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DOI
10.1109/TAC.2019.2939639

Publication date
2020

Document Version
Accepted author manuscript

Published in
IEEE Transactions on Automatic Control

Citation (APA)
De Persis, C., & Grammatico, S. (2020). Continuous-time integral dynamics for a class of aggregative games with coupling constraints. *IEEE Transactions on Automatic Control*, *65*(5), 2171-2176. https://doi.org/10.1109/TAC.2019.2939639

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2019.2939639, IEEE Transactions on Automatic Control

Continuous-time integral dynamics for a class of aggregative games with coupling constraints

Claudio De Persis and Sergio Grammatico

Abstract—**We consider continuous-time equilibrium seeking in a class of aggregative games with strongly convex cost functions and affine coupling constraints. We propose simple, semi-decentralized integral dynamics and prove their global asymptotic convergence to a variational generalized aggregative or Nash equilibrium. The proof is based on Lyapunov arguments and invariance techniques for differential inclusions.**

Index Terms—**Aggregative game theory, Multi-agent systems, Decentralized control, Projected dynamical systems.**

I. Introduction

Aggregative game theory [2] is a mathematical framework to model inter-dependent optimal decision making problems for a set of noncooperative agents, where the decision of each agent is affected by some aggregate effect of all the agents. Motivated by application domains where this aggregative feature arises, e.g. demand side management and network congestion control [3], equilibrium seeking in aggregative games is currently an active research area.

Existence and uniqueness of (Nash) equilibria in (aggregative) games has been comprehensively studied, especially in close connection with variational inequalities [4], [5, §12]. Distributed and semi-decentralized algorithms [6], [7], [3], [8] have been proposed as *discrete-time* dynamics that converge to an equilibrium of the game, e.g. Nash or aggregative equilibrium, under appropriate technical assumptions and sufficient conditions on the problem data. Specifically, one can characterize the desired equilibria as the zeros of a monotone operator, e.g. via the concatenation of interdependent Karush–Kuhn–Tucker operators, and formulate an equivalent fixed-point problem, to be solved via fixed-point iterations with guaranteed global asymptotic convergence [3], [8].

Within the literature on equilibrium seeking for aggregative games with coupling constraints, almost all solution methods are algorithms in *discrete time*, where tuning the step size is typically a hard task, or it requires global information, usually unavailable in multi-agent game setups. Instead, in this paper, we address the aggregative equilibrium seeking problem via *continuous-time* dynamics. Game equilibrium seeking algorithms in continuous time can be in fact used as controllers for continuous-time processes [9], [10], without the challenges of interconnecting a discrete-time algorithm with continuous-time dynamics [11]. In turn, the use of continuous-time algorithms as optimal feedback controllers opens up the possibility to directly study disturbance rejection and robustness to time-varying uncertainties [12].

Inspired by passivity arguments [13], our contribution is to provide simple primal-dual, integral, semi-decentralized dynamics for the computation of generalized aggregative and Nash equilibria. Our contribution is complementary to that in [14], which proposes continuous-time, *distributed* dynamics for generalized Nash equilibrium seeking in aggregative games with coupling *equality* constraints. Differently from ours, the dynamics in [14] require supplementary discontinuous sign consensus algorithms to estimate the aggregate strategy, a suitable initialization of dual and auxiliary variables and an off-line, non-parallelizable calculation of the gain parameters.

To handle both local and global *inequality* constraints, we propose equilibrium seeking dynamics that are characterized as the dynamics of a projected dynamical system [15]. Thus, we exploit invariance arguments for differential inclusions with maximally monotone set-valued right-hand side, and apply it to our primal-dual projected dynamics [16], [17]. Our main technical contribution is to prove global asymptotic convergence of the proposed dynamics to a generalized (primal-dual) equilibrium of the aggregative game, under some technical assumptions on the problem data, mainly, convexity of constraints and strong convexity of the local cost functions which implies strict monotonicity of the game mapping. Compared to our preliminary work [1], in this paper, we consider aggregative games with *coupling constraints*, propose *primal-dual* dynamics, and discuss convergence to both generalized aggregative equilibria and generalized Nash equilibria under less restrictive assumptions.

The paper is organized as follows. We introduce and mathematically characterize the problem setup in Section II. We propose the equilibrium seeking dynamics and present the main result in Section III. Technical discussions and corollaries are in Section IV. The proofs are given in the Appendix.

Notation and definitions: $\mathbf{0}$ denotes a matrix/vector with all elements equal to 0. $\otimes$ denotes the Kronecker product. Given N vectors $x_1, \ldots, x_N \in \mathbb{R}^n$, we define $\boldsymbol{x} := \mathrm{col}\,(x_1, \ldots, x_N) = \left[x_1^\top, \ldots, x_N^\top\right]^\top$, $\boldsymbol{x}_{-i} := \mathrm{col}\,(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)$, and $\mathrm{avg}(\boldsymbol{x}) := \frac{1}{N}\sum_{i=1}^N x_i$. Let the set $\mathcal{S} \subseteq \mathbb{R}^n$ be non-empty. The mapping $\iota_{\mathcal{S}} : \mathbb{R}^n \to \{0, \infty\}$ denotes the indicator function, i.e., $\iota_{\mathcal{S}}(x) = 0$ if $x \in \mathcal{S}$, ∞ otherwise. The set-valued mapping $\mathcal{N}_{\mathcal{S}} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ denotes the normal cone operator. The set-valued mapping $\mathcal{T}_{\mathcal{S}} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ denotes the tangent cone operator. The mapping

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$\mathrm{proj}_{\mathcal{S}}(\cdot) := \mathrm{argmin}_{y\in\mathcal{S}} \|y-\cdot\| : \mathbb{R}^n \to \mathcal{S}$ denotes the projection operator; $\Pi_{\mathcal{S}}(x,v) := \lim_{h\to 0^+} \frac{1}{h}\left(\mathrm{proj}_{\mathcal{S}}(x+hv)-x\right)$ denotes the projection of the vector $v\in\mathbb{R}^n$ onto the tangent cone of $\mathcal{S}$ at $x\in\mathcal{S}$, i.e., $\Pi_{\mathcal{S}}(x,\cdot) = \mathrm{proj}_{\mathcal{T}_{\mathcal{S}}(x)}(\cdot)$. Let $A:\mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued mapping. $\mathrm{zer}(A) := \{z\in\mathrm{dom}(A) \mid 0\in A(z)\}$ denotes the set of zeros of A. A is (strictly) monotone if, for all $x,y\in\mathbb{R}^n$ $(x\neq y)$, and $\xi\in A(x)$, $\zeta\in A(y)$, $(\xi-\zeta)^\top(x-y)\geq 0$ (>0); it is $\mu-$strongly monotone, $\mu>0$, if for all $x,y\in\mathbb{R}^n$, and $\xi\in A(x)$, $\zeta\in A(y)$, $(\xi-\zeta)^\top(x-y)\geq \mu\|x-y\|^2$. For a function $f:\mathbb{R}^n\to\overline{\mathbb{R}}$, $\mathrm{dom}(f):=\{x\in\mathbb{R}^n \mid f(x)<\infty\}$; $\partial f:\mathrm{dom}(f)\rightrightarrows\mathbb{R}^n$ denotes its subdifferential set-valued mapping, defined as $\partial f(x) := \{v\in\mathbb{R}^n \mid f(z)\geq f(x)+v^\top(z-x) \text{ for all } z\in\mathrm{dom}(f)\}$; if f is convex and differentiable at x, then $\partial f(x) = \{\nabla f(x)\}$. Given a nonempty, closed convex set $C\subseteq\mathbb{R}^n$, $\partial\iota_C(x) = \mathcal{N}_C(x)$ for every $x\in C$, and $\partial\iota_C(x)=\varnothing$ otherwise [18, §23]. Given a closed convex set $C\subseteq\mathbb{R}^n$ and a (set-valued) mapping $F:C\rightrightarrows\mathbb{R}^n$, the (generalized) variational inequality problem, denoted by $\mathrm{VI}(C,F)$ ($\mathrm{GVI}(C,F)$), is the problem to find $x^*\in C$ such that $\inf_{y\in C}\sup_{\varphi\in F(x^*)} (y-x^*)^\top\varphi\geq 0$.

II. Mathematical background: Aggregative games and variational equilibria

A. Aggregative games with affine coupling constraints

An aggregative game with coupling constraints is denoted by a triplet $\mathcal{G}_{\text{agg}} = (\mathcal{I}, (J_i)_{i\in\mathcal{I}}, (\mathcal{X}_i)_{i\in\mathcal{I}})$, where $\mathcal{I} := \{1,\ldots,N\}$ is the index set of N decision makers, or agents, $\left(J_i:\mathbb{R}^n\times\mathbb{R}^n\to\overline{\mathbb{R}}\right)_{i\in\mathcal{I}}$ is an ordered set of cost functions and $\left(\mathcal{X}_i:\mathbb{R}^{n(N-1)}\rightrightarrows\mathbb{R}^n\right)_{i\in\mathcal{I}}$ is an ordered set of set-valued mappings that represent coupled constraint sets. For each $i\in\mathcal{I}$, we assume an affine structure for the coupling constraints:

$$\mathcal{X}_i(\boldsymbol{x}_{-i}) := \{y\in\Omega_i \mid A_i y + \textstyle\sum_{j\in\mathcal{I}\setminus\{i\}} A_j x_j \leq b\},$$

for some set $\Omega_i\subseteq\mathbb{R}^n$ and matrices $A_1,\ldots,A_N\in\mathbb{R}^{m\times n}$.

In aggregative games, the aim of each agent $i\in\mathcal{I}$ is to minimize its cost function $J_i(x_i,\mathrm{avg}(\boldsymbol{x}))$ that depends on the local decision variable and on the average among the decision variables of all agents, i.e., $\mathrm{avg}(\boldsymbol{x}) := \frac{1}{N}\sum_{i=1}^N x_i$. Formally, in this paper, we consider aggregative games represented by the following collection of inter-dependent problems:

$$\begin{cases} \mathcal{P}_i(\sigma,\boldsymbol{x}_{-i}): \min\limits_{x_i\in\Omega_i} J_i(x_i,\sigma) \text{ s.t. } A\boldsymbol{x}-b\leq 0, \quad \forall i\in\mathcal{I} \\ \sigma = \mathrm{avg}(\boldsymbol{x}) \end{cases} \tag{1}$$

where $A\boldsymbol{x} := [A_1,\ldots,A_N]\,\boldsymbol{x} = A_i x_i + \sum_{j\neq i} A_j x_j$.

The optimization problems $\mathcal{P}_i(\sigma,\boldsymbol{x}_{-i})$, $i\in\mathcal{I}$, in (1) are parametric in σ and $\boldsymbol{x}_{-i}$. Note also that the decision variable x_i affects the cost J_i only via its first argument. For given σ and $\boldsymbol{x}_{-i}$, let $x_i^\star(\sigma,\boldsymbol{x}_{-i})$ be the set of optimal solutions to $\mathcal{P}_i(\sigma,\boldsymbol{x}_{-i})$, i.e., $x_i^\star(\sigma,\boldsymbol{x}_{-i}) := \mathrm{argmin}_{y_i\in\Omega_i} J_i(y_i,\sigma)$ s.t. $A_i y_i + \sum_{j\neq i} A_j x_j - b\leq 0$. As notion of solution to (1), which we call *generalized aggregative equilibrium* (GAE), we consider a set of optimal responses to the average, namely, a set of decision variables such that each is optimal given the average among all the decision variables and the coupling constraints. Formally, a GAE is a collection of vectors $\bar{\boldsymbol{x}} = (\bar{x}_i)_{i\in\mathcal{I}}$ such that $\bar{x}_i\in x_i^\star(\bar{\sigma},\bar{\boldsymbol{x}}_{-i})$ for all $i\in\mathcal{I}$, and $\bar{\sigma} = \frac{1}{N}\sum_{i\in\mathcal{I}}\bar{x}_i = \mathrm{avg}(\bar{\boldsymbol{x}})$, as defined next.

Definition 1: Generalized aggregative equilibrium. A set of decision variables $\boldsymbol{x}^* = \mathrm{col}(x_1^*,\ldots,x_N^*)\in\mathbb{R}^{nN}$ is a generalized aggregative equilibrium (GAE) of the game in (1) if, for all $i\in\mathcal{I}$,

$$x_i^*\in\underset{y\in\Omega_i}{\mathrm{argmin}}\; J_i(y,\mathrm{avg}(\boldsymbol{x}^*)) \text{ s.t. } A_i y + \sum_{j\neq i} A_j x_j^* \leq b.$$

□

Throughout the paper, we postulate the following technical assumptions.

Standing Assumption 1: Continuity, compactness, convexity. The sets $\{\Omega_i\}_{i\in\mathcal{I}}$ are non-empty, compact and convex. The set $\boldsymbol{X} := \boldsymbol{\Omega}\cap\boldsymbol{C}$, where $\boldsymbol{\Omega} := \Omega_1\times\ldots\times\Omega_N$ and $\boldsymbol{C} := \{\boldsymbol{x}\in\mathbb{R}^{nN} \mid A\boldsymbol{x}\leq b\}$, is non-empty and satisfies Slater's constraint qualification. □

Standing Assumption 2: Strong convexity, Lipschitz continuity. For all $i\in\mathcal{I}$, and $\xi,\zeta\in\mathbb{R}^n$, the function $J_i(\cdot,\zeta)$ is continuously differentiable and $\mu-$strongly convex, and the mapping $\mathrm{col}(\nabla_{x_1}J_1(x_1,\cdot),\ldots,\nabla_{x_N}J_N(x_N,\cdot))$ is $\ell-$Lipschitz continuous, where $\mu>\ell>0$. □

B. Game mapping

A fundamental mapping in game equilibrium problems is the mapping that collects the gradients with respect to the local decision variable. Since we are interested in generalized aggregative equilibria, rather than generalized Nash equilibria, together with semi-decentralized equilibrium seeking dynamics, let us define the following game mapping:

$$F(\boldsymbol{x},\sigma) := \begin{bmatrix} \mathrm{col}\left((\nabla_{x_i}J_i(x_i,\sigma))_{i\in\mathcal{I}}\right) \\ k\left(\sigma-\mathrm{avg}(\boldsymbol{x})\right)\end{bmatrix}, \tag{2}$$

where $k>0$ is a design parameter, and σ is a control variable. Throughout the paper, we choose the design parameter k such that the game mapping F in (2) is strongly monotone.

Standing Assumption 3: Parameter choice. The gain parameter $k>0$ in (2) is chosen such that

$$k\in\left(2\mu-\ell-2\sqrt{\mu(\mu-\ell)},\, 2\mu-\ell+2\sqrt{\mu(\mu-\ell)}\right). \tag{3}$$

□

We note that the interval for k in Standing Assumption 3 is non-empty, thanks to $\mu>\ell$ in Standing Assumption 2.

Proposition 1: There exists $\epsilon>0$ such that the mapping F in (2) is $\epsilon-$strongly monotone. □

C. Variational and operator-theoretic characterization

In this subsection, we show that a special GAE is the solution to a representative variational inequality and, equivalently, the zero of a monotone operator.

Lemma 1: The variational inequality $\mathrm{VI}(\boldsymbol{X}\times\mathbb{R}^n, F)$, with $\boldsymbol{X}$ as in Standing Assumption 1 and F as in (2), has a unique solution $(\boldsymbol{x}^*,\sigma^*)$, where $\boldsymbol{x}^*$ is a GAE of the game in (1), called variational GAE (v-GAE). □

Our aim is to design semi-decentralized dynamics that converge to the unique solution to $\mathrm{VI}(\boldsymbol{X}\times\mathbb{R}^n, F)$, which in view of Lemma 1 generates a v-GAE. Thus, in order to decouple the coupling constraints of the game, $A\boldsymbol{x}\leq b$ in (1), we adopt duality theory for equilibrium problems. We start from the definition of the Lagrangian functions, $\{L_i\}_{i\in\mathcal{I}}$, one for each agent $i \in \mathcal{I}$:

$$L_i\left(x_i, \sigma, \lambda_i\right) := J_i(x_i, \sigma) + \iota_{\Omega_i}(x_i) + \lambda_i{}^\top (A\boldsymbol{x} - b)\,, \quad (4)$$

where λ_i is a dual variable. Then, for each $i \in \mathcal{I}$, we introduce the Karush–Kuhn–Tucker (KKT) system associated with the optimization problems in (1):

$$\forall i \in \mathcal{I}: \begin{cases} 0 \in \nabla_{x_i} J_i(x_i, \sigma) + \partial\iota_{\Omega_i}(x_i) + A_i^\top \lambda_i \\ 0 = \sigma - \mathrm{avg}(\boldsymbol{x}) \\ 0 \leq \lambda_i \perp -(A\boldsymbol{x} - b) \geq 0\,, \end{cases} \quad (5)$$

where $\{\lambda_i\}_{i\in\mathcal{I}}$ are the dual variables, one vector for each agent $i \in \mathcal{I}$, associated with the coupling constraint, and $0 \leq \lambda_i \perp -(A\boldsymbol{x} - b) \geq 0$ represents the complementarity condition. We anticipate that in (5), the first two equations will allow us to recover semi-decentralized dynamics later on.

Inspired by [19, Th. 9, Def. 3] and [8], we show that the v-GAE is associated with a solution to the KKT system in (5) with equal dual variables, $\lambda_i = \lambda$ for all $i \in \mathcal{I}$. Thus, let us extend the space of the decision variables of the aggregative game and define the extended version of the game mapping,

$$F_{\text{ext}}(\boldsymbol{x}, \sigma, \lambda) := \begin{bmatrix} F(\boldsymbol{x}, \sigma) \\ b \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & A^\top \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -A & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \sigma \\ \lambda \end{bmatrix}, \quad (6)$$

which has a fundamental role in the characterization of the equilibrium. Specifically, the solution of the KKT system is a zero of a (maximally) monotone operator that contains the extended game mapping in (6) and that generates a v-GAE. These arguments are formalized in the following result:

Proposition 2: Variational/Operator-theoretic characterization. The following statements are equivalent:

(i) $\boldsymbol{x}^*$ is a v-GAE of the game in (1);
(ii) the triplet $(\boldsymbol{x}^*, \mathrm{avg}(\boldsymbol{x}^*), \mathbf{1}_N \otimes \lambda^*)$ solves the KKT system in (5), for some $\lambda^* \in \mathbb{R}^m_{\geq 0}$.
(iii) $(\boldsymbol{x}^*, \mathrm{avg}(\boldsymbol{x}^*), \lambda^*) \in \mathrm{zer}(\mathcal{N}_{\boldsymbol{\Omega}\times\mathbb{R}^n\times\mathbb{R}^m_{\geq 0}} + F_{\text{ext}})$, for some $\lambda^* \in \mathbb{R}^m_{\geq 0}$. □

Remark 1: Maximal monotonicity. The operator in Proposition 2 (iii), sum of maximally monotone operators [20, Def. 12.5, Ex. 12.7] is maximally monotone [20, Cor. 12.18]. □

III. Continuous-time integral dynamics for generalized aggregative equilibrium seeking

For asymptotically reaching the v-GAE, we consider the following continuous-time integral dynamics:

$$\forall i \in \mathcal{I}: \quad \begin{aligned} \dot{x}_i &= \Pi_{\Omega_i}\left(x_i\,, -\nabla_{x_i} J_i(x_i, \sigma) - A_i^\top \lambda\right) \\ \dot{\sigma} &= k\left(\mathrm{avg}(\boldsymbol{x}) - \sigma\right) \\ \dot{\lambda} &= \Pi_{\mathbb{R}^m_{\geq 0}}\left(\lambda, A\boldsymbol{x} - b\right). \end{aligned} \quad (7)$$

where $k > 0$ is the gain parameter.

Equivalently, in collective projected-vector form, the dynamics in (7) read as

$$\begin{bmatrix} \dot{\boldsymbol{x}} \\ \dot{\sigma} \\ \dot{\lambda} \end{bmatrix} = \Pi_{\boldsymbol{\Omega}\times\mathbb{R}^n\times\mathbb{R}^m_{\geq 0}}\left(\begin{bmatrix} \boldsymbol{x} \\ \sigma \\ \lambda \end{bmatrix}, \begin{bmatrix} -F(\boldsymbol{x}, \sigma) + \begin{bmatrix} -A^\top\lambda \\ \mathbf{0} \end{bmatrix} \\ A\boldsymbol{x} - b \end{bmatrix}\right). \quad (8)$$

Remark 2: Semi-decentralized structure. The computation and information exchange in (7) are semi-decentralized: each agent performs decentralized computations, namely, projected-pseudo-gradient steps, and does not exchange information with other agents. A central control unit, which does not participate in the game, collects aggregative information, $\mathrm{avg}(\boldsymbol{x}(t))$ and $A\boldsymbol{x}(t) - b$, and broadcasts two signals, $\sigma(t)$ and $\lambda(t)$, to the agents playing the aggregative game. In turn, the dynamics of the broadcast signal $\sigma(t)$ are driven by the average among all the decision variables, $\mathrm{avg}(\boldsymbol{x}(t))$, while the dynamics of the signal $\lambda(t)$ are driven by the coupling-constraint violation, $A\boldsymbol{x}(t) - b$. Unlike distributed coordination schemes, this semi-decentralized structure prevents that the noncooperative agents are imposed to exchange information. □

First, we show that the $\boldsymbol{x}-$part of an equilibrium for the dynamics in (8) is a v-GAE, in view of Proposition 2 (iii).

Proposition 3: The following statements are equivalent:

(i) $\left(\bar{\boldsymbol{x}}, \bar{\sigma}, \bar{\lambda}\right)$ is an equilibrium for the dynamics in (8);
(ii) $\left(\bar{\boldsymbol{x}}, \bar{\sigma}, \bar{\lambda}\right) \in \mathrm{zer}\left(\mathcal{N}_{\boldsymbol{\Omega}\times\mathbb{R}^n\times\mathbb{R}^m_{\geq 0}} + F_{\text{ext}}\right)$. □

In view of Proposition 3, we can directly analyze the convergence of the projected dynamics in (8) to an equilibrium. Let us introduce a quadratic function, V, which is used later on to obtain a Lyapunov function.

Lemma 2: Consider the function

$$V(\boldsymbol{x}, \sigma, \lambda) := \tfrac{1}{2}\left\|\boldsymbol{x} - \boldsymbol{x}'\right\|^2 + \tfrac{1}{2}\left\|\sigma - \sigma'\right\|^2 + \tfrac{1}{2}\left\|\lambda - \lambda'\right\|^2\,, \quad (9)$$

where $(\boldsymbol{x}, \sigma, \lambda), (\boldsymbol{x}', \sigma', \lambda')$ are arbitrary vectors in $\boldsymbol{\Omega} \times \mathbb{R}^n \times \mathbb{R}^m_{\geq 0}$. It holds that

$$\begin{aligned} \dot{V}(\boldsymbol{x}, \sigma, \lambda) :=&\ \nabla V(\boldsymbol{x}, \sigma, \lambda)^\top \left[\begin{smallmatrix} \dot{\boldsymbol{x}} \\ \dot{\sigma} \\ \dot{\lambda} \end{smallmatrix}\right] \\ \leq&\ \nabla V(\boldsymbol{x}, \sigma, \lambda)^\top \begin{bmatrix} -F(\boldsymbol{x}, \sigma) + \left[\begin{smallmatrix} -A^\top\lambda \\ \mathbf{0} \end{smallmatrix}\right] \\ A\boldsymbol{x} - b \end{bmatrix}, \end{aligned} \quad (10)$$

where $\left[\begin{smallmatrix} \dot{\boldsymbol{x}} \\ \dot{\sigma} \\ \dot{\lambda} \end{smallmatrix}\right]$ stands for the right-hand side in (8). □

We are now ready to establish our main global asymptotic convergence result. The proof, given in Appendix A, is based on invariance arguments for differential inclusions with maximal monotone set-valued right-hand side.

Theorem 1: Convergence to variational generalized aggregative equilibrium. Let $\boldsymbol{x}^*$ be the v-GAE of the game in (1). For any initial condition $(\boldsymbol{x}_0, \sigma_0, \lambda_0) \in \boldsymbol{\Omega} \times \mathbb{R}^n \times \mathbb{R}^m_{\geq 0}$, there exists a unique solution to (8) starting from $(\boldsymbol{x}_0, \sigma_0, \lambda_0)$, which satisfies (8) almost everywhere, remains in $\boldsymbol{\Omega} \times \mathbb{R}^n \times \mathbb{R}^m_{\geq 0}$, is bounded for all time, and converges to $\{\boldsymbol{x}^*\} \times \{\mathrm{avg}\,(\boldsymbol{x}^*)\} \times \{\bar{\lambda}\}$, a Lyapunov stable equilibrium of (8). □

IV. Technical discussions

A. On generalized Nash equilibria

We recall that a Nash equilibrium is a set of strategies where each is optimal given the other strategies, as formalized next.

Definition 2: Generalized Nash equilibrium. A set of decision variables $\boldsymbol{x}^* = \mathrm{col}\,(x_1^*, \ldots, x_N^*) \in \boldsymbol{X}$ is a generalized

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2019.2939639, IEEE Transactions on Automatic Control

Nash equilibrium (GNE) of the game in (1) if, for all $i \in \mathcal{I}$,

$$\begin{array}{rll} x_i^* \in & \underset{y \in \Omega_i}{\operatorname{argmin}} & J_i\left(y, \frac{1}{N} y + \frac{1}{N}\sum_{j\neq i} x_j^*\right) \\ & \text{s.t.} & A_i y + \sum_{j \neq i} A_j x_j^* \leq b. \end{array} \quad (11)$$

□

Remark 3: A GNE in Definition 2 differs from a GAE in Definition 1, since in the latter, each decision variable is optimal given the average among the decision variables of all agents that enter as second argument of the cost functions. Under our regularity assumptions, the distance between the variational GNE (v-GNE) and the v-GAE tends to zero as N tends to infinity, see the limit arguments in [3], [21], [22]. □

If we aim at computing a GNE, rather than a GAE, then the definition of game mapping shall be changed into

$$F_{\mathrm{N}}(\boldsymbol{x}, \sigma) := \begin{bmatrix} \operatorname{col}\left(\left(\nabla_{x_i} J_i(x_i, \sigma) + \frac{1}{N}\nabla_\sigma J_i(x_i,\sigma)\right)_{i=1}^N\right) \\ k\left(\sigma - \operatorname{avg}(\boldsymbol{x})\right) \end{bmatrix}, \quad (12)$$

since, for each agent i, the variable x_i enters as local decision variable in both the first and the second argument of the cost function J_i. Analogously to (8), possible continuous-time generalized Nash equilibrium seeking dynamics are

$$\begin{bmatrix} \dot{\boldsymbol{x}} \\ \dot{\sigma} \\ \dot{\lambda} \end{bmatrix} = \Pi_{\boldsymbol{\Omega} \times \mathbb{R}^n \times \mathbb{R}^m_{\geq 0}} \left(\begin{bmatrix} \boldsymbol{x} \\ \sigma \\ \lambda \end{bmatrix}, \begin{bmatrix} -F_{\mathrm{N}}(\boldsymbol{x},\sigma) + \begin{bmatrix} -A^\top \lambda \\ \mathbf{0} \end{bmatrix} \\ A\boldsymbol{x} - b \end{bmatrix} \right). \quad (13)$$

Convergence to a v-GNE of the above dynamics then follows under the following assumption.

Assumption 1: For all $i \in \mathcal{I}$, the mapping $x_i \mapsto \nabla_{x_i} J_i(x_i, \sigma) + \frac{1}{N}\nabla_\sigma J_i(x_i,\sigma)$ from (12) is $\mu_{\mathrm{N}}-$strongly monotone; the mapping $\sigma \mapsto \operatorname{col}\left(\left(\nabla_{x_i} J_i(x_i,\sigma) + \frac{1}{N}\nabla_\sigma J_i(x_i,\sigma)\right)_{i\in\mathcal{I}}\right)$ from (12) is $\ell_{\mathrm{N}}-$Lipschitz continuous, where $\mu_{\mathrm{N}} > \ell_{\mathrm{N}} > 0$. □

Under Assumption 1, the gain parameter k chosen as in (3) with $(\mu_{\mathrm{N}}, \ell_{\mathrm{N}})$ in place of (μ, ℓ) guarantees strong monotonicity of the game mapping $F_{\mathrm{N}}(\boldsymbol{x}, \sigma)$ and an analogous analysis as in Subsection II-C and Section III leads to the following result.

Corollary 1: Convergence to generalized Nash equilibrium. Let $\boldsymbol{x}_{\mathrm{N}}^*$ be the v-GNE of the game in (11), let Assumption 1 hold and k be chosen as in (3) with $(\mu_{\mathrm{N}}, \ell_{\mathrm{N}})$ in place of (μ, ℓ). Then, for any initial condition $(\boldsymbol{x}_0, \sigma_0, \lambda_0) \in \boldsymbol{\Omega} \times \mathbb{R}^n \times \mathbb{R}^m_{\geq 0}$, there exists a unique solution to (13) starting from $(\boldsymbol{x}_0, \sigma_0, \lambda_0)$, which satisfies (8) almost everywhere, remains in $\boldsymbol{\Omega} \times \mathbb{R}^n \times \mathbb{R}^m_{\geq 0}$, is bounded for all time, and converges to $\{\boldsymbol{x}_{\mathrm{N}}^*\} \times \{\operatorname{avg}(\boldsymbol{x}_{\mathrm{N}}^*)\} \times \{\overline{\lambda}\}$, a Lyapunov stable equilibrium of (13). □

B. On the case of cost functions with separable structure

In this subsection, let us consider a separable structure for the cost functions, i.e.,

$$\forall i \in \mathcal{I}: \; J_i(x_i, \sigma) = f_i(x_i) + (C_i \sigma)^\top x_i, \quad (14)$$

for some $\eta-$strongly convex functions $\{f_i\}_{i\in\mathcal{I}}$ and $n \times n$ matrices $\{C_i\}_{i\in\mathcal{I}}$. The condition in Standing Assumption 2 is then satisfied if

$$\mu := \eta > \max_{i \in \mathcal{I}} \|C_i\| =: \ell.$$

We note that this condition is less restrictive than the one in [1, Prop. 1], which requires homogeneous matrices $\{C_i\}_{i\in\mathcal{I}}$, and the one in [23, Th. 2]. Importantly, we emphasize that the condition does not depend on N, which is desirable for large number of agents [1, §IV].

For the v-GNE problem, with the separable structure in (14), we have that $\nabla_{x_i} J_i(x_i,\sigma) + \frac{1}{N}\nabla_\sigma J_i(x_i,\sigma) = \nabla_{x_i} f_i(x_i) + C_i\sigma + \frac{1}{N} C_i^\top x_i$. Thus, Assumption 1 is satisfied if $\eta I_n + \frac{1}{N}\frac{C_i + C_i^\top}{2} > \|C_i\| \, I_n$, for all $i \in \mathcal{I}$, with $\ell_{\mathrm{N}} = \|C_i\|$ and μ_{N} such that $\eta I_n + \frac{1}{N}\frac{C_i+C_i^\top}{2} \geq \mu_{\mathrm{N}} I_n > \|C_i\| \, I_n$.

V. Conclusion

In aggregative games with affine coupling constraints, continuous-time integral dynamics with semi-decentralized computation and information exchange can ensure asymptotic convergence to a generalized aggregative or Nash equilibrium, under mild regularity and strict monotonicity assumptions.

Appendix A: Proofs

Proof of Proposition 1: For ease of notation, let $D(\boldsymbol{x},\sigma) := \operatorname{col}\left((\nabla_{x_i} J_i(x_i,\sigma))_{i\in\mathcal{I}}\right)$. F is ϵ-strongly monotone iff, for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^{nN}$ and $\sigma, \tau \in \mathbb{R}^n$, we have

$$\begin{aligned} &(F(\boldsymbol{x},\sigma) - F(\boldsymbol{y},\tau))^\top(\operatorname{col}(\boldsymbol{x},\sigma) - \operatorname{col}(\boldsymbol{y},\tau)) = \\ &(D(\boldsymbol{x},\sigma) - D(\boldsymbol{y},\tau))^\top(\boldsymbol{x}-\boldsymbol{y}) + k(\sigma - \operatorname{avg}(\boldsymbol{x}) - \tau + \operatorname{avg}(\boldsymbol{y}))^\top(\sigma-\tau) \\ &\geq \epsilon\|\boldsymbol{x}-\boldsymbol{y}\|^2 + \epsilon\|\sigma-\tau\|^2, \end{aligned} \quad (15)$$

i.e.,

$$\begin{aligned} &(D(\boldsymbol{x},\sigma) - D(\boldsymbol{y},\sigma))^\top(\boldsymbol{x}-\boldsymbol{y}) + (D(\boldsymbol{y},\sigma) - D(\boldsymbol{y},\tau))^\top(\boldsymbol{x}-\boldsymbol{y}) \\ &+ k\|\sigma-\tau\|^2 - k(\operatorname{avg}(\boldsymbol{x}) - \operatorname{avg}(\boldsymbol{y}))^\top(\sigma-\tau) \\ &\geq \epsilon\|\boldsymbol{x}-\boldsymbol{y}\|^2 + \epsilon\|\sigma-\tau\|^2. \end{aligned} \quad (16)$$

Since $D(\cdot,\sigma)$ is $\mu-$strongly monotone, $D(\boldsymbol{y},\cdot)$ is $\ell-$Lipschitz continuous, and the mapping $\operatorname{avg}(\cdot)$ is $1-$Lipschitz continuous, by Cauchy-Schwartz inequality it is sufficient to have

$$(\mu-\epsilon)\|\boldsymbol{x}-\boldsymbol{y}\|^2 - (\ell+k)\|\sigma-\tau\|\,\|\boldsymbol{x}-\boldsymbol{y}\| + (k-\epsilon)\|\sigma-\tau\|^2 \geq 0,$$

for some $\epsilon > 0$. The inequality above is true if the associated discriminant is negative, i.e., $k^2 - 2(2\mu - \ell - 2\epsilon)k + \ell^2 + 4\epsilon(\mu-\epsilon) < 0$. Since $\epsilon > 0$ can be chosen arbitrarily small, it suffices to have $k^2 - 2(2\mu-\ell)k + \ell^2 < 0$. The interval in (3) guarantees the fulfilment of the latter quadratic inequality. ■

Proof of Lemma 1: Since the set $\boldsymbol{X} \times \mathbb{R}^n := \boldsymbol{K}$ is closed and convex, and the mapping F is strongly monotone by Proposition 1, the variational inequality $\mathrm{VI}(\boldsymbol{K}, F)$ has unique solution, $(\boldsymbol{x}^*, \sigma^*)$ [4, Th. 2.3.3 (b)]. By definition, the solution satisfies the inequality $0 \leq (\boldsymbol{x} - \boldsymbol{x}^*)^\top \left(\operatorname{col}((\nabla_{x_i} J_i(x_i^*, \sigma^*))_{i\in\mathcal{I}})\right) + k(\sigma-\sigma^*)^\top(\sigma^* - \operatorname{avg}(\boldsymbol{x}^*))$, for all $\boldsymbol{x} \in \boldsymbol{X}$, $\sigma \in \mathbb{R}^n$. In particular, for $\boldsymbol{x} = \boldsymbol{x}^*$, it holds that $0 \leq (\sigma - \sigma^*)^\top(\sigma^* - \operatorname{avg}(\boldsymbol{x}^*))$, for all $\sigma \in \mathbb{R}^n$, hence $\sigma^* = \operatorname{avg}(\boldsymbol{x}^*)$, otherwise, for $\sigma = \operatorname{avg}(\boldsymbol{x}^*) \neq \sigma^*$, we would reach a false statement, $\|\sigma^* - \operatorname{avg}(\boldsymbol{x}^*)\|^2 \leq 0$. Thus, the solution $(\boldsymbol{x}^*, \operatorname{avg}(\boldsymbol{x}^*))$ satisfies the inequality $0 \leq (\boldsymbol{x} - \boldsymbol{x}^*)^\top \left(\operatorname{col}((\nabla_{x_i} J_i(x_i^*, \operatorname{avg}(\boldsymbol{x}^*)))_{i\in\mathcal{I}})\right)$, for all $(\boldsymbol{x}, \sigma) \in \boldsymbol{K}$. Moreover, for each i, since $J_i(\cdot, \sigma)$ is convex, we have that $(x_i - x_i^*)^\top \nabla_{x_i} J_i(x_i^*, \operatorname{avg}(\boldsymbol{x}^*)) \leq$

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2019.2939639, IEEE Transactions on Automatic Control

$J_i(x_i, \text{avg}(\boldsymbol{x}^*)) - J_i(x_i^*, \text{avg}(\boldsymbol{x}^*))$, for all $x_i \in \mathbb{R}^n$. Thus, $0 \leq \sum_{i\in\mathcal{I}} \{J_i(x_i, \text{avg}(\boldsymbol{x}^*)) - J_i(x_i^*, \text{avg}(\boldsymbol{x}^*))\}$, for all $(\boldsymbol{x}, \sigma) \in \boldsymbol{K}$. Take an arbitrary $i \in \mathcal{I}$. The last inequality holds for $\boldsymbol{x} = (x_i, \boldsymbol{x}_{-i}^*)$ such that $(\boldsymbol{x}, \text{avg}(\boldsymbol{x})) \in \boldsymbol{K}$, i.e., $J_i(x_i^*, \text{avg}(\boldsymbol{x}^*)) \leq J_i(x_i, \text{avg}(\boldsymbol{x}^*))$, for all $x_i \in \Omega_i$ and $A_i x_i + \sum_{j\neq i} A_j x_j^* \leq b$. Thus, by Definition 1, $\boldsymbol{x}^*$ is a GAE. ■

Proof of Proposition 2: By [24, (1.1), (2.8)], we have that $(\boldsymbol{x}^*, \sigma^*)$ is the solution to VI$(\boldsymbol{X} \times \mathbb{R}^n, F)$ if and only if $\boldsymbol{0} \in F(\boldsymbol{x}^*, \sigma^*) + \mathcal{N}_{\boldsymbol{X}\times\mathbb{R}^n}(\boldsymbol{x}^*, \sigma^*) = F(\boldsymbol{x}^*, \sigma^*) + \mathcal{N}_{\boldsymbol{\Omega}\times\mathbb{R}^n}(\boldsymbol{x}^*, \sigma^*) + \mathcal{N}_{\boldsymbol{C}\times\mathbb{R}^n}(\boldsymbol{x}^*, \sigma^*)$, i.e., $\inf_{y\in\boldsymbol{C}\times\mathbb{R}^n} \sup_{\varphi\in F(\boldsymbol{x}^*,\sigma^*)+\mathcal{N}_{\boldsymbol{\Omega}\times\mathbb{R}^n}} \left(y - \left[\begin{smallmatrix}\boldsymbol{x}^*\\ \sigma^*\end{smallmatrix}\right]\right)^\top \varphi \geq 0$. By [24, Th. 3.1], the latter GVI holds if and only if there exists $\lambda^* \in \mathbb{R}^m_{\geq 0}$ such that

$$\begin{aligned} &0 \in F(\boldsymbol{x}^*, \sigma^*) + \partial\iota_{\boldsymbol{\Omega}}(\boldsymbol{x}^*) \times \{\boldsymbol{0}\} + \left[\begin{smallmatrix}A^\top\lambda^*\\ \boldsymbol{0}\end{smallmatrix}\right] \\ &0 \in -(A\boldsymbol{x} - b) + \mathcal{N}_{\mathbb{R}^m_{\geq 0}}(\lambda^*) \end{aligned} \tag{17}$$

The inclusion $0 \in \mathcal{N}_{\mathbb{R}^m_{\geq 0}}(\lambda^*) - (A\boldsymbol{x} - b)$ is equivalent to the complementarity condition $0 \leq \lambda \perp -(A\boldsymbol{x} - b) \geq 0$ [20, Ex. 6.13]. Hence, (17) is equivalent to

$$\begin{aligned} &0 \in F(\boldsymbol{x}^*, \sigma^*) + \partial\iota_{\boldsymbol{\Omega}}(\boldsymbol{x}^*) \times \{\boldsymbol{0}\} + \left[\begin{smallmatrix}A^\top\lambda^*\\ \boldsymbol{0}\end{smallmatrix}\right] \\ &0 \leq \lambda^* \perp -(A\boldsymbol{x}^* - b) \geq 0\,, \end{aligned} \tag{18}$$

which is (5) with $\lambda_i^* = \lambda^*$ for all $i \in \mathcal{I}$. ■

For ease of notation, next, we use $\xi := \text{col}(\boldsymbol{x}, \sigma)$, $\xi^* := \text{col}(\boldsymbol{x}^*, \sigma^*)$, $\bar{\xi} := \text{col}(\bar{\boldsymbol{x}}, \bar{\sigma})$ and $\Xi := \boldsymbol{\Omega} \times \mathbb{R}^n$.

Proof of Proposition 3: By Moreau's decomposition,

$$\begin{aligned} \boldsymbol{0} &= \Pi_\Xi \left(\bar{\xi}, -F(\bar{\xi}) + \left[\begin{smallmatrix}-A^\top\bar{\lambda}\\ \boldsymbol{0}\end{smallmatrix}\right]\right) \\ &= -F(\bar{\xi}) + \left[\begin{smallmatrix}-A^\top\bar{\lambda}\\ \boldsymbol{0}\end{smallmatrix}\right] - \text{proj}_{\mathcal{N}_{\boldsymbol{\Omega}\times\mathbb{R}^n}(\bar{\xi})}\left(-F(\bar{\xi}) + \left[\begin{smallmatrix}-A^\top\bar{\lambda}\\ \boldsymbol{0}\end{smallmatrix}\right]\right) \end{aligned}$$

and $\boldsymbol{0} = \Pi_{\mathbb{R}^m_{\geq 0}}\left(\bar{\lambda}, A\bar{\boldsymbol{x}} - b\right) = A\bar{\boldsymbol{x}} - b - \text{proj}_{\mathcal{N}_{\mathbb{R}^m_{\geq 0}}(\bar{\lambda})}(A\bar{\boldsymbol{x}} - b)$. The proof then follows immediately. ■

Proof of Lemma 2: The proof follows the steps of [13, Proof of Lemma 6]. Since $\nabla V(\xi, \lambda)^\top = \text{col}\left(\xi - \xi', \lambda - \lambda'\right)^\top$, for all vectors $\boldsymbol{u}$, by Moreau's decomposition theorem, we have

$$\begin{aligned} &(\xi - \xi')^\top \Pi_\Xi\left(\xi, -F(\xi) + \boldsymbol{u}\right) = \\ &\quad (\xi - \xi')^\top \left[-F(\xi) + \boldsymbol{u} - \text{proj}_{\mathcal{N}_\Xi(\xi)}(-F(\xi) + \boldsymbol{u})\right]. \end{aligned}$$

By definition of the normal cone $\mathcal{N}_\Xi(\xi)$, we have that $-(\xi - \xi')^\top \text{proj}_{\mathcal{N}_\Xi(\xi)}(-F(\xi) + \boldsymbol{u}) \leq 0$, and in turn

$$(\xi - \xi')^\top \Pi_\Xi\left(\xi, -F(\xi) + \boldsymbol{u}\right) \leq (\xi - \xi')^\top \left(-F(\xi) + \boldsymbol{u}\right). \tag{19}$$

With similar arguments, we can show that

$$(\lambda - \lambda')^\top \Pi_{\mathbb{R}^m_{\geq 0}}\left(\lambda, A\boldsymbol{x} - b\right) \leq (\lambda - \lambda')^\top \left(A\boldsymbol{x} - b\right). \tag{20}$$

The proof follows by summing up the inequalities in (19) with $\boldsymbol{u} = \left[\begin{smallmatrix}-A^\top\lambda\\ \boldsymbol{0}\end{smallmatrix}\right]$ and (20). ■

Proof of Theorem 1: The dynamics in (8) represent a projected dynamical system with discontinuous right-hand side [15], for which existence and uniqueness of the solution under our assumptions is known, see Lemma 3 in Appendix B. The proof uses invariance arguments for differential inclusions with maximally monotone right-hand side [16]. First, we note that F_{ext} in (6) is continuous and monotone. Then, we consider a zero of $\mathcal{N}_{\Xi\times\mathbb{R}^m_{\geq 0}} + F_{\text{ext}}$ (Proposition 2 (iii)), (ξ^*, λ^*), and, bearing in mind Lemma 2, define the Lyapunov function $W(\xi, \lambda) := \frac{1}{2}\|\xi - \xi^*\|^2 + \frac{1}{2}\|\lambda - \lambda^*\|^2$. We show next that

$$\nabla W(z)^\top F_{\text{ext}}(z^*) = \begin{bmatrix}\xi - \xi^*\\ \lambda - \lambda^*\end{bmatrix}^\top F_{\text{ext}}(\xi^*, \lambda^*) \geq 0 \tag{21}$$

for all $z = (\xi, \lambda) \in \Xi \times \mathbb{R}^m_{\geq 0}$. By Propositions 2, 3,

$$\begin{aligned} \boldsymbol{0} &= \Pi_\Xi\left(\begin{bmatrix}\boldsymbol{x}^*\\ \sigma^*\end{bmatrix}, -F(\boldsymbol{x}^*, \sigma^*) + \left[\begin{smallmatrix}-A^\top\lambda^*\\ \boldsymbol{0}\end{smallmatrix}\right]\right) \\ \boldsymbol{0} &= \Pi_{\mathbb{R}^m_{\geq 0}}\left(\lambda^*, A\boldsymbol{x}^* - b\right)\,, \end{aligned} \tag{22}$$

therefore, we have $0 = -\nabla W(\xi, \lambda)^\top \left[\begin{smallmatrix}\dot{\xi}\\ \dot{\lambda}\end{smallmatrix}\right]^*$, where $\left[\begin{smallmatrix}\dot{\xi}\\ \dot{\lambda}\end{smallmatrix}\right]^*$ stands for the right-hand side of (22). By Lemma 2, we immediately obtain (21):

$$0 = -\nabla W(\xi, \lambda)^\top \left[\begin{smallmatrix}\dot{\xi}\\ \dot{\lambda}\end{smallmatrix}\right]^* \leq \nabla W(\xi, \lambda)^\top F_{\text{ext}}(\xi^*, \lambda^*).$$

Consequently, we have that

$$\begin{aligned} \nabla W(z)^\top \dot{z} &\leq -\nabla W(z)^\top F_{\text{ext}}(z) \\ &\leq -\nabla W(z)^\top [F_{\text{ext}}(z) - F_{\text{ext}}(z^*)] \leq 0, \end{aligned}$$

by the monotonicity of F_{ext}. We conclude that W is not increasing along the trajectories of (8). By radial unboundedness of W, for any initial condition z_0, the corresponding solution is bounded and therefore the associated ω-limit set $\Lambda(z_0)$ is non-empty, compact, invariant and attractive. Moreover, by definition of the ω-limit set, W is constant on $\Lambda(z_0)$. Thus, any solution $\zeta(\cdot)$ with initial condition in $\Lambda(z_0)$ must satisfy $\dot{W}(\zeta(t)) = 0$, that is $\Lambda(z_0)$ is contained in the set of points satisfying $\nabla W(z)^\top F_{\text{ext}}(z) = 0$. We then study the set $\mathcal{O} = \left\{z \in \Xi \times \mathbb{R}^m_{\geq 0} \mid \nabla W(z)^\top F_{\text{ext}}(z) = 0\right\}$. For all $\text{col}(\xi, \lambda) \in \mathcal{O}$, it holds:

$$\begin{aligned} \nabla W(\xi, \lambda)^\top F_{\text{ext}}(\xi, \lambda) &= \left[\begin{smallmatrix}\xi-\xi^*\\ \lambda-\lambda^*\end{smallmatrix}\right]^\top F_{\text{ext}}(\xi, \lambda) \\ &= (\xi - \xi^*)^\top \left(F(\xi) + \left[\begin{smallmatrix}A^\top\lambda\\ \boldsymbol{0}\end{smallmatrix}\right]\right) - (\lambda - \lambda^*)^\top (A\boldsymbol{x} - b)\,. \end{aligned} \tag{23}$$

By Proposition 2, we have that $F_{\text{ext}}(\xi^*, \lambda^*) + \left[\begin{smallmatrix}v^*\\ \boldsymbol{0}\end{smallmatrix}\right] = \boldsymbol{0}$ for some $v^* \in \mathcal{N}_{\boldsymbol{\Omega}}(\boldsymbol{x}^*)$, hence $F(\xi^*) + \left[\begin{smallmatrix}A^\top\lambda^*\\ \boldsymbol{0}\end{smallmatrix}\right] + \left[\begin{smallmatrix}v^*\\ \boldsymbol{0}\end{smallmatrix}\right] = \boldsymbol{0}$ and ${\lambda^*}^\top(A\boldsymbol{x}^* - b) = 0$. Therefore, for all $\text{col}(\xi, \lambda) \in \mathcal{O}$,

$$\begin{aligned} 0 = (\xi - \xi^*)^\top \Big(&F(\boldsymbol{x}, \sigma) - F(\boldsymbol{x}^*, \sigma^*) - \left[\begin{smallmatrix}v^*\\ \boldsymbol{0}\end{smallmatrix}\right] + \\ &\left[\begin{smallmatrix}A^\top(\lambda-\lambda^*)\\ \boldsymbol{0}\end{smallmatrix}\right]\Big) - (\lambda - \lambda^*)^\top (A\boldsymbol{x} - b)\,. \end{aligned} \tag{24}$$

Now, we observe that $(\lambda - \lambda^*)^\top (A\boldsymbol{x}^* - b) = \underbrace{\lambda^\top}_{\geq 0} \underbrace{(A\boldsymbol{x}^* - b)}_{\leq 0} - \underbrace{{\lambda^*}^\top (A\boldsymbol{x}^* - b)}_{=0} \leq 0$, and in turn

$$\begin{aligned} 0 \geq \left[\begin{smallmatrix}\boldsymbol{x}-\boldsymbol{x}^*\\ \sigma-\sigma^*\end{smallmatrix}\right]^\top \Big(&F(\boldsymbol{x}, \sigma) - F(\boldsymbol{x}^*, \sigma^*) - \left[\begin{smallmatrix}v^*\\ \boldsymbol{0}\end{smallmatrix}\right] + \\ &\left[\begin{smallmatrix}A^\top(\lambda-\lambda^*)\\ \boldsymbol{0}\end{smallmatrix}\right]\Big) - (\lambda - \lambda^*)^\top A\left(\boldsymbol{x} - \boldsymbol{x}^*\right) \\ = (\xi - \xi^*)^\top &\left(F(\boldsymbol{x}, \sigma) - F(\boldsymbol{x}^*, \sigma^*) - \left[\begin{smallmatrix}v^*\\ \boldsymbol{0}\end{smallmatrix}\right]\right) \geq 0. \end{aligned} \tag{25}$$

The last inequality holds because, by Standing Assumption 3, $\left(F(\boldsymbol{x}, \sigma) - F(\boldsymbol{x}^*, \sigma^*)\right)^\top (\xi - \xi^*) \geq 0$ and, by the definition of normal cone, ${v^*}^\top(\boldsymbol{x} - \boldsymbol{x}^*) \leq 0$. Thus, we obtain

$$\left(F(\boldsymbol{x}, \sigma) - F(\boldsymbol{x}^*, \sigma^*)\right)^\top (\xi - \xi^*) = {v^*}^\top(\boldsymbol{x} - \boldsymbol{x}^*) = 0. \tag{26}$$

From (26), due to Standing Assumption 3, we conclude that $\boldsymbol{x} = \boldsymbol{x}^*$ and $\sigma = \sigma^* = \mathrm{avg}(\boldsymbol{x}^*)$. From (24) and (26), we obtain $0 = (\boldsymbol{x} - \boldsymbol{x}^*)^\top A^\top (\lambda - \lambda^*) - (\lambda - \lambda^*)^\top (A\boldsymbol{x} - b)$, hence $\lambda^\top (A\boldsymbol{x}^* - b) = 0$. The latter implies $(\lambda' - \lambda)^\top (A\boldsymbol{x}^* - b) \leq 0$ for all $\lambda' \in \mathbb{R}^m_{\geq 0}$, i.e., $A\boldsymbol{x}^* - b \in \mathcal{N}_{\mathbb{R}^m_{\geq 0}}(\lambda)$, or, equivalently, $\mathbf{0} = \Pi_{\mathbb{R}^m_{\geq 0}}(\lambda, A\boldsymbol{x}^* - b)$. The latter and the identity $\xi = \xi^*$ established before returns that (ξ^*, λ) is a zero of $\mathcal{N}_{\Xi \times \mathbb{R}^m_{\geq 0}} + F_{\mathrm{ext}}$, hence an equilibrium of (8), and this concludes the characterisation of $\mathcal{O}$.

We finally show that convergence is to an equilibrium point of (8). By Lemma 3 in Appendix B, the solution to (8) is the same as the solution to $-\dot{z} \in F_{\mathrm{ext}}(z) + \mathcal{N}_{\Xi \times \mathbb{R}^m_{\geq 0}}(z)$, where the right-hand side of the differential inclusion is maximally monotone by Remark 1. We can then apply [25, Ch. 3, Sec. 2, Th. 1], [16, Th. 2.2, (C1), (C3)], to conclude that every equilibrium of (8) is Lyapunov stable and that, if the solution has an ω-limit point at an equilibrium, then the solution converges to that equilibrium. Now, from the arguments in the first part of the proof, the non-empty and invariant ω-limit set $\Lambda(\mathrm{col}(\xi_0, \lambda_0))$ is contained in $\mathcal{O}$. Since points of $\mathcal{O}$ are equilibria of (8), then the ω-limit set $\Lambda(\mathrm{col}(\xi_0, \lambda_0))$ is a singleton with an equilibrium (ξ^*, λ) to which the solution converges. This concludes the proof. ■

Proof of Corollary 1: Analogous to the proof of Theorem 1, namely, with F_{N} in (12), in place of F in (2). ■

Appendix B: Projected dynamical systems

We consider a generic projected dynamical system

$$\dot{z} = \Pi_K(z, -F(z)) \tag{27}$$

where $K \subseteq \mathbb{R}^n$ is a non-empty, closed and convex set. The dynamic behavior of (27) is well-studied for continuous, hypomonotone mappings F.

Definition 3: Hypomonotonicity. A mapping $F: \mathbb{R}^n \to \mathbb{R}^n$ is hypomonotone if there exists $\beta \geq 0$ such that

$$(z - z')^\top (F(z) - F(z')) \geq -\beta \|z - z'\|^2$$

for all $z, z' \in \mathbb{R}^n$. □

In view of [17], [15], we recall next some equivalent formulations of the projected dynamical system in (27).

Lemma 3 (from [17, Th. 1]): Let F in (27) be continuous and hypomonotone. For any initial condition $z_0 \in K$, the differential inclusion

$$-\dot{z}(t) \overset{\text{a.e.}}{\in} F(z(t)) + \mathcal{N}_K(z(t)). \tag{28}$$

has a unique solution $z(t)$ that belongs to K for almost all $t \geq 0$. Furthermore, the evolution variational inequality

$$z(t) \overset{\text{a.e.}}{\in} K, \ t \geq 0, \ \inf_{v \in K} \langle \dot{z}(t) + F(z(t)), v - z(t) \rangle \overset{\text{a.e.}}{\geq} 0, \tag{29}$$

and the projected dynamical system

$$\dot{z}(t) \overset{\text{a.e.}}{=} \mathrm{proj}_{\mathcal{T}_K(z(t))}(-F(z(t))) = \Pi_K(z(t), -F(z(t)))$$

have the same solution as to (28). □

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