

The ℓ^s -boundedness of a family of integral operators on UMD banach function spaces

Lorist, Emiel

DOI

[10.1007/978-3-030-10850-2_20](https://doi.org/10.1007/978-3-030-10850-2_20)

Publication date

2019

Document Version

Accepted author manuscript

Published in

Positivity and Noncommutative Analysis

Citation (APA)

Lorist, E. (2019). The ℓ^s -boundedness of a family of integral operators on UMD banach function spaces. In G. Buskes, M. de Jeu, P. Dodds, A. Schep, F. Sukochev, & J. van Neerven (Eds.), *Positivity and Noncommutative Analysis: Festschrift in Honour of Ben de Pagter on the Occasion of his 65th Birthday* (pp. 365-379). (Trends in Mathematics). Birkhäuser. https://doi.org/10.1007/978-3-030-10850-2_20

Important note

To cite this publication, please use the final published version (if applicable).
Please check the document version above.

Copyright

Other than for strictly personal use, it is not permitted to download, forward or distribute the text or part of it, without the consent of the author(s) and/or copyright holder(s), unless the work is under an open content license such as Creative Commons.

Takedown policy

Please contact us and provide details if you believe this document breaches copyrights.
We will remove access to the work immediately and investigate your claim.

THE ℓ^s -BOUNDEDNESS OF A FAMILY OF INTEGRAL OPERATORS ON UMD BANACH FUNCTION SPACES

EMIEL LORIST

Dedicated to Ben de Pagter on the occasion of his 65th birthday.

ABSTRACT. We prove the ℓ^s -boundedness of a family of integral operators with an operator-valued kernel on UMD Banach function spaces. This generalizes and simplifies the earlier work by Gallarati, Veraar and the author [12], where the ℓ^s -boundedness of this family of integral operators was shown on Lebesgue spaces. The proof is based on a characterization of ℓ^s -boundedness as weighted boundedness by Rubio de Francia.

1. INTRODUCTION

Over the past decades there has been a lot of interest in the L^p -maximal regularity of PDEs. Maximal L^p -regularity of the abstract Cauchy problem

$$(1.1) \quad \begin{cases} u'(t) + Au(t) = f(t), & t \in (0, T] \\ u(0) = x, \end{cases}$$

where A is a closed operator on a Banach space X , means that for all $f \in L^p((0, T]; X)$ the solution u has “maximal regularity”, i.e. both u' and Au are in $L^p((0, T]; X)$. Maximal L^p -regularity can for example be used to solve quasi-linear and fully nonlinear PDEs by linearization techniques combined with the contraction mapping principle, see e.g. [1, 8, 30, 36].

In the breakthrough work of Weis [40, 41], an operator theoretic characterization of maximal L^p -regularity on UMD Banach spaces was found in terms of the \mathcal{R} -boundedness of the resolvents of A on a sector. \mathcal{R} -boundedness is a random boundedness condition on a family of operators which is a strengthening of uniform boundedness. We refer to [7, 21] for more information on \mathcal{R} -boundedness.

In [13, 14] Gallarati and Veraar developed a new approach to maximal L^p -regularity for the case where the operator A in (1.1) is time-dependent and $t \mapsto A(t)$ is merely assumed to be measurable. In this new approach \mathcal{R} -boundedness is once again one of the main tools. For their approach the \mathcal{R} -boundedness of the family of integral operators $\{I_k : k \in \mathcal{K}\}$ on $L^p(\mathbb{R}; X)$

2010 *Mathematics Subject Classification*. Primary: 42B20; Secondary: 42B25 46E30.

Key words and phrases. ℓ^s -boundedness, Integral operator, Banach function space, Muckenhoupt weights, Hardy–Littlewood maximal operator, UMD.

The author is supported by the VIDI subsidy 639.032.427 of the Netherlands Organisation for Scientific Research (NWO).

is required. Here I_k is defined for $f \in L^p(\mathbb{R}; X)$ as

$$I_k f(t) := \int_{-\infty}^t k(t-s)T(t,r)f(r) dr, \quad t \in \mathbb{R},$$

where $T(t,s)$ is the two-parameter evolution family associated to $A(t)$ and \mathcal{K} contains all kernels $k \in L^1(\mathbb{R})$ such that $|k| * |g| \leq Mg$ for all simple $g : \mathbb{R} \rightarrow \mathbb{C}$.

In the literature there are many \mathcal{R} -boundedness results for integral operators, see [21, Chapter 8] for an overview. However none of these are applicable to the operator family of $\{I_k : k \in \mathcal{K}\}$. Therefore in [12] Gallarati, Veraar and the author show a sufficient condition for the \mathcal{R} -boundedness of $\{I_k : k \in \mathcal{K}\}$ on $L^p(\mathbb{R}; X)$ in the special case where $X = L^q$. This is done through the notion of ℓ^s -boundedness, which states that for all finite sequences $(I_{k_j})_{j=1}^n$ in $\{I_k : k \in \mathcal{K}\}$ and $(x_j)_{j=1}^n$ in X we have

$$\left\| \left(\sum_{j=1}^n |I_{k_j} x_j|^s \right)^{1/s} \right\|_X \lesssim \left\| \left(\sum_{j=1}^n |x_j|^s \right)^{1/s} \right\|_X.$$

For $s = 2$ this notion coincides with \mathcal{R} -boundedness as a consequence of the Kahane-Khintchine inequalities.

Our main contribution is the generalization of the main result in [12] to the setting of UMD Banach function spaces X . For the proof we will follow the general scheme of [12] with some simplifications. As in case $X = L^q$, for any UMD Banach function space the notions of ℓ^2 -boundedness and \mathcal{R} -boundedness coincide, so the following theorem in particular implies the \mathcal{R} -boundedness of $\{I_k : k \in \mathcal{K}\}$.

Theorem 1.1. *Let X be a UMD Banach function space and $p \in (1, \infty)$. Let $T : \mathbb{R} \times \mathbb{R} \rightarrow \mathcal{L}(X)$ be such that the family of operators*

$$\{T(t,r) : t, r \in \mathbb{R}\}$$

is ℓ^s -bounded for all $s \in (1, \infty)$. Then $\{I_k : k \in \mathcal{K}\}$ is ℓ^s -bounded on $L^p(\mathbb{R}; X)$ for all $s \in (1, \infty)$.

We will prove Theorem 1.1 in a more general setting in Section 3. In particular we allow weights in time, which in applications for example allow rather rough initial values (see e.g. [23, 26, 31, 37]).

For certain UMD Banach function spaces the ℓ^s -boundedness assumption in Theorem 1.1 can be checked by weighted extrapolation techniques, see Corollary 3.5 and Remark 3.6.

Notation. For a measure space (S, μ) we denote the space of all measurable functions by $L^0(S)$. We denote the Lebesgue measure of a Borel set $E \in \mathcal{B}(\mathbb{R}^d)$ by $|E|$. For Banach spaces X and Y we denote the vector space of bounded linear operators from X to Y by $\mathcal{L}(X, Y)$ and we set $\mathcal{L}(X) := \mathcal{L}(X, X)$. For an operator family $\Gamma \subset \mathcal{L}(X, Y)$ we set $\Gamma^* := \{T^* : T \in \Gamma\}$. For $p \in [1, \infty]$ we let $p' \in [1, \infty]$ be such that $\frac{1}{p} + \frac{1}{p'} = 1$.

Throughout the paper we write $C_{a,b,\dots}$ and $\phi_{a,b,\dots}$ to denote a constant and a nondecreasing function on $[1, \infty)$ respectively, which only depend on the parameters a, b, \dots and the dimension d and which may change from line to line.

Acknowledgement. The author would like to thank Mark Veraar for carefully reading the draft version of this paper.

2. PRELIMINARIES

2.1. Banach function spaces. Let (S, μ) be a σ -finite measure space. An order ideal X of $L^0(S)$ equipped with a norm $\|\cdot\|_X$ is called a *Banach function space* if it has the following properties:

- (i) *Compatibility:* If $\xi, \eta \in L^0(S)$ with $|\xi| \leq |\eta|$, then $\|\xi\|_X \leq \|\eta\|_X$
- (ii) *Weak order unit:* There is a $\xi \in X$ with $\xi > 0$.
- (iii) *Fatou property:* If $0 \leq \xi_n \uparrow \xi$ for $(\xi_n)_{n=1}^\infty$ in X , $\xi \in L^0(S)$ and $\sup_{n \in \mathbb{N}} \|\xi_n\|_X < \infty$, then $\xi \in X$ and $\|\xi\|_X = \sup_{n \in \mathbb{N}} \|\xi_n\|_X$.

A Banach function space is called *order continuous* if for any sequence $0 \leq \xi_n \uparrow \xi \in X$ we have $\|\xi_n - \xi\|_X \rightarrow 0$. Every reflexive Banach function space is order continuous. Order continuity ensures that the dual of X is also a Banach function space. For a thorough introduction to Banach function spaces we refer to [28, section 1.b] or [3, Chapter 1].

A Banach function space X is said to be *p -convex* for $p \in [1, \infty]$ if

$$\left\| \left(\sum_{j=1}^n |\xi_j|^p \right)^{1/p} \right\|_X \leq \left(\sum_{j=1}^n \|\xi_j\|_X^p \right)^{1/p}$$

for all $\xi_1, \dots, \xi_n \in X$ with the sums replaced by suprema if $p = \infty$. The defining inequality for p -convexity often includes a constant, but X can always be renormed such that this constant equals 1. If a Banach function space is p -convex for some $p \in [1, \infty]$, then X is also q -convex for all $q \in [1, p]$.

For a p -convex Banach function space X we can define another Banach function space by

$$X^p := \{ |\xi|^p \operatorname{sgn} \xi : \xi \in X \} = \{ \xi \in L^0(S) : |\xi|^{1/p} \in X \}$$

equipped with the norm $\|\xi\|_{X^p} := \left\| |\xi|^{1/p} \right\|_X^p$. We refer the interested reader to [28, section 1.d] for an introduction to p -convexity.

2.2. ℓ^s -boundedness. Let X and Y be Banach function spaces and let $\Gamma \subseteq \mathcal{L}(X, Y)$ be a family of operators. We say that Γ is *ℓ^s -bounded* if for all finite sequences $(T_j)_{j=1}^n$ in Γ and $(x_j)_{j=1}^n$ in X we have

$$\left\| \left(\sum_{j=1}^n |T_j x_j|^s \right)^{1/s} \right\|_Y \leq C \left\| \left(\sum_{j=1}^n |x_j|^s \right)^{1/s} \right\|_X.$$

with the sums replaced by suprema if $s = \infty$. The least admissible constant C will be denoted by $[\Gamma]_{\ell^s}$.

Implicitly ℓ^s -boundedness is a classical tool in harmonic analysis for operators on L^p -spaces (see e.g. [16, Chapter V] and [17, 18]). For Banach function spaces the notion was introduced in [40] under the name \mathcal{R}_s -boundedness, underlining its connection to the more well-known notion of \mathcal{R} -boundedness. An extensive study of ℓ^s -boundedness can be found in [24] and for a comparison between ℓ^2 -boundedness and \mathcal{R} -boundedness we refer to [25].

Lemma 2.1. *Let X and Y be Banach function spaces and let $\Gamma \subseteq \mathcal{L}(X, Y)$.*

- (i) Let $1 \leq s_0 < s_1 \leq \infty$ and assume that X and Y are order continuous. If Γ is ℓ^{s_0} - and ℓ^{s_1} -bounded, then Γ is ℓ^s -bounded for all $s \in [s_0, s_1]$ with $[\Gamma]_{\ell^s} \leq \max\{[\Gamma]_{\ell^{s_0}}, [\Gamma]_{\ell^{s_1}}\}$
- (ii) Let $s \in [1, \infty]$ and assume that Γ is ℓ^s -bounded. Then the adjoint family Γ^* is $\ell^{s'}$ -bounded with $[\Gamma^*]_{\ell^{s'}} = [\Gamma]_{\ell^s}$

Proof. Lemma 2.1(i) follows from Calderón's theory of complex interpolation of vector-valued function spaces, see [6] or [24, Proposition 2.14]. Lemma 2.1(ii) is direct from the identification $X(\ell_n^s)^* = X^*(\ell_n^{s'})$, see [28, Section 1.d] or [24, Proposition 2.17] \square

The following characterization of ℓ^s -boundedness for $s \in [1, \infty)$ will be one of the key ingredients of our main result. This characterization relating ℓ^s -boundedness to a certain weighted boundedness comes from the work of Rubio de Francia [16, 38, 39].

Proposition 2.2. *Let $s \in [1, \infty)$ and let X and Y be s -convex order continuous Banach function spaces over (S_X, μ_X) and (S_Y, μ_Y) respectively. Let $\Gamma \subseteq \mathcal{L}(X)$ and take $C > 0$. Then the following are equivalent:*

- (i) Γ is ℓ^s -bounded with $[\Gamma]_{\ell^s} \leq C$.
- (ii) For all nonnegative $u \in (Y^s)^*$, there exists a nonnegative $v \in (X^s)^*$ with $\|v\|_{(Y^s)^*} \leq \|u\|_{(X^s)^*}$ and

$$\left(\int_{S_Y} |T(\xi)|^s u \, d\mu_Y \right)^{1/s} \leq C \left(\int_{S_X} |\xi|^s v \, d\mu_X \right)^{1/s}$$

for all $\xi \in X$ and $T \in \Gamma$.

Proof. The statement is a combination of [39, Lemma 1, p. 217] and [16, Theorem VI.5.3], which for $X = Y$ is proven [2, Lemma 3.4]. The statement for $X \neq Y$ can be extracted from the proof of [2, Lemma 3.4] and can in full detail be found in [29, Proposition 6.1.3] \square

2.3. Muckenhoupt weights. A locally integrable function $w : \mathbb{R}^d \rightarrow (0, \infty)$ is called a *weight*. For $p \in (1, \infty)$ and a weight w we let $L^p(w)$ be the space of all $f \in L^0(\mathbb{R}^d)$ such that

$$\|f\|_{L^p(w)} := \left(\int_{\mathbb{R}^d} |f|^p w \right)^{1/p} < \infty.$$

We will say that a weight w lies in the *Muckenhoupt class* A_p and write $w \in A_p$ if it satisfies

$$[w]_{A_p} := \sup_Q \frac{1}{|Q|} \int_Q w \cdot \left(\frac{1}{|Q|} \int_Q w^{1-p'} \right)^{p-1} < \infty,$$

where the supremum is taken over all cubes $Q \subseteq \mathbb{R}^d$ with sides parallel to the coordinate axes.

Lemma 2.3. *Let $p \in (1, \infty)$ and $w \in A_p$.*

- (i) $w \in A_q$ for all $q \in (p, \infty)$ with $[w]_{A_q} \leq [w]_{A_p}$.
- (ii) $w^{1-p'} \in A_{p'}$ with $[w]_{A_p}^{1/p} = [w^{1-p'}]_{A_{p'}}^{1/p'}$.
- (iii) $w \in A_{p-\varepsilon}$ for $\varepsilon = \frac{1}{\phi_p([w]_{A_p})}$ with $[w]_{A_{p-\varepsilon}} \leq \phi_p([w]_{A_p})$.

The first two properties of Lemma 2.3 follow directly from the definition. The third is for example proven in [18, Exercise 9.2.4]. For a more thorough introduction to Muckenhoupt weights we refer to [18, Chapter 9].

2.4. The UMD property. A Banach space X is said to have the UMD property if the martingale difference sequence of any finite martingale in $L^p(\Omega; X)$ is unconditional for some (equivalently all) $p \in (1, \infty)$. We will work with UMD Banach function spaces, of which standard examples include reflexive Lebesgue, Lorentz and Orlicz spaces. In this Festschrift it is shown that reflexive Musielak-Orlicz spaces, so in particular reflexive variable Lebesgue spaces, have the UMD property, see [27]. The UMD property implies reflexivity, so in particular L^1 and L^∞ do not have the UMD property. For a thorough introduction to the theory of UMD Banach spaces we refer to [5, 20].

For an order continuous Banach function space X over (S, μ) there is also a characterization of the UMD property in terms of the *lattice Hardy–Littlewood maximal operator*, which for simple functions $f: \mathbb{R}^d \rightarrow X$ is given by

$$\widetilde{M}f(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| \, dy, \quad x \in \mathbb{R}^d$$

where the supremum is taken pointwise in S and over all cubes $Q \subseteq \mathbb{R}^d$ with sides parallel to the coordinate axes (see [15] or [19, Lemma 5.1] for a detailed definition of \widetilde{M}). It is a deep result by Bourgain [4] and Rubio de Francia [39] that X has the UMD property if and only if \widetilde{M} is bounded on $L^p(\mathbb{R}^d; X)$ and $L^p(\mathbb{R}^d; X^*)$ for some (equivalently all) $p \in (1, \infty)$. For weighted L^p -spaces we have the following proposition, which was proven in [15]. The increasing dependence on $[w]_{A_p}$ is shown in [19, Corollary 5.3].

Proposition 2.4. *Let X be a UMD Banach function space, $p \in (1, \infty)$ and $w \in A_p$. Then for all $f \in L^p(w; X)$ we have*

$$\|\widetilde{M}f\|_{L^p(w; X)} \leq \phi_{X,p}([w]_{A_p}) \|f\|_{L^p(w; X)}.$$

The UMD property of a Banach function space X also implies that X^q has the UMD property for a $q > 1$, which is a deep result by Rubio de Francia [39, Theorem 4].

Proposition 2.5. *Let X be a UMD Banach function space. Then there is a $p > 1$ such that X is p -convex and X^q is a UMD Banach function space for all $q \in [1, p]$.*

3. INTEGRAL OPERATORS WITH AN OPERATOR-VALUED KERNEL

Before turning to our main result on the ℓ^s -boundedness of a family of integral operators on $L^p(w; X)$ with operator-valued kernels, we will first study the ℓ^s -boundedness of a family of convolution operators on $L^p(w; X)$ with scalar-valued kernels. For this define

$$\mathcal{K} := \{k \in L^1(\mathbb{R}^d) : |k| * |f| \leq Mf \text{ a.e. for all simple } f: \mathbb{R}^d \rightarrow \mathbb{C}\}.$$

As an example any radially decreasing $k \in L^1(\mathbb{R}^d)$ with $\|k\|_{L^1(\mathbb{R}^d)} \leq 1$ is an element of \mathcal{K} . For more examples see [17, Chapter 2] and [34, Proposition 4.6].

Let X be a Banach function space. For a kernel $k \in \mathcal{K}$ and a simple function $f: \mathbb{R}^d \rightarrow X$ we define

$$T_k f := k * f = \int_{\mathbb{R}^d} k(x-y)f(y) dy.$$

As

$$\|T_k f\|_X \leq |k| * \|f\|_X \leq M(\|f\|_X),$$

and since the Hardy-Littlewood maximal operator M is bounded on $L^p(w)$ for all $p \in (1, \infty)$ and $w \in A_p$, T_k extends to a bounded linear operator on $L^p(w; X)$ by density. This argument also shows that the family of convolution operators given by $\Gamma := \{T_k : k \in \mathcal{K}\}$ is uniformly bounded on $L^p(w; X)$.

If X is a UMD Banach function space we can say more. The following lemma was first developed by van Neerven, Veraar and Weis in [33, 34] in connection to stochastic maximal regularity. As in [33, 34], the endpoint case $s = 1$ will play a major role in the proof of our main theorem in the next section.

Proposition 3.1. *Let X be a UMD Banach function space, $s \in [1, \infty]$, $p \in (1, \infty)$ and $w \in A_p$. Then $\Gamma = \{T_k : k \in \mathcal{K}\}$ is ℓ^s -bounded on $L^p(w; X)$ with*

$$[\Gamma]_{\ell^s} \leq \phi_{X,p}([w]_{A_p}).$$

The proof is a weighted variant of [34, Theorem 4.7], which for the special case where X is an iterated Lebesgue space is presented in [12, Proposition 3.6]. For convenience of the reader we sketch the proof in the general case.

Proof. As X is reflexive and therefore order-continuous, \widetilde{M} is well-defined on $L^p(w; X)$ and we have $T_k f \leq \widetilde{M}f$ pointwise a.e. for all simple $f: \mathbb{R}^d \rightarrow X$.

If $s = \infty$ take simple functions $f_1, \dots, f_n \in L^p(w; X)$ and $k_1, \dots, k_n \in \mathcal{K}$. Using Proposition 2.4 we have

$$\begin{aligned} \left\| \sup_{1 \leq j \leq n} |T_{k_j} f_j| \right\|_{L^p(w; X)} &\leq \left\| \sup_{1 \leq j \leq n} \widetilde{M} f_j(x) \right\|_{L^p(w; X)} \\ &\leq \left\| \widetilde{M} \left(\sup_{1 \leq j \leq n} |f_j| \right)(x) \right\|_{L^p(w; X)} \\ &\leq \phi_{X,p}([w]_{A_p}) \left\| \sup_{1 \leq j \leq n} |f_j| \right\|_{L^p(w; X)}. \end{aligned}$$

The result now follows by the density of simple functions in $L^p(w; X)$.

If $s = 1$ we use duality. Note that since X is reflexive we have $L^p(w; X)^* = L^{p'}(w'; X^*)^*$ with $w' = w^{1-p'}$ under the duality pairing

$$(3.1) \quad \langle f, g \rangle_{L^p(w; X), L^{p'}(w'; X^*)} = \int_{\mathbb{R}^d} \langle f(x), g(x) \rangle_{X, X^*} dx$$

by Lemma 2.3(ii) and [20, Corollary 1.3.22]. One can routinely check that $T_k^* = T_{\tilde{k}}$ with $\tilde{k}(x) = k(-x)$ and that $k \in \mathcal{K}$ if and only if $\tilde{k} \in \mathcal{K}$. Since X^* is also a UMD Banach function space (see [20, Proposition 4.2.17]) we know from the case $s = \infty$ that the adjoint family Γ^* is ℓ^∞ -bounded on $L^{p'}(\mathbb{R}^d, w'; X^*)$, so the result follows by Lemma 2.1(ii). Finally if $s \in (1, \infty)$ the result follows by Lemma 2.1(i). \square

With these preparations done we can now introduce the family of integral operators with operator-valued kernel that we will consider. Let X and Y be a Banach function space and let \mathcal{T} be a family of operators $\mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathcal{L}(X, Y)$ such that $(x, y) \mapsto T(x, y)\xi$ is measurable for all $T \in \mathcal{T}$ and $\xi \in X$. The integral operators that we will consider are for simple $f : \mathbb{R}^d \rightarrow X$ given by

$$I_{k,T}f(x) = \int_{\mathbb{R}^d} k(x-y)T(x,y)f(y) dy$$

with $k \in \mathcal{K}$ and $T \in \mathcal{T}$. If $\|T(x,y)\|_{\mathcal{L}(X,Y)} \leq C$ for all $T \in \mathcal{T}$ and $x, y \in \mathbb{R}^d$, we have

$$\|I_{k,T}f\|_X \leq C |k| * \|f\|_X \leq C M(\|f\|_X).$$

So as before $I_{k,T}$ extends to a bounded linear operator from $L^p(w; X)$ to $L^p(w; Y)$ for all $p \in (1, \infty)$ and $w \in A_p$, and

$$\mathcal{I}_{\mathcal{T}} := \{I_{k,T} : k \in \mathcal{K}, T \in \mathcal{T}\}$$

is uniformly bounded. For the details see [12, Lemma 3.9].

If X and Y are Hilbert spaces, this implies that $\mathcal{I}_{\mathcal{T}}$ is also ℓ^2 -bounded from $L^2(\mathbb{R}^d; X)$ to $L^2(\mathbb{R}^d; Y)$, as these notions coincide on Hilbert spaces. However if X and Y are not Hilbert spaces, but a UMD Banach function space or if we move to weighted L^p -spaces, the ℓ^2 -boundedness of $\mathcal{I}_{\mathcal{T}}$ is a lot more delicate.

Our main theorem is a quantitative and more general version of Theorem 1.1 in the introduction:

Theorem 3.2. *Let X and Y be a UMD Banach function spaces and let $p, s \in (1, \infty)$. Let \mathcal{T} be a family of operators $\mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathcal{L}(X, Y)$ such that*

- (i) $(x, y) \mapsto T(x, y)\xi$ is measurable for all $T \in \mathcal{T}$ and $\xi \in X$.
- (ii) The family of operators $\tilde{\mathcal{T}} := \{T(x, y) : T \in \mathcal{T}, x, y \in \mathbb{R}^d\}$ is ℓ^σ -bounded for all $\sigma \in (1, \infty)$.

Then $\mathcal{I}_{\mathcal{T}}$ is ℓ^s -bounded from $L^p(w; X)$ to $L^p(w; Y)$ for all $w \in A_p$ with

$$\begin{aligned} [\mathcal{I}_{\mathcal{T}}]_{\ell^s} &\leq \phi_{X,Y,p}([w]_{A_p}) \max\{[\tilde{\mathcal{T}}]_{\ell^\sigma}, [\tilde{\mathcal{T}}]_{\ell^{\sigma'}}\}, & \sigma &= 1 + \frac{1}{\phi_{p,s}[w]_{A_p}} \\ &\leq \phi_{X,Y,\mathcal{T},p,s}([w]_{A_p}). \end{aligned}$$

We will first prove a result assuming the ℓ^s -boundedness of $\tilde{\mathcal{T}}$ for a fixed $s \in [1, \infty)$.

Proposition 3.3. *Fix $1 \leq s \leq r < p < \infty$ and let X and Y be s -convex Banach function spaces such that X^s has the UMD property. Let \mathcal{T} be a family of operators $\mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathcal{L}(X, Y)$ such that*

- (i) $(x, y) \mapsto T(x, y)\xi$ is measurable for all $T \in \mathcal{T}$ and $\xi \in X$.
- (ii) The family of operators $\tilde{\mathcal{T}} := \{T(x, y) : T \in \mathcal{T}, x, y \in \mathbb{R}^d\}$ is ℓ^s -bounded.

Then $\mathcal{I}_{\mathcal{T}}$ is ℓ^s -bounded from $L^p(w; X)$ to $L^p(w; Y)$ for all $w \in A_{p/s}$ with

$$[\mathcal{I}_{\mathcal{T}}]_{\ell^s} \leq \phi_{X,p,r}([w]_{A_{p/s}}) [\tilde{\mathcal{T}}]_{\ell^s}.$$

Proof. Let (S_X, μ_X) and (S_Y, μ_Y) be the measure spaces associated to X and Y respectively. For $j = 1, \dots, n$ take $I_j \in \mathcal{I}_{\mathcal{T}}$ and let $k_j \in \mathcal{K}$ and $T_j \in \mathcal{T}$ be such that $I_j = I_{k_j, T_j}$. Fix simple functions $f_1, \dots, f_n \in L^p(w; X)$ and note that

$$(3.2) \quad \left\| \left(\sum_{j=1}^n |I_j f_j|^s \right)^{1/s} \right\|_{L^p(w; Y)} = \left\| \sum_{j=1}^n |I_j f_j|^s \right\|_{L^{p/s}(w; Y^s)}^{1/s}.$$

Fix $x \in \mathbb{R}^d$, then by Hahn-Banach we can find a nonnegative $u_x \in (Y^s)^*$ with $\|u_x\|_{(X^s)^*} = 1$ such that

$$(3.3) \quad \left\| \sum_{j=1}^n |I_j f_j(x)|^s \right\|_{Y^s} = \sum_{j=1}^n \int_{S_Y} |I_j f_j(x)|^s u_x \, d\mu_Y.$$

With Proposition 2.2 we can then find a nonnegative $v_x \in (X^s)^*$ with $\|v_x\|_{(X^s)^*} \leq 1$ such that

$$(3.4) \quad \int_{S_Y} |T_j(x, y) \xi|^s v_x \, d\mu_Y \leq [\tilde{\mathcal{T}}]_{\ell^s} \int_{S_X} |\xi|^s v_x \, d\mu_X$$

for $j = 1, \dots, n$, $y \in \mathbb{R}^d$ and $\xi \in X$. Since $\|k_j\|_{L^1(\mathbb{R}^d)} \leq 1$ by [34, Lemma 4.3], Holder's inequality yields

$$(3.5) \quad |I_j f_j(x)|^s \leq \int_{\mathbb{R}^d} |k_j(x-y)| |T_j(x, y) f_j(y)|^s \, dy.$$

Applying (3.5) and (3.4) successively we get

$$\begin{aligned} \sum_{j=1}^n \int_{S_Y} |I_j f_j(x)|^s u_x \, d\mu_Y &\leq \sum_{j=1}^n \int_{S_Y} \int_{\mathbb{R}^d} |k_j(x-y)| |T_j(x, y) f_j(y)|^s \, dy u_x \, d\mu_Y \\ &= \sum_{j=1}^n \int_{\mathbb{R}^d} |k_j(x-y)| \int_{S_Y} |T_j(x, y) f_j(y)|^s u_x \, d\mu_Y \, dy \\ &\leq [\tilde{\mathcal{T}}]_{\ell^s} \sum_{j=1}^n \int_{S_X} \int_{\mathbb{R}^d} |k_j(x-y)| |f_j(y)|^s \, dy v_x \, d\mu_X \\ &\leq [\tilde{\mathcal{T}}]_{\ell^s} \left\| \sum_{j=1}^n (|k_j| * |f_j|^s)(x) \right\|_{X^s}, \end{aligned}$$

using duality and $\|v_x\|_{(X^s)^*} \leq 1$ in the last step. We can now use the ℓ^1 -boundedness result of Proposition 3.1, since $(X^s)^*$ has the UMD property by [21, Proposition 4.2.17]. Combined with (3.2) and (3.3) we obtain

$$\begin{aligned} \left\| \left(\sum_{j=1}^n |I_j f_j|^s \right)^{1/s} \right\|_{L^p(w; Y)} &\leq [\tilde{\mathcal{T}}]_{\ell^s} \left\| \sum_{j=1}^n |k_j| * |f_j|^s \right\|_{L^{p/s}(w; X^s)}^{1/s} \\ &\leq \phi_{X, p/s}([w]_{A_{p/s}}) [\tilde{\mathcal{T}}]_{\ell^s} \left\| \sum_{j=1}^n |f_j|^s \right\|_{L^{p/s}(w; X^s)}^{1/s} \\ &\leq \phi_{X, p, r}([w]_{A_{p/s}}) [\tilde{\mathcal{T}}]_{\ell^s} \left\| \left(\sum_{j=1}^n |f_j|^s \right)^{1/s} \right\|_{L^p(w; X)}, \end{aligned}$$

where we can pick the increasing function ϕ in the last step independent of s , since the increasing function in Proposition 3.1 depends continuously on p . This can for example be seen by writing out the exact dependence on p in Theorem 2.4 using [19, Theorem 1.3] and [32, Theorem 3.1]. \square

Using this preparatory proposition, we will now prove Theorem 3.2.

Proof of Theorem 3.2. Let $w \in A_p$. We shall prove the theorem in three steps.

Step 1. First we shall prove the theorem very small $s > 1$. By Proposition 2.5 we know that there exists a $\sigma_{X,Y} \in (1,p)$ such that X and Y are s -convex and X^s has the UMD property for all $s \in [1, \sigma_X]$. By Lemma 2.3(iii) we can then find a $\sigma_{p,w} \in (1, \sigma_{X,Y}]$ such that for all $s \in [1, \sigma_{p,w}]$

$$[w]_{A_{p/s}} \leq [w]_{A_{p/\sigma_{p,w}}} \leq \phi_p([w]_{A_p})$$

Let $\sigma_1 = \min\{\sigma_{X,Y}, \sigma_{p,w}\}$, then by Proposition 3.3 we know that $\mathcal{I}_{\mathcal{T}}$ is ℓ^s -bounded from $L^p(w; X)$ to $L^p(w; Y)$ for $s \in (1, \sigma_1]$ with

$$(3.6) \quad [\mathcal{I}_{\mathcal{T}}]_{\ell^s} \leq \phi_{X,p,\sigma_{X,Y}}([w]_{A_{p/s}}) [\tilde{\mathcal{T}}]_{\ell^s} \leq \phi_{X,Y,p}([w]_{A_p}) [\tilde{\mathcal{T}}]_{\ell^s}.$$

Step 2. Now we use a duality argument to prove the theorem for large $s < \infty$. As noted in the proof of Proposition 3.1, we have $L^p(w; X)^* = L^{p'}(w'; X^*)$ with $w' = w^{1-p'}$ under the duality pairing as in (3.1) and similarly for Y . Furthermore X^* and Y^* have the UMD property.

It is routine to check that under this duality $I_{k,T}^* = I_{\tilde{k},\tilde{T}}$ with $\tilde{k}(x) = k(-x)$ and $\tilde{T}(x,y) = T^*(y,x)$ for any $I_{k,T} \in \mathcal{I}_{\mathcal{T}}$. Trivially $\tilde{k} \in \mathcal{K}$ if and only if $k \in \mathcal{K}$ and by Proposition 3.1(ii) the adjoint family $\tilde{\mathcal{T}}^*$ is $\ell^{\sigma'}$ -bounded with

$$[\tilde{\mathcal{T}}^*]_{\ell^{\sigma'}} = [\tilde{\mathcal{T}}]_{\ell^{\sigma}}$$

for all $\sigma \in (1, \infty)$. Therefore, it follows from step 1 that there is a $\sigma_2 > 1$ such that $\mathcal{I}_{\mathcal{T}}^*$ is ℓ^s -bounded from $L^{p'}(w'; Y^*)$ to $L^{p'}(w'; X^*)$ for all $s \in (1, \sigma_2]$. Using Proposition 3.1(ii) again, we deduce that $\mathcal{I}_{\mathcal{T}}$ is ℓ^s -bounded from $L^p(w; X)$ to $L^p(w; Y)$ for all $s \in [\sigma_2', \infty)$ with

$$(3.7) \quad [\mathcal{I}_{\mathcal{T}}]_{\ell^s} = [\mathcal{I}_{\mathcal{T}}^*]_{\ell^{s'}} \leq \phi_{X,Y,p}([w]_{A_p}) [\tilde{\mathcal{T}}]_{\ell^s}.$$

Step 3. We can finish the prove by an interpolation argument for $s \in (\sigma_1, \sigma_2')$. By Proposition 2.2(i) we get for $s \in (\sigma_1, \sigma_2')$ that $\mathcal{I}_{\mathcal{T}}$ is ℓ^s -bounded from $L^p(w; X)$ to $L^p(w; Y)$ with

$$(3.8) \quad [\mathcal{I}_{\mathcal{T}}]_{\ell^s} \leq \phi_{X,Y,p}([w]_{A_p}) \max\left\{[\tilde{\mathcal{T}}]_{\ell^{\sigma_1}}, [\tilde{\mathcal{T}}]_{\ell^{\sigma_2'}}\right\}.$$

Now note that by Lemma 2.3 there is a $\sigma \in (1, \infty)$ such that $\sigma < \sigma_1, \sigma_2$ and $\sigma < s < \sigma'$ and

$$\sigma = 1 + \frac{1}{\phi_{p,s}([w]_{A_p})}.$$

Thus combining (3.6), (3.7) and (3.8) we obtain

$$[\mathcal{I}_{\mathcal{T}}]_{\ell^s} \leq \phi_{X,Y,p}([w]_{A_p}) \max\left\{[\tilde{\mathcal{T}}]_{\ell^{\sigma}}, [\tilde{\mathcal{T}}]_{\ell^{\sigma'}}\right\} \leq \phi_{X,Y,T,p,s}([w]_{A_p}),$$

using the fact that $t \mapsto \max\left\{[\tilde{\mathcal{T}}]_{\ell^t}, [\tilde{\mathcal{T}}]_{\ell^{t'}}\right\}$ is increasing for $t \rightarrow 1$ by Proposition 2.2(i). This proves the theorem. \square

Remark 3.4.

- From Theorem 3.2 one can also conclude that $\mathcal{I}_{\mathcal{T}}$ is \mathcal{R} -bounded, since \mathcal{R} - and ℓ^2 -boundedness coincide if X and Y have the UMD property, see e.g. [21, Theorem 8.1.3].
- The UMD assumptions in Theorem 3.2 are necessary. Indeed already if $X = Y$, $w = 1$ and if $\tilde{\mathcal{T}}$ only contains the identity operator, it is shown in [22] that the ℓ^2 -boundedness of $\mathcal{I}_{\mathcal{T}}$ implies the UMD property of X .
- The main result of [12] is Theorem 3.2 for the special case $X = Y = L^q(S)$. In applications to systems of PDEs one needs Theorem 3.2 on $L^q(S; \mathbb{C}^n)$ with $s = 2$, see e.g. [13]. This could be deduced from the proof of [12, Theorem 3.10], by replacing absolute values by norms in \mathbb{C}^n . In our more general statement the case $L^q(S; \mathbb{C}^n)$ is included, since $L^q(S; \mathbb{C}^n)$ is a UMD Banach function space over $S \times \{1, \dots, n\}$

If $X = Y$ is a rearrangement invariant Banach function space on \mathbb{R}^e , we can check the ℓ^σ -boundedness of $\tilde{\mathcal{T}}$ for all $\sigma \in (1, \infty)$ by weighted extrapolation. Examples of such Banach function spaces are Lebesgue, Lorentz and Orlicz spaces. See [28, Section 2.a] for an introduction to rearrangement invariant Banach function spaces.

Corollary 3.5. *Let X be a rearrangement invariant UMD Banach function space on \mathbb{R}^e and let $p, s \in (1, \infty)$. Let \mathcal{T} be a family of operators $\mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathcal{L}(X)$ such that*

- (i) $(x, y) \mapsto T(x, y)\xi$ is measurable for all $T \in \mathcal{T}$ and $\xi \in X$.
- (ii) For some $q \in (1, \infty)$ and all $v \in A_q$ we have

$$\sup_{T \in \mathcal{T}, x, y \in \mathbb{R}^d} \|T(x, y)\|_{\mathcal{L}(L^q(v))} \leq \phi_{\mathcal{T}, q}([v]_{A_q})$$

Then $\mathcal{I}_{\mathcal{T}}$ is ℓ^s -bounded on $L^p(w; X)$ for all $w \in A_p$ with

$$[\mathcal{I}_{\mathcal{T}}]_{\ell^s} \leq \phi_{X, Y, \mathcal{T}, p, q, s}([w]_{A_p}).$$

Note that in Corollary 3.5 we need that $T(x, y)$ is well-defined on $L^q(v)$ for all $T \in \mathcal{T}$ and $x, y \in \mathbb{R}^d$. This is indeed the case, since $X \cap L^q(v)$ is dense in $L^q(v)$.

Proof. Let Y be the linear span of

$$\{\mathbf{1}_K \xi : K \subseteq \mathbb{R}^e \text{ compact}, \xi \in X \cap L^\infty(\mathbb{R}^e)\}.$$

Then $Y \subseteq L^q(v)$ for all $v \in A_p$ and Y is dense in X by order continuity. Define

$$\mathcal{F} := \{(|T(x, y)\xi|, |\xi|) : T \in \mathcal{T}, x, y \in \mathbb{R}^d, \xi \in Y\}.$$

Note that X has upper Boyd index $q_X < \infty$ by the UMD property (see [21, Proposition 7.4.12] and [28, Section 2.a]). So we can use the extrapolation result for Banach function spaces in [11, Theorem 2.1] to conclude that for $\sigma \in (1, \infty)$

$$\left\| \left(\sum_{j=1}^n |T_j(x_j, y_j)\xi_j|^\sigma \right)^{1/\sigma} \right\|_X \leq C_{\mathcal{T}, q} \left\| \left(\sum_{j=1}^n |\xi_j|^\sigma \right)^{1/\sigma} \right\|_X$$

for any $T_j \in \mathcal{T}$, $x_j, y_j \in \mathbb{R}^d$ and $\xi_j \in Y$ for $j = 1, \dots, n$. By the density this extends to $\xi_j \in X$, so

$$\{T(x, y) : x, y \in \mathbb{R}^d, T \in \mathcal{T}\}$$

is ℓ^σ -bounded for all $\sigma \in (1, \infty)$. Therefore the corollary follows from Theorem 3.2. \square

Remark 3.6.

- A sufficient condition for the weighted boundedness assumption in Corollary 3.5 is that $T(x, y)\xi \leq CM\xi$ for all $T \in \mathcal{T}$, $x, y \in \mathbb{R}^d$ and $\xi \in L^q(\mathbb{R}^e)$, which follows directly from [18, Theorem 9.1.9].
- Corollary 3.5 holds more generally for UMD Banach function spaces X such that the Hardy-Littlewood maximal operator is bounded on both X and X^* (see [10, Theorem 4.6]). For example the variable Lebesgue spaces $L^{p(\cdot)}$ satisfy this assumption if $p_+, p_- \in (1, \infty)$ and $p(\cdot)$ satisfies a certain continuity condition, see [9, 35].
- The conclusion of Corollary 3.5 also holds for $X(v)$ for all $v \in A_{p_X}$ where p_X is the lower Boyd index of X and $X(v)$ is a weighted version of X , see [11, Theorem 2.1].

REFERENCES

- [1] H. Amann. Maximal regularity and quasilinear parabolic boundary value problems. In *Recent advances in elliptic and parabolic problems*, pages 1–17. World Sci. Publ., Hackensack, NJ, 2005.
- [2] A. Amenta, E. Lorist, and M. C. Veraar. Rescaled extrapolation for vector-valued functions. Accepted for publication in *Publicacions Matemàtiques*, 2017.
- [3] C. Bennett and R. Sharpley. *Interpolation of operators*, volume 129 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1988.
- [4] J. Bourgain. Extension of a result of Benedek, Calderón and Panzone. *Ark. Mat.*, 22(1):91–95, 1984.
- [5] D. L. Burkholder. Martingales and singular integrals in Banach spaces. In *Handbook of the geometry of Banach spaces, Vol. I*, pages 233–269. North-Holland, Amsterdam, 2001.
- [6] A. P. Calderón. Intermediate spaces and interpolation, the complex method. *Studia Math.*, 24:113–190, 1964.
- [7] P. Clément, B. de Pagter, F. A. Sukochev, and H. Witvliet. Schauder decompositions and multiplier theorems. *Studia Math.*, 138(2):135–163, 2000.
- [8] P. Clément and J. Prüss. An operator-valued transference principle and maximal regularity on vector-valued L_p -spaces. In *Evolution equations and their applications in physical and life sciences (Bad Herrenalb, 1998)*, volume 215 of *Lecture Notes in Pure and Appl. Math.*, pages 67–87. Dekker, New York, 2001.
- [9] D. Cruz-Uribe, A. Fiorenza, and C. J. Neugebauer. The maximal function on variable L^p spaces. *Ann. Acad. Sci. Fenn. Math.*, 28(1):223–238, 2003.
- [10] D. V. Cruz-Uribe, J. M. Martell, and C. Pérez. *Weights, extrapolation and the theory of Rubio de Francia*, volume 215 of *Operator Theory: Advances and Applications*. Birkhäuser/Springer Basel AG, Basel, 2011.
- [11] G. P. Curbera, J. García-Cuerva, J. M. Martell, and C. Pérez. Extrapolation with weights, rearrangement-invariant function spaces, modular inequalities and applications to singular integrals. *Adv. Math.*, 203(1):256–318, 2006.
- [12] C. Gallarati, E. Lorist, and M. C. Veraar. On the ℓ^s -boundedness of a family of integral operators. *Rev. Mat. Iberoam.*, 32(4):1277–1294, 2016.
- [13] C. Gallarati and M. C. Veraar. Evolution families and maximal regularity for systems of parabolic equations. *Adv. Differential Equations*, 22(3-4):169–190, 2017.

- [14] C. Gallarati and M. C. Veraar. Maximal regularity for non-autonomous equations with measurable dependence on time. *Potential Anal.*, 46(3):527–567, 2017.
- [15] J. García-Cuerva, R. Macías, and J. L. Torrea. The Hardy-Littlewood property of Banach lattices. *Israel J. Math.*, 83(1-2):177–201, 1993.
- [16] J. García-Cuerva and J. L. Rubio de Francia. *Weighted norm inequalities and related topics*, volume 116 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1985. Notas de Matemática, 104.
- [17] L. Grafakos. *Classical Fourier analysis*, volume 249 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2008.
- [18] L. Grafakos. *Modern Fourier analysis*, volume 250 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2009.
- [19] T. S. Hänninen and E. Lorist. Sparse domination for the lattice Hardy–Littlewood maximal operator. *Proc. Amer. Math. Soc.*, 146(13), 2018.
- [20] T. P. Hytönen, J. M. A. M. van Neerven, M. C. Veraar, and L. Weis. *Analysis in Banach Spaces. Volume I: Martingales and Littlewood-Paley Theory*, volume 63 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer, 2016.
- [21] T. P. Hytönen, J. M. A. M. van Neerven, M. C. Veraar, and L. Weis. *Analysis in Banach spaces. Volume II: Probabilistic methods and operator theory*, volume 67 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer, 2017.
- [22] N. J. Kalton, E. Lorist, and L. Weis. Euclidean structures. In preparation.
- [23] M. Köhne, J. Prüss, and M. Wilke. On quasilinear parabolic evolution equations in weighted L_p -spaces. *J. Evol. Equ.*, 10(2):443–463, 2010.
- [24] P. Kunstmann and A. Ullmann. \mathcal{R}_s -sectorial operators and generalized Triebel-Lizorkin spaces. *J. Fourier Anal. Appl.*, 20(1):135–185, 2014.
- [25] S. Kwapien, M. C. Veraar, and L. Weis. R -boundedness versus γ -boundedness. *Ark. Mat.*, 54(1):125–145, 2016.
- [26] N. Lindemulder. Maximal regularity with weights for parabolic problems with inhomogeneous boundary data. arXiv:1702.02803, 2017.
- [27] N. Lindemulder, M. C. Veraar, and I. S. Yaruslavitsev. The UMD property for Musielak–Orlicz spaces. In *Positivity and noncommutative analysis – Festschrift in honour of Ben de Pagter on the occasion of his 65th birthday*, Trends in Mathematics. Birkhäuser Verlag, 2019.
- [28] J. Lindenstrauss and L. Tzafriri. *Classical Banach spaces. II*, volume 97 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer-Verlag, Berlin-New York, 1979.
- [29] E. Lorist. Maximal functions, factorization, and the \mathcal{R} -boundedness of integral operators. Master’s thesis, Delft University of Technology, Delft, the Netherlands, 2016.
- [30] A. Lunardi. *Analytic semigroups and optimal regularity in parabolic problems*, volume 16 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Verlag, Basel, 1995.
- [31] M. Meyries and R. Schnaubelt. Maximal regularity with temporal weights for parabolic problems with inhomogeneous boundary conditions. *Math. Nachr.*, 285(8-9):1032–1051, 2012.
- [32] K. Moen. Sharp weighted bounds without testing or extrapolation. *Arch. Math. (Basel)*, 99(5):457–466, 2012.
- [33] J. M. A. M. van Neerven, M. C. Veraar, and L. Weis. Stochastic maximal L^p -regularity. *Ann. Probab.*, 40(2):788–812, 2012.
- [34] J. M. A. M. van Neerven, M. C. Veraar, and L. Weis. On the R -boundedness of stochastic convolution operators. *Positivity*, 19(2):355–384, 2015.
- [35] A. Nekvinda. Hardy-Littlewood maximal operator on $L^{p(x)}(\mathbb{R})$. *Math. Inequal. Appl.*, 7(2):255–265, 2004.
- [36] J. Prüss. Maximal regularity for evolution equations in L_p -spaces. *Conf. Semin. Mat. Univ. Bari*, (285):1–39 (2003), 2002.
- [37] J. Prüss and G. Simonett. Maximal regularity for evolution equations in weighted L_p -spaces. *Arch. Math. (Basel)*, 82(5):415–431, 2004.
- [38] J. L. Rubio de Francia. Factorization theory and A_p weights. *Amer. J. Math.*, 106(3):533–547, 1984.

- [39] J. L. Rubio de Francia. Martingale and integral transforms of Banach space valued functions. In *Probability and Banach spaces (Zaragoza, 1985)*, volume 1221 of *Lecture Notes in Math.*, pages 195–222. Springer, Berlin, 1986.
- [40] L. Weis. A new approach to maximal L_p -regularity. In *Evolution equations and their applications in physical and life sciences (Bad Herrenalb, 1998)*, volume 215 of *Lecture Notes in Pure and Appl. Math.*, pages 195–214. Dekker, New York, 2001.
- [41] L. Weis. Operator-valued Fourier multiplier theorems and maximal L_p -regularity. *Math. Ann.*, 319(4):735–758, 2001.

DELFT INSTITUTE OF APPLIED MATHEMATICS, DELFT UNIVERSITY OF TECHNOLOGY,
P.O. BOX 5031, 2600 GA DELFT, THE NETHERLANDS
E-mail address: e.lorist@tudelft.nl