-step QRW read

$$\begin{aligned} \mathbb{E} \left[ \left( \frac{X_n}{2n} \right)^r \right] &\rightarrow \int_{\Omega} h^r d\mu = \frac{1}{8\pi} \int_0^{2\pi} \sum_{i=1}^4 (h_j(k)/2)^r dk \\ &= \frac{1}{8\pi} \int_0^{2\pi} \left( \frac{\frac{3}{2} \cos(\frac{3}{2}k)}{2\sqrt{2 - \sin^2(\frac{3}{2}k)}} + \frac{\frac{1}{2} \cos(\frac{1}{2}k)}{2\sqrt{2 - \sin^2(\frac{1}{2}k)}} \right)^r \\ &\quad + \left( -\frac{\frac{3}{2} \cos(\frac{3}{2}k)}{2\sqrt{2 - \sin^2(\frac{3}{2}k)}} - \frac{\frac{1}{2} \cos(\frac{1}{2}k)}{2\sqrt{2 - \sin^2(\frac{1}{2}k)}} \right)^r \\ &\quad + \left( -\frac{\frac{3}{2} \cos(\frac{3}{2}k)}{2\sqrt{2 - \sin^2(\frac{3}{2}k)}} + \frac{\frac{1}{2} \cos(\frac{1}{2}k)}{2\sqrt{2 - \sin^2(\frac{1}{2}k)}} \right)^r \\ &\quad + \left( \frac{\frac{3}{2} \cos(\frac{3}{2}k)}{2\sqrt{2 - \sin^2(\frac{3}{2}k)}} - \frac{\frac{1}{2} \cos(\frac{1}{2}k)}{2\sqrt{2 - \sin^2(\frac{1}{2}k)}} \right)^r dk. \end{aligned}$$

We have divided over  $2n$  to compensate for the 2 steps. Note that all terms cancel for odd  $r$ , such that we can also write the moments as

$$\begin{cases} 0 & , r \text{ odd} \\ \frac{1}{4\pi} \int_0^{2\pi} \left( \frac{\frac{3}{2} \cos(\frac{3}{2}k)}{2\sqrt{2 - \sin^2(\frac{3}{2}k)}} + \frac{\frac{1}{2} \cos(\frac{1}{2}k)}{2\sqrt{2 - \sin^2(\frac{1}{2}k)}} \right)^r + \left( \frac{\frac{3}{2} \cos(\frac{3}{2}k)}{2\sqrt{2 - \sin^2(\frac{3}{2}k)}} - \frac{\frac{1}{2} \cos(\frac{1}{2}k)}{2\sqrt{2 - \sin^2(\frac{1}{2}k)}} \right)^r dk & , r \text{ even.} \end{cases}$$

Recall that the moments of the 1-step QRW are given by

$$\begin{aligned} \mathbb{E} \left[ \left( \frac{X_n}{n} \right)^r \right] &\rightarrow \int_{\Omega} h^r d\mu = \frac{1}{4\pi} \int_0^{2\pi} \sum_{i=1}^2 (h_j(k))^r dk \\ &= \frac{1}{4\pi} \int_0^{2\pi} \left( \frac{\cos(k)}{\sqrt{2 - \sin^2(k)}} \right)^r + \left( -\frac{\cos(k)}{\sqrt{2 - \sin^2(k)}} \right)^r dk. \end{aligned}$$

Here the terms cancel for odd  $r$  too:

$$\begin{cases} 0 & , r \text{ odd} \\ \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{\cos(k)}{\sqrt{2 - \sin^2(k)}} \right)^r dk & , r \text{ even} \end{cases}$$

These results are plotted in Figure 2 for comparison.

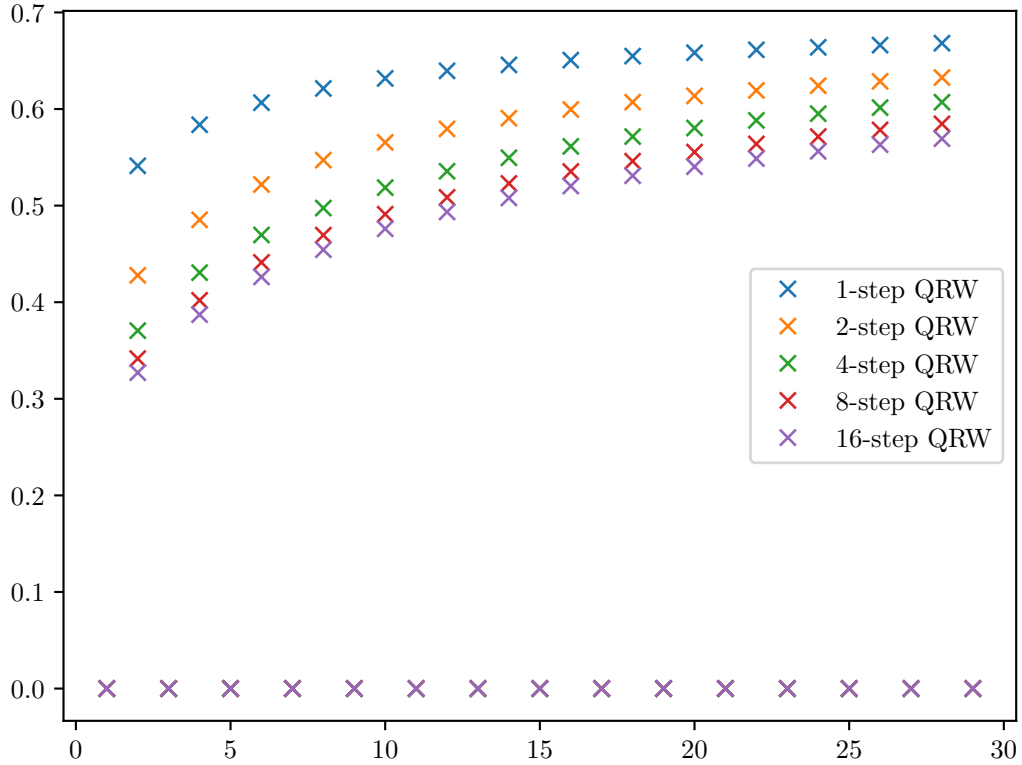


Figure 2. The first 50 moments of the limit distribution for the first 5 multiple-step QRW's. More precisely, for  $i = 1, \dots, 50$  the  $i^{\text{th}}$  root of the  $i^{\text{th}}$  moment was plotted. That is  $\lim_{n \rightarrow \infty} \mathbb{E}[(X_n/sn)^r]^{1/r}$  for an  $s$ -step QRW.



### 3.5 General case

For  $s > 2$ , or better  $s = 2^{m-1}$  where  $m > 2 \in \mathbb{N}$ , we would like to find a decomposition of  $\hat{U}(k)$  similar to

$$\hat{U}_4(k) = \left( \begin{pmatrix} e^{\frac{3}{2}ik} & 0 \\ 0 & e^{-\frac{3}{2}ik} \end{pmatrix} H \right) \otimes \left( \begin{pmatrix} e^{\frac{1}{2}ik} & 0 \\ 0 & e^{-\frac{1}{2}ik} \end{pmatrix} H \right).$$

We will try to find a general decomposition by first considering  $\hat{S}_{2^m}$  for the cases  $m = 2, 3$ . In the decomposition above the right hand matrix was chosen because the powers on the diagonal of  $\hat{S}$  decrease with steps of one, except in the middle. Let us try a similar approach for  $m = 3$ .

$$\begin{aligned} \hat{S}_8 &= \begin{pmatrix} e^{4ik} & & & & & & & \\ & \ddots & & & & & & \\ & & e^{ik} & & & & & \\ & & & e^{-ik} & & & & \\ & & & & \ddots & & & \\ & & & & & e^{-4ik} & & \\ & & & & & & & \end{pmatrix} \\ &= \begin{pmatrix} e^{\frac{7}{2}ik} & & & & & & & \\ & e^{\frac{3}{2}ik} & & & & & & \\ & & e^{-\frac{3}{2}ik} & & & & & \\ & & & e^{-\frac{7}{2}ik} & & & & \\ & & & & & & & \end{pmatrix} \otimes \begin{pmatrix} e^{\frac{1}{2}ik} & 0 \\ 0 & e^{-\frac{1}{2}ik} \end{pmatrix} \end{aligned}$$

This leaves us with a matrix on the left whose powers on the diagonal decrease with steps of two, except in the middle. We will therefore we try write the remaining matrix as a tensor product of a matrix with  $\begin{pmatrix} e^{ik} & 0 \\ 0 & e^{-ik} \end{pmatrix}$ , which also has powers on the diagonal decreasing with steps of two. We find the decomposition

$$\hat{S}_8 = \begin{pmatrix} e^{\frac{5}{2}ik} & 0 \\ 0 & e^{-\frac{5}{2}ik} \end{pmatrix} \otimes \begin{pmatrix} e^{ik} & 0 \\ 0 & e^{-ik} \end{pmatrix} \otimes \begin{pmatrix} e^{\frac{1}{2}ik} & 0 \\ 0 & e^{-\frac{1}{2}ik} \end{pmatrix}.$$

In general, when we use this approach for the decomposition of higher orders we will find after each iteration a matrix on the left whose powers on the diagonal decrease with steps twice as big as before. So our decomposition is of the form

$$\begin{aligned} \hat{S}_{2^m} &= \begin{pmatrix} e^{aik} & 0 \\ 0 & e^{bik} \end{pmatrix} \\ &\otimes \underbrace{\begin{pmatrix} e^{2^{m-3}ik} & 0 \\ 0 & e^{-2^{m-3}ik} \end{pmatrix} \otimes \begin{pmatrix} e^{2^{m-4}ik} & 0 \\ 0 & e^{-2^{m-4}ik} \end{pmatrix} \otimes \dots \otimes \begin{pmatrix} e^{\frac{1}{2}ik} & 0 \\ 0 & e^{-\frac{1}{2}ik} \end{pmatrix}}_{m-1} \end{aligned}$$

or

$$\hat{S}_{2^m} = \begin{pmatrix} e^{aik} & 0 \\ 0 & e^{bik} \end{pmatrix} \otimes \bigotimes_{j=-1}^{m-3} \begin{pmatrix} e^{2^j ik} & 0 \\ 0 & e^{-2^j ik} \end{pmatrix}.$$

The top left entry of  $\hat{S}_{2^m}$  equals  $e^{2^{m-1}ik}$ , therefore  $a + \sum_{j=-1}^{m-3} 2^j = 2^{m-1} \implies a = 2^{m-1} - (2^{m-2} - \frac{1}{2}) = \frac{1}{2} + 2^{m-2}$ . In a similar way we find that  $b = -(\frac{1}{2} + 2^{m-2})$ . Now we can apply the mixed product property to obtain our result:

$$\begin{aligned} \hat{U}_{2^m}(k) &= \hat{S}_{2^m} H_{2^m} \\ &= \left( \begin{pmatrix} e^{(1/2+2^{m-2})ik} & 0 \\ 0 & e^{-(1/2+2^{m-2})ik} \end{pmatrix} H \right) \otimes \bigotimes_{j=-1}^{m-3} \left( \begin{pmatrix} e^{2^j ik} & 0 \\ 0 & e^{-2^j ik} \end{pmatrix} H \right) \end{aligned}$$

and thus we find for the random variables associated with this matrix

$$h_j(k) = \pm \frac{(1/2 + 2^{m-2}) \cos((1/2 + 2^{m-2})k)}{\sqrt{2 - \sin^2((1/2 + 2^{m-2})k)}} + \sum_{n=-1}^{m-3} \pm \frac{2^n \cos(2^n k)}{\sqrt{2 - \sin^2(2^n k)}}.$$

The expression is a sum of  $m$  terms in which the terms are separated by a  $\pm$ -sign. Also, we have  $2^m$  different eigenvalues of  $\hat{U}_{2^m}$ , so  $j$  can take  $2^m$  values. Each choice of  $j$  corresponds to a different choice of  $+$  or  $-$  from the  $\pm$ -signs. Now we consider an initial state with  $X_0 = 0$  and some  $w_i \in \{w_m, \dots, w_{-m}\}$ , so  $\psi_0(x, i) = \mathbb{1}_{(0, i)}$ , then  $\hat{\psi}_0(k, i) = 1$  and  $\hat{\psi}_0(k, j) = 0$  for  $j \neq i$ . In which case

$$\mu = |\langle \psi_0(k), v_j(k) \rangle|^2 \frac{dk}{2\pi} = |v_{ji}(k)|^2 \frac{dk}{2\pi}.$$

If we instead consider a random initial coin state, that is all  $w_i \in \{w_m, \dots, w_{-m}\}$  with equal probabilities, then

$$\mu = \frac{1}{2s} \sum_{i \in \sigma(s)} = |v_{ji}(k)|^2 \frac{dk}{2\pi} = \frac{1}{2s} |v_j|^2 \frac{dk}{2\pi} = \frac{1}{2s} \frac{dk}{2\pi} = \frac{1}{2^m} \frac{dk}{2\pi}.$$

So now the expression for the limit of the moments of  $X_n/n$  with this choice of initial state becomes

$$\mathbb{E} \left[ \left( \frac{X_n}{n} \right)^r \right] \rightarrow \int_0^{2\pi} \frac{1}{2^m} \sum_{j=1}^{2^m} h_j(k)^r \frac{dk}{2\pi}.$$

Alternatively, we can write this expression as

$$\mathbb{E} \left[ \left( \frac{X_n}{n} \right)^r \right] \rightarrow \int_0^{2\pi} \frac{1}{2^m} \sum_{\eta \in \{\pm 1\}^m} h(\eta, k)^r \frac{dk}{2\pi},$$

where

$$h(\eta, k) = \sum_{n=1}^m \eta(n) \alpha(n, k),$$

and

$$\alpha(n, k) = \begin{cases} \frac{2^{n-2} \cos(2^{n-2}k)}{\sqrt{2 - \sin^2(2^{n-2}k)}} & , n = 1, \dots, m-1 \\ \frac{(1/2 + 2^{n-2}) \cos((1/2 + 2^{n-2})k)}{\sqrt{2 - \sin^2((1/2 + 2^{n-2})k)}} & , n = m. \end{cases}$$

Note that these are the moments of a random variable  $Y_m = h(\eta, k)$  on  $\Omega = \mathbb{K} \times \{\pm 1\}^m$  with measure  $\mu = \frac{1}{2^m} \frac{dk}{2\pi}$ . By Theorem 1 we conclude that  $X_n/n \xrightarrow{d} Y_m$ . This result shows that the limiting random variable of the long-range QRW is a sum over functions  $\alpha(n, k)$  evaluated at a random point in  $[0, 2\pi]$  multiplied with a random sign.

The result shows that the limiting random variable of a long-range QRW is different for each range. This finding illustrates another difference between the QRW and the classical random walk whose limiting random variable does not depend on the step-distribution.

### 3.6 Limit of infinite range

Now that we have identified the limiting random variable for a  $s$ -step QRW with the Hadamard coin  $Y_m$ . We will examine the behaviour of this random variable as the number of steps  $s$  increases. For the purposes of this calculation, it is convenient to write  $Y_m$  in the following form:

$$\frac{Y_m}{2^{m-1}} = \frac{1}{2} \sum_{n=1}^{m-1} \eta_n \phi(2^{n-2}U) 2^{-(m-n)} + (2^{-m} + \frac{1}{2}) \eta_m \phi((\frac{1}{2} + 2^{m-2})U),$$

where  $(\eta_n)_{n=1}^m$  is a sequence of iid Bernoulli random variables taking values  $\pm 1$  with equal probability,  $U$  is uniform on  $[0, 1]$  and  $\phi(x) = \frac{\cos(2\pi x)}{\sqrt{2 - \sin^2(2\pi x)}}$ . Note that we have also normalised the limiting random variable by the maximum amount of shifts in this QRW. Let  $(\xi_i)_{i=1}^\infty$  be a sequence of iid Bernoulli random variables taking values in  $\{0, 1\}$  with equal probability. Then  $U \stackrel{d}{=} k(\xi) := \sum_{i=1}^\infty \xi_i 2^{-i}$  and

$$2^n U \bmod 1 \stackrel{d}{=} k(\theta_n \xi), \quad n \geq 0$$

with  $\theta_n$  the shift of  $n$  to the left, i.e.  $(\theta_n(\xi))_m = \xi_{m+n}$ . And because  $\phi$  has period 1 we can write

$$\phi(2^n U) \stackrel{d}{=} \phi(k(\theta_n \xi)), \quad n \geq 0.$$

Hence:

$$\frac{Y_m}{2^{m-1}} \stackrel{d}{=} \eta_1 \phi(\frac{1}{2} k(\xi)) 2^{-m} + \frac{1}{2} \sum_{n=2}^{m-1} \eta_n \phi(k(\theta_{n-2} \xi)) 2^{-(m-n)} + (2^{-m} + \frac{1}{2}) \eta_m \phi((\frac{1}{2} + 2^{m-2}) k(\xi)).$$

We can change the summation index to  $i = m - n$  to obtain

$$\frac{Y_m}{2^{m-1}} \stackrel{d}{=} \eta_1 \phi(\frac{1}{2} k(\xi)) 2^{-m} + \frac{1}{2} \sum_{i=1}^{m-2} \eta_{m-i} \phi(k(\theta_{m-i-2} \xi)) 2^{-i} + (2^{-m} + \frac{1}{2}) \eta_m \phi((\frac{1}{2} + 2^{m-2}) k(\xi)).$$

Note that  $\frac{Y_m}{2^{m-1}}$  is a function on  $(\eta_1, \eta_{m-1}, \eta_{m-2}, \dots, \eta_2, \eta_m, \xi, \theta_{m-3}\xi, \theta_{m-4}\xi, \dots, \theta_0\xi, \xi)$  which equals in distribution  $(\eta_1, \eta_3, \eta_4, \dots, \eta_m, \eta_2, \theta_m\xi, \theta_3\xi, \theta_4\xi, \dots, \theta_m\xi, \theta_m\xi)$ . So we find that  $\frac{Y_m}{2^{m-1}}$  equals in distribution

$$\frac{Y_m}{2^{m-1}} \stackrel{d}{=} \eta_1 \phi(\frac{1}{2} k(\theta_m \xi)) 2^{-m} + (2^{-m} + \frac{1}{2}) \eta_2 \phi((\frac{1}{2} + 2^{m-2}) k(\theta_m \xi)) + \frac{1}{2} \sum_{i=3}^m \eta_i \phi(k(\theta_i \xi)) 2^{-i}.$$

Or in terms of  $U$  and  $\eta$ :

$$\frac{Y_m}{2^{m-1}} \stackrel{d}{=} \eta_1 \phi(\frac{1}{2} (2^m U \bmod 1)) 2^{-m} + (2^{-m} + \frac{1}{2}) \eta_2 \phi((\frac{1}{2} + 2^{m-2}) (2^m U \bmod 1)) + \frac{1}{2} \sum_{i=3}^m \eta_i \phi(2^i U) 2^{-i}.$$

As  $m$  tends to infinity the first term will vanish and the last term converges to  $\frac{1}{2} \sum_{i=3}^\infty \eta_i \phi(2^i U) 2^{-i}$ . In fact, the division by  $2^{m-1}$  ensures the convergence of the last term. Another expression of the same leading order,  $2^m$ , would also suffice. If we consider the classical analogue of the QRW with range  $s = 2^{m-1}$ , a different choice of division must be made to obtain convergence. For the classical random walk that takes steps uniformly distributed in  $\{-s, \dots, -1, 1, \dots, s\}$ , the division should be of leading order  $\sqrt{2^m}$ . See Appendix C.8 for a derivation. Much like the ballistic scaling of the QRW, the appropriate scaling correction for the range of the QRW is

the square of the classical correction. Histograms of  $\frac{Y_m}{2^{m-1}}$  for various values of  $m$  are plotted in Figure 4. Histograms of the convergent term are plotted in Figure 3. These show that the convergent term takes values between  $-0.1$  and  $0.1$ . This relatively small range might be caused by interference of the signs brought about by the  $\eta_i$ 's. Their distribution appears symmetric, unsurprisingly. In Figure 4a we see that in case of  $m = 1$  we retrieve the expected probability density as calculated in Section 1.5. Figure 4b shows a more complicated histogram, perhaps due to the fact that the first term still plays a significant role for this low value of  $m$ . For slightly higher values of  $m$  we can see in Figures 4 that the peaks in the histogram move more towards the middle, thereby narrowing the bowl-shaped middle section and the range of the histogram. The narrowing of the range is compliant with Figure 2, where we see that the moments decrease as the range increases.

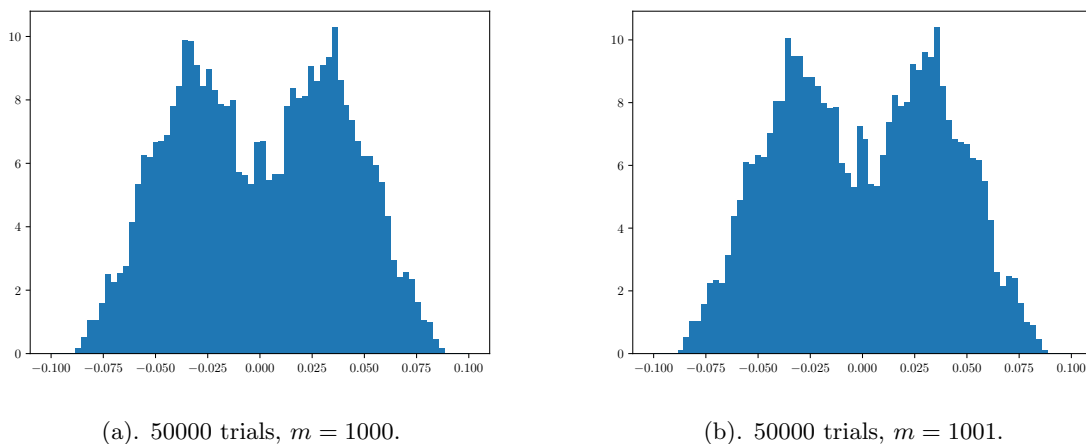
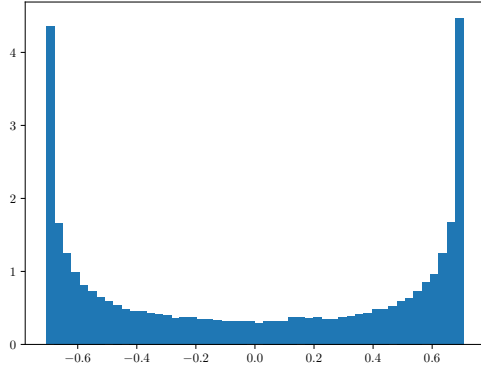
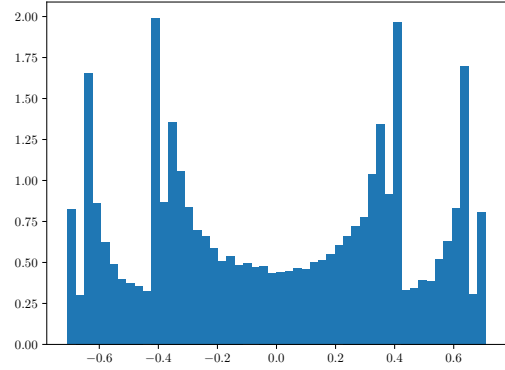


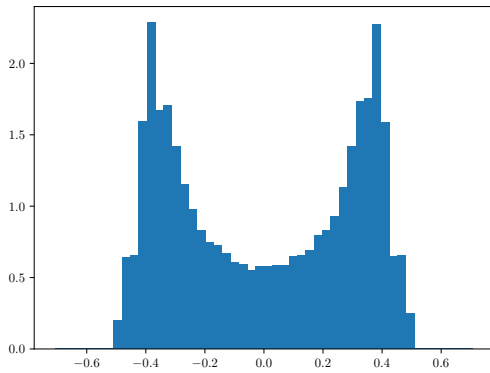
Figure 3. Histograms of the convergent term  $\frac{1}{2} \sum_{i=3}^m \eta_i \phi(2^i U) 2^{-i}$ . The  $y$ -axis has been scaled such that the total area in the histogram equals 1.



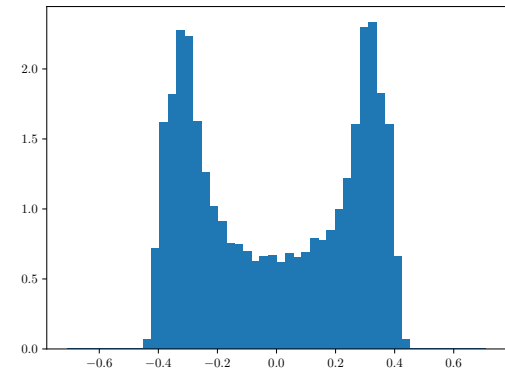
(a). 50000 trials.  $m = 1, s = 1$ .



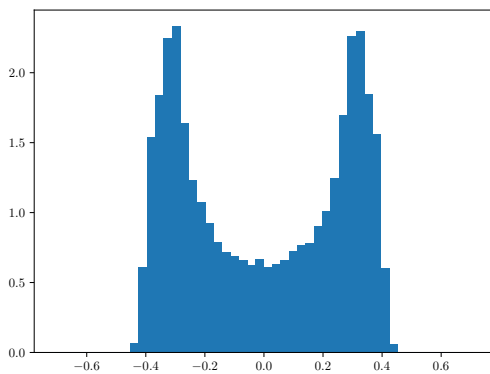
(b). 50000 trial.  $m = 2, s = 2$ .



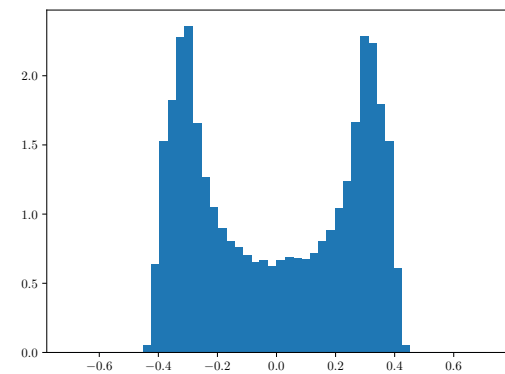
(c). 50000 trials.  $m = 4, s = 8$ .



(d). 50000 trials.  $m = 8, s = 128$ .



(e). 50000 trials.  $m = 12, s = 2048$ .



(f). 50000 trials.  $m = 20, s = 524288$ .

Figure 4. Histograms of  $Y_m/2^{m-1}$ . The  $y$ -axis has been scaled such that the total area in the histogram equals 1.

## 4 Further extensions; Traps

So far we have encountered the QRW on the integers in Section 1, which we extended to a long-range QRW in Section 3. There are however many more variations on a QRW worth examining. In this section we introduce a QRW with traps. We start with some background information on the problem in Section 4.1. In Section 4.2 we analyse an elementary setting, a trapped QRW on a 3-cycle, which may serve as a starting point for further research.

### 4.1 Context and classical trapped random walk

The concept arises from the classical random walk. In a trapped classical random walk on the integers, some integers are assigned as traps. These traps may also be randomly distributed. For instance, one can assign a Bernoulli random variable to each integer that decides whether it becomes a trap site. After the traps are assigned, a classical walk is initiated as usual. However if at some point the walker arrives at a trap site, the walk ends. Clearly, the trapped random walk does not necessarily continue indefinitely like the regular random walk. A topic of interest in this trapped classical random walk is the survival probability. That is the probability that the walker will live through  $n$  steps. For the classical random walk on the integers where traps are randomly distributed this probability scales as  $e^{-cn^{1/3}}$  for large  $n$ , the Donkser-Varadhan (DV) regime [18]. More generally, on a  $d$ -dimensional lattice the survival probability scales as  $e^{-cn^{d/(d+2)}}$  for large  $n$  [19]. Intuitively, in order to survive a walker should find a large trap-free area. Such area's are however exponentially rare. The question arises whether a QRW would perform better or worse than a classical random in an environment with traps. In other words, how does the survival probability of a QRW compare to the classical case. And, how does the ballistic scaling of the QRW affect the survival probability. It has been shown in [20] that for a QRW evolving in continuous time the survival probability scales as  $e^{-cn^{1/4}}$ . Hence, a slower decay of the survival probability compared to the classical case. Numerical simulations have been made in [21] that suggest a similar behaviour in the discrete case. It would be interesting to analytically derive such a result. Before this behaviour can be examined, the formalism of a trapped QRW must be formulated. Say we have a set  $T \subset \mathbb{Z}$  of traps. In the QRW there is not really a walker in the sense that a QRW does not produce a sequence of walker positions, but rather a sequence of probability measures. In terms of probability measures, the key property of a trap would be that it removes all probability of being at the trap site. So we define the QRW with traps as follows: The first part of each iteration remains the same, so the operator  $U$  is applied to a state  $\psi$ . After this unitary operation the state is projected on all non-trap sites by the operator  $p_T : \ell^2(\mathbb{Z})^2 \rightarrow \ell^2(\mathbb{Z})^2$  as

$$\begin{pmatrix} \psi(x, 1) \\ \psi(x, -1) \end{pmatrix} \mapsto \begin{pmatrix} \psi(x, 1)\mathbb{1}_{\mathbb{Z} \setminus T} \\ \psi(x, -1)\mathbb{1}_{\mathbb{Z} \setminus T} \end{pmatrix}.$$

### 4.2 Trapped QRW on a 3-cycle

In this section we analyse an elementary case; instead of a QRW on the integers we consider a QRW on a 3-cycle with one trap. In this variation on a QRW the coin state does not determine a shift to the right or left, but a clockwise shift or counter-clockwise shift. In this simple setting

we can write a state as

$$\psi = \begin{pmatrix} \psi_{1,\circlearrowleft} \\ \psi_{1,\circlearrowright} \\ \psi_{2,\circlearrowleft} \\ \psi_{2,\circlearrowright} \\ \psi_{3,\circlearrowleft} \\ \psi_{3,\circlearrowright} \end{pmatrix}.$$

Let us for simplicity take the Hadamard coin. It acts only on the coin state, and therefore consists of blocks on the diagonal:

$$H_3 = \begin{pmatrix} H & 0 & 0 \\ 0 & H & 0 \\ 0 & 0 & H \end{pmatrix}$$

To figure out what the shift operation is, we examine its effect on  $\psi_{1,\circlearrowleft}$ . As we can see in Figure 5a  $\psi_{1,\circlearrowleft}$  should move to  $\psi_3$  and should keep its internal state, so  $\psi_{3,\circlearrowleft}$ . Considering each of the elements of  $\psi$  we find that the shift operator equals

$$S_3 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

And the total evolution is given by  $U_3 = S_3 H_3$ . With the introduction of a trap at site 3, it is no longer necessary to consider  $\psi_3$ . We simply have to alter the shift operation to make sure that sites 1 and 2 do not receive anything from the trap site. See Figure 5b. The state may be written as

$$\psi = \begin{pmatrix} \psi_{1,\circlearrowleft} \\ \psi_{1,\circlearrowright} \\ \psi_{2,\circlearrowleft} \\ \psi_{2,\circlearrowright} \end{pmatrix}.$$

The resulting coin operation and shift are then

$$H_{T,3} = \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix}$$

and

$$S_{T,3} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

The total evolution becomes

$$U_{T,3} = S_{T,3} H_{T,3} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}.$$

The powers of  $U_{T,3}$  take a simple form which allows us to examine the  $n$ -th state of the trapped QRW on a 3-cycle. To see this, we first calculate

$$(\sqrt{2}U_{T,3})^2 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad (\sqrt{2}U_{T,3})^3 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix} = \sqrt{2}U_{T,3}.$$

So the powers of  $U_{T,3}$  for arbitrary  $n \in \mathbb{N}$  are

$$U_{T,3}^{2n} = 2^{-n} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad U_{T,3}^{2n+1} = 2^{-(n+1/2)} \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}.$$

Suppose we start from a state

$$\psi_0 = \begin{pmatrix} \psi_{1,\circ} \\ \psi_{1,\circ} \\ \psi_{2,\circ} \\ \psi_{2,\circ} \end{pmatrix}.$$

After  $n$  steps the state equals

$$\psi_{2n} = 2^{-n} \begin{pmatrix} \psi_{1,\circ} - \psi_{1,\circ} \\ 0 \\ 0 \\ \psi_{2,\circ} + \psi_{2,\circ} \end{pmatrix}, \quad \psi_{2n+1} = 2^{-(n+1/2)} \begin{pmatrix} \psi_{1,\circ} + \psi_{1,\circ} \\ 0 \\ 0 \\ \psi_{2,\circ} - \psi_{2,\circ} \end{pmatrix}.$$

Finally, let us calculate the probability that the walker is still at sites 1 or 2. This equals  $\mathbb{P}_n(X_n = 1 \vee X_n = 2) = |\psi_n|^2$ :

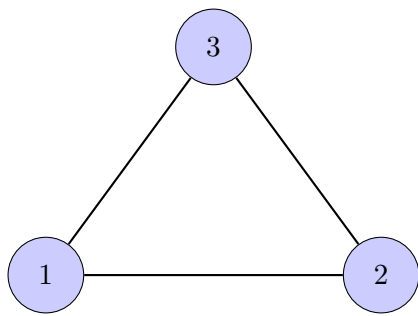
$$\mathbb{P}_{2n} = \frac{|\psi_{1,\circ} - \psi_{1,\circ}|^2 + |\psi_{2,\circ} + \psi_{2,\circ}|^2}{2^n}, \quad \mathbb{P}_{2n+1} = \frac{|\psi_{1,\circ} + \psi_{1,\circ}|^2 + |\psi_{2,\circ} - \psi_{2,\circ}|^2}{2^{n+1/2}}.$$

Consider an initial state where the total probability to find the walker equals 1, for instance

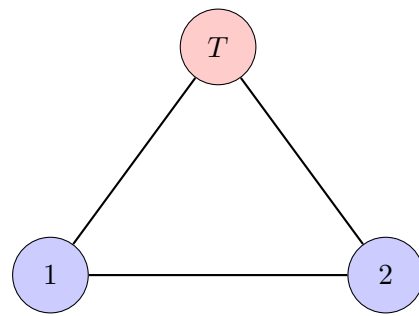
$$\psi_0 = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{pmatrix}.$$

We find in this case  $\mathbb{P}_n = 2^{-n}$ . For comparison, it is a straightforward exercise to calculate the surviving probability in the classical case with equal transition probabilities to either side. Assuming that the walk starts at site 1 or 2, the only way for the walker to survive is by hopping between sites 1 and 2. Thereby never visiting the trap site. The survival probability after  $n$  steps therefore equals  $\mathbb{P}_n = 2^{-n}$ . So in the simple topology of a 3-cycle the survival probability of the classical random walk and the QRW behave the same.





(a). 3-cycle.



(b). 3-cycle with trap.

Figure 5. Topology of the 3-cycle QRW with and without trap.

## 5 Conclusions

A long-range extension was made of the QRW which allows shifts from  $-s$  up to  $s$  by extending the coin state from  $\mathbb{C}^2$  to  $\mathbb{C}^{2s}$ . The coin operator was changed to a unitary  $2s \times 2s$  matrix, consistent with the coin state. Both the coin operator and shift operator take a simple form in the Fourier transform space, as  $2s \times 2s$  matrices. As with the single-step QRW, the unitary evolution guarantees an eigendecomposition of any state. Parseval's theorem allows for a calculation of the moments of the random variable  $X_n/n$  in an  $s$  step QRW. By the method of moments we have proven that  $X_n/n$  converges in distribution to  $h(Z)$ . Here  $Z$  is a random variable taking values in  $\mathbb{K} \times \{1, \dots, 2s\}$  with distribution proportional to the projection of the initial state on the eigenvectors of the evolution matrix and  $h(k, j) = \frac{D\lambda_j}{\lambda_j(k)}$  with  $\lambda_j$  the eigenvalues of the evolution matrix.

The concept of a fair Hadamard coin was generalised to a coin of higher order by taking tensor powers of Hadamard coins. The behaviour of the limiting random variable was investigated first for a 2-step QRW with Hadamard coin. The eigenvalues of the evolution matrix were calculated by a tensor product decomposition. The resulting moments were calculated for an initial position at position 0 with random coin state. The odd moments were found to be 0 as expected. The 2-step moments did not equal the 1-step moment, suggesting that a long-range QRW differs in distribution from the single-step QRW.

The same approach of the tensor product decomposition of the evolution matrix was applied to the general case of a  $s$ -step QRW. An alternative characterisation of the limiting random variable was identified, which consists of a sum over functions evaluated at a random point multiplied by a random sign. The limiting random variable was found to be different for each range, revealing yet another difference between the QRW and its classical counterpart. It appears that the limiting random variable of a QRW does depend on the single-step distribution, whereas it does not in the classical case.

The behaviour of this random variable was investigated for a long-range by rewriting a uniform random variable into a sum of Bernoulli distributed digits. This yielded an expression for the limiting random variable consisting of three dependent parts: A fastly decaying part, a non-convergent part and a convergent part. Histograms of the limiting random variable were created which were in agreement with the probability density for the single-step QRW. The histograms for longer ranges showed a decreased range and more smoothed peaks.

Finally, a starting point for the investigation of trapped QRW's was presented; a QRW on a 3-cycle with one trap. A calculation of the survival time showed that it scales with  $2^{-n}$ , just like its classical counterpart. Further research may be done to investigate the behaviour of survival time in more complex topologies.

## A Higher dimensions

Like the classical random walk a QRW can be extended into higher dimensions. This extension is the topic of the following section based on [4].

### A.1 Concept & limiting distribution

The one-dimensional QRW can be extended to an arbitrary number of dimensions by taking a state of the form

$$\psi = \begin{pmatrix} \psi(\mathbf{x}, d) \\ \vdots \\ \psi(\mathbf{x}, -d) \end{pmatrix}$$

for a  $d$ -dimensional walk. Here we use  $\psi(\cdot, i) \in \ell^2(\mathbb{Z}^d)$  with  $i \in \{-d, \dots, -1, 1, \dots, d\} = \sigma(d)$  to form a probability distribution on  $\mathbb{Z}^d$  given by

$$\mathbb{P}(\mathbf{x}) = \sum_{i \in \sigma(d)} \langle e_{\mathbf{x}} w_i, \psi \rangle = \sum_{i \in \sigma(d)} |\psi(\mathbf{x}, i)|^2$$

where  $\{e_{\mathbf{x}}\}_{\mathbf{x} \in \mathbb{Z}^d}$  are the usual basis elements of  $\ell^2(\mathbb{Z}^d)$  and  $\{w_i = |i\rangle\}_{i \in \sigma(d)}$  the basis elements of  $\mathbb{C}^{2d}$ . Note that the total state is an element of the space  $\ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^{2d}$ . Again,  $\ell^2(\mathbb{Z}^d)$  can be thought of as the position part and  $\mathbb{C}^{2d}$  as the coin state. We need  $2d$  components to allow for all  $2d$  steps in the directions of the unit vectors  $\{\mathbf{e}_i\}_{i \in \sigma(d)}$  where we mean  $\mathbf{e}_{-i} = -\mathbf{e}_i$  for positive  $i$ . In order to describe the dynamics for the  $d$ -dimensional walk it is convenient to write the state as

$$\sum_{i \in \sigma(d)} \sum_{\mathbf{x} \in \mathbb{Z}^d} \psi(\mathbf{x}, i) e_{\mathbf{x}} \otimes w_i.$$

The first part of the dynamics is a unitary operator  $A$  acting on the coin state. The second part is a shift in the direction of a unit vector, depending on the coin state. With  $\tau_m : \ell^2(\mathbb{Z}^d) \rightarrow \ell^2(\mathbb{Z}^d)$  we denote the shift to the right such that for  $p = \sum_{\mathbf{x} \in \mathbb{Z}^d} a_{\mathbf{x}} e_{\mathbf{x}}$  we have  $\tau_{\mathbf{m}} p = \sum_{\mathbf{x} \in \mathbb{Z}^d} a_{\mathbf{x}} e_{\mathbf{x} + \mathbf{m}}$ . We define the shift  $S : \ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^{2d} \rightarrow \ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^{2d}$  on the total state by:

$$S(p \otimes w_i) = \tau_{\mathbf{e}_i} p \otimes w_i$$

Again we define  $U$  by  $U = S(I \otimes A)$  with  $I$  the identity on  $\ell^2(\mathbb{Z}^d)$ , such that the time evolution becomes

$$\psi_n = U^n \psi_0$$

where  $\psi_0$  is the initial state. With this state at time  $n$  we associate a random vector  $\mathbf{X}_n$  taking values in  $\mathbb{Z}^d$  with probability as described above. As with the one-dimensional QRW we will perform calculations in the Fourier-transform space, which is now  $L^2(\mathbb{K}^d) = \widehat{\ell^2(\mathbb{Z}^d)}$ . For  $\psi(\mathbf{x}, i) \in \ell^2(\mathbb{Z}^d)$  we have  $\hat{\psi}(\mathbf{k}, i) = \sum_{\mathbf{x} \in \mathbb{Z}^d} e^{i\mathbf{k} \cdot \mathbf{x}} \psi(\mathbf{x}, i)$  and for the total state we have

$$\hat{\psi} = \begin{pmatrix} \hat{\psi}(\mathbf{k}, d) \\ \vdots \\ \hat{\psi}(\mathbf{k}, -d) \end{pmatrix}.$$

In the Fourier-transform space, the shift  $S$  becomes

$$\begin{aligned}
\widehat{(S\psi)}(\mathbf{k}) &= \begin{pmatrix} \widehat{\psi(\mathbf{x} - \mathbf{e}_d, d)} \\ \vdots \\ \widehat{\psi(\mathbf{x} + \mathbf{e}_d, -d)} \end{pmatrix} = \begin{pmatrix} \sum_{\mathbf{x} \in \mathbb{Z}^d} e^{i\mathbf{k} \cdot \mathbf{x}} \psi(\mathbf{x} - \mathbf{e}_d, d) \\ \vdots \\ \sum_{\mathbf{x} \in \mathbb{Z}^d} e^{i\mathbf{k} \cdot \mathbf{x}} \psi(\mathbf{x} + \mathbf{e}_d, -d) \end{pmatrix} = \begin{pmatrix} \sum_{\mathbf{x} \in \mathbb{Z}^d} e^{i\mathbf{k} \cdot (\mathbf{x} + \mathbf{e}_d)} \psi(\mathbf{x}, d) \\ \vdots \\ \sum_{\mathbf{x} \in \mathbb{Z}^d} e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{e}_d)} \psi(\mathbf{x}, -d) \end{pmatrix} \\
&= \begin{pmatrix} e^{i\mathbf{k} \cdot \mathbf{e}_d} \sum_{\mathbf{x} \in \mathbb{Z}^d} e^{i\mathbf{k} \cdot \mathbf{x}} \psi(\mathbf{x}, d) \\ \vdots \\ e^{-i\mathbf{k} \cdot \mathbf{e}_d} \sum_{\mathbf{x} \in \mathbb{Z}^d} e^{i\mathbf{k} \cdot \mathbf{x}} \psi(\mathbf{x}, -d) \end{pmatrix} = \begin{pmatrix} e^{i\mathbf{k} \cdot \mathbf{e}_d} \hat{\psi}(\mathbf{k}, d) \\ \vdots \\ e^{-i\mathbf{k} \cdot \mathbf{e}_d} \hat{\psi}(\mathbf{k}, -d) \end{pmatrix} \\
&= \begin{pmatrix} e^{i\mathbf{k} \cdot \mathbf{e}_d} & & \emptyset \\ & \ddots & \\ \emptyset & & e^{-i\mathbf{k} \cdot \mathbf{e}_d} \end{pmatrix} \hat{\psi}(\mathbf{k}).
\end{aligned}$$

Such that a step is described by

$$\widehat{U\psi} = \widehat{SA\psi} = \begin{pmatrix} e^{i\mathbf{k} \cdot \mathbf{e}_d} & & \emptyset \\ & \ddots & \\ \emptyset & & e^{-i\mathbf{k} \cdot \mathbf{e}_d} \end{pmatrix} A \hat{\psi}(\mathbf{k}) = \hat{U}(\mathbf{k}) \hat{\psi}(\mathbf{k}),$$

with

$$\hat{U}(\mathbf{k}) = \begin{pmatrix} e^{i\mathbf{k} \cdot \mathbf{e}_d} & & \emptyset \\ & \ddots & \\ \emptyset & & e^{-i\mathbf{k} \cdot \mathbf{e}_d} \end{pmatrix} A.$$

We denote by  $v_1(\mathbf{k}), \dots, v_{2d}(\mathbf{k})$  and  $\lambda_1(\mathbf{k}), \dots, \lambda_{2d}(\mathbf{k})$  the orthonormal eigenvectors and corresponding eigenvalues of  $\hat{U}(\mathbf{k})$ . This allows us to write the initial state  $\hat{\psi}_0(\mathbf{k})$  as

$$\hat{\psi}_0(\mathbf{k}) = \sum_{j=1}^{2d} \langle v_j(\mathbf{k}), \hat{\psi}_0(\mathbf{k}) \rangle v_j(\mathbf{k})$$

and the state at time  $n$  as

$$\hat{\psi}_n(\mathbf{k}) = \sum_{j=1}^{2d} \lambda_j^n(\mathbf{k}) \langle v_j(\mathbf{k}), \hat{\psi}_0(\mathbf{k}) \rangle v_j(\mathbf{k}).$$

By the method of moments we will show that each component  $X_{i,n}$  of the random vector  $\mathbf{X}_n$  converges in distribution to some random variable as  $n \rightarrow \infty$ . By definition the moments of  $X_{i,n}$  are given by

$$\mathbb{E}[X_{i,n}^r] = \langle \psi_n(\mathbf{x}), X_i^r \psi_n(\mathbf{x}) \rangle.$$

In order to use Parseval's theorem for the  $d$ -dimensional Fourier transform we need to find an appropriate expression for the inner product in  $\ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^{2d}$ . Let  $\psi, \phi \in \ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^{2d}$  and let us write them as  $\psi = \sum_{j \in \sigma(d)} \psi_j w_j$  and  $\phi = \sum_{k \in \sigma(d)} \phi_k w_k$ . The inner product can then be written as

$$\begin{aligned}
\langle \psi, \phi \rangle &= \left\langle \sum_{j \in \sigma(d)} \psi_j w_j, \sum_{k \in \sigma(d)} \phi_k w_k \right\rangle = \sum_{j \in \sigma(d)} \sum_{k \in \sigma(d)} \langle \psi_j w_j, \phi_k w_k \rangle \\
&= \sum_{j \in \sigma(d)} \sum_{k \in \sigma(d)} \langle \psi_j, \phi_k \rangle \langle w_j, w_k \rangle = \sum_{j \in \sigma(d)} \langle \psi_j, \phi_j \rangle.
\end{aligned}$$

At this point we can use Parseval's theorem to obtain

$$\langle \psi, \phi \rangle = \sum_{j \in \sigma(d)} \int_{\mathbb{K}^d} \widehat{\psi}_j \widehat{\phi}_j \frac{dk}{(2\pi)^d} = \int_{\mathbb{K}^d} \langle \widehat{\psi}, \widehat{\phi} \rangle_{\mathbb{C}^{2d}} \frac{dk}{(2\pi)^d}.$$

And the moments of  $\mathbb{E}[X_{i,n}^r]$  of  $X_{i,n}$  now read

$$\mathbb{E}[X_{i,n}^r] = \int_{\mathbb{K}^d} \langle \widehat{\psi}_n, \widehat{X_i^r \psi_n} \rangle_{\mathbb{C}^{2d}} \frac{dk}{(2\pi)^d}.$$

In order to obtain a more explicit expression, we take a closer look at  $\widehat{X_i^r \psi_n}$  or to begin with  $\widehat{X_i \psi_n}$ :

$$\begin{aligned} \widehat{X_i \psi_n} &= \sum_{\mathbf{x} \in \mathbb{Z}^d} x_i e^{i\mathbf{k} \cdot \mathbf{x}} = \sum_{\mathbf{x} \in \mathbb{Z}^d} x_i e^{i(k_1 x_1 + \dots + k_d x_d)} \\ &= -i \frac{d}{dk_i} \sum_{\mathbf{x} \in \mathbb{Z}^d} e^{i(k_1 x_1 + \dots + k_d x_d)} = -i \frac{d}{dk_i} \sum_{\mathbf{x} \in \mathbb{Z}^d} e^{i\mathbf{k} \cdot \mathbf{x}} \\ &= -i \frac{d}{dk_i} \widehat{\psi}_n \end{aligned}$$

Thus we find  $\widehat{X_i^r \psi_n} = D_i^r \widehat{\psi}_n$  with  $D_i = -i \frac{d}{dk_i}$ . Next we can calculate  $D_i^r \widehat{\psi}_n$  which yields

$$D_i^r \widehat{\psi}_n = \sum_{j=1}^{2d} n(n-1) \dots (n-r+1) \lambda_j(\mathbf{k})^{n-r} (D_i \lambda_j)^r \langle v_j(\mathbf{k}), \widehat{\psi}_0 \rangle v_j(\mathbf{k}) + \mathcal{O}(n^{r-1}),$$

and

$$\frac{D_i^r \widehat{\psi}_n}{n^r} = \sum_{j=1}^{2d} \lambda_j(\mathbf{k})^{n-r} (D_i \lambda_j)^r \langle v_j(\mathbf{k}), \widehat{\psi}_0 \rangle v_j(\mathbf{k}) + \mathcal{O}(n^{-1}).$$

We can use this result to calculate the moments  $\mathbb{E}[(X_{i,n}/n)^r]$  of  $X_{i,n}/n$ .

$$\begin{aligned} \mathbb{E} \left[ \left( \frac{X_{i,n}}{n} \right)^r \right] &= \int_{\mathbb{K}^d} \left\langle \widehat{\psi}_n, \frac{D_i^r \widehat{\psi}_n}{n^r} \right\rangle \frac{dk}{(2\pi)^d} \\ &= \int_{\mathbb{K}^d} \sum_{j=1}^{2d} \lambda_j(\mathbf{k})^{n-r} (D_i \lambda_j)^r \langle v_j(\mathbf{k}), \widehat{\psi}_0 \rangle \langle \widehat{\psi}_n, v_j(\mathbf{k}) \rangle \frac{dk}{(2\pi)^d} + \mathcal{O}(n^{-1}) \\ &= \int_{\mathbb{K}^d} \sum_{j=1}^{2d} \left( \frac{D_i \lambda_j(\mathbf{k})}{\lambda_j} \right)^r |\langle v_j(\mathbf{k}), \widehat{\psi}_0(\mathbf{k}) \rangle|^2 \frac{dk}{(2\pi)^d} + \mathcal{O}(n^{-1}) \end{aligned}$$

We recognise in this last expression the moments of a random variable. Indeed, take  $\Omega = \mathbb{K}^d \times \{1, 2, \dots, 2d\}$  and let  $\mu$  be the probability measure on  $\Omega$  defined by  $|\langle v_j(\mathbf{k}), \widehat{\psi}_0(\mathbf{k}) \rangle|^2 \frac{dk}{(2\pi)^d}$ . Now define  $h_i(\mathbf{k}, j) = \frac{D_i \lambda_j}{\lambda_j}$ . Then  $h_i : \mathbb{K} \times \{1, 2, \dots, 2d\} \rightarrow \mathbb{R}$  is a random variable.

With this interpretation we can write

$$\mathbb{E} \left[ \left( \frac{X_{i,n}}{n} \right)^r \right] \rightarrow \int_{\Omega} h_i^r d\mu \quad \text{as } n \rightarrow \infty.$$

And the method of moments guarantees that

$$\frac{X_{i,n}}{n} \xrightarrow{d} Y_i = h_i(Z),$$

with  $Z$  a random element of  $\Omega$  whose distribution is given by  $\mu$ . Here ‘ $\xrightarrow{d}$ ’ denotes convergence in distribution.

This result shows that the components  $X_{i,n}/n$  of the random vector  $\mathbf{X}_n/n$  converge in distribution to some random variable  $Y_i$  as  $n \rightarrow \infty$ . However, this does not necessarily mean that  $\mathbf{X}_n/n$  converges to the random vector  $\mathbf{Y}$  as  $n \rightarrow \infty$ . In order to prove that it does, we will use the theorem below.

**Theorem 4.** *Consider a sequence  $\mathbf{X}_n = (X_{1,n}, X_{2,n}, \dots, X_{d,n}), n \geq 1$ , of random vectors, and let  $\mathbf{Y} = (Y_1, Y_2, \dots, Y_d)$  be a random vector. If*

$$\sum_{j=1}^d c_j X_{j,n} \xrightarrow{d} \sum_{j=1}^d c_j Y_{j,n} \quad \text{as } n \rightarrow \infty$$

for all  $\mathbf{c} = (c_1, \dots, c_d) \in \mathbb{R}^d$ , then  $\mathbf{X}_n \xrightarrow{d} \mathbf{Y}$ .

We will show that the sequence  $\mathbf{X}_n$  of random vectors satisfies the conditions for the theorem above in the case of  $d = 2$ . Then, we will argue that the result extends to arbitrary  $d$ . As a first step we compute

$$\begin{aligned} \mathbb{E} \left[ \left( \frac{c_1 X_{1,n} + c_2 X_{2,n}}{n} \right)^r \right] &= \frac{1}{n^r} \sum_{p=0}^r \binom{n}{p} c_1^{r-p} c_2^p \mathbb{E}[X_{1,n}^{r-p} X_{2,n}^p] \\ &= \frac{1}{n^r} \sum_{p=0}^r \binom{n}{p} c_1^{r-p} c_2^p \langle \psi_n, X_1^{r-p} X_2^p \psi_n \rangle \\ &= \frac{1}{n^r} \sum_{p=0}^r \binom{n}{p} c_1^{r-p} c_2^p \int_{\mathbb{K}^2} \langle \hat{\psi}_n, D_1^{r-p} D_2^p \hat{\psi}_n \rangle \frac{dk}{(2\pi)^2}, \end{aligned}$$

and furthermore

$$\begin{aligned} D_2^p \hat{\psi}_n(\mathbf{k}) &= \sum_{j=1}^4 n^p \lambda_j(\mathbf{k})^{n-p} (D_2 \lambda_j)^p \langle v_j(\mathbf{k}), \hat{\psi}_0, \rangle v_j(\mathbf{k}) + \mathcal{O}(n^{p-1}) \\ D_1^{r-p} [D_2^p \hat{\psi}_n](\mathbf{k}) &= \sum_{j=1}^4 n^r \lambda_j(\mathbf{k})^{n-r} (D_1 \lambda_j)^{r-p} (D_2 \lambda_j)^p \langle v_j(\mathbf{k}), \hat{\psi}_0, \rangle v_j(\mathbf{k}) + \mathcal{O}(n^{r-1}). \end{aligned}$$

We can substitute this result to find

$$\begin{aligned}
& \mathbb{E} \left[ \left( \frac{c_1 X_{1,n} + c_2 X_{2,n}}{n} \right)^r \right] \rightarrow \\
& \frac{1}{n^r} \sum_{p=0}^r \binom{n}{p} c_1^{r-p} c_2^p \int_{\mathbb{K}^2} \langle \hat{\psi}_n, \sum_{j=1}^4 n^r \lambda_j(\mathbf{k})^{n-r} (D_1 \lambda_j)^{r-p} (D_2 \lambda_j)^p \langle v_j(\mathbf{k}), \hat{\psi}_0, \rangle v_j(\mathbf{k}) \rangle \frac{d\mathbf{k}}{(2\pi)^2} \\
& = \sum_{p=0}^r \binom{n}{p} c_1^{r-p} c_2^p \int_{\mathbb{K}^2} \sum_{j=1}^4 \lambda_j(\mathbf{k})^{-r} (D_1 \lambda_j)^{r-p} (D_2 \lambda_j)^p |\langle v_j(\mathbf{k}), \hat{\psi}_0, \rangle|^2 \frac{d\mathbf{k}}{(2\pi)^2} \\
& = \sum_{p=0}^r \binom{n}{p} c_1^{r-p} c_2^p \int_{\mathbb{K}^2} \sum_{j=1}^4 \left( \frac{D_1 \lambda_j}{\lambda_j} \right)^{r-p} \left( \frac{D_2 \lambda_j}{\lambda_j} \right)^p |\langle v_j(\mathbf{k}), \hat{\psi}_0, \rangle|^2 \frac{d\mathbf{k}}{(2\pi)^2} \\
& = \int_{\mathbb{K}^2} \sum_{j=1}^4 \left\{ \sum_{p=0}^r \binom{n}{p} c_1^{r-p} c_2^p h_1(\mathbf{k}, j)^{r-p} h_2(\mathbf{k}, j)^p \right\} |\langle v_j(\mathbf{k}), \hat{\psi}_0, \rangle|^2 \frac{d\mathbf{k}}{(2\pi)^2} \\
& = \int_{\mathbb{K}^2} \sum_{j=1}^4 \{c_1 h_1(\mathbf{k}, j) + c_2 h_2(\mathbf{k}, j)\}^r |\langle v_j(\mathbf{k}), \hat{\psi}_0, \rangle|^2 \frac{d\mathbf{k}}{(2\pi)^2} \\
& = \mathbb{E} [(c_1 Y_1 + c_2 Y_2)^r].
\end{aligned}$$

As before, the method of moments guarantees that

$$c_1 X_{1,n} + c_2 X_{2,n} \xrightarrow{d} c_1 Y_{1,n} + c_2 Y_{2,n} \quad \text{as } n \rightarrow \infty,$$

as desired. So indeed,  $\mathbf{X}_n/n$  converges in distribution to the random vector  $\mathbf{Y}$  as  $n \rightarrow \infty$ . The same argument extends to the case of  $d > 2$  where one can use the multinomial theorem instead of the binomial theorem to calculate the expectation  $\mathbb{E} [(c_1 X_{1,n} + \dots + c_d X_{d,n}/n)^r]$ .

## A.2 Hadamard coin in two dimensions

In this section we focus on the Hadamard coin of order 4 to illustrate the results of the previous section in the two dimensional case. Recall from Section 3.3 that the Hadamard coin of order 4 is given by

$$H_4 = H \otimes H = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

And the operator that evolves the walk is now  $\hat{U}_4 = \hat{S}H_4$  where

$$\hat{S} = \begin{pmatrix} e^{ik_2} & 0 & 0 & 0 \\ 0 & e^{ik_1} & 0 & 0 \\ 0 & 0 & e^{-ik_1} & 0 \\ 0 & 0 & 0 & e^{-ik_2} \end{pmatrix},$$

which can alternatively be written as

$$\hat{S} = \begin{pmatrix} e^{i(k_1+k_2)/2} & 0 \\ 0 & e^{-i(k_1+k_2)/2} \end{pmatrix} \otimes \begin{pmatrix} e^{i(k_2-k_1)/2} & 0 \\ 0 & e^{-i(k_2-k_1)/2} \end{pmatrix}.$$

This yields for the total evolution

$$\hat{U}_4 = \left( \left( \begin{array}{cc} e^{i(k_1+k_2)/2} & 0 \\ 0 & e^{-i(k_1+k_2)/2} \end{array} \right) \otimes \left( \begin{array}{cc} e^{i(k_2-k_1)/2} & 0 \\ 0 & e^{-i(k_2-k_1)/2} \end{array} \right) \right) (H \otimes H),$$

and by the mixed-product property

$$\hat{U}_4 = \left( \left( \begin{array}{cc} e^{i(k_1+k_2)/2} & 0 \\ 0 & e^{-i(k_1+k_2)/2} \end{array} \right) H \right) \otimes \left( \left( \begin{array}{cc} e^{i(k_2-k_1)/2} & 0 \\ 0 & e^{-i(k_2-k_1)/2} \end{array} \right) H \right).$$

If we now define  $k_+ = (k_1 + k_2)/2$  and  $k_- = (k_2 - k_1)/2$  we see that the evolution corresponds to  $\hat{U}_4 = \hat{U}(k_+) \otimes \hat{U}(k_-)$ . Its eigenvalues are products of the eigenvalues of  $\hat{U}(k_+)$  and  $\hat{U}(k_-)$  such that the corresponding random variable equals  $h_i(\mathbf{k}) = h_i(k_+) + h_i(k_-)$  with  $h_i(k_+) = \lambda(k_+)^{-1} D_i \lambda(k_+)$  and  $h_i(k_-) = \lambda(k_-)^{-1} D_i \lambda(k_-)$ . The calculation of  $h_i(k_+)$  is exactly the same as that of  $h(k)$  in the one dimensional QRW except for a factor  $\frac{1}{2}$  as result of the chain rule for  $(k_1 + k_2)/2$ .  $h_i(k_-)$  receives an additional factor  $(-1)^i$  due to the minus sign in  $(k_2 - k_1)/2$ . So in total

$$h_i(\mathbf{k}, j) = \pm \frac{\cos(k_+)}{2\sqrt{2 - \sin^2(k_+)}} \pm (-1)^i \frac{\cos(k_-)}{2\sqrt{2 - \sin^2(k_-)}}.$$

## B Proof of the method of moments

In Section 1.4 we have found the limiting distribution of  $X_n/n$  as  $n \rightarrow \infty$ . Here we have used that convergence of moments implies convergence in distribution (Theorem 1). In this section we proof this claim. We start with a weak formulation of this implication (Theorem 5) which is restricted to bounded random variables. After that we proceed towards the main result of the section, Theorem 1.

**Lemma 1.** *Let  $X$  be a random variable and  $X_n$  a sequence of random variables such that for each  $k \in \mathbb{N}$*

$$\mathbb{E}[X_n^k] \rightarrow \mathbb{E}[X^k],$$

*then if  $p$  is a polynomial:*

$$\mathbb{E}[p(X_n)] \rightarrow \mathbb{E}[p(X)]$$

*Proof.* Suppose  $\epsilon > 0$ . Let  $p(x) = a_1 x + \dots + a_m x^m$ . Define  $a = \max\{a_1, \dots, a_m\}$ . For each  $k$  we have  $\mathbb{E}[X_n^k] \rightarrow \mathbb{E}[X^k]$ , so

$$\forall k \exists n_k : n \geq n_k \implies |\mathbb{E}[X_n^k - X^k]| < \epsilon/ma.$$

Pick  $n_0 = \max\{n_1, \dots, n_m\}$ . Then  $\forall n \geq n_0$ :

$$\begin{aligned} |\mathbb{E}[p(X_n)] - \mathbb{E}[p(X)]| &= |\mathbb{E}[a_1(X_n - X) + \dots + a_m(X_n^m - X^m)]| \\ &= |a_1 \mathbb{E}[(X_n - X)] + \dots + a_m \mathbb{E}[(X_n^m - X^m)]| \\ &\leq a_1 |\mathbb{E}[(X_n - X)]| + \dots + a_m |\mathbb{E}[(X_n^m - X^m)]| \\ &\leq a |\mathbb{E}[(X_n - X)]| + \dots + a |\mathbb{E}[(X_n^m - X^m)]| \\ &\leq a\epsilon/ma + \dots + a\epsilon/ma = \epsilon \end{aligned}$$

Hence,

$$\mathbb{E}[p(X_n)] \rightarrow \mathbb{E}[p(X)]. \quad \square$$



**Theorem 5.** Let  $X$  be a random variable and  $X_n$  a sequence of random variables such that for each  $k \in \mathbb{N}$

$$\mathbb{E}[X_n^k] \rightarrow \mathbb{E}[X^k],$$

with  $X, X_n \in [a, b]$  for each  $n$ . then if  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  is bounded and continuous:

$$\mathbb{E}[\phi(X_n)] \rightarrow \mathbb{E}[\phi(X)]$$

*Proof.* Suppose  $\epsilon > 0$ . Let  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  be a bounded and continuous function. By the Stone-Weierstrass theorem [22] the polynomials are dense in the bounded and continuous functions on  $[a, b]$ :

$$\exists p : \|\phi - p\|_\infty < \epsilon/3.$$

Where  $\|\cdot\|_\infty$  is the supremum-norm on  $[a, b]$ . Lemma 1 guarantees that for this  $p$

$$\exists n_0 \in \mathbb{N} : n \geq n_0 \implies |\mathbb{E}[p(X_n)] - \mathbb{E}[p(X)]| < \epsilon/3.$$

Then  $\forall n \geq n_0$ :

$$\begin{aligned} |\mathbb{E}[\phi(X_n)] - \mathbb{E}[\phi(X)]| &= |\mathbb{E}[\phi(X_n)] - \mathbb{E}[p(X_n)] + \mathbb{E}[p(X_n)] - \mathbb{E}[p(X)] + \mathbb{E}[p(X)] - \mathbb{E}[\phi(X)]| \\ &= |\mathbb{E}[\phi(X_n) - p(X_n)] + \mathbb{E}[p(X_n) - p(X)] + \mathbb{E}[p(X) - \phi(X)]| \\ &\leq |\mathbb{E}[\phi(X_n) - p(X_n)]| + |\mathbb{E}[p(X_n) - p(X)]| + |\mathbb{E}[p(X) - \phi(X)]| \\ &\leq \|\phi - p\|_\infty + \epsilon/3 + \|\phi - p\|_\infty \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon \end{aligned}$$

Hence,

$$\mathbb{E}[\phi(X_n)] \rightarrow \mathbb{E}[\phi(X)]. \quad \square$$

Next, we extend Theorem 5 to Theorem 1, which does not require that  $X_n \in [a, b]$ .

**Definition 2.** A random variable  $X$  is called *tight*, if for every  $\epsilon > 0$  there exists a compact  $K \subset \mathbb{R}$  such that

$$\mathbb{P}(X \notin K) < \epsilon.$$

**Theorem 6** (Prokhorov [23]). If a sequence of random variables  $X_n$  is tight, then it has a subsequence  $X_{n_i}$  that converges in distribution to some random variable  $Y$ .

**Theorem 7** (Hausdorff [7]). Let  $X, Y$  be random variables of bounded support such that

$$\mathbb{E}[X^k] = \mathbb{E}[Y^k]$$

for each  $k \in \mathbb{N}$ . Then  $X \stackrel{d}{=} Y$ .

**Theorem 8.** Let  $X_n$  be a sequence of random variables that converges in distribution to  $Y$  and for each  $k \in \mathbb{N}$

$$\mathbb{E}[X_n^k] \rightarrow \mathbb{E}[X^k].$$

Then, if  $X$  is a bounded random variable <sup>1</sup>,  $X \stackrel{d}{=} Y$ .

---

<sup>1</sup>Bounded support is a sufficient condition, but it is not necessary. In fact we require  $X$  to be uniquely determined by its moments. More generally, this requirement is met when  $\exists t > 0 : \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathbb{E}[X^n] < \infty$  [24].

*Proof.* Let  $X$  be bounded. Denote by  $\mathbb{P}, \mathbb{P}_n$  and  $\mathbb{P}^*$  the probability measures of  $X, X_n$  and  $Y$  respectively. We will show that for each  $k \in \mathbb{N}$ ,  $\int x^k d\mathbb{P} = \int x^k d\mathbb{P}^*$ . That is  $X \stackrel{d}{=} Y$ . Pick  $k \in \mathbb{N}$ . Let  $f_k(x, M) : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a continuous function satisfying

$$f_k(x, M) \begin{cases} = x^k, & x \in [-M, M] \\ \leq x^k, & x \in [-M-1, M+1] \\ 0, & \text{otherwise.} \end{cases}$$

Let  $\epsilon > 0$  be given. By hypothesis,

$$\mathbb{E}[X_n^k] \rightarrow \mathbb{E}[X^k]$$

or

$$\int x^k d\mathbb{P}_n \rightarrow \int x^k d\mathbb{P}.$$

So there exists an  $n_0$  such that

$$\left| \int x^k d\mathbb{P} - \int x^k d\mathbb{P}_n \right| < \epsilon/3$$

for all  $n \geq n_0$ . Also, by Cauchy-Schwarz we find

$$\left| \int x^k d\mathbb{P}_n - \int f_k(x, M) d\mathbb{P}_n \right| \leq \mathbb{E}[X_n^k \mathbb{1}_{|X_n| > M}] \leq \sqrt{\mathbb{E}[X_n^{2k}]} \sqrt{\mathbb{P}(|X_n| > M)}.$$

Now it follows from Chebyshev's inequality that

$$\left| \int x^k d\mathbb{P}_n - \int f_k(x, M) d\mathbb{P}_n \right| \leq \sqrt{\mathbb{E}[X_n^{2k}]} \sqrt{\frac{\mathbb{E}[X_n^{2k}]}{M^2}} = \frac{\mathbb{E}[X_n^{2k}]}{M}.$$

But  $\mathbb{E}[X_n^{2k}]$  converges to  $\mathbb{E}[X^{2k}]$  and is therefore bounded, say by  $c$ . Thus

$$\left| \int x^k d\mathbb{P}_n - \int f_k(x, M) d\mathbb{P}_n \right| \leq \frac{c}{M}.$$

Now pick  $M_0 > 3c/\epsilon$ , then

$$\left| \int x^k d\mathbb{P}_n - \int f_k(x, M) d\mathbb{P}_n \right| \leq \epsilon/3.$$

Furthermore, because  $f_k$  is bounded and continuous, and  $X_n$  converges in distribution to  $Y$  we have some  $n_M$  such that

$$\left| \int f_k(x, M) d\mathbb{P}_n - \int f_k(x, M) d\mathbb{P}^* \right| < \epsilon/3$$

for all  $n \geq n_M$ . Let  $M > M_0$  and pick  $n \geq \max\{n_0, n_M\}$ , then

$$\begin{aligned} & \left| \int x^k d\mathbb{P} - \int f_k(x, M) d\mathbb{P}^* \right| \\ & \leq \left| \int x^k d\mathbb{P} - \int x^k d\mathbb{P}_n + \int x^k d\mathbb{P}_n - \int f_k(x, M) d\mathbb{P}_n + \int f_k(x, M) d\mathbb{P}_n - \int f_k(x, M) d\mathbb{P}^* \right| \\ & \leq \left| \int x^k d\mathbb{P} - \int x^k d\mathbb{P}_n \right| + \left| \int x^k d\mathbb{P}_n - \int f_k(x, M) d\mathbb{P}_n \right| + \left| \int f_k(x, M) d\mathbb{P}_n - \int f_k(x, M) d\mathbb{P}^* \right| \\ & \leq \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon. \end{aligned}$$

Hence

$$\lim_{M \rightarrow \infty} \int f_k(x, M) d\mathbb{P}^* = \int x^k d\mathbb{P}.$$

Also, by choice of  $f_k$  and the monotone convergence theorem

$$\lim_{M \rightarrow \infty} \int f_k(x, M) d\mathbb{P}^* = \int x^k d\mathbb{P}^*.$$

We conclude that  $\int x^k d\mathbb{P} = \int x^k d\mathbb{P}^*$ , and because  $X$  is bounded and therefore characterised by its moments (Theorem 7) we find that  $X \stackrel{d}{=} Y$ .  $\square$

**Lemma 2.** *If every subsequence  $a_{n_i}$  of a sequence  $a_n$  has a sub-subsequence  $a_{n_{i_j}}$  that converges to  $a$ , then  $a_n$  converges to  $a$ .*

*Proof.* The following is a proof by contradiction. Let  $a_n$  be a sequence such that each subsequence  $a_{n_i}$  has a sub-subsequence  $a_{n_{i_j}}$  that converges to  $a$ . Suppose  $a_n$  does not converge to  $a$ . Then there exists some  $\epsilon > 0$  such that

$$\forall m \exists n \geq m : |a_n - a| \geq \epsilon.$$

We construct a subsequence  $a_{n_i}$  of  $a_n$  as follows. Pick  $n_1$  such that  $|a_{n_1} - a| \geq \epsilon$ . Now pick  $n_2 \geq n_1 + 1 : |a_{n_2} - a| \geq \epsilon$ , and in general pick  $n_i \geq n_{i-1} + 1 : |a_{n_i} - a| \geq \epsilon$ . For each element of the subsequence we have  $|a_{n_i} - a| \geq \epsilon$ , so it does not have a sub-subsequence that converges to  $a$ . But this is a contradiction.  $\square$

**Theorem 1.** *Let  $X$  be a random variable of bounded support and  $X_n$  a sequence of random variables such that for each  $k \in \mathbb{N}$*

$$\mathbb{E}[X_n^k] \rightarrow \mathbb{E}[X^k].$$

*Then, if  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  is bounded and continuous:*

$$\mathbb{E}[\phi(X_n)] \rightarrow \mathbb{E}[\phi(X)]$$

*Proof.* Let  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  be a bounded and continuous function and  $X_{n_i}$  a subsequence of  $X_n$ . By Chebyshev's inequality we have for each  $i$ , and arbitrary  $A > 0$

$$\mathbb{P}(X_{n_i} \notin [-A, A]) \leq \frac{\mathbb{E}[X_{n_i}^2]}{A^2}.$$

But the subsequence  $\mathbb{E}[X_{n_i}^2]$  converges to  $\mathbb{E}[X^2]$  and is therefore bounded, say by  $c$ , hence

$$\mathbb{P}(X_{n_i} \notin [-A, A]) \leq \frac{c}{A^2}.$$

Given an  $\epsilon > 0$  we can pick  $A > \sqrt{c/\epsilon}$ , such that

$$\mathbb{P}(X_{n_i} \notin [-A, A]) < \epsilon,$$

thus the subsequence  $X_{n_i}$  is tight. By Prokhorov's theorem we now find a sub-subsequence  $X_{n_{i_j}}$  of  $X_{n_i}$  that converges in distribution to some random variable  $Y$ . But since this sub-subsequence is a subsequence of  $X_n$ , we have for each  $k \in \mathbb{N}$

$$\mathbb{E}[X_{n_{i_j}}^k] \rightarrow \mathbb{E}[X^k].$$

Now it follows from Theorem 8 that  $X \stackrel{d}{=} Y$ , hence  $X_{n_{i_j}}$  converges in distribution to  $X$ . But the subsequence  $X_{n_i}$  was arbitrary, thus every subsequence of  $X_n$  has a sub-subsequence  $X_{n_{i_j}}$  that converges in distribution to  $X$ , so

$$\mathbb{E}[\phi(X_{n_{i_j}})] \rightarrow \mathbb{E}[\phi(X)].$$

By Lemma 2 we find that

$$\mathbb{E}[\phi(X_n)] \rightarrow \mathbb{E}[\phi(X)].$$

Hence  $X_n$  converges in distribution to  $X$ . □

## C Derivations and calculations

### C.1 Degenerate eigenvalues

In Section 1.3 we stated that the degenerate eigenvalues of the evolution operator in the Fourier transform space lead to a trivial evolution. To show that this, we can use the eigenvectors of  $\hat{U}$  as a basis to write

$$\begin{aligned} \hat{U}(k)\psi &= \hat{U}(k)(\langle \hat{\psi}, v_1 \rangle v_1 + \langle \hat{\psi}, v_2 \rangle v_2) \\ &= \lambda_1(k)\langle \hat{\psi}, v_1 \rangle v_1 + \lambda_2(k)\langle \hat{\psi}, v_2 \rangle v_2 = \lambda(k)\hat{\psi}. \end{aligned}$$

So  $\hat{U}$  is diagonal:

$$\hat{U} = \begin{pmatrix} \lambda(k) & 0 \\ 0 & \lambda(k) \end{pmatrix}.$$

Therefore  $A$  must be diagonal as well. With a diagonal coin operator there is no interference between the two components of the state. Meaning that starting from an initial state  $\psi_0 = p \otimes |1\rangle + q \otimes |-1\rangle$  the  $n$ -th state becomes  $\psi_n = a^n \tau_n p \otimes |1\rangle + b^n \tau_{-n} q \otimes |-1\rangle$  with  $|a| = |b| = 1$ . So a diagonal coin is totally biased in the sense that evolution with a diagonal coin simply shifts the  $|1\rangle$  and  $|-1\rangle$  parts right and left respectively (up to a phase factor).

### C.2 Inner product relation

Here we derive the following inner product relation used in Section 1.4:

$$\langle \psi, \phi \rangle_{\ell^2(\mathbb{Z}) \otimes \mathbb{C}^2} = \int_0^{2\pi} \langle \hat{\psi}, \hat{\phi} \rangle_{\mathbb{C}^2} \frac{dk}{2\pi}.$$

Let us first rewrite the state  $\psi$  as

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_{-1} \end{pmatrix},$$

where the subscript 1 or  $-1$  refers to the coin state instead of time to avoid overcrowded notation. For  $\psi, \phi \in \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$  we now have

$$\begin{aligned}
\langle \psi, \phi \rangle_{\ell^2(\mathbb{Z}) \otimes \mathbb{C}^2} &= \langle \psi_1 w_1 + \psi_{-1} w_{-1}, \phi_1 w_1 + \phi_{-1} w_{-1} \rangle_{\ell^2(\mathbb{Z}) \otimes \mathbb{C}^2} \\
&= \langle \psi_1 w_1, \phi_1 w_1 \rangle_{\ell^2(\mathbb{Z}) \otimes \mathbb{C}^2} + \langle \psi_1 w_1, \phi_{-1} w_{-1} \rangle_{\ell^2(\mathbb{Z}) \otimes \mathbb{C}^2} \\
&\quad + \langle \psi_{-1} w_{-1}, \phi_1 w_1 \rangle_{\ell^2(\mathbb{Z}) \otimes \mathbb{C}^2} + \langle \psi_{-1} w_{-1}, \phi_{-1} w_{-1} \rangle_{\ell^2(\mathbb{Z}) \otimes \mathbb{C}^2} \\
&= \langle \psi_1, \phi_1 \rangle_{\ell^2(\mathbb{Z})} \langle w_1, w_1 \rangle_{\mathbb{C}^2} + \langle \psi_1, \phi_{-1} \rangle_{\ell^2(\mathbb{Z})} \langle w_1, w_{-1} \rangle_{\mathbb{C}^2} \\
&\quad + \langle \psi_{-1}, \phi_1 \rangle_{\ell^2(\mathbb{Z})} \langle w_{-1}, w_1 \rangle_{\mathbb{C}^2} + \langle \psi_{-1}, \phi_{-1} \rangle_{\ell^2(\mathbb{Z})} \langle w_{-1}, w_{-1} \rangle_{\mathbb{C}^2} \\
&= \langle \psi_1, \phi_1 \rangle_{\ell^2(\mathbb{Z})} + \langle \psi_{-1}, \phi_{-1} \rangle_{\ell^2(\mathbb{Z})}.
\end{aligned}$$

We can now use Parseval's theorem to write

$$\begin{aligned}
\langle \psi, \phi \rangle_{\ell^2(\mathbb{Z}) \otimes \mathbb{C}^2} &= \int_0^{2\pi} \bar{\psi}_1 \hat{\phi}_1 \frac{dk}{2\pi} + \int_0^{2\pi} \bar{\psi}_{-1} \hat{\phi}_{-1} \frac{dk}{2\pi} \\
&= \int_0^{2\pi} \bar{\psi}_1 \hat{\phi}_1 + \bar{\psi}_{-1} \hat{\phi}_{-1} \frac{dk}{2\pi}.
\end{aligned}$$

We conclude

$$\langle \psi, \phi \rangle_{\ell^2(\mathbb{Z}) \otimes \mathbb{C}^2} = \int_0^{2\pi} \langle \hat{\psi}, \hat{\phi} \rangle_{\mathbb{C}^2} \frac{dk}{2\pi}.$$

### C.3 Leibniz rule

In Section 1.4 we used an expression for  $D^r \hat{\psi}_n$  which we prove here.

$$\begin{aligned}
D^r \hat{\psi}_n &= (-i)^r \frac{d^r}{dk^r} \hat{\psi}_n = (-i)^r \frac{d^r}{dk^r} \sum_{j=1}^2 \lambda_j(k)^n \langle v_j(k), \hat{\psi}_0 \rangle v_j(k) \\
&= (-i)^r \sum_{j=1}^2 \langle v_j(k), \hat{\psi}_0 \rangle \frac{d^r}{dk^r} (\lambda_j^n v_j)(k)
\end{aligned}$$

Using the Leibniz rule we can further write  $\frac{d^r}{dk^r} (\lambda_j^n v_j)(k)$  as

$$\frac{d^r}{dk^r} (\lambda_j^n v_j)(k) = \sum_{s=0}^r \binom{r}{s} \frac{d^{r-s}}{dk^{r-s}} (\lambda_j^n)(k) \frac{d^s}{dk^s} (v_j)(k).$$

Note that the term  $\frac{d^{r-s}}{dk^{r-s}} (\lambda_j^n)(k)$  is a power series in  $n$  for fixed  $k$ . As a consequence  $\frac{d^r}{dk^r} (\lambda_j^n v_j)(k)$  is a power series in  $n$  for fixed  $k$  as well, and then so is  $D^r \hat{\psi}_n$ . The term for  $s = 0$  contains the leading term in the power series  $\frac{d^r}{dk^r} (\lambda_j^n v_j)(k)$ :

$$\begin{aligned}
\frac{d^r}{dk^r} (\lambda_j^n v_j)(k) &= n(n-1) \dots (n-r+1) \lambda_j(k)^{n-r} \frac{d^r \lambda_j}{dk^r} v_j(k) + \mathcal{O}(n^{r-1}) \\
&= n^r \lambda_j(k)^{n-r} \frac{d^r \lambda_j}{dk^r} v_j(k) + \mathcal{O}(n^{r-1}).
\end{aligned}$$

Now we substitute this result and find

$$D^r \hat{\psi}_n = (-i)^r \sum_{j=1}^2 \langle v_j(k), \hat{\psi}_0 \rangle n^r \lambda_j(k)^{n-r} \frac{d^r \lambda_j}{dk^r} v_j(k) + \mathcal{O}(n^{r-1}).$$

## C.4 Division

In Section 1.5 we used the result

$$h_j(k) = \pm \frac{\cos(k)}{\sqrt{2 - \sin^2(k)}},$$

which we prove here. To evaluate this fraction we make use of the identity

$$\frac{a + bi}{c + di} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i,$$

with

$$a = \frac{\cos(k)}{\sqrt{2}}, \quad b = \pm \frac{\sin(k) \cos(k)}{\sqrt{4 - 2 \sin^2(k)}}, \quad c = \pm \sqrt{1 - 1/2 \sin^2(k)}, \quad d = \frac{\sin(k)}{\sqrt{2}}.$$

The denominator reduces to

$$c^2 + d^2 = 1 - \frac{1}{2} \sin^2(k) + \frac{1}{2} \sin^2(k) = 1.$$

So the real part equals

$$\begin{aligned} ac + bd &= \pm \frac{1}{2} \cos(k) \sqrt{2 - \sin^2(k)} \pm \frac{\sin^2(k) \cos(k)}{2\sqrt{2 - \sin^2(k)}} \\ &= \pm \frac{1}{\sqrt{2 - \sin^2(k)}} \left[ \cos(k) - \frac{1}{2} \sin^2(k) \cos(k) + \frac{1}{2} \sin^2(k) \cos(k) \right] \\ &= \pm \frac{\cos(k)}{\sqrt{2 - \sin^2(k)}}. \end{aligned}$$

The imaginary part yields

$$bc - ad = \frac{1}{2} \sin(k) \cos(k) - \frac{1}{2} \sin(k) \cos(k) = 0,$$

hence

$$h_j(k) = \pm \frac{\cos(k)}{\sqrt{2 - \sin^2(k)}}.$$

## C.5 Biased coin

Here we repeat the calculation of the limiting probability density for a biased coin

$$A = \begin{pmatrix} \sqrt{\rho} & \sqrt{1 - \rho} \\ \sqrt{1 - \rho} & -\sqrt{\rho} \end{pmatrix}.$$

In this case we have

$$\hat{U}(k) = \begin{pmatrix} e^{ik} & 0 \\ 0 & e^{-ik} \end{pmatrix} A = \begin{pmatrix} e^{ik} & 0 \\ 0 & e^{-ik} \end{pmatrix} \begin{pmatrix} \sqrt{\rho} & \sqrt{1 - \rho} \\ \sqrt{1 - \rho} & -\sqrt{\rho} \end{pmatrix} = \begin{pmatrix} e^{ik} \sqrt{\rho} & e^{ik} \sqrt{1 - \rho} \\ e^{-ik} \sqrt{1 - \rho} & -e^{-ik} \sqrt{\rho} \end{pmatrix}.$$

It's eigenvalues are given by

$$\begin{aligned} \begin{vmatrix} e^{ik}\sqrt{\rho} - \lambda & e^{ik}\sqrt{1-\rho} \\ e^{-ik}\sqrt{1-\rho} & -e^{-ik}\sqrt{\rho} - \lambda \end{vmatrix} &= (e^{ik}\sqrt{\rho} - \lambda)(-e^{-ik}\sqrt{\rho} - \lambda) - 1 + \rho = 0 \\ &-\rho - \lambda e^{ik}\sqrt{\rho} + \lambda e^{-ik}\sqrt{\rho} + \lambda^2 - 1 + \rho = 0 \\ &\lambda^2 - 2i\sqrt{\rho}\sin(k)\lambda - 1 = 0 \\ \implies \lambda_j(k) &= i\sqrt{\rho}\sin(k) \pm \sqrt{1 - \rho\sin^2(k)}. \end{aligned}$$

Furthermore,

$$D\lambda_j = -i\frac{d}{dk}\lambda_j(k) = \sqrt{\rho}\cos(k) \pm \frac{\rho\sin(k)\cos(k)}{\sqrt{1-\rho\sin^2(k)}}i.$$

The random variable  $h$  now equals

$$h(j, k) = \frac{D\lambda_j}{\lambda_j(k)}.$$

To evaluate this fraction we make use of the identity

$$\frac{a + bi}{c + di} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i,$$

with

$$a = \sqrt{\rho}\cos(k), \quad b = \pm \frac{\rho\sin(k)\cos(k)}{\sqrt{1-\rho\sin^2(k)}}, \quad c = \pm\sqrt{1-\rho\sin^2(k)}, \quad d = \sqrt{\rho}\sin(k).$$

The denominator reduces to

$$c^2 + d^2 = 1 - \rho\sin^2(k) + \rho\sin^2(k) = 1.$$

So the real part equals

$$\begin{aligned} ac + bd &= \pm\sqrt{\rho}\cos(k)\sqrt{1-\rho\sin^2(k)} \pm \sqrt{\rho}\frac{\rho\sin^2(k)\cos(k)}{\sqrt{1-\rho\sin^2(k)}} \\ &= \pm\frac{\sqrt{\rho}}{\sqrt{1-\rho\sin^2(k)}} [(1-\rho\sin^2(k))\cos(k) + \rho\sin^2(k)\cos(k)] \\ &= \pm\frac{\cos(k)}{\sqrt{\rho^{-1} - \sin^2(k)}}. \end{aligned}$$

The imaginary part yields

$$bc - ad = \rho\sin(k)\cos(k) - \rho\sin(k)\cos(k) = 0,$$

hence

$$h_j(k) = \pm\frac{\cos(k)}{\sqrt{\rho^{-1} - \sin^2(k)}}.$$

Again we calculate the cumulative density function for the random initial state:

$$\begin{aligned}\mathbb{P}(Y \leq y) &= \int_{h^{-1}([-\infty, y])} d\mu = 2 \int_{\cos(k)/\sqrt{\rho^{-1}-1+\cos^2(k)} \leq y} \frac{dk}{4\pi} \\ &= \int_{\cos^{-1}(\sqrt{\rho^{-1}-1}/\sqrt{1-y^2})}^{2\pi - \cos^{-1}(\sqrt{\rho^{-1}-1}y/\sqrt{1-y^2})} \frac{dk}{2\pi} \\ &= 1 - \frac{1}{\pi} \cos^{-1} \left( \frac{\sqrt{\rho^{-1}-1}y}{\sqrt{1-y^2}} \right).\end{aligned}$$

Finally we can take the derivative with respect to  $y$  to obtain the probability density function:

$$\frac{d}{dy} \left[ 1 - \frac{1}{\pi} \cos^{-1} \left( \frac{\sqrt{\rho^{-1}-1}y}{\sqrt{1-y^2}} \right) \right] = -\frac{1}{\pi} \frac{d}{du} \cos^{-1}(u) \frac{d}{dy} \left( \frac{\sqrt{\rho^{-1}-1}y}{\sqrt{1-y^2}} \right)$$

where  $u = \frac{\sqrt{\rho^{-1}-1}y}{\sqrt{1-y^2}}$ . So

$$f(y) = \frac{1}{\pi} \frac{1}{\sqrt{1 - \frac{(\rho^{-1}-1)y^2}{1-y^2}}} \frac{\sqrt{\rho^{-1}-1}}{(1-y^2)^{3/2}} dy = \frac{\sqrt{1-\rho} dy}{\pi \sqrt{\rho - y^2(1-y^2)}}.$$

## C.6 Rotating frame

Here we derive the rotating frame approach used in Section 2.2. We consider a Hamiltonian acting on a two-level system consisting of a constant part  $H_0$  and a time-dependent part  $V(t)$ :

$$H(t) = H_0 + V(t).$$

Furthermore, we consider for this Hamiltonian its evolution operator  $U(t)$  defined by  $|\psi(t)\rangle = U(t)|\psi(0)\rangle$  or equivalently  $i\hbar\dot{U}(t) = H(t)U(t)$ . To see that these formulations are indeed equivalent simply substitute either in the Schrödinger equation. Next write  $U(t)$  as the product of some rotation  $R(t)$  and a remaining part  $U'(t)$ , so  $U(t) = R(t)U'(t)$ . We substitute this representation of  $U(t)$  into  $i\hbar\dot{U}(t) = H(t)U(t)$  to find

$$i\hbar\dot{R}(t)U' + i\hbar R(t)\dot{U}' = H_0R(t)U' + V(t)R(t)U'.$$

We then pick  $R(t)$  such that  $i\hbar\dot{R}(t) = H_0R(t)$  which cancels to terms of the equation. This differential equation is solved by  $R(t) = e^{-iH_0t/\hbar}$ . We are left with

$$i\hbar R(t)\dot{U}'(t) = V(t)R(t)U'(t),$$

and after multiplication from the left with  $R^\dagger(t)$ :

$$i\hbar\dot{U}'(t) = R^\dagger(t)V(t)R(t)U'(t).$$

Note that this equation defines  $U'(t)$  as the evolution operator of the Hamiltonian  $H_{rot} := R^\dagger(t)V(t)R(t)$  which we will call the Hamiltonian in the rotated frame. Denote by  $|\phi(t)\rangle$  the wave-function corresponding to this Hamiltonian with  $|\phi(0)\rangle = |\psi(0)\rangle$ . Then

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle = R(t)U'(t)|\phi(0)\rangle = R(t)|\phi(t)\rangle.$$

So to find  $|\psi(t)\rangle$  we need to solve for the Hamiltonian in the rotated frame and then apply the rotation  $R(t)$ .



## C.7 Division for long range

In Section 3.4 we stated that the random variables  $h_j(k)$  related to matrices of the form

$$\frac{1}{\sqrt{2}} \begin{pmatrix} e^{aik} & e^{aik} \\ e^{aik} & -e^{-aik} \end{pmatrix}$$

are given by

$$h_j(k) = \pm \frac{a \cos(ak)}{\sqrt{2 - \sin^2(ak)}}.$$

Here we show this calculation. Again we use that

$$\frac{\alpha + \beta i}{\gamma + \delta i} = \frac{\alpha\gamma + \beta\delta}{\gamma^2 + \delta^2} + \frac{\beta\gamma - \alpha\delta}{\gamma^2 + \delta^2} i$$

with

$$\alpha = a \frac{\cos(ak)}{\sqrt{2}}, \quad \beta = \pm a \frac{\sin(ak) \cos(ak)}{\sqrt{4 - 2 \sin^2(ak)}}, \quad \gamma = \pm \sqrt{1 - \sin^2(ak)/2}, \quad \delta = \frac{\sin(ak)}{\sqrt{2}}.$$

The denominator reduces to

$$\gamma^2 + \delta^2 = 1 - \frac{1}{2} \sin^2(ak) + \frac{1}{2} \sin^2(ak) = 1,$$

and the imaginary part yields

$$\beta\gamma - \alpha\delta = a \frac{\sin(ak) \cos(ak)}{2} - a \frac{\sin(ak) \cos(ak)}{2} = 0.$$

Therefore all that is left is

$$\begin{aligned} \alpha\gamma + \beta\delta &= \frac{a}{2} \cos(ak) \sqrt{2 - \sin^2(ak)} \pm \frac{a \sin^2(ak) \cos(ak)}{2 \sqrt{2 - \sin^2(ak)}} \\ &= \pm \frac{a}{\sqrt{2 - \sin^2(ak)}} \left[ \cos(ak) \left( 1 - \frac{\sin^2(ak)}{2} \right) \pm \frac{1}{2} \sin^2(ak) \cos(ak) \right] \\ h_j(k) &= \pm \frac{a \cos(ak)}{\sqrt{2 - \sin^2(ak)}}. \end{aligned}$$

## C.8 Scale correction

Here we show that the appropriate scale correction of the classical random walk taking steps equally distributed in  $\{-s, \dots, -1, 1, \dots, s\} = \{-2^{m-1}, \dots, -1, 1, \dots, 2^{m-1}\}$  is of leading order  $\sqrt{2^m}$ . We consider the random variable  $X_n^{(m)} = \sum_{i=1}^n A_i^{(m)}$  where the  $A_i$ 's are iid random variables taking values in  $\{-2^{m-1}, \dots, -1, 1, \dots, 2^{m-1}\}$  uniformly. The central limit theorem [5] guarantees that

$$\frac{X_n^{(m)}}{\sqrt{n \text{Var}[A_i]}} \rightarrow \mathcal{N}(0, 1), \quad \text{as } n \rightarrow \infty.$$

We will show that  $\sqrt{\text{Var}[A_i]}$  is indeed of leading order  $\sqrt{2^m}$ .

$$\text{Var}[A_i] = \mathbb{E}[A_i^2] = \frac{2}{2^m} \sum_{i=0}^{m-1} 4^i = 2^{-(m-1)} \frac{1 - 4^m}{1 - 4} = \mathcal{O}(2^{2m} 2^{-m}) = \mathcal{O}(2^m)$$

Hence  $\sqrt{\text{Var}[A_i]} = \mathcal{O}(\sqrt{2^m})$ .

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