The burning number conjecture

on cat-constructs and trees with a single degree-2 vertex

by

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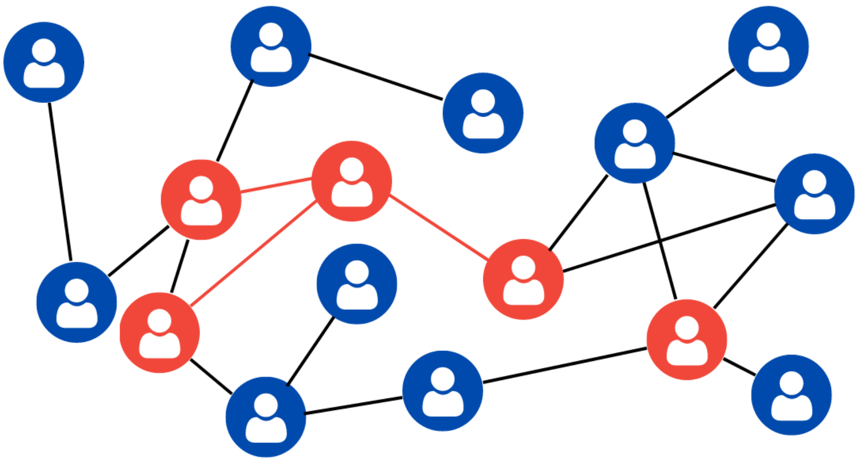

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Summary

Where people gather, they communicate. In the last decade the graph burning model was developed to model the spread of information between people. Graph burning is a process that is done in rounds with the aim of spreading information to every connected person in a network. Every round one new source of information may be appointed and information spreads from people who have received it, to all of their connections, just like fire spreads. The burning number of a graph, denoted by $b(G)$, is the parameter that quantifies the speed of this spread of information.

It has been conjectured that the burning number for a connected graph on n vertices is at most $\lceil\sqrt{n}\rceil$. This has been proven for many types of graphs, but not all. Moreover, not much is known about the relation between the number of sources that are used to burn a graph and the burning number. In this light, we prove the burning number conjecture for cat-constructs. Cat-constructs are trees obtained from a path graph P_n by adding at most two vertices to subtrees of P_n. These subtrees are induced by closed balls which have the vertices of a burning sequence as their centers and a chosen radius. We show the burning sequence of a cat-construct may contain one fewer source than its burning number if the number of vertices for the cat-construct is more than the first square bigger than n. With this result we show that adding a vertex as a leaf to these cat-constructs and appointing it as a source results in the proof of the burning number conjecture for certain 3-caterpillars.

Finally, we prove the burning number conjecture for trees with a single degree-2 vertex. For this proof we use a method similar to the method developed by Murakami (2023) for proving the burning number conjecture for homeomorphically irreducible trees.

Contents

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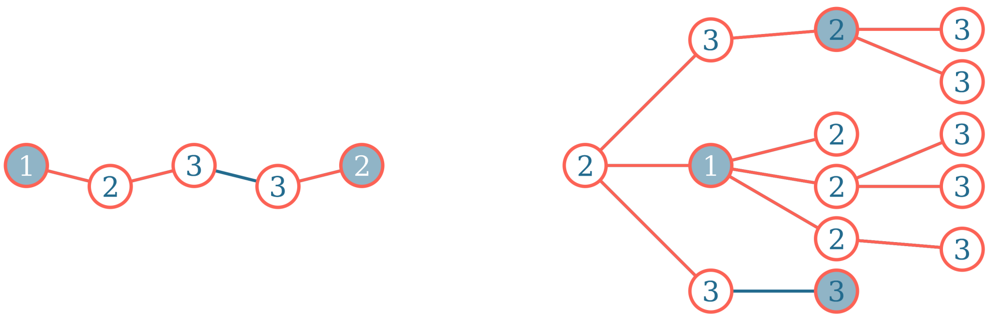
Introduction

Graph burning models the spread of information between people. It is done in rounds. Initially, all vertices are unburned. Every round one vertex may be chosen as a source. This source is burned in that round. Every round the fire spreads from burned vertices to all neighbour vertices, making these vertices burned as well. A vertex cannot be unburned. This process continues until all vertices are burned. The minimal number of rounds it takes for all vertices of a graph to be burned is called the burning number, denoted by $b(G)$.



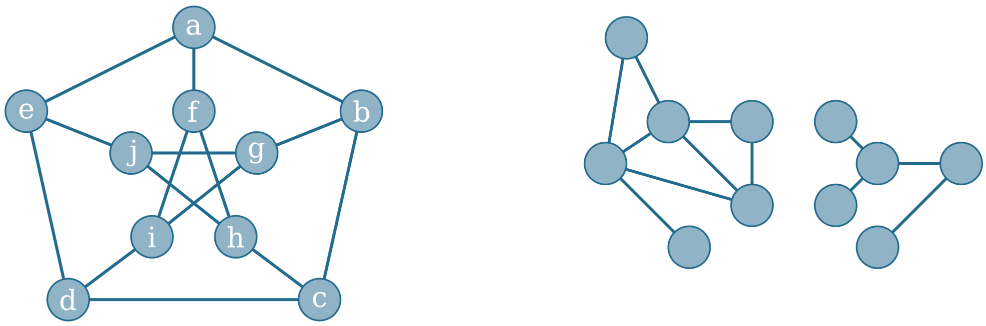
(a) A path graph that is burned in 3 rounds with 2 sources in grey. The numbers denote the round in which the vertex is burned.

(b) A tree that is burned in 3 rounds with 3 sources. The numbers denote the round in which the vertex is burned.

Figure 1.1: Examples of burned graphs.

1.1. Basic concepts in graph theory

We first introduce some basic concepts in graph theory important to graph burning. Graph burning is currently being researched on simple graphs. Janssen (2020) proved that graph burning can be naturally generalized to directed graphs. Therefore, the focus is currently on simple graphs. A **simple graph** G is a pair of sets (V, E) where V is the set of vertices and E the set of edges such that E is a subset of all two-element subsets of V. We denote the number of vertices in G by $|G|$, the set of vertices of G by $V(G)$ and the set of edges of G by $E(G)$. Furthermore, graphs are denoted by capital letters, vertices are denoted by lowercase letters and edges are denoted by the two letters of the vertices that are the endpoints of the edge. For example, in a graph G, for two vertices $u \in V$ and $v \in V$, uv is an edge in G if $uv \in E(G)$. Vertices in a graph may be labeled if we want to distinguish between the vertices. For example, if a graph depicts the connection between people we can label the vertices with the names of the people that they represent. If it does not matter what label the vertex holds then the vertices will not have a label. A **subgraph** of G is a graph $G' = (V', E')$ such that V' is a subset of V and E' is a subset of E such that it contains only edges of E that have both endpoints in V'. Such a subgraph is **induced** by V' if it contains exactly the edges between the vertices in V' that were originally also in E. Two vertices $u, v \in V$ are called **neighbours** if $uv \in E$. Every vertex $v \in V$ has a **degree**, denoted by $\deg(v)$ which equals the number of vertices that are adjacent to v. Between any two vertices u, v in a graph there exists a **path** if there exists a sequence of unique vertices $u, v_1, ..., v_k, v$ such that $uv_1 \in E$, $v_k v \in E$ and for every $i \in \{1, ..., k-1\}$, $v_i v_{i+1} \in E$. A path's **length** is equal to the number of edges in the path. When a path exists between every pair of vertices in a graph, then a graph is **connected**. For this thesis, graph burning is considered only on simple connected graphs. An example of a connected and a disconnected graph is shown in Figure 1.2.



(a) A connected graph.

(b) A disconnected graph.

Figure 1.2: Examples of a connected and a disconnected graph.

An important type of connected graph that will be the focal point of this thesis is a tree. A **tree** is a connected graph without **cycles**. A cycle is defined as a path of which the start and endpoint are the same. Every tree has at least one **leaf**, a vertex v with $\deg(v) = 1$ or $\deg(v) = 0$ if the tree consists of a single vertex. All vertices in trees that have degree more than 1 are **internal vertices**. Finally, denote the **distance between two vertices** u, v in a graph G by $d(u, v)$, which is the length of a shortest path between these two vertices.

More background information on graph theory may be found in Diestel (2017) and for more advanced information on graph theory, Yadav (2023) may be consulted.

1.2. Graph burning

We will now go into more detail on what graph burning is formally. Graph burning is a process on an arbitrary simple connected graph G. In G a sequence of vertices $S = (x_1, x_2, ..., x_k)$ is chosen to be burned as **sources** in the order of the sequence. In the round after a source is burned all neighbours of the source and any other burned vertex also get burned. A vertex cannot be unburned. So in round 1, x_1 is burned as a source. In round 2, all adjacent vertices of x_1 are burned and x_2 is burned as a source. In round i, all adjacent vertices of burned vertices are burned and vertex x_i is burned as a source. This process stops after all vertices are burned. If at the end of round k all vertices of G are burned, then S is called a **burning sequence**. The **length of a burning sequence** equals the number sources in the burning sequence. The **burning number of** G, $b(G)$, equals the length of a burning sequence of shortest length in G. Multiple burning sequences of equal length may exist for a graph. Therefore, a burning sequence does not necessarily have a unique length.

1.3. The burning number conjecture

Graph burning is a concept in graph theory that was developed in the last decade. Bonato et al. (2016) showed that a **path graph** P_n on n vertices has burning number exactly $\lceil\sqrt{n}\rceil$. A path graph P_n is a graph that consists of n vertices such that all vertices are part of one unique path and appear in the path sequence only once. An example of a path graph on 5 vertices in an unburned and burned state is shown in Figure 1.3.

(a) A path graph P_5 on 5 vertices.

(b) A path graph P_5 on 5 vertices that is fully burned. The grey vertex is the source in the burning sequence and the numbers indicate in which round the vertex is burned.

Figure 1.3: An example of a path graph P_5 on 5 vertices in an unburned and burned state.

Following this proof Bonato et al. (2016) conjectured that the burning number of all graphs must be smaller equal to $\lceil\sqrt{n}\rceil$.

Conjecture 1.1 (Bonato et al. 2016)**.** *Let G be a connected graph on n vertices. Then $b(G) \leq \lceil\sqrt{n}\rceil$.*

Bonato et al. (2016) also proved that if the burning number conjecture is true for trees, it is true for all graphs.

Corollary 1.1 (Bonato et al. 2016)**.** *For a graph G we have that*

$$b(G) = \min\{b(T) : T \text{ is a spanning tree of } G\}.$$

A **spanning tree** T of a graph G is a tree that consists of all vertices of G and the edges of T are a subset of the edges of G. Examples of a spanning tree in a graph are shown in Figure 1.4.

(a) A spanning tree with burning number equal to 3.

(b) A spanning tree with burning number equal to 2.

Figure 1.4: An example of a spanning tree of a graph depicted by red edges and connected vertices. The blue edges are part of the original graph, but not of the spanning tree.

1.4. Contributions

In this thesis, we prove the burning number conjecture holds for cat-constructs in Chapter 3. Cat-constructs are trees constructed by adding at most two vertices to subtrees of a path graph P_n. These subtrees are induced by closed balls with the vertices of a burning sequence as their centers and a chosen radius. The definition for cat-constructs is found in Chapter 3. We further prove cat-constructs may be burned by a burning sequence, with the number of sources equal to $\lceil\sqrt{n}\rceil$, which is the burning number for P_n. Finally, we will show that the burning number conjecture holds for certain 3-caterpillars that can be constructed from cat-constructs by adding a vertex as a leaf.

In Chapter 4, we prove the burning number conjecture for trees with a single degree 2 vertex. An example of a tree with a single degree 2 vertex is shown in Figure 1.5.

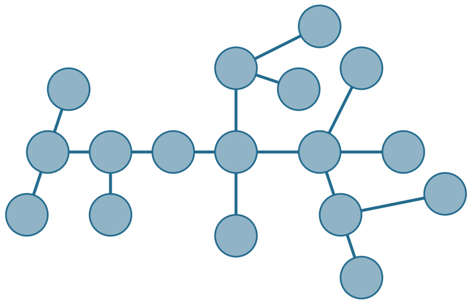


Figure 1.5: A tree with a single degree-2 vertex.

2
Literature review

Although the concept of graph burning is relatively new, many articles can be found on the subject. The burning number conjecture has been researched on different topics such as computational complexity, algorithms to determine the burning number for a graph and proving the conjecture for certain types of graphs. We will discuss some articles on these subjects.

2.1. Computational complexity

It was proven by Bonato et al. (2015) that finding an optimal burning sequence in a graph is an NP-complete problem. A burning sequence is considered optimal when a graph is fully burned by the sequence in a minimal number of rounds. The computational complexity for graph burning was further proven in 2020 by Gupta et al. (2021), where the burning graph problem was extended to connected interval graphs, permutation graphs and several other geometric graph classes. Gorain et al. (2023) proved once more that the burning problem is NP-complete on connected interval graphs and permutation graphs and studied the burning properties of grids. They proved the lower bound of the burning number of a grid of length l and width b is at least $(l \times b)^{\frac{1}{3}}$.

2.2. Deterministic algorithms

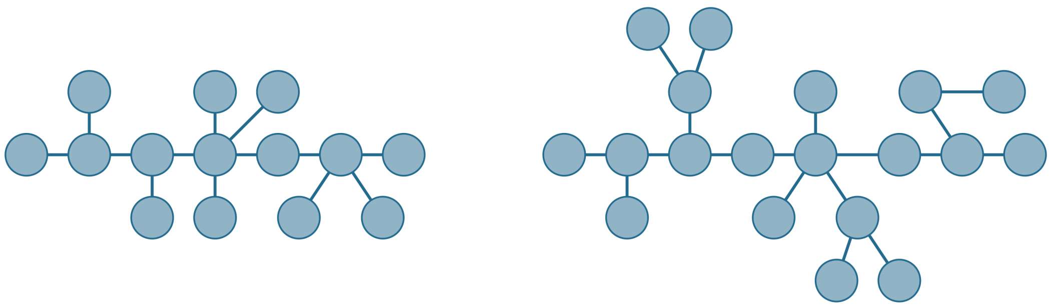
Bonato and Kamali (2019) introduced a 2-approximation algorithm for trees and a 3-approximation algorithm for graphs in general. The exact value of the burning number has been found for algorithms for graph products (Mitsche et al. 2018), the Petersen graph (Sim et al. 2017), theta graph (Liu et al. 2019), dense and tree-like graph (Kamali et al. 2019) and grid graphs (Bonato et al. 2020). These types of graphs are not subjects for this thesis. Therefore, a description is not included, but may be found in Diestel (2017) and Yadav (2023). More recently, Farokh et al. (2020) introduced 6 new heuristics for burning graphs which they tested on datasets, which are used for solving other NP-hard problems in graph theory. Examples of such problem are the independent set problem and the maximum clique problem. The latest research on an $O(mn + kn^2)$ greedy heuristic for graph burning, with n the number of vertices of the graph, m the number of edges and k a guess on the burning number, was published by García-Díaz and Cornejo-Acosta (2024). Their research explored advantages and limitations of such a heuristic for the burning graph problem.

2.3. Burning number conjecture on specific types of graphs

In Chapter 1 we introduced the burning number conjecture by Bonato et al. (2016). In the same article Bonato et al. (2016) proved that it suffices to prove the burning number conjecture for trees. Many proofs for different types of trees on the burning number conjecture have been published, such as a proof on path graphs by Bonato et al. (2016), generalized Petersen graphs by Sim et al. (2017), spider graphs by Bonato and Lidbetter (2017) and large enough graphs with minimum degree 3 or 4 by Bastide et al. (2022). Furthermore, the burning number conjecture was proven for caterpillars, 2-caterpillars and p-caterpillars with at least $2\lceil\sqrt{n}\rceil - 1$ leaves by Hiller et al. (2019). For homeomorphically irreducible trees (HITs) and graphs that contain a homeomorphically irreducible spanning tree (HIST) the conjecture was proven by Murakami (2023). The burning number conjecture has also been proven to hold asymptotically such that $b(G) \leq (1 + o(1))\sqrt{n}$ by Norin and Turcotte (2022).

Since caterpillars, 2-caterpillars and HITs are important graph types for this thesis, these will be discussed in more detail below.

Caterpillars are trees such that when all leaves are deleted, a path remains. A p-caterpillar is a tree with a central (longest) path such that all leaves are at a shortest distance to the path of at most p. The proof of the burning number conjecture for caterpillars and 2-caterpillars by Hiller et al. (2019) is a proof by induction. It consists of a recursive algorithm where at every step a part of the tree is burned off, such that the number of vertices left in the tree is at most the next smaller square number. This proof is rather long and complex and for caterpillars a shorter proof has since been found by Liu et al. (2020). We will use the result from Hiller et al. (2019) in our proof on the burning number conjecture for trees with a single degree 2 vertex in Chapter 4. An example of a caterpillar and a 2-caterpillar is given in Figure 3.1.



(a) An example of a caterpillar.

(b) An example of a 2-caterpillar.

Figure 2.1: An example of a caterpillar and a 2-caterpillar.

HITs are trees that have no degree 2 vertices. Figure 2.2 shows an example of a HIT. In the proof for HITs, Murakami (2023) searches an arbitrary HIT on n vertices for a large enough subtree such that burning one vertex as a source results in burning the whole subtree. This means that a single source is needed to burn the full subtree and the propagation of the fire burns part of the remainder of the tree as well. They then continue to burn the remainder of the tree (which is another subtree) with the remaining number of sources available and they use the fact that the vertex that was burned in round 1 also burns part of the leftover subtree. This results in burning the full tree with at most $\lceil\sqrt{n}\rceil$ sources such that the burning number for HITs is at most $\lceil\sqrt{n}\rceil$. We will use a similar method to prove the burning number conjecture for trees with a single degree 2 vertex in Chapter 4.

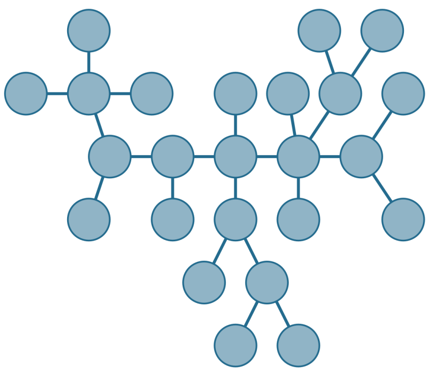


Figure 2.2: An example of a HIT.

3
Caterpillar construction

The burning number conjecture for 2-caterpillars was proven by Hiller et al. (2019). In this chapter, we introduce a different approach to proving the burning number conjecture for certain subsets of 2-caterpillars, which we will call cat-constructs (short for caterpillar constructs). With this approach we prove the burning number conjecture holds, such that there is a burning sequence with the number of sources at most the burning number minus 1 for cat-constructs of large enough size.

Before we define cat-constructs, we describe the action of adding a vertex to a (sub)tree. The **addition of a vertex x to a tree T_1** to obtain T_1', is considered to be done in two ways, in this thesis:

1. Adding a leaf x to an existing vertex $v \in V(T)$ such that $T' = (V(T) \cup \{x\}, E(T) \cup \{xv\})$.
2. Subdividing an edge $uw \in E(T)$ such that $T' = (V(T) \cup \{x\}, E(T) \cup \{ux, xw\} \backslash \{uw\})$.

If T_1 is a subtree of T then a vertex x that is added to T_1 is also added to T. Formally we consider **adding a vertex x to a subtree** T_1 to be done in three ways

1. Adding a leaf x to an existing vertex $v \in V(T_1)$ such that x is added as a leaf to $v \in V(T)$ such that $T' = (V(T) \cup \{x\}, E(T) \cup \{xv\})$.
2. Adding a leaf x to an existing vertex $v \in V(T_1)$ such that an edge $uv \in E(T)$, where $u \notin V(T_1)$, is subdivided. This results in $T' = (V(T) \cup \{x\}, E(T) \cup \{ux, xv\} \backslash \{uv\})$.
3. Subdividing an edge $uw \in E(T_1)$ such that edge $uw \in E(T)$ is subdivided, such that $T' = (V(T) \cup \{x\}, E(T) \cup \{ux, xw\} \backslash \{uw\})$.

Note that T_1' is a subtree of T' for each of these ways of adding a vertex to a subtree T_1. **Adding 2 vertices sequentially** is defined as adding the first vertex to a tree T, to obtain T' and then adding the second vertex to T', to obtain T^*. Suppose we have a tree T with subtrees T_1 and T_2. Assume we add a vertex x to T_1 and T in the second way and we add a vertex y to T_2 and T in the second way. It could occur that a single edge uv is subdivided, to add x and y to T. In that case we consider uv to be subdivided, such that we obtain $T' = (V(T) \cup \{x, y\}, E(T) \cup \{ux, xy, yv\} \backslash \{uv\})$.

We now introduce the definition for a closed ball on a graph that will be used in this chapter based on the definition of closed balls on a graph that was used by Bastide et al. (2022). We define a **closed ball on a graph with center v_i and radius** r as the set of all vertices $u \in V(T)$ of a graph T that satisfy $d_T(v_i, u) \leq r$. We denote such a closed ball by $B^T(v_i, r)$. All vertices in a closed ball are at distance less than or equal to r from its center in a given graph. Therefore, it can be easily seen, that all these vertices can be burned in $r + 1$ rounds by assigning the center as the only source. Thus, if a graph G admits a burning sequence $S = (v_1, v_2, ..., v_k)$ that has k sources, such that it burns G in k rounds, we find the union of the closed balls with centers v_i and radii $k - i$ contains all vertices of G.

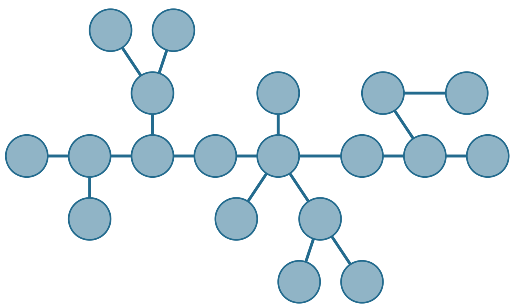
We will now define a cat-construct in Definition 3.1. Note that for a path graph P_n on n vertices, it has been proven by Bonato et al. (2016) that $b(P_n) = \lceil \sqrt{n} \rceil$. Therefore, we can find a burning sequence that consists of $\lceil\sqrt{n}\rceil$ sources which burns P_n in $\lceil\sqrt{n}\rceil$ rounds. Note that such a burning sequence may not be optimal for P_n.

Definition 3.1 (cat-construct). *Let $P_n = (V, E)$ be a path graph on n vertices and let $S = (v_1, v_2, ..., v_{\lceil\sqrt{n}\rceil})$ be a burning sequence such that it burns P_n in $\lceil\sqrt{n}\rceil$ rounds with $\lceil\sqrt{n}\rceil$ sources. For every i in $\{1, 2, ..., \lceil\sqrt{n}\rceil\}$ let $B^{P_n}(v_i, \lceil\sqrt{n}\rceil - i)$ be closed balls with center v_i and radius $\lceil\sqrt{n}\rceil - i$ such that*

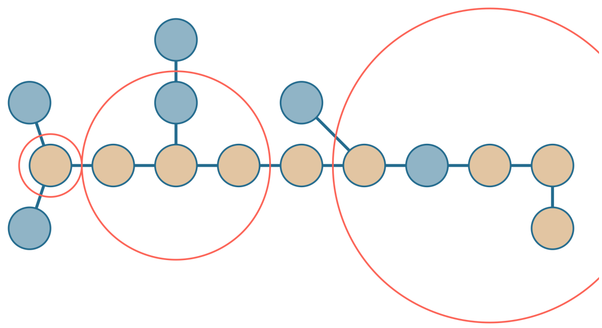
$$V = \bigcup_{i=1}^{\lceil\sqrt{n}\rceil} B^{P_n}(v_i, \lceil\sqrt{n}\rceil - i).$$

Let T_i' denote the i-th subtree of P_n, induced by $B^{P_n}(v_i, \lceil\sqrt{n}\rceil - i)$. Any tree T obtained from P_n, by sequentially adding for every i at most 2 vertices to T_i', is called a cat construct.

Note that P_n is a subtree of the cat-construct that is constructed from it. Furthermore, any cat-construct is a subtree of a 2-caterpillar. A cat-construct may therefore be a path graph itself with at most $n + 2k$ vertices. **The distance from a vertex v to a subtree F in a tree** T is defined as the shortest distance from v to any vertex in F. We define a **longest path in a tree** as a path with maximal length such that there is no path in the tree that is of greater length. A longest path in a tree is, in general, not necessarily unique. The distance from leaves to a longest path in a cat-construct can clearly be at most 2, because for every i at most 2 vertices are added sequentially to T_i'. So a cat-construct can be a 2-caterpillar. However, not every 2-caterpillar is a cat-construct. If the number of vertices that are not on a single longest path in the 2-caterpillar is bigger than $2\lceil\sqrt{n}\rceil$ then we can easily see it cannot be a cat-construct. However if this is not the case then the steps for creating a cat-construct would have to be retraced which is hard to do.



(a) An example of a 2-caterpillar that is not a cat-construct.



(b) An example of a 2-caterpillar that is a cat-construct. The gold vertices were originally in the path from which the caterpillar was constructed. The red circles depict the closed balls that were used in the construction process.

Figure 3.1: Two examples of 2-caterpillars where one is a cat-construct and the other is not.

We will now show, with Theorem 3.1, that the burning number for any cat-construct T on m vertices, obtained from a path graph P_n on n vertices, is at most $\lceil\sqrt{n}\rceil+1$. Furthermore, T has a burning sequence of $\lceil\sqrt{n}\rceil$ sources.

Theorem 3.1. *Let T be a cat-construct on m vertices, obtained from a path graph P_n on n vertices. Then $b(T) \leq \lceil\sqrt{n}\rceil + 1$ with a burning sequence that consists of $\lceil\sqrt{n}\rceil$ sources.*

Proof. We will construct an arbitrary cat-construct T, by the process from Definition 3.1.
Let $P_n = (V, E)$ be a path graph on n vertices and let $S = (v_1, v_2, ..., v_k)$ be a burning sequence for P_n, such that $\lceil\sqrt{n}\rceil = k$. Denote the closed balls with center v_i in $V(P_n)$ and radius $k-i$ by $B^{P_n}(v_i, k-i)$. Note that we have $V(P_n) = \bigcup_{i=1}^{k} B^{P_n}(v_i, k-i)$. For all $i \in \{1, 2, ..., k\}$ we add at most 2 vertices sequentially to every i-th subtree T_i' of P_n induced by $B^{P_n}(v_i, k-i)$. We thereby obtain T_i from T_i' and cat-construct T from P_n. Let $B^{T_i'}(v_i, k-i)$ and $B^{T}(v_i, k-i)$ denote closed balls with centers v_i in $V(T_i')$ and v_i in $V(T)$ and radius $k - i$.

We consider three cases:

1. Without loss of generality assume for some i that no vertices are added to T_i'. Note that for all vertices u in $V(T_i')$ it holds that $u \in B^{P_n}(v_i, k-i) \subseteq B^{T_i}(v_i, k-i) \subseteq B^T(v_i, k-i)$. Then for all such i we choose a ball $B^T(q_i, k-i+1)$ with center $q_i = v_i$ and radius $k-i+1$. Note that $B^T(v_i, k-i+1) \subseteq B^T(q_i, k-i+1)$.

2. Without loss of generality assume for some i that one vertex is added to T_i'. Then there can be one vertex a in $V(T_i)$ that is not contained in $B^{T_i}(v_i, k-i)$ and $B^T(v_i, k-i)$. Vertex a is not necessarily the added vertex. Since only one vertex was added to T_i' the distance from v_i to a in T_i and T must be equal to $k-i+1$. Therefore, for all such i, we choose a ball $B^T(q_i, k-i+1)$ with center $q_i = v_i$ and radius $k-i+1$ such that all vertices in $V(T_i)$ are contained in $B^T(q_i, k-i+1)$.

3. Without loss of generality assume for some i that 2 vertices are added sequentially to T_i'. Then there can be at most 2 vertices w and y in $V(T_i)$ that are not contained in $B^{T_i}(v_i, k-i)$ and $B^T(v_i, k-i)$. Note that w and y are not necessarily the added vertices. If w and y are contained within $B^T(v_i, k-i)$, we choose a ball as in case 1 which contains all vertices of T_i.

   Without loss of generality, if w is not contained within $B^T(v_i, k-i)$, but y is contained within $B^T(v_i, k-i)$, we choose a ball as in case 2 which contains all vertices of T_i.

   Suppose w and y are not contained in $B^{T_i}(v_i, k-i)$ and $B^T(v_i, k-i)$. We either have $d_{T_i}(v_i, w) = d_{T_i}(v_i, y) = k-i+1$ or without loss of generality $d_{T_i}(v_i, w) = k-i+1$ and $d_{T_i}(v_i, y) = k-i+2$. Suppose $d_{T_i}(v_i, w) = d_{T_i}(v_i, y) = k-i+1$ holds. Then we choose a ball as in case 2 which contains all vertices of T_i.
   Suppose $d_{T_i}(v_i, w) = k-i+1$ and $d_{T_i}(v_i, y) = k-i+2$ holds. Then w and y are on the same path from v_i to y. Let q_i be the neighbour of v_i on the path from v_i to y. As $d_{T_i}(v_i, q_i) = 1$ and q_i is closer to y than v_i we now have $d_{T_i}(q_i, y) = k-i+1$. For all other vertices u in $V(T_i)$ we now have $d_{T_i}(q_i, u) \leq k-i+1$. Then we choose a ball $B^T(q_i, k-i+1)$ with center q_i and radius $k-i+1$ such that $V(T_i) \subseteq B^T(q_i, k-i+1)$.

We therefore choose for every i a ball $B^T(q_i, k-i+1)$ with its center q_i as discussed in the three cases and radius $k-i+1$. We then have $V(T) = \bigcup_{i=1}^{k} B^T(q_i, k-i+1)$. We can therefore find a burning sequence $S_T = (q_1, q_2, ...q_k)$ such that S_T contains $k = \lceil\sqrt{n}\rceil$ sources and all vertices in T are burned within $k+1 = \lceil\sqrt{n}\rceil + 1$ rounds. We have thus found a burning sequence that contains $k = \lceil\sqrt{n}\rceil$ sources and for the burning number it holds that $b(T) \leq \lceil\sqrt{n}\rceil + 1$ as required.

□

We have now proven that every cat-construct T on m vertices, obtained from a path graph P_n on n vertices, has a burning sequence of $\lceil\sqrt{n}\rceil$ vertices, such that $b(T) \leq \lceil\sqrt{n}\rceil + 1$. With this result we prove Corollary 3.1, 3.2 and 3.3.

Corollary 3.1. *Let P_n be a path graph on n vertices, where n is square, and let T be a cat-construct, obtained from P_n, such that $|T| = m > n$. Then $b(T) \leq \lceil\sqrt{m}\rceil$.*

Proof. Let T be such a cat-construct. By Theorem 3.1 we have $b(T) \leq \lceil\sqrt{n}\rceil + 1$. Since n is square and $m > n$, it then holds that $b(T) \leq \lceil\sqrt{n}\rceil + 1 = \lceil\sqrt{m}\rceil$. □

Corollary 3.2. *Let P_n be a path graph on n vertices, where n is not square. Let T be a cat-construct on m vertices, obtained from P_n. If $|T| = m > (\lceil\sqrt{n}\rceil)^2$, then we get $b(T) \leq \lceil\sqrt{m}\rceil$.*

Proof. Let T be such a cat-construct. By Theorem 3.1, we have $b(T) \leq \lceil\sqrt{n}\rceil + 1$. It is given that n is not square and we have $|T| > (\lceil\sqrt{n}\rceil)^2$, which is the first square bigger than n. Note that the number of vertices in T can be at most $n + 2\lceil\sqrt{n}\rceil$. Therefore, $|T|$ can never be greater than the first square bigger than $(\lceil\sqrt{n}\rceil)^2$. Thus, for the burning number it must hold that $b(T) \leq \lceil\sqrt{n}\rceil + 1 = \lceil\sqrt{m}\rceil$. □

Corollary 3.3. *Let T be a cat-construct on m vertices, obtained from a path graph P_n on n vertices. Suppose $|T| = m > (\lceil\sqrt{n}\rceil)^2 - 1$. Let w be a vertex that is added as a leaf to T, to obtain a tree T_w. Then $b(T_w) \leq \lceil\sqrt{m}\rceil$.*

Proof. Suppose T is such a cat-construct and w is a vertex that is added as a leaf to T, to obtain T_w. Since $V(T_w) = V(T) \cup \{w\}$, we have $|T_w| = |T| + 1 > (\lceil\sqrt{n}\rceil)^2$, which is the first square bigger than n. Since T is a cat-construct, by Theorem 3.1, we find a burning sequence $S_T = (q_1, q_2, ..., q_{\lceil\sqrt{n}\rceil})$ that contains $\lceil\sqrt{n}\rceil$ sources and we have that $b(T) \leq \lceil\sqrt{n}\rceil + 1 = \lceil\sqrt{m}\rceil$. We choose for a burning sequence S_{T_w} for T_w the vertices from S_T and add w as a source, to be burned in round $\lceil\sqrt{n}\rceil + 1$. So we have found a burning sequence $S_{T_w} = (q_1, q_2, ..., q_{\lceil\sqrt{n}\rceil}, w)$, such that all vertices of T_w are burned in $\lceil\sqrt{n}\rceil + 1$ rounds and. For the burning number, it then holds that $b(T_w) \leq \lceil\sqrt{n}\rceil + 1 = \lceil\sqrt{m}\rceil$. □

Remark 3.1. *Since a cat-construct T may be a 2-caterpillar, adding a leaf w to T may result in a 3-caterpillar. Therefore corollary 3.3 shows the burning number conjecture holds for certain 3-caterpillars.*

We will now prove for cat-constructs on m vertices, obtained from a path graph P_n on n vertices, that $b(T) \leq \lceil\sqrt{m}\rceil$ such that it has a burning sequence that contains $\lceil\sqrt{n}\rceil$ sources.

Theorem 3.2. *Let T be a cat-construct on m vertices obtained from P_n on n vertices. Then $b(T) \leq \lceil\sqrt{m}\rceil$ and T has a burning sequence, that consists of $\lceil\sqrt{n}\rceil$ sources.*

Proof. Let T be such a cat-construct. Suppose n is square. Then, by Corollary 3.1, we have $b(T) \leq \lceil\sqrt{m}\rceil$. Note that in Corollary 3.1 the burning sequence from Theorem 3.1 for T remains unchanged. So we have a burning sequence for T, that consists of $\lceil\sqrt{n}\rceil = \lceil\sqrt{m}\rceil - 1$ vertices.

Suppose n is not square. By Corollary 3.2, if $|T| = m > (\lceil\sqrt{n}\rceil)^2$, we have $b(T) \leq \lceil\sqrt{m}\rceil$. Note that in this corollary Theorem 3.1 is also used so there exists a burning sequence that consists of $\lceil\sqrt{n}\rceil$ sources.

Now suppose we have $|T| = m < (\lceil\sqrt{n}\rceil)^2$. Then $\lceil\sqrt{n}\rceil = \lceil\sqrt{m}\rceil$. Note that a cat-construct is either a path, caterpillar or 2-caterpillar and the burning conjecture for these types of trees holds. Thus we have that $b(T) \leq \lceil\sqrt{n}\rceil$ and there must exist a burning sequence with exactly $\lceil\sqrt{n}\rceil$ vertices.

We conclude that $b(T) \leq \lceil\sqrt{m}\rceil$ and T has a burning sequence that consists of $\lceil\sqrt{n}\rceil$ sources.

□

We have now proven that the burning number conjecture holds for cat-constructs, obtained from a path P_n. In particular we have seen that for cat-constructs with a number of vertices m larger than the first square after n there exists a burning sequence with $\lceil\sqrt{m}\rceil - 1$ sources. We suspect this is also true for cat-constructs with a number of vertices less than the first square after n so we introduce Conjecture 3.1.

Conjecture 3.1. *A cat-construct T on m vertices, obtained from a path graph P_n on n vertices, has a burning sequence with $\lceil\sqrt{m}\rceil - 1$ sources.*

4

Trees with single degree 2 vertex

Murakami (2023) proved that the burning number conjecture holds for homeomorphically irreducible trees (HITs) trees that contain homeomorphically irreducible spanning trees (HISTS). A tree with a single degree-2 vertex can be viewed as a HIT with one added vertex that subdivides an edge between 2 distinct vertices. Therefore we will use multiple lemmas and the theorem on the burning number conjecture for HITs by Murakami (2023) for the proof of the burning number conjecture for trees with a single degree-2 vertex. We will now introduce these lemmas, the theorem and key concepts that were introduced in the article by Murakami (2023). Furthermore, we introduce the theorems on the burning number conjecture for paths, proven by Bonato et al. (2016) and caterpillars and 2-caterpillars, proven by Hiller et al. (2019).

We now prove trees with a single degree vertex on less than or equal to 12 vertices are paths, caterpillars or 2-caterpillars in Lemma 4.1.

Lemma 4.1. *Let T be a tree with a single degree-2 vertex on $n \leq 12$ vertices. Then T is either a path, a 1-caterpillar or a 2-caterpillar.*

Proof. Let T be a tree on $n \leq 12$ vertices with a single degree-2 vertex x. First we show the longest possible path P in T is of length 6. Suppose for a contradiction that P were of length at least 7. In that case P contains 8 vertices. Then, there would be at least 5 internal vertices of degree at least 3. So we would have at least 5 more leaves that are not on path P. We now add up the number of these internal vertices and leaves and the 3 remaining vertices in P. This results in $|T| \geq 5 + 5 + 3 = 13$. Thus the longest possible path P in T is of length 6.

We will now show T cannot be a p-caterpillar with $p \geq 3$. For any p-caterpillar with $p > 3$ there exists a subtree which is a 3-caterpillar. Therefore it suffices to show T cannot be a 3-caterpillar. Note that all trees are p-caterpillars with $p \geq 0$. If $p = 0$ then the p-caterpillar is a path.

Suppose for a contradiction that T is a 3-caterpillar. Then T has at least one vertex x at distance 3 from a central longest path P. Therefore, for any vertex u in $V(P)$ we have $d_T(x, u) \geq 3$. Let a be the vertex on P such that $d_T(x, a) = 3$. Then the leaves of P must be at distance at least 3 from a since P is a longest path in T. Thus P must be of length at least $3 + 3 + 1 = 7$. We have therefore arrived at a contradiction.

Since all trees are p-caterpillars and trees with a single degree-2 vertex cannot have $p \geq 3$ it remains to show there exists a path, a 1-caterpillar and a 2-caterpillar on at most 12 vertices for this type of trees.

Suppose we have a tree T with a single degree-2 vertex on $n = 3$ vertices. We then have a single degree-2 vertex with 2 neighbours that are not connected to each other and must be leaves. Therefore, a tree on $n = 3$ vertices is a path.

Suppose we have a tree T with a single degree-2 vertex on $n = 5$ vertices. Let x be the degree-2 vertex. Then x has precisely 2 neighbours u and v. u and v are vertices of degree 1 or degree at least 3. Since there are 5 vertices in T and x cannot have any other neighbours, either u or v is a vertex of degree at least 3. Without loss of generality suppose u is a vertex of degree at least 3. Then u has exactly $5 - 3 = 2$ neighbours y and z. There exists no edge between y and z since T is a tree. Thus z cannot be on the path from v to y. More precisely the path from v to y goes through the vertices

v, x, u, y. We have $d(z, u) = 1$ and the distance from z to any vertex in the path from v to y then equals at least 1. Thus T is a 1-caterpillar for $n = 5$.

Suppose we have a tree T with a single degree-2 vertex on $n = 11$ vertices. Let P be the longest path in T such that it is of length 5. Then P contains 6 vertices of which 4 are internal vertices. Then P has 4 internal vertices of degree at least 3 and there must be at least 4 leaves not on the path. Let x be the degree-2 vertex in T. Furthermore, let x be connected to an internal vertex of P that is at distance at least 2 from both leaves of P. Adding up the number of vertices we have $6 + 4 + 1 = 11$ vertices in T. One of the leaves u must be connected to x, otherwise x cannot be a degree-2 vertex. Then u is at distance 2 from the path P. Thus T is a 2-caterpillar.

We conclude a tree T with a single degree-2 vertex on $n \leq 12$ vertices can be either a path, a caterpillar or a 2-caterpillar.

□

We will now introduce the lemmas and theorem by Murakami (2023) that are important for the proof of the burning number conjecture for trees with a single degree-2 vertex. We first introduce an important concept that is used in the lemmas. For a graph G on $n \geq 2$ vertices with edge xy the component that contains x upon removing xy from $E(G)$ is denoted by $G_x(xy)$.

Lemma 4.2 (Murakami 2023). *Let $n \geq 6$. Any tree T on n vertices contains a vertex x with neighbours $v_1, ..., v_k$ such that $|T_x(xv_k)| \geq 2\lceil\sqrt{n}\rceil - 1$ and $|T_{v_i}(xv_i)| < 2\lceil\sqrt{n}\rceil - 1$ for $i \in [k-1] = \{1, 2, ..., k-1\}$.*

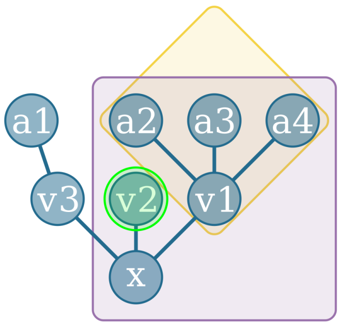


Figure 4.1: An example of a tree on $n = 7$ vertices where a vertex x has been found such that $|T_x(xv_3)| \geq 5$ and $|T_i(xv_i)| < 5$ for $i \in \{1, 2\}$. $T_x(xv)$ is the subtree within the purple area. $T_{v_1}(xv_1)$ is the subtree within the yellow area and $T_{v_2}(xv_2)$ is the subtree within the green area.

In Figure 4.1 we see an example of the result from Lemma 4.2. We prove a similar result for when trees contain a single degree-2 vertex in Lemma 4.3 Furthermore, we prove Lemma 4.4 with a method similar to the proof by Murakami (2023) for Lemma 4.2.

Lemma 4.3. *Let T be a tree on $n \geq 13$ vertices with exactly one vertex x of degree 2. Then $|T_x(xv)| \geq 2\lceil\sqrt{n}\rceil - 1$ or $|T_x(xu)| \geq 2\lceil\sqrt{n}\rceil - 1$ with u and v the distinct neighbours of x.*

Proof. Suppose we have such a tree T on $n \geq 13$ vertices with x the degree-2 vertex. Furthermore, suppose for a contradiction that $|T_x(xv)| < 2\lceil\sqrt{n}\rceil - 1$ and $|T_x(xu)| < 2\lceil\sqrt{n}\rceil - 1$. $|T_x(xv)| + |T_x(xu)| = n+1 < 4\lceil\sqrt{n}\rceil - 2$. This inequality does not hold for $n \geq 13$. Therefore, we have arrived at a contradiction.

□

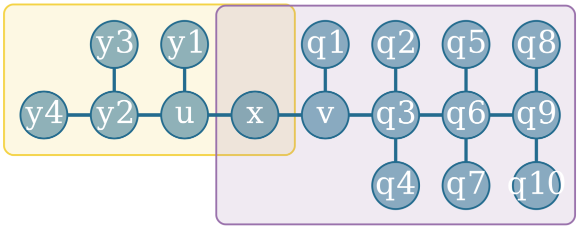


Figure 4.2: An example of a tree with a single degree-2 vertex on $n = 17$ vertices where $T_x(xv)$ is in the yellow area and $T_x(xu)$ is in the purple are. So we have $|T_x(xv)| < 9$ and $|T_x(xu)| \geq 9$.

In Figure 4.2 a tree with a single degree-2 vertex on 17 vertices is shown such that Lemma 4.3 holds.

Lemma 4.4. *Let T be a tree on $n \geq 13$ vertices with exactly one vertex x of degree 2, with u, v distinct neighbours of x. Suppose $|T_x(xv)| \geq |T_u(xu)|$ and $|T_u(xu)| \geq 2\lceil\sqrt{n}\rceil - 1$ then $T_u(xu)$ contains a vertex y with neighbours $q_1, ..., q_k$ such that $|T_y(yq_k)| \geq 2\lceil\sqrt{n}\rceil - 1$ and $|T_{q_i}(yq_i)| < 2\lceil\sqrt{n}\rceil - 1$ for $i \in [k-1] = \{1, 2, ..., k-1\}$.*

Proof. Let T be such a tree. Suppose $|T_x(xv)| \geq |T_u(xu)|$ and $|T_u(xu)| \geq 2\lceil\sqrt{n}\rceil - 1$. Let the neighbours of u be denoted by $q_1, ..., q_m$ where $q_m = x$. Given $|T_u(xu)| \geq 2\lceil\sqrt{n}\rceil - 1$, if $|T_{q_i}(uq_i)| < 2\lceil\sqrt{n}\rceil - 1$ for $i \in [m-1]$, then we are done.
Otherwise, for some $i \in [m-1]$ we have that $|T_{q_i}(uq_i)| \geq 2\lceil\sqrt{n}\rceil - 1$. Without loss of generality assume q_1 is such a neighbour. Now let the neighbours of q_1 be denoted by $w_1, ..., w_j$ such that $w_j = u$. If $|T_{w_i}(q_1 w_i)| < 2\lceil\sqrt{n}\rceil - 1$ for all $i \in [j-1]$ we are done. Otherwise we repeat this process.
Since $T_u(xu)$ has a finite amount of vertices this process must terminate. □

Remark 4.1. *Using the notation in the proof for Lemma 4.4, the single degree-2 vertex x is not contained within $|T_y(yq_k)|$, because $T_y(yq_k) \subset T_u(xu)$ and $x \notin V(T_u(xu))$. Moreover, since $T_{q_i}(yq_i) \subset T_y(yq_k)$, it then follows that $x \notin V(T_{q_i}(yq_i))$.*

An important lemma in the proof of the burning conjecture on HITs is Lemma 4.5. This Lemma is also important for the proof of the burning number conjecture on trees with a single degree-2 vertex.

Lemma 4.5 (Murakami 2023)**.** *Let $k \in \mathbb{N}^{>0}$. Let T be a HIT where $|T| \leq 2k - 1$. Then T contains at most $k - 2$ internal vertices.*

For the next two lemmas we introduce two more concepts. A **modified burning sequence** M of a graph G is a burning sequence where the first vertex x_1 is burned simultaneously with another set of vertices $U \subseteq V(G)$ in round 1. In all rounds $i \neq 1$, graph burning occurs in the traditional way for a burning sequence. So all vertices of G are burned after k rounds. M is denoted by $(U \cup \{x_1\}, x_2, ..., x_k)$. The length of a shortest modified burning sequence for some set $U \subseteq V(G)$ is then called the **modified burning number** $b^U(G)$ of G.

Lemma 4.6 (Murakami 2023)**.** *Let T be a tree with one degree-2 vertex v, and let T' be the HIT obtained from T by smoothing v. Then $b^{\{v\}}(T) \leq b(T')$.*

For the proof of the burning number conjecture for trees with a single degree-2 vertex, Lemma 4.6 can be generalized into Lemma 4.7. We prove this lemma by a method similar to the proof for the original Lemma by Murakami (2023).

Lemma 4.7. *Let T be a tree on n vertices with $m \leq n - 2$ degree-2 vertices $v_1, ..., v_m$. Let v_a be such a degree-2 vertex where $v_a \neq v_m$. Let T' be the tree with $m - 1$ degree-2 vertices that is obtained by smoothing v_a. Then $b^{\{v_m\}}(T) \leq b(T')$.*

Proof. Let $S = (x_1, ..., x_k)$ be a (not necessarily optimal) burning sequence for T'. We will show by a proof of contradiction that $M = (\{v_m, x_1\}, x_2, ..., x_k)$ is a modified burning sequence for T. From which we can conclude that any burning sequence for T' yields a modified burning sequence of the same length for T, and therefore $b^{\{v_m\}}(T) \leq b(T')$.
Assume for a contradiction that M is not a modified burning sequence for T. Then we find a vertex p in T that is not burned at the end of round k. Thus $p \neq x_i$ or $p \neq v_m$. Which means p is not one of the sources of the modified burning sequence. In T', p is burned at the end of round k since S is a burning sequence for T'. Assume p is burned in round i for some $i \leq k$. Then we can find a source x_j with $j < i$ such that $d_{T'}(p, x_j) = i - j$.
Since p does not get burned in T, the distance for p and x_j in T must be greater than $i - j$. So we have $d_T(p, x_j) > i - j$. Then v_a must be on the path from p to x_j, because this is the only vertex in $V(T)$ that is not in $V(T')$. Then $d_T(p, x_j) = d_{T'}(p, x_j) + 1 = i - j + 1$, which implies that $d_T(p, v_m) \leq i - j$. Then p must be burned in T in round at most $1 + i - j$ as v_m is burned in round 1. Then p must be burned in T at the end of round k as $1 + j - i \leq k$. Thus we have arrived at the required contradiction. □

Finally we introduce Theorem 4.1 on the burning number conjecture for HITs by Murakami (2023), Theorem 4.2 by Bonato et al. (2016) and Theorems 4.3 and 4.4 by Hiller et al. (2019) which are used in our proof of the burning number conjecture for trees with a single degree-2 vertex.

Theorem 4.1 (Murakami 2023)**.** *Let T be a HIT on n vertices. Then $b(T) \leq \lceil\sqrt{n}\rceil$.*

Theorem 4.2 (Bonato et al. 2016)**.** *Let P_n be a path on n vertices. Then $b(T) \leq \lceil\sqrt{n}\rceil$.*

Theorem 4.3 (Hiller et al. 2019)**.** *Let T be a 1-caterpillar on n vertices. Then $b(T) \leq \lceil\sqrt{n}\rceil$.*

Theorem 4.4 (Hiller et al. 2019)**.** *Let T be a 2-caterpillar on n vertices. Then $b(T) \leq \lceil\sqrt{n}\rceil$.*

We will now prove the burning number conjecture for trees with a single degree-2 vertex.

Theorem 4.5. *Let T be a tree on n vertices with exactly one vertex of degree-2. Then $b(T) \leq \lceil\sqrt{n}\rceil$.*

Proof. We prove by induction on the number of vertices n.
For the base cases we show the burning number for trees with a single degree-2 vertex on $n \leq 12$ vertices is at most equal to $\lceil\sqrt{n}\rceil$. By Lemma 4.1 we know that all trees T with a single degree-2 vertex on $n \leq 12$ vertices are either paths, 1-caterpillars or 2-caterpillars. Therefore by Theorem 4.2, 4.3 and 4.4 we have $b(T) \leq \lceil\sqrt{n}\rceil$.
We may now assume the induction hypothesis that for any tree with exactly one degree-2 vertex on less than n vertices the theorem holds.

Let T be a tree on $n > 12$ vertices and let x be the vertex in T of degree 2 with neighbours u and v. Then by Lemma 4.3 $|T_x(xv)| \geq 2\lceil\sqrt{n}\rceil - 1$ or $|T_x(xu)| \geq 2\lceil\sqrt{n}\rceil - 1$. Assume without loss of generality that $|T_x(xv)| \geq 2\lceil\sqrt{n}\rceil - 1$ and $|T_x(xv)| \geq |T_x(xu)|$. Note that $V(T_x(xu)) = V(T_v(xv) \cup \{x\})$ and $V(T_x(xv)) = V(T_u(xu) \cup \{x\})$. We consider two cases:

1. $|T_u(xu)| < 2\lceil\sqrt{n}\rceil - 1$ and
2. $|T_u(xu)| \geq 2\lceil\sqrt{n}\rceil - 1$.

Case 1: $|T_u(xu)| < 2\lceil\sqrt{n}\rceil - 1$.
Since x is a leaf in $T_x(xv)$, $T_x(xv)$ is a HIT. Note $V(T_u(xu)) \cup \{x\} = V(T_x(xv))$ implies $|T_x(xv)| = |T_u(xu)| + 1$. Since $V(T_x(xv)) = V(T_u(xu)) \cup \{x\}$ is a HIT with x as a leaf, its internal vertices are the same as the internal vertices of $T_u(xu)$. As $|T_u(xu)| < 2\lceil\sqrt{n}\rceil - 1$, $|V(T_u(xu)) \cup \{x\}| \leq 2\lceil\sqrt{n}\rceil - 1$. So by lemma 4.5 and since $V(T_u(xu)) \cup \{x\} = V(T_x(xv))$, $T_x(xv)$ has at most $\lceil\sqrt{n}\rceil - 2$ internal vertices. Since x is a leaf in $T_x(x_v)$, the longest path from x to any other vertex in $T_x(xv)$ is then of length at most $\lceil\sqrt{n}\rceil - 1$. By choosing x as the first source to be burned in T, $T_x(xv)$ will be fully burned by round $\lceil\sqrt{n}\rceil$ without needing any other sources. It remains to show that $b(T_v(xv)) \leq \lceil\sqrt{n}\rceil - 1$.
$T_v(xv)$ is either a HIT or a tree with a single degree-2 vertex, where v is the degree-2 vertex. Note $|T_v(xv)| + |T_x(xv)| = n$ and we assumed before $|T_x(xv)| \geq 2\lceil\sqrt{n}\rceil - 1$. So $|T_v(xv)| = n - |T_x(xv)| \leq n - 2\lceil\sqrt{n}\rceil - 1 \leq (\lceil\sqrt{n}\rceil - 1)^2$.
Suppose $T_v(xv)$ is a HIT. Then by Theorem 4.1 we have $b(T_v(xv)) \leq \lceil\sqrt{n}\rceil - 1$.

Suppose $T_v(xv)$ is a tree with a single degree-2 vertex, where v is the degree-2 vertex. Then by the induction hypothesis $b(T_v(xv)) \leq \lceil\sqrt{n}\rceil - 1$. Thus the theorem holds for $|T_u(xu)| < 2\lceil\sqrt{n}\rceil - 1$.

Case 2: $|T_u(xu)| \geq 2\lceil\sqrt{n}\rceil - 1$.
By Lemma 4.4 we find $y \in V(T_u(xu))$ with neighbours $q_1, \ldots, q_k$ such that $|T_y(yq_k)| \geq 2\lceil\sqrt{n}\rceil - 1$ and $|T_{q_i}(yq_i)| < 2\lceil\sqrt{n}\rceil - 1$. By Remark 4.1 we know that $x \notin V(T_y(yq_k))$ and $x \notin V(T_{q_i}(yq_i))$.

We will now show that the distance from y to any vertex in $V(T_{q_i}(yq_i))$ for i in $\{1, \ldots, k-1\}$ is at most $\lceil\sqrt{n}\rceil - 1$. Let T_i be the subtree of T induced by $V(T_{q_i}(yq_i)) \cup \{y\}$. Then y is a leaf in T_i and every T_i is a HIT. We have $|T_{q_i}(yq_i)| < 2\lceil\sqrt{n}\rceil - 1$ and $|T_i| = |T_{q_i}(yq_i)| + 1$. Therefore, we find $|T_i| \leq 2\lceil\sqrt{n}\rceil - 1$. Then by Lemma 4.5, we know that T_i has at most $\lceil\sqrt{n}\rceil - 1$ internal vertices. By choosing y as a source in round 1, all vertices in every T_i will then be fully burned by round $\lceil\sqrt{n}\rceil$. Note that $T_y(yq_k)$ is a subtree of T that is induced by $\bigcup_{i=1}^{k-1} V(T_i)$. So by choosing y as a source in round 1, $T_y(yq_k)$ will be fully burned by round $\lceil\sqrt{n}\rceil$.

Since y is a source in round 1, it remains to show that $b^{\{q_k\}}(T_{q_k}(yq_k)) \leq \lceil\sqrt{n}\rceil - 1$.
$T_{q_k}(yq_k)$ is either a tree with a single degree-2 vertex x or a tree with two degree-2 vertices, x and q_k. Suppose $T_{q_k}(yq_k)$ is a tree with a single degree-2 vertex. We can apply the induction hypothesis since the number of vertices for this tree is less than n. So if we have $|T_{q_k}(yq_k)| \leq (\lceil\sqrt{n}\rceil - 1)^2$, then it holds that $b(T_{q_k}(yq_k)) \leq \lceil\sqrt{n}\rceil - 1$. Furthermore, $b^{\{q_k\}}(T_{q_k}(yq_k)) \leq b(T_{q_k}(yq_k))$ always holds. So we would then have that $b^{\{q_k\}}(T_{q_k}(yq_k)) \leq \lceil\sqrt{n}\rceil - 1$ as required. We now show this is the case. Since $|T_y(yq_k)| + |T_{q_k}(yq_k)| = n$, we know $|T_{q_k}(yq_k)| = n - |T_y(yq_k)| \leq n - 2\lceil\sqrt{n}\rceil + 1 = (\lceil\sqrt{n}\rceil - 1)^2$ as we wanted.

Suppose $T_{q_k}(yq_k)$ is a tree with two degree-2 vertices. Let T' denote the tree with a single degree-2 vertex x obtained by smoothing q_k in $T_{q_k}(yq_k)$. From Lemma 4.7 we have $b^{\{q_k\}}(T_{q_k}(yq_k)) \leq b(T')$. Furthermore, we have that $|T'| < |T_{q_k}(yq_k)|$. Since T' is a tree with a single degree-2 vertex we can also apply the induction hypothesis in this case. We then get $|T'| < |T_{q_k}(yq_k)| = n - |T_y(yq_k)| \leq 2\lceil\sqrt{n}\rceil - 1 \leq (\lceil\sqrt{n}\rceil - 1)^2$. Thus $b^{\{q_k\}}(T_{q_k}(yq_k)) \leq b(T') \leq \lceil\sqrt{n}\rceil - 1$. Since y is a source in round 1 we know q_k is burned in round 2. Thus we have $b^{\{q_k\}}(T_{q_k}(yq_k)) \leq \lceil\sqrt{n}\rceil - 1$.

Thus we can conclude that $b(T) \leq \lceil\sqrt{n}\rceil$ for trees with a single degree-2 vertex. □

We have now proven the burning number conjecture for trees with a single degree-2 vertex.

5

Conclusion & Discussion

The goal for this thesis was to find proofs for the burning number conjecture that had not been found yet. In chapter 3 we have shown that the burning number conjecture holds for cat-constructs, obtained from a path P_n, such that a cat-construct on m vertices may be burned in $\lceil\sqrt{m}\rceil$ rounds. A cat-construct is obtained from P_n by adding at most 2 vertices to subtrees of P_n induced by closed balls that have the vertices of a burning sequence of P_n as their centers with a chosen radius. These cat-constructs are burned with a burning sequence that consists of $\lceil\sqrt{m}\rceil - 1$ sources if they have more vertices than the first square bigger than n. A proof for needing one fewer source on cat-constructs smaller than the first square after n has not yet been found. We have seen that adding one vertex as a leaf to a cat-construct, which counts as a source in the last round of the burning of the graph, upholds the burning number conjecture. Thereby, we have shown that for some 3-caterpillars, which can be constructed by adding a vertex to a cat-construct, the burning number conjecture holds. Further research into the number of sources that are needed to burn a graph, could be beneficial to proving the full burning number conjecture.

In chapter 4 we have shown that the burning number conjecture holds for trees with a single degree-2 vertex, by adjusting the method from the article by Murakami (2023) to fit this situation. By increasing the number of degree-2 vertices in the trees, step by step, eventually the full conjecture could possibly be proven by induction. For a next step in proving the burning number conjecture, it would be interesting to look at trees with 2 degree-2 vertices and create a similar proof. There are two aspects that need to be addressed when looking at this next step. The first aspect is that the subtrees which can be burned with one source are more difficult to find in trees with more degree-2 vertices. Therefore, it is harder to prove that they exist. The second aspect is that a new proof for the size of the subtrees must be found for trees with more degree-2 vertices. In order to prove the burning number conjecture with this method, the lemmas on the size of the subtrees have to be generalized.

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