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Ergodic theory and hydrodynamic limit for run-and-tumble particle processes

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"Ergodic theory and hydrodynamic limit for run-and-tumble particle processes"

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Abstract

In this thesis we will study the ergodic measures and the hydrodynamic limit of independent run-and-tumble particle processes, i.e., an interacting particle system for particles with an internal energy source, which makes them move in a preferred direction that changes at random times. We start by providing some basic concepts and theory of Markov processes and interacting particle systems. Afterwards, we define our model on the particle state space $\mathbb{Z}^d \times S$, with S a finite space of internal states, by giving its generator, and we prove a duality result with a similar process which we will use repeatedly throughout this thesis. Then we show that the product Poisson measures with constant parameter are ergodic, and are also the only ergodic probability measures for this process in the space of so-called tempered measures, i.e., measures with bounded factorial moments. Lastly we prove the hydrodynamic limit of this process on $\mathbb{Z} \times S$ by showing that the evolution of the macroscopic density is a weak solution to a PDE.

Preface

First of all, as agreed, I would like to thank my parents for their endless support throughout my studies and personal life. I could not have asked for better parents.

With this report, I have finished the master Applied Mathematics at the University of Technology in Delft. When I had to make a choice of study, I went for mathematics because I was good at it at the time. I enjoyed the first two years of my bachelor, apart from the courses in probability theory and statistics. It was in my third year, when I studied a semester abroad in Australia and the only courses available were those in probability and statistics, that I started to appreciate the measure theoretic approach towards probability theory. From that point on, I decided this was the direction I wanted to take.

At the beginning of my master's, Prof. Redig gave a presentation about the track Applied Probability and spend some time talking about interacting particle systems. I was immediately intrigued, so I went to talk with him about this subject and together we made a list of suitable courses to take. Two years later I asked him to be my supervisor for my master thesis, and this is when he introduced me to run-and-tumble particles.

I would like to thank Prof. Redig for his enthusiasm, knowledge and sense of humor during his time as my supervisor. In addition, I would like to thank him for helping me with my quest for a PhD position, and ultimately, when one became available, giving me the chance to further help him with his research.

Furthermore, I would like to thank the rest of my thesis committee for their eagerness to join, my colleagues at Stanislas for making the combination of work and studies feasible, and my friends, family and boyfriend for reminding me there is more to life than just mathematics.

I hope you enjoy reading this thesis.

Hidde van Wiechen

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Contents

Chapter 1

Introduction

In this thesis we will study the dynamics and hydrodynamic limits of independent run-and-tumble particle processes. The particles in this process, which are also referred to as active particles, follow a motion consisting of two components.

The first component is a symmetric random walk. This component models a "passive" diffusion caused by random collisions with surrounding molecules.

The second component is a motion in a direction that is determined by the internal state of the particle. We will refer to this component as the active part of the motion, and it is what causes the particle to "run". However, the internal state of a particle will also change at random times, causing the particle to "tumble" towards a new direction.

Model

Here we will discuss the general model of a run-and-tumble particle process considered in this thesis. Let $V := G \times S$, with G a countable set and S a finite set. We will view G as the space of possible positions of a particle and S as the internal state space. Furthermore, let $T : V \to G$ be a function. This value of this function $T(x, \sigma)$ is the preferred direction of a particle at position $x \in G$ with internal state $\sigma \in S$, i.e., the active part of the motion sends a particle at (x, σ) to $(T(x, \sigma), \sigma)$.

A *run-and-tumble particle* is a Markov process of the form $\{(X_t, \sigma_t)\}_{t \geq 0}$ on V with the following three types of transitions.

1. With rate κ, the position performs a symmetric random walk on G
2. With rate λ, the particle jumps in its preferred direction according to the function T.
3. With rate γ, the internal state changes uniformly on S. (Generalizations are possible)

In this report, we will look at multiple (possibly infinitely many) run-and-tumble particles, all moving independently on the space of positions $G = \mathbb{Z}^d$. on $\mathbb{Z}^d$ we can actually define a "direction" in which a particle moves with a certain "velocity" by taking $S \subset \mathbb{Z}^d$ a finite subset and letting $T(x, \sigma) = (x + \sigma, \sigma)$. A simple case for this is by taking $V = \mathbb{Z} \times \{-1, 1\}$. Then the active motion of the particle equals moving one space to the right or left if its internal state is 1 or -1 respectively.

Motivation

A run-and-tumble particle system falls in the category of a larger area, that both physicists and biologists have taken an interest in, called *active matter*. The theory of active matter is about the motion of organisms with an energy source, which is able to use this energy to move in a certain direction [20]. Examples of active matter include flocks of birds, schools of fish, but also a human crowd [6]. However, the dynamics of run-and-tumble particles were inspired by certain bacteria, such as *E. coli*, which are able to perform a self-propelled motion [27].

Another example of objects where the run-and-tumble dynamics are observed is in *motor proteins* [11]. These “motors”, which can be found in any animal body, are involved in the transport of proteins and vesicles. By converting energy into motion, they move along the filaments of the cytoskeleton, which are long, interconnected strings. At any crossroad, the motor protein can switch from string to string, changing its direction.

Lately, active matter and run-and-tumble particles are being studied extensively. Examples of recent articles include Soto and Golestanian, where mutually excluding run-and-tumble particles were being considered [25], and Le Doussal et al., where they studied two independent run-and-tumble particles [17]. However, not a lot is known about systems of infinitely many run-and-tumble particles moving independently, hence they are the subject of this thesis.

Duality

Duality is a term that appears in many branches of mathematics. It is usually a way to reformulate a problem into a different (“dual”) problem that is either easier to solve or helps to better understand the original problem. In the theory of interacting particle systems, or more generally the theory of Markov processes, duality of a process $\{\eta_t, t \geq 0\}$, with initial value η, consists of a dual process $\{\xi_t, t \geq 0\}$, with initial value ξ, and a duality function D such that

$$\mathbb{E}\left[D(\xi, \eta_t)\right] = \mathbb{E}\left[D(\xi_t, \eta)\right].$$

In Chapter 3, we will prove a duality result for the run-and-tumble particle system, which we will use repeatedly throughout the rest of this thesis, where the dual process only consists of finitely many particles. We will prove this result in two different ways; firstly through straight-forward calculations, by applying the generators of the processes to the duality function. Secondly we introduce a so-called deterministic system, which corresponds to a Markov process, and show that there is a duality result between the run-and-tumble particle system and its deterministic system. Afterwards, we will show that this result is equivalent to the duality result which we are after.

Ergodic theory

In the study of dynamical systems, ergodic theory looks at average asymptotic behavior. The notion of ergodicity began with the Boltzmann hypothesis, stating that “for large systems of interacting particles in equilibrium, the time average along a single trajectory equals the space average” [8]. It turned out that this hypothesis was false, and conditions on the measure of the space were needed. The measures for which the hypothesis was true were called the *ergodic measures*.

It turned out that the ergodic measures also play a strong part in the structure of the invariant measures of a process, i.e., the measures that are preserved under the process. Namely, the set of invariant measures is a convex set with the ergodic measures as extremal points, as we will see in Section 2.2. Therefore, knowing all the ergodic measures can tell a lot about the invariant measures.

In Chapter 4 we will show that the product Poisson measures with constant parameter are ergodic and we show that in a certain subspace of probability measures, called the tempered measures, these are the only ergodic measures. To prove this, we use the method of Kuoch and Redig in [16] where they prove a similar result for the Symmetric Inclusion Process. In their proof, they needed that there is a successful coupling starting from any two configurations η and ξ, i.e. there exists a joint process $\{(\xi_t; \eta_t), t \geq 0\}$ such that ξ_t and η_t eventually coincide. Therefore, we will look at a successful coupling of two configurations of run-and-tumble particles in Section 4.2.

Hydrodynamic limit

The hydrodynamic limit focusses on the "macro scale" of particle densities instead of the "micro scale" of particle configurations, i.e., instead of looking at all the particles individually, we perform a rescaling of the particle state space with a factor $\frac{1}{N}$, and turn it into a continuum by taking N to infinity. This way, due to some sort of extended version of the law of large numbers, we end up with a macroscopic density. After this rescaling of space, we can ask ourselves how this density evolves under the run-and-tumble dynamics. In order to let these dynamics have an effect on the macro-scale, we will also need a rescaling of time with a factor N^2. However, this will will speed up some parts of the process too much, and we will have to perform the following change of parameters (which we will motivate later); $\lambda \to \frac{\lambda}{N}$ and $\gamma \to \frac{\gamma}{N^2}$. The evolution of the density under these dynamics is what we call the hydrodynamic limit of the system.

In Section 5, we set out to prove the hydrodynamic limit of the run-and-tumble particle system on the one-dimensional space $G = \mathbb{Z}$. We do this by following the proof Seppäläinen in [23], in which he showed that the evolution of the Exclusion Process is described by the heat equation. For the run-and-tumble particle system, we will see that this evolution will be described by a system of PDEs, one for every internal state in S, which are all dependent. Afterwards, we will also find a PDE for describing the evolution of the total density of particles, i.e. the density of particles where there is no subdivision of particles in regards to the internal state.

Structure of this thesis

In Chapter 2 we begin with the necessary preliminary knowledge of this thesis. Section 2.1 starts with the basics of Markov processes, Markov semigroups and Markov generators, and Section 2.2 looks at invariant and ergodic measures. Experienced readers on Markov processes can skip the first two sections. Section 2.3 introduces the concept of duality and self-duality, and a useful theorem for proving duality results. In Section 2.4 we look at empirical measures and define convergence to density profiles, along with some examples. Afterwards we give the mathematical meaning of a hydrodynamic limit. Lastly, in Section 2.5 we define the Simple Symmetric Exclusion Process and apply all the mathematical concepts we have introduced in this chapter to this process.

In Chapter 3 we show a duality result for the run-and-tumble particle system with a dual process. In Section 3.1 we introduce the result and proof it on the level of generators. Afterwards, in Section 3.2, we prove duality of the run-and-tumble particle system with a deterministic system, which will help us prove the required duality result. We end this chapter with a duality result between the two deterministic systems of the run-and-tumble particle system and the dual process.

In Chapter 4 we look at ergodic properties of product Poisson measures. Section 4.1 starts by showing an application of Doob's theorem, stating that Poisson measures are preserved under independent Markovian particle processes. This will show that product Poisson measures with constant parameters are invariant. The rest of the section is dedicated to showing that these measures are also ergodic by showing they are mixing. Afterwards In Section

4.2 we prove that we can achieve a successful coupling between finite configurations of run-and-tumble particles. We will use this result in Section 4.3 where we show that all bounded harmonic functions only depend on the number of particles, and we end with the proof that the product Poisson measures with constant parameter are the only ergodic measures in the space of tempered measures.

In Chapter 5 we prove the hydrodynamic result of the run-and-tumble particle system on $\mathbb{Z} \times S$. In Section 5.1 we state the result and in the sections 5.2 - 5.6 we prove this in the same way as it is done in [23]. Then in Section 5.7 we define the local equilibrium measures and show that the evolution of these measures corresponds to the hydrodynamic limit. Lastly, in Section 5.8 we look at the evolution of the total density of particles.

Chapter 2

Interacting Particle System Theory

In this chapter, we will give an overview of the background material and math concepts needed to study the run-and-tumble particle process. At the end of this chapter, we will first apply this material in the context of a simpler model, namely the Exclusion Process. Much of the theory discussed here is derived from the following sources [8, 18, 21]. For a reader that is already familiar with Markov semigroups, Markov generators, and invariant and ergodic measures, the first two sections of this chapter can be skipped.

2.1 Markov processes

We start with some basics of the theory of Markov processes.

2.1.1 Markov processes and Markov semigroups

Markov processes

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space with filtration $\{\mathscr{F}_t, t \geq 0\}$ and $(E, \mathscr{E})$ a measurable space. We start with the definition of a stochastic process.

Definition 2.1. A *stochastic process* is a family $X = \{X_t, t \geq 0\}$ of random elements $X_t : \Omega \to E$. We say that a stochastic process is *adapted* to the filtration $\{\mathscr{F}_t, t \geq 0\}$ if for every $t \geq 0$ we have that X_t is $\mathscr{F}_t$-measurable.

Remark 2.1. While every random element X_t of the stochastic process is a function from Ω to E, the distinction between these two spaces is often irrelevant. Therefore we tend to work with the case where $(\Omega, \mathscr{F}) = (E, \mathscr{E})$ and every X_t is the identity. Unless mentioned otherwise, we from now on assume that this is the case.

If we want the starting value of the process, i.e., the value of X_0, to follow a probability distribution μ on $(\Omega, \mathscr{F})$, then we associate the process X with the law $\mathbb{P}_\mu := \mathbb{P}(\cdot | X_0 \sim \mu)$, and let $\mathbb{E}_\mu$ be the corresponding expectation. However, in the case that $\mu = \delta_x$ for some $x \in \Omega$ and δ the Dirac-measure, i.e., with probability 1 the process starts from x, we will simply write $\mathbb{P}_x$ and $\mathbb{E}_x$.

A Markov process is a special kind of stochastic process.

Definition 2.2. An adapted stochastic process X is called a *Markov process* if for every bounded measurable function $f : \Omega \to \mathbb{R}$ and $0 \leq s < t$, the following holds,

$$\mathbb{E}\left[f(X_t) \middle| \mathscr{F}_s\right] = \mathbb{E}\left[f(X_t) \middle| X_s\right]. \tag{2.1}$$

We furthermore call the Markov process *homogeneous* if for any $s > 0$ the following process $\{X_{t+s}, t \geq 0\}$ starting from $X_s = x$ has the same distributions as $\{X_t, t \geq 0\}$ starting from $X_0 = x$.

Intuitively, the property 2.1 tells us that a Markov process is a stochastic process that is memoryless: the future does not depend on the whole past, but only on the current state. This property is also called the *Markov property.*

Markov semigroups

Definition 2.3. Let $(F(\Omega), ||\cdot||_\infty)$ be a complete, real-valued function space. A family $\{S_t, t \geq 0\}$ of bounded linear operators $S_t : F(\Omega) \to F(\Omega)$ is called a *Markov semigroup* if for all $f \in F(\Omega)$ and $s, t \geq 0$,

(S_1). $S_0 f = f$,

(S_2). $S_{t+s} f = S_t(S_s f)$,

(S_3). $\lim_{t\downarrow 0} ||S_t f - f||_\infty = 0$,

(S_4). $S_t \mathbb{1} = \mathbb{1}$,

(S_5). If $f \geq 0$ then $S_t f \geq 0$,

(S_6). $||S_t f||_\infty \leq ||f||_\infty$.

The property (S_2) is called the *semigroup property.* It says that applying the operator S_t after S_s is the same as applying S_{t+s}. (S_3) is also called *strong continuity.* This property says that the semigroup is right-continuous at 0 with respect to the norm $||\cdot||_\infty$. And lastly (S_6) says that S_t is a *contraction.*

Remark 2.2. If we combine the semigroup property with the strong continuity, we can conclude that the path $\{S_t f, t \geq 0\}$ is continuous in t for each $f \in F(\Omega)$.

As the names would suggest, theres is a connection between a Markov semigroup and a Markov process. For $X = \{X_t, t \geq 0\}$ a homogeneous Markov process that is right-continuous, we want to define the the following family of operators $\{S_t, t \geq 0\}$ acting on functions $f : \Omega \to \mathbb{R}$ as follows

$$S_t f(x) = \mathbb{E}_x \left[f(X_t) \right]. \tag{2.2}$$

The first question that arises from this definition is on which function space $F(\Omega)$ we can define these operators. This entirely depends on the state space Ω that we are working with.

1. If Ω is a compact metric space, then we can take $F(\Omega) = C(\Omega)$.

2. If Ω is a locally compact metric space, then we can take $F(\Omega) = C_0(\Omega)$, i.e., the space of continuous functions that vanish at infinity, or $F(\Omega) = C_b(\Omega)$, i.e., the space of bounded continuous functions.

3. If Ω is a general discrete, measurable space, then we can take $F(\Omega) = \mathscr{B}(\Omega)$, i.e., the space of bounded measurable functions. Note however that while are working in a discrete measurable space, we have that $\mathscr{B}(\Omega) = C_b(\Omega)$, since every set in Ω is open.

One can verify that for each case, our choice of function space is a Banach space when equipped with the infinity norm $||\cdot||_\infty$, defined by

$$||f||_\infty := \sup_{x\in\Omega} |f(x)|.$$

Proposition 2.1. *The family of operators $\{S_t, t \geq 0\}$ as defined in 2.2 is a Markov semigroup on $(F(\Omega), ||\cdot||_\infty)$.*

Proof. The linearity of the operators comes from the linearity of the expectation, and the properties (S_1) and $(S_4) - (S_6)$ follow directly from the definition of S_t, so we only have to prove the semigroup property and the strong continuity of $\{S_t, t \geq 0\}$.

In order to prove the semigroup property, we use the tower property of conditional expectation and the Markov property of X to see that for every $x \in \Omega$ we have the following:

$$S_{t+s}f(x) = \mathbb{E}_x[f(X_{t+s})] = \mathbb{E}_x\big[\mathbb{E}[f(X_{t+s})|\mathscr{F}_t]\big] = \mathbb{E}_x\big[\mathbb{E}[f(X_{t+s})|X_t]\big]. \tag{2.3}$$

Now note that by the homogeneity of X

$$\mathbb{E}[f(X_{t+s})|X_t] = \mathbb{E}_{X_t}[f(X_s)] = S_s f(X_t),$$

so by filling this back into 2.3 we indeed find that

$$S_{t+s}f(x) = \mathbb{E}_x[f(X_{t+s})] = \mathbb{E}_x\big[S_s f(X_t)\big] = S_t\left(S_s f(x)\right).$$

For the strong continuity, note first of all that $f \in F(\Omega)$ is continuous for every possible function space $F(\Omega)$ that we have considered. Furthermore, if the process starts from $X_0 = x$, by the right-continuity of X we have that $\lim_{t\downarrow 0} X_t = x$. Therefore, by the dominated convergence theorem, we find that

$$\lim_{t\downarrow 0} S_t f(x) = \lim_{t\downarrow 0} \mathbb{E}_x[f(X_t)] = f(x),$$

which shows the pointwise right-continuity in 0. The proof to go from here to uniform right-continuity can be found in Section 1 of chapter IX in [28]. □

By Proposition 2.1 we have shown that for every Markov process $\{X_t, t \geq 0\}$ there exists a Markov semigroup $\{S_t, t \geq 0\}$given by 2.2. It turns out that the other direction is also true, and the connection between a Markov semigroup and Markov process is even one-to-one. This result can be found in the book of Liggett [18, Theorem 1.5]

Theorem 2.2. *Suppose $\{S_t, t \geq 0\}$ is a Markov semigroup on $(F(\Omega), ||\cdot||_\infty)$, then there exists a unique homogeneous, right-continuous Markov process $\{X_t, t \geq 0\}$ such that*

$$S_t f(x) = \mathbb{E}_x[f(X_t)]$$

for all $f \in F(\Omega), t \geq 0$ and $x \in \Omega$.

2.1.2 Markov generators and the Hille-Yosida Theorem

The semigroup property of the Markov semigroup motivates the existence of an operator $L : F(\Omega) \to F(\Omega)$, such that the informal relation '$S_t = e^{tL}$' holds. By this relation, L would be defined as follows:

$$Lf = \lim_{t\downarrow 0} \frac{S_t f - f}{t} \tag{2.4}$$

This operator is known as the *(infinitesimal) generator* of the Markov semigroup $\{S_t, t \geq 0\}$, since it determines the behavior in the future of the corresponding Markov process for an infinitesimal time interval, and intuitively we can in this way generate the whole process via the Markov property.

Example 2.1. In this example, we will look at the continuous-time Markov chain $\{X_t, t \geq 0\}$ with "transition rates", i.e., we define the rates $c(x, y) > 0$ for every $x, y \in \Omega$, and set $c_x = \sum_{y\in\Omega} c(x, y)$. Here we assume that $c_x < \infty$ for all $x \in \Omega$. If at any point in time t, the process is at $X_t = x$, then it will wait an exponential time, distributed by $\exp(c_x)$, before jumping to another point, where it will jump to $y \in \Omega$ with probability $\pi(x, y) = \frac{c(x,y)}{c_x}$.

We will now compute the generator L as it is given in 2.4 for these type of Markov processes. First note that the probability that more than one jump is made in the time interval $[0,t]$ can be bounded above by Ct^2 for some $C > 0$. If we then denote K_t as the number of jumps made in $[0,t]$, then by the law of total expectation, we have that for any $x \in \Omega$,

$$\begin{aligned} S_t f(x) - f(x) &= \sum_{k=0}^{\infty} \mathbb{E}_x \left[f(X_t) | K_t = k \right] \mathbb{P}(K_t = k) - f(x) \\ &= \mathbb{E}_x \left[f(X_t) | K_t = 1 \right] \mathbb{P}(K_t = 1) + \mathbb{E}_x \left[f(X_t) | K_t = 0 \right] \mathbb{P}(K_t = 0) - f(x) + \mathcal{O}(t^2) \\ &= \left(1 - e^{-c_x t}\right) \left[\sum_{y \in \Omega} \frac{c(x,y)}{c_x} f(y) \right] + e^{-c_x t} f(x) - f(x) + \mathcal{O}(t^2) \\ &= \frac{1 - e^{-c_x t}}{c_x} \left[\sum_{y \in \Omega} c(x,y) \big(f(y) - f(x) \big) \right] + \mathcal{O}(t^2) \end{aligned}$$

Now since $\lim_{x \downarrow 0} \frac{1-e^{-x}}{x} = 1$, we find that

$$Lf(x) = \lim_{t \downarrow 0} \frac{S_t f(x) - f(x)}{t} = \sum_{y \in \Omega} c(x,y) \big(f(y) - f(x) \big) \tag{2.5}$$

Remark 2.3. If we furthermore assume that our state space $\Omega = \{x_1, x_2, ..., x_n\}$ is finite, then every function $f \in F(\Omega)$ can be represented as a column vector $(f(x_i))_{1 \le i \le n}$, and the generator L is the following matrix

$$L = \begin{pmatrix} -n & c(x_1, x_2) & \cdots & c(x_1, x_n) \\ c(x_2, x_1) & -n & \cdots & c(x_2, x_n) \\ \vdots & \vdots & \ddots & \vdots \\ c(x_n, x_1) & c(x_n, x_2) & \cdots & -n \end{pmatrix}$$

In Example 2.1, we see that the (possibly) infinite sum given in 2.5 might not exist for all $f \in F(\Omega)$, and hence the generator is not defined on the whole of $F(\Omega)$. One example of a generator where this is even more clear, is the following:

Example 2.2. Let $\Omega = \mathbb{R}$, then Ω is a locally compact space, so for our functions space we can choose $F(\Omega) = C_b(\Omega)$. Now define the Markov semigroup $\{S_t, t \ge 0\}$ by the deterministic operator that for every $t \ge 0$ sends $f \in C_b(\Omega)$ to the following function

$$S_t f(x) = f(x+t),$$

for every $x \in \Omega$. Then

$$Lf(x) = \lim_{t \downarrow 0} \frac{f(x+t) - f(x)}{t} = f'(x),$$

but only if $f'(x)$ exists for every $x \in \Omega$, i.e., f must be continuously differentiable. But this is not the case for every $f \in C_b(\Omega)$.

In order to avoid this problem, we restrict the functions that L can work on by defining the domain of the generator L as follows:

$$D(L) := \left\{ f \in F(\Omega) : \lim_{t \downarrow 0} \frac{S_t f - f}{t} \text{ exists} \right\}. \tag{2.6}$$

Markov generators formalized

We can formalize the theory we have discussed above by giving the definition of a Markov generator.

Definition 2.4. Let $D(L) \subset F(\Omega)$, then the operator $L : D(L) \to F(\Omega)$ is called a *Markov generator* if the following properties hold:

(G_1). $\mathbb{1} \in D(L)$ and $L\mathbb{1} = 0$,

(G_2). $D(L)$ is dense in $F(\Omega)$,

(G_3). L is a closed operator, i.e., $\{(f, Lf) : f \in D(L)\}$ is closed.

(G_4). $\mathscr{R}(I - \lambda L) = F(\Omega)$ for all $\lambda \geq 0$,

(G_5). If $f \in D(L), \lambda \geq 0$ and $(I - \lambda L)f = g$, then

$$\min_{x\in\Omega} f(x) \geq \min_{x\in\Omega} g(x).$$

By an application of the Hille-Yosida theorem, there exists a one-to-one correspondence between a Markov semigroup and a Markov generator [21, Theorem 2.2].

Theorem 2.3.

1. *For every Markov semigroup $\{S_t, t \geq 0\}$, the operator L given by 2.4 is a Markov generator with domain $D(L)$ as given in 2.6.*

2. *For every Markov generator L, the process $\{S_t, t \geq 0\}$ given by*

$$S_t = \lim_{n\to\infty} \left(I - \frac{t}{n} L\right)^{-n} \tag{2.7}$$

is a Markov semigroup.

3. *For $f \in D(L)$, we have that $S_t f \in D(L)$ and*

$$\frac{d}{dt} S_t f = S_t L f = L S_t f. \tag{2.8}$$

Moreover, $S_t f$ is the unique solution to this equation.

We see that we can ultimately obtain the following diagram, showing the relations between a Markov process, semigroup and generator.

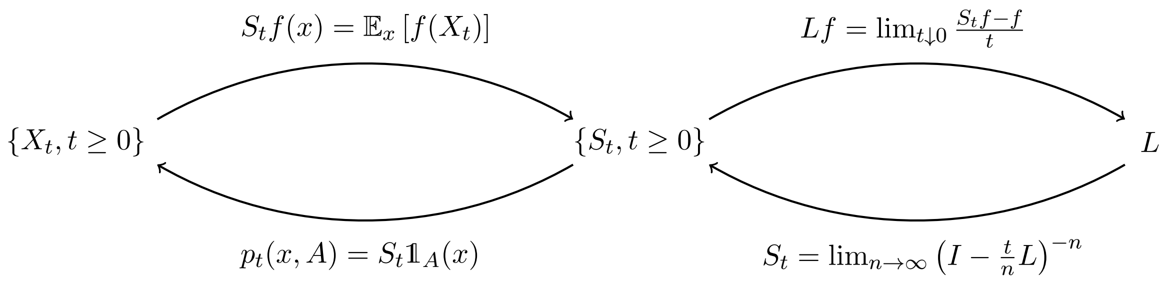

Figure 2.1: one-to-one correspondence of Markov process, semigroup and generator.

Cores

For a given Markov generator L, it is not always easy to characterize the whole domain $D(L)$ explicitly, as it is given in 2.6. However, it is usually enough to only look at the operator L defined on a certain subspace $\mathscr{D} \subset D(L)$, called a core.

Definition 2.5. A set $\mathscr{D} \subset D(L)$ is a *core* of the generator L if the closure of the following restriction $L\big|_{\mathscr{D}}$ is again the generator L, i.e., the graph $\{(f, Lf) : f \in \mathscr{D}\}$ is dense in the closed graph $\{(f, Lf) : f \in D(L)\}$. This can be .. for all (f, Lf) with $f \in D(L)$, there exists a sequence $\{(f_n, Lf_n), n \in \mathbb{N}\}$ with all $f_n \in \mathscr{D}$ such that $(f_n, Lf_n) \to (f, Lf)$.

Example 2.3. Let K be a finite set. If we consider a continuous-time Markov chain on the compact state space $\Omega = K^{\mathbb{Z}}$, with transition rates $c(x, y)$ for all $x, y \in \Omega$, then as we have seen in Example 2.1, the generator L is given by

$$Lf(x) = \sum_{y \in \Omega} c(x, y)\big(f(y) - f(x)\big), \tag{2.9}$$

where $D(L)$ is the set of all $f \in \mathscr{B}(\Omega)$ for which $||Lf||_\infty < \infty$. However it is easier to assume that f is a *local* function, i.e., functions that only depend on finitely many coordinates in $\mathbb{Z}$. If this is the case, then the sum in 2.9 is a finite sum, hence $f \in D(L)$. By an application of the Stone-Weierstrass theorem, it can also be shown that the local functions are dense in $C(\Omega)$, so if we now take

$$\mathscr{D} := \{f \in C(\Omega) : f \text{ is a local function}\},$$

then $\mathscr{D}$ is a core of L.

Trotter-Kurtz theorem

The theorem of Trotter-Kurtz tells us that convergence of a Markov generator on a core implies the convergence of the corresponding Markov semigroup and process.

Theorem 2.4. *Let $\big(\{X_t^n, t \geq 0\}\big)_{n \in \mathbb{N}}$, X be Markov processes on a compact space Ω, with corresponding Markov semigroups $\big(\{S_t^n, t \geq 0\}\big)_{n \in \mathbb{N}}$, $\{S_t, t \geq 0\}$ and generators $\big(L^n\big)_{n \in \mathbb{N}}$, L, respectively. Furthermore, let $\mathscr{D}$ be a core for L, then the following are equivalent:*

- *for all $f \in \mathscr{D}$ there is a sequence $(f^n)_{n \in \mathbb{N}}$ with $f^n \in D(L^n)$ such that $f^n \to f$ and $L^n f^n \to Lf$.*
- *$S_t^n f \to S_t f$ for every $f \in F(\Omega)$, uniformly for $t \in [0, T]$.*
- *if $X_0^n \to X_0$ in distribution, then $X^n \to X$ in distribution in D_Ω (path space, see appendix A.3.2).*

Proof. The proof can be found in [14, Theorem 19.25] □

2.2 Invariant and ergodic measures

For a Markov process $\{X_t, t \geq 0\}$ on Ω, the initial state can be given by a distribution $X_0 \sim \mu$ with $\mu \in \mathscr{P}(\Omega)$, the space of Borel probability measures on Ω (see Appendix A.3). Let $\{S_t, t \geq 0\}$ be the Markov semigroup corresponding to the Markov process $\{X_t, t \geq 0\}$ and define $\mu S_t \in \mathscr{P}(\Omega)$ as the distribution of the process at time t, i.e., $X_t \sim \mu S_t$, then this is the unique probability measure such that for all $f \in F(\Omega)$ we have that

$$\int f d\mu S_t = \int S_t f d\mu.$$

In this section we will look at starting distributions μ that are invariant and ergodic with respect to the process X.

2.2.1 Invariant measures

The notion of invariance in mathematics means that it remains unchanged after a certain operation. In the case of Markov processes, this leads to the following definition:

Definition 2.6. Let $\{S_t, t \geq 0\}$ be a Markov semigroup and let $\mu \in \mathscr{P}(\Omega)$, then we say that μ is *invariant* (with respect to $\{S_t, t \geq 0\}$) if

$$\int f d\mu S_t = \int f d\mu$$

for all $t \geq 0$ and $f \in F(\Omega)$. We denote the set of invariant measures by $\mathscr{I}$.

From this definition, we see that if μ is an invariant measure, then the distribution of the process remains the same, i.e., $X_t \sim \mu$ for all $t \geq 0$.

Proposition 2.5. *$\mathscr{I}$ is convex. If furthermore Ω is a compact, then $\mathscr{I}$ is also non-empty and compact.*

Proof. Let $\mu_1, \mu_2 \in \mathscr{I}$, and for $\lambda \in (0,1)$ define the probability measure $\mu_\lambda \in \mathscr{P}(\Omega)$ as $\mu_\lambda = \lambda\mu_1 + (1-\lambda)\mu_2$. Furthermore, let $t \geq 0$ and $f \in F(\Omega)$, then

$$\int f d\mu_\lambda S_t = \lambda \int f d\mu_1 S_t + (1-\lambda) \int f d\mu_2 S_t = \lambda \int f d\mu_1 + (1-\lambda) \int f d\mu_2 = \int f d\mu_\lambda.$$

So indeed $\mu_\lambda \in \mathscr{I}$ for all $\lambda \in (0,1)$, and so $\mathscr{I}$ is convex.

Now assume that Ω is compact. Then we have that $\mathscr{P}(\Omega)$ is also compact. To see this, we will use that by Theorem A.7 $\mathscr{P}(\Omega)$ is compact if it is tight, i.e., for all $\varepsilon > 0$, there exists a compact set $K \subset \Omega$ such that for all $\mu \in \mathscr{P}(\Omega)$ we have that $\mu(K) \geq 1 - \varepsilon$. But this is trivial since Ω itself is compact.

The non-emptyness of $\mathscr{I}$ now follows from the so-called "Bogolioubov-Krylov" argument. Let $\nu \in \mathscr{P}(\Omega)$, and define the measures

$$\mu_T := \frac{1}{T}\int_0^T \nu S_s ds,$$

then we have that for every $f \in F(\Omega)$,

$$\begin{aligned}\left|\int S_t f d\mu_T - \int f d\mu_T\right| &= \frac{1}{T}\left|\int_0^T \int S_{t+s} f d\nu ds - \int_0^T \int S_s f d\nu ds\right| \\ &\leq \frac{1}{T}\left(\int_0^t \int |S_s f| d\nu ds + \int_T^{T+t} \int |S_s f| d\nu ds\right) \\ &\leq \frac{2t||f||_\infty}{T},\end{aligned} \tag{2.10}$$

where we have used the semigroup property of $\{S_t, t \geq 0\}$ and the fact that S_s is a contraction for every $s \geq 0$. We see here that if we let $T \to \infty$, then the right-hand side of 2.10 goes to zero, hence any limit point of a subsequence $\mu_{T_n} \to \mu$ is invariant. Such a limit point exists since $\mathscr{P}(\Omega)$ is compact.

Finally, in order to show that the space $\mathscr{I}$ is compact, all we have to show is that it is closed in the weak topology on $\mathscr{P}(\Omega)$, since $\mathscr{P}(\Omega)$ is compact. Let $\{\mu_n, n \in \mathbb{N}\}$ be a sequence of probability measures in $\mathscr{I}$ such that $\mu_n \xrightarrow{w} \mu$ with $\mu \in \mathscr{P}(\Omega)$, i.e., for all $f \in C(\Omega)$ we have that

$$\int f d\mu = \lim_{n\to\infty} \int f d\mu_n.$$

If we have that $f \in C(\Omega)$ then also $S_t f \in C(\Omega)$. Therefore, since every $\mu_n \in \mathscr{I}$,

$$\int S_t f d\mu = \lim_{n\to\infty} \int S_t f d\mu_n = \lim_{n\to\infty} f d\mu_n = \int f d\mu,$$

hence $\mu \in \mathscr{I}$, and so $\mathscr{I}$ is closed. □

Proposition 2.6. *$\mu \in \mathscr{I}$ if and only if for all $f \in D(L)$,*

$$\int Lf d\mu = 0. \tag{2.11}$$

Proof. Assume that $\mu \in \mathscr{I}$, then for every $f \in D(L)$ we have that

$$\int Lf d\mu = \int \lim_{t\downarrow 0} \frac{1}{t}(S_t f - f) d\mu = \lim_{t\downarrow 0} \frac{1}{t}\left(\int S_t f d\mu - \int f d\mu\right) = 0,$$

where we could interchange the limit and integral since the convergence from $\frac{S_t f - f}{t} \to Lf$ is uniform, so we can apply the dominated convergence theorem.

Conversely, suppose that 2.11 holds, then by 2.8, we have that for all $f \in D(L)$

$$\int (S_t f - f) d\mu = \int \int_0^t (LS_s f) ds d\mu = \int_0^t \left(\int LS_s f d\mu\right) ds = 0,$$

where we have used Fubini for the second equality. This implies that

$$\int S_t f d\mu = \int f d\mu,$$

for all $f \in D(L)$. Now since $D(L)$ is dense in $F(\Omega)$, for any $f \in F(\Omega)$ we can take a sequence $\{f_n, n \in \mathbb{N}\} \subset D(L)$ such that $f_n \to f$ uniformly. Furthermore, since S_t is a contraction, we have that

$$\sup_{n\in\mathbb{N}} ||S_t f_n||_\infty \leq \sup_{n\in\mathbb{N}} ||f_n||_\infty < \infty.$$

So by the dominated convergence theorem, we have that

$$\int S_t f d\mu = \lim_{n\to\infty} \int S_t f_n d\mu = \lim_{n\to\infty} \int f_n d\mu = \int f d\mu,$$

which shows that $\mu \in \mathscr{I}$. □

Of course, as we have discussed earlier, it is not always easy to determine the whole set $D(L)$. Therefore it is again useful to turn to a core $\mathscr{D}$ for this property.

Proposition 2.7. *Let $\mathscr{D}$ be a core for the generator L, then $\mu \in \mathscr{I}$ if 2.11 holds for all $f \in \mathscr{D}$.*

Proof. Since $\mathscr{D}$ is dense in $D(L)$, for every $f \in D(L)$ there exists a sequence $\{f_n, n \in \mathbb{N}\}$ such that $f_n \to f$ and $Lf_n \to Lf$, both uniformly. Therefore, again by the dominated convergence theorem we have that

$$\int Lf d\mu = \lim_{n\to\infty} \int Lf_n d\mu = 0,$$

which shows that 2.11 holds for all $f \in D(L)$, and so $\mu \in \mathscr{I}$. □

Reversibility

A notion even stronger than invariance is that of reversibility.

Definition 2.7. A measure $\mu \in \mathscr{P}(\Omega)$ is called *reversible* (with respect to $\{S_t, t \geq 0\}$) if for all $f, g \in F(\Omega)$.

$$\int (S_t f) g d\mu = \int f(S_t g) d\mu.$$

By putting $g \equiv 1$ we can conclude that a reversible measure is invariant. The main idea behind a reversible measure is that, while an invariant measure remains the same moving forward in time, a reversible measure also has this property while moving backward in time. This is why, if a process is following a reversible measure, we often say the process is in "equilibrium".

Reversibility can also be shown by looking at the generator and turning to a core $\mathscr{D}$.

Proposition 2.8. *$\mu \in \mathscr{P}(\Omega)$ is reversible if and only if for all $f, g \in \mathscr{D}$,*

$$\int g(Lf) d\mu = \int (Lg) f d\mu.$$

Proof. The proof is similar to that of Proposition 2.6 and Proposition 2.7 □

2.2.2 Ergodic measures

If we have a measure $\mu \in \mathscr{P}(\Omega)$ and take $f \in F(\Omega)$, then since $||f||_\infty < \infty$ we have that $f \in L^p(\Omega, \mu)$ for $p \geq 1$. It turns out that for $\mu \in \mathscr{I}$, we can even extend the semigroup $\{S_t, t \geq 0\}$ to $L^p(\Omega, \mu)$ by the following proposition.

Proposition 2.9. *Let $\mu \in \mathscr{I}$, then S_t is a contraction with respect to $|| \cdot ||_p$ for each $p \geq 1$ and $t \geq 0$.*

Proof. Let $p \geq 1$ and $t \geq 0$. For any $x \in \Omega$ and $f \in C(\Omega) \cap L^p(\Omega, \mu)$, Jensen's inequality tells us that

$$|(S_t f)(x)|^p = |\mathbb{E}_x [f(X_t)]|^p \leq \mathbb{E}_x [|f(X_t)|^p] = (S_t |f|^p)(x).$$

Now by the invariance of the measure μ we see that

$$||S_t f||_p = \int |S_t f|^p d\mu \leq \int S_t |f|^p d\mu = \int |f|^p d\mu = ||f||_p,$$

which proves the proposition. □

Since for any set $A \in \mathscr{F}$ we have that $\mathbb{1}_A \in L^p(\Omega, \mu)$, by the above theorem we see that the semigroup $\{S_t, t \geq 0\}$ can also work on indicator functions. With that in mind, we can now define what it means for a set $A \in \mathscr{F}$ to be invariant.

Definition 2.8. We call a set $A \in \mathscr{F}$ *invariant* (with respect to $\{S_t, t \geq 0\}$) if for all $t \geq 0$ we have that $S_t \mathbb{1}_A = \mathbb{1}_A$ a.s.. Similarly, we call a function $f \in L^p(\Omega, \mu)$ *invariant* if for all $t \geq 0$ we have that $S_t f = f$.

The invariance of a set A implies that if the corresponding Markov process $\{X_t, t \geq 0\}$ has a starting value $X_0 \in A$, then $X_t \in A$ for all $t \geq 0$ with probability 1, i.e., it (almost) never leaves the set A. Note that it is also easy to see that if A is invariant, then also A^c is invariant.

Definition 2.9. A probability measure $\mu \in \mathscr{I}$ is *ergodic* (with respect to $\{S_t, t \geq 0\}$) if for all invariant sets $A \in \mathscr{F}$ either $\mu(A) = 1$ or $\mu(A) = 0$.

From this definition follows the following equivalence.

Proposition 2.10. *$\mu \in \mathscr{I}$ is ergodic if and only if for any $p \geq 1$, all invariant functions $f \in L^p(\Omega, \mu)$ are μ-a.s. constant.*

Proof. Suppose that μ is ergodic and $f \in L^p(\Omega, \mu)$ is an invariant function. Define the sets B_a for $a \in \mathbb{R}$ as follows:

$$B_a := \{x \in \Omega : f(x) \leq a\}.$$

Since $S_t f = f$ we have that B_a is an invariant set, which by ergodicity of μ implies that $\mu(B_a) = 0$ or $\mu(B_a) = 1$. Now define

$$a_0 := \inf \{a \in \mathbb{R} : \mu(B_a^c) = 0\},$$

and note that

$$\{x \in \Omega : f(x) = a_0\} = B_{a_0} \setminus \bigcup_{n=1}^{\infty} B_{a_0 - \frac{1}{n}}.$$

Since $\mu(B_{a_0}^c) = 0$ and $\mu(B_{a_0-\frac{1}{n}}) = 0$ for all $n \geq 1$, we find that

$$\mu(\{x \in \Omega : f(x) = a_0\}) = 1,$$

hence f is μ-a.s. constant.

For the other direction, if all invariant functions in $L^p(\Omega, \mu)$ are μ-a.s. constant and $A \in \mathscr{F}$ is an invariant set, then since $\mathbb{1}_A \in L^p(\Omega, \mu)$ either $\mathbb{1}_A = 1$ or $\mathbb{1}_A = 0$ μ-a.s.. This implies that $\mu(A) = 1$ or $\mu(A) = 0$, so indeed μ is ergodic. □

Birkhoff ergodic theorem

It was Birkhoff who first proved that the Boltzmann hypothesis, stating that the time-average of a process equals the space-average, needed ergodicity of the measures [4].

Theorem 2.11. *Let $\mu \in \mathscr{I}$ be ergodic, then for any $f \in L^1(\Omega, \mu)$ we have that μ-a.s.*

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T S_t f(x) dt = \int f d\mu$$

Proof. The idea of the proof can be found in [8, Chapter 3]. □

Extremal points of $\mathscr{I}$

In Proposition 2.5, we have seen that the set of invariant measures is convex. It turns out that the extreme points of this set are exactly the ergodic measures. Before we prove this, we will need to prove two lemmas.

Lemma 2.12. *If $\mu \in \mathscr{I}$ and $A \in \mathscr{F}$ is an invariant set, then for the corresponding Markov process $\{X_t, t \geq 0\}$ with $X_0 \sim \mu$ we have that $\mathbb{1}_A(X_0) = \mathbb{1}_A(X_t)$ $\mathbb{P}$-a.s. for all $t \geq 0$. As a consequence, if furthermore $\mu(A) > 0$, then $\mu(\cdot|A) \in \mathscr{I}$.*

Proof. For $A \in \mathscr{F}$ we have that

$$\mathbb{E}_\mu \left[(\mathbb{1}_A(X_0) - \mathbb{1}_A(X_t))^2 \right] = \mathbb{E}_\mu \Big[\mathbb{1}_A(X_0) + \mathbb{1}_A(X_t) - 2\mathbb{1}_A(X_0)\mathbb{1}_A(X_t) \Big] \tag{2.12}$$

In general, we have that $\mathbb{E}_\mu \big[\mathbb{1}_A(X_0) \big] = \mu(A)$, and if A is now invariant, then

$$\mathbb{E}_\mu \Big[\mathbb{1}_A(X_t) \Big] = \int \mathbb{E}_x \Big[\mathbb{1}_A(X_t) \Big] d\mu(x) = \int S_t \mathbb{1}_A(x) d\mu(x) = \int \mathbb{1}_A(x) d\mu(x) = \mu(A),$$

and similarly

$$\mathbb{E}_\mu\Big[\mathbb{1}_A(X_0)\mathbb{1}_A(X_t)\Big] = \int \mathbb{1}_A(x)S_t\mathbb{1}_A(x)d\mu(x) = \int \mathbb{1}_A(x)d\mu(x) = \mu(A).$$

By filling this back into 2.12, we find that

$$\mathbb{E}_\mu\left[(\mathbb{1}_A(X_0) - \mathbb{1}_A(X_t))^2\right] = \mu(A) + \mu(A) - 2\mu(A) = 0,$$

hence $\mathbb{1}_A(X_0) = \mathbb{1}_A(X_t)$ $\mathbb{P}$-a.s.

If we furthermore have that $\mu(A) > 0$, then for all $f \in F(\Omega)$, we have that

$$\int S_t f d\mu(\cdot|A) = \int (S_t f)\cdot \mathbb{1}_A d\mu = \mathbb{E}_\mu\Big[f(X_t)\mathbb{1}_A(X_0)\Big].$$

By now filling in that $\mathbb{1}_A(X_0) = \mathbb{1}_A(X_t)$ and using that $\mu \in \mathscr{I}$, we see that

$$\int S_t f d\mu(\cdot|A) = \mathbb{E}_\mu\Big[f(X_t)\mathbb{1}_A(X_t)\Big] = \int S_t(f\cdot \mathbb{1}_A)d\mu = \int f\mathbb{1}_A d\mu = \int f d\mu(\cdot|A),$$

i.e., $\mu(\cdot|A) \in \mathscr{I}$. □

Lemma 2.13. *If $\mu, \mu_1 \in \mathscr{I}$, μ is ergodic and $\mu_1 << \mu$, i.e., $\mu(B) = 0$ implies that $\mu_1(B) = 0$ for $B \in \mathscr{F}$, then $\mu = \mu_1$.*

Proof. If μ is ergodic, then by Theorem 2.11 we find that for any $A \in \mathscr{F}$ and μ-a.s.,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T S_t\mathbb{1}_A(x)dt = \int \mathbb{1}_A d\mu = \mu(A),$$

i.e., if we define the following set

$$C_A := \left\{x \in \Omega : \lim_{T\to\infty}\frac{1}{T}\int_0^T S_t\mathbb{1}_A(x)dt = \mu(A)\right\},$$

then $\mu(C_A) = 1$. By the fact that $\mu_1 << \mu$, we then also have $\mu_1(C_A) = 1$. Since $\mu_1 \in \mathscr{I}$, for every $T > 0$ we have the following,

$$\frac{1}{T}\int_0^T\int S_t\mathbb{1}_A d\mu_1 dt = \frac{1}{T}\int_0^T\int \mathbb{1}_A d\mu_1 dt = \mu_1(A).$$

On the other hand, by the dominated convergence theorem and Fubini, we also find that

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\int S_t\mathbb{1}_A d\mu_1 dt = \int \lim_{T\to\infty}\frac{1}{T}\int_0^T S_t\mathbb{1}_A dt d\mu_1 = \int \mu(A)d\mu_1 = \mu(A).$$

So we find that $\mu(A) = \mu_1(A)$, and since we have taken $A \in \mathscr{F}$ arbitrary, we can conclude that $\mu = \mu_1$. □

Proposition 2.14. *$\mu \in \mathscr{I}$ is ergodic if and only if it is an extremal point of $\mathscr{I}$.*

Proof. Assume that $\mu \in \mathscr{I}$ is ergodic but not an extremal point of $\mathscr{I}$, i.e., there exists $\mu_1, \mu_2 \in \mathscr{I}$ such that $\mu_1, \mu_2 \neq \mu$ and

$$\mu = \lambda\mu_1 + (1-\lambda)\mu_2$$

for some $\lambda \in (0,1)$. By this relation, we see that $\mu_1 << \mu$, and so by applying Lemma 2.13 we then find that $\mu = \mu_1$, which is a contradiction. Therefore μ is an extremal point of $\mathscr{I}$.

For the other direction, assume that $\mu \in \mathscr{I}$ is not ergodic, i.e., there is an invariant set $A \in \mathscr{F}$ with $0 < \mu(A) < 1$. Then we can write μ as the decomposition

$$\mu = \mu(\cdot|A)\mu(A) + \mu(\cdot|A^c)\mu(A^c).$$

By Lemma 2.12, we see that both $\mu(\cdot|A), \mu(\cdot|A^c) \in \mathscr{I}$, which means that μ is not an extremal point of $\mathscr{I}$. □

Proposition 2.14 shows us that knowing the ergodic measures of a process can tell a lot about the invariant measures. Especially if Ω is compact, since by Proposition 2.5 this tells us that $\mathscr{I}$ is also compact, and so by the Krein-Milman theorem, $\mathscr{I}$ is equal to the closed convex hull of its extremal points, i.e., we can write every $\mu \in \mathscr{I}$ as the convex combination of ergodic measures.

Mixing

The following property is an even stronger one than ergodicity.

Definition 2.10. A probability measure $\mu \in \mathscr{I}$ is *mixing* (with respect to $\{S_t, t \geq 0\}$) if for all $f, g \in L^2(\Omega, \nu)$ we have

$$\lim_{t\to\infty} \int g(S_t f) d\mu = \int g d\mu \int f d\mu$$

If we would fill in $f = \mathbb{1}_A$ and $g = \mathbb{1}_B$ for some arbitrary set $A, B \in \mathscr{F}$, then this equality can be written as

$$\lim_{t\to\infty} \mu(B \cap S_t^{-1} A) = \mu(B)\mu(A),$$

which looks fairly similar to an independence result of A and B. This is what the notion of mixing really does, it turns any set $A \in \mathscr{F}$ into something fairly independent of an arbitrary set $B \in \mathscr{F}$.

Proposition 2.15. *$\mu \in \mathscr{I}$ is ergodic if it is mixing.*

Proof. Let $f \in L^2(\Omega, \mu)$ be an invariant function. By writing $g = f$ in the definition of mixing, we see that

$$\int f^2 d\mu = \lim_{t\to\infty} \int f(S_t f) d\mu = \left(\int f d\mu\right)^2,$$

i.e., $\mathbb{E}[f^2] = (\mathbb{E}[f])^2$. So f has zero variance, therefore it is a.s. constant. By Proposition 2.10, we then see that μ is ergodic □

2.3 Duality

Duality is an important tool in the analysis of Interacting Particle Systems. It allows us to associate a so-called dual process to our process of interest, such that a problem regarding the original process can be reformulated as a problem in terms of the, often much simpler, dual process. We start with the general definition of duality.

Definition 2.11. Suppose X and Y are Markov processes on the state spaces Ω_1 and Ω_2, and with Markov semigroups $\{S_t, t \geq 0\}$ and $\{\widehat{S}_t, t \geq 0\}$ respectively, then X and Y are said to be *dual* to one another with respect to the bounded measurable function $D : \Omega_1 \times \Omega_2 \to \mathbb{R}$ if

$$S_t D(\cdot, y)(x) = \widehat{S}_t D(x, \cdot)(y),$$

for all $x \in \Omega_1$ and $y \in \Omega_2$. In this case we call D the *duality function*.

However, it is often much easier to show duality with respect to the generator of the process.

Proposition 2.16. *Let L and $\widehat{L}$ be the generators of the Markov processes X and Y from Definition 2.11 respectively. If $D(\cdot, y), S_t D(\cdot, y) \in D(L)$ for all $y \in \Omega_2$, $t \geq 0$ and $D(x, \cdot), \widehat{S}_t D(x, \cdot) \in D(\widehat{L})$ for all $x \in \Omega_1$, $t \geq 0$, then X is dual to Y with duality function D if*

$$LD(\cdot, y)(x) = \widehat{L} D(x, \cdot)(y),$$

for all $x \in \Omega_1$ and $y \in \Omega_2$.

Proof. The proof can be found in [13, Proposition 1.2]. □

In the case of Proposition 2.16 we will say, with abuse of notation, that L and $\widehat{L}$ are dual to one another with respect to duality function D.

2.3.1 Self-duality

A Markov process X on Ω is self-dual if (as the name suggests) it is dual to itself. Usually however, for the 'dual process' we do not look at the whole set Ω but only at a subset.

Definition 2.12. A Markov process X is called *self-dual* if there exists a subset $\Omega' \subset \Omega$ and a duality function $D : \Omega' \times \Omega \to \mathbb{R}$ such that

$$S_t D(\cdot, y)(x) = S_t D(x, \cdot)(y),$$

for all $x \in \Omega'$ and $y \in \Omega$.

The real power of the self-duality lies in the choice of the subset Ω'. This subset is usually chosen in such a way that the dynamics of the process would be easier to work with. In Section 2.5 we will look at a process that is self-dual and see why this is a useful property.

2.4 Density profiles and hydrodynamic limits

For this section, we look at particle configurations on $\mathbb{Z}$ where the particles are indistinguishable, i.e., we set $\Omega = \mathbb{N}_0^{\mathbb{Z}}$ and view $\eta \in \Omega$ as a set of particles on the lattice $\mathbb{Z}$ where for every $x \in \mathbb{Z}$ the value $\eta(x)$ is the number of particles at site x.

In this section, we discuss how to pass from the "micro world" of particle configurations on Ω to the "macro world" of density profiles on $\mathbb{R}$.

2.4.1 Density profiles

To go from the micro-world to the macro-world we introduce a scaling parameter N. The rescaling is than as follows: for any macro-point $x \in \mathbb{R}$ the corresponding micro-point is $\lfloor xN \rfloor \in \mathbb{Z}$. More precisely, we define, for $\{\eta^N : N \in \mathbb{N}\}$ a sequence of configurations in Ω, the sequence of measures $\{\pi^N, N \in \mathbb{N}\}$ on $\frac{1}{N}\mathbb{Z}$ as follows:

$$\pi^N = \frac{1}{N} \sum_{x \in \mathbb{Z}} \eta^N(x) \delta_{\frac{x}{N}}.$$

We call these measures the *empirical mesures*. If we let $N \to \infty$ then, under suitable conditions, these measures will converge to some measure in $\mathbb{R}$. We can denote these measures by $\rho(x)dx$ with $\rho : \mathbb{R} \to \mathbb{R}_{\geq 0}$ some non-negative function. We will call such a function a (*macroscopic*) *density profile*.

Definition 2.13. Let $\rho : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be a density profile.

1. A sequence of configurations $\{\eta^N, N \in \mathbb{N}\}$ is said to *correspond* to ρ if for all smooth test functions $\phi \in C_c^\infty(\mathbb{R})$

$$\int \phi(x) d\pi^N(x) \to \int \phi(x)\rho(x)dx, \qquad \text{as } N \to \infty.$$

2. A sequence of probability measures $\{\nu^N, N \in \mathbb{N}\}$ is said to *correspond* to ρ if for the sequence of stochastic variables $\{\eta^N, N \in \mathbb{N}\}$, where $\eta^N \sim \nu^N$ for every N, and for all smooth test functions $\phi \in C_c^\infty(\mathbb{R})$

$$\int \phi(x) d\pi^N(x) \xrightarrow{\mathbb{P}} \int \phi(x)\rho(x)dx, \qquad \text{as } N \to \infty.$$

The convergence in the first part of the definition is also called *vague convergence* (see Appendix A.4), and the convergence in the second part is some sort of vague convergence in probability. In both cases we will denote it as

$$\lim_{N\to\infty} \pi^N = \rho(x)dx.$$

Example 2.4. Let $\eta \in \Omega$ be the configuration such that $\eta(x) = 1$ for all $x \geq 0$ and $\eta(x) = 0$ for all $x < 0$, then this corresponds to the density profile $\rho(x) = \mathbb{1}_{x\geq 0}$. Indeed, by the Riemann approximation of the integral we see that

$$\int \phi(x)d\pi^N(x) = \frac{1}{N}\sum_{x\in\mathbb{Z}} \phi(\tfrac{x}{N})\mathbb{1}_{x\geq 0} \to \int \phi(x)\mathbb{1}_{x\geq 0}dx.$$

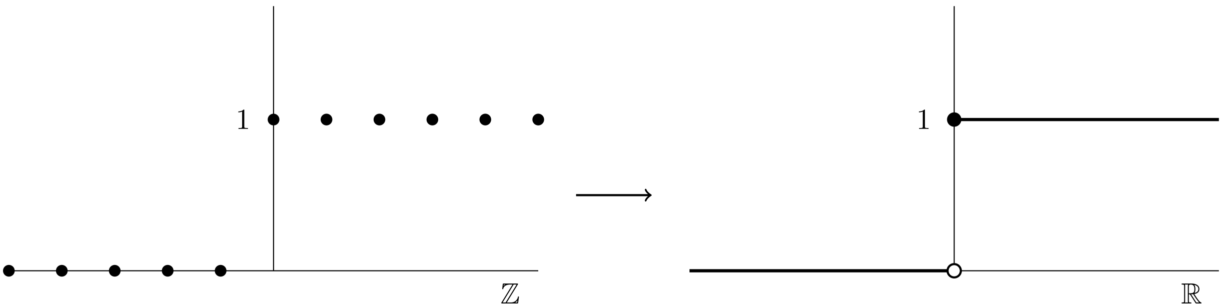

Figure 2.2: Transition from micro-configuration η (left) to macro-density ρ (right).

It is important to notice that for a given density profile $\rho(x)$, there is not just one corresponding configuration $\eta \in \Omega$. It is therefore not a one-to-one mapping but a "many-to-one" mapping. To see this, we will give another example of a configuration corresponding to the density profile from Example 2.4.

Example 2.5. Let $\eta \in \Omega$ be the configuration such that $\eta(x) = 2$ for all even $x \geq 0$ and $\eta(x) = 0$ otherwise, then this also corresponds to the density profile $\rho(x) = \mathbb{1}_{x\geq 0}$. To see this, note that since ϕ is continuous, we have that $\lim_{N\to\infty} \left|\phi(\frac{x}{N}) - \phi(\frac{x+1}{N})\right| = 0$, and so

$$\begin{aligned}\int \phi(x)d\pi^N(x) &= \frac{1}{N}\sum_{x\in 2\mathbb{Z}} 2\phi(\tfrac{x}{N})\mathbb{1}_{x\geq 0}\\ &= \frac{1}{N}\sum_{x\in 2\mathbb{Z}} \left(\phi(\tfrac{x}{N}) + \phi(\tfrac{x+1}{N})\right)\mathbb{1}_{x\geq 0} + \frac{1}{N}\sum_{x\in 2\mathbb{Z}} \left(\phi(\tfrac{x}{N}) - \phi(\tfrac{x+1}{N})\right)\mathbb{1}_{x\geq 0}\\ &\to \int \phi(x)\mathbb{1}_{x\geq 0}dx + 0.\end{aligned}$$

Other configurations corresponding to the same density profile include $\eta \in \Omega$ for which $\eta(x) \neq 1$ for finitely many $x \geq 0$ and $\eta(x) = 0$ otherwise.

Lastly, we will also look at examples of probability distributions corresponding to the same density profile.

Example 2.6. If for the configuration η, we let $\eta(x)$ follow a Poisson(1) distribution, independently for every $x \geq 0$, and set $\eta(x) = 0$ for $x < 0$, then this again corresponds to the density profile $\rho(x) = \mathbb{1}_{x\geq 0}$. To see this, we first look at the expectation

$$\mathbb{E}\left[\int \phi(x)d\pi^N(x)\right] = \frac{1}{N}\sum_{x\in\mathbb{Z}} \mathbb{E}\left[\eta(x)\right]\phi(\tfrac{x}{N}) = \frac{1}{N}\sum_{x\in\mathbb{Z}} \phi(\tfrac{x}{N})\mathbb{1}_{x\geq 0} \to \int \phi(x)\mathbb{1}_{x\geq 0}dx. \tag{2.13}$$

For the variance, since all the $\eta(x)$ are independent, the variance of the sum is the sum of the variances, hence

$$\mathrm{Var}\left[\int \phi(x)d\pi^N(x)\right] = \sum_{x\in\mathbb{Z}} \mathrm{Var}\left[\frac{1}{N}\eta(x)\phi(\tfrac{x}{N})\right] = \frac{1}{N^2}\sum_{x\in Z}\phi^2(\tfrac{x}{N})\mathbb{1}_{x\geq 0}.$$

Since we already know by the Riemann integral of $\phi^2(x)$ that

$$\frac{1}{N}\sum_{x\in Z}\phi^2(\tfrac{x}{N})\mathbb{1}_{x\geq 0} \to \int \phi^2(x)\mathbb{1}_{x\geq 0}dx,$$

we see that the variance approximates this integral times $\frac{1}{N}$, for very large N, which will in turn go to zero as $N\to\infty$, i.e.,

$$\mathrm{Var}\left[\int \phi(x)d\pi^N(x)\right] \approx \frac{1}{N}\int \phi^2(x)\mathbb{1}_{x\geq 0}dx \to 0. \tag{2.14}$$

Combining 2.13 and 2.14 gives us that

$$\int \phi(x)d\pi^N(x) \xrightarrow{L^2} \int \phi(x)\mathbb{1}_{x\geq 0}dx,$$

and since convergence in L^2 implies convergence in probability, we indeed find that

$$\int \phi(x)d\pi^N(x) \xrightarrow{\mathbb{P}} \int \phi(x)\mathbb{1}_{x\geq 0}dx.$$

By the same calculations, we can generalize this example to any probability distribution ν with the property that if $\eta\sim\nu$, then $\mathbb{E}[\eta(x)] = 1$ for all $x\geq 0$ and $\mathbb{E}[\eta(x)] = 0$ otherwise, and $\mathrm{Var}[\eta(x)] < \infty$ for all $x\in\mathbb{Z}$.

2.4.2 Hydrodynamic limits

The main idea of hydrodynamic limits is to see how the density profile of the microscopic configurations $\{\eta_t, t\geq 0\}$ with initial distribution ν changes over time. It is sometimes the case that this density profile follows a partial differential equation. This is rather interesting if you think about it in the following way: on the microscopic scale level of configurations the dynamics are random, but on the macroscopic scale of densities the behavior is completely deterministic.

The first thing to consider is the rescaling of time. It is often the case that in a fixed time t, a single particle of a Markov process of configurations can only move a distance of order $\sqrt{t}$ away from the origin. For example, think about a rate-1 random walker. However, since we are rescaling space with a factor $\frac{1}{N}$ to get a density profile, we will need to rescale time with a factor N^2 in order to let enough particles move far enough to actually have a significant impact on the macroscopic density profile, because a random walk moves a distance of order N in a time N^2. Therefore we will be looking at the time-rescaled Markov process $\{\eta_{N^2t}, t\geq 0\}$. The hydrodynamic limit is now concerning with the rescaled versions of the empirical measures

$$\pi_t^N := \frac{1}{N}\sum_{x\in\mathbb{Z}} \eta_{N^2t}(x)\delta_{\frac{x}{N}} \tag{2.15}$$

and looks at the behavior of the corresponding density profile ρ_t defined by

$$\lim_{N\to\infty}\pi_t^N = \rho_t(x)dx,$$

i.e., for all $t\geq 0, \phi\in C_c^\infty(\mathbb{R})$ and $\varepsilon > 0$,

$$\lim_{N\to\infty}\mathbb{P}\left(\left|\int \phi d\pi_t^N - \int \phi(x)\rho_t(x)dx\right| \geq \varepsilon\right) = 0.$$

We see this behavior as the *macro-evolution* of the process.

2.5 Simple Symmetric Exclusion Process

In this section we will really look at an example of an Interacting Particle System, namely the Exclusion Process. This process can be seen as a number of particles moving around on a lattice, where only a finite number of particles are allowed to be on the same site at the same time. This example can be seen as a paradigm for the rest of this thesis.

For our example, we will take the lattice to be $\mathbb{Z}^d$, and we will set the maximum number of particles at every site to be one. In this case, the state space of this process will be the set of configurations $\Omega = \{0,1\}^{\mathbb{Z}^d}$, where for $\eta \in \Omega$ we say there is a particle at state $x \in \mathbb{Z}^d$ if $\eta(x) = 1$ and no particle if $\eta(x) = 0$.

Furthermore we will assume that the particles can only move to a nearest neighbor with rate 1, given that there is not already a particle at this site. This specific example is called the *Simple Symmetric Exclusion Process* (SSEP).

Most of the the theory of this chapter is derived from [18, 21].

2.5.1 Generator of SSEP

In order to define the generator of the SSEP, we first need to further understand what happens when a particle is about to jump. Every particle wants to jump from its starting site with rate 1 to a nearest neighbor, but if there is already a particle at this site, no jump will happen. It turns out that this is the same interaction as exchanging the occupancies of the two sites in question with rate 1. Figure 2.3 shows the effect of exchanging the occupancies. If there is only one particle present at the neighboring sites, then this is the same as a jump occurring from one site to the other, and if there are two or no particles present, then this is the same as no jump occurring.

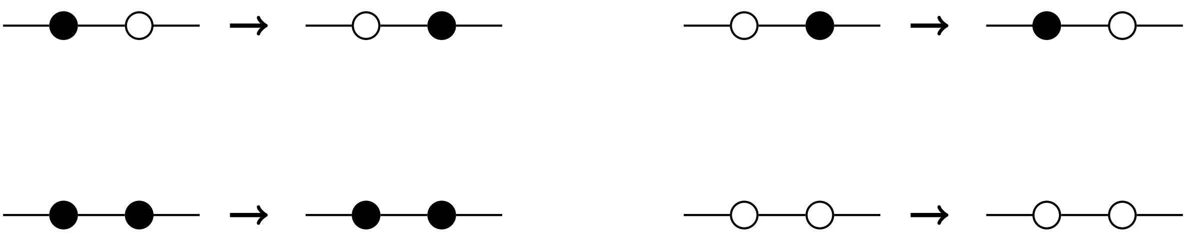

Figure 2.3: The effect of exchanging occupancies of two sites for the four possible cases of there being a particle present (black) or not present (white) at each site.

With this in mind, for $\eta \in \Omega$ we now define $\eta^{x,y}$ as the configuration in Ω obtained by interchanging the occupancies at x and y, i.e.,

$$\eta^{x,y}(z) := \begin{cases} \eta(z) & \text{if } z \notin \{x,y\}, \\ \eta(x) & \text{if } z = y, \\ \eta(y) & \text{if } z = x. \end{cases}$$

If we furthermore denote by $x \sim y$ the relation that $x, y \in \mathbb{Z}^d$ are nearest neighbors, then we can define the generator of the SSEP working on a local function $f \in C(\Omega)$ as

$$Lf(\eta) = \sum_{x \sim y} f(\eta^{x,y}) - f(\eta). \tag{2.16}$$

The set of local functions is also a core for this generator.

2.5.2 Duality of SSEP

Here we will discuss the duality of the SSEP. It turns out that this process is self-dual. As in Definition 2.12, we will need a subset $\Omega' \subset \Omega$ and a duality function $D : \Omega \times \Omega' \to \mathbb{R}$. The subset Ω' has to be chosen in such a way that the process does become easier to analyze in a certain way. We therefore take

$$\Omega' = \Omega_f := \left\{ \xi \in \Omega : \xi(x) \neq 0 \text{ for finitely many } x \in \mathbb{Z}^d \right\}, \tag{2.17}$$

i.e., Ω_f is the subset of configurations in Ω with only finitely many particles.

Before we give the duality result, we first introduce the following notation. For $\eta, \xi \in \Omega$ we say that the relation $\xi \leq \eta$ holds if for any $x \in \mathbb{Z}^d$, $\eta(x) = 1$ implies that $\xi(x) = 1$, i.e., wherever there is a particle in η there is also a particle in ξ.

Theorem 2.17. *The SSEP is self-dual with $\Omega_f \subset \Omega$ defined as in 2.17 and duality function $D : \Omega_f \times \Omega \to \mathbb{R}$ defined as*

$$D(\xi, \eta) := \prod_{x;\xi(x)=1} \eta(x) = \mathbb{1}_{\xi \leq \eta}.$$

Proof. According to Proposition 2.16, all we have to show is that for any $\eta \in \Omega$ and $\xi \in \Omega_f$ we have that

$$LD(\xi, \cdot)(\eta) = LD(\cdot, \eta)(\xi).$$

If we take L as defined in 2.16, since we only consider finite configurations $\xi \in \Omega_f$, we have that both sums $LD(\cdot, \xi)(\eta)$ and $LD(\eta, \cdot)(\xi)$ are finite, hence they exists. Therefore we only have to show that

$$\sum_{x \sim y} D(\xi, \eta^{x,y}) - D(\xi, \eta) = \sum_{x \sim y} D(\xi^{x,y}, \eta) - D(\xi, \eta).$$

It turns out that this is fairly trivial, since $\mathbb{1}_{\xi \leq \eta^{x,y}} = \mathbb{1}_{\xi^{x,y} \leq \eta}$, which can be seen by considering all the possible cases for $\eta(x), \eta(y), \xi(x)$ and $\xi(y)$. □

This duality result is used quite often to prove result regarding the SSEP, since it helps to reformulate a problem for the general case, with possibly infinitely many particles, to the case with only finitely many particles. In the next bit, we will see how this can be used to prove the ergodicity of measures.

2.5.3 Invariance and ergodicity of product Bernoulli measures

For an arbitrary function $\rho : \mathbb{Z} \to [0, 1]$, we define the ν_ρ as the product Bernoulli measures with density profile ρ, i.e.,

$$\nu_\rho := \bigotimes_{x \in \mathbb{Z}^d} \text{Ber}\big(\rho(x)\big). \tag{2.18}$$

In this part we will prove that these measures are invariant and even ergodic if ρ is a constant. We will eventually make use of the duality result that we have proven in the previous part.

Proposition 2.18. *For all $\rho \in [0, 1]$ constant, ν_ρ is an invariant measure for the SSEP.*

Proof. Let f be a local function on Ω, then

$$\int L f d\nu_\rho = \sum_{x \sim y} \int \big(f(\eta^{x,y}) - f(\eta)\big) d\nu_\rho(\eta),$$

where we could interchange the integral and sum because f is a local function, so the sum only has finitely many non-zero elements. The claim now follows from the fact that the product Bernoulli measure with a constant parameter is invariant under permutations and then applying Proposition 2.7. □

Using the same method as above, we can even prove that the measures ν_ρ are reversible if ρ is constant.

Proposition 2.19. *For all $\rho \in [0,1]$ constant, ν_ρ is a reversible measure for the SSEP.*

Proof. Let f and g be a local functions on Ω, then

$$\int g(Lf)d\nu_\rho = \sum_{x\sim y}\int \big(g(\eta)f(\eta^{x,y}) - g(\eta)f(\eta)\big)d\nu_\rho(\eta).$$

By again using that ν_ρ is invariant under permutations, and the fact that interchanging occupancies twice is the same as doing nothing, we have

$$\int g(\eta)f(\eta^{x,y})d\nu_\rho = \int g(\eta^{x,y})f(\eta)d\nu_\rho.$$

Therefore, we indeed find that

$$\int g(Lf)d\nu_\rho = \sum_{x\sim y}\int \big(g(\eta^{x,y})f(\eta) - g(\eta)f(\eta)\big)d\nu_\rho(\eta) = \int (Lg)fd\nu_\rho,$$

so by Proposition 2.8, we see that ν_ρ is reversible. □

To prove the ergodicity of ν_ρ, we will actually show that it is mixing. Then by Proposition 2.15, we indeed find that it is ergodic. Before we prove the mixing property however, we will need the following.

Lemma 2.20. *For $\xi, \xi' \in \Omega_f$ finite configurations and $\{\xi'_t, t \geq 0\}$ the SSEP starting from ξ', we have that*

$$\lim_{t\to\infty} \mathbb{P}\big(\xi \perp \xi'_t\big) = 1.$$

Proof. Since ξ, ξ' are finite configurations, we can write $\xi = \sum_{i=1}^n \delta_{x_i}$ and $\xi' = \sum_{j=1}^m \delta_{y_j}$ for some $x_i, y_j \in \mathbb{Z}^d$ and $n, m \in \mathbb{N}$. We furthermore write $\xi'_t = \sum_{j=1}^m \delta_{y_j(t)}$, where every $\{y_j(t), t \geq 0\}$ is the processes corresponding to the individual particles of ξ'_t starting from y_j respectively. This then tells us that

$$\mathbb{P}\big(\xi \not\perp \xi'_t\big) = \mathbb{P}(y_j(t) = x_i, \text{for some } i,j) \leq \sum_{i,j}\mathbb{P}(y_j(t) = x_i),$$

All we have to show now is that $\mathbb{P}(y_j(t) = x_i) \to 0$ as $t \to \infty$, for every i, j. To see this, note that every process $\{y_j(t), t \geq 0\}$ is irreducible but not positive recurrent, which follows from the fact that a d-dimensional random walk is also not positive recurrent. These two properties combined tell us that the Markov process does not have a stationary measure. A consequence of this is that for all $x \in \mathbb{Z}^d$,

$$\mathbb{P}(y_j(t) = x) \to 0$$

as $t \to \infty$, which ultimately proves the lemma. □

Proposition 2.21. *For all $\rho \in [0,1]$ constant, ν_ρ is mixing with respect to the SSEP.*

Proof. Before we prove the mixing property for functions in $L^2(\Omega,\mu)$, we first prove it for the duality functions $D(\cdot,\xi)$ and $D(\cdot,\xi')$, with $\xi,\xi' \in \Omega$ some finite configurations. First note that

$$\int D(\xi,\eta)d\nu_\rho(\eta) = \rho^{|\xi|}, \qquad \int D(\xi',\eta)d\nu_\rho(\eta) = \rho^{|\xi'|}. \tag{2.19}$$

Now, by self-duality we have that

$$\int D(\xi,\eta)S_t D(\xi',\cdot)(\eta)d\nu_\rho(\eta) = \int D(\xi,\eta)S_t D(\cdot,\eta)(\xi')d\nu_\rho(\eta),$$

where since the SSEP has conserves the number of particles, we can write for any $\eta \in \Omega$,

$$S_t D(\cdot,\eta)(\xi') = \sum_{|\xi''|=|\xi'|} p_t(\xi',\xi'')D(\xi'',\eta),$$

i.e., we sum over all configurations $\xi'' \in \Omega$ that have the same number of particles as ξ', and $\{p_t(\cdot,\cdot)\}_{t\geq 0}$ are the transition probabilities of the SSEP. By Fubini we therefore find that

$$\int D(\xi,\eta)S_t D(\xi',\cdot)(\eta)d\nu_\rho(\eta) = \sum_{|\xi''|=|\xi'|} p_t(\xi',\xi'') \int D(\xi,\eta)D(\xi'',\eta)d\nu_\rho(\eta)$$

We now have two complementary cases. The first case is where the two configurations ξ and ξ'' do not have any particles in common. We will denote this relation by $\xi \perp \xi''$. Notice that in this case

$$D(\xi,\eta)D(\xi'',\eta) = D(\xi+\xi'',\eta).$$

We therefore find that

$$\begin{aligned}\sum_{|\xi''|=|\xi'|} p_t(\xi',\xi'') &\int D(\xi,\eta)D(\xi'',\eta)d\nu_\rho(\eta)\cdot \mathbb{1}_{\xi\perp\xi''}\\ &= \sum_{|\xi''|=|\xi'|} p_t(\xi',\xi'') \int D(\xi+\xi',\eta)d\nu_\rho(\eta)\mathbb{1}_{\xi\perp\xi''} \\ &= \mathbb{P}\big(\xi \perp \xi'_t\big)\cdot \rho^{|\xi|+|\xi'|},\end{aligned} \tag{2.20}$$

with $\{\xi'(t), t\geq 0\}$ the SSEP starting from ξ'. On the other hand we have that

$$\sum_{|\xi''|=|\xi'|} p_t(\xi',\xi'') \int D(\xi,\eta)D(\xi'',\eta)d\nu_\rho(\eta)\cdot \mathbb{1}_{\xi\not\perp\xi''} \leq \mathbb{P}\big(\xi \not\perp \xi'(t)\big) = 1-\mathbb{P}\big(\xi\perp\xi'_t\big), \tag{2.21}$$

since $D(\xi,\eta)\leq 1$ for all $\xi,\eta\in\Omega$. By combining everything, we find that

$$\begin{aligned}\lim_{t\to\infty}&\left|\int D(\xi,\cdot)S_t D(\xi',\cdot)d\nu_\rho - \int D(\xi,\cdot)d\nu_\rho \int D(\xi',\cdot)d\nu_\rho\right| \\ &\leq \lim_{t\to\infty}\left|\mathbb{P}\big(\xi\perp\xi'_t\big)\cdot\rho^{|\xi|+|\xi'|} - \rho^{|\xi|+|\xi'|}\right| + 1 - \mathbb{P}\big(\xi\perp\xi'_t\big)\end{aligned}$$

The claim now follows from Lemma 2.20.

By linearity of the integral and the Markov semigroup $\{S_t, t\geq 0\}$, it is clear that the mixing property also holds for the linear combinations

$$p_n = \sum_{i=1}^n a_i D(\xi_{1,i},\cdot), \qquad q_m = \sum_{j=1}^m b_j D(\xi_{2,j},\cdot),$$

with all $\xi_{1,i},\xi_{2,j}\in\Omega$ finite configurations. Furthermore, we have that every local function can be represented as p_n above, that the local functions are dense in $C(\Omega)$ (by Stone-Weierstrass), and that $C(\Omega)$ is dense in $L^2(\Omega,\mu)$ since Ω is compact. Therefore, the proof for the general case $f,g\in L^2(\Omega,\mu)$ now follows from a density argument. In Section 4.1.2, we show this in detail for the run-and-tumble particle process. □

From the mixing property follows the ergodicity of ν_ρ. It turns out that these measures are the only ergodic measures for the SSEP.

Theorem 2.22. *Let $\nu \in \mathscr{I}$ be ergodic with respect to the SSEP, then there exists a $\rho \in [0,1]$ such that $\nu = \nu_\rho$.*

Proof. The idea of the proof of this theorem can be found in [21, Chapter 4]. In Chapter 4 we will show a similar result for the run-and-tumble particle process. □

Since we are working on a compact space Ω, this means by Proposition 2.14 that we now know the full set of invariant measures. Namely, $\mu \in \mathscr{I}$ if and only if it is a convex combination of the set of probability measures $\{\nu_\rho, \rho \in [0,1]\}$.

2.5.4 Hydrodynamic limit

For this part we will look at the one-dimensional SSEP, i.e., $\Omega = \{0,1\}^{\mathbb{Z}}$. Furthermore, let $\rho : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be a density profile and define the following probability measures on Ω:

$$\nu_\rho^N := \bigotimes_{x\in\mathbb{Z}} \mathrm{Ber}\big(\rho(\tfrac{x}{N})\big). \tag{2.22}$$

By similar calculations as in Example 2.6, it can be shown that these probability distributions correspond to the density profile $\rho(x)$.

Theorem 2.23. *Let $\{\eta^N, N \in \mathbb{N}\}$ be a sequence of configurations such that $\eta^N \sim \nu_\rho^N$ for all $N \in \mathbb{N}$, and let π_t^N be defined as in 2.15, then*

$$\lim_{N\to\infty} \pi_t^N = \rho_t(x)dx,$$

where $\rho_t(x)$ is a weak solution to the heat equation, i.e.,

$$\frac{\partial}{\partial t}\rho_t(x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}\rho_t(x),$$

with initial condition

$$\rho_0(x) = \rho(x).$$

Proof. The proof can be found in [23, Chapter 8]. We will use this line of proof to show the hydrodynamic result for the run-and-tumble particle process in Chapter 5. □

Apart from the hydrodynamic limit for the SSEP, it is also interesting to look at the evolution of local equilibrium distributions. For $y \in \mathbb{Z}$, define the operator $\theta_y : \Omega \to \Omega$ as $\theta_y\eta(x) = \eta(x+y)$ for any $\eta \in \Omega$ and $x \in \mathbb{Z}$. Then define the operator $\tau_y : C(\Omega) \to C(\Omega)$ such as

$$\tau_y f(\eta) = f(\theta_y \eta),$$

for all $f \in C(\Omega)$.

Definition 2.14. Let $\rho : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be a density profile, then a sequence of probability measures $\{\nu^N : N \in \mathbb{N}\}$ is a *local equilibrium distribution* for the SSEP, associated to the density profile ρ, if for all $y \in \mathbb{Z}$ and local functions $f \in C(\Omega)$, we have that

$$\lim_{N\to\infty} \int \tau_{\lfloor yN \rfloor} f d\nu^N = \int f d\nu_{\rho(y)},$$

with $\nu_{\rho(y)}$ as defined in 2.18 with the constant $\rho(y)$.

The idea of a local equilibrium distribution is that for a small area (small compared to $\frac{1}{N}$) around the macropoint $y \in \mathbb{R}$, the distribution of particles is very close to that of $\nu_{\rho(y)}$. This measure is an invariant measure according to Proposition 2.18, so in this area we therefore have some sort of "local equilibrium".

First of all, we have that the probability measures defined in 2.22 are local equilibrium distributions. To see this, we will prove the following proposition.

Proposition 2.24. *For any $y \in \mathbb{Z}$ and local function $f \in C(\Omega)$, we have that*

$$\lim_{N\to\infty} \int \tau_{\lfloor yN \rfloor} f d\nu_\rho^N = \int f d\nu_{\rho(y)},$$

Proof. We will prove that for all finite configurations $\xi \in \Omega_f$ we have that

$$\int \tau_{\lfloor yN \rfloor} D(\xi, \cdot) d\nu_\rho^N \to \int D(\xi, \cdot) d\nu_{\rho(y)}.$$

Then, because every local function can be written as a linear combinations of duality functions, we obtain the general result.

We will write $\xi = \sum_{i=1}^n \delta_{x_i}$ for $x_i \in \mathbb{Z}$, then

$$\tau_{\lfloor yN \rfloor} D(\xi, \eta) = \tau_{\lfloor yN \rfloor} \prod_{i=1}^n \eta(x_i) = \prod_{i=1}^n \eta(x_i + \lfloor yN \rfloor).$$

Therefore we find that

$$\lim_{N\to\infty} \int \tau_{\lfloor yN \rfloor} D(\xi, \cdot) d\nu_\rho^N = \lim_{N\to\infty} \prod_{i=1}^n \rho\left(\tfrac{x_i + \lfloor yN \rfloor}{N}\right) = \prod_{i=1}^n \rho(y) = \int D(\xi, \cdot) d\nu_{\rho(y)},$$

which proves the proposition. □

We then have the following evolution of the local equilibrium distributions.

Theorem 2.25. *For any given density profile $\rho : \mathbb{R} \to \mathbb{R}_{\geq 0}$ we have that*

$$\lim_{N\to\infty} \left| \int f d\nu_\rho^N S_{N^2 t} - \int f d\nu_{\rho_t}^N \right| = 0,$$

where $\rho_t(x)$ is a weak solution to the heat equation with initial condition $\rho_0(x) = \rho(x)$.

More generally, if $\{\nu^N : N \in \mathbb{N}\}$ is a local equilibrium distribution associated to the density profile ρ, then $\{\nu^N S_{N^2 t} : N \in \mathbb{N}\}$ is a local equilibrium distribution associated to the density profile ρ_t, where ρ_t solves the heat equation.

Proof. A proof of this theorem can be found in [19] □

Chapter 3

Duality for run-and-tumble particles

Now that we have the necessary prior knowledge about Markov processes, we can finally define the interacting particle system from Section 1. Since we are working with the state space $\Omega = \mathbb{N}_0^V$, with $V = \mathbb{Z}^d \times S$, we will define the Markov semigroup $\{S_t, t \geq 0\}$ of the run-and-tumble particle process on functions $f \in \mathscr{B}(\Omega)$, and the generator $\mathbb{L}$ on local functions, i.e., functions that only depend on finitely many coordinates in $\mathbb{Z} \times S$. This generator can be written as the sum of three individual parts. Namely, for f a local function, we have

$$\mathbb{L}f = \lambda L_a f + \gamma L_i f + \kappa \mathcal{L} f, \tag{3.1}$$

where $\lambda, \gamma, \kappa \in \mathbb{R}_{\geq 0}$ are constants and the three operators L_a, L_i and $\mathcal{L}$ are again Markov generators.

- L_a is the *active part* of the generator. It gives the particles of a configuration $\eta \in \Omega$ its active motion ("run") in a certain direction by letting a particle move from its initial position, say (x, σ), to its new position $(x + \sigma, \sigma)$. This comes down to the following generator,

$$L_a f(\eta) = \sum_{x,\sigma} \eta(x, \sigma) \left[f(\eta - \delta_{(x,\sigma)} + \delta_{(x+\sigma,\sigma)}) - f(\eta) \right]. \tag{3.2}$$

- L_i is the *internal part* of the generator. This part describes the transitions from one internal state $\sigma \in S$ to some other state $\sigma' \in S$ at each location $x \in \mathbb{Z}^d$, so

$$L_i f(\eta) = \sum_{x,\sigma \neq \sigma'} \eta(x, \sigma) c_x(\sigma, \sigma') \left[f(\eta - \delta_{(x,\sigma)} + \delta_{(x,\sigma')}) - f(\eta) \right]. \tag{3.3}$$

  Here the rates $c_x(\sigma, \sigma')$ can either restrict or encourage the jump between two internal states $\sigma, \sigma' \in S$ at position $x \in \mathbb{Z}$. We will require these rates to be symmetric, i.e., $c_x(\sigma, \sigma')$. In most examples, we will choose the uniform rates $c_x(\sigma, \sigma') = 1$ everywhere.

- $\mathcal{L}$ is the *diffusion generator*. This is the random walk generator

$$\mathcal{L} f(\eta) = \sum_{x \neq y, \sigma} \eta(x, \sigma) c(x, y) \left[f(\eta - \delta_{(x,\sigma)} + \delta_{(y,\sigma)}) - f(\eta) \right], \tag{3.4}$$

  where we require the transition rates $c(x, y)$ to be symmetric again. Furthermore, for existence we require that $\sup_x \sum_y c(x, y) < \infty$ and that $c(x, y) = 0$ if $x - y > R$ for some predetermined $R \in \mathbb{R}$. The easiest example is where $c(x, y) = \frac{1}{2}$ if x and y are nearest neighbors, and 0 otherwise.

Remark 3.1. In order to save some space, we will from now on use the following notation,

$$\eta^{(x_1,\sigma_1),(x_2,\sigma_2)} := \eta - \delta_{(x_1,\sigma_1)} + \delta_{(x_2,\sigma_2)}.$$

Remark 3.2. The number of parameters in 3.1 is redundant. If we assume $\kappa > 0$, then we can also write

$$\mathbb{L}'f := \frac{1}{\kappa}\mathbb{L}f = \lambda' L_a f + \gamma' L_i f + \mathcal{L}f$$

where $\lambda' = \frac{\lambda}{\kappa}$ and $\gamma' = \frac{\gamma}{\kappa}$. This shows that up to a rescaling of time only two parameters are left.

In this chapter, we will prove a duality result of this process by showing that a duality result holds for the three generators independently. Afterwards, we will look at an alternative approach of proving the duality by introducing a so-called deterministic system, and see how this can be used in general for independently moving particles.

3.1 The duality result

To understand the duality, we will first look at a graphical representation of a single run-and-tumble particle on $\mathbb{Z}$ with internal state $\{-1, 1\}$ in Figure 3.1, where for simplicity we have taken $\kappa = 0$. Here we see for every site $x \in \mathbb{Z}$ two internal state, where the particle moves to the right if it is in the blue (right) internal state, and it moves to the left if it is in the red (left) internal state, both with exponential waiting times. Furthermore, the particle switches between the two internal states along the black arrows in-between, which again have exponential waiting times. We can now determine the path of the run-and-tumble particle by following the arrows from top to bottom.

For the dual particle, we want it to follow the same path, but now from the bottom to the top. Therefore, we want the arrows to occur with the same rates, but now in the opposite direction. This lead to the following generator $\widehat{\mathbb{L}}$ for the dual process, defined on local functions f on Ω as follows:

$$\widehat{\mathbb{L}}f := \lambda \widehat{L}_a f + \gamma L_i f + \kappa \mathcal{L} f. \tag{3.5}$$

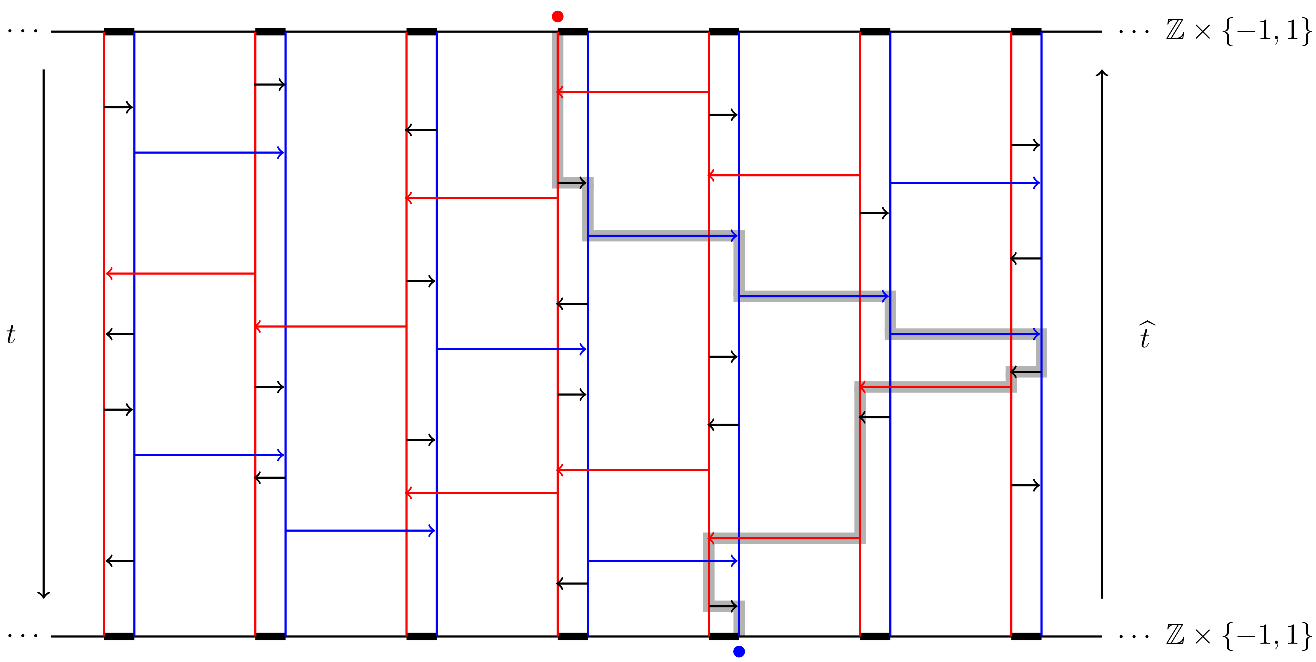

Figure 3.1: Graphical representation of a single run-and-tumble particle with $\kappa = 0$, following the arrows from top to bottom, and a dual particle, following the same path against the arrows from bottom to top.

Here $\widehat{L}_i$ and $\mathcal{L}$ are the same internal and diffusion generators as given in 3.3 and 3.4, and $\widehat{L}_a$ is the active operator that sends a particle in the opposite direction, i.e.,

$$\widehat{L}_a f(\eta) = \sum_{x,\sigma} \eta(x,\sigma) \left[f(\eta^{(x,\sigma),(x-\sigma,\sigma)}) - f(\eta) \right].$$

In order to make the dual process easier to analyze, we will only define it on the space of finite configurations $\xi \in \Omega_f$, as defined in 2.17. To achieve duality between the two generators, we still need a duality function. Define for $k, n \in \mathbb{N}$ the function $d(k,n)$ as follows,

$$d(k,n) := \begin{cases} \frac{n!}{(n-k)!} & \text{if } k \leq n, \\ 0 & \text{if } k > n. \end{cases} \tag{3.6}$$

We then have that the function $\mathscr{D} : \Omega_f \times \Omega \to \mathbb{R}$, defined as

$$\mathscr{D}(\xi,\eta) = \prod_{v \in V} d\big(\xi(v), \eta(v)\big), \tag{3.7}$$

is the duality function. The duality result between $\mathbb{L}$ and $\widehat{\mathbb{L}}$ now follows from the following theorem.

Theorem 3.1. *1. L_a is dual to $\widehat{L}_a$ with duality function $\mathscr{D}$.*

2. L_i is self-dual with duality function $\mathscr{D}$.

3. $\mathcal{L}$ is self-dual with duality function $\mathscr{D}$.

Remark 3.3. The duality result is the reason why we require the transition rates $c_x(\sigma,\sigma')$ and $c(x,y)$ in 3.3 and 3.4 to be symmetric. Without this property, the self-duality of these two generators with respect to $\mathscr{D}$ will not hold. In the proof of the self-duality of L_i we will see why this is the case.

Remark 3.4. From now on we will write $\{\widehat{S}_t, t \geq 0\}$ as the Markov semigroup corresponding to the Markov process generated by $\widehat{\mathbb{L}}$. Furthermore we will write $\{\widehat{p}_t(v,v') : v, v' \in V, t \geq 0\}$ as the corresponding transition probabilities.

Remark 3.5. A direct consequence of the duality result is that for any $v, v' \in V$ we have that

$$p_t(v,v') = \widehat{p}_t(v',v).$$

Indeed, by putting $\eta = \delta_v$ an $\xi = \delta_{v'}$ in 3.7 we have that $\mathscr{D}(\xi,\eta) = \mathbb{1}_{v=v'}$, and hence $S_t\mathscr{D}(\xi,\cdot)(\eta) = p_t(v,v')$ and $\widehat{S}_t\mathscr{D}(\cdot,\eta)(\xi) = \widehat{p}_t(v',v)$.

Before we get to the proof of Theorem, 3.1, we will first prove the following lemma that will help us with our calculations.

Lemma 3.2. *for $k,l,m,n \in \mathbb{N}$ we have that*

$$k\frac{d(k-1,m)d(l+1,n)}{d(k,m)d(l,n)} - n\frac{d(k,m+1)d(l,n-1)}{d(k,m)d(l,n)} = l - n$$

Proof. By a straightforward calculation, we find that

$$\begin{aligned} k\frac{d(k-1,m)d(l+1,n)}{d(k,m)d(l,n)} &- n\frac{d(k,m+1)d(l,n-1)}{d(k,m)d(l,n)} \\ &= \frac{k\frac{m!\cdot n!}{(m-k+1)!(n-l-1)!} - n\frac{(m+1)!(n-1)!}{(m-k+1)!(n-l-1)!}}{\frac{m!\cdot n!}{(m-k)!(n-l)!}} \\ &= (k-m-1)\cdot\frac{n-l}{m-k+1} \\ &= l-n, \end{aligned}$$

which proves the lemma ∎

The rest of this section will be dedicated to proving the individual parts of Theorem 3.1.

3.1.1 Duality of L_a

We will first prove the duality result of the active part of the generator, i.e., we will show that the following relation

$$L_a\mathscr{D}(\xi,\cdot)(\eta) = \widehat{L}_a\mathscr{D}(\cdot,\eta)(\xi),$$

holds. Since we are only interested in finite configurations $\xi \in \Omega$, it is enough to show that for every $\sigma \in S$ we have the following equality,

$$\sum_x \eta(x,\sigma)\left[\mathscr{D}(\xi,\eta^{(x,\sigma),(x+\sigma,\sigma)}) - \mathscr{D}(\xi,\eta)\right] = \sum_x \xi(x,\sigma)\left[\mathscr{D}(\xi^{(x,\sigma),(x-\sigma,\sigma)},\eta) - \mathscr{D}(\xi,\eta)\right]. \tag{3.8}$$

In order to prove this, we have to make a distinction between two cases, namely the case where $\mathscr{D}(\xi,\eta) > 0$ and where $\mathscr{D}(\xi,\eta) = 0$.

The case where $\mathscr{D}(\xi,\eta) > 0$

If $\mathscr{D}(\xi,\eta) > 0$, then 3.8 is equivalent to showing that

$$\sum_x \eta(x,\sigma)\frac{\mathscr{D}(\xi,\eta^{(x,\sigma),(x+\sigma,\sigma)})}{\mathscr{D}(\xi,\eta)} - \xi(x+\sigma,\sigma)\frac{\mathscr{D}(\xi^{(x+\sigma,\sigma),(x,\sigma)},\eta)}{\mathscr{D}(\xi,\eta)} + \xi(x+\sigma,\sigma) - \eta(x,\sigma) = 0.$$

First of all, it is useful to note that for any $x \in \mathbb{Z}^d$, we have $\mathscr{D}(\xi,\eta^{(x,\sigma),(x+\sigma,\sigma)}) = 0$ if and only if $\mathscr{D}(\xi^{(x+\sigma,\sigma),(x,\sigma)},\eta) = 0$. If we assume they are both zero for some $y \in \mathbb{Z}^d$, then we get that

$$\begin{aligned}\eta(y,\sigma)\frac{\mathscr{D}(\xi,\eta^{(y,\sigma),(y+\sigma,\sigma)})}{\mathscr{D}(\xi,\eta)} - \xi(y+\sigma,\sigma)\frac{\mathscr{D}(\xi^{(y+\sigma,\sigma),(y,\sigma)},\eta)}{\mathscr{D}(\xi,\eta)} + \xi(y+\sigma,\sigma) - \eta(y,\sigma)& \\ = \xi(y+\sigma,\sigma) - \eta(y,\sigma).&\end{aligned}$$

Since $\mathscr{D}(\xi,\eta) > 0$ we know that $\xi(x,\sigma) \leq \eta(x,\sigma)$ for all $x \in \mathbb{Z}^d$ and $\sigma \in S$, and $\mathscr{D}(\xi^{(y+\sigma,\sigma),(y,\sigma)},\eta) = 0$ tells us that $\xi(y,\sigma)+1 > \eta(y,\sigma)$. Combining these two facts gives us that $\xi(y,\sigma) = \eta(y,\sigma)$, therefore,

$$\begin{aligned}\eta(y,\sigma)\frac{\mathscr{D}(\xi,\eta^{(y,\sigma),(y+\sigma,\sigma)})}{\mathscr{D}(\xi,\eta)} - \xi(y+\sigma,\sigma)\frac{\mathscr{D}(\xi^{(y+\sigma,\sigma),(y,\sigma)},\eta)}{\mathscr{D}(\xi,\eta)} + \xi(y+\sigma,\sigma) - \eta(y,\sigma)& \\ = \xi(y+\sigma,\sigma) - \xi(y,\sigma).&\end{aligned} \tag{3.9}$$

On the other hand, if we have that $\mathscr{D}(\xi,\eta^{(z,\sigma),(z+\sigma,\sigma)}) > 0$ for some $z \in \mathbb{Z}^d$, then we must also have that $\mathscr{D}(\xi^{(z+\sigma,\sigma),(z,\sigma)},\eta) > 0$. For convenience we will now write

$$\begin{aligned}\xi(z,\sigma) &= k, & \eta(z+\sigma) &= m,\\ \xi(z+\sigma,\sigma) &= l, & \eta(z+\sigma,\sigma) &= n.\end{aligned}$$

By using Lemma 3.2, we then get the following:

$$\begin{aligned}\eta(z,\sigma)\frac{\mathscr{D}(\xi,\eta^{(z,\sigma),(z+\sigma,\sigma)})}{\mathscr{D}(\xi,\eta)} - \xi(z+\sigma,\sigma)\frac{\mathscr{D}(\xi^{(z+\sigma,\sigma),(z,\sigma)},\eta)}{\mathscr{D}(\xi,\eta)}& \\ = m\frac{d(k,m-1)d(l,n+1)}{d(k,m)d(l,n)} - l\frac{d(k+1,m)d(l-1,n)}{d(k,m)d(l,n)}& \\ = m-k& \\ = \eta(z,\sigma) - \xi(z,\sigma),&\end{aligned}$$

so

$$\begin{aligned}\eta(z,\sigma)\frac{\mathscr{D}(\xi,\eta^{(z,\sigma),(z+\sigma,\sigma)})}{\mathscr{D}(\xi,\eta)} - \xi(z+\sigma,\sigma)\frac{\mathscr{D}(\xi^{(z,\sigma),(z-\sigma,\sigma)},\eta)}{\mathscr{D}(\xi,\eta)} &+ \xi(z+\sigma,\sigma) - \eta(z,\sigma) \\ &= \eta(z,\sigma) - \xi(z,\sigma) + \xi(z+\sigma,\sigma) - \eta(z,\sigma) \\ &= \xi(z+\sigma,\sigma) - \xi(z,\sigma).\end{aligned} \tag{3.10}$$

If we combine 3.9 and 3.10, we see that

$$\begin{aligned}\sum_x \eta(x,\sigma)\frac{\mathscr{D}(\xi,\eta^{(x,\sigma),(x+\sigma,\sigma)})}{\mathscr{D}(\xi,\eta)} - \xi(x+\sigma,\sigma)\frac{\mathscr{D}(\xi^{(x+\sigma,\sigma),(x,\sigma)},\eta)}{\mathscr{D}(\xi,\eta)} &+ \xi(x+\sigma,\sigma) - \eta(x,\sigma) \\ &= \sum_x \xi(x+\sigma,\sigma) - \xi(x,\sigma) \\ &= 0,\end{aligned}$$

where the last equality holds since ξ is a finite configuration.

The case where $\mathscr{D}(\xi,\eta) = 0$

If $\mathscr{D}(\xi,\eta) = 0$, to show that 3.8 holds, it is sufficient to show that

$$\eta(x,\sigma)\mathscr{D}(\xi,\eta^{x,\sigma}) - \xi(x+\sigma,\sigma)\mathscr{D}(\xi^{x+\sigma,\sigma},\eta) = 0 \tag{3.11}$$

holds for every $x \in \mathbb{Z}^d$. As before, we know that for any x we have that $\mathscr{D}(\xi,\eta^{(x,\sigma),(x+\sigma,\sigma)}) = 0$ if and only if $\mathscr{D}(\xi^{(x+\sigma,\sigma),(x,\sigma)},\eta) = 0$. If they are both zero for some $x \in \mathbb{Z}$, then trivially

$$\eta(x,\sigma)\mathscr{D}(\xi,\eta^{(x,\sigma),(x+\sigma,\sigma)}) - \xi(x+\sigma,\sigma)\mathscr{D}(\xi^{(x+\sigma,\sigma),(x,\sigma)},\eta) = 0.$$

So assume that $\mathscr{D}(\xi^{(x+\sigma,\sigma),(x,\sigma)},\eta) > 0$. We write again

$$\begin{aligned}\xi(x,\sigma) &= k, & \eta(x,\sigma) &= m, \\ \xi(x+\sigma,\sigma) &= l, & \eta(x+\sigma,\sigma) &= n.\end{aligned}$$

then 3.11 is equivalent to showing that

$$m \cdot d(k,m-1)d(l,n+1) - l \cdot d(k+1,m)d(l-1,n) = 0.$$

$\mathscr{D}(\xi^{(x+\sigma,\sigma),(x,\sigma)},\eta) > 0$ implies that $\xi(x+\sigma,\sigma) - 1 \leq \eta(x,\sigma)$ for all $y \in \mathbb{Z}^d$, and since$\mathscr{D}(\xi,\eta) = 0$ we have that $\xi(x+\sigma,\sigma) > \eta(x,\sigma)$, so by combining these two facts, we see that $\xi(x+\sigma,\sigma) = \eta(x+\sigma,\sigma) + 1$, i.e., $l = n+1$. Therefore

$$\begin{aligned}m \cdot d(k,m-1)d(l,n+1) &- l \cdot d(k+1,m)d(l-1,n) \\ &= m \cdot d(k,m-1)d(l,l) - l \cdot d(k+1,m)d(l-1,l-1) \\ &= m \cdot \frac{(m-1)!}{(m-k-1)!} \cdot l! - l \cdot \frac{m!}{(m-k-1)!} \cdot (l-1)! \\ &= 0.\end{aligned}$$

We see that 3.11 indeed holds, and this concludes the proof of the duality.

3.1.2 Duality of L_i

In order to prove the self-duality of L_i, i.e.,

$$L_i\mathscr{D}(\xi,\cdot)(\eta) = L_i\mathscr{D}(\cdot,\eta)(\xi),$$

we will show that for every $x \in \mathbb{Z}^d$ and $\sigma, \sigma' \in S$ we have that

$$\begin{aligned}\xi(x,\sigma)c_x(\sigma,\sigma')\left[\mathscr{D}(\xi^{(x,\sigma),(x,\sigma')},\eta) - \mathscr{D}(\xi,\eta)\right] + \xi(x,\sigma')c_x(\sigma',\sigma)\left[\mathscr{D}(\xi^{(x,\sigma'),(x,\sigma)},\eta) - \mathscr{D}(\xi,\eta)\right] \\ = \eta(x,\sigma)c_x(\sigma,\sigma')\left[\mathscr{D}(\xi,\eta^{(x,\sigma),(x,\sigma')}) - \mathscr{D}(\xi,\eta)\right] + \eta(y,\sigma)c_x(\sigma',\sigma)\left[\mathscr{D}(\xi,\eta^{(x,\sigma'),(x,\sigma)}) - \mathscr{D}(\xi,\eta)\right].\end{aligned} \tag{3.12}$$

We will only focus on the case where $\mathscr{D}(\xi,\eta) > 0$, $\mathscr{D}(\xi^{(x,\sigma),(x,\sigma')},\eta) > 0$ and $\mathscr{D}(\xi,\eta^{(x,\sigma),(x,\sigma')}) > 0$. The proof of the other cases will then work similarly as those of L_a. We write

$$\begin{aligned} \xi(x,\sigma) &= k, & \eta(x,\sigma) &= m, \\ \xi(x,\sigma') &= l, & \eta(x,\sigma') &= n. \end{aligned}$$

By the symmetry of the transition rates $c_x(\sigma,\sigma') = c_x(\sigma',\sigma)$, 3.12 comes down to showing that

$$\begin{aligned} k\big[d(k-1,m)d(l+1,n) - d(k,m)d(l,n)\big] + l\big[d(k+1,m)d(l-1,n) - d(k,m)d(l,n)\big] \\ = m\big[d(k,m-1)d(l,n+1) - d(k,m)d(l,n)\big] + n\big[d(k,m+1)d(l,n-1) - d(k,m)d(l,n)\big], \end{aligned}$$

which, after dividing everything by $d(k,m)d(l,n)$ and some reordering, is the same as

$$\begin{aligned} k\frac{d(k-1,m)d(l+1,n)}{d(k,m)d(l,n)} - n\frac{d(k,m+1)d(l,n-1)}{d(k,m)d(l,n)} + n - k \\ = m\frac{d(k,m-1)d(l,n+1)}{d(k,m)d(l,n)} - l\frac{d(k+1,m)d(l-1,n)}{d(k,m)d(l,n)} + l - m. \end{aligned} \tag{3.13}$$

Using Lemma 3.2, we have that

$$k\frac{d(k-1,m)d(l+1,n)}{d(k,m)d(l,n)} - n\frac{d(k,m+1)d(l,n-1)}{d(k,m)d(l,n)} = l - n,$$

and similarly

$$m\frac{d(k,m-1)d(l,n+1)}{d(k,m)d(l,n)} - l\frac{d(k+1,m)d(l-1,n)}{d(k,m)d(l,n)} = m - k.$$

Filling these back into 3.14, we indeed get our result.

Remark 3.6. Notice that if $c_x(\sigma,\sigma') \neq c_x(\sigma',\sigma)$ for some $x \in \mathbb{Z}^d$ and $\sigma, \sigma' \in S$, then 3.12 would become

$$\begin{aligned} kc_x(\sigma,\sigma')\frac{d(k-1,m)d(l+1,n)}{d(k,m)d(l,n)} - nc_x(\sigma',\sigma)\frac{d(k,m+1)d(l,n-1)}{d(k,m)d(l,n)} + n - k \\ = mc_x(\sigma,\sigma')\frac{d(k,m-1)d(l,n+1)}{d(k,m)d(l,n)} - lc_x(\sigma',\sigma)\frac{d(k+1,m)d(l-1,n)}{d(k,m)d(l,n)} + l - m. \end{aligned} \tag{3.14}$$

However, this equation is not always true, and therefore the duality would not hold.

3.1.3 Duality of $\mathcal{L}$

Similar to the case of L_i, the self-duality of $\mathcal{L}$ is a consequence of the symmetry property of the rates $c(x,y) = c(y,x)$ for all $x, y \in \mathbb{Z}^d$.

3.2 Duality with the deterministic system

While the proof of the duality result that we have given above is correct, it is computational. Here we will give an alternative approach to prove the duality of the processes generated by $\mathbb{L}$ and $\widehat{\mathbb{L}}$ respectively. We will do this by introducing the so-called "deterministic system".

The deterministic system is a semigroup $\{\psi_t, t \geq 0\}$ on bounded operators $g : \mathscr{B}(V) \to \mathbb{R}$, which lets a function $f \in \mathscr{B}(V)$ follow the motion determined by the Kolmogorov Backward equation of the run-and-tumble particle process. i.e., for every $t \geq 0$ and $g : \mathscr{B}(V) \to \mathbb{R}$ we define

$$(\psi_t g)(f) := g(f_t), \qquad \text{with for all } v \in V\text{: } f_t(v) := \sum_{v' \in V} p_t(v, v') f(v'). \tag{3.15}$$

First we will show that we have a duality result between this deterministic system and a run-and-tumble process $\{\eta_t, t \geq 0\}$

Proposition 3.3. *Let $\eta \in \Omega$ and $f \in \mathscr{B}(V)$ such that $f(v) \neq 1$ for only finitely many $v \in V$, then*

$$S_t \mathcal{D}(f, \cdot)(\eta) = \psi_t \mathcal{D}(\cdot, \eta)(f),$$

where

$$\mathcal{D}(f, \eta) = \prod_{v \in V} f(v)^{\eta(v)}.$$

Proof. Note that the particles of the process $\{\eta_t, t \geq 0\}$ all move independently. Therefore, if we label positions of the particles in η_t as $v_i(t)$, we find that

$$S_t \mathcal{D}(f, \cdot)(\eta) = \mathbb{E}_\eta \mathcal{D}(f, \eta_t) = \mathbb{E}_\eta \prod_{v \in V} f(v)^{\eta_t(v)} = \mathbb{E}_\eta \prod_i f(v_i(t)) = \prod_i \mathbb{E}_{v_i} f(v_i(t)).$$

By the definition of f_t, found in 3.15, we see that $\mathbb{E}_{v_i} f(v_i(t)) = f_t(v_i)$, therefore

$$S_t \mathcal{D}(f, \cdot)(\eta) = \prod_i f_t(v_i) = \prod_{v \in V} f_t(v)^{\eta(v)} = \mathcal{D}(f_t, \eta) = \psi_t \mathcal{D}(\cdot, \eta)(f).$$

which completes the proof. □

Single dual particle case

From this duality result with the deterministic system, we can recover the duality from Theorem 3.1. We will first do this for the case where we only have a single dual particle.

Proposition 3.4. *For $\eta \in \Omega$ and $v_i \in V$ we have that*

$$S_t \mathscr{D}(\delta_{v_i}, \cdot)(\eta) = \widehat{S}_t \mathscr{D}(\cdot, \eta)(\delta_{v_i}).$$

Proof. From Proposition 3.3, we find that

$$\mathbb{E}_\eta \left[\prod_{v \in V} f(v)^{\eta_t(v)} \right] = \prod_{v \in V} \left(\sum_{v' \in V} p_t(v, v') f(v') \right)^{\eta(v)} \tag{3.16}$$

By now taking the derivative with respect to the variable $f(v_i)$, with $v_i \in V$, and afterwards filling in $f(v) = 1$ for all $v \in V$, then on the left-hand side we get

$$\frac{\partial}{\partial f(v_i)} \mathbb{E}_\eta \left[\prod_{v \in V} f(v)^{\eta_t(v)} \right] \Bigg|_{f \equiv 1} = \mathbb{E}_\eta \left[\frac{\partial}{\partial f(v_i)} \prod_{v \in V} f(v)^{\eta_t(v)} \right] \Bigg|_{f \equiv 1} = \mathbb{E}_\eta[\eta_t(v_i)]$$

and on the right hand-side

$$\begin{aligned}
&\frac{\partial}{\partial f(v_i)} \prod_{v\in V} \left(\sum_{v'\in V} p_t(v,v') f(v') \right)^{\eta(v)} \Bigg|_{f\equiv 1} \\
&= \sum_{v\in V} \frac{\partial}{\partial f(v_i)} \left(\sum_{v'\in V} p_t(v,v') f(v') \right)^{\eta(v)} \prod_{v''\neq v} \left(\sum_{v'\in V} p_t(v'',v') f(v') \right)^{\eta(v'')} \Bigg|_{f\equiv 1} \\
&= \sum_{v\in V} p_t(v,v_i)\eta(v).
\end{aligned} \tag{3.17}$$

Now write $\{\widehat{p}_t(v,v') : v,v' \in V, t\geq 0\}$ as the family of transition probabilities of a single particle process generated by the dual generator $\widehat{\mathbb{L}}$, as given in 3.5. Now let $v=(x,\sigma)$ and $v'=(x',\sigma')$ be two particles in V. If $v=v'$, then

$$\mathbb{L}\mathbb{1}_{\{v'\}}(v) = \widehat{\mathbb{L}}\mathbb{1}_{\{v\}}(v') = -3\mathbb{1}_{v=v'} = -3,$$

and if $v\neq v'$, then we have that

$$\mathbb{L}\mathbb{1}_{\{v'\}}(v) = \mathbb{1}_{\{(x',\sigma')\}}(x+\sigma,\sigma) + \mathbb{1}_{\{(x',\sigma')\}}(x,\sigma') + \mathbb{1}_{\{(x',\sigma')\}}(x',\sigma)$$

and

$$\widehat{\mathbb{L}}\mathbb{1}_{\{v\}}(v') = \mathbb{1}_{\{(x,\sigma)\}}(x'-\sigma',\sigma') + \mathbb{1}_{\{(x,\sigma)\}}(x',\sigma) + \mathbb{1}_{\{(x,\sigma)\}}(x,\sigma'),$$

so we can conclude that

$$\mathbb{L}\mathbb{1}_{\{v'\}}(v) = \widehat{\mathbb{L}}\mathbb{1}_{\{v\}}(v').$$

By Proposition 2.16, we then find that for a run-and-tumble particle $v(t)$ and a dual particle $v'(t)$ starting from v and v' respectively, we have that

$$\mathbb{E}_v\left[\mathbb{1}_{\{v'\}}(v(t))\right] = \widehat{\mathbb{E}}_{v'}\left[\mathbb{1}_{\{v\}}(v'(t))\right],$$

which implies that $p_t(v,v') = \widehat{p}_t(v',v)$.

By now going back to 3.17 and filling in this equality, it follows that

$$\frac{\partial}{\partial f(v_i)} \prod_{v\in V} \left(\sum_{v'\in V} p_t(v,v') f(v') \right)^{\eta(v)} \Bigg|_{f\equiv 1} = \sum_{v\in V} p_t(v,v_i)\eta(v) = \sum_{v\in V} \widehat{p}_t(v_i,v)\eta(v) = \widehat{\mathbb{E}}_{v_i}\big(\eta(v_i(t))\big)$$

where we have used $\widehat{\mathbb{E}}_{v_i}$ to denote the expectation of a particle, starting from v_i, under the process generated by $\widehat{\mathbb{L}}$. Since for any $x\in\mathbb{N}_0$ we have that $d(x,1)=x$, we indeed find that

$$S_t\mathscr{D}(\delta_{v_i},\cdot)(\eta) = \mathbb{E}_\eta\big[\mathscr{D}(\delta_{v_i},\eta_t)\big] = \widehat{\mathbb{E}}_{\delta_{v_i}}\big[\mathscr{D}(\delta_{v_i(t)},\eta)\big] = \widehat{S}_t\mathscr{D}(\cdot,\eta)(\delta_{v_i})$$

which is what we wanted to prove. □

Alternative proof of the duality result

We will now give the idea of how we can recover the full duality result from the duality in Proposition 3.3.

Theorem 3.5. *For any $\eta\in\Omega$ and $\xi\in\Omega_f$ we have that*

$$S_t\mathscr{D}(\xi,\cdot)(\eta) = \widehat{S}_t\mathscr{D}(\cdot,\eta)(\xi).$$

Proof. We will look at n particles at position v_i, i.e., $\xi = n\delta_{v_i}$. By taking the n-th order derivative with respect to $f(v_i)$ on the left-hand side of 3.16 and afterwards setting $f \equiv 1$ again, we find that

$$\frac{\partial^n}{\partial f(v_i)^n}\mathbb{E}_\eta\left[\prod_{v\in V} f(v)^{\eta_t(v)}\right]\Bigg|_{f\equiv 1} = \mathbb{E}_\eta\left[\frac{\partial^n}{\partial f(v_i)^n}\prod_{v\in V} f(v)^{\eta_t(v)}\right]\Bigg|_{f\equiv 1} = \mathbb{E}_\eta\big[d(\eta_t(v_i), n)\big]. \quad (3.18)$$

For the right-hand side we have that

$$\begin{aligned}
&\frac{\partial^n}{\partial f(v_i)^n}\prod_{v\in V}\left(\sum_{v'\in V} p_t(v,v')f(v')\right)^{\eta(v)}\Bigg|_{f\equiv 1} \\
&= \sum_{m=1}^{n}\sum_{\substack{v^{(1)},\dots,v^{(m)}\in V\\ v^{(i)}\neq v^{(j)}}}\sum_{k_1+\dots+k_m=n}\binom{n}{k_1,\dots,k_m}\prod_{j=1}^{m}\frac{\partial^{k_j}}{\partial f(v_i)^{k_j}}\left(\sum_{v'\in V} p_t(v^{(j)},v')f(v')\right)^{\eta(v^{(j)})}\Bigg|_{f\equiv 1} \\
&= \sum_{m=1}^{n}\sum_{\substack{v^{(1)},\dots,v^{(m)}\in V\\ v^{(i)}\neq v^{(j)}}}\sum_{k_1+\dots+k_m=n}\binom{n}{k_1,\dots,k_m}\prod_{j=1}^{m} d\left(\eta(v^{(j)}),k_j\right)p_t\left(v^{(j)},v_i\right)^{k_j} \\
&= \sum_{m=1}^{n}\sum_{\substack{v^{(1)},\dots,v^{(m)}\in V\\ v^{(i)}\neq v^{(j)}}}\sum_{k_1+\dots+k_m=n}\binom{n}{k_1,\dots,k_m}\prod_{j=1}^{m} d\left(\eta(v^{(j)}),k_j\right)\widehat{p}_t\left(v_i,v^{(j)}\right)^{k_j},
\end{aligned}$$

where we have again used that $p_t(v,v') = \widehat{p}_t(v',v)$ for all $v,v' \in V$. Notice that this last line is the expected value of $D(\eta,\xi_t)$ if $\{\xi_t, t\geq 0\}$ starts from ξ and is generated by $\widehat{\mathbb{L}}$, i.e., we find that

$$\frac{\partial^n}{\partial f(v_i)^n}\prod_{v\in V}\left(\sum_{v'\in V} p_t(v,v')f(v')\right)^{\eta(v)}\Bigg|_{f\equiv 1} = \widehat{\mathbb{E}}_\xi\big[\mathscr{D}(\eta,\xi_t)\big]. \quad (3.19)$$

Combining 3.18 and 3.19, we indeed find that

$$\mathbb{E}_\eta\big[\mathscr{D}(\eta_t,\xi)\big] = \widehat{\mathbb{E}}_\xi\big[\mathscr{D}(\eta,\xi_t)\big].$$

If we look at $\xi \in \Omega_f$ any finite configuration of particles, i.e., $\xi = \sum_{i=1}^n \delta_{v_i}$ for some $n \in \mathbb{N}$ and $v_i \in V$, then the duality result can be found by taking the derivative with respect to each $f(v_i)$ on both the left-hand side and right-hand side of the equation 3.16. □

3.2.1 The dual deterministic system

We have already shown that from the duality result between the deterministic system and the process itself, we can recover the duality given in Theorem 3.1. Here we will show that we can go even further. We introduce the dual deterministic system, which is the semigroup $\{\widehat{\psi}_t, t \geq 0\}$ given by

$$(\widehat{\psi}_t g)(\widehat{f}) := g(\widehat{f}_t), \qquad \text{with for all } v\in V\text{: } \widehat{f}_t(v) := \sum_{v'\in V}\widehat{p}_t(v,v')\widehat{f}(v').$$

for all $g : \mathscr{B}(V) \to \mathbb{R}$ and $f \in \mathscr{B}(V)$.

It turns out that we have a duality result between the two deterministic systems, and that from this result we can again recover the duality between the two processes generated by $\mathbb{L}$ and $\widehat{\mathbb{L}}$.

Proposition 3.6. *Let* $f, \widehat{f} \in \mathscr{B}(V)$ *be such that* $\widehat{f}(v) \neq 0$ *for only finitely many* $v \in V$, *then*

$$\psi_t \mathrm{D}(\cdot, \widehat{f})(f) = \widehat{\psi}_t \mathrm{D}(f, \cdot)(\widehat{f}),$$

where

$$\mathrm{D}(f, \widehat{f}) = \exp\left(\sum_{v \in V} f(v)\widehat{f}(v)\right).$$

Proof. First of all

$$\psi_t \mathrm{D}(\cdot, \widehat{f})(f) = \exp\left(\sum_{v \in V}\sum_{v' \in V} p_t(v, v')f(v')\widehat{f}(v)\right),$$

and since $p_t(v, v') = \widehat{p}_t(v', v)$,

$$\widehat{\psi}_t \mathrm{D}(f, \cdot)(\widehat{f}) = \exp\left(\sum_{v \in V}\sum_{v' \in V} \widehat{p}_t(v, v')\widehat{f}(v')f(v)\right) = \exp\left(\sum_{v \in V}\sum_{v' \in V} p_t(v', v)\widehat{f}(v')f(v)\right)$$

where since $\widehat{f}(v) \neq 0$ for only finitely many $v \in V$, we can interchange the finite and infinite sum, which then tells us that indeed $\psi_t \mathrm{D}(\cdot, \widehat{f})(f) = \widehat{\psi}_t \mathrm{D}(f, \cdot)(\widehat{f})$. □

We see that we end up with the diagram of dualities given in Figure 3.2.

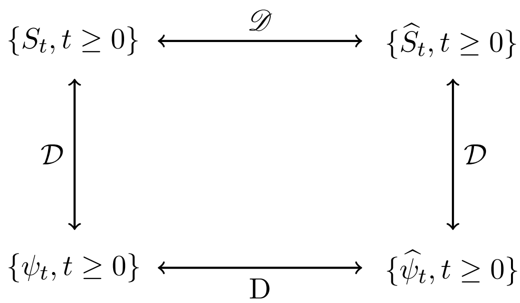

Figure 3.2: Duality results between the Markov semigroups and deterministic processes with the corresponding duality functions.

Remark 3.7. It can be shown that all the dualities in Figure 3.2 are equivalent to one another. Results of this type can be used to simplify proofs for duality results of independent particle systems.

Chapter 4

Ergodic theory of the run-and-tumble particle process

In the last chapter we have proven a duality result of the run-and-tumble particle process. Just as in the case of the Exclusion Process in Section 2.5, we will use this result repeatedly in this chapter to determine the ergodic measures of the process. For this chapter, we assume that the

For $\rho : V \to \mathbb{R}_{\geq 0}$ an arbitrary bounded function, define the probability distribution μ_ρ on Ω as the product Poisson measures with density profile ρ, i.e.,

$$\mu_\rho = \bigotimes_{v \in V} \text{Pois}\big(\rho(v)\big). \tag{4.1}$$

4.1 Invariance and ergodicity of product Poisson measures

4.1.1 Propagation of product Poisson measures

If we take μ_ρ as the initial distribution of the process generated by $\mathbb{L}$ as given in 3.1, then one can wonder about the time evolution of this distribution $\mu_\rho S_t$. It turns out that this distribution will again be a product Poisson distribution μ_{ρ_t}, where the density profile ρ_t can be attained by letting the dual Markov semigroup $\{\widehat{S}_t, t \geq 0\}$ work on the initial density profile ρ, i.e., the following theorem holds.

Theorem 4.1. *Let μ_ρ be as in 4.1, then for any $t \geq 0$ we have that $\mu_\rho S_t = \mu_{\rho_t}$, with $\rho_t(v) = \widehat{S}_t \rho(v)$ for all $v \in V$.*

This result is an application of Doob's theorem, which says that independent Markovian particle evolutions preserve Poisson measures. A more general result of this can be found e.g. in [5].

Proof of Theorem 4.1. Using duality, we see that for any $\xi \in \Omega_f$ we have that

$$\begin{aligned}
\int \mathscr{D}(\xi, \cdot) d\mu_\rho S_t &= \int S_t \mathscr{D}(\xi, \cdot)(\eta) d\mu_\rho(\eta) \\
&= \int \widehat{S}_t \mathscr{D}(\cdot, \eta)(\xi) d\mu_\rho(\eta) \\
&= \int \widehat{\mathbb{E}}_\xi \left[\mathscr{D}(\xi_t, \eta)\right] d\mu_\rho(\eta) \\
&= \widehat{\mathbb{E}}_\xi \left[\int \mathscr{D}(\xi_t, \eta) d\mu_\rho(\eta)\right],
\end{aligned}$$

where we could interchange the integral and expectation in the last equality by Fubini. Since ξ is a finite configuration, we can write $\xi = \sum_{i=1}^{n} \delta_{v_i}$, for some $n \in \mathbb{N}$ and $v_i \in \Omega$ for all i. By conservation of the number particles, this means that $\xi_t = \sum_{i=1}^{n} \delta_{v_i(t)}$, with $\{v_i(t), t \geq 0\}$ the path of a dual run-and-tumble particle, starting from $v_i(0) := v_i$. Using Proposition A.1, this gives us that

$$\int \mathscr{D}(\xi, \cdot) d\mu_\rho S_t = \widehat{\mathbb{E}}_\xi \left[\int \mathscr{D}(\xi_t, \eta) d\mu_\rho(\eta) \right] = \widehat{\mathbb{E}}_\xi \left[\prod_{i=1}^{n} \rho(v_i(t)) \right]. \tag{4.2}$$

We know that the dual particles $v_i(t)$ all move independently, therefore we find that

$$\int \mathscr{D}(\xi, \cdot) d\mu_\rho S_t = \prod_{i=1}^{n} \widehat{\mathbb{E}}_{v_i} \left[\rho(v_i(t)) \right] = \prod_{i=1}^{n} \widehat{S}_t \rho(v_i). \tag{4.3}$$

By using Proposition A.1 in the reverse direction this time, we see that $\mu_\rho S_t$ is again a product Poisson distribution with density profile $\rho_t(v) = \widehat{S}_t \rho(v)$. □

From Theorem 4.1 we can immediately conclude the following

Corollary 4.2. *If ρ is constant, then μ_ρ is S_t-invariant.*

4.1.2 Ergodicity of Poisson measures

In the previous section we have seen that if μ_ρ is defined as in 4.1 and ρ is constant, then μ_ρ is an invariant measure for the run-and-tumble particle process. In this section we will see that these measures are also ergodic. To prove this we will actually show a stronger result, namely that the measures are mixing.

Theorem 4.3. *if ρ is constant, then μ_ρ is mixing with respect to S_t.*

Proof. For this proof, we will work with a similar method as the proof of Proposition 2.21, i.e., we will first look at the case where $g = \mathscr{D}(\xi, \cdot)$ and $f = \mathscr{D}(\xi', \cdot)$, with $\xi, \xi' \in \Omega_f$ finite configurations. By duality we have that

$$S_t \mathscr{D}(\xi', \cdot)(\eta) = \widehat{S}_t \mathscr{D}(\cdot, \eta)(\xi') = \sum_{|\xi''| = |\xi'|} \widehat{p}_t(\xi', \xi'') \mathscr{D}(\xi'', \eta), \tag{4.4}$$

We can view this infinite sum as an integral and so, since $\widehat{p}_t(\xi', \xi'') \mathscr{D}(\xi, \eta) \mathscr{D}(\xi'', \eta)$ is a non-negative, measurable function, Fubini then tells us that

$$\int \mathscr{D}(\xi, \cdot) \big(S_t \mathscr{D}(\xi', \cdot)\big) d\mu_\rho = \sum_{|\xi''| = |\xi'|} p_t(\xi', \xi'') \int \mathscr{D}(\xi, \eta) \mathscr{D}(\xi'', \eta) d\mu_\rho(\eta).$$

We now again have the two complementary cases, namely the case where $\xi \perp \xi''$ or where $\xi \not\perp \xi''$. Since the function $\mathscr{D}$ is a product over the whole particle state space V, the relation $\xi \perp \xi''$ implies that

$$\mathscr{D}(\xi, \eta) \mathscr{D}(\xi'', \eta) = \mathscr{D}(\xi + \xi'', \eta).$$

Therefore, by Proposition A.1 we find that, just like in 2.20,

$$\begin{aligned} &\sum_{|\xi''| = |\xi'|} p_t(\xi', \xi'') \int \mathscr{D}(\xi, \eta) \mathscr{D}(\xi'', \eta) d\mu_\rho(\eta) \cdot \mathbb{1}_{\xi' \perp \xi''} \\ &\qquad = \sum_{|\xi''| = |\xi'|} p_t(\xi', \xi'') \int \mathscr{D}(\xi + \xi'', \eta) d\mu_\rho(\eta) \cdot \mathbb{1}_{\xi \perp \xi''} \\ &\qquad = \widehat{\mathbb{P}} \left(\xi \perp \xi'_t \right) \rho^{|\xi| + |\xi'|}, \end{aligned} \tag{4.5}$$

On the other hand, if $\xi' \not\leq \xi''$, then similarly as in 2.21, we can simply upper bound the whole thing,

$$\sum_{|\xi''|=m} p_t(\xi',\xi'') \int \mathscr{D}(\xi,\eta)\mathscr{D}(\xi'',\eta)d\mu_\rho(\eta) \cdot \mathbb{1}_{\xi' \not\leq \xi''} \leq C_{\xi,\xi'}\big(1 - \mathbb{P}(\xi \perp \xi'_t)\big). \tag{4.6}$$

Here however, we have the constant

$$C_{\xi,\xi'} := \sup_{|\xi''|=|\xi'|} \left\{ \int \mathscr{D}(\xi,\eta)\mathscr{D}(\xi'',\eta)d\mu_\rho(\eta) \right\}. \tag{4.7}$$

This constant is finite due to the fact that by the Hölder inequality,

$$\int \mathscr{D}(\xi,\eta)\mathscr{D}(\xi'',\eta)d\mu_\rho(\eta) = ||\mathscr{D}(\xi,\cdot)\mathscr{D}(\xi'',\cdot)||_1 \leq ||\mathscr{D}(\xi,\cdot)||_2 \cdot ||\mathscr{D}(\xi'',\cdot)||_2,$$

and $||\mathscr{D}(\xi,\cdot)||_2 < \infty$ and $||\mathscr{D}(\xi'',\cdot)||_2 < \infty$, which is evident by the fact that if $|\xi| = n$ for some $n \in \mathbb{N}$, then $\mathscr{D}(\xi,\cdot)^2$ is at most a (multivariate) polynomial of order $2n$. Therefore, by Lemma A.2, we can find finite configurations $\xi_1, \xi_2, ..., \xi_{2n}$ such that

$$\mathscr{D}(\xi,\cdot)^2 = \sum_{k=1}^{2n} \mathscr{D}(\xi_k,\cdot).$$

Since we have that $\int \mathscr{D}(\xi_k,\eta)d\mu_\rho(\eta) < \infty$ for all k, we indeed find that $||\mathscr{D}(\xi,\cdot)||_2 < \infty$, and by a similar argument we have that $||\mathscr{D}(\xi'',\cdot)||_2 < \infty$.

Combining 4.5 and 4.6 gives us that

$$\begin{aligned}\lim_{t\to\infty} &\left| \int \mathscr{D}(\xi,\cdot)(S_t\mathscr{D}(\xi',\cdot))d\mu_\rho - \int \mathscr{D}(\xi,\cdot)d\mu_\rho \int \mathscr{D}(\xi',\cdot)d\mu_\rho \right| \\ &\leq \lim_{t\to\infty} \left| \widehat{\mathbb{P}}\left(\xi' \perp \xi_t\right) \rho^{|\xi|+|\xi'|} - \rho^{|\xi|+|\xi'|} \right| + C_{\xi,\xi'}\left(1 - \mathbb{P}(\xi' \perp \xi_t)\right)\end{aligned}$$

The convergence now follows from a result similar to Lemma 2.20 for the dual run-and-tumble particle process. This result can be proven in the exact same way.

This concludes the proof for the case where $g = \mathscr{D}(\xi,\cdot)$ and $f = \mathscr{D}(\xi',\cdot)$, and by some standard computations, we can also find that it holds for

$$g = \sum_{i=1}^{p} \mathscr{D}(\xi_i,\cdot), \qquad f = \sum_{j=1}^{q} \mathscr{D}(\xi'_j,\cdot),$$

where $\xi_i, \xi'_j \in \Omega_f$ for all i,j. Therefore, by Lemma A.2, the statement also holds for the (multivariate) polynomials.

The rest of the proof now follows from a density argument. We start by taking two arbitrary functions $f, g \in L^2(\Omega, \mu_\rho)$. It is a known fact that if a measure has a moment generating function, then the polynomials are dense in L^2, see e.g. ([2, Corollary 2.3.3]). Since this is the case for our measure μ_ρ, we can take two sequences of polynomials $\{p_n\}_{n\in\mathbb{N}}$ and $\{q_n\}_{n\in\mathbb{N}}$ such that

$$||f - p_n||_2 \to 0, \qquad ||g - q_n||_2 \to 0$$

as $n \to \infty$. We start with the following.

$$\begin{aligned}&\left| \int g(S_tf)d\mu_\rho - \int fd\mu_\rho \int gd\mu_\rho \right| \\ &\quad \leq \left| \int g(S_tf)d\mu_\rho - \int q_n(S_tp_n)d\mu_\rho \right| + \left| \int q_n(S_tp_n)d\mu_\rho - \int p_nd\mu_\rho \int q_nd\mu_\rho \right| \\ &\quad + \left| \int p_nd\mu_\rho \int q_nd\mu_\rho - \int fd\mu_\rho \int gd\mu_\rho \right|.\end{aligned} \tag{4.8}$$

Here we see that the second term goes to zero, as we let t go to infinity, by what we have already proven. We therefore need to show that the first and the last term can be bounded, uniformly for $t \geq 0$, in such a way that it still goes to zero as n goes to infinity. The first term can be upper bounded as follows.

$$\begin{aligned}
&\left|\int g(S_t f)d\mu_\rho - \int q_n(S_t p_n)d\mu_\rho\right| \\
&\quad\leq \left|\int g(S_t f)d\mu\rho - \int g(S_t p_n)d\mu_\rho\right| + \left|\int g(S_t p_n)d\mu\rho - \int q_n(S_t p_n)d\mu_\rho\right| \\
&\quad\leq \int \left|g\big(S_t(f-p_n)\big)\right| d\mu_\rho + \int |(g-q_n)(S_t p_n)|\, d\mu_\rho \\
&\quad= \left|\left|g\big(S_t(f-p_n)\big)\right|\right|_1 + ||(g-q_n)(S_t p_n)||_1.
\end{aligned}$$

By now applying Hölder's inequality twice and using the fact that S_t is a contraction on $L^2(\Omega, \mu_\rho)$ by Proposition 2.9, we find that

$$\left|\int g(S_t f)d\mu_\rho - \int q_n(S_t p_n)d\mu_\rho\right| \leq ||g||_2 \cdot ||f-p_n||_2 + ||g-q_n||_2 \cdot ||p_n||_2.$$

Here we have that $||g||_2 < \infty$ and $||p_n||_2 \to ||f||_2$, so the right-hand side will go to zero as we take $n \to \infty$.

For the last term of 4.8, we have that

$$\begin{aligned}
&\left|\int p_n d\mu_\rho \int q_n d\mu_\rho - \int f d\mu_\rho \int g d\mu_\rho\right| \\
&\leq \left|\int p_n d\mu_\rho \int q_n d\mu_\rho - \int f d\mu_\rho \int q_n d\mu_\rho\right| + \left|\int f d\mu_\rho \int q_n d\mu_\rho - \int f d\mu_\rho \int g d\mu_\rho\right| \\
&\leq \int |f-p_n| d\mu_\rho \int |q_n| d\mu_\rho + \int |f| d\mu_\rho \int |g-q_n| d\mu_\rho \\
&= ||f-p_n||_1 \cdot ||q_n||_1 + ||f||_1 \cdot ||g-q_n||_1.
\end{aligned}$$

Since we are working on a probability space, $p_n \xrightarrow{L^2} f$ implies $p_n \xrightarrow{L^1} f$, and so also $q_n \xrightarrow{L^1} g$, therefore this upper bound will also go to zero as $n \to \infty$.

From this we can conclude that

$$\lim_{t\to\infty} \left|\int g(S_t f)d\mu_\rho - \int f d\mu_\rho \int g d\mu_\rho\right| = 0,$$

hence μ_ρ is mixing with respect to $\{S_t, t \geq 0\}$. □

By Proposition 2.15 we now know that the measures μ_ρ are ergodic with respect to the Markov semigroup $\{S_t, t \geq 0\}$. The logical follow-up question is if these are the only ergodic measures for this process. Using the same method as Kuoch and Redig in [16], we will see that this is the case if we restrict our space of probability measures in the following way:

$$\mathscr{P}_{temp}(\Omega) := \left\{\mu \in \mathscr{P}(\Omega) : \text{for any } n \in \mathbb{N}, \sup_{|\xi|=n} \int \mathscr{D}(\xi, \cdot)d\mu < \infty\right\} \tag{4.9}$$

We call the probability measures in this set *tempered measures*. The fact that $\mu_\rho \in \mathscr{P}_{temp}(\Omega)$ for every $\rho \geq 0$ is evident from Proposition A.1. Our main goal this section is then to prove the following theorem.

Theorem 4.4. *Let $\mu \in \mathscr{P}_{temp}(\Omega)$ be ergodic with respect to the run-and-tumble particle system, then there exists a constant $\rho \geq 0$ such that $\mu = \mu_\rho$.*

Before we do this, we will first need to introduce some theory about coupling.

4.2 Coupling of run-and-tumble particles

Coupling is a method of comparing two random elements with one another, and it is used in many different areas of probability theory [12]. We start with its definition.

Definition 4.1. Let $\{\xi_t, t \geq 0\}$ and $\{\eta_t, \geq 0\}$ be two stochastic processes. A *coupling* is a pair $(\hat{\xi}_t, \hat{\eta}_t)$ such that $\xi_t \overset{d}{=} \hat{\xi}_t$ and $\eta_t \overset{d}{=} \hat{\eta}_t$ for all $t \geq 0$.

We are interested in when a coupling is so-called "successful". This means at some point in time, the two parts of the coupling meet and never split up.

Definition 4.2. We say there exists a *succesful coupling* of the processes $\{\xi_t, t \geq 0\}$ and $\{\eta_t, t \geq 0\}$ if there exists a coupling $\{(\hat{\xi}_t; \hat{\eta}_t), t \geq 0\}$ such that

$$\mathbb{P}_{\xi;\eta}\Big(\exists \tau > 0 \ \forall t \geq \tau : \hat{\xi}_t = \hat{\eta}_t\Big) = 1.$$

we call the corresponding τ the *coupling time* of $\{\xi_t, t \geq 0\}$ and $\{\eta_t, t \geq 0\}$.

To actually achieve a successful coupling, all we need is for $\tau = \min\{t \geq 0 : \xi_t = \eta_t\}$ to be $\mathbb{P}$-a.s. finite. From that point on, by the graphical representation of the run-and-tumble particle process given in Figure 3.1, we can let the particles in the coupling $(\hat{\xi}_{\tau+t}, \hat{\eta}_{\tau+t})$ follow the same arrows, so they will stay together for all $t > 0$.

In this section, we will prove that a successful coupling of independent run-and-tumble particles exists. We will do this by first looking at the very simple case where we couple two particles on $V = \mathbb{Z} \times \{-1, 1\}$. After that we will prove that a successful coupling exists for two particles in the general 1-dimensional case, and use this to prove it for any two finite configurations with the same number of particles in the 1-dimensional case.

4.2.1 Coupling of two particles on $\mathbb{Z} \times \{-1, 1\}$

Let $\{(X_t, \sigma_t^X), t \geq 0\}$ and $\{(Y_t, \sigma_t^Y), t \geq 0\}$ be two particles in the state space $\mathbb{Z} \times \{-1, 1\}$, with starting positions $(x, \sigma^x), (y, \sigma^y)$ respectively, following the process generated by the generator for one run-and-tumble particle, i.e.,

$$\begin{aligned} Lf(x,\sigma) = \lambda\big(&f(x+\sigma,\sigma) - f(x,\sigma)\big) \\ &+ \gamma\big(f(x,-\sigma) - f(x,\sigma)\big) \\ &+ \kappa\big(f(x+1,\sigma) + f(x-1,\sigma) - 2f(x,\sigma)\big). \end{aligned}$$

Note that here we have taken the simple random walk generator as the diffusion generator, and the rates $c_x(1,-1) = c_x(-1,1) = 1$. This is mainly for a more convenient graphical representation, since the proof of a successful coupling works the same for any symmetric random walk generator and symmetric rates $c_x(1,-1) = c_x(-1,1)$.

Now define the following process $\{(Z_t, \sigma_t), t \geq 0\}$ on $\mathbb{Z} \times \{-1, 0, 1\}$ as $Z_t = X_t - Y_t$ and $\sigma_t = \frac{1}{2}(\sigma_t^X - \sigma_t^Y)$. Note that this process is a continuous-time Markov chain that can graphically described as in Figure 4.1.

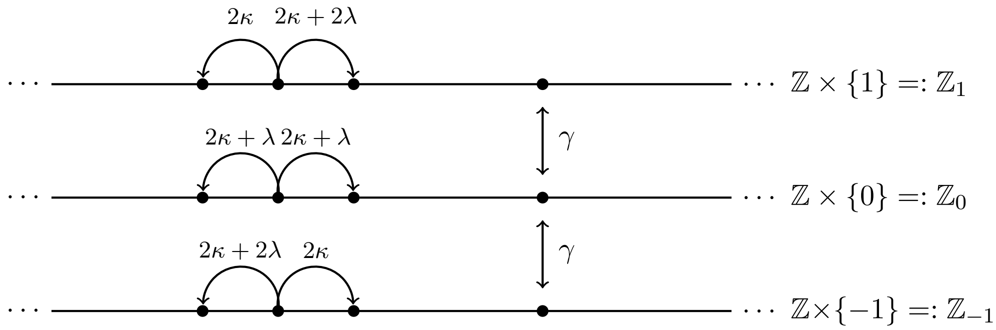

Figure 4.1: Graphical representation of the motion of (Z_t, σ_t).

A succesfull coupling of $\{(X_t, \sigma_t^X), t \geq 0\}$ and $\{(Y_t, \sigma_t^Y), t \geq 0\}$ is now equivalent to proving that $(Z_\tau, \sigma_\tau) = (0, 0)$ for some a.s. finite time $\tau > 0$, i.e., we need to show that the random walk $\{(Z_t, \sigma_t), t \geq 0\}$ is recurrent. We will first show this for the even simpler case where $\kappa = 0$.

The case where $\kappa = 0$

If $\kappa = 0$, then the graphical description of Figure 4.1 becomes the following:

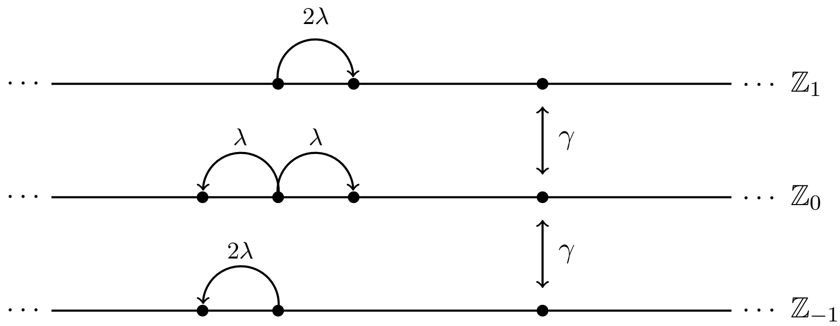

Figure 4.2: Graphical representation of the motion of (Z_t, σ_t) when $\kappa = 0$.

Since it is clear that the Markov process $\{\sigma_t, t \geq 0\}$ is recurrent, we can assume without loss of generality that we start from the point $(Z_0, 0) \in \mathbb{Z}_0$. From this point, with probability $\frac{\gamma}{2\lambda+2\gamma}$, the next jump that occurs takes us to $\mathbb{Z}_1$. Once it goes up there, it will make a random number of jumps to the right before returning to $\mathbb{Z}_0$. Since all the jumps are i.i.d., where a jump to the right happens with probability $\frac{2\lambda}{2\lambda+\gamma}$ and down with probability $\frac{\gamma}{2\lambda+\gamma}$, this number follows a Geometric distribution with parameter $\frac{\gamma}{2\lambda+\gamma}$.

By the same reasoning, if we start at $\mathbb{Z}_0$, the next jump takes us to $\mathbb{Z}_{-1}$ with probability $\frac{\gamma}{2\lambda+2\gamma}$. There it will also make a number of jumps to the left, which is is Geometrically distributed with parameter $\frac{\gamma}{2\lambda+\gamma}$, before returning to $\mathbb{Z}_0$.

Now let X be distributed as follows:

– X is uniform on $\{-1, 1\}$ with probability $\frac{2\lambda}{2\lambda+2\gamma}$,

– X is uniform on $\left\{-\text{Geo}\left(\frac{\gamma}{2\lambda+\gamma}\right), \text{Geo}\left(\frac{\gamma}{2\lambda+\gamma}\right)\right\}$ with probability $\frac{2\gamma}{2\lambda+2\gamma}$.

Define the random variables $X_k \sim X$ i.i.d. for all $k \in \mathbb{N}$ and consider the the following discrete random walk $\{Z_n, n \in \mathbb{N}_0\}$ on $\mathbb{Z}$,

$$Z_n = Z_0 + \sum_{k=1}^{n} X_k.$$

From the definition of the random variable X, we can easily deduce that $\mathbb{E}[X] = 0$ and $\mathbb{E}[|X|] < \infty$. By the result of Chung and Fuchs on random walks on $\mathbb{Z}$ given in [7], this tells us that the discrete random walk $\{Z_n, n \in \mathbb{N}_0\}$ is recurrent.

We can view $\{Z_n, n \in \mathbb{N}_0\}$ as the discrete version of the Markov process $\{(Z_t, \sigma_t), t \geq 0\}$, where it is only interested in the times where the process is at $\mathbb{Z}_0$.

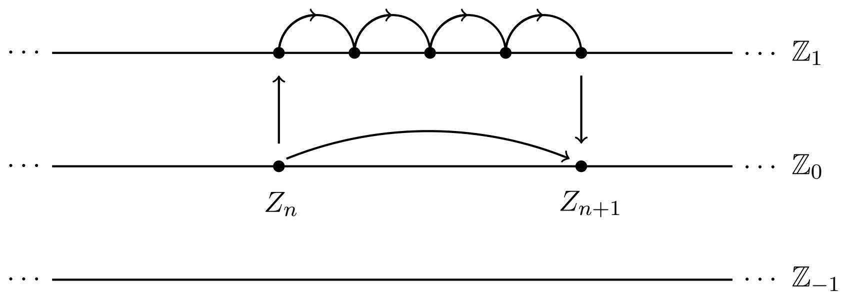

Figure 4.3: Example of a jump of the discrete process $\{Z_n, n \in \mathbb{N}\}$.

By this observation, it is clear that the recurrence of $\{Z_n, n \in \mathbb{N}_0\}$ implies the recurrence of $\{(Z_t, \sigma_t), t \geq 0\}$. Therefore we can conclude that we have a successful coupling of the two particles $\{(X_t, \sigma_t^X), t \geq 0\}$ and $\{(Y_t, \sigma_t^Y), t \geq 0\}$.

The case where $\kappa \neq 0$

For $\kappa \neq 0$ the same strategy works, but the distribution of X is different. Whereas in the previous version, the jumps in $\mathbb{Z}_1$ were only to the right, now they can also be to the left. From Figure 4.1 we can deduce that the number of jumps taken before returning to $\mathbb{Z}_0$ is still a Geometric distribution, but now with parameter $\frac{\gamma}{4\kappa+2\lambda+\gamma}$. Set $N \sim \text{Geo}\left(\frac{\gamma}{4\kappa+2\lambda+\gamma}\right)$ and let m be distributed as follows

$$\mathbb{P}(m = 1) = \frac{2\kappa + 2\lambda}{4\kappa + 2\lambda}, \qquad \mathbb{P}(m = -1) = \frac{2\kappa}{4\kappa + 2\lambda}.$$

By now setting the random variables $m_i \sim m$ be i.i.d. for all $i \in \mathbb{N}$ and

$$S = \sum_{k=0}^{\infty} \sum_{i=1}^{k} m_i \mathbb{1}\{N = k\},$$

we can define X as follows

- X is uniform on $\{-1, 1\}$ with probability $\frac{4\kappa+2\lambda}{4\kappa+2\lambda+2\gamma}$,
- X is uniform on $\{-S, S\}$ with probability $\frac{2\gamma}{4\kappa+2\lambda+2\gamma}$.

It is clear that X is symmetric, and since $|S| \leq N$, we also have that $\mathbb{E}[|X|] < \infty$. Therefore, if we define $\{Z_n, n \geq \mathbb{N}_0\}$ in the same way as before, it is again recurrent and we attain a successful coupling of $\{(X_t, \sigma_t^X), t \geq 0\}$ and $\{(Y_t, \sigma_t^Y), t \geq 0\}$.

4.2.2 Coupling of two particles on $\mathbb{Z} \times S$

Now we consider the general case where the internal state space S an arbitrary finite subset of $\mathbb{Z}$. It turns out that through a similar method as the one described in the previous case, we can again achieve a successful coupling of two particles.

Let (X_t, σ_t^X) and (Y_t, σ_t^Y) be two particles in $\mathbb{Z} \times S$ following the process generated by

$$Lf(x,\sigma) = \lambda\big(f(x+\sigma,\sigma) - f(x,\sigma)\big) \\ + \gamma \sum_{\sigma' \in S} \big(f(x,\sigma') - f(x,\sigma)\big) \\ + \kappa\big(f(x+1,\sigma) + f(x-1,\sigma) - 2f(x,\sigma)\big).$$

Define the Markov process (Z_t, σ_t) on $\mathbb{Z} \times S^2$ as $Z_t = X_t - Y_t$ and $\sigma_t = (\sigma_t^X, \sigma_t^Y)$. Since S is a finite set we can order the elements, i.e., we can set $S = \{\sigma_1, \sigma_2, .., \sigma_n\}$ for some $n \in \mathbb{N}$. We see that for any $z \in \mathbb{Z}$, the active part of the process $(Z_t, \sigma_1, \sigma_2)$ can be graphically described as

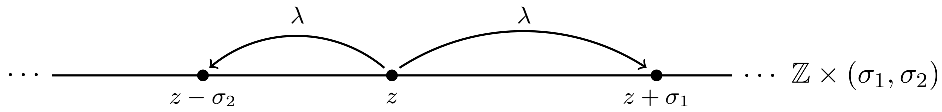

Figure 4.4: Rates of the active part on $\mathbb{Z} \times (\sigma_1, \sigma_2)$.

while for $(Z_t, \sigma_2, \sigma_1)$ we have

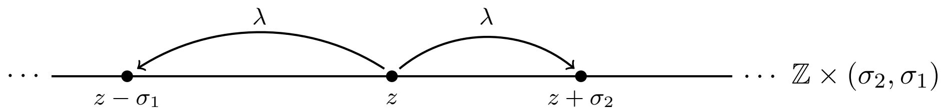

Figure 4.5: Rates of the active part on $\mathbb{Z} \times (\sigma_2, \sigma_1)$.

For the internal part, we have that S^2 is a finite set and the Markov process (σ_t^X, σ_t^Y) is irreducible, since for any two $(\sigma_{i_1}, \sigma_{j_1}), (\sigma_{i_2}, \sigma_{j_2}) \in S^2$, we can get from one to the other with positive probability, e.g. through the following path

$$(\sigma_{i_1}, \sigma_{j_1}) \xrightarrow{\gamma} (\sigma_{i_2}, \sigma_{j_1}) \xrightarrow{\gamma} (\sigma_{i_2}, \sigma_{j_2}).$$

This tells us that the process (σ_t^X, σ_t^Y) is recurrent. Therefore we can assume without loss of generality that $(\sigma_0^X, \sigma_0^Y) = (\sigma_1, \sigma_1)$, and if we then look at the first jump of the internal state

$$\varsigma_1 := \min\{t > 0 : (\sigma_t^X, \sigma_t^Y) \neq (\sigma_1, \sigma_1)\},$$

then the following stopping time

$$\tau_1 := \min\{t > \varsigma_1 : (\sigma_t^X, \sigma_t^Y) = (\sigma_1, \sigma_1)\}$$

is a.s. finite.

Now we look at the distribution of the position Z_{τ_1}. The claim is that it is symmetric around its starting position Z_0. To see this, we take an arbitrary excursion on the internal space S^2

$$(\sigma_1, \sigma_1) \to (\sigma_{i_1}, \sigma_{j_1}) \to (\sigma_{i_2}, \sigma_{j_2}) \to ... \to (\sigma_{i_m}, \sigma_{j_m}) \to (\sigma_1, \sigma_1)$$

where $m \in \mathbb{N}$. Then we note that with equal probability, the following excursion

$$(\sigma_1, \sigma_1) \to (\sigma_{j_1}, \sigma_{i_1}) \to (\sigma_{j_2}, \sigma_{i_2}) \to ... \to (\sigma_{j_m}, \sigma_{i_m}) \to (\sigma_1, \sigma_1)$$

could take place. With Figures 4.4 and 4.5 in mind, it indeed follows that $Z_{\tau_1} = Z_0 + X + R$, where X is a symmetric random variable given by the active motion of the particles, and R

is a symmetric random variable given by the symmetric random walk generated by $\mathcal{L}$ from 3.4 up until time τ_1.

Now define, recursively for $n \geq 2$, the stopping times

$$\varsigma_n := \min\{t > \tau_{n-1} : (\sigma_t^X, \sigma_t^Y) \neq (\sigma_1, \sigma_1)\},$$

and

$$\tau_n := \min\{t > \varsigma_n : (\sigma_t^X, \sigma_t^Y) = (\sigma_1, \sigma_1)\}.$$

Defining the discrete process $Z = \{Z_n, n \in \mathbb{N}\}$ as $Z_n := Z_{\tau_n}$ and taking the random variables $X_k \sim X$ and $R_k \sim R$ for all $k \in \mathbb{N}$, we find that

$$Z_n = Z_0 + \sum_{k=1}^{n} X_k + R_k,$$

i.e., Z is a symmetric random walk on $\mathbb{Z}$. The recurrence of Z now follows from the following lemma, combined with the result of Chung and Fuchs [7].

Lemma 4.5. *We have that* $\mathbb{E}\big[|X+R|\big] < \infty$.

Proof. Let M be the length of the excursion on S^2 up until τ_1, and define

$$J_{\max} := \max\{|\sigma|, |x-y| : \sigma \in \ S,\ x, y \in \mathbb{Z} \text{ with } c(x,y) > 0\}.$$

This $J_{\max}$ is the distance of the largest possible jump between two sites (from both the active motion and the random walk). Furthermore, define $N \sim \text{Geo}\left(\frac{2\gamma(|S|-1)}{4\kappa+2\lambda+2\gamma(|S|-1)}\right)$ and set $N_k \sim N$ for all $k \in \mathbb{N}$. These N_k resemble the number of jumps on $\mathbb{Z}$ the two particles will take with internal states (σ_i, σ_j) before one particle switches to another internal state. We then have that

$$|X+R| \leq \sum_{m=1}^{\infty} \sum_{k=1}^{m} J_{\max} N_k \mathbb{1}_{\{M=m\}},$$

so, since M is independent of every N_k, we have that

$$\mathbb{E}\big[|X+R|\big] \leq \sum_{m=1}^{\infty} \sum_{k=1}^{m} J_{\max} \mathbb{E}\big[N_k \mathbb{1}_{\{M=m\}}\big] = \sum_{m=1}^{\infty} m \cdot J_{\max} \mathbb{E}\big[N\big] \mathbb{P}(M=m) = J_{\max} \mathbb{E}[N] \mathbb{E}[M].$$

Here it is clear that $\mathbb{E}[N] < \infty$, and the fact that $\mathbb{E}[M] < \infty$ follows from the fact that M is a first-hitting time for a Markov chain on the finite state space S^2, which has a finite first moment. This indeed shows that $\mathbb{E}\big[|X+R|\big] < \infty$. □

From the recurrence of Z, it follows that there exists an $r \in \mathbb{N}$ such that τ_r is a.s. finite, and $Z_{\tau_r} = 0$. Since we defined the stopping time τ_r in such a way that $\sigma_{\tau_r}^X = \sigma_{\tau_r}^Y = \sigma_1$, we have that

$$(X_{\tau_r}, \sigma_{\tau_r}^X) = (Y_{\tau_r}, \sigma_{\tau_r}^Y),$$

therefore we indeed have a successful coupling.

4.2.3 Coupling finitely many particles on $\mathbb{Z} \times S$

Now that we have proven that we have a successful coupling of two run-and-tumble particles, we can extend to a successful coupling of two configurations of particles $\{\xi_t, t \geq 0\}$ and $\{\zeta_t, t \geq 0\}$. Of course, due to preservation of particles, the two configurations have to contain the same number of particles, say $n \in \mathbb{N}$. The successful coupling now follows from a method called the Ornstein-coupling (See [12, Chapter 3]). This method works as follows:

First we write the two configurations as two n-dimensional vectors, where the components are independent run-and-tumble particles, i.e.,

$$\xi_t = \begin{pmatrix} (X_t^{(1)}, \sigma_t^{(X,1)}) \\ (X_t^{(2)}, \sigma_t^{(X,2)}) \\ \vdots \\ (X_t^{(n)}, \sigma_t^{(X,n)}) \end{pmatrix}, \qquad \zeta_t = \begin{pmatrix} (Y_t^{(1)}, \sigma_t^{(Y,1)}) \\ (Y_t^{(2)}, \sigma_t^{(Y,2)}) \\ \vdots \\ (Y_t^{(n)}, \sigma_t^{(Y,n)}) \end{pmatrix}.$$

Then we let the processes evolve until the first coördinates of the two vectors coincide, i.e., $(X_t^{(1)}, \sigma_t^{(X,1)}) = (Y_t^{(1)}, \sigma_t^{(Y,1)})$ for some time $t > 0$. The time until this happens is a.s. finite by what we have already proven, and since all the particles in the vectors are independent, we can from there on let the first coördinates remain together while the other coördinates continue to evolve without being affected.

Afterwards, we wait until the second coördinates of the two vectors coincide, and so on, until the point that all coördinates coincide. Since there are only finitely many particles, this will still happen in an a.s. finite time. We therefore achieve a successful coupling of the processes $\{\xi_t, t \geq 0\}$ and $\{\zeta_t, t \geq 0\}$.

4.2.4 Coupling of two particles on $\mathbb{Z}^d \times S$

A successful coupling of two run-and-tumble particles on $\mathbb{Z}^d \times S$ for $d \geq 3$ has not yet been found. For $d = 2$ one could copy the method in Section 4.2.2 and show that $\mathbb{E}\left[|X + R|^2\right] < \infty$ holds, but for $d \geq 3$ this would not work since a random walk in $\mathbb{Z}^d$ would not be recurrent.

In the case of a simple random walk, a successful coupling does exists. This is achieved, much like in the coupling of finitely many particles in $\mathbb{Z}$ above, by an Ornstein-coupling. For details on this, see [12, Section 3.2]. This method would not work for run-and-tumble particles since all the coordinates are dependent of the internal state. Therefore we can not couple the internal states first, so after we have coupled the first coordinates, the internal state between the particles can still differ and split them up again.

There are also examples of two run-and-tumble particles where no successful coupling exists. For instance, set $\kappa = 0$ and let $S = \{e_1, -e_1\}$ with e_1 the basis vector, then a successful coupling is impossible if the particles differ in any of the other coordinates.

However, one would still suspect that, if $\kappa > 0$ and the random walk is simple, a successful coupling could take place. Evidence of this claim has nonetheless not yet been found.

4.3 Full ergodic theory

In Section 4.1.2 we have already seen that the measure μ_ρ as defined in 4.1 is ergodic for the run-and-tumble particle process if $\rho \geq 0$ is a constant. In this section we will prove Theorem 4.4 which says that, in the space of tempered measures defined in 4.9, these measures are the only ergodic measures. In order to prove this theorem, we will need some knowledge of harmonic functions. For this section, we will assume that we are in a scenario where a successful coupling exists.

Definition 4.3. Let $\{\xi_t, t \geq 0\}$ be a dual run-and-tumble process starting from $\xi \in \Omega_f$, then a function $g : \Omega_f \to \mathbb{R}$ is *harmonic* if for every $t \geq 0$ it satisfies

$$\widehat{\mathbb{E}}_\xi\big[g(\xi_t)\big] = g(\xi).$$

Proposition 4.6. *if μ is an invariant measure for the Markov semigroup $\{S_t, t \geq 0\}$, then the following function,*

$$\widehat{\mu}(\xi) = \int \mathscr{D}(\xi, \eta) d\mu(\eta) \tag{4.10}$$

is harmonic.

Proof. By Fubini we have that

$$\widehat{\mathbb{E}}_\xi\left[\widehat{\mu}(\xi_t)\right] = \widehat{\mathbb{E}}_\xi\left[\int \mathscr{D}(\xi_t,\eta)d\mu(\eta)\right] = \int \widehat{\mathbb{E}}_\xi\left[\mathscr{D}(\xi_t,\eta)\right]d\mu(\eta) = \int \widehat{S}_t\mathscr{D}(\cdot,\eta)(\xi)d\mu(\eta).$$

Now, by using duality and the fact that μ is invariant, we find that

$$\int \widehat{S}_t\mathscr{D}(\cdot,\eta)(\xi)d\mu(\eta) = \int S_t\mathscr{D}(\xi,\cdot)(\eta)d\mu(\eta). = \int \mathscr{D}(\xi,\eta)d\mu(\eta) = \widehat{\mu}(\xi),$$

so $\widehat{\mu}$ is indeed harmonic. □

Now we will use the result from the previous section, where we have proven that there exists a successful coupling between two arbitrary configurations in Ω_f with the same number of particles. The following theorem now gives us a useful consequence of this property with respect to bounded harmonic functions. For this theorem, define Ω_n as the configurations in Ω_f with n particles, i.e.,

$$\Omega_n = \{\xi \in \Omega_f : |\xi| = n\}.$$

Theorem 4.7. *If there exists a successful coupling, then bounded harmonic functions g on Ω_n are constant.*

Proof. Let g be a bounded harmonic function and take $\xi,\zeta \in \Omega_n$. By the succesful coupling, there exists a stopping time τ with $\mathbb{P}(\tau < \infty) = 1$ such that $\mathbb{P}(\xi_t = \zeta_t) = 1$ for all $t \geq \tau$. Since g is harmonic, we have that

$$|g(\xi) - g(\zeta)| = \left|\widehat{\mathbb{E}}_\xi\left[g(\xi_t)\right] - \widehat{\mathbb{E}}_\zeta\left[g(\zeta_t)\right]\right| = \left|\widehat{\mathbb{E}}_{\xi;\zeta}\left[g(\xi_t) - g(\zeta_t)\right]\right|$$

If $t \geq \tau$, then $\widehat{\mathbb{E}}_{\xi;\zeta}\left[g(\xi_t) - g(\zeta_t)\right] = 0$, so we only have to look at the case where $t < \tau$. Furthermore, since g is bounded we get the upper bound $g(\xi_t) - g(\zeta_t) \leq 2||g||_\infty$. Therefore we find that

$$|g(\xi) - g(\zeta)| = \left|\widehat{\mathbb{E}}_{\xi;\zeta}\left[\left(g(\xi_t) - g(\zeta_t)\right)\mathbb{1}_{\{t<\tau\}}\right]\right| \leq 2||g||_\infty \mathbb{P}(t < \tau)$$

Due to the fact that τ is a.s. finite, by now letting $t \to \infty$ we see that $|g(\xi) - g(\zeta)| = 0$. This proves the theorem. □

This theorem tells us that if $g : \Omega_f \to \mathbb{R}$ is a bounded harmonic function, then it is a function that only depends on the number of particles in the configurations $\xi \in \Omega_f$, i.e., there exists a function $f : \mathbb{N} \to \mathbb{R}$ such that for all $\xi \in \Omega_f$ we have that $g(\xi) = f(|\xi|)$. If we assume that $\mu \in \mathscr{P}_{temp}(\Omega)$, then by definition we have that $\widehat{\mu}$, as given in 4.10, is a bounded harmonic function. We therefore can find an $f : \mathbb{N} \to \mathbb{R}$ such that

$$\int \mathscr{D}(\xi,\eta)d\mu(\eta) = f(|\xi|). \tag{4.11}$$

We are now ready to give the proof of Theorem 4.4.

Proof of Theorem 4.4. Assume that $\mu \in \mathscr{P}_{temp}(\Omega)$ is an ergodic measure with respect to $\{S_t, t \geq 0\}$. In this proof we will show that the function $f : \mathbb{N} \to \mathbb{R}$ as given in 4.11 has the property that for any $\xi, \xi' \in \Omega_f$,

$$f(|\xi| + |\xi'|) = f(|\xi|) \cdot f(|\xi'|). \tag{4.12}$$

If this is the case, then we know that the function f is of the form $f(|\xi|) = \rho^{|\xi|}$ for some $\rho \geq 0$. By Proposition A.1, this then finishes the proof.

In order to prove that 4.12 holds, we will look at the following limit,

$$\lim_{T\to\infty} \int \mathscr{D}(\xi,\eta)\cdot \frac{1}{T}\int_0^T S_t\mathscr{D}(\xi',\eta)dtd\mu(\eta). \tag{4.13}$$

This limit will help us since, by the fact that μ is ergodic, the Birkhoff ergodic theorem (Theorem 2.11) tells us that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T S_t\mathscr{D}(\xi',\eta)dt = \int \mathscr{D}(\xi',\eta)d\mu(\eta) = f(|\xi'|)$$

By now applying the dominated Convergence Theorem to 4.13, we then find that

$$\lim_{T\to\infty} \int \mathscr{D}(\xi,\eta)\cdot \frac{1}{T}\int_0^T S_t\mathscr{D}(\xi',\eta)dtd\mu(\eta) = \int \mathscr{D}(\xi,\eta)\cdot f(|\xi'|)d\mu(\eta) = f(|\xi|)\cdot f(|\xi'|).$$

Therefore, if we can show that we also have

$$\lim_{T\to\infty} \int \mathscr{D}(\xi,\eta)\cdot \frac{1}{T}\int_0^T S_t\mathscr{D}(\xi',\eta)dtd\mu(\eta) = f(|\xi|+|\xi'|),$$

then by the uniqueness of limits, we indeed get our result. The rest of the proof is now dedicated to showing that the above limit holds.

Using 4.4 together with Fubini, we find that

$$\int \mathscr{D}(\xi,\eta)\cdot \frac{1}{T}\int_0^T S_t\mathscr{D}(\xi',\eta)dtd\mu(\eta) = \sum_{\xi''\in\Omega_f} \frac{1}{T}\int_0^T \widehat{p}_t(\xi',\xi'')\int \mathscr{D}(\xi,\eta)\mathscr{D}(\xi'',\eta)d\mu(\eta)dt. \tag{4.14}$$

Just like in the proof of Proposition 2.21 and Theorem 4.3, we now look at the two complementary cases $\xi''\perp\xi$ and $\xi''\not\perp\xi$. If $\xi''\perp\xi$, then

$$\begin{aligned}
&\sum_{\xi''\perp\,\xi} \frac{1}{T}\int_0^T \widehat{p}_t(\xi',\xi'')\int \mathscr{D}(\xi,\eta)\mathscr{D}(\xi'',\eta)d\mu(\eta)dt \\
&\qquad = \sum_{\xi''\perp\,\xi} \frac{1}{T}\int_0^T \widehat{p}_t(\xi',\xi'')\int \mathscr{D}(\xi+\xi'',\eta)d\mu(\eta)dt \\
&\qquad = f(|\xi|+|\xi'|)\cdot \sum_{\xi''\perp\,\xi} \frac{1}{T}\int_0^T \widehat{p}_t(\xi',\xi'')dt \\
&\qquad = f(|\xi|+|\xi'|)\cdot \frac{1}{T}\int_0^T \widehat{\mathbb{P}}(\xi\perp\xi'_t)dt,
\end{aligned} \tag{4.15}$$

where we have used Fubini for the last equality.

Now we again use the fact that $\widehat{\mathbb{P}}(\xi\perp\xi'_t)\to 1$ as $t\to\infty$, to see that for any $\varepsilon>0$ there exists an $S>0$ such that for all $t>S$ we have that

$$|\widehat{\mathbb{P}}(\xi\perp\xi'_t)-1|<\varepsilon.$$

Therefore, we can divide the integral in the last line of 4.15 into two parts, and in this way show that we have the following,

$$\begin{aligned}
\lim_{T\to\infty}\left|\frac{1}{T}\int_0^T \widehat{\mathbb{P}}(\xi\perp\xi'_t)dt-1\right| &= \lim_{T\to\infty}\left|\frac{1}{T}\int_0^T \left(\widehat{\mathbb{P}}(\xi\perp\xi'_t)-1\right)dt\right| \\
&\le \lim_{T\to\infty}\frac{1}{T}\left|\int_0^S \left(\widehat{\mathbb{P}}(\xi\perp\xi'_t)-1\right)dt\right| + \frac{1}{T}\int_S^T |\widehat{\mathbb{P}}(\xi\perp\xi'_t)-1|dt \\
&< \lim_{T\to\infty}\frac{1}{T}\left|\int_0^S \left(\widehat{\mathbb{P}}(\xi\perp\xi'_t)-1\right)dt\right| + \frac{T-S}{T}\varepsilon \\
&= \varepsilon.
\end{aligned}$$

Since this is true for all $\varepsilon > 0$ we find that $\frac{1}{T}\int_0^T \widehat{\mathbb{P}}(\xi \perp \xi'_t)dt \to 1$, which by filling this back into 4.15, gives us that

$$\lim_{T\to\infty} \sum_{\xi''\perp\,\xi} \frac{1}{T}\int_0^T \widehat{p}_t(\xi',\xi'') \int \mathscr{D}(\xi,\eta)\mathscr{D}(\xi'',\eta)d\mu(\eta)dt = f(|\xi|+|\xi'|). \tag{4.16}$$

Now for the second case, if $\xi'' \not\perp \xi$ then through a similar calculation as in 4.15, we find that

$$\lim_{T\to\infty} \sum_{\xi''\not\perp\xi} \frac{1}{T}\int_0^T \widehat{p}_t(\xi',\xi'') \int \mathscr{D}(\xi,\eta)\mathscr{D}(\xi'',\eta)d\mu(\eta)dt \leq \lim_{T\to\infty} C_{\xi,\xi'} \cdot \frac{1}{T}\int_0^T \mathbb{P}(\xi \not\perp \xi'_t)dt = 0 \tag{4.17}$$

where we can take $C_{\xi,\xi'}$ as the constant defined in 4.7

Filling 4.16 and 4.17 into 4.14, we ultimately get that

$$\lim_{T\to\infty} \int \mathscr{D}(\xi,\eta)\cdot\frac{1}{T}\int_0^T S_t\mathscr{D}(\xi',\eta)dtd\mu(\eta) = f(|\xi|+|\xi'|),$$

which ends the proof □

Chapter 5

Hydrodynamic limits of run-and-tumble particles

The main idea of density profiles and hydrodynamic limits is given in Section 2.4. For the run-and-tumble particles this is a little bit different because we are working on the particle state space $V = \mathbb{Z} \times S$ instead of just $\mathbb{Z}$. We deal with that problem by looking at a density profile indexed by $\sigma \in S$. To see how this work, let $\{\eta^N : N \in \mathbb{N}\}$ be a sequence of configurations in Ω, then we will define the empirical measures $\{\pi_\sigma^N : N \in \mathbb{N}\}$ as follows,

$$\pi_\sigma^N := \frac{1}{N} \sum_{x \in \mathbb{Z}} \eta^N(x, \sigma) \delta_{\frac{x}{N}}.$$

We then say that the density profile $\rho(x, \sigma) : \mathbb{Z} \times S \to \mathbb{R}_{\geq 0}$ corresponds to $\{\eta^N : N \in \mathbb{N}\}$ if for all $\sigma \in S$, we have the following convergence, as given in Definition 2.13,

$$\lim_{N \to \infty} \pi_\sigma^N = \rho(x, \sigma) dx.$$

The question of the hydrodynamic limit is now what we can say about the macroscopic evolution of the density profiles $\rho(x, \sigma)$ for every $\sigma \in S$, i.e., if we look at the run-and-tumble particle process $\{\eta_t, t \geq 0\}$ with some initial distribution μ, then what can we say about $\rho_t(x, \sigma)$ defined by

$$\lim_{N \to \infty} \pi_{\sigma,t}^N = \rho_t(x, \sigma) dx,$$

where $\pi_{\sigma,t}^N$ is given by

$$\pi_{\sigma,t}^N := \frac{1}{N} \sum_{x \in \mathbb{Z}} \eta_{N^2 t}^N(x, \sigma) \delta_{\frac{x}{N}}.$$

Afterwards, we will look at the behavior of the overall macroscopic density, i.e., we will look at the evolving density profile $\rho_t(x) : \mathbb{R} \to \mathbb{R}$ defined by

$$\rho_t(x) = \sum_{\sigma \in S} \rho_t(x, \sigma).$$

This density profile gives us the density of all particles at time t and location $x \in \mathbb{Z}$, disregarding their internal states $\sigma \in S$.

Remark 5.1. In the scenario described above we will need a rescaling of the parameters found in the generator 3.1. Since the active part L_a generates a process where the particles only walk in one direction with rate 1, speeding this process up with a factor N^2 will send the particles off to infinity too fast. In order to prevent this, we set $\lambda \to \frac{\lambda}{N}$. Furthermore, the internal part L_i only lets the particles move around the internal state space S, and therefore it does not need to be sped up at all, therefore we set $\gamma \to \frac{\gamma}{N^2}$. We then write $\{S_{N^2t}, t \geq 0\}$ as the Markov semigroup generated by the process

$$\mathbb{L}_N = \kappa N^2 \mathcal{L} + \lambda N L_a + \gamma L_i.$$

Furthermore, for this chapter, we again assume that the random walk is simple and that $c_x(\sigma, \sigma') = 1$ for all $x \in \mathbb{Z}$ and $\sigma, \sigma' \in S$.

5.1 The statement

For a given bounded and continuous density profile $\rho : \mathbb{R} \times S \to \mathbb{R}$, we will define the following probability measures on Ω,

$$\mu_\rho^N := \bigotimes_{(x,\sigma)\in V} \mathrm{Pois}\big(\rho(\tfrac{x}{N}, \sigma)\big).$$

We will then look at the run-and-tumble processes $\{\eta^N_{N^2t} : t \geq 0\}$ starting from $\eta^N \sim \mu^N_\rho$ for all $N \in \mathbb{N}$. It is easy to see that $\{\eta^N : N \in \mathbb{N}\}$ corresponds to the density profile ρ. In this chapter we will show that the hydrodynamics of the run-and-tumble particle process are given by the following PDE for every $\sigma \in S$,

$$\frac{\partial}{\partial t}\rho_t(x,\sigma) = \frac{\kappa}{2}\frac{\partial^2}{\partial x^2}\rho_t(x,\sigma) - \lambda\sigma\frac{\partial}{\partial x}\rho_t(x,\sigma) + \gamma\sum_{\sigma'\neq\sigma}\big(\rho_t(x,\sigma') - \rho_t(x,\sigma)\big), \tag{5.1}$$

with initial condition

$$\rho_0(x,\sigma) = \rho(x,\sigma). \tag{5.2}$$

This statement comes down to the following theorem.

Theorem 5.1. *For every $t \geq 0, \sigma \in S, \varepsilon > 0$ and test function $\phi \in C_c^\infty(\mathbb{R}\times S)$, we have that*

$$\lim_{N\to\infty} \mathbb{P}\left(\left|\frac{1}{N}\sum_{x\in\mathbb{Z}} \eta^N_{N^2t}(\tfrac{x}{N},\sigma)\phi(\tfrac{x}{N},\sigma) - \int \phi(x,\sigma)\rho_t(x,\sigma)dx\right| \geq \varepsilon\right) = 0, \tag{5.3}$$

where $\rho_t(x,\sigma)$ is a weak solution to 5.1 with initial condition 5.2.

Remark 5.2. Notice that this equation also depends on all the other $\rho_t(x,\sigma')$ with $\sigma' \neq \sigma$, i.e., we get a system of PDEs for the density profiles that all depend on each other. This can be represented as follows,

$$\frac{\partial}{\partial t}\rho_t(x,\cdot) = A\rho_t(x,\cdot) + \gamma \mathrm{C}\rho_t(x,\cdot), \qquad \rho_0(x,\sigma) = \rho(x,\sigma) \text{ for all } x \in \mathbb{R}, \sigma \in S \tag{5.4}$$

Here $\rho_t(x,\cdot)$ is the column vector where the entries are the individual density profiles for every $\sigma \in S$, i.e., $\rho_t(x,\cdot) = \big(\rho_t(x,\sigma)\big)_{\sigma\in S}$, the operator A is the differential operator

$$A\rho_t(x,\cdot) = \frac{\kappa}{2}\frac{\partial^2}{\partial x^2}\rho_t(x,\cdot) - \lambda\frac{\partial}{\partial x}\big(\boldsymbol{\sigma}\circ\rho_t(x,\cdot)\big) \tag{5.5}$$

where the operation $\circ$ is the hadamard product, i.e., termwise multiplication, $\boldsymbol{\sigma} = (\sigma)_{\sigma\in S}$, and lastly C is the following matrix,

$$\mathrm{C} = \begin{pmatrix} -(|S|-1) & 1 & \cdots & 1 \\ 1 & -(|S|-1) & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & -(|S|-1) \end{pmatrix}. \tag{5.6}$$

In order to prove Theorem 5.1, we will use the method that is used by Seppäläinen in [23, Chapter 8] to prove the hydrodynamics of the Exclusion Process. This is done by proving convergence of the empirical measures $\pi^N_{\sigma,t}$ in the path space. Namely, every $\bar{\pi}^N_\sigma = \{\pi^N_{\sigma,t} : t \geq 0\}$ is a path in the path space $D_{\mathbf{M}}$, with $\mathbf{M}$ the set of Borel measures on $\mathbb{R}$. Therefore, we also have that $\bar{\boldsymbol{\pi}}^N = (\bar{\pi}^N_\sigma)_{\sigma\in S}$ is a path in the path space $D_{\mathbf{M}^S}$, with $\mathbf{M}^S$ the Borel measures on $\mathbb{R} \times S$.

We now say that path $\bar{\boldsymbol{\alpha}} \in D_{\mathbf{M}^S}$ is a *weak solution* to 5.4 if it satisfies the initial condition for all $\sigma \in S$, i.e.,

$$\bar{\alpha}_\sigma(0) := \rho_0(x,\sigma)dx,$$

and for all $t \geq 0$ and $\phi \in C_c^\infty(\mathbb{R} \times S)$, we have that

$$\bar{\boldsymbol{\alpha}}(t,\phi) - \bar{\boldsymbol{\alpha}}(0,\phi) - \int_0^t \left[\bar{\boldsymbol{\alpha}}(s, A^*\phi) + \gamma \mathrm{C}\bar{\boldsymbol{\alpha}}(s,\phi)\right] ds = 0,$$

where we define

$$\bar{\boldsymbol{\alpha}}(t,\phi) = \int \phi \bar{\boldsymbol{\alpha}}(t),$$

and

$$A^*\phi(x,\cdot) := \frac{\kappa}{2}\frac{\partial^2}{\partial x^2}\phi(x,\cdot) + \lambda \frac{\partial}{\partial x}\big(\boldsymbol{\sigma} \circ \phi(x,\cdot)\big).$$

Using a similar method as in [1], we can prove that there exists a unique weak solution to 5.4. Theorem 5.1 is now a corollary of the convergence of $\bar{\boldsymbol{\pi}}^N$ to this weak solution, therefore we will show that the following result holds.

Theorem 5.2. *Let $\bar{\boldsymbol{\alpha}} \in D_{\mathbf{M}^S}$ be the unique weak solution to* 5.4*, then*

$$\lim_{N\to\infty} \mathbb{P}\big(s_{\mathbf{M}^S}(\bar{\boldsymbol{\pi}}^N, \bar{\boldsymbol{\alpha}}) \geq \varepsilon\big) = 0,$$

where $s_{\mathbf{M}^S}$ is the Skorokhod distance as defined in A.6.

In this chapter, we will first prove some results needed for the proof of 5.2. In Section 5.6, we will give this proof and show how we can use this result to prove Theorem 5.1.

5.2 Introducing the Dynkin martingale

For convenience, we will from now on use the following notation for $\phi \in C_c^\infty(\mathbb{R} \times S)$,

$$\pi^N_{\sigma,t}(\phi) := \int \phi(\cdot,\sigma) d\pi^N_{\sigma,t} = \frac{1}{N}\sum_{x\in\mathbb{Z}} \phi(\tfrac{x}{N},\sigma)\eta^N_{N^2t}(x,\sigma).$$

If we introduce the Dynkin martingale, as it is explained in Appendix A.2, we get that the following stochastic process

$$M^N_{\sigma,t} = \pi^N_{\sigma,t}(\phi) - \pi^N_{\sigma,0}(\phi) - \int_0^t \mathbb{L}_N \pi^N_{\sigma,s}(\phi) ds \tag{5.7}$$

is a martingale. In this section we will then prove the following proposition.

Proposition 5.3. *For all $t \geq 0, N \in \mathbb{N}$ and $\sigma \in S$, we have that*

$$M^N_{\sigma,t} = \pi^N_{\sigma,t}(\phi) - \pi^N_{\sigma,0}(\phi) - \int_0^t \left[\pi^N_{\sigma,s}(A^*_\sigma\phi) + \gamma \sum_{\sigma'\neq\sigma}\left[\pi^N_{\sigma',s}(\phi) - \pi^N_{\sigma,s}(\phi)\right]\right] ds + \mathcal{O}(tN^{-1}), \tag{5.8}$$

where the equality is in probability, and the operator A^*_σ is the differential operator given by

$$A^*_\sigma = \frac{\kappa}{2}\frac{\partial^2}{\partial x^2} + \lambda\sigma\frac{\partial}{\partial x}.$$

Proof. We start with the following calculation of $\mathbb{L}_N \pi^N_{\sigma,s}(\phi)$,

$$\begin{aligned}\mathbb{L}_N \pi^N_{\sigma,s}(\phi) &= \kappa N^2 \frac{1}{N} \sum_{x\in\mathbb{Z}} \phi(\tfrac{x}{N},\sigma) \left[\eta^N_{N^2 s}(x+1,\sigma) + \eta^N_{N^2 s}(x-1,\sigma) - 2\eta^N_{N^2 s}(x,\sigma)\right] \\ &+ \lambda N \frac{1}{N} \sum_{x\in\mathbb{Z}} \phi(\tfrac{x}{N},\sigma) \left[\eta^N_{N^2 s}(x-\sigma,\sigma) - \eta^N_{N^2 s}(x,\sigma)\right] \\ &+ \gamma \frac{1}{N} \sum_{x\in\mathbb{Z}} \phi(\tfrac{x}{N},\sigma) \sum_{\sigma'\neq\sigma} \left[\eta^N_{N^2 s}(x,\sigma') - \eta^N_{N^2 s}(x,\sigma)\right].\end{aligned}$$

By now reordering the summation terms, we find that

$$\begin{aligned}\mathbb{L}_N \pi^N_{\sigma,s}(\phi) &= \kappa N \sum_{x\in\mathbb{Z}} \left[\phi(\tfrac{x+1}{N},\sigma) + \phi(\tfrac{x-1}{N},\sigma) - 2\phi(\tfrac{x}{N},\sigma)\right] \eta^N_{N^2 s}(x,\sigma) \\ &+ \lambda \sum_{x\in\mathbb{Z}} \left[\phi(\tfrac{x+\sigma}{N},\sigma) - \phi(\tfrac{x}{N},\sigma)\right] \eta^N_{N^2 s}(x,\sigma) \\ &+ \gamma \sum_{\sigma'\neq\sigma} \frac{1}{N} \sum_{x\in\mathbb{Z}} \phi(\tfrac{x}{N},\sigma) \left[\eta^N_{N^2 s}(x,\sigma') - \eta^N_{N^2 s}(x,\sigma)\right].\end{aligned} \tag{5.9}$$

In this equation, there are forward and backward differences of the function ϕ around the points $\frac{x}{N}$ for every $x \in \mathbb{Z}$. By writing out the truncated Taylor series of $\phi(\frac{x+1}{N}), \phi(\frac{x-1}{N})$ and $\phi(\frac{x+\sigma}{N})$ around these points, we have that

$$\begin{aligned}\phi(\tfrac{x+1}{N},\sigma) &= \phi(\tfrac{x}{N},\sigma) + \frac{1}{N}\phi'(\tfrac{x}{N},\sigma) + \frac{1}{2N^2}\phi''(\tfrac{x}{N},\sigma) + \frac{1}{6N^3}\phi'''(y_1,\sigma), \\ \phi(\tfrac{x-1}{N},\sigma) &= \phi(\tfrac{x}{N},\sigma) - \frac{1}{N}\phi'(\tfrac{x}{N},\sigma) + \frac{1}{2N^2}\phi''(\tfrac{x}{N},\sigma) + \frac{1}{6N^3}\phi'''(y_2,\sigma), \\ \phi(\tfrac{x+\sigma}{N},\sigma) &= \phi(\tfrac{x}{N},\sigma) + \frac{1}{N}\sigma\phi'(\tfrac{x}{N},\sigma) + \frac{1}{2N^2}\sigma^2\phi''(y_3,\sigma).\end{aligned} \tag{5.10}$$

for some $y_1, y_2, y_3 \in \mathbb{R}$. By filling these back into 5.9, we find that

$$\begin{aligned}\mathbb{L}_N \pi^N_{\sigma,s}(\phi) &= \kappa \frac{1}{N} \sum_{x\in\mathbb{Z}} \frac{1}{2}\phi''(\tfrac{x}{N},\sigma)\eta^N_{N^2 s}(x,\sigma) \\ &+ \lambda \frac{1}{N} \sum_{x\in\mathbb{Z}} \sigma\phi'(\tfrac{x}{N},\sigma)\eta^N_{N^2 s}(x,\sigma) \\ &+ \gamma \sum_{\sigma'\neq\sigma} \frac{1}{N} \sum_{x\in\mathbb{Z}} \phi(\tfrac{x}{N},\sigma) \left[\eta^N_{N^2 s}(x,\sigma') - \eta^N_{N^2 s}(x,\sigma)\right] \\ &+ R_1(\phi, N, s, \sigma)\end{aligned}$$

which can be written as

$$\mathbb{L}_N \pi^N_{\sigma,s}(\phi) = \pi^N_{\sigma,s}(A^*_\sigma \phi) + \gamma \sum_{\sigma'\neq\sigma} \left[\pi^N_{\sigma',s}(\phi) - \pi^N_{\sigma,s}(\phi)\right] + R_1(\phi, N, s, \sigma). \tag{5.11}$$

Here $R_1(\phi, N, s, \sigma)$ is a rest term that has all the truncation terms from the Taylor series in 5.10 in it. However, since $\phi \in C^\infty_c(\mathbb{R})$, we also have that $\phi'', \phi''' \in C^\infty_c(\mathbb{R})$, i.e., they are uniformly bounded. Therefore we can upper bound this rest term by

$$R_1(\phi, N, s, \sigma) \leq C_1 \frac{1}{N^2} \sum_{x\in\mathbb{Z}} (||\phi'''||_\infty + ||\phi''||_\infty||)\eta^N_{N^2 s}(x,\sigma), \tag{5.12}$$

for some $C_1 \in \mathbb{R}$. By now taking the limit of $N \to \infty$, we find that for every $0 \leq s \leq t$ that

$$R_1(\phi, N, s, \sigma) \xrightarrow{\mathbb{P}} 0,$$

i.e., $R(\phi, N, s, \sigma) = \mathcal{O}(N^{-1})$ in probability. Therefore, by filling 5.11 into 5.7, we find that

$$\begin{aligned} M^N_{\sigma,t} &= \pi^N_{\sigma,t}(\phi) - \pi^N_{\sigma,0}(\phi) - \int_0^t \left(\pi^N_{\sigma,s}(A^*_\sigma \phi) + \gamma \sum_{\sigma' \neq \sigma} \left[\pi^N_{\sigma',s}(\phi) - \pi^N_{\sigma,s}(\phi) \right] + \mathcal{O}(N^{-1}) \right) ds \\ &= \pi^N_{\sigma,t}(\phi) - \pi^N_{\sigma,0}(\phi) - \int_0^t \left(\pi^N_{\sigma,s}(A^*_\sigma \phi) + \gamma \sum_{\sigma' \neq \sigma} \left[\pi^N_{\sigma',s}(\phi) - \pi^N_{\sigma,s}(\phi) \right] \right) ds + \mathcal{O}(tN^{-1}), \end{aligned}$$

which finishes the proof. □

5.3 Vanishing martingale

On the right-hand side of 5.8 we can already recognize the PDE given in 5.1. What we need to show now is that the Dynkin martingale on the right-hand side $M^N_{\sigma,t}$, given in 5.7, vanishes as $N \to \infty$. In order to see that this is true, notice first of all that clearly for any $N \in \mathbb{N}$ we have that $\mathbb{E}\left[M^N_{\sigma,0}\right] = 0$. By the fact that a martingale has a constant expectation, this tells us that $\mathbb{E}\left[M^N_{\sigma,t}\right] = 0$ for all $t \geq 0$ and $N \in \mathbb{N}$. Therefore, the Dynkin martingale vanishes as a consequence of the following lemma.

Lemma 5.4. *For any* $0 < T < \infty$,

$$\lim_{N \to \infty} \mathbb{E}\left[\sup_{t \in [0,T]} \left(M^N_{\sigma,t} \right)^2 \right] = 0.$$

Proof. We can apply Doob's maximal inequality to find that

$$\mathbb{E}\left[\sup_{t \in [0,T]} \left(M^N_t \right)^2 \right] = \mathbb{E}\left[\left(\sup_{t \in [0,T]} M^N_{\sigma,t} \right)^2 \right] \leq 4\mathbb{E}\left[\left(M^N_{\sigma,T} \right)^2 \right]$$

where by the Itô isometry we have that

$$\mathbb{E}\left[\left(M^N_{\sigma,T} \right)^2 \right] = \mathbb{E}\left[\left(\int_0^T dM^N_{\sigma,t} \right)^2 \right] = \mathbb{E}\left[\int_0^T d\left\langle M^N_\sigma \right\rangle_t \right] = \mathbb{E}\left[\left\langle M^N_\sigma \right\rangle_T \right].$$

Here $\left\langle M^N_\sigma \right\rangle$ is the quadratic variation process of M^N_σ, as explained in Appendix A.2.1, which by Theorem A.6 is given by

$$\left\langle M^N_\sigma \right\rangle_T = \int_0^T \left[\mathbb{L}_N \left(\pi^N_{\sigma,s}(\phi) \right)^2 - 2\pi^N_{\sigma,s}(\phi) \cdot \mathbb{L}_N \pi^N_{\sigma,s}(\phi) \right] ds.$$

Now, in general, for a generator L of the form $Lf(\eta) = \sum_{\eta' \in \Omega} c(\eta, \eta')\big(f(\eta') - f(\eta)\big)$, we have that

$$\begin{aligned} Lf^2(\eta) - 2f(\eta)Lf(\eta) &= \sum_{\eta' \in \Omega} c(\eta, \eta')\big(f^2(\eta') - f^2(\eta)\big) - 2\sum_{\eta' \in \Omega} c(\eta, \eta')\big(f(\eta')f(\eta) - f^2(\eta)\big) \\ &= \sum_{\eta' \in \Omega} c(\eta, \eta')\big(f(\eta') - f(\eta)\big)^2. \end{aligned}$$

In our case this would mean that

$$\begin{aligned}
&\mathbb{L}_N \left(\pi^N_{\sigma,s}(\phi)\right)^2 - 2\pi^N_{\sigma,s}(\phi)\cdot \mathbb{L}_N \pi^N_{\sigma,s}(\phi) \\
&\quad = \kappa N^2 \sum_{x\in\mathbb{Z}} \eta^N_{N^2 s}(x,\sigma)\frac{1}{N^2}\Big(\left[\phi(\tfrac{x+1}{N},\sigma)-\phi(\tfrac{x}{N},\sigma)\right]^2 + \left[\phi(\tfrac{x-1}{N},\sigma)-\phi(\tfrac{x}{N},\sigma)\right]^2 \Big) \\
&\qquad + \lambda N \sum_{x\in\mathbb{Z}} \eta^N_{N^2 s}(x,\sigma)\frac{1}{N^2}\left[\phi(\tfrac{x+\sigma}{N},\sigma)-\phi(\tfrac{x}{N},\sigma)\right]^2 \\
&\qquad + \gamma \sum_{\sigma'\neq\sigma}\frac{1}{N^2}\sum_{x\in\mathbb{Z}} \phi(\tfrac{x}{N},\sigma)^2 \left[\eta^N_{N^2 s}(x,\sigma') + \eta^N_{N^2 s}(x,\sigma)\right].
\end{aligned}$$

Just like in 5.10, we can find the following truncated Taylor series,

$$\begin{aligned}
\phi(\tfrac{x+1}{N},\sigma) &= \phi(\tfrac{x}{N},\sigma) + \frac{1}{N}\phi'(\tfrac{x}{N},\sigma) + \frac{1}{2N^2}\phi''(y_4,\sigma) \\
\phi(\tfrac{x-1}{N},\sigma) &= \phi(\tfrac{x}{N},\sigma) - \frac{1}{N}\phi'(\tfrac{x}{N},\sigma) + \frac{1}{2N^2}\phi''(y_5,\sigma)
\end{aligned}$$

for some $y_4, y_5 \in \mathbb{R}$. Using these, along with the last truncated Taylor series found in 5.10, we find that

$$\begin{aligned}
&\mathbb{L}_N \left(\pi^N_{\sigma,s}(\phi)\right)^2 - 2\pi^N_{\sigma,s}(\phi)\cdot \mathbb{L}_N \pi^N_{\sigma,s}(\phi) \\
&\quad = \kappa \sum_{x\in\mathbb{Z}} \eta^N_{N^2 s}(x,\sigma)\left(\left[\frac{1}{N}\phi'(\tfrac{x}{N},\sigma)+\frac{1}{2N^2}\phi''(y_4,\sigma)\right]^2 + \left[-\frac{1}{N}\phi'(\tfrac{x}{N},\sigma)+\frac{1}{2N^2}\phi''(y_5,\sigma)\right]^2\right) \\
&\qquad + \lambda\frac{1}{N}\sum_{x\in\mathbb{Z}} \eta^N_{N^2 s}(x,\sigma)\left(\frac{1}{N}\sigma\phi'(\tfrac{x}{N},\sigma)+\frac{1}{2N^2}\sigma^2\phi''(y_3,\sigma)\right)^2 \\
&\qquad + \gamma \sum_{\sigma'\neq\sigma}\frac{1}{N^2}\sum_{x\in\mathbb{Z}} \phi(\tfrac{x}{N},\sigma)^2 \left[\eta^N_{N^2 s}(x,\sigma') + \eta^N_{N^2 s}(x,\sigma)\right] \\
&\quad = \kappa\frac{4}{N^2}\sum_{x\in\mathbb{Z}} \eta^N_{N^2 s}(x,\sigma)\phi'(\tfrac{x}{N},\sigma)^2 \\
&\qquad + \lambda\frac{2}{N^3}\sum_{x\in\mathbb{Z}} \eta^N_{N^2 s}(x,\sigma)\phi'(\tfrac{x}{N},\sigma)^2 \\
&\qquad + \gamma \sum_{\sigma'\neq\sigma}\frac{1}{N^2}\sum_{x\in\mathbb{Z}} \phi(\tfrac{x}{N},\sigma)^2 \left[\eta^N_{N^2 s}(x,\sigma') + \eta^N_{N^2 s}(x,\sigma)\right] \\
&\qquad + R_2(\phi,N,s),
\end{aligned} \tag{5.13}$$

where we have used $(a+b)^2 \leq 2a^2 + 2b^2$ numerous times, and by a similar reasoning as for $R_1(\phi, N, s)$ in 5.12, we have that there exists some $C_2 \in \mathbb{R}$ such that

$$R_2(\phi,N,s) \leq C_2\frac{1}{N^4}\sum_{x\in\mathbb{Z}} ||\phi''||^2_\infty \eta^N_{N^2 s}(x,\sigma). \tag{5.14}$$

From 5.13 and 5.14 we can deduce that, as $N\to\infty$,

$$\mathbb{L}_N \left(\pi^N_{\sigma,s}(\phi)\right)^2 - 2\pi^N_{\sigma,s}(\phi)\cdot \mathbb{L}_N \pi^N_{\sigma,s}(\phi) \xrightarrow{\mathbb{P}} 0.$$

Therefore, by Fubini and the dominated convergence theorem, we see that

$$\lim_{N\to\infty}\mathbb{E}\left[\langle M^N_\sigma\rangle_T\right] = \lim_{N\to\infty}\int_0^T \mathbb{E}\left[\mathbb{L}_N \left(\pi^N_{\sigma,s}(\phi)\right)^2 - 2\pi^N_{\sigma,s}(\phi)\cdot \mathbb{L}_N \pi^N_{\sigma,s}(\phi)\right] ds = 0,$$

and the result follows. □

5.4 Tightness

For a fixed $N \in \mathbb{N}$ and $\sigma \in S$, we have that the process $\bar{\pi}_\sigma^N := \{\pi_{\sigma,t}^N : t \geq 0\}$ has its paths in the path-space $D_{\mathbf{M}}$, with $\mathbf{M}$ the Borel measures on $\mathbb{R}$. We define Q_σ^N to be the probability distribution of this process, i.e., for a Borel set $B \subset D_{\mathbf{M}}$

$$Q_\sigma^N(B) := \mathbb{P}\left(\bar{\pi}_\sigma^N \in B\right)$$

Proposition 5.5. *The sequence of probability measures $\{Q_\sigma^N : N \in \mathbb{N}\}$ is tight in $D_{\mathbf{M}}$.*

Before we get to the proof of this proposition, we will first show that the following inequality holds.

Lemma 5.6. *Let $\{\eta_t^N : N \in \mathbb{N}, t \geq 0\}$ be a sequence of run-and-tumble particles with every η_t^N a configuration of particles on $\frac{1}{N}\mathbb{Z} \times S$ and initial conditions $\eta_0^N \sim \mu_\rho^N$. Then for all $N \in \mathbb{N}$, $t \geq 0$ and $x, y \in \frac{1}{N}\mathbb{Z}$ we have that*

$$\mathbb{E}_{\mu_\rho^N}\left[\eta_t^N(x,\sigma)\right] \leq ||\rho||_\infty \tag{5.15}$$

$$\mathbb{E}_{\mu_\rho^N}\left[\eta_t^N(x,\sigma)^2\right] \leq ||\rho||_\infty^2 + ||\rho||_\infty \tag{5.16}$$

Proof. The first inequality follows directly from Theorem 4.1 and the fact that the operator $\widehat{S}_t$ is a contraction. For the other inequality, we have that

$$\mathbb{E}_{\mu_\rho^N}\left[\eta_t^N(x,\sigma)^2\right] = \int S_t\mathscr{D}(2\delta_{(x,\sigma)}, \cdot)(\eta^N)d\mu_\rho^N(\eta^N) + \int S_t\mathscr{D}(\delta_{(x,\sigma)}, \cdot)(\eta^N)d\mu_\rho^N(\eta^N)$$

Now by duality we have that

$$\int S_t\mathscr{D}(2\delta_{(x,\sigma)}, \cdot)(\eta^N)d\mu_\rho^N(\eta^N) = \int \widehat{S}_t\mathscr{D}(\cdot, \eta^N)(2\delta_{(x,\sigma)})d\mu_\rho^N(\eta^N)$$

and

$$\int S_t\mathscr{D}(\delta_{(x,\sigma)}, \cdot)(\eta^N)d\mu_\rho^N(\eta^N) = \int \widehat{S}_t\mathscr{D}(\cdot, \eta^N)(\delta_{(x,\sigma)})d\mu_\rho^N(\eta^N).$$

So if we then let $X^{(1)}$ and $X^{(2)}$ be two independently moving particles generated by the dual run-and-tumble process on $\frac{1}{N}\mathbb{Z} \times S$, both starting from (x,σ), then by Proposition A.1 we have that

$$\begin{aligned}
\mathbb{E}_{\mu_\rho^N}\left[\eta_t^N(x)^2\right] &\leq \widehat{\mathbb{E}}_{(x,\sigma),(x+y,\sigma)}\left[\int \mathscr{D}(\delta_{X_t^{(1)}} + \delta_{X_t^{(2)}}, \eta^N)d\mu_\rho^N(\eta) + \int \mathscr{D}(\delta_{X_t^{(1)}}, \eta^N)d\mu_\rho^N(\eta)\right] \\
&= \rho(X_t^{(1)})\rho(X_t^{(2)}) + \rho(X_t^{(1)}) \\
&\leq ||\rho||_\infty^2 + ||\rho||_\infty
\end{aligned}$$

Which proves the lemma. □

Proof of Proposition 5.5. By Theorem A.9 and Lemma A.10, one way of proving tightness of the sequence $\{Q_\sigma^N : N \in \mathbb{N}\}$ is by showing that the following two assertions hold:

1. For every $\varepsilon > 0$ and $t \geq 0$ there exists a compact set $K \subset \mathbf{M}$ such that
$$\liminf_{N\to\infty} \mathbb{P}(\pi_{\sigma,t}^N \in K) \geq 1 - \varepsilon.$$

2. For every $\varepsilon > 0$ and $0 < T < \infty$ there exists a $\delta > 0$ such that
$$\limsup_{N\to\infty} \mathbb{P}(\omega(\bar{\pi}_\sigma^N, 2\delta, T+1) \geq \varepsilon) \leq \varepsilon,$$
with
$$\omega(\alpha, \delta, T) = \sup\{d_{\mathbf{M}}\big(\alpha(s), \alpha(t)\big) : s, t \in [0, T], |t - s| < \delta\},$$

We start by proving the first one. Fix $\varepsilon > 0$ and $t \geq 0$, and for some $A > 0$ let K_A be the following set

$$K_A = \left\{\mu \in \mathbf{M} : \mu([-k,k]) \leq A(2k+1)k^2 \text{ for all } k \in \mathbb{N}\right\}.$$

By Theorem A.14 we see that K_A is relatively compact, and the compactness of K_A then follows from the fact that K_A is sequentially closed. By Markov's inequality we now have that

$$\begin{aligned}
\mathbb{P}(\pi^N_{\sigma,t}([-k,k]) \geq A(2k+1)k) &\leq \frac{1}{A(2k+1)k^2}\mathbb{E}\left[\pi^N_{\sigma,t}([-k,k])\right] \\
&= \frac{1}{A(2k+1)k^2 N} \sum_{x\in[-kN,kN]\cap\mathbb{Z}} \mathbb{E}\left[\eta^N_{N^2t}(x,\sigma)\right] \\
&\leq \frac{1}{A(2k+1)k^2 N}(2k+1)N||\rho||_\infty \\
&= \frac{1}{Ak^2}||\rho||_\infty.
\end{aligned}$$

Here we have used the inequality in 5.15. We therefore have that

$$\mathbb{P}(\pi^N_{\sigma,t} \notin K_A) \leq \sum_{k=1}^{\infty} \mathbb{P}(\pi^N_{\sigma,t}([-k,k]) \geq A(2k+1)k) \leq \frac{1}{A}||\rho_0(\cdot,\sigma)||_\infty \sum_{k=1}^{\infty}\frac{1}{k^2} < \infty.$$

By now taking A big enough, we then have that for all $N \in \mathbb{N}$ that $\mathbb{P}(\pi^N_{\sigma,t} \notin K_A) < \varepsilon$, which proves the first part.

For the second part, by filling in the definition of the metric $d_{\mathbf{M}}$ as found in A.8, we find that

$$\begin{aligned}
&\omega(\bar{\pi}^N_\sigma, 2\delta, T+1) \\
&\quad = \sup_{\substack{s,t\in[0,T+1]\\|t-s|<2\delta}} \sum_{j=1}^{\infty} 2^{-j}\left(1 \wedge \left|\pi^N_{\sigma,N^2t}(\phi_j) - \pi^N_{\sigma,N^2s}(\phi_j)\right|\right) \\
&\quad \leq 2^{-m} + \sum_{j=1}^{m} \sup_{\substack{s,t\in[0,T+1]\\|t-s|<2\delta}} 2^{-j}\left(1 \wedge \left|\pi^N_{\sigma,N^2t}(\phi_j) - \pi^N_{\sigma,N^2s}(\phi_j)\right|\right) \\
&\quad \leq 2^{-m} + \sum_{j=1}^{m} \sup_{\substack{s,t\in[0,T+1]\\|t-s|<2\delta}} \left|\pi^N_{\sigma,N^2t}(\phi_j) - \pi^N_{\sigma,N^2s}(\phi_j)\right|.
\end{aligned} \tag{5.17}$$

Here we have taken m arbitrarily, so the first term can be made as small as we want. We now want to show that the expecation of the sum vanishes as we let $N \to \infty$ and $\delta \downarrow 0$. Afterwards, the claim can be shown by using the Markov inequality.

By reintroducing the Dynkin martingale found in 5.7, we have that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{\substack{s,t\in[0,T+1]\\|t-s|<2\delta}}\left|\pi^N_{\sigma,t}(\phi_j)-\pi^N_{\sigma,s}(\phi_j)\right|^2\right]\\
&\quad\le 2\mathbb{E}\left[\sup_{\substack{s,t\in[0,T+1]\\|t-s|<2\delta}}\left|M^N_{\sigma,t}-M^N_{\sigma,s}\right|^2\right]+2\mathbb{E}\left[\sup_{\substack{s,t\in[0,T+1]\\|t-s|<2\delta}}\left|\int_s^t\mathbb{L}_N\pi^N_{\sigma,r}(\phi_j)dr\right|^2\right]\\
&\quad\le 4\mathbb{E}\left[\sup_{\substack{s,t\in[0,T+1]\\|t-s|<2\delta}}\left(M^N_{\sigma,t}\right)^2+\left(M^N_{\sigma,s}\right)^2\right]+2\mathbb{E}\left[\sup_{\substack{s,t\in[0,T+1]\\|t-s|<2\delta}}\left|\int_s^t\mathbb{L}_N\pi^N_{\sigma,r}(\phi_j)dr\right|^2\right]\\
&\quad\le 8\mathbb{E}\left[\sup_{t\in[0,T+1]}\left(M^N_{\sigma,t}\right)^2\right]+2\mathbb{E}\left[\sup_{\substack{s,t\in[0,T+1]\\|t-s|<2\delta}}\left|\int_s^t\mathbb{L}_N\pi^N_{\sigma,r}(\phi_j)dr\right|^2\right]
\end{aligned}\tag{5.18}$$

where we have used the fact that $(a+b)^2\le 2(a^2+b^2)$ twice. By Lemma 5.4, the first term goes to zero as $N\to\infty$. For the second term, by filling in 5.11 we find that

$$\begin{aligned}
&\left|\int_s^t\mathbb{L}_N\pi^N_{\sigma,r}(\phi_j)dr\right|^2=\left[\int_s^t\left(\pi^N_{\sigma,r}(A^*_\sigma\phi_j)+\gamma\sum_{\sigma'\neq\sigma}\left[\pi^N_{\sigma',r}(\phi_j)-\pi^N_{\sigma,r}(\phi_j)\right]+R_1(\phi_j,N,r)\right)dr\right]^2\\
&\le 3\left[\int_s^t\pi^N_{\sigma,r}(A^*_\sigma\phi_j)dr\right]^2+3\left[\int_s^t\gamma\sum_{\sigma'\neq\sigma}\left[\pi^N_{\sigma',r}(\phi_j)-\pi^N_{\sigma,r}(\phi_j)\right]dr\right]^2+3\left[\int_s^t R_1(\phi_j,N,r)dr\right]^2
\end{aligned}\tag{5.19}$$

where we have used the fact that $(a+b+c)^3\le 3(a^2+b^2+c^2)$. By the upper bound on $R_1(\phi_j,N,r)$ in 5.12, we can see that the last term vanishes when $N\to\infty$.

For the other two terms, we have that

$$\left[\int_s^t\pi^N_{\sigma,r}(A^*_\sigma\phi_j)dr\right]^2=\frac{1}{N^2}\left[\int_s^t\sum_{x\in\mathbb{Z}}A^*_\sigma\phi_j(\tfrac{x}{N})\cdot\eta^N_{N^2r}(x,\sigma)dr\right]^2,$$

and

$$\left[\int_s^t\gamma\sum_{\sigma'\neq\sigma}\left[\pi^N_{\sigma',r}(\phi_j)-\pi^N_{\sigma,r}(\phi_j)\right]dr\right]^2=\frac{1}{N^2}\left[\sum_{\sigma'\neq\sigma}\int_s^t\gamma\sum_{x\in\mathbb{Z}}\phi_j(\tfrac{x}{N})\left(\eta^N_{N^2r}(x,\sigma')-\eta^N_{N^2r}(x,\sigma)\right)dr\right]^2.$$

Since ϕ_j only has a compact support, there exists a $k\in\mathbb{N}$ such that $\operatorname{supp}(\phi_j)\in[-k,k]$. It is easy to see that the supports of ϕ'_j and ϕ''_j are also inside this interval $[-k,k]$, therefore we do not have to sum over all possible $x\in\mathbb{Z}$, but we only have to sum over all $x\in[-kN,kN]$. Furthermore, by the Cauchy-Schwarz inequality, we have in general that,

$$\left(\sum_{i\in I}a_i\right)^2=\left(\sum_{i\in I}1\cdot a_i\right)^2\le\left(\left(\sum_{i\in I}1\right)^{\frac12}\cdot\left(\sum_{i\in I}a_i^2\right)^{\frac12}\right)^2=|I|\cdot\sum_{i\in I}a_i^2.$$

Using this in our situation gives us that

$$\begin{aligned}
\left[\int_s^t\pi^N_{\sigma,r}(A^*_\sigma\phi_j)dr\right]^2&\le\frac{2k+1}{N}\sum_{x\in[-kN,kN]}\left[\int_s^t A^*_\sigma\phi_j(\tfrac{x}{N})\cdot\eta^N_{N^2r}(x,\sigma)dr\right]^2\\
&\le\frac{2k+1}{N}\sum_{x\in[-kN,kN]}\int_s^t\left(A^*_\sigma\phi_j(\tfrac{x}{N})\right)^2dr\cdot\int_s^t\left(\eta^N_{N^2r}(x,\sigma)\right)^2dr,
\end{aligned}\tag{5.20}$$

where we have used the Hölder inequality for the last line. Similarly, we also find that

$$\begin{aligned}
&\left[\int_s^t \gamma \sum_{\sigma'\neq\sigma} \left[\pi^N_{\sigma',r}(\phi_j) - \pi^N_{\sigma,r}(\phi_j)\right] dr\right]^2 \\
&\leq \frac{2k+1}{N} \sum_{x\in[-kN,kN]} \int_s^t \gamma^2 \phi_j(\tfrac{x}{N})^2 dr \cdot \left[\sum_{\sigma'\neq\sigma} \int_s^t \left(\eta^N_{N^2r}(x,\sigma')\right)^2 dr - \int_s^t \left(\eta^N_{N^2r}(x,\sigma)\right)^2 dr\right]
\end{aligned} \tag{5.21}$$

Combining the bounds in 5.20 and 5.21 with 5.19 and using the fact that ϕ_j, ϕ_j' and ϕ_j'' are bounded, there exists a constant $C(\phi_j, T)$ such that for $|t-s| < 2\delta$ we have that

$$\left|\int_s^t \mathbb{L}_N \pi^N_{\sigma,r}(\phi_j) dr\right|^2 \leq \frac{2k+1}{N} \sum_{x\in[-kN,kN]} \delta C(\phi_j,T) \left[2\int_s^t \left(\eta^N_{N^2r}(x,\sigma)\right)^2 dr + \int_s^t \left(\eta^N_{N^2r}(x,\sigma')\right)^2 dr\right]$$

Therefore we find that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{\substack{s,t\in[0,T+1]\\|t-s|<2\delta}} \left|\int_s^t \mathbb{L}_N \pi^N_{\sigma,r}(\phi_j) dr\right|^2\right] \\
&\leq \frac{2k+1}{N}\delta C(\phi_j,T) \sum_{x\in[-kN,kN]} \mathbb{E}\left[\sup_{\substack{s,t\in[0,T+1]\\|t-s|<2\delta}} \left(2\int_s^t \left(\eta^N_{N^2r}(x,\sigma)\right)^2 dr + \int_s^t \left(\eta^N_{N^2r}(x,\sigma')\right)^2 dr\right)\right] \\
&\leq \frac{2k+1}{N}\delta C(\phi_j,T) \sum_{x\in[-kN,kN]} \mathbb{E}\left[2\int_0^{T+1} \left(\eta^N_{N^2r}(x,\sigma)\right)^2 dr + \int_0^{T+1} \left(\eta^N_{N^2r}(x,\sigma')\right)^2 dr\right] \\
&\leq \frac{2k+1}{N}\delta C(\phi_j,T) \sum_{x\in[-kN,kN]} 2\int_0^{T+1} \mathbb{E}\left[\left(\eta^N_{N^2r}(x,\sigma)\right)^2\right] dr + \int_0^{T+1} \mathbb{E}\left[\left(\eta^N_{N^2r}(x,\sigma')\right)^2\right] dr \\
&\leq \frac{2k+1}{N}\delta C(\phi_j,T) \sum_{x\in[-kN,kN]} 3(T+1)(||\rho||^2_\infty + ||\rho||_\infty) \\
&\leq 3\delta(2k+1)^2 C(\phi_j,T)(T+1)(||\rho||^2_\infty + ||\rho||_\infty),
\end{aligned}$$

where we have used the inequality in 5.16 to upper bound $\mathbb{E}\left[\left(\eta^N_{N^2r}(x,\sigma)\right)^2\right]$ and $\mathbb{E}\left[\left(\eta^N_{N^2r}(x,\sigma')\right)^2\right]$. By filling this back into 5.18, we find that

$$\begin{aligned}
&\lim_{\delta\downarrow 0} \limsup_{N\to\infty} \mathbb{E}\left[\sup_{\substack{s,t\in[0,T+1]\\|t-s|<2\delta}} \left|\pi^N_{\sigma,t}(\phi_j) - \pi^N_{\sigma,s}(\phi_j)\right|^2\right] \\
&\qquad \leq \lim_{\delta\downarrow 0} 24\delta(2k+1)^2 C(\phi_j,T)(T+1)(||\rho||^2_\infty + ||\rho||_\infty) \\
&\qquad = 0.
\end{aligned} \tag{5.22}$$

So, by going back to 5.17 and using the Markov inequality, we get the following:

$$\mathbb{P}(\omega(\bar{\pi}^N_\sigma, 2\delta, T+1) \geq \varepsilon) \leq \frac{1}{\varepsilon}\left(2^{-m} + \sum_{j=1}^m \mathbb{E}\left[\sup_{\substack{s,t\in[0,T+1]\\|t-s|<2\delta}} \left|\pi^N_{\sigma,t}(\phi_j) - \pi^N_{\sigma,s}(\phi_j)\right|\right]\right)$$

by now taking m such that $2^{-m} < \varepsilon^2$ and using 5.22 we see that

$$\lim_{\delta\downarrow 0}\limsup_{N\to\infty}\mathbb{P}(\omega(\bar{\pi}_\sigma^N, 2\delta, T+1) \geq \varepsilon) < \varepsilon,$$

which ultimately proves the tightness result. □

Now we define the sequence of probability measures $\{Q^N : N \in \mathbb{N}\}$ to be the distribution for the path process $\bar{\boldsymbol{\pi}}^N = \left(\bar{\pi}_\sigma^N\right)_{\sigma\in S}$, i.e., for any $B \subset D_{\mathbf{M}^S}$ we have

$$Q^N(B) = \mathbb{P}(\bar{\boldsymbol{\pi}}^N \in B).$$

This sequence the property that for any $N \in \mathbb{N}$ and cylindrical set $B = \bigtimes_{\sigma\in S} B_\sigma$, where every $B_\sigma \subset D_{\mathbf{M}}$, we have that

$$Q^N(B) = \prod_{\sigma\in S} Q_\sigma^N(B_\sigma).$$

Therefore, Proposition 5.5 has the following corollary.

Corollary 5.7. *The sequence of probability measures $\{Q^N : N \in \mathbb{N}\}$ is tight in $D_{\mathbf{M}^S}$.*

Proof. Set $\varepsilon > 0$. By definition A.4, the tightness of $\{Q_\sigma^N : N \in \mathbb{N}\}$ implies the existence of a compact set $K_\sigma \in D_{\mathbf{M}}$ such that for all $N \in \mathbb{N}$ we have that

$$Q_\sigma^N(K_\sigma) \geq \left(1-\varepsilon\right)^{\frac{1}{|S|}}.$$

Since this is true for all $\sigma \in S$, by taking $K = \bigtimes_{\sigma\in S} K_\sigma$, we have that $K \subset D_{\mathbf{M}^S}$ is compact, and

$$Q^N(K) = \prod_{\sigma\in S} Q_\sigma^N(K_\sigma) \geq 1-\varepsilon,$$

hence $\{Q^N : N \in \mathbb{N}\}$ is tight. □

5.5 Coinciding Limit points

From Theorem A.7 we know that the tightness of the sequence of probability measures $\{Q^N : N \in \mathbb{N}\}$ leads to the compactness of its closure, which in turn implies that there exists a subsequence $\{Q^{N_k} : k \in \mathbb{N}\}$ that converges weakly, i.e., $Q^{N_k} \xrightarrow{w} Q$ for some probability measure $Q \in \mathscr{P}(D_{\mathbf{M}^S})$. In this section we will prove that Q-a.s. the paths $\boldsymbol{\alpha} \in D_{\mathbf{M}^S}$ are continuous and weak solutions to 5.4.

Firstly, we define the metric $d_{\mathbf{M}^S}$ on $\mathbf{M}^S$ as

$$d_{\mathbf{M}^S}(\boldsymbol{\mu}, \boldsymbol{\nu}) = \max_{\sigma\in S} d_{\mathbf{M}}(\mu_\sigma, \nu_\sigma), \tag{5.23}$$

for all $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathbf{M}^S$, where $d_{\mathbf{M}}$ is the metric on $\mathbf{M}$, defined in A.8. To now prove the continuity result, define the function $G : D_{\mathbf{M}^S} \to [0,1]$ as follows

$$G(\boldsymbol{\alpha}) := \sup_{t\geq 0} e^{-t} d_{\mathbf{M}^S}\big(\boldsymbol{\alpha}(t), \boldsymbol{\alpha}(t^-)\big),$$

with $\boldsymbol{\alpha}(t^-)$ the left limit of the path $\boldsymbol{\alpha}$ at time t, i.e., $\lim_{r\uparrow t}\boldsymbol{\alpha}(r)$. Furthermore, we set $C_{\mathbf{M}^S} \subset D_{\mathbf{M}^S}$ as the set of the paths that are continuous.

It is important to note that the function G is bounded, since by definition of the metric, found in A.8, we have that $d_{\mathbf{M}} \leq 1$. The function G can also characterize wether a path is continuous in the following way:

Proposition 5.8. *$\boldsymbol{\alpha} \in C_{\mathbf{M}^S}$ if and only if $G(\boldsymbol{\alpha}) = 0$. Furthermore, G is continuous*

Proof. Let $\boldsymbol{\alpha} \in C_{\mathbf{M}^S}$, then it is clear that $\boldsymbol{\alpha}(t^-) = \boldsymbol{\alpha}(t)$ so we indeed find that $G(\boldsymbol{\alpha}) = 0$. For the other direction, assume that $G(\boldsymbol{\alpha}) = 0$, then $d_{\mathbf{M}}\big(\boldsymbol{\alpha}(t), \boldsymbol{\alpha}(t^-)\big) = 0$ for all $t \geq 0$, i.e., $\boldsymbol{\alpha}$ is everywhere left-continuous. Since every $\boldsymbol{\alpha} \in D_{\mathbf{M}^S}$ is automatically right-continuous by definition of $D_{\mathbf{M}^S}$, this proofs our first claim.

To see that G is continuous, we use a result found in [9]. Here it is shown that the metric

$$s^*(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \inf_{\lambda \in \Lambda} \left\{ \sup_{0 \leq t \leq T} |\boldsymbol{\alpha}(t) - \boldsymbol{\beta}(\lambda(t))| + \sup_{0 \leq t \leq T} |t - \lambda(t)| \right\}$$

is equivalent to the metric $s(\alpha, \beta)$ given in A.6, and therefore also induces the Skorokhod topology. Under this new metric, the following inequalities are satisfied:

$$G(\boldsymbol{\alpha}) \leq G(\boldsymbol{\beta}) + s^*(\boldsymbol{\alpha}, \boldsymbol{\beta}),$$

$$G(\boldsymbol{\beta}) \leq G(\boldsymbol{\alpha}) + s^*(\boldsymbol{\alpha}, \boldsymbol{\beta}).$$

Therefore, if we take $\boldsymbol{\alpha}, \boldsymbol{\beta} \in D_{\mathbf{M}}$ such that $s^*(\boldsymbol{\alpha}, \boldsymbol{\beta}) < \varepsilon$, then $|G(\boldsymbol{\alpha}) - G(\boldsymbol{\beta})| < \varepsilon$, which proves the continuity of G. □

With these properties of G, we are now able to prove that every path $\boldsymbol{\alpha} \in D_{\mathbf{M}^S}$ is Q-a.s. continuous.

Lemma 5.9. $Q(C_{\mathbf{M}^S}) = 1$

Proof. Since the function G is bounded, we can apply the dominated convergence theorem to find that

$$\mathbb{E}^Q[G] = \lim_{k \to \infty} E^{Q^{N_k}}[G] = \lim_{k \to \infty} E^{N_k}[G(\bar{\boldsymbol{\pi}}^{N_k})],$$

where since $d_{\mathbf{M}} \leq 1$ we have that for any $T > 0$ that

$$G\left(\bar{\boldsymbol{\pi}}^{N_k}\right) = \sup_{t \geq 0} e^{-t} d_{\mathbf{M}^S}\left(\boldsymbol{\pi}_t^{N_k}, \boldsymbol{\pi}_{t^-}^{N_k}\right) \leq \sup_{0 \leq t \leq T} d_{\mathbf{M}^S}\left(\boldsymbol{\pi}_t^{N_k}, \boldsymbol{\pi}_{t^-}^{N_k}\right) + e^{-T},$$

and so

$$\mathbb{E}^Q[G] \leq \max_{\sigma \in S} \lim_{k \to \infty} \mathbb{E}\left[\sup_{\substack{0 \leq s,t \leq T \\ |t-s| < \delta}} d_{\mathbf{M}}\left(\pi_{\sigma,t}^{N_k}, \pi_{\sigma,s}^{N_k}\right)\right] + e^{-T}.$$

Similarly as in 5.22 the expectation vanishes as $k \to \infty$ for any $T > 0$, and since we have taken T arbitrarily, we find that $\mathbb{E}^Q[G] = 0$, i.e., $G = 0$ Q-a.s., so by Proposition 5.8 we have that Q-almost every $\boldsymbol{\alpha} \in D_{\mathbf{M}^S}$ is continuous. □

Now for $\phi \in C_c^\infty(\mathbb{R} \times S), \delta > 0$ and $T > 0$, we consider the following set:

$$H(\phi, \delta, T) := \left\{ \boldsymbol{\alpha} \in D_{\mathbf{M}^S} : \sup_{0 \leq t < T} \left|\left| \boldsymbol{\alpha}(t, \boldsymbol{\phi}) - \boldsymbol{\alpha}(0, \boldsymbol{\phi}) - \int_0^t \left[\boldsymbol{\alpha}(s, A\boldsymbol{\phi}) + \gamma \mathrm{C} \boldsymbol{\alpha}(s, \boldsymbol{\phi}) \right] ds \right|\right|_S \leq \delta \right\}.$$

Here the operator A and the matrix C are the ones defined in 5.5 and 5.6 respectively, and the S-norm is defined as $||\boldsymbol{\alpha}(t, \phi)||_S = \max_{\sigma \in S} |\alpha_\sigma(t, \phi)|$. We can see this $H(\phi, \delta, T)$ as the $\boldsymbol{\alpha} \in D_{\mathbf{M}^S}$ that, for a specific test function ϕ, almost satisfy the PDE from 5.4, up to time T. If we can show that $Q\big(H(\phi, \delta, T)\big) = 1$, then since we can take δ arbitrarily small and T arbitrarily large, we can conclude that Q-almost every path $\boldsymbol{\alpha} \in D_{\mathbf{M}^S}$ satisfies the PDE for a given test function.

Afterwards, the result has to be extended to all $\phi \in C_c^\infty(\mathbb{R}\times S)$ simultaneously. We can do this by turning to a countable set $\{\phi_j, j \in \mathbb{N}\} \subset C_c^\infty(\mathbb{R}\times S)$ such that for any $\phi \in C_c^\infty(\mathbb{R}\times S)$ there exists a subsequence $\{\phi_{j_k}, k \in \mathbb{N}\}$ such that for any $\boldsymbol{\alpha} \in D_{\mathbf{M}^S}$ and $t \geq 0$, $\boldsymbol{\alpha}(t, \phi_{j_k}) \to \boldsymbol{\alpha}(t,\phi)$ and

$$\boldsymbol{\alpha}(t, A\phi_{j_k}) + \gamma C\boldsymbol{\alpha}(s, \phi_{j_k}) \to \boldsymbol{\alpha}(t, A\phi) + \gamma C\boldsymbol{\alpha}(s,\phi)$$

uniformly. The proof that such a countable set exists is given in [23].

So all we need to show is that $Q\big(H(\phi,\delta,T)\big) = 1$. Before we can do this, we must first show that the set $H(\phi,\delta,T)$ is closed.

Lemma 5.10. *For any $\phi \in C_c^\infty(\mathbb{R}\times S), \delta > 0$ and $T > 0$, the set $H(\phi,\delta,T)$ is closed under the Skorokhod topology of the path space $D_{\mathbf{M}^S}$.*

Proof. Let $\{\boldsymbol{\alpha}_n, n \in \mathbb{N}\} \subset H(\phi,\delta,T)$ be a sequence such that $\boldsymbol{\alpha}_n \to \boldsymbol{\alpha}$ for some $\boldsymbol{\alpha} \in D_{\mathbf{M}}$. Our aim is to show that $\boldsymbol{\alpha} \in H(\phi,\delta,T)$, i.e., we have to show that for all $t \in [0,T)$

$$\left|\left|\boldsymbol{\alpha}(t,\phi) - \boldsymbol{\alpha}(0,\phi) - \int_0^t \Big[\boldsymbol{\alpha}(s, A\boldsymbol{\phi}) + \gamma \mathrm{C}\boldsymbol{\alpha}(s,\boldsymbol{\phi})\Big]ds\right|\right|_S \leq \delta. \tag{5.24}$$

According to Lemma A.8, there exists a sequence $\{\lambda_n, n \in \mathbb{N}\} \subset \Lambda$ such that $\theta(\lambda_n) \to 0$ and

$$d_{\mathbf{M}^S}\big(\boldsymbol{\alpha}_n(\lambda_n(t)), \boldsymbol{\alpha}(t)\big) \to 0. \tag{5.25}$$

By plugging in the definition of θ, found in A.5, we find that

$$\sup_{0\leq s<t}\left|\log\left(\frac{\lambda_n(t)-\lambda_n(s)}{t-s}\right)\right| \to 0, \tag{5.26}$$

which implies that λ_n converges uniformly to the identity function on the interval $[0,T]$, so in particular we find that for $n \in \mathbb{N}$ large enough, $\lambda_n(t) < T$ for all $t \in [0,T)$. Therefore, since $\boldsymbol{\alpha}_n \in H(\phi,\delta,T)$ we see that

$$\left|\left|\boldsymbol{\alpha}_n(\lambda_n(t),\phi) - \boldsymbol{\alpha}_n(0,\phi) - \int_0^{\lambda_n(t)} \Big[\boldsymbol{\alpha}_n(s, A\boldsymbol{\phi}) + \gamma \mathrm{C}\boldsymbol{\alpha}_n(s,\boldsymbol{\phi})\Big]ds\right|\right|_S \leq \delta. \tag{5.27}$$

If we now show that the left-hand side of 5.27 converges to the left-hand side of 5.24, then we have our desired result. First of all, by 5.25 we already have that

$$\boldsymbol{\alpha}_n(\lambda_n(t),\phi) \to \boldsymbol{\alpha}(t,\phi),$$

and since every $\lambda_n : [0,\infty) \to [0,\infty)$ is bijective and continuous by definition of Λ, we must have that $\lambda_n(0) = 0$ for all $n \in \mathbb{N}$, so we also get that

$$\boldsymbol{\alpha}_n(0,\phi) \to \boldsymbol{\alpha}(0,\phi).$$

All that is left now is the convergence

$$\lim_{n\to\infty}\int_0^{\lambda_n(t)} \Big[\boldsymbol{\alpha}_n(s, A\phi) + \gamma \mathrm{C}\boldsymbol{\alpha}_n(s,\boldsymbol{\phi})\Big]ds = \int_0^t \Big[\boldsymbol{\alpha}(s, A\boldsymbol{\phi}) + \gamma \mathrm{C}\boldsymbol{\alpha}(s,\boldsymbol{\phi})\Big]ds.$$

First of all, by the Lipschitz continuity given in 5.26 we know that that λ_n' is defined at least Lebesgue-a.e. with $\lambda_n'(t) \to 1$ for all $t \in [0,T)$, and that λ_n is absolutely continuous. Furthermore, since $\lambda_n : [0,\infty) \to [0,\infty)$ is bijective with $\lambda_n(0) = 0$, it is also monotonically increasing. By these properties, we can apply the change-of-variables formula (for details, see [22, Theorem 7.26]). This tells us that

$$\int_0^{\lambda_n(t)} \Big[\boldsymbol{\alpha}_n(s, A\boldsymbol{\phi}) + \gamma \mathrm{C}\boldsymbol{\alpha}_n(s,\boldsymbol{\phi})\Big] ds = \int_0^t \Big[\boldsymbol{\alpha}_n(\lambda_n(s), A\boldsymbol{\phi}) + \gamma \mathrm{C}\boldsymbol{\alpha}_n(\lambda_n(s),\boldsymbol{\phi})\Big] \lambda_n'(s) ds \tag{5.28}$$

By Proposition A.13, we can find an $m = m(\phi)$ and another test function $\phi_m \in C_c^\infty(\mathbb{R}\times S)$ such that for n big enough

$$||\boldsymbol{\alpha}_n(\lambda_n(s), A\phi)||_S \leq ||A\phi||_\infty \cdot \Big(||\boldsymbol{\alpha}(s,\phi_m)||_S + 2^m d_{\mathbf{M}^S}\big(\boldsymbol{\alpha}_n(\lambda_n(s)), \boldsymbol{\alpha}(s)\big)\Big),$$

so also for some n_1 big enough, we find that $\sup_{n\geq n_1} ||\boldsymbol{\alpha}_n(\lambda_n(s), A\phi)||_S < \infty$. With the same proposition, we can also find that there exists an n_2 big enough such that $\sup_{n\geq n_2} ||\gamma \mathrm{C}\boldsymbol{\alpha}_n(\lambda_n(s),\boldsymbol{\phi})||_S < \infty$, hence we can apply the dominated convergence theorem to find that

$$\lim_{n\to\infty} \int_0^t \Big[\boldsymbol{\alpha}_n(\lambda_n(s), A\boldsymbol{\phi}) + \gamma \mathrm{C}\boldsymbol{\alpha}_n(\lambda_n(s),\boldsymbol{\phi})\Big] \lambda_n'(s) ds = \int_0^t \Big[\boldsymbol{\alpha}(s, A\boldsymbol{\phi}) + \gamma \mathrm{C}\boldsymbol{\alpha}(s,\boldsymbol{\phi})\Big] ds.$$

Together with 5.28, this now proves the claim. □

Corollary 5.11. *For any $\phi \in C_c^\infty(\mathbb{R}\times S), \delta > 0$ and $T > 0$ we have that $Q\big(H(\phi,\delta,T)\big) = 1$*

Proof. Now that we have shown that the set $H(\phi,\delta,T)$ is closed, we can apply the Portmanteau Theorem (Theorem A.11) to see that

$$\begin{aligned} Q\big(H(\phi,\delta,T)\big) &\geq \limsup_{k\to\infty} Q^{N_k}\big(H(\phi,\delta,T)\big) \\ &= \limsup_{k\to\infty} \mathbb{P}\left(\sup_{0\leq t<T} \left|\left|\boldsymbol{\pi}_t^{N_k}(\phi) - \boldsymbol{\pi}_0^{N_k}(\phi) - \int_0^t \Big[\boldsymbol{\pi}_s^{N_k}(A\phi)ds + \gamma C \boldsymbol{\pi}_s^{N_k}(\phi)\Big] ds\right|\right|_S \leq \delta\right) \\ &= \limsup_{k\to\infty} \mathbb{P}\left(\max_{\sigma\in S}\sup_{0\leq t<T} \left|M_{\sigma,t}^{N_k} + \mathcal{O}(tN)k^{-1})\right| \leq \delta\right) \end{aligned}$$

Here we have used Proposition 5.3 for the last equality.By Lemma 5.4 and the Markov inequality we then have that

$$\mathbb{P}\left(\max_{\sigma\in S}\sup_{0\leq t<T} \left|M_{\sigma,t}^{N_k} + \mathcal{O}(tN_k^{-1})\right| > \delta\right) \leq \frac{2}{\delta^2}\left(\max_{\sigma\in S}\mathbb{E}\left[\sup_{0\leq t<T}\left(M_{\sigma,t}^{N_k}\right)^2\right] + \mathcal{O}(T^2 N_k^{-2})\right) \to 0,$$

so $Q\big(H(\phi,\delta,T)\big) = 1$. □

From this section, we can conclude that Q-a.s. every path $\boldsymbol{\alpha} \in D_{\mathbf{M}^S}$ is continuous and a weak solution to 5.4. Since we took an arbitrary weak limit point Q of the sequence $\{Q^N : N \in \mathbb{N}\}$ we actually find that $Q^N \xrightarrow{w} \delta_{\bar{\boldsymbol{\alpha}}}$, with $\bar{\boldsymbol{\alpha}}$ the unique weak solution. We now have the results needed to prove the main theorems.

5.6 Proof of hydrodynamics

We will first give the proof of Theorem 5.2

Proof of Theorem 5.2. . The proof of this theorem is now an application of the Portmanteau Theorem (Theorem A.11). Define $B_\varepsilon(\bar{\boldsymbol{\alpha}})$ as the open ball around $\bar{\boldsymbol{\alpha}} \in D_{\mathbf{M}^S}$ of radius $\varepsilon > 0$ with respect to the Skorokhod distance $s_{\mathbf{M}^S}$, i.e.,

$$B_\varepsilon(\bar{\boldsymbol{\alpha}}) := \{\boldsymbol{\alpha} \in D_{\mathbf{M}^S} : s_{\mathbf{M}^S}(\boldsymbol{\alpha},\bar{\boldsymbol{\alpha}}) < \varepsilon\},$$

then clearly its complement $B_\varepsilon(\bar{\boldsymbol{\alpha}})^c$ is closed. By the Portmanteau Theorem, we then have that

$$\limsup_{N\to\infty} \mathbb{P}\big(s_{\mathbf{M}^S}(\bar{\boldsymbol{\pi}}^N, \bar{\boldsymbol{\alpha}}) \geq \varepsilon\big) = \limsup_{N\to\infty} Q^N\big(B_\varepsilon(\bar{\boldsymbol{\alpha}})^c\big).$$

Now, since $Q^N \xrightarrow{w} \delta_{\bar{\boldsymbol{\alpha}}}$, we have that for every $\varepsilon > 0$,

$$\limsup_{N\to\infty} Q^N\big(B_\varepsilon(\bar{\boldsymbol{\alpha}})^c\big) = 0,$$

which proves the theorem. □

Now we can finally prove the main result.

Proof of Theorem 5.1. Notice that 5.3 can be written in terms of $\bar{\boldsymbol{\pi}}^N$ and $\bar{\boldsymbol{\alpha}}$ (the unique weak solution of 5.4 in the path space $D_{\mathbf{M}^S}$) in the following way,

$$\lim_{N\to\infty} \mathbb{P}\Big(||\boldsymbol{\pi}_t^N(\phi) - \bar{\boldsymbol{\alpha}}(t,\phi)||_S \geq \varepsilon\Big) = 0, \tag{5.29}$$

for every $\varepsilon > 0$ and $\phi \in C_c^\infty(\mathbb{R} \times S)$.

Define the set $B(t,\phi)$ as follows,

$$B(t,\phi) = \big\{\boldsymbol{\alpha} \in D_{\mathbf{M}^S} : ||\boldsymbol{\alpha}(t,\phi) - \bar{\boldsymbol{\alpha}}(t,\phi)||_S \geq \varepsilon\big\},$$

then 5.29 is equivalent to

$$\lim_{N\to\infty} Q^N\big(B(t,\phi)\big) = 0.$$

If we can now show that $B(t,\phi)$ is a Q-continuity set for every $t \geq 0$ and $\phi \in C_c^\infty(\mathbb{R} \times S)$, i.e., $Q\big(\partial B(t,\phi)\big) = 0$, then by the Portmanteau Theorem we find that

$$\lim_{N\to\infty} Q^N\big(B(t,\phi)\big) = Q\big(B(t,\phi)\big) = 0,$$

where the last equality follows from Theorem 5.2.

To prove that $B(t,\phi)$ is a Q-continuity set, all we have to show is that $\bar{\boldsymbol{\alpha}} \notin \partial B(t,\phi)$, i.e., there is no sequence $\{\boldsymbol{\alpha}_n : n \in \mathbb{N}\} \subset B(t,\phi)$ such that $\boldsymbol{\alpha}_n \to \bar{\boldsymbol{\alpha}}$ in the Skorokhod topology. So assume that such a sequence does exists. Since $\bar{\boldsymbol{\alpha}}$ is Q-a.s. continuous by Lemma 5.9, by an application of Lemma A.8 and the definition of $d_{\mathbf{M}^S}$, as given in 5.23, we find that for all $t \geq 0$ and $\phi \in C_c^\infty(\mathbb{R} \times S)$,

$$\boldsymbol{\alpha}_n(t,\phi) \to \bar{\boldsymbol{\alpha}}(t,\phi).$$

This means that there exists n big enough such that $\boldsymbol{\alpha}_n \notin B(t,\phi)$, which gives us a contradiction. Therefore $B(t,\phi)$ is a Q-continuity set, and the proof is finished. □

5.7 Propagation of local equilibrium

For this section we will look at the evolution of local equilibrium distributions. Before we do this, let us first define what it means for a sequence of probability distributions $\{\mu^N : N \in \mathbb{N}\}$ to be a local equilibrium distribution for the run-and-tumble particle process

For $\eta \in \Omega$, define $\theta : \Omega \to \Omega$ as $\theta_y\eta(x,\sigma) = \eta(x+y,\sigma)$ for every $x \in \mathbb{Z}$ and $\sigma \in S$. Then define $\tau_y : \mathscr{B}(\Omega) \to \mathscr{B}(\Omega)$ as follows,

$$\tau_y f(\eta) = f(\theta_y\eta).$$

Definition 5.1. Let $\rho : \mathbb{R} \times S \to \mathbb{R}_{\geq 0}$ be a density profile, then a sequence of probability measures $\{\mu^N : N \in \mathbb{N}\}$ is a *local equilibrium distribution* for the run-and-tumble particle process, associated to the density profile ρ, if for all $y \in \mathbb{Z}$ and local (multivariate) polynomial f, we have that

$$\lim_{N\to\infty} \int \tau_{\lfloor yN \rfloor} f d\mu^N = \int f d\mu_{\rho(y,\cdot)},$$

with $\mu_{\rho(y,\cdot)}$ as defined in 4.1 with the function $\rho(y,\cdot) : S \to \mathbb{R}_{\geq} 0$.

Proposition 5.12. *For any $y \in \mathbb{Z}$ and local polynomial f, we have that*

$$\lim_{N\to\infty} \int \tau_{\lfloor yN \rfloor} f d\mu^N_\rho = \int f d\mu_{\rho(y,\cdot)}$$

Proof. Notice that by Lemma A.2 we only have to prove it for $f(\eta) = \mathscr{D}(\xi, \eta)$ where $\xi \in \Omega_f$ is any finite configuration. By writing $\xi = \sum_{i=1}^m \delta_{(x_i,\sigma_i)}$, we have that

$$\tau_{\lfloor yN \rfloor}\mathscr{D}(\xi,\eta) = \tau_{\lfloor yN \rfloor}\prod_{i=1}^m \eta(x_i,\sigma_i) = \prod_{i=1}^m \eta(x_i + \lfloor yN \rfloor, \sigma_i),$$

and so by applying Proposition A.1 twice, we find that

$$\lim_{N\to\infty} \int \tau_{\lfloor yN \rfloor}\mathscr{D}(\xi,\cdot) d\mu^N_\rho = \lim_{N\to\infty}\prod_{i=1}^m \rho(\tfrac{x_i+\lfloor yN \rfloor}{N}, \sigma_i) = \prod_{i=1}^m \rho(y,\sigma_i) = \int \mathscr{D}(\xi,\cdot) d\mu_{\rho(y,\cdot)},$$

which finishes the proof. □

Theorem 5.13. *Let $\rho : \mathbb{R} \times S \to \mathbb{R}_{\geq} 0$ be a smooth and bounded density profile, with bounded derivatives, then for every $t \geq 0$ and every local polynomial f, we have that*

$$\lim_{N\to\infty} \left| \int f d\mu^N_\rho S_{N^2 t} - \int f d\mu^N_{\rho_t} \right| = 0,$$

where ρ_t is the solution to 5.4, with initial condition $\rho_0 = \rho$.

Proof. By Theorem 4.1, we know that for any $N \in \mathbb{N}$ we have that $\mu^N_\rho S_{N^2 t} = \mu_{\rho^N_t}$, where $\rho^N_t(\frac{x}{N},\sigma) = \widehat{S}_{N^2 t}\rho(\frac{x}{N},\sigma)$, i.e.,

$$\rho^N_t(\tfrac{x}{N},\sigma) = \widehat{\mathbb{E}}_{\frac{x}{N},\sigma}\left[\rho\left(\tfrac{X_{N^2 t}}{N}, \sigma_t\right)\right].$$

By now going from a macropoint $x \in \mathbb{R}$ to its corresponding micropoint $x \to \lfloor xN \rfloor$, we find that

$$\lim_{N\to\infty} \rho^N_t\left(\tfrac{\lfloor xN \rfloor}{N},\sigma\right) = \lim_{N\to\infty} \widehat{\mathbb{E}}_{\frac{\lfloor xN \rfloor}{N},\sigma}\left[\rho\left(\tfrac{X_{N^2 t}}{N}, \sigma_t\right)\right].$$

To calculate this limit, we turn to the generator $\widehat{\mathbb{L}}_N$ of the single dual particle $\left(\frac{X_{N^2 t}}{N}, \sigma_t\right)$ working on the function ρ,

$$\begin{aligned}\widehat{\mathbb{L}}_N \rho(\tfrac{x}{N},\sigma) = {} & \kappa N^2\big(\rho(\tfrac{x+1}{N},\sigma) + \rho(\tfrac{x-1}{N},\sigma) - 2\rho(\tfrac{x}{N},\sigma)\big) \\ & + \lambda N\big(\rho(\tfrac{x-\sigma}{N},\sigma) - \rho(\tfrac{x}{N},\sigma)\big) \\ & + \gamma \sum_{\sigma' \neq \sigma} \rho(\tfrac{x}{N},\sigma') - \rho(\tfrac{x}{N},\sigma).\end{aligned}$$

Making use of Taylor series again,

$$\rho(\tfrac{x+1}{N},\sigma) = \rho(\tfrac{x}{N},\sigma) + \frac{1}{N}\frac{\partial}{\partial x}\rho(\tfrac{x}{N},\sigma) + \frac{1}{N^2}\frac{\partial^2}{\partial x^2}\rho(\tfrac{x}{N},\sigma) + \frac{1}{N^3}\frac{\partial^3}{\partial x^3}\rho(y_1,\sigma),$$
$$\rho(\tfrac{x-1}{N},\sigma) = \rho(\tfrac{x}{N},\sigma) - \frac{1}{N}\frac{\partial}{\partial x}\rho(\tfrac{x}{N},\sigma) + \frac{1}{N^2}\frac{\partial^2}{\partial x^2}\rho(\tfrac{x}{N},\sigma) - \frac{1}{N^3}\frac{\partial^3}{\partial x^3}\rho(y_2,\sigma),$$
$$\rho(\tfrac{x-\sigma}{N},\sigma) = \rho(\tfrac{x}{N},\sigma) - \frac{1}{N}\sigma\frac{\partial}{\partial x}\rho(\tfrac{x}{N},\sigma) + \frac{1}{N^2}\sigma^2\frac{\partial^2}{\partial x^2}\rho(y_3,\sigma),$$

we find that

$$\widehat{\mathbb{L}}_N\rho(\tfrac{x}{N},\sigma) = \kappa\frac{\partial^2}{\partial x^2}\rho(\tfrac{x}{N},\sigma) - \lambda\sigma\frac{\partial}{\partial x}\rho(\tfrac{x}{N},\sigma) + \gamma\sum_{\sigma'\neq\sigma}\rho(\tfrac{x}{N},\sigma') - \rho(\tfrac{x}{N},\sigma) + R(N,\rho),$$

where

$$R(N,\rho) \leq \frac{2}{N}\left(\left|\left|\frac{\partial^3}{\partial x^3}\rho(\cdot,\sigma)\right|\right|_\infty + \left|\left|\frac{\partial^2}{\partial x^2}\rho(\cdot,\sigma)\right|\right|_\infty\right) \to 0.$$

This tells us that the generator $\widehat{\mathbb{L}}_N$ converges to the generator of the PDE in 5.4. By the theorem of Trotter-Kurtz (Theorem 2.4), we therefore find that

$$\lim_{N\to\infty}\widehat{\mathbb{E}}_{\frac{\lfloor xN\rfloor}{N},\sigma}\left[\rho\left(\tfrac{X_{N^2t}}{N},\sigma_t\right)\right] = \rho_t(x,\sigma),$$

where ρ_t solves the PDE.

So, let $f = \mathscr{D}(\xi,\cdot)$ for some $\xi = \sum_{i=1}^m \delta_{(x_i,\sigma_i)}$ some finite configuration, then by Proposition A.1 we have that

$$\lim_{N\to\infty}\left|\int f d\mu_\rho^N S_{N^2t} - \int f d\mu_{\rho_t}^N\right| = \lim_{N\to\infty}\left|\prod_{i=1}^m \rho_t^N(\tfrac{x_i}{N},\sigma_i) - \prod_{i=1}^m \rho_t(\tfrac{x_i}{N},\sigma_i)\right| = 0,$$

which proves the theorem. □

5.8 The equation of the total density

For this section we will look at the case where $S = \{-1,1\}$, In this case the system of PDEs given by 5.1 becomes the following.

$$\begin{cases}\dfrac{\partial}{\partial t}\rho_t(x,1) = \dfrac{\kappa}{2}\dfrac{\partial^2}{\partial x^2}\rho_t(x,1) - \lambda\dfrac{\partial}{\partial x}\rho_t(x,1) + \gamma\big(\rho_t(x,-1) - \rho_t(x,1)\big), \\[2ex] \dfrac{\partial}{\partial t}\rho_t(x,-1) = \dfrac{\kappa}{2}\dfrac{\partial^2}{\partial x^2}\rho_t(x,-1) + \lambda\dfrac{\partial}{\partial x}\rho_t(x,-1) + \gamma\big(\rho_t(x,1) - \rho_t(x,-1)\big).\end{cases}$$

We then define the following sum and difference functions,

$$\rho_t(x) = \rho_t(x,1) + \rho_t(x,-1), \qquad \Delta_t(x) = \rho_t(x,1) - \rho_t(x,-1).$$

By taking the first derivative in time for both of these functions we find that

$$\begin{aligned}\frac{\partial}{\partial t}\rho_t(x) &= \frac{\kappa}{2}\frac{\partial^2}{\partial x^2}\big(\rho_t(x,1) - \rho_t(x,-1)\big) - \lambda\frac{\partial}{\partial x}\big(\rho_t(x,1) - \rho_t(x,-1)\big) \\ &= \frac{\kappa}{2}\frac{\partial^2}{\partial x^2}\rho_t(x) - \lambda\frac{\partial}{\partial x}\Delta_t(x),\end{aligned} \tag{5.30}$$

and

$$\begin{aligned}\frac{\partial}{\partial t}\Delta_t(x) &= \frac{\kappa}{2}\frac{\partial^2}{\partial x^2}\big(\rho_t(x,1)-\rho_t(x,-1)\big) - \lambda\frac{\partial}{\partial x}\big(\rho_t(x,1)+\rho_t(x,-1)\big) + 2\gamma\big(\rho_t(x,-1)-\rho_t(x,1)\big)\\ &= \left(\frac{\kappa}{2}\frac{\partial^2}{\partial x^2} - 2\gamma\right)\Delta_t(x) - \lambda\frac{\partial}{\partial x}\rho_t(x).\end{aligned} \tag{5.31}$$

If we now also look at the second derivative in time for $\rho_t(x)$ and filling in the first derivatives found above, we find that

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\rho_t(x) &= \frac{\kappa}{2}\frac{\partial^2}{\partial x^2}\frac{\partial}{\partial t}\rho_t(x) - \lambda\frac{\partial}{\partial x}\frac{\partial}{\partial t}\Delta_t(x)\\ &= \frac{\kappa^2}{4}\frac{\partial^4}{\partial x^4}\rho_t(x) + \lambda^2\frac{\partial^2}{\partial x^2}\rho_t(x) + \left(\kappa\frac{\partial^2}{\partial x^2} - 2\gamma\right)\lambda\frac{\partial}{\partial x}\Delta_t(x).\end{aligned} \tag{5.32}$$

By 5.30 we have that

$$\lambda\frac{\partial}{\partial x}\Delta_t(x) = \frac{\kappa}{2}\frac{\partial^2}{\partial x^2}\rho_t(x) - \frac{\partial}{\partial t}\rho_t(x),$$

so by filling this back into 5.32 we find that

$$\frac{\partial^2}{\partial t^2}\rho_t(x) = -\frac{\kappa^2}{4}\frac{\partial^4}{\partial x^4}\rho_t(x) + (\lambda^2+\gamma\kappa)\frac{\partial^2}{\partial x^2}\rho_t(x) - 2\gamma\frac{\partial}{\partial t}\rho_t(x) + \kappa\frac{\partial^2}{\partial x^2}\frac{\partial}{\partial t}\rho_t(x),$$

which can be rewritten as

$$\left(\frac{\partial}{\partial t} + 2\gamma - \kappa\frac{\partial^2}{\partial x^2}\right)\frac{\partial}{\partial t}\rho_t(x) = -\frac{\kappa^2}{4}\frac{\partial^4}{\partial x^4}\rho_t(x) + (\lambda^2+\gamma\kappa)\frac{\partial^2}{\partial x^2}\rho_t(x) \tag{5.33}$$

Through similar steps we can also derive that the difference function Δ_t follows the exact same PDE, i.e.,

$$\left(\frac{\partial}{\partial t} + 2\gamma - \kappa\frac{\partial^2}{\partial x^2}\right)\frac{\partial}{\partial t}\Delta_t(x) = -\frac{\kappa^2}{4}\frac{\partial^4}{\partial x^4}\Delta_t(x) + (\lambda^2+\gamma\kappa)\frac{\partial^2}{\partial x^2}\Delta_t(x) \tag{5.34}$$

Remark 5.3. Since these PDEs have a second derivative in time, we both need initial values for the function itself $\rho_0(x)$ as for the first derivative in time $\frac{\partial}{\partial t}\rho_0(x)$. Intuitively this makes sense for our model, since we will need to know how many particles will move in which direction at $t=0$.

5.8.1 Connection to other PDEs

If we set $\kappa = 0$, i.e., we assume there is no general diffusion of the particles, then the system of PDEs from 5.30 and 5.31 becomes the following

$$\begin{cases}\dfrac{\partial}{\partial t}\rho_t(x) = -\lambda\dfrac{\partial}{\partial x}\Delta_t(x),\\[2ex] \dfrac{\partial}{\partial t}\Delta_t(x) = -2\gamma\Delta_t(x) - \lambda\dfrac{\partial}{\partial x}\rho_t(x).\end{cases}$$

This system is a special case of the *Telegrapher's equations*, which are being used to model the current and the voltage along a transmission line. By the methods we have used above, twe can combine these equations to make the following two independent but equivalent PDEs

$$\begin{cases}\left(\dfrac{\partial}{\partial t} + 2\gamma\right)\dfrac{\partial}{\partial t}\rho_t(x) = \lambda^2\dfrac{\partial^2}{\partial x^2}\rho_t(x),\\[2ex] \left(\dfrac{\partial}{\partial t} + 2\gamma\right)\dfrac{\partial}{\partial t}\Delta_t(x) = \lambda^2\dfrac{\partial^2}{\partial x^2}\Delta_t(x).\end{cases}$$

If we furthermore assume that $\gamma = 0$, i.e., the particles can never change the direction in which they are moving, then these PDEs will become equal to the wave equation, which is the model of a vibrating string,

$$\frac{\partial^2}{\partial t^2}\rho_t(x) = \lambda^2 \frac{\partial^2}{\partial x^2}\rho_t(x).$$

The general solution of the wave equation is the following consist of a right- and a left-moving wave, i.e., there exist functions $F, G \in C(\mathbb{R})$ such that

$$\rho_t(x) = F(x + \lambda t) + G(x - \lambda t).$$

This relation is quite intuitive to our model, since the right-moving wave will be caused by particles that can only move to the right, and similarly the left-moving wave will be caused by particles moving to the left.

Chapter 6

Summary and concluding remarks

In this thesis we have looked at three distinct properties of a run-and-tumble particle system. In Chapter 3 we have proved a duality result in two different ways, in Chapter 4 we looked at the ergodic measures of the process, and in Chapter 5 we have proved the hydrodynamic limit. In this Chapter we will discuss all these subjects separately.

Duality

In Chapter 3 we have proven a duality result between the run-and-tumble particle process and the same process, but where only the active jumps occur in the opposite direction. For the dual process we only looked at finite configurations ξ. The duality function corresponding to this result is equal to

$$\mathscr{D}(\xi, \eta) = \prod_{v \in V} d\big(\xi(v), \eta(v)\big),$$

where

$$d(k, n) := \begin{cases} \frac{n!}{(n-k)!} & \text{if } k \leq n, \\ 0 & \text{if } k > n. \end{cases}$$

We have given two different proofs of this.

Main line of proof

The first proof was on the level of generators. Through straight-forward calculations and taking into account all the possible cases, we have shown that

$$\mathbb{L}\mathscr{D}(\xi, \cdot)(\eta) = \widehat{\mathbb{L}}\mathscr{D}(\cdot, \eta)(\xi).$$

Since this proof was rather cumbersome, we also used an alternative approach.

We defined another process $\{\psi_t, t \geq 0\}$ as

$$(\psi_t g)(f) := g(f_t), \qquad \text{with for all } v \in V\colon\ f_t(v) := \sum_{v' \in V} p_t(v, v') f(v'),$$

and called it the deterministic system. Here $p_t(v, v')$ are the transition probabilities of a single run-and-tumble particle. We showed that there is a duality result between this system and the run-and-tumble particle process, with the following duality function,

$$\mathcal{D}(f, \eta) = \prod_{v \in V} f(v)^{\eta(v)}.$$

Afterwards we proved that, by simply taking the right derivatives, we could go from this duality result to the duality result which we were after.

Lastly we also introduced the dual deterministic system, given by

$$(\widehat{\psi}_t g)(\widehat{f}) := g(\widehat{f}_t), \qquad \text{with for all } v \in V\text{: } \widehat{f}_t(v) := \sum_{v' \in V} \widehat{p}_t(v,v')\widehat{f}(v'),$$

where $\widehat{p}_t(v,v')$ are te transition probabilities of a single dual run-and-tumble particle. The final duality result we then showed was that between the two deterministic systems, with duality function

$$\mathrm{D}(f,\widehat{f}) = \exp\left(\sum_{v\in V} f(v)\widehat{f}(v)\right).$$

It turned out that all these duality results are equivalent to one another.

Concluding remarks

For the proofs of the duality result between the deterministic system and the run-and-tumble particle process and of the duality result between the two deterministic systems, the only two properties of the processes that we used were that the particles move independently, and that $p_t(v,v') = \widehat{p}_t(v',v)$ for all $v, v' \in V$. For any two processes with these properties, by the same steps we could prove the same results.

Ergodic theory

In Chapter 4 we defined the following product Poisson measures

$$\mu_\rho = \bigotimes_{v\in V} \mathrm{Pois}\big(\rho(v)\big), \tag{6.1}$$

where ρ was some bounded, non-negative function. Using duality, we could show that μ_ρ was an invariant measure for the run-and-tumble particle process if and only if ρ was constant. Afterwards we showed that these measures were also ergodic if ρ was constant by showing they are mixing. The proof of this relied on the fact that the particles spread out under the run-and-tumble dynamics.

Then we turned to the space tempered measures, i.e., measures μ such that for all $n \in \mathbb{N}$, $\sup_{|\xi|=n} \int \mathscr{D}(\xi,\eta)d\mu(\eta)$ is finite. Using the same method as Kuoch and Redig in [16], we proved that in this subspace of measures, the measures μ_ρ with ρ constant are the only ergodic measures. first showed that the product Poisson measures were the only ergodic measures in this space by.

Main line of proof

We started by showing that we have a successful coupling of finite configurations. To prove this we first started with a successful coupling of two particles (X_t, σ_t^X) and (Y_t, σ_t^Y) in $\mathbb{Z}\times S$. We defined the process $Z_t = X_t - Y_t$ and for some $\sigma_1 \in S$, we defined the following stopping times in a recursive manner,

$$\varsigma_n := \min\{t > \tau_{n-1} : (\sigma_t^X, \sigma_t^Y) \neq (\sigma_1, \sigma_1)\},$$

and

$$\tau_n := \min\{t > \varsigma_n : (\sigma_t^X, \sigma_t^Y) = (\sigma_1, \sigma_1)\},$$

with $\tau_0 = 0$. We set $\{Z_n, n \in \mathbb{N}\}$ as the discrete random walk defined as $Z_n = Z_{\tau_n}$ and we have showed that this is a symmetric random walk on $\mathbb{Z}$, where the transition probabilities have a finite first moment. Therefore, by a result from Chung and Fuchs in [7], this walk is recurrent and hence there is a successful coupling. Afterwards, we have shown that a successful coupling of finitely many particles holds by performing a component-wise Ornstein-coupling.

We have used this coupling to show that for any tempered measure μ, the function

$$\widehat{\mu}(\xi) = \int \mathscr{D}(\xi, \eta) d\mu(\eta),$$

working on finite configurations ξ, only depends on the number of particles $|\xi|$, i.e., there is a function $f : \mathbb{N} \to \mathbb{R}$ such that $\mu(\xi) = f(|\xi|)$. To prove that any ergodic tempered measure μ is a product Poisson measure with some constant parameter ρ, it was enough to show that $f(|\xi|) = \rho^{|\xi|}$, since this is a characterizing property of the product Poisson measures. This was done by showing that for any two finite configurations ξ, ξ',

$$f(|\xi| + |\xi'|) = \lim_{T\to\infty} \int \mathscr{D}(\xi, \eta) \cdot \frac{1}{T} \int_0^T S_t \mathscr{D}(\xi', \eta) dt d\mu(\eta) = f(|\xi|) \cdot f(|\xi'|),$$

which followed from both the Birkhoff ergodic theorem and the fact that particles spread out under the run-and-tumble dynamics.

Concluding remarks

We set out to find all the ergodic measures of the run-and-tumble particle system, but underway we met with some difficulties. The first one was the coupling of particles in $\mathbb{Z}^d$ with $d \geq 3$. In the case of a symmetric random walk, if we have a coupling in one dimension, we can find a coupling d dimensions by performing a component-wise Ornstein-coupling (see [12, Section 3.2]). For two general d dimensional run-and-tumble particles this does not necessarily work. Namely, even if the first coordinates of the two particles are coupled at some point, the internal states can still be different and eventually decouple the first coordinates. We also gave an example of two run-and-tumble particles where a successful coupling is not possible, so the question of how and when a successful coupling exists for $d \geq 3$ is still an open problem.

Another problem we encountered was that, since we are working on $\Omega = \mathbb{N}^V$ where the number of particles at any site is unbounded, we could only prove our result on the space of tempered measures, since we needed the functions $\widehat{\mu}(\xi) = \int \mathscr{D}(\xi, \eta) d\mu(\eta)$ to be bounded. Outside of this space there might still be other ergodic measures, and hence we might not yet know the full space of invariant measures.

Hydrodynamic limit

In Chapter 5 we set out to proof the hydrodynamic limit for the run-and-tumble particle system on the particle state space $\mathbb{Z} \times S$. We only did this for the following initial conditions: we defined the probability measures

$$\mu_\rho^N := \bigotimes_{(x,\sigma)\in V} \mathrm{Pois}\big(\rho(\tfrac{x}{N}, \sigma)\big), \tag{6.2}$$

and let the sequence of configurations $\{\eta^N, n \in \mathbb{N}\}$ such that $\eta^N \sim \mu_\rho^N$. We did a rescaling of space for every internal state in S using the empirical measures, defined as

$$\pi_\sigma^N := \frac{1}{N} \sum_{x\in\mathbb{Z}} \eta^N(x, \sigma) \delta_{\frac{x}{N}}.$$

where δ is the Dirac measure, and then $\lim_{N\to\infty} \pi_\sigma^N \to \rho(x, \sigma)dx$ (in the manner defined as in Definition 2.13). Afterwards we also performed a rescaling of time with a factor N^2, and then we needed the following change of parameters, $\lambda \to \frac{\lambda}{N}$ and $\gamma \to \frac{\gamma}{N^2}$, which gave the generator

$$\mathbb{L}_N = \kappa N^2 \mathcal{L} + \lambda N L_a + \gamma L_i.$$

The hydrodynamic limit then became a question of the convergence of the following empirical measures,

$$\pi^N_{\sigma,t} := \frac{1}{N} \sum_{x \in \mathbb{Z}} \eta^N_{N^2 t}(x,\sigma) \delta_{\frac{x}{N}}.$$

It turned out that these converge to a weak solution, $\rho_t(x,\sigma)$, of a system of PDEs where for every $\sigma \in S$ we had

$$\frac{\partial}{\partial t} \rho_t(x,\sigma) = \frac{\kappa}{2} \frac{\partial^2}{\partial x^2} \rho_t(x,\sigma) - \lambda\sigma \frac{\partial}{\partial x} \rho_t(x,\sigma) + \gamma \sum_{\sigma' \neq \sigma} \big(\rho_t(x,\sigma') - \rho_t(x,\sigma)\big) \tag{6.3}$$

with initial value $\rho_0(x,\sigma) = \rho(x,\sigma)$. We proved this using the method by Seppäläinen in [23].

Main line of proof

This proof used the Dynkin martingale from Apendix A.2.2, i.e.,

$$M^N_{\sigma,t} = \pi^N_{\sigma,t}(\phi) - \pi^N_{\sigma,0}(\phi) - \int_0^t \mathbb{L}_N \pi^N_{\sigma,s}(\phi) ds,$$

where $\pi^N_{\sigma,t}(\phi) := \int \phi(\cdot,\sigma) d\pi^N_{\sigma,t}$. We then showed that we had the following equality

$$M^N_{\sigma,t} = \pi^N_{\sigma,t}(\phi) - \pi^N_{\sigma,0}(\phi) - \int_0^t \left[\pi^N_{\sigma,s}(A^*_\sigma \phi) + \gamma \sum_{\sigma' \neq \sigma} \left[\pi^N_{\sigma',s}(\phi) - \pi^N_{\sigma,s}(\phi) \right] \right] ds + \mathcal{O}(tN^{-1}),$$

where $A^*_\sigma = \frac{\kappa}{2} \frac{\partial^2}{\partial x^2} + \lambda\sigma \frac{\partial}{\partial x}$. Afterwards we proved that the quadratic variation of $M^N_{\sigma,t}$ vanishes (and hence the martingale also goes to 0 in probability). We used this result to actually prove convergence in the path-space; using a tightness argument, we showed that the path-space measures of $\bar{\boldsymbol{\pi}}^N = \left(\bar{\pi}^N_\sigma\right)_{\sigma \in S}$, where $\bar{\pi}^N_\sigma = \{\pi^N_{\sigma,t} : t \geq 0\}$, converged to the Dirac measure of the unique weak solution of the system of PDEs given by 6.3. This we used to prove the hydrodynamic limit.

Further results

After we had proven the hydrodynamic limits, we also looked at the propagation of local equilibrium measures. Using convergence of generators, we showed that the local equilibrium measures μ^N_ρ in 6.2 converge weakly, under the run-and-tumble dynamics, to μ_{ρ_t} as in 6.2, where ρ_t solves the hydrodynamic limit with initial condition ρ.

Lastly we have looked at the evolution of the total density of particles for the case where $S = \{-1, 1\}$, i.e., we have looked at $\rho_t(x) = \rho_t(x,1) + \rho_t(x,-1)$. By using the results of the hydrodynamic limit, and also introducing the difference function $\Delta_t(x) = \rho_t(x,1) - \rho_t(x,-1)$ we were able to find that the total density adheres to the following PDE,

$$\left(\frac{\partial}{\partial t} + 2\gamma - \kappa \frac{\partial^2}{\partial x^2} \right) \frac{\partial}{\partial t} \rho_t(x) = -\frac{\kappa^2}{4} \frac{\partial^4}{\partial x^4} \rho_t(x) + (\lambda^2 + \gamma\kappa) \frac{\partial^2}{\partial x^2} \rho_t(x),$$

and so does $\Delta_t(x)$.

Concluding remarks

In this thesis we have only looked at the case where the particles are in $\mathbb{Z} \times S$. Seppäläinen gives in his book a method of proving the hydrodynamic limit for particles in $\mathbb{Z}^d \times S$. However, more steps and assumptions are needed for this proof, including proving the so-called "gradient condition" [23, Section 8.2]. Hence, further research could be done in proving the hydrodynamic limit for run-and-tumble particles in higher dimensions.

Appendix A

A.1 Factorial Poisson moments

It is a well known fact that the Poisson distribution is a solution to a moment problem, i.e., $X \sim \text{Pois}(\lambda)$ for some $\lambda > 0$ if and only if X has the following moments,

$$\mathbb{E}[X^n] = \sum_{k=0}^{n} \left\{ n \atop k \right\} \lambda^k,$$

with $\left\{ n \atop k \right\}$ the Stirling numbers given by

$$\left\{ n \atop k \right\} = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n.$$

In this section we will show a similar result for the factorial moments of the poisson distribution. We will immediately prove this for the product poisson distribution, given by 4.1.

Proposition A.1. *Let ρ be a density profile and μ be an arbitrary probability measure on V, then $\mu = \mu_\rho$, with μ_ρ defined as in 4.1, if and only if for every finite configuration $\xi \in \Omega_f$ given by $\xi = \sum_{i=1}^{n} \delta_{v_i}$, with $v_i \in V$ for all i, the equality*

$$\int \mathscr{D}(\xi, \eta) d\mu(\eta) = \prod_{i=1}^{n} \rho(v_i) \tag{A.1}$$

holds, with the function $\mathscr{D}$ defined as in 3.7.

Before we prove this theorem, we first give a standard result about the Stirling numbers. The proof of this will be omitted from this report.

Lemma A.2. *for $m, n \in \mathbb{N}$, one has*

$$\sum_{k=0}^{n} \left\{ n \atop k \right\} d(k, m) = m^n,$$

with $d(k, m)$ defined as in 3.6.

Proof of Proposition A.1. Let $\xi \in \Omega$ be a finite configuration. Since μ_ρ is a product Poisson measure, the $\eta(v)$ are independent for all $v \in V$, therefore

$$\int \mathscr{D}(\xi, \eta) d\mu_\rho(\eta) = \prod_{v \in V} \int d(\xi(v), \eta(v)) d\mu_\rho(\eta) = \prod_{v \in V} \sum_{k=\xi(v)}^{\infty} d(\xi(v), k) \mathbb{P}(\eta(v) = k). \tag{A.2}$$

If we now look at the infinite series, we find that

$$\begin{aligned}\sum_{k=\xi(v)}^{\infty} d(\xi(v),k)\mathbb{P}(\eta(v)=k) &= \sum_{k=\xi(v)}^{\infty} \frac{k!}{(k-\xi(v))!}\frac{\rho(v)^k}{k!}e^{-\rho(v)} \\ &= \rho(v)^{\xi(v)}e^{-\rho(v)}\sum_{k=0}^{\infty}\frac{\rho(v)^k}{k!} \\ &= \rho(v)^{\xi(v)}.\end{aligned}$$

Filling this back into A.2 indeed gives us the following equality.

$$\int \mathcal{D}(\xi,\eta)d\mu_\rho(\eta) = \prod_{v\in V}\rho(v)^{\xi(v)} = \prod_{i=1}^{n}\rho(v_i).$$

For the other direction, assume A.1 holds for all finite $\xi \in \Omega$. By Lemma A.2, we have that for any $v \in V$ and $n \in \mathbb{N}$

$$\eta(v)^n = \sum_{k=0}^{n} \left\{ n \atop k \right\} d\big(k\delta_v, \eta(v)\big).$$

Therefore, if we take $v_1, ..., v_m \in V$ and $n_1, ..., n_m \in \mathbb{N}$, then

$$\begin{aligned}\int \prod_{i=1}^{m} \eta(v_i)^{n_i} d\nu(\eta) &= \int \prod_{i=1}^{m}\sum_{k_i=0}^{n_i} \left\{ n_i \atop k_i \right\} d(k_i\delta_{v_i}, \eta(v_i))d\nu(\eta) \\ &= \sum_{k_1,...,k_m} \int \prod_{i=1}^{m} \left\{ n_i \atop k_i \right\} d(k_i\delta_{v_i}, \eta(v_i))d\nu(\eta) \\ &= \sum_{k_1,...,k_m} \left(\prod_{i=1}^{m}\left\{ n_i \atop k_i \right\}\right)\cdot \int D\left(\sum_{i=1}^{m} k_i\delta_{v_i}, \eta\right) d\nu(\eta).\end{aligned}$$

Now we can use A.1 to see that

$$\int \prod_{i=1}^{m}\eta(v_i)^{n_i}d\nu(\eta) = \sum_{k_1,...,k_m}\prod_{i=1}^{m}\left\{ n_i \atop k_i \right\}\rho(v_i)^{k_i} = \prod_{i=1}^{m}\sum_{k_i=1}^{n_i}\left\{ n_i \atop k_i \right\}\rho(v_i)^{k_i},$$

where the sum is also the n_i'th moment of a Poisson distribution with parameter $\rho(v_i)$, i.e., the following holds,

$$\sum_{k_i=1}^{n_i}\left\{ n_i \atop k_i \right\}\rho(v_i)^{k_i} = \int \eta(v_i)^{n_i}d\mu_\rho(\eta).$$

Again using the independence of all $\eta(v_i)$ under μ_ρ, we find that

$$\int \prod_{i=1}^{m}\eta(v_i)^{n_i}d\nu(\eta) = \int \prod_{i=1}^{m}\eta(v_i)^{n_i}d\mu_\rho(\eta),$$

thus we see that the moments of ν and μ_ρ agree. □

The proof is not quite finished, since we only showed that if A.1 holds then the moments of ν and μ_ρ agree. However it turns out that this is enough to conclude that $\nu = \mu_\rho$. To see this, we use the following result by Kleiber and Stoyanov, given in [15].

Theorem A.3. *Suppose F is the distribution function of the n-dimensional random vector $(X_1, ..., X_n)$ and denote by F_j the distribution function of the random variable X_j for all j. Then F is uniquely determined by its moments, which we will refer to as M-det, if and only if all F_j are.*

This theorem tells us that if the Poisson distribution is M-det, then any random vector $(X_1, ..., X_n)$ of independent Poisson random variables is as well. Then, by using the Kolmogorov extension theorem, we can conclude that μ_ρ is M-det.

In order to see that a Poisson distribution is M-det, we use Carleman's condition.

Carleman's Condition. *Let ν be a measure on $\mathbb{R}_{\geq 0}$ such that all moments $m_n := \int_0^\infty x^n d\nu(x)$ are finite. Then ν is M-det if*

$$\sum_{n=1}^{\infty} m_n^{-\frac{1}{2n}} = \infty.$$

An easy computation shows that this condition is satisfied if ν is a Poisson measure with parameter $\lambda > 0$ (and if $\lambda = 0$, uniqueness is trivial). It turns out however that if the moment generating function of a measure exists, then Carleman's Condition is automatically satisfied. Since this is the case for a Poisson measure, we are done.

A.2 The Dynkin martingale

In this section we will introduce the notions of a martingale and its quadratic variation and prove that for a given Markov process there exists a corresponding martingale, which is called the Dynkin martingale. Next we will show that this martingale is square integrable and that there exists a simple formula for its quadratic variation.

A.2.1 Quadratic variation of martingales

Definition A.1. A continuous stochastic process $M = \{M_t, t \geq 0\}$ on $(\Omega, \mathscr{F}, \mathbb{P})$ is called a *martingale* with respect to a filtration $\{\mathscr{F}_t, t \geq 0\}$ if it is an $\mathscr{F}_t$-adapted process in $L^1(\Omega, \mathscr{F}, \mathbb{P})$ such that the so-called *martingale property* is satisfied, i.e., for all $0 \leq s \leq t$ we have the following:

$$\mathbb{E}[M_t | \mathscr{F}_s] = M_s$$

The main idea of a martingale is that it has a constant expectation and that the best prediction for the future at any given point in time is the present state. Before we can define the quadratic variation, we need the following norm on partitions of intervals.

Definition A.2. Let $P = \{t_0, t_1, ..., t_n\}$ be a partition of the interval $[0, t]$ such that we have $0 = t_0 < t_1 < ... < t_n = t$. The *mesh size* of such a partition is a norm $|| \cdot ||_m$ that is defined as follows

$$||P||_m = \max_{1 \leq k \leq n} |t_k - t_{k-1}|.$$

While the quadratic variation can be defined for any real-valued stochastic process, we will only give its definition for continuous martingales.

Definition A.3. Let M be a continuous real-valued martingale on $(\Omega, \mathscr{F}, \mathbb{P})$, then the *quadratic variation* of M is a stochastic process $\langle M \rangle = \{\langle M \rangle_t, t \geq 0\}$ defined by the following convergence in probability

$$\langle M \rangle_t := \lim_{||P||_m \to 0} \sum_{k=1}^{n} (X_{t_k} - X_{t_{k-1}})^2,$$

Theorem A.4. *Let M be a continuous real-valued martingale on $(\Omega, \mathscr{F}, \mathbb{P})$ such that every $M_t \in L^2(\Omega, \mathscr{F}, \mathbb{P})$ for all $t \geq 0$ and $M_0 = 0$, then the quadratic variation $\langle M \rangle$ is the a.s-unique predictable process starting at zero such that $M^2 - \langle M \rangle$ is a martingale.*

Proof. The proof can be found in [26, Proposition 3.8]. □

A.2.2 The Dynkin martingale

Theorem A.5. *Let $X = \{X_t, t \geq 0\}$ be an $\mathscr{F}_t$-adapted Markov process on $(\Omega, \mathscr{F}, \mathbb{P})$, generated by L. Then for any $f \in D(L)$, the process M defined as*

$$M_t := f(X_t) - f(X_0) - \int_0^t Lf(X_s)ds$$

is a real-valued martingale on $(\Omega, \mathscr{F}, \mathbb{P})$ with respect to the filtration $\{\mathscr{F}_t, t \geq 0\}$.

Proof. The adaptiveness of the process follows from the fact that X_t is $\mathscr{F}_t$-adapted. To see that M_t is in $L^1(\Omega, \mathscr{F}, \mathbb{P})$, we have that

$$\mathbb{E}[|M_t|] \leq 2||f||_\infty + t||Lf||_\infty < \infty.$$

So all we have to show is the martingale property. Let $0 \leq s \leq t$, then

$$\begin{aligned}\mathbb{E}[M_t - M_s|\mathscr{F}_s] &= \mathbb{E}\left[f(X_t) - f(X_s) - \int_s^t Lf(X_r)dr \;|\mathscr{F}_s\right] \\ &= \mathbb{E}\left[f(X_t)|\mathscr{F}_s\right] - \mathbb{E}[f(X_s)|\mathscr{F}_s] - \int_0^{t-s} \mathbb{E}\left[Lf(X_{r+s})|\mathscr{F}_s\right] dr,\end{aligned}$$

where we have used Fubini in the last equality. Now since $\{X_t, t \geq 0\}$ is a Markov process with respect to the filtration $\{\mathscr{F}_t, t \geq 0\}$ the expectation given the whole past $\mathscr{F}_s$ is equal to the expectation given X_s, i.e.,

$$\begin{aligned}\mathbb{E}[M_t - M_s|\mathscr{F}_s] &= \mathbb{E}\left[f(X_t)|X_s\right] - \mathbb{E}[f(X_s)|X_s] - \int_0^{t-s} \mathbb{E}\left[Lf(X_{r+s})|X_s\right] dr \\ &= S_{t-s}f(X_s) - f(X_s) - \int_0^{t-s} S_r Lf(X_s)dr.\end{aligned}$$

By the Hille-Yosida theorem, we then have that

$$\int_0^{t-s} S_r Lf(X_s)dr = \int_0^{t-s} \frac{\partial}{\partial r} S_r f(X_s)dr = S_{t-s}f(X_s) - f(X_s).$$

Hence we indeed find that

$$\mathbb{E}[M_t - M_s|\mathscr{F}_s] = 0,$$

therefore M is a martingale with respect to $\{\mathscr{F}_t, t \geq 0\}$. □

The martingale we have introduced in this theorem is called the *Dynkin martingale.* If we make some further assumptions on the generator L, we get a formula for the quadratic variation of this process.

Theorem A.6. *If L is the generator of the Markov process X such that $f \in D(L)$ implies that $f^2 \in D(L)$, and take $f \in D(L)$, then the quadratic variation of the Dynkin martingale is given by*

$$\langle M \rangle_t = \int_0^t \left[Lf^2(X_s) - 2f(X_s)Lf(X_s)\right] ds.$$

Proof. Set the stochastic process A as the process in the theorem, i.e.,

$$A_t := \int_0^t \left[Lf^2(X_s) - 2f(X_s)Lf(X_s)\right] ds.$$

Since A_t is an integral from 0 to t of a uniformly bounded function of X_s, we have that $A_0 = 0$ and that the process A is $\mathscr{F}_t$-adapted and left-continuous, hence predictable. The process M^2 is given by

$$M_t^2 = f(X_t)^2 - 2f(X_t)\int_0^t Lf(X_s)ds + \left(\int_0^t Lf(X_s)ds\right)^2 - f(X_0)M_t. \tag{A.3}$$

We then have that

$$\mathbb{E}\left[|M_t^2|\right] \leq ||f||_\infty^2 + 2t||f||_\infty||Lf||_\infty + t||Lf||_\infty^2 + ||f||_\infty\mathbb{E}[|M_t|] < \infty,$$

so $M_t \in L^2(\Omega, \mathscr{F}, \mathbb{P})$ for all $t \geq 0$. Therefore by Theorem A.4, all we have to show is that $M^2 - A$ is a martingale. Notice that from A.3 we can write

$$M_t^2 = f^2(X_t) - 2M_t\int_0^t Lf(X_s)ds - \left(\int_0^t Lf(X_s)ds\right)^2 - f(X_0)M_t. \tag{A.4}$$

We now use the fact that

$$2M_t\int_0^t Lf(X_s)ds + \left(\int_0^t Lf(X_s)ds\right)^2 = \int_0^t d\left(2M_r\int_0^r Lf(X_s)ds + \left(\int_0^r Lf(X_s)ds\right)^2\right).$$

By the product and chain rule, we have that

$$\begin{aligned} &d\left(2M_r\int_0^r Lf(X_s)ds + \left(\int_0^r Lf(X_s)ds\right)^2\right) \\ &\qquad = 2\int_0^r Lf(X_s)dsdM_r + 2Lf(X_r)M_rdr + 2Lf(X_r)\int_0^r Lf(X_s)dsdr, \end{aligned}$$

so we find that

$$\begin{aligned} &2M_t\int_0^t Lf(X_s)ds + \left(\int_0^t Lf(X_s)ds\right)^2 \\ &\qquad = 2\int_0^t\int_0^r Lf(X_s)dsdM_r + \int_0^t 2Lf(X_r)\left(M_r + \int_0^r Lf(X_s)ds\right)dr \\ &\qquad = 2\int_0^t\int_0^r Lf(X_s)dsdM_r + \int_0^t 2f(X_s)Lf(X_s)ds \end{aligned}$$

Filling this back into A.4 gives us

$$\begin{aligned} M_t^2 &= f^2(X_t) - 2\int_0^t\int_0^r Lf(X_s)dsdM_r - \int_0^t 2f(X_s)Lf(X_s)ds - f(X_0)M_t \\ &= f^2(X_t) - \int_0^t Lf^2(X_s)ds - 2\int_0^t\int_0^r Lf(X_s)dsdM_r + \int_0^t Lf^2(X_s) - 2f(X_s)Lf(X_s)ds - f(X_0)M_t, \end{aligned}$$

where the third term is equal to A_t, i.e.,

$$M_t^2 - A_t = f^2(X_t) - \int_0^t Lf^2(X_s)ds - 2\int_0^t\int_0^r Lf(X_s)dsdM_r - f(X_0)M_t.$$

Notice how the first two terms are the Dynkin martingale corresponding to the function $f^2 \in D(L)$. Furthermore, the term $f(X_0)M_t$ is a martingale and the stochastic integral $\int_0^t\int_0^r Lf(X_s)dsdM_r$ is also a martingale, therefore $M_t^2 - A_t$ is a sum of martingales, which is again a martingale. □

A.3 Path-space tightness

Here we will define the notion of tightness, and we will ultimately give a characterization of a tight sequence of probability measures on path-spaces. Throughout this section, we will assume (Y, d) to be a complete, seperable metric space.

A.3.1 Tightness of probability measures

We define $\mathscr{P}(Y)$ as the space of Borel probability measures on Y. We equip $\mathscr{P}(Y)$ with the so-called *Prokhorov metric*, defined by

$$r(\mu, \nu) = \inf \left\{ \varepsilon > 0 : \nu(K) \leq \mu(K^{(\varepsilon)}) + \varepsilon \text{ for very closed } K \subset Y \right\},$$

where $K^{(\varepsilon)}$ is the ε-neighborhood of F, i.e.,

$$K^{(\varepsilon)} := \{x \in Y : d(x, y) < \varepsilon \text{ for some } y \in K\}.$$

We have that a sequence of probability measures $\{\mu_n : n \in \mathbb{N}\}$ converges in this metric to a distribution μ if and only if it converges weakly, i.e., for all $f \in C_b(Y)$

$$\int f d\mu_n \to \int f d\mu.$$

We shall write this as $\mu_n \xrightarrow{w} \mu$

Definition A.4. A set $U \subset \mathscr{P}(Y)$ is called *tight* if for every $\varepsilon > 0$ there exists a compact set $K \subset Y$ such that for all $\mu \in U$ we have that $\mu(K) \geq 1 - \varepsilon$.

By the fact that (Y, d) is a complete and seperable metric space, we get that $(\mathscr{P}(Y), r)$ is one as well. That means that we get the following theorem as an application of *Prokhorov's Theorem.*

Theorem A.7. *If $(\mathscr{P}(Y), r)$ is a complete and seperable, then a collection $U \subset \mathscr{P}(Y)$ is tight if and only if the closure of U is compact in $(\mathscr{P}(Y), r)$.*

Proof. The proof can be found in [3, section 5] □

From this theorem we find that if we have a sequence of probability measures $\{\mu_n : n \in \mathbb{N}\}$ that is tight, then there exists a subsequence that $\{\mu_{n_k} : k \in \mathbb{N}\}$ such that $\mu_{n_k} \xrightarrow{w} \mu$ for some $\mu \in \mathscr{P}(Y)$.

A.3.2 Path spaces and the Skorokhod topology

We define the set D_Y as the following:

$$D_Y = \{\alpha : [0, \infty) \to Y \mid \omega \text{ is right-continuous and has left-limits}\}.$$

We see D_Y as a path space of stochastic processes taking values in Y. We also refer to the $\alpha \in D_Y$ as being *càdlàg*, which comes from the french "continue à droite, limite à gauche" (translation: "continuous on the right, limit on the left").

The space D_Y is metrizable in the following way. First we assume without loss of generality that the metric d on Y satisfies that $d(x, y) \leq 1$ for all $x, y \in Y$ (otherwise we can take the metric $d \wedge 1$). If we define Λ as the collection of strictly increasing, bijective Lipschitz functions $\lambda : [0, \infty) \to [0, \infty)$ satisfying

$$\theta(\lambda) := \sup_{0 \leq s < t} \left| \log \left(\frac{\lambda(t) - \lambda(s)}{t - s} \right) \right| < \infty, \tag{A.5}$$

and for $\alpha, \beta \in D_Y$, $\lambda \in \Lambda$ and $0 < u < \infty$ we set

$$s_Y(\alpha,\beta,\lambda,u) = \sup_{t\geq 0} d\big(\alpha(t\wedge u), \beta(\lambda(t)\wedge u)\big),$$

then we can define the following metric on D_Y:

$$s_Y(\alpha,\beta) = \inf_{\lambda\in\Lambda}\left\{\theta(\lambda)\wedge\int_0^\infty e^{-u}s(\alpha,\beta,\lambda,u)du\right\}. \tag{A.6}$$

This metric is called the *Skorokhod distance* between two paths α and β, and it was introduced by P. Billingsly [3]. It is named this way since it induces the *Skorokhod topology*, which was introduced by A.V. Skorokhod [24] in order to study the convergence in distribution of stochastic jump-processes, e.g. continuous-time Markov chains.

Convergence under this metric can be characterized by the following lemma.

Lemma A.8. *Let $\{\alpha_n, n\in\mathbb{N}\}\subset D_Y$ and $\alpha\in D_Y$, then $s_Y(\alpha_n,\alpha)\to 0$ if and only if there exists a sequence $\{\lambda_n, n\in\mathbb{N}\}\subset\Lambda$ such that $\gamma(\lambda_n)\to 0$ and*

$$\sup_{0\leq t\leq T} d\big(\alpha_n(\lambda_n(t)),\alpha(t)\big)\to 0$$

for all $T>0$.

Proof. The proof can be found in [23, Lemma A.2]. □

Again by the fact that (Y,d) is a complete, seperable metric space, we find that (D_Y, s) is one as well, therefore we are able to apply Theorem A.7 for the metric space $(\mathscr{P}(D_Y), r)$. The following theorem then gives us a characterization of tight sequences of probability measures on $\mathscr{P}(D_Y)$.

Theorem A.9. *The sequence of probability measures $\{Q_n : n\in\mathbb{N}\}\subset\mathscr{P}(D_Y)$ is tight if and only if the following two conditions are satisfied:*

1. *For every $\varepsilon>0$ and $t\geq 0$ there exists a compact set $K\subset Y$ such that*

$$\liminf_{n\to\infty} Q_n\{\alpha\in D_Y : \zeta(t)\in K\}\geq 1-\varepsilon.$$

2. *For every $\varepsilon>0$ and $0<T<\infty$ there exists a $\delta>0$ such that*

$$\limsup_{n\to\infty} Q_n\{\alpha\in D_Y : \omega'(\zeta,\delta,T)\geq\varepsilon\}\leq\varepsilon,$$

where

$$\omega'(\alpha,\delta,T) = \inf_P \sup\big\{d\big(\alpha(s),\alpha(t)\big) : s,t\in[t_{i-1},t_i) \text{ for some } i\big\}$$

with $P=\{t_0,t_1,...,t_n\}$ ranging over all the partitions with $0=t_1<...<t_{n-1}<T\leq t_n$ and $\min_i|t_i-t_{i-1}|>\delta$.

Proof. The proof can be found in [10, Corollary 7.4] □

Instead of working with ω', in our proof tightness proof in Section 5.4 we work with the simpler function

$$\omega(\alpha,\delta,T) := \sup\{d\big(\alpha(s),\alpha(t)\big) : s,t\in[0,T], |t-s|<\delta\}, \tag{A.7}$$

which has the following useful property

Lemma A.10. *If $\delta \leq \frac{1}{2}$ then we have the following inequalities:*

$$\omega'(\alpha, \delta, T) \leq \omega(\alpha, 2\delta, T + 2\delta) \leq \omega(\alpha, 2\delta, T + 1).$$

Proof. Let $k = \lceil \frac{T}{2\delta} \rceil$, then we have that $T \leq k \cdot 2\delta \leq T + 2\delta$. Now, the following partition

$$P = \{t_i = i \cdot 2\delta : 1 \leq i \leq k\},$$

satisfies the requirements of Theorem A.9. But clearly for this partition we have

$$\sup \left\{d\big(\alpha(s), \alpha(t)\big) : s, t \in [t_{i-1}, t_i) \text{ for some } i\right\} \leq \sup \left\{d\big(\alpha(s), \alpha(t)\big) : s, t \in [0, T+2\delta], |t-s| < \delta\right\},$$

so we find that

$$\omega'(\alpha, \delta, T) \leq \omega(\alpha, 2\delta, T + 2\delta).$$

The second inequality of the lemma follows from the fact that $\omega(\alpha, \delta, T)$ is an increasing function of T. □

A.3.3 Portmanteau Theorem

If we have weak convergence $\mu_n \xrightarrow{w} \mu$ of some sequence of probability measures, then the Portmanteau Theorem gives us a number of useful properties of these measures

Theorem A.11. *Let $\{\mu_n, n \in \mathbb{N}\} \subset \mathscr{P}(Y)$ and $\mu \in \mathscr{P}(Y)$, then the following are equivalent:*

1. $\mu_n \xrightarrow{w} \mu$,
2. $\int f d\mu_n \to \int f d\mu$ *for all bounded, uniformly continuous f,*
3. $\limsup_{n\to\infty} \mu_n(K) \leq \mu(K)$ *for all closed $K \subset Y$,*
4. $\liminf_{n\to\infty} \mu_n(O) \geq \mu(O)$ *for all closed $O \subset Y$,*
5. $\lim_{n\to\infty} \mu_n(A) = \mu(A)$ *for all μ-continuity sets $A \subset Y$.*

Here a set $A \subset Y$ is called a μ-continuity set if $\mu(\partial A) = 0$.

Proof. The proof can be found in [3, Theorem 2.1]. □

A.4 Borel measures on $\mathbb{R}$ and vague convergence

We denote $\mathbf{M}$ as the set of Borel measures on $\mathbb{R}$, i.e., the set of $[0, \infty]$-valued measures μ on the Borel sets $\mathscr{B}(\mathbb{R})$ with the property that $\mu(B) < \infty$ for any bounded Borel set B. We can induce this metric with the following metric:

$$d_{\mathbf{M}}(\mu, \nu) := \sum_{j=1}^{\infty} 2^{-j} \left(1 \wedge \left| \int \phi_j d\mu - \int \phi_j d\nu \right| \right), \tag{A.8}$$

with $\{\phi_j, j \geq 1\}$ a certain sequence of test functions, i.e., $\phi_j \in C_c^\infty(\mathbb{R})$ (for more information, see [23, section A.10]).

Proposition A.12. $(\mathbf{M}, d_{\mathbf{M}})$ *is a complete, seperable metric space*

Proof. The proof can be found in [23, Proposition A.26]. □

Convergence with respect to this metric comes down to the following: Let $\{\mu_n, n \in \mathbb{N}\}$ be a sequence of Borel measures, then $d_{\mathbf{M}}(\mu_n, \mu) \to 0$ as $n \to \infty$ for some $\mu \in \mathbf{M}$ if and only if for all $\phi \in C_c^\infty(\mathbb{R})$ we have

$$\int \phi d\mu_n \to \int \phi d\mu.$$

A proof of this can be found in [23, Lemma A.24]. This type of convergence is called *vague convergence.*

If we have vague convergence of the Borel measures μ_n, we have the following proposition that will help us bound $\int \phi d\mu_n$ for n big enough

Proposition A.13. *Suppose that $d_{\mathbf{M}}(\mu_n, \mu) \to 0$. For any test function $\phi \in C_c^\infty(\mathbb{R})$, there exists and $m = m(\phi)$ and another test fuction $\phi_m \in C_c^\infty(\mathbb{R})$ such that for any $n \in \mathbb{N}$ with $d_{\mathbf{M}}(\mu_n, \mu) < 2^{-m}$*

$$\left|\int \phi d\mu_n\right| \leq ||\phi||_\infty \cdot \left(\int \phi_m d\mu + 2^m d_{\mathbf{M}}(\mu_n, \mu)\right)$$

A.4.1 Relative Compactness

Definition A.5. In a metric space (Y, d), we say that a set $V \subset Y$ is *relatively compact* if its closure $\overline{V}$ is compact.

In the metric space $(\mathbf{M}, d_{\mathbf{M}})$ we have a very useful characterization for relative compactness of a set.

Theorem A.14. *A set $V \in \boldsymbol{M}$ is relatively compact if and only if for all compact sets $K \in \mathscr{B}(\mathbb{R})$ we have that*

$$\sup_{\mu \in V} \mu(K) < \infty.$$

Proof. The proof can be found in [23, Proposition A.25]. □

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