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Central Extensions of Lie Groups Preserving a Differential Form

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Let M be a manifold with a closed, integral $(k+1)$-form ω, and let G be a Fréchet–Lie group acting on (M,ω). As a generalization of the Kostant–Souriau extension for symplectic manifolds, we consider a canonical class of central extensions of $\mathfrak{g}$ by $\mathbb{R}$, indexed by $H^{k-1}(M,\mathbb{R})^*$. We show that the image of $H_{k-1}(M,\mathbb{Z})$ in $H^{k-1}(M,\mathbb{R})^*$ corresponds to a lattice of Lie algebra extensions that integrate to smooth central extensions of G by the circle group $\mathbb{T}$. The idea is to represent a class in $H_{k-1}(M,\mathbb{Z})$ by a weighted submanifold (S,β), where β is a closed, integral form on S. We use transgression of differential characters from S and M to the mapping space $C^\infty(S,M)$ and apply the Kostant–Souriau construction on $C^\infty(S,M)$.

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1 Introduction

In this paper, we investigate the integrability of a natural class of Lie algebra extensions arising from an *exact* action of a (possibly infinite-dimensional) Lie group G on an n-dimensional manifold M, endowed with a closed, integral $(k+1)$-form ω.

To describe this class of extensions, recall that the action of a Lie group G with Lie algebra $\mathfrak{g}$ is called exact if it preserves ω and, for all $X \in \mathfrak{g}$, the insertion of the fundamental vector field X_M in ω is exact, that is, $i_{X_M}\omega = \mathrm{d}\psi_X$ for some $(k-1)$-form ψ_X. The fact that ψ_X is not uniquely determined gives rise to a natural central extension

$$H^{k-1}(M,\mathbb{R}) \to \widehat{\mathfrak{g}} \xrightarrow{\pi} \mathfrak{g}, \tag{1}$$

which is canonically associated to the action of $\mathfrak{g}$ on (M,ω). The Lie algebra $\widehat{\mathfrak{g}}$ is defined by

$$\widehat{\mathfrak{g}} := \{(X,[\psi_X]) \in \mathfrak{g} \times \overline{\Omega}^{k-1}(M) \ : \ i_{X_M}\omega = \mathrm{d}\psi_X\}, \tag{2}$$

where $[\psi_X]$ denotes the class of ψ_X in $\overline{\Omega}^{k-1}(M) := \Omega^{k-1}(M)/\mathrm{d}\Omega^{k-2}(M)$. The Lie bracket on $\widehat{\mathfrak{g}}$ is given by $[(X,[\psi_X]),(Y,[\phi_Y])] := ([X,Y],[i_{X_M}i_{Y_M}\omega])$ (cf. [23, Thm 13]).

Although we do not take the "higher category" point of view in this paper, let us briefly mention that the central extension (1) is part of an L_∞-algebra arising naturally in multisymplectic geometry [29, 36], where it provides the higher analogue of a momentum map [6]. It is connected to the "quantomorphism n-group" of a higher prequantum bundle [10] and, for $k = 2$, can be interpreted as the Lie 2-algebra of observables for a classical bosonic string coupled to a B-field [1].

We are interested in the integrability of those extensions of $\mathfrak{g}$ by $\mathbb{R}$ that factor through the central extension (1). Indeed, for every nonzero $\lambda \in H^{k-1}(M,\mathbb{R})^*$, the quotient $\mathfrak{g}_\lambda := \widehat{\mathfrak{g}}/\mathrm{Ker}(\lambda)$ is a central extension of $\mathfrak{g}$ by $H^{k-1}(M,\mathbb{R})/\ker(\lambda)$, which is identified with $\mathbb{R}$ using the functional λ. The question, then, is to determine elements of $H^{k-1}(M,\mathbb{R})^*$ for which the corresponding Lie algebra extension $\mathbb{R} \to \widehat{\mathfrak{g}}_\lambda \to \mathfrak{g}$ integrates to a smooth Lie group extension $\mathbb{T} \to \widehat{G}_\lambda \to G$.

It is instructive to consider the classical case where $k = 1$ and (M,ω) is a symplectic manifold. Here, the G-action is exact if it is (weakly) Hamiltonian, and the central extension canonically associated to the Hamiltonian action is the *Kostant–Souriau extension* ([31, §II.11], [17])

$$H^0(M,\mathbb{R}) \to \widehat{\mathfrak{g}} \xrightarrow{\pi} \mathfrak{g}. \tag{3}$$

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If (M,ω) is connected and prequantizable, then the image of $H_0(M,\mathbb{Z})$ in $H^0(M,\mathbb{R})^*$ yields a lattice of integrable Lie algebra extensions. The corresponding Lie group extensions are obtained by pulling back the group extension $\mathbb{T} \to \mathrm{Aut}(P,\nabla) \to \mathrm{Ham}(M)$ along the Hamiltonian action $\iota\colon G \to \mathrm{Ham}(M,\omega)$, where (P,∇) is a prequantum bundle whose curvature class is an integral multiple of $[\omega] \in H^2(M,\mathbb{R})$ (cf. [4, §2.4]). Note that even if $\mathbb{R} \to \widehat{\mathfrak{g}}_\lambda \to \mathfrak{g}$ is a trivial extension, the corresponding group extension may still be nontrivial. In general, this will depend on the choice of prequantum line bundle (P,∇).

A more representative example is the case where $k = n-1$ and (M,ω) is a manifold with a volume form. Here, the central extension associated to the exact divergence-free action of $\mathfrak{g}$ on (M,ω) is the *Lichnerowicz extension* [19]

$$H^{n-2}(M,\mathbb{R}) \to \widehat{\mathfrak{g}} \xrightarrow{\pi} \mathfrak{g}. \tag{4}$$

If M is compact, then the group $G = \mathrm{Diff}_{\mathrm{ex}}(M,\omega)$ of exact volume-preserving diffeomorphisms is a Fréchet–Lie group. Its action on (M,ω) is exact by definition, and the image of $H_{n-2}(M,\mathbb{Z})$ in $H^{n-2}(M,\mathbb{R})^*$ yields a lattice of integrable $\mathbb{R}$-extensions for the Lie algebra $\mathfrak{g} = \mathfrak{X}_{\mathrm{ex}}(M,\omega)$ of exact divergence-free vector fields. The corresponding group extensions are the *Ismagilov central extensions* [14, Sec. 25.3]. In [13], they were constructed for integral volume forms ω using a prequantum line bundle over the (infinite dimensional) nonlinear Grassmannian $\mathrm{Gr}_{n-2}(M)$ of compact, oriented submanifolds of codimension 2.

The main result of this paper is Theorem 5.4, where we prove a generalization of the above results for arbitrary k. If G is a connected Fréchet–Lie group with an exact action $\alpha\colon G\times M \to M$ on a compact manifold (M,ω), and if ω is an integral $(k+1)$-form on M, then the image of $H_{k-1}(M,\mathbb{Z})$ in $H^{k-1}(M,\mathbb{R})^*$ yields a lattice of integrable Lie algebra extensions of $\mathfrak{g}$ by $\mathbb{R}$.

In fact, the requirements that M be compact and that G be connected can both be relaxed. For classes in $H_{k-1}(M,\mathbb{Z})$ that can be represented by a closed, oriented submanifold S, we show in Corollary 5.5 that the compactness assumption on M is not needed. And rather than requiring G to be connected, we require the weaker property that every $\alpha_g\colon M \to M$ is homotopic to the identity.

Moreover, in this setting, the appropriate analogue of an exact action is expressed in terms of *differential characters*. Differential characters (or Cheeger–Simons characters) are generalizations of principal circle bundles with connection. In particular, the *curvature* of a differential character h of degree k is a closed, integral $(k+1)$-form ω. We give a brief overview of the theory of differential characters in

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Section 2 and refer the reader to [3, 8] for further details. Rather than requiring the action of G to be exact, we require the condition that $\alpha_g^* h = h$ for all $g \in G$, where h is a given differential character with curvature ω. In general, different characters with the same curvature may give rise to different group extensions with the same Lie algebra extension.

The above results rely heavily on the machinery of *transgression* for differential characters. Let S be a compact, oriented manifold, and consider the canonical evaluation and projection maps:

$$\begin{tikzcd} M & C^\infty(S,M) \times S \arrow[l, "\mathrm{ev}"'] \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & S \\ & C^\infty(S,M). & \end{tikzcd}$$

On the level of differential forms, the *hat product* $\alpha \widehat{*} \beta$ of $\alpha \in \Omega^{\ell+1}(M)$ and $\beta \in \Omega^{r+1}(S)$ is the differential form on the mapping space $C^\infty(S,M)$ of degree $2+\ell+r-\dim S$ obtained by pulling back α and β to $C^\infty(S,M) \times S$, and subsequently integrating over the fiber S, see [34]. If α and β are closed, and the degrees satisfy $\ell + r = \dim S$, then the hat product $\alpha \widehat{*} \beta$ yields a closed 2-form on $C^\infty(S,M)$. In this case, every smooth action of a Lie algebra $\mathfrak{g}$ on $C^\infty(S,M)$ preserving $\alpha \widehat{*} \beta$ gives rise to a continuous 2-cocycle τ on $\mathfrak{g}$ defined by

$$\tau(X,Y) = (\alpha \widehat{*} \beta)_\Phi (X_\Phi, Y_\Phi), \tag{5}$$

where $\Phi \in C^\infty(S,M)$, $X, Y \in \mathfrak{g}$, and X_Φ denotes the value of the fundamental vector field at Φ generated by the action of $X \in \mathfrak{g}$ on $C^\infty(S,M)$.

To construct central Lie group extensions, we refine the above setting by additional data. Thus, we assume that α and β are curvature forms of differential characters $h \in \widehat{H}^\ell(M,\mathbb{T})$ and $g \in \widehat{H}^r(S,\mathbb{T})$, respectively. Such differential characters exist if and only if the closed forms α and β are integral. The hat product $h \widehat{*} g$ of the differential characters h and g is a differential character on $C^\infty(S,M)$ defined in complete analogy to the hat product of differential forms (cf. equation (22)). In particular, the curvature of $h \widehat{*} g$ is $\alpha \widehat{*} \beta$. If $\ell + r = \dim S$, then $h \widehat{*} g$ is a differential character of degree 1, which can be represented as the holonomy of a principal circle bundle $\mathcal{P}$ over $C^\infty(S,M)$, endowed with a connection ∇ whose curvature is $\alpha \widehat{*} \beta$ (see Theorem 4.2). Consider a smooth action of a Fréchet–Lie group G on $C^\infty(S,M)$, and assume that this action preserves $h \widehat{*} g$. In particular, the G-action leaves $\alpha \widehat{*} \beta$ invariant. If, furthermore, G preserves the connected component $C^\infty(S,M)_\Phi$ of a map Φ, then we

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obtain a central extension

$$\mathbb{T} \to \widehat{G} \to G,$$

where $\widehat{G}$ is the group of automorphisms of $(\mathcal{P}, \nabla)$ covering the G-action on $C^\infty(S, M)_\Phi$. This is a *smooth* central extension of Lie groups, and the associated Lie algebra extension is characterized by the cohomology class of the cocycle τ of (5).

In Theorem 5.1, we apply this to the case where the G-action on $C^\infty(S, M)$ is induced by an action on the finite-dimensional manifolds S or M, preserving all the relevant data. This immediately yields Theorem 5.4 on lattices of integrable Lie algebra extensions. The more involved example of the action of the current group $C^\infty(S, G)$ on $C^\infty(S, M)$ will be treated elsewhere [9].

Notation:

We write $\mathbb{T} = \{z \in \mathbb{C}^\times : |z| = 1\}$ for the circle group that we identify with $\mathbb{R}/\mathbb{Z}$. Accordingly, we write $\exp_{\mathbb{T}}(t) = e^{2\pi i t}$ for its exponential function. For a smooth manifold M, we call a curve $(\varphi_t)_{0 \le t \le 1}$ in $\mathrm{Diff}(M)$ *smooth* if the map

$$[0, 1] \times M \to M^2, \quad (t, m) \mapsto (\varphi_t(m), \varphi_t^{-1}(m))$$

is smooth. We write $\mathrm{Diff}(M)_0 \subseteq \mathrm{Diff}(M)$ for the normal subgroup of those diffeomorphisms φ for which there exists a smooth curve from id_M to φ. All finite dimensional manifolds M are required to be 2nd countable.

2 Differential Characters and Line Bundles

We introduce differential characters following [8] and [3]. In this section, M denotes a locally convex smooth manifold for which the de Rham isomorphism holds. (See [18, Thm. 34.7] for a de Rham Theorem in this context and sufficient criteria for it to hold.) Let $C_k(M)$ be the group of smooth singular k-chains, and let $Z_k(M)$ and $B_k(M)$ denote the groups of k-cycles and k-boundaries, so that $H_k(M) := Z_k(M)/B_k(M)$ is the k-th smooth singular homology group.

A *differential character (Cheeger–Simons character) of degree* k is a group homomorphism $h : Z_k(M) \to \mathbb{T}$ for which there exists a differential form $\omega \in \Omega^{k+1}(M)$ such that $h(\partial\sigma) = \exp_{\mathbb{T}}\left(\int_\sigma \omega\right)$ holds for every $\sigma \in C_{k+1}(M)$. Then ω is uniquely determined by h. It is called the *curvature of* h and will be denoted by $\mathrm{curv}(h)$. We

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write

$$\widehat{H}^k(M,\mathbb{T}) \subseteq \mathrm{Hom}(Z_k(M),\mathbb{T})$$

for the group of differential characters of degree k. (In [3], this group is denoted $\widehat{H}^{k+1}(M,\mathbb{Z})$. In this sense, our notations are compatible, although the degree is shifted by 1. Our convention follows the original one introduced by Cheeger and Simons in [8].) It contains the cohomology group $H^k(M,\mathbb{T}) \cong \mathrm{Hom}(H_k(M),\mathbb{T}) \subseteq \mathrm{Hom}(Z_k(M),\mathbb{T})$ as the subgroup of differential characters that vanish on k-boundaries. This is the subgroup of differential characters with curvature zero. For $\omega = \mathrm{curv}(h)$, the relation $1 = h(\partial\varsigma) = \exp_{\mathbb{T}}\left(\int_\varsigma \omega\right)$ for every $\varsigma \in Z_{k+1}(M)$ implies that ω belongs to the abelian group

$$\Omega^{k+1}_{\mathbb{Z}}(M) := \left\{\omega \in \Omega^{k+1}(M) : \int_{Z_{k+1}(M)} \omega \subseteq \mathbb{Z}\right\}$$

of forms with integral periods. Note that such forms are automatically closed. We thus get an exact sequence

$$0 \to H^k(M,\mathbb{T}) \hookrightarrow \widehat{H}^k(M,\mathbb{T}) \xrightarrow{\mathrm{curv}} \Omega^{k+1}_{\mathbb{Z}}(M) \to 0. \tag{6}$$

A differential character h of degree k also determines a *characteristic class* $\mathrm{ch}(h) \in H^{k+1}(M,\mathbb{Z})$. In order to define $\mathrm{ch}(h)$, let $\widetilde{h}\colon Z_k(M) \to \mathbb{R}$ be a homomorphism that lifts h (such a homomorphism exists because $Z_k(M)$ is a free abelian group). Consider the cocycle $\widehat{h}\colon C_{k+1}(M) \to \mathbb{Z}$ given by

$$\widehat{h}(c) := \int_c \mathrm{curv}(h) - \widetilde{h}(\partial c)$$

and define $\mathrm{ch}(h)$ to be the class of $\widehat{h}$ in $H^{k+1}(M,\mathbb{Z})$. The image of $\mathrm{ch}(h)$ in $H^{k+1}(M,\mathbb{R})$ coincides with the class of $\mathrm{curv}(h)$, when we identify $\mathbb{R}$-valued cohomology and de Rham cohomology via the de Rham isomorphism. In particular, $\mathrm{curv}(h)$ is exact if and only if the image of $\mathrm{ch}(h)$ in $H^{k+1}(M,\mathbb{R})$ is trivial. To each differential form $\alpha \in \Omega^k(M)$ one assigns a differential character $i(\alpha)$ defined by

$$i(\alpha)(c) := \exp_{\mathbb{T}}\left(\int_c \alpha\right) \tag{7}$$

for $c \in Z_k(M)$. Note that

$$\mathrm{curv}(i(\alpha)) = \mathrm{d}\alpha. \tag{8}$$

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The kernel of i is $\Omega^k_{\mathbb{Z}}(M)$. We thus obtain an exact sequence, cf. [3, Eq. 30],

$$0 \to \Omega^k_{\mathbb{Z}}(M) \to \Omega^k(M) \xrightarrow{\ i\ } \widehat{H}^k(M,\mathbb{T}) \xrightarrow{\ \mathrm{ch}\ } H^{k+1}(M,\mathbb{Z}) \to 0. \tag{9}$$

Differential characters of degree 1 can be realized as the holonomy of a principal circle bundle $P \to M$ with connection form $\theta \in \Omega^1(P)$. Indeed, the holonomy map h assigns to each piecewise smooth 1-cycle $c \in Z_1(M)$ an element $h(c) \in \mathbb{T}$. If the connection has curvature $\omega \in \Omega^2(M)$, then $h(\partial\sigma) = \exp_{\mathbb{T}}(\int_\sigma \omega)$ for all $\sigma \in Z_2(M)$, so that $h \in \widehat{H}^1(M,\mathbb{T})$ is a differential character whose curvature is the same as that of the connection, $\mathrm{curv}(h) = \omega$. The class $\mathrm{ch}(h) \in H^2(M,\mathbb{Z})$, which does not depend on θ, is the *1st Chern class* of the bundle P. In this paper, we will often need the following well-known result.

Proposition 2.1. The map $(P,\theta) \mapsto h$ that assigns to a principal circle bundle P with connection θ its holonomy map h defines an isomorphism from the group of equivalence classes of pairs (P,θ) modulo bundle automorphisms onto the group $\widehat{H}^1(M,\mathbb{T})$ of differential characters of degree 1.

This can be derived from the fact that principal circle bundles with connection are classified by Deligne cohomology [4, Thm. 2.2.12], which is an alternative model for differential cohomology, cf. [3, Sec. 5.2]. We give a direct proof in Appendix B.

3 Stabilizers of a Differential Character

For any manifold M, the right action of the diffeomorphism group $\mathrm{Diff}(M)$ on $\widehat{H}^k(M,\mathbb{T})$ by pull-back

$$\varphi^* h(c) := h(\varphi \circ c), \quad \text{for} \varphi \in \mathrm{Diff}(M), \quad h \in \widehat{H}^k(M,\mathbb{T}) \tag{10}$$

extends to the exact sequence (6) of abelian groups:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^k(M,\mathbb{T}) & \longrightarrow & \widehat{H}^k(M,\mathbb{T}) & \xrightarrow{\mathrm{curv}} & \Omega^{k+1}_{\mathbb{Z}}(M) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi^*} & & \downarrow{\scriptstyle \varphi^*} & & \downarrow{\scriptstyle \varphi^*} & & \\
0 & \longrightarrow & H^k(M,\mathbb{T}) & \longrightarrow & \widehat{H}^k(M,\mathbb{T}) & \xrightarrow{\mathrm{curv}} & \Omega^{k+1}_{\mathbb{Z}}(M) & \longrightarrow & 0.
\end{array}$$

Let $\mathrm{Diff}(M)_0$ be the subgroup of $\mathrm{Diff}(M)$ consisting of those diffeomorphisms φ for which there exists a smooth curve from id_M to φ. Note that the action of $\mathrm{Diff}(M)_0$ on $H^k(M,\mathbb{T})$ is

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trivial: as $\varphi \circ c - c$ is a boundary for every $c \in Z_k(M)$, we find $(\varphi^* a - a)(c) = a(\varphi \circ c - c) = 1$ for all $a \in H^k(M, \mathbb{T})$.

Denote the stabilizer group of $\omega \in \Omega^{k+1}_{\mathbb{Z}}(M)$ by

$$\mathrm{Diff}(M, \omega) := \{\varphi \in \mathrm{Diff}(M) : \varphi^* \omega = \omega\}, \tag{11}$$

and the stabilizer Lie algebra by

$$\mathfrak{X}(M, \omega) := \{X \in \mathfrak{X}(M) : L_X \omega = 0\}. \tag{12}$$

Remark 3.1. Of particular relevance is the case where $\omega \in \Omega^{k+1}(M)$ is nondegenerate, in the sense that $\omega^\flat \colon T_m M \to \mathrm{Alt}^k(T_m M, \mathbb{R}), v \mapsto i_v \omega_m$ is injective for all $m \in M$. For $k = 1$ and $k = \dim(M) - 1$, this leads to the groups of *symplectic* and *volume-preserving* transformations, respectively. If M is compact, then the symplectic and volume-preserving diffeomorphism groups $\mathrm{Diff}(M, \omega)$ are Fréchet Lie groups, with Lie algebra $\mathfrak{X}(M, \omega)$ consisting of *symplectic* or *divergence free* vector fields [18, Thm. 43.7, 43.12].

3.1 Stabilizer groups

For a differential character $h \in \widehat{H}^k(M, \mathbb{T})$, we denote its stabilizer by

$$\mathrm{Diff}(M, h) := \{\varphi \in \mathrm{Diff}(M) : \varphi^* h = h\}. \tag{13}$$

Remark 3.2. Let $h \in \widehat{H}^1(M, \mathbb{T})$ be the differential character defined by the holonomy of the principal $\mathbb{T}$-bundle $(P, \theta) \to (M, \omega)$ with curvature ω. Then $\varphi \in \mathrm{Diff}(M)$ satisfies $\varphi^* h = h$ if and only if, for every smooth loop γ in M, the holonomy of the loop γ coincides with the holonomy of $\varphi \circ \gamma$. Since this is equivalent to the existence of a connection-preserving lift $\widetilde{\varphi} \in \mathrm{Aut}(P, \theta)$ by [26, Thm. 2.7], one can view $\mathrm{Diff}(M, h)$ as the group of *liftable* diffeomorphisms if $k = 1$.

Consider the left action $(\varphi, h) \mapsto \varphi^{-1*} h$ of $\mathrm{Diff}(M, \omega)$ on $\widehat{H}^k(M, \mathbb{T})$. By (6), the preimage $\mathrm{curv}^{-1}(\omega)$ is a principal homogeneous space (a torsor) for the abelian group $H^k(M, \mathbb{T})$. For every h with $\mathrm{curv}(h) = \omega$, we thus get a 1-cocycle

$$\mathrm{Flux}_h \colon \mathrm{Diff}(M, \omega) \to H^k(M, \mathbb{T}), \quad \mathrm{Flux}_h(\varphi) = (\varphi^{-1})^* h - h\,. \tag{14}$$

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Note that the cohomology class of this cocycle Flux_h in $H^1(\mathrm{Diff}(M,\omega), H^k(M,\mathbb{T}))$ is independent of the choice of h. The kernel of Flux_h is the subgroup $\mathrm{Diff}(M,h)$, so that we obtain an exact sequence

$$1 \to \mathrm{Diff}(M,h) \to \mathrm{Diff}(M,\omega) \xrightarrow{\mathrm{Flux}_h} H^k(M,\mathbb{T})\,, \tag{15}$$

where Flux_h is only a cocycle and not necessarily a group homomorphism. Since $\mathrm{Diff}(M)_0$ acts trivially on $H^k(M,\mathbb{T})$, the restriction of Flux_h to the intersection $\mathrm{Diff}(M,\omega)\cap\mathrm{Diff}(M)_0$ is a group homomorphism. As this homomorphism does not depend on the choice of h with $\mathrm{curv}(h) = \omega$, we denote the homomorphism by Flux_ω, and its kernel by

$$\overline{\mathrm{Diff}}_{\mathrm{ex}}(M,\omega) := \mathrm{Diff}(M,h) \cap \mathrm{Diff}(M)_0.$$

Since $\mathrm{Diff}(M)_0$ leaves the characteristic class of h invariant, the image of Flux_ω is contained in the kernel of $\mathrm{ch}\,|_{H^k(M,\mathbb{T})}$, which can be identified with the Jacobian torus

$$J^k(M) := H^k(M,\mathbb{R})/H^k(M,\mathbb{R})_{\mathbb{Z}} \cong \mathrm{Hom}(H_k(M),\mathbb{R})/\,\mathrm{Hom}(H_k(M),\mathbb{Z})$$

the quotient of $H^k(M,\mathbb{R})$ by the image of $H^k(M,\mathbb{Z})$. Summarizing the above discussion, we obtain:

Proposition 3.3. For any manifold M and $\omega \in \Omega^{k+1}_{\mathbb{Z}}(M)$, we have an exact sequence *of groups*

$$1 \to \overline{\mathrm{Diff}}_{\mathrm{ex}}(M,\omega) \longrightarrow \mathrm{Diff}(M,\omega) \cap \mathrm{Diff}(M)_0 \xrightarrow{\mathrm{Flux}_\omega} J^k(M)\,. \tag{16}$$

Remark 3.4. To define the group $\mathrm{Diff}_{\mathrm{ex}}(M,\omega)$ of *exact ω-preserving diffeomorphisms*, note that by a straightforward calculation, the restriction of Flux_ω to $\mathrm{Diff}(M,\omega)_0$ satisfies

$$\mathrm{Flux}_\omega(\varphi) = \left[\int_0^1 i_{\delta^l\varphi_t}\omega\,\mathrm{d}t\right],$$

where $(\varphi_t)_{0\le t\le 1}$ is a smooth path in $\mathrm{Diff}(M,\omega)$ from $\varphi_0 = \mathrm{id}_M$ to $\varphi_1 = \varphi$. This shows that the restriction of Flux_ω to $\mathrm{Diff}(M,\omega)_0$ recovers the flux homomorphism defined in [2, 5], cf. also [26, Prop. 1.8]. If we define the exact ω-preserving diffeomorphism group by

$$\mathrm{Diff}_{\mathrm{ex}}(M,\omega) := \overline{\mathrm{Diff}}_{\mathrm{ex}}(M,\omega) \cap \mathrm{Diff}(M,\omega)_0 = \mathrm{Diff}(M,h) \cap \mathrm{Diff}(M,\omega)_0, \tag{17}$$

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then we obtain the exact sequence of groups

$$1 \to \mathrm{Diff}_{\mathrm{ex}}(M,\omega) \longrightarrow \mathrm{Diff}(M,\omega)_0 \xrightarrow{\mathrm{Flux}_\omega} J^k(M)\,. \tag{18}$$

The elements of $\mathrm{Diff}_{\mathrm{ex}}(M,\omega)$ are called exact ω-preserving diffeomorphisms. In general, the group $\overline{\mathrm{Diff}}_{\mathrm{ex}}(M,\omega)$ may be strictly larger than the group $\mathrm{Diff}_{\mathrm{ex}}(M,\omega)$ of exact ω-preserving diffeomorphisms.

3.2 Stabilizer Lie algebras

We now continue with the stabilizers at the infinitesimal level. The Lie algebra homomorphism

$$\mathrm{flux}_\omega : \mathfrak{X}(M,\omega) \to H^k(M,\mathbb{R}), \quad \mathrm{flux}_\omega(X) = [i_X\omega] \tag{19}$$

is the *infinitesimal flux* cocycle whose kernel is the ideal

$$\mathfrak{X}_{\mathrm{ex}}(M,\omega) := \{X \in \mathfrak{X}(M,\omega) : i_X\omega \text{ is exact}\}.$$

If $h \in \widehat{H}^k(M,\mathbb{T})$ has curvature ω, then the groups $\mathrm{Diff}(M,h)$, $\overline{\mathrm{Diff}}_{\mathrm{ex}}(M,\omega)$, and $\mathrm{Diff}_{\mathrm{ex}}(M,\omega)$ have $\mathfrak{X}_{\mathrm{ex}}(M,\omega)$ as their Lie algebra in the sense that, for every smooth curve $(\varphi_t)_{t\in[0,1]}$ in $\mathrm{Diff}(M)$ with $\varphi_0 = \mathrm{id}_M$, the curve (φ_t) is contained in the group if and only if its logarithmic derivative

$$\delta^l\varphi_t := T(\varphi_t)^{-1}\tfrac{d}{dt}\varphi_t \in \mathfrak{X}(M) \tag{20}$$

takes values in $\mathfrak{X}_{\mathrm{ex}}(M,\omega)$. Accordingly, the exact sequence of Lie algebras associated to (15), (16), and (18) is

$$0 \to \mathfrak{X}_{\mathrm{ex}}(M,\omega) \longrightarrow \mathfrak{X}(M,\omega) \xrightarrow{\mathrm{flux}_\omega} H^k(M,\mathbb{R})\,. \tag{21}$$

Proposition 3.5. Let $h \in \widehat{H}^k(M,\mathbb{T})$ be a differential character with curvature ω. All three groups $\mathrm{Diff}(M,h)$, $\overline{\mathrm{Diff}}_{\mathrm{ex}}(M,\omega)$, and $\mathrm{Diff}_{\mathrm{ex}}(M,\omega)$ have the same smooth arc-component and the same Lie algebra $\mathfrak{X}_{\mathrm{ex}}(M,\omega)$.

Remark 3.6. For manifolds M with trivial k-th homology, the relations between the above groups simplify. If $H_k(M) = \{0\}$, then $\mathrm{Diff}(M,h) = \mathrm{Diff}(M,\omega)$, as the last term in (15) vanishes. If $H_k(M)$ is a torsion group, then $J^k(M) = 0$, so $\overline{\mathrm{Diff}}_{\mathrm{ex}}(M,\omega) = \mathrm{Diff}(M,\omega) \cap \mathrm{Diff}(M)_0$ by (16), and $\mathrm{Diff}_{\mathrm{ex}}(M,\omega) = \mathrm{Diff}(M,\omega)_0$ by (18).

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Remark 3.7. If ω is nondegenerate and $k = 1$ or $k = n - 1$, then the infinitesimal flux homomorphism flux_ω is surjective.

(i) For $k = 1$, the form ω is symplectic, $\mathfrak{X}_{\mathrm{ex}}(M, \omega)$ is the Lie algebra $\mathfrak{X}_{\mathrm{ham}}(M, \omega)$ of *Hamiltonian vector fields*, and the group $\mathrm{Ham}(M, \omega)$ of *Hamiltonian diffeomorphisms* coincides with the identity component $\mathrm{Diff}_{\mathrm{ex}}(M, \omega)_0$.

(ii) For $k = n - 1$, ω is a volume form, so we get the Lie algebra of *exact divergence free vector fields* $\mathfrak{X}_{\mathrm{ex}}(M, \omega)$, and the group of *exact volume-preserving diffeomorphisms* $\mathrm{Diff}_{\mathrm{ex}}(M, \omega)_0$.

3.3 Fréchet–Lie groups

If M is a compact manifold with symplectic form ω, then the group $\mathrm{Diff}(M, \omega)$ of symplectomorphisms is a Fréchet–Lie group acting smoothly on M. If $\mu \in \Omega^d(M)$ is a volume form, then the same holds for the group $\mathrm{Diff}(M, \mu)$ of volume-preserving diffeomorphisms [18, §43]. From the following proposition, we see in particular that $\mathrm{Ham}(M, \omega) \subseteq \mathrm{Diff}(M, \omega)_0$ and $\mathrm{Diff}_{\mathrm{ex}}(M, \mu) \subseteq \mathrm{Diff}(M, \mu)_0$ are embedded Fréchet–Lie subgroups.

Proposition 3.8. Let M be a compact manifold, and let $h \in \widehat{H}^k(M, \mathbb{T})$ be a differential character with curvature $\omega \in \Omega^{k+1}_{\mathbb{Z}}(M)$. Let G be a Fréchet–Lie group with Lie algebra $\mathfrak{g}$, and let $i\colon G \to \mathrm{Diff}(M, \omega)$ be a group homomorphism that induces a smooth action $G \times M \to M$ and hence a Lie algebra homomorphism $i_*\colon \mathfrak{g} \to \mathfrak{X}(M, \omega)$. Then $G_{\mathrm{ex}} := \{g \in G : \mathrm{Flux}_h(i(g)) = 0\}$ is an embedded Lie subgroup of G with Lie algebra $\mathfrak{g}_{\mathrm{ex}} := \{\xi \in \mathfrak{g} : [i_{i_*(\xi)}\omega] = 0\}$.

Proof. Since G acts smoothly on M, the homomorphism $i\colon G \to \mathrm{Diff}(M, \omega)$ is a morphism of diffeological spaces. Since the same holds for the flux cocycle $\mathrm{Flux}_h\colon \mathrm{Diff}(M, \omega) \to H^k(M, \mathbb{T})$, the cocycle $\mathrm{Flux}_h \circ i\colon G \to H^k(M, \mathbb{T})$ is a morphism of diffeological spaces as well. Since M is compact, $H^k(M, \mathbb{T})$ is a finite dimensional Lie group. In particular, G and $H^k(M, \mathbb{T})$ are both objects in the category of Fréchet manifolds, which constitutes a full subcategory of the category of diffeological spaces [20]. It follows that $\mathrm{Flux}_h \circ i\colon G \to H^k(M, \mathbb{T})$ is smooth, with derivative $\mathrm{flux}_\omega \circ i_*\colon \mathfrak{g} \to H^k(M, \mathbb{R})$ at the identity. Since $H^k(M, \mathbb{R})$ is finite dimensional, we conclude from Glöckner's Implicit Function Theorem [12] (or more precisely, from the Regular Value Theorem [25, Thm. III.11] derived from this) that the kernel

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G_{ex} of $\mathrm{Flux}_h \circ i$ is a split embedded submanifold of G. It is a Lie subgroup, because $\mathrm{Flux}_h \circ i$ is a cocycle. The corresponding Lie algebra $\mathfrak{g}_{\mathrm{ex}}$ is the kernel of $\mathrm{flux}_\omega \circ i_*$. ■

4 Transgression of Differential Characters

Let S be a compact, oriented manifold, and let M be a finite dimensional manifold. We now describe a natural way to combine differential characters from S and from M to a differential character on the mapping space $C^\infty(S,M)$, which carries a natural Fréchet manifold structure [18]. For this, we consider the canonical evaluation and projection maps:

$$\begin{tikzcd} M & C^\infty(S,M) \times S \arrow[l, "\mathrm{ev}"'] \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & S \\ & C^\infty(S,M). & \end{tikzcd}$$

We define the hat product $h \mathbin{\widehat{*}} g$ of $h \in \widehat{H}^\ell(M,\mathbb{T})$ and $g \in \widehat{H}^r(S,\mathbb{T})$ as the differential character on $C^\infty(S,M)$ of degree $1+\ell+r-\dim S$ obtained by pulling back h and g to $C^\infty(S,M)\times S$ and integrating over the fiber S:

$$h \mathbin{\widehat{*}} g = \fint_S \mathrm{ev}^* h * \mathrm{pr}_2^* g. \tag{22}$$

Here, $*$ denotes the product $\widehat{H}^\ell(Q,\mathbb{T})\times\widehat{H}^r(Q,\mathbb{T}) \to \widehat{H}^{\ell+r+1}(Q,\mathbb{T})$ of differential characters, see [3, Def. 28], and $\fint_S\colon \widehat{H}^\star(Q\times S,\mathbb{T}) \to \widehat{H}^{\star-\dim S}(Q,\mathbb{T})$ denotes fiber integration, see [3, Def. 38]. Since the curvature map intertwines these operations with the wedge product and with fiber integration of differential forms, respectively, we obtain

$$\mathrm{curv}(h \mathbin{\widehat{*}} g) = \fint_S \mathrm{ev}^* \mathrm{curv}(h) \wedge \mathrm{pr}_2^* \mathrm{curv}(g). \tag{23}$$

This expression coincides with the hat product $\mathrm{curv}\, h \mathbin{\widehat{*}} \mathrm{curv}\, g$ of ordinary differential forms as introduced in [34].

Note that $\mathrm{Diff}(M)$ acts on $C^\infty(S,M)$ by $A(\phi)(f) = \phi\circ f$ and that $\mathrm{Diff}(S)$ acts by $B(\psi)(f) = f\circ\psi^{-1}$. Let $\mathrm{Diff}(S)_+$ be the subgroup of $\mathrm{Diff}(S)$ that consists of orientation-preserving diffeomorphisms.

Proposition 4.1. The hat product is equivariant with respect to $\mathrm{Diff}(M)$ and $\mathrm{Diff}(S)_+$, that is, $A(\phi)^*(h \mathbin{\widehat{*}} g) = (\phi^* h) \mathbin{\widehat{*}} g$ and $B(\psi)^*(h \mathbin{\widehat{*}} g) = h \mathbin{\widehat{*}} (\psi^* g)$.

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Proof. Note that fiber integration as well as the product of differential characters are natural operations [3, Def. 28 and 38]. The 1st equality follows from commutativity of the diagram

$$\begin{tikzcd} M & C^\infty(S,M)\times S \arrow[l, "\mathrm{ev}"'] \arrow[r, "\mathrm{pr}_2"] \arrow[d, "A(\varphi)\times\mathrm{id}"] & S \arrow[d, "\mathrm{id}"] \\ M \arrow[d, "\varphi"] & C^\infty(S,M)\times S \arrow[l, "\mathrm{ev}"'] \arrow[r, "\mathrm{pr}_2"] & S\,. \\ M & & \end{tikzcd} \tag{24}$$

The 2nd equality is derived from a similar commutative diagram

$$\begin{tikzcd} M & C^\infty(S,M)\times S \arrow[l, "\mathrm{ev}"'] \arrow[r, "\mathrm{pr}_2"] \arrow[d, "B(\psi)\times\psi"] & S \arrow[d, "\psi"] \\ M \arrow[d, "\mathrm{id}"] & C^\infty(S,M)\times S \arrow[l, "\mathrm{ev}"'] \arrow[r, "\mathrm{pr}_2"] & S\,, \\ M & & \end{tikzcd} \tag{25}$$

together with the invariance of fiber integration under orientation-preserving diffeomorphisms. ■

Assume the degrees satisfy the relation $\ell + r = \dim S$. Then, by Proposition 2.1, the resulting differential character $h \widehat{*} g \in \widehat{H}^1(C^\infty(S,M),\mathbb{T})$ can be represented by a circle bundle $\mathcal{P}$ over $C^\infty(S,M)$ with connection ∇. From (23), one sees that the curvature of ∇ is $\alpha \widehat{*} \beta \in \Omega^2_{\mathbb{Z}}(C^\infty(S,M))$, where $\alpha = \mathrm{curv}(h)$ and $\beta = \mathrm{curv}(g)$. By Proposition 4.1, the action $A(\phi)$ of $\phi \in \mathrm{Diff}(M,h)$ on $C^\infty(S,M)$ preserves the differential character $h \widehat{*} g$ and thus the holonomy of $(\mathcal{P},\nabla)$. The same holds for the action $B(\psi)$ of $\psi \in \mathrm{Diff}_+(S,g)$. By Proposition 2.1, the pull-back bundles $A(\phi)^*(\mathcal{P},\nabla)$ and $B(\psi)^*(\mathcal{P},\nabla)$ are isomorphic to $(\mathcal{P},\nabla)$, so there exist connection-preserving bundle automorphisms $\widehat{A}(\phi)\colon \mathcal{P} \to \mathcal{P}$ and $\widehat{B}(\psi)\colon \mathcal{P} \to \mathcal{P}$ that cover $A(\phi)$ and $B(\psi)$. The following theorem summarizes the discussion so far.

Theorem 4.2. Let M and S be finite dimensional manifolds, with S compact and oriented. Let $h \in \widehat{H}^\ell(M,\mathbb{T})$ and $g \in \widehat{H}^r(S,\mathbb{T})$ be differential characters with curvature forms $\alpha \in \Omega^{\ell+1}_{\mathbb{Z}}(M)$ and $\beta \in \Omega^{r+1}_{\mathbb{Z}}(S)$, respectively. Assume that $\ell + r = \dim S$. Then the differential character $h \widehat{*} g \in \widehat{H}^1(C^\infty(S,M),\mathbb{T})$ is the holonomy of a circle bundle $\mathcal{P}$ over $C^\infty(S,M)$ with connection ∇ and curvature $\alpha \widehat{*} \beta \in \Omega^2_{\mathbb{Z}}(C^\infty(S,M))$. Moreover, for every $\phi \in \mathrm{Diff}(M,h)$ and $\psi \in \mathrm{Diff}_+(S,g)$, there exist bundle automorphisms $\widehat{A}(\phi)$ and $\widehat{B}(\psi)$ in $\mathrm{Aut}(\mathcal{P},\nabla)$ that cover $A(\phi), B(\psi) \in \mathrm{Diff}(C^\infty(S,M))$.

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5 Central Extensions of $\mathrm{Diff}_{\mathrm{ex}}(M,\alpha)$ and $\mathrm{Diff}_{\mathrm{ex}}(S,\beta)$

We continue in the setting of Theorem 4.2 and use the bundle $(\mathcal{P},\nabla)$ to construct central extensions of $\mathrm{Diff}_{\mathrm{ex}}(M,\alpha)$ and $\mathrm{Diff}_{\mathrm{ex}}(S,\beta)$. For $\Phi \in C^\infty(S,M)$, consider the restriction $\mathcal{P}_\Phi \to C^\infty(S,M)_\Phi$ of $\mathcal{P} \to C^\infty(S,M)$ to the connected component of Φ. Denote by $\mathrm{Diff}(M,h,[\Phi]) \subseteq \mathrm{Diff}(M,h)$ and $\mathrm{Diff}(S,g,[\Phi]) \subseteq \mathrm{Diff}_+(S,g)$ the subgroups that fix the homotopy class $[\Phi] \in \pi_0(C^\infty(S,M))$. By continuity, these contain $\mathrm{Diff}_{\mathrm{ex}}(M,\alpha) \subseteq \mathrm{Diff}(M,h)$ and $\mathrm{Diff}_{\mathrm{ex}}(S,\beta) \subseteq \mathrm{Diff}(S,g)$. The pull-back of the group extension

$$\mathbb{T} \to \mathrm{Aut}(\mathcal{P}_\Phi,\nabla) \to \mathrm{Diff}(C^\infty(S,M)_\Phi)$$

along the action $A\colon \mathrm{Diff}(M,h,[\Phi]) \to \mathrm{Diff}(C^\infty(S,M)_\Phi)$ yields a central extension

$$\mathbb{T} \to \widehat{\mathrm{Diff}}(M,h,[\Phi]) \to \mathrm{Diff}(M,h,[\Phi])\,. \tag{26}$$

Similarly, the pull-back along $B\colon \mathrm{Diff}(S,g,[\Phi]) \to \mathrm{Diff}(C^\infty(S,M)_\Phi)$ yields the central extension

$$\mathbb{T} \to \widehat{\mathrm{Diff}}(S,g,[\Phi]) \to \mathrm{Diff}(S,g,[\Phi])\,. \tag{27}$$

5.1 Smooth Lie group extensions

We would like to say that these are *smooth* extensions of Lie groups. But, unfortunately, we are not aware of any Fréchet–Lie group structure on $\mathrm{Diff}(M,h,[\Phi])$ and $\mathrm{Diff}(S,g,[\Phi])$ except when α and β are either symplectic or volume forms and M is compact.

In general, we therefore formulate the smoothness requirement as follows. Consider a smooth action of a Fréchet–Lie group G on M that preserves h and $[\Phi]$; similarly, let H be a Fréchet–Lie group that acts smoothly on S and preserves g as well as $[\Phi]$. We denote the corresponding group homomorphisms by $i_G\colon G \to \mathrm{Diff}(M,h,[\Phi])$ and $i_H\colon H \to \mathrm{Diff}(S,g,[\Phi])$. By Theorem A.1, the pull-back along i_G and i_H of the central extensions (26) and (27) then yields *smooth* Lie group extensions

$$\mathbb{T} \to \widehat{G} \to G \quad \text{and} \quad \mathbb{T} \to \widehat{H} \to H, \tag{28}$$

respectively.

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Let $a\colon \mathfrak{X}_{\mathrm{ex}}(M,\alpha) \to T_\Phi C^\infty(S,M)$ and $b\colon \mathfrak{X}_{\mathrm{ex}}(S,\beta) \to T_\Phi C^\infty(S,M)$ be the infinitesimal versions at Φ of the actions A and B, respectively. That is, we have

$$a(X) = X \circ \Phi, \qquad b(v) = -\Phi_* v. \tag{29}$$

From Theorem 4.2 and equation (23) (cf. also [34, Eq. 2]), one finds that the curvature of ∇ at $\Phi \in C^\infty(S,M)$ is $(\alpha \mathbin{\widehat{*}} \beta)_\Phi \in \mathrm{Alt}^2(T_\Phi C^\infty(S,M))$, with

$$(\alpha \mathbin{\widehat{*}} \beta)_\Phi(U,V) = \int_S \Phi^*\big(i_V i_U(\alpha \circ \Phi)\big) \wedge \beta\,. \tag{30}$$

Here, $\Phi^*\big(i_U i_V(\alpha \circ \Phi)\big)$ is the differential $(r-1)$-form on S assigning to $v_1,\dots,v_{r-1} \in T_s S$ the value

$$\alpha_{\Phi(s)}\big(U(s), V(s), \Phi_* v_1, \dots, \Phi_* v_{r-1}\big).$$

The pull-back along a and b of the curvature $(\alpha \mathbin{\widehat{*}} \beta)_\Phi$ yields the Lie algebra 2-cocycles τ on $\mathfrak{X}_{\mathrm{ex}}(M,\alpha)$ and ν on $\mathfrak{X}_{\mathrm{ex}}(S,\beta)$ given by

$$\tau(X,Y) = \int_S (\Phi^* i_Y i_X \alpha) \wedge \beta\,, \qquad \nu(u,v) = \int_S (i_v i_u \Phi^* \alpha) \wedge \beta\,. \tag{31}$$

For the former identity, we use that, for $U = a(X)$ and $V = a(Y)$, Eq. (29) yields $\Phi^*\big(i_V i_U(\alpha \circ \Phi)\big) = \Phi^* i_Y i_X \alpha$. For the latter, note that the fundamental vector fields $U = b(u)$ and $V = b(v)$ give $\Phi^*\big(i_V i_U(\alpha \circ \Phi)\big) = i_v i_u \Phi^* \alpha$ according to Eq. (29). The Lie algebra cocycles on $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{h} = \mathrm{Lie}(H)$ corresponding to Lie group extensions (28) are given by the pull-back along the Lie algebra actions $i_{\mathfrak{g}}\colon \mathfrak{g} \to \mathfrak{X}_{\mathrm{ex}}(M,\alpha)$ and $i_{\mathfrak{h}}\colon \mathfrak{h} \to \mathfrak{X}_{\mathrm{ex}}(S,\beta)$ of the cocycles τ and ν. In summary, we obtain the following.

Theorem 5.1. Let M and S be finite dimensional manifolds, and let $h \in \widehat{H}^\ell(M,\mathbb{T})$ and $g \in \widehat{H}^r(S,\mathbb{T})$ be differential characters with curvature $\alpha \in \Omega^{\ell+1}_{\mathbb{Z}}(M)$ and $\beta \in \Omega^{r+1}_{\mathbb{Z}}(S)$, respectively. Assume that S is compact and oriented and that $\ell + r = \dim(S)$. Let $i_G\colon G \to \mathrm{Diff}(M,h,[\Phi])$ and $i_H\colon H \to \mathrm{Diff}(S,g,[\Phi])$ be Fréchet–Lie groups that act smoothly on M and S, in such a way that h, g and $[\Phi] \in \pi_0(C^\infty(S,M))$ are preserved. Then the pull-back to G and H of (26) and (27) yields smooth central extensions

$$\mathbb{T} \to \widehat{G} \to G\,, \qquad \mathbb{T} \to \widehat{H} \to H \tag{32}$$

of Fréchet–Lie groups. The corresponding Lie algebra 2-cocycles are given by the pull-back along the infinitesimal action $i_{\mathfrak{g}}\colon \mathfrak{g} \to \mathfrak{X}_{\mathrm{ex}}(M,\alpha)$ and $i_{\mathfrak{h}}\colon \mathfrak{h} \to \mathfrak{X}_{\mathrm{ex}}(S,\beta)$ of the

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continuous Lie algebra 2-cocycles

$$\tau\colon \bigwedge^2 \mathfrak{X}_{\mathrm{ex}}(M,\alpha) \to \mathbb{R}, \quad \tau(X,Y) = \int_S (\Phi^* i_Y i_X \alpha) \wedge \beta\,, \tag{33}$$

$$\nu\colon \bigwedge^2 \mathfrak{X}_{\mathrm{ex}}(S,\beta) \to \mathbb{R}, \quad \nu(u,v) = \int_S (i_v i_u \Phi^* \alpha) \wedge \beta = \int_S \Phi^* \alpha \wedge i_v i_u \beta\,. \tag{34}$$

Note that although the Lie algebra cocycles τ and ν (and, hence, the corresponding Lie algebra extensions) depend only on the forms α and β, the group extensions will in general depend on the choice of integrating differential characters h and g.

5.2 Extensions at the Lie algebra level

Theorem 5.1 yields cocycles τ and ν on the Lie algebras $\mathfrak{X}_{\mathrm{ex}}(M,\alpha)$ and $\mathfrak{X}_{\mathrm{ex}}(S,\beta)$, respectively, that give rise to Lie group extensions. We will now show that the Lie algebra extensions obtained in this way all factor through a common type of central extension.

Let $\Omega^k_{\mathrm{ex}}(M)$ denote the space of exact k-forms, and let

$$\overline{\Omega}^k(M) := \Omega^k(M)/\Omega^k_{\mathrm{ex}}(M)\,.$$

For every closed $(k+1)$-form ω, we define

$$\widehat{\mathfrak{X}}_{\mathrm{ex}}(M,\omega) := \{\big(v,[\psi_v]\big) \in \mathfrak{X}_{\mathrm{ex}}(M,\omega) \times \overline{\Omega}^{k-1}(M) : i_v\omega = \mathrm{d}\psi_v\} \tag{35}$$

and denote the projection on the 1st factor by π. If ω is non-degenerate, then $\widehat{\mathfrak{X}}_{\mathrm{ex}}(M,\omega)$ is canonically isomorphic to $\overline{\Omega}^k(M)$.

Proposition 5.2. Let M be a finite dimensional manifold, and let ω be a closed $(k+1)$-form on M. With the Lie bracket on $\widehat{\mathfrak{X}}_{\mathrm{ex}}(M,\omega)$ defined by

$$\big[\big(v,[\psi_v]\big),\big(w,[\psi_w]\big)\big] := \big([v,w],[i_v i_w \omega]\big) = \big([v,w],[L_v\psi_w]\big)\,, \tag{36}$$

the projection π yields a central extension

$$0 \to H^{k-1}(M,\mathbb{R}) \longrightarrow \widehat{\mathfrak{X}}_{\mathrm{ex}}(M,\omega) \xrightarrow{\pi} \mathfrak{X}_{\mathrm{ex}}(M,\omega) \to 0\,, \tag{37}$$

of Fréchet–Lie algebras.

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Proof. (cf. [23, Thm. 13], [27, Rem. 1.11]) To see that (36) defines an element of $\widehat{\mathfrak{X}}_{\mathrm{ex}}(M,\omega)$, note that $L_v\omega = 0$ and that $i_w\omega$ is closed, so that

$$i_{[v,w]}\omega = L_v i_w\omega - i_w L_v\omega = \mathrm{d}(i_v i_w\omega) + i_v \mathrm{d}(i_w\omega) = \mathrm{d}(i_v i_w\omega)\,.$$

For the 2nd equality in (36), note that $L_v\psi_w = i_v i_w\omega + \mathrm{d}(i_v\psi_w)$.

By the de Rham isomorphism, a $(k-1)$-form γ is exact if and only if it integrates to zero on all closed cycles. It follows that $\Omega^{k-1}_{\mathrm{ex}}(M)$ is a closed subspace of the Fréchet space $\Omega^{k-1}(M)$, and so the quotient $\overline{\Omega}^{k-1}(M)$ is a Fréchet space by [30, Thm. 1.41]. Since $\widehat{\mathfrak{X}}_{\mathrm{ex}}(M,\omega)$ is the kernel of the continuous linear map

$$\mathfrak{X}(M)\times\overline{\Omega}^{k-1}(M)\to\Omega^k(M),\quad (v,[\psi])\mapsto i_v\omega-\mathrm{d}\psi\,,$$

it is a closed subspace of a Fréchet space, hence it is Fréchet itself. The continuity of the bracket and projection follows from the explicit description.

It remains to show that the bracket is a Lie bracket. It is manifestly skew-symmetric. For the Jacobi identity, it suffices to show that the form

$$i_u i_{[v,w]}\omega + i_v i_{[w,u]}\omega + i_w i_{[u,v]}\omega$$

is exact. This expression equals

$$\begin{aligned}&(i_v i_{[w,u]} - i_w i_{[v,u]} - i_{[v,w]} i_u)\,\omega = (i_v L_w - i_w L_v - i_{[v,w]})\,\mathrm{d}\psi_u\\ &= (L_v L_w - L_w L_v - L_{[v,w]})\,\psi_u - \mathrm{d}(i_v L_w - i_w L_v - i_{[v,w]})\,\psi_u\,,\end{aligned}$$

where we used the formulæ $i_{[X,Y]} = L_X i_Y - i_Y L_X$, $i_X\omega = \mathrm{d}\psi_X$, and $L_X\omega = 0$ in the 1st equality, and $L_X\mathrm{d} = \mathrm{d}L_X$ and $i_X\mathrm{d} = L_X - \mathrm{d}i_X$ in the 2nd. Since $L_{[v,w]} = L_v L_w - L_w L_v$, we find

$$i_u i_{[v,w]}\omega + i_v i_{[w,u]}\omega + i_w i_{[u,v]}\omega = \mathrm{d}(i_w L_v - i_v L_w + i_{[v,w]})\,\psi_u\,,$$

so that it defines the zero class in $\overline{\Omega}^{k-1}(M)$ as required. ■

Every continuous linear functional $\lambda \in \widehat{\mathfrak{X}}_{\mathrm{ex}}(M,\omega)^*$ gives rise to a central extension $\mathfrak{X}_{\mathrm{ex}}(M,\omega)\oplus_\lambda\mathbb{R}$, with Lie bracket

$$[v\oplus x, w\oplus y] = [v,w]\oplus\lambda\big(([v,w],[i_v i_w\omega])\big)\,. \tag{38}$$

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The isomorphism class of the extension obtained in this way depends only on the restriction of λ to $H^{k-1}(M,\mathbb{R})$, as the homomorphism $(\pi,\lambda)\colon \widehat{\mathfrak{X}}_{\mathrm{ex}}(M,\omega) \to \mathfrak{X}_{\mathrm{ex}}(M,\omega) \oplus_\lambda \mathbb{R}$ drops to an isomorphism between $\widehat{\mathfrak{X}}_{\mathrm{ex}}(M,\omega)/\ker(\lambda|_{H^{k-1}(M,\mathbb{R})})$ and $\mathfrak{X}_{\mathrm{ex}}(M,\omega) \oplus_\lambda \mathbb{R}$. This gives rise to a map

$$\Xi_\omega\colon H^{k-1}(M,\mathbb{R})^* \to H^2(\mathfrak{X}_{\mathrm{ex}}(M,\omega),\mathbb{R}). \tag{39}$$

Summarizing, the restriction of λ to $H^{k-1}(M,\mathbb{R})$ determines the one-dimensional central extension, while $\lambda \in \widehat{\mathfrak{X}}_{\mathrm{ex}}(M,\omega)^*$ provides this extension with a splitting.

The central extension $\pi\colon \widehat{\mathfrak{X}}_{\mathrm{ex}}(M,\omega) \to \mathfrak{X}_{\mathrm{ex}}(M,\omega)$ is in general not universal. Its significance lies in the fact that it captures the central extensions corresponding to the cocycles (33) and (34) from Theorem 5.1. For the cocycle τ on $\mathfrak{X}_{\mathrm{ex}}(M,\alpha)$, note that $\pi^*\tau = \mathrm{d}_{\mathrm{CE}}\sigma_\beta$, where d_{CE} denotes the differential of the Chevalley–Eilenberg complex, π is the central extension $\pi\colon \widehat{\mathfrak{X}}_{\mathrm{ex}}(M,\alpha) \to \mathfrak{X}_{\mathrm{ex}}(M,\alpha)$ of $\mathfrak{X}_{\mathrm{ex}}(M,\alpha)$ by $H^{\ell-1}(M,\mathbb{R})$, and σ_β is the 1-cochain on $\widehat{\mathfrak{X}}_{\mathrm{ex}}(M,\alpha)$ given by

$$\sigma_\beta\colon \widehat{\mathfrak{X}}_{\mathrm{ex}}(M,\alpha) \to \mathbb{R}, \qquad \sigma_\beta\big((X,[\psi_X])\big) = \int_S (\Phi^*\psi_X)\wedge\beta\,. \tag{40}$$

Similarly, the cocycle ν of (34) trivializes when it is pulled back along the extension $\pi\colon \widehat{\mathfrak{X}}_{\mathrm{ex}}(S,\beta) \to \mathfrak{X}_{\mathrm{ex}}(S,\beta)$, as $\pi^*\nu = \mathrm{d}_{\mathrm{CE}}\bar{\sigma}_\alpha$ for the Lie algebra 1-cochain

$$\bar{\sigma}_\alpha\colon \widehat{\mathfrak{X}}_{\mathrm{ex}}(S,\beta) \to \mathbb{R}, \qquad \bar{\sigma}_\alpha\big((u,[\psi_u])\big) = \int_S (\Phi^*\alpha)\wedge\psi_u\,. \tag{41}$$

Proposition 5.3. The isomorphism class of the Lie algebra extension of $\mathfrak{X}_{\mathrm{ex}}(M,\alpha)$ corresponding to the 2-cocycle τ is determined by the restriction of σ_β to $H^{\ell-1}(M,\mathbb{R}) \subseteq \widehat{\mathfrak{X}}_{\mathrm{ex}}(M,\alpha)$. Similarly, the isomorphism class of the Lie algebra extension of $\mathfrak{X}_{\mathrm{ex}}(S,\beta)$ corresponding to ν is determined by the restriction of $\bar{\sigma}_\alpha$ to $H^{r-1}(S,\mathbb{R})$. In other words, the following diagrams commute:

$$\begin{tikzcd}
\Omega^{r+1}_{\mathbb{Z}}(S) \arrow[r, "\beta\mapsto[\tau]"] \arrow[dr, "\beta\mapsto\sigma_\beta"'] & H^2(\mathfrak{X}_{\mathrm{ex}}(M,\alpha),\mathbb{R}) \\
& H^{\ell-1}(M,\mathbb{R})^* \arrow[u, "\Xi_\alpha"']
\end{tikzcd}
\qquad
\begin{tikzcd}
H^2(\mathfrak{X}_{\mathrm{ex}}(S,\beta),\mathbb{R}) & \Omega^{\ell+1}_{\mathbb{Z}}(M) \arrow[l, "\alpha\mapsto[\nu]"'] \arrow[dl, "\alpha\mapsto\bar{\sigma}_\alpha"] \\
H^{r-1}(S,\mathbb{R})^* \arrow[u, "\Xi_\beta"] &
\end{tikzcd}$$

In general, not every central extension of Lie algebras integrates to a central extension of Lie groups, see for example, [21]. Combining Proposition 5.3 with Theorem 5.1, we identify a lattice of integrable classes.

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Theorem 5.4 (Compact version). Let M be a compact orientable n-dimensional manifold, and let $h \in \widehat{H}^k(M,\mathbb{T})$ be a differential character with curvature $\omega \in \Omega^{k+1}(M)_{\mathbb{Z}}$. Let G be a Fréchet–Lie group, and let $i\colon G \to \mathrm{Diff}(M,h)$ be a smooth action on M such that, for every $g \in G$, the diffeomorphism $i(g)\colon M \to M$ preserves h and is homotopic to the identity. Finally, let

$$H^{k-1}(M,\mathbb{R}) \to \widehat{\mathfrak{g}} \to \mathfrak{g}$$

be the pull-back of the central extension (37) along the Lie algebra homomorphism $i_{\mathfrak{g}}\colon \mathfrak{g} \to \mathfrak{X}_{\mathrm{ex}}(M,\omega)$. Then the image of the integral classes $H^{n-k+1}(M,\mathbb{R})_{\mathbb{Z}} \subseteq H^{k-1}(M,\mathbb{R})^*$ under the map $i_{\mathfrak{g}}^* \circ \Xi_\omega\colon H^{k-1}(M,\mathbb{R})^* \to H^2(\mathfrak{g},\mathbb{R})$ consists of classes in $H^2(\mathfrak{g},\mathbb{R})$ that are integrable to central Lie group extensions of G by $\mathbb{T}$.

Proof. For $[\beta] \in H^{n-k+1}(M,\mathbb{R})_{\mathbb{Z}}$, choose a differential character $g \in \widehat{H}^{n-k}(M,\mathbb{T})$ with $\mathrm{curv}(g) = \beta$. By Theorem 5.1 with $M = S$, $\Phi = \mathrm{Id}$, and $\alpha = \omega$, we obtain a central Lie group extension

$$\mathbb{T} \to \widehat{G} \to G. \tag{42}$$

(Indeed, the diffeomorphism $i(g)$ preserves $[\mathrm{Id}] \in \pi_0(C^\infty(M,M))$ if it is homotopic to the identity.) The corresponding Lie algebra cocycle is the pull-back along $i_{\mathfrak{g}}\colon \mathfrak{g} \to \mathfrak{X}_{\mathrm{ex}}(M,\omega)$ of the 2-cocycle $\tau(X,Y) = \int_M (i_Y i_X \omega) \wedge \beta$ on $\mathfrak{X}_{\mathrm{ex}}(M,\omega)$ (cf. equation (33)). By Proposition 5.3, the isomorphism class of the cocycle τ is determined by the restriction of σ_β to $H^{k-1}(M,\mathbb{R}) \subseteq \widehat{\mathfrak{X}}_{\mathrm{ex}}(M,\omega)$. This is the pairing with $[\beta] \in H^{n-k+1}(M,\mathbb{R})_{\mathbb{Z}}$. In terms of the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^{k-1}(M,\mathbb{R}) \arrow[r] \arrow[d, equal] & \widehat{\mathfrak{X}}_{\mathrm{ex}}(M,\omega) \arrow[r, "\pi"] & \mathfrak{X}_{\mathrm{ex}}(M,\omega) \arrow[r] & 0 \\
0 \arrow[r] & H^{k-1}(M,\mathbb{R}) \arrow[r] & \widehat{\mathfrak{g}} \arrow[u, "\widehat{i_{\mathfrak{g}}}"] \arrow[r, "\pi_{\mathfrak{g}}"] & \mathfrak{g} \arrow[u, "i_{\mathfrak{g}}"] \arrow[r] & 0,
\end{tikzcd}$$

we have $\pi_{\mathfrak{g}}^* i_{\mathfrak{g}}^* \tau = \mathrm{d}_{\mathrm{CE}}(\widehat{i_{\mathfrak{g}}}^* \sigma_\beta)$. Since $\widehat{i_{\mathfrak{g}}}$ is the identity on $H^{k-1}(M,\mathbb{R})$, the restriction of $\pi_{\mathfrak{g}}^* i_{\mathfrak{g}}^* \tau$ to $H^{k-1}(M,\mathbb{R})$ is the same as the restriction of σ_β, namely the pairing with $[\beta]$. ■

In order to extend Theorem 5.4 to manifolds M that are not necessarily compact or orientable, we use "singular" functionals on $\widehat{\mathfrak{X}}_{\mathrm{ex}}(M,\omega)$. Suppose that $z \in H_{k-1}(M,\mathbb{Z})$ can be represented as $z = \Phi_*[S]$, where $[S]$ is the fundamental class of a closed, oriented manifold S and $\Phi\colon S \to M$ is a smooth map. Then, with $\beta = 1$, equation (40) yields the linear functional $\sigma(X,\psi_X) = \int_S \Phi^* \psi_X$ on $\widehat{\mathfrak{X}}_{\mathrm{ex}}(M,\omega)$, whose restriction to $H^k(M,\mathbb{R})$ is the

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pairing with $\Phi_*[S]$. This yields a way of building group extensions for those classes in $H_{k-1}(M,\mathbb{Z})$ that can be represented by a closed, oriented, smooth manifold.

Corollary 5.5 (Noncompact version). Let M be a manifold of dimension n, and let $h \in \widehat{H}^k(M,\mathbb{T})$ be a differential character with curvature $\omega \in \Omega^{k+1}(M)_{\mathbb{Z}}$. Assume that a class in $H_{k-1}(M,\mathbb{Z})$ can be realized as the push-forward $\Phi_*[S]$ of the fundamental class of a closed, oriented $(k-1)$-manifold S along a smooth map $\Phi\colon S \to M$. Let G be a Fréchet–Lie group, and let $i\colon G \to \mathrm{Diff}(M,h)$ be a smooth action on M such that, for all $g \in G$, $i(g)\colon M \to M$ preserves h, and $i(g)^*\Phi$ is homotopic to Φ. Then the Lie algebra extension of $\mathfrak{g}$ corresponding to the image of $\Phi_*[S]$ under the map $i_{\mathfrak{g}}^* \circ \Xi_\omega\colon H^{k-1}(M,\mathbb{R})^* \to H^2(\mathfrak{g},\mathbb{R})$ is integrable to a central Lie group extension of G by $\mathbb{T}$.

In general, not every class $z \in H_{k-1}(M,\mathbb{Z})$ can be represented as $z = \Phi_*[S]$. By [7, Thm. 9.1], the classes with this property are *Steenrod representable*; they constitute the image of the bordism group under the natural homomorphism $\Omega_{k-1}(M) \to H_{k-1}(M,\mathbb{Z})$ that takes $[f\colon S \to M]$ to $f_*[S]$. This image has been extensively investigated by Thom in [32, especially §2.11]. If $H_\bullet(M,\mathbb{Z})$ is finitely generated and has no odd torsion, then *every* class is Steenrod representable by [7, Thm. 15.2]. In this case, Corollary 5.5 is simply a generalization of Theorem 5.4 to noncompact manifolds. Without any assumptions on $H_\bullet(M,\mathbb{Z})$, one can still show that every class admits an odd multiple that is Steenrod representable [7, Thm. 15.3]. In particular, Corollary 5.5 yields a lattice of full rank in $H^{k-1}(M,\mathbb{R})^*$ that gives rise to integrable Lie algebra extensions, regardless whether $H_\bullet(M,\mathbb{Z})$ has odd torsion or not.

If M is compact and orientable, then the Lie algebra extensions of Corollary 5.5 are isomorphic to the ones in Theorem 5.4 by Poincaré duality, although the group extensions may well be different.

5.3 The volume-preserving diffeomorphism group

Let μ be an integral volume form on an n-dimensional compact manifold M. Then the extension (37) is the *Lichnerowicz extension* ([19], [28, Sec. 10])

$$H^{n-2}(M,\mathbb{R}) \to \overline{\Omega}^{n-2}(M) \to \mathfrak{X}_{\mathrm{ex}}(M,\mu). \tag{43}$$

By Theorem 5.4, the elements in the integral lattice $H_{n-2}(M,\mathbb{Z}) \subseteq H^{n-2}(M,\mathbb{R})^*$ give rise to Lie group extensions of $\mathrm{Diff}_{\mathrm{ex}}(M,\mu)$.

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In the remainder of this section, we illustrate how different geometric realizations of these integral elements of $H^{n-2}(M,\mathbb{R})^*$ give rise to different representatives of the same integrable classes in $H^2(\mathfrak{X}_{\mathrm{ex}}(M,\mu),\mathbb{R})$.

5.3.1 *Singular Lichnerowicz cocycles*

Let h be a differential character with curvature μ. Choose a closed manifold S of dimension $n-2$, and apply Theorem 5.1 with $\alpha = \mu \in \Omega^n(M)_{\mathbb{Z}}$ and $\beta = k \in \Omega^0(S)_{\mathbb{Z}}$. This yields a smooth central extension

$$\mathbb{T} \to \widehat{\mathrm{Diff}}_{\mathrm{ex}}(M,\mu) \to \mathrm{Diff}_{\mathrm{ex}}(M,\mu) \tag{44}$$

of the exact volume-preserving diffeomorphism group, integrating the *singular Lichnerowicz cocycle* on $\mathfrak{X}_{\mathrm{ex}}(M,\mu)$

$$\tau(X,Y) = k\int_S \Phi^* i_Y i_X \mu. \tag{45}$$

These group extensions are closely related to *Ismagilov's central extensions* [14, Sec. 25.3] of the Lie group $\mathrm{Diff}_{\mathrm{ex}}(M,\mu)$.

5.3.2 *Lichnerowicz cocycles*

As for the Hamiltonian actions in §1, we can change the role of α and β to obtain (possibly nonequivalent) group extensions corresponding to cohomologous Lie algebra cocycles. If ω is a closed, integral 2-form on M, then we can apply Theorem 5.1 to $S = M$, with $\alpha = k\omega$, $\beta = \mu$, and $\Phi\colon S \to M$ the identity map. For every choice of differential characters g and h with $\mathrm{curv}(g) = \mu$ and $\mathrm{curv}(h) = k\omega$, we then obtain a central extension of $\mathrm{Diff}_{\mathrm{ex}}(M,\mu)$ that integrates the *Lichnerowicz cocycle* on $\mathfrak{X}_{\mathrm{ex}}(M,\mu)$,

$$\nu(X,Y) = k\int_M \omega(X,Y)\,\mu. \tag{46}$$

If $\Phi_*[S] \in H_{n-2}(M,\mathbb{Z})$ is Poincaré dual to $-[\omega] \in H^2(M,\mathbb{R})_{\mathbb{Z}}$, then the singular Lichnerowicz cocycle τ is cohomologous to ν [16, 33].

5.3.3 *Other representations*

Let N be an $(n-1)$-dimensional manifold. Consider a differential character F of degree 0 on N with curvature 1-form ρ. The character F can be viewed as a $\mathbb{T}$-valued function $F\colon N \to \mathbb{T}$ on N and, from this viewpoint, ρ is then the logarithmic derivative δF of F. Let $\Psi\colon N \to M$ be a smooth map. By Theorem 5.1 (with $S = N$, $\alpha = \mu$, $\beta = \delta F$, and $\Phi = \Psi$),

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there exists a smooth central extension of $\mathrm{Diff}_{\mathrm{ex}}(M,\mu,[\Psi])$ that integrates the following Lie algebra cocycle on $\mathfrak{X}_{\mathrm{ex}}(M,\mu)$:

$$\kappa(X,Y)=\int_N \Psi^*(i_Y i_X \mu)\wedge \delta F. \tag{47}$$

If the functional on $H^{n-2}(M,\mathbb{R})$ given by

$$[\alpha]\mapsto \int_N \Psi^*\alpha\wedge\delta F \tag{48}$$

coincides with the one given by $\Phi_*[S]\in H_{n-2}(M,\mathbb{Z})$ or by $-[\omega]\in H^2(M,\mathbb{R})_{\mathbb{Z}}$, then the cocycle κ is cohomologous to the Lichnerowicz cocycles τ and ν, respectively. In the context of hydrodynamics, the pair (N,β) corresponds to a codimension one singular membrane in the ideal fluid on M. We refer to [11] for further details.

5.3.4 *Outlook*

On the Lie algebra level, the central $\mathbb{R}$-extensions of $\mathfrak{X}_{\mathrm{ex}}(M)$ constructed from functionals $\lambda\in H^{n-2}(M,\mathbb{R})^*$ all descend from a *single* central extension of $\mathfrak{X}_{\mathrm{ex}}(M)$ by $H^{n-2}(M,\mathbb{R})$, namely the Lichnerowicz extension (43). It is natural to ask whether, at the group level, the central $\mathbb{T}$-extensions constructed above descend from a single central Lie group extension of $\mathrm{Diff}_{\mathrm{ex}}(M)$ by $H^{n-2}(M,\mathbb{T})$.

If M is a three-dimensional compact, connected, orientable manifold, we believe that this is indeed the case. To construct the required extension, let $h\in\widehat{H}^2(M,\mathbb{T})$ be a differential character whose curvature is the volume form μ. Since h is of degree 2, the principal circle bundle $\mathcal{P}\to C^\infty(S^1,M)$ can be viewed as the bundle obtained from transgression of the $U(1)$-bundle gerbe corresponding to h (cf. [4]). In fact, since the circle bundle $\mathcal{P}\to C^\infty(S^1,M)$ is obtained by transgression of a bundle gerbe, it is not only endowed with a connection, but also with a *fusion structure* [35]. It is therefore natural to consider the group of automorphisms of $\mathcal{P}\to C^\infty(S^1,M)$ that preserve both the connection and—in an appropriate sense—also the fusion structure. Since the group of *vertical* automorphisms of this type is isomorphic to $H^1(M,\mathbb{T})$, this is a natural candidate for a central Lie group extension. We are currently pursuing this line of reasoning in joint work with Peter Kristel.

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A Central Extensions of Non-Connected Groups

In this appendix, we show how to extend [26, Thm. 3.4] concerning the Lie group structure on the central extension of a connected Lie group G, obtained by pull-back of the prequantization central extension, to non-connected Lie groups. The results also work for locally convex manifolds.

Let Z be a connected abelian Lie group of the form $\mathfrak{z}/\Gamma$, where $\mathfrak{z}$ is a Mackey complete vector space and $\Gamma \subseteq \mathfrak{z}$ is a discrete subgroup. Furthermore, let $q : \mathcal{P} \to \mathcal{M}$ be a Z-bundle over the connected locally convex manifold $\mathcal{M}$, and let θ be a connection 1-form with curvature ω. We further assume that $\sigma : G \to \mathrm{Diff}(\mathcal{M}, \omega)$ defines a smooth action of the Lie group G on $\mathcal{M}$ whose image lies in the group $\mathrm{Diff}_{\mathrm{hol}}(\mathcal{M})$ of holonomy-preserving diffeomorphism. According to [26], this implies that $\sigma(G)$ lies in the image of $\mathrm{Aut}(\mathcal{P}, \theta)$, so that the pull-back

$$\widehat{G} := \sigma^* \mathrm{Aut}(\mathcal{P}, \theta) = G \times_{\mathrm{Diff}_{\mathrm{hol}}(\mathcal{M})} \mathrm{Aut}(\mathcal{P}, \theta)$$

of the extension $\mathrm{Aut}(\mathcal{P}, \theta) \to \mathrm{Diff}_{\mathrm{hol}}(\mathcal{M})$ is a central Z-extension of G. The following theorem is a generalization of [26, Thm. 3.4] to non-connected Lie groups.

Theorem A.1. If $\mathcal{M}$ is connected and G not necessarily connected with $\sigma(G) \subset \mathrm{Diff}_{\mathrm{hol}}(\mathcal{M})$, then $\widehat{G}$ carries a Lie group structure for which it is a central Lie group extension by Z. Moreover, the action of $\widehat{G}$ on $\mathcal{P}$ defined by $(g, \varphi) \cdot p = \varphi(p)$ for $g \in G$ and $\varphi \in \mathrm{Aut}(\mathcal{P}, \theta)$ is smooth.

Proof. Pick $m_0 \in \mathcal{M}$ and let $\sigma^{m_0} : G \to \mathcal{M}, g \mapsto \sigma(g)m_0$ be the corresponding smooth orbit map. We endow $\widehat{G}$ with the smooth manifold structure obtained by identifying it with the Z-bundle $(\sigma^{m_0})^*\mathcal{P} = G \times_{\mathcal{M}} \mathcal{P}$ over G; [26, Lemma 3.2] also works for non-connected groups G. In [26, Thm. 3.4], we have already seen that we thus obtain a Lie group structure on the central extension $\widehat{G}_0$ of the identity component G_0 of G. We now show that the smooth structure extends to all of $\widehat{G}$.

The elements of $\widehat{G}$ are of the form $\widehat{g} = (g, \widehat{\sigma_g})$, where the quantomorphism $\widehat{\sigma}_g$ projects to σ_g. Since every $\widehat{g} \in \widehat{G}$ acts as an automorphism on $\mathcal{P}$, the left multiplication by $\widehat{g} = (g, \widehat{\sigma_g})$ in $\widehat{G} \cong (\sigma^{m_0})^*\mathcal{P} \subset G \times \mathcal{P}$ is smooth. It therefore remains to show that the inner automorphisms of $\widehat{G}$ are smooth. Fix $\widehat{g}_0 = (g_0, \widehat{\sigma_{g_0}})$ in $\widehat{G}$. For $\widehat{g} = (g, \widehat{\sigma_g}) \in \widehat{G}$ we then have

$$\widehat{g}_0 \widehat{g} \widehat{g}_0^{-1} = (g_0 g g_0^{-1}, \widehat{\sigma_{g_0}} \widehat{\sigma}_g \widehat{\sigma_{g_0}}^{-1}).$$

Pick a point $y_0 \in \mathcal{P}_{m_0}$. We have to show that the map

$$\widehat{G} \to \mathcal{P}, \quad (g, \widehat{\sigma_g}) \mapsto \widehat{\sigma_{g_0}} \widehat{\sigma}_g \widehat{\sigma_{g_0}}^{-1}(y_0)$$

is smooth. This will follow if all orbit maps for $\widehat{G}$ on $\mathcal{P}$ are smooth, but this is ensured by the smoothness of the action of the connected Lie group $\widehat{G}_0$ on $\mathcal{P}$ ([26]). ■

B Circle Bundles and Differential Characters

In this appendix, we give a direct proof of the well-known correspondence between equivalence classes of principal circle bundles with connection and differential characters of degree 1. Alternatively, this can be derived from the fact that principal circle bundles with connection are classified by Deligne cohomology [4, Thm. 2.2.12], which is a different model for differential cohomology, cf. [3, Sec. 5.2].

Proposition 2.1 The map $(P, \theta) \mapsto h$ that assigns to a principal circle bundle P with connection θ its holonomy map h defines an isomorphism from the group of equivalence classes of pairs (P, θ) modulo bundle automorphisms onto the group $\widehat{H}^1(M, \mathbb{T})$ of differential characters of degree 1.

Proof. If two principal $\mathbb{T}$-bundles with connection P and Q over M have the same holonomy $h\colon Z_1(M) \to \mathbb{T}$, then we show that they are isomorphic by trivializing $\overline{P} \otimes_{\mathbb{T}} Q$, where $\overline{P}$ has the same underlying manifold as P but inverse $\mathbb{T}$-action. A flat section $s\colon M \to \overline{P} \otimes_{\mathbb{T}} Q$ is obtained as follows. Choose a base point m_i in every connected component M_i of M, and choose $p_i \in \overline{P} \otimes_{\mathbb{T}} Q$ that project to m_i. (The section will depend on these choices in a controlled way.) For a point $m \in M$, choose a piecewise smooth path $\gamma\colon [0,1] \to M$ from $\gamma(0) = m_i$ to $\gamma(1) = m$, and let $\overline{\gamma}$ be its horizontal lift to $\overline{P} \otimes_{\mathbb{T}} Q$ with initial point $\overline{\gamma}(0) = p_i$. Then $s(m) := \overline{\gamma}(1)$. The fact that the holonomies coincide guarantees that $s(m)$ does not depend on the choice of path, and s is smooth because in a local trivialization it takes the form $s(x) = (x, \exp(\int_{\underline{0x}} A))$, where $\underline{0x}$ is a straight line from 0 to x and A is the 1-form that represents the connection in the given trivialization.

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It remains to show that for every $h \in \widehat{H}^1(M,\mathbb{T})$, we can construct a principal $\mathbb{T}$-bundle $P \to M$ whose connection represents h. Consider the groupoid $PM \rightrightarrows M$ of piecewise smooth paths, and let $\mathcal{G} \rightrightarrows M$ be the groupoid

$$\mathcal{G} := (PM \times \mathbb{T})/\sim, \text{where}(\gamma_1, z_1) \sim (\gamma_2, z_2) \;\Leftrightarrow\; h(\gamma_1 * \gamma_2^{-1}) = z_1^{-1} z_2\,.$$

To see that $\mathcal{G}$ is a Lie groupoid, we define charts as follows: for $x, y \in M$, choose open convex coordinate neighborhoods $U_x, U_y \subseteq M$ centered at these points. The chart is labeled by a path γ from x to y

$$\Phi_\gamma : U_x \times U_y \times \mathbb{T} \to \mathcal{G}, \quad (u, v, z) \mapsto [\underline{ux} * \gamma * \underline{yv}, z], \tag{B1}$$

where $\underline{ux}$ denotes the straight path from $u \in U_x$ to x in the coordinate neighborhood U_x. To verify smoothness of the chart transition maps, first consider the case where γ and γ' both start at x and end at y. Then the chart transition $T_{\gamma\gamma'} = \Phi_{\gamma'}^{-1} \circ \Phi_\gamma$ takes the form

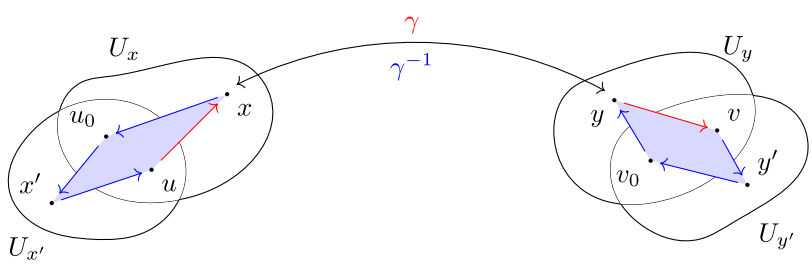
$$\begin{aligned} T_{\gamma\gamma'} : U_x \times U_y \times \mathbb{T} &\to U_x \times U_y \times \mathbb{T}, \\ (u, v, z) &\mapsto (u, v, h(\underline{ux} * \gamma * \underline{yv} * \underline{vy} * \gamma'^{-1} * \underline{xu}) \cdot z). \end{aligned} \tag{B2}$$

Since h is invariant under thin homotopies, this expression simplifies to $T_{\gamma\gamma'}(u, v, z) = (u, v, h(\gamma * \gamma'^{-1}) \cdot z)$, which is smooth.

Secondly, consider the case where γ goes from x to y and γ' from x' to y'. By the above argument, we can choose the path γ' to be $\gamma' = \underline{x'u_0} * \underline{u_0x} * \gamma * \underline{yv_0} * \underline{v_0y'}$, where u_0 and v_0 are fixed but otherwise arbitrary points in $U_x \cap U_{x'}$ and $U_y \cap U_{y'}$, respectively. A similar calculation as above shows that $T_{\gamma\gamma'}(u, v, z) = (u, v, h(\eta_{u,v}) \cdot z)$, where

$$\eta_{u,v} := \underline{ux} * \gamma * \underline{yv} * \underline{vy'} * \underline{y'v_0} * \underline{v_0y} * \gamma^{-1} * \underline{xu_0} * \underline{u_0x'} * \underline{x'u}. \tag{B3}$$

Up to thin homotopy $\eta_{u,v}$ is the boundary of the two diamonds with edges u_0, x, u, x' and v_0, y, v, y', and thus $h(\eta_{u,v})$ is given by integration of curv(h) over the diamonds. Since this operation depends smoothly on u and v, the transition map $T_{\gamma\gamma'}$ is smooth.



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If one chooses a base point m_i in every connected component of M, then the union $P := \bigsqcup_{m_i} s^{-1}(m_i)$ of the source fibers of $\mathcal{G}$ is a principal $\mathbb{T}$-bundle over M. The groupoid homomorphism $\mathcal{F} : PM \to \mathcal{G}, \mathcal{F}(\gamma) = [\gamma, 1]$ satisfies $\mathcal{F}(\sigma) = [\underline{mm}, h(\sigma)]$ if σ starts and ends at the same point m. For a smooth path $c\colon [0,1] \to M$ with $c(0) = m$ and $\frac{d}{dt}\big|_{t=0} c = v$, let $c_t\colon [0,1] \to M$ be defined by $c_t(s) = c(ts)$. Define the horizontal lift of v as $\frac{d}{dt}\big|_{t=0} \mathcal{F}(c_t) \cdot [\gamma, z] = \frac{d}{dt}\big|_{t=0} [\gamma * c_t, z] \in T_{[\gamma,z]}P$. This connection has holonomy h. ■

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