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On stability of a harmonic oscillator with a delayed feedback system

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"On stability of a harmonic oscillator with a delayed feedback system"

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Abstract

In this thesis the stability type of $y = 0$ is being considered for the delay differential equation $y''(t) + ay(t) + by(t-1) = 0$ with $a, b \in \mathbb{R}$. It is already known that $y = 0$ is stable when $b = 0$ and $a > 0$ and unstable when $b = 0$ and $a < 0$. The aim of this project is to determine the stability of $y = 0$ for all $a, b \in \mathbb{R}$. First, the general stability theory for delay differential equations was highlighted before giving an in-depth stability analysis of the equation $y''(t) + ay(t) + by(t-1) = 0$. It turns out that a theorem of Pontryagin (1908 -1988) is really helpful for answering these stability questions. Due to this theorem all values for a and b are determined such that $y = 0$ is asymptotically stable for $y''(t) + ay(t) + by(t-1) = 0$. However, this does not cover the stability type of $y = 0$ for all $a, b \in \mathbb{R}$. So more analysis was done in order to give a full answer of the stability problem. The full answer was not achieved as there are still values for a and b where the stability is unknown. Finally, numerical solutions of $y''(t) + ay(t) + by(t-1) = 0$ are shown to confirm the results that are obtained.

Contents

1 Introduction

Differential equations appear naturally in many areas of science. They are widely used to describe real-life phenomena such as mechanical systems, spreading of diseases and population growth. A differential equation is an equation that relates a function with its derivatives. The equations

$$\frac{dy}{dt} = 3y$$

and

$$\frac{d^5y}{dt^5} = e^{-y^2}\frac{d^3y}{dt^3} + 4t$$

are both examples of differential equations.

An ordinary differential equation (ODE) is a relationship containing an (unknown) function of one variable t, its derivatives and some given function(s) of t. So, in particular, the above-mentioned equations are actually called ordinary differential equations.

However, ordinary differential equations are not always suitable to describe various events as precise as possible. Using ODE models in fields such as control engineering of mechanical systems, population biology and epidemiology might give results that do not coincide with the reality. In reality there are many processes that include aftereffect phenomena in their inner dynamics. Therefore, the structure of ordinary differential equations needs to be slightly altered in order to obtain results that coincide with the real outcomes.

In the field of control engineering of mechanical systems aftereffect phenomena are very common. Systems with such an aftereffect phenomenon are called delay systems. An example that will be studied is a harmonic oscillator with such a delay system. The equation describing this oscillator is given by

$$y''(t) + ay(t) + by(t-1) = 0, \quad a, b \neq 0. \tag{1}$$

Figure 1: Mass spring system with delay in left spring.

Intuitively, one can consider a mass spring system with two springs (Figure 1). However, in this case the left spring is made of a material that causes a delayed response of one time unit before the spring stretches again when the spring is pushed in. So the left spring has a delayed feedback with delay equal to 1, whereas the right spring is made of a material with no such delay. The term $by(t-1)$ corresponds, therefore, with the left spring. Obviously, the term $ay(t)$ corresponds with the right spring. The constants a and b depend on the mass m and the spring constants of the two springs. Not all $a, b \neq 0$ will physically make sense. Nevertheless, in this thesis all $a, b \neq 0$ will be taken into consideration for the sake of completeness.

Equation (1) is called a delay differential equation (DDE). It is a type of differential equation in which the derivative of the (unknown) function is given in terms of the function value at previous times. If $b = 0$, then we obtain the ordinary differential equation $y''(t) + ay(t) = 0$. It is known that the general solution of this ODE is $y(t) = c_1 e^{t\sqrt{-a}} + c_2 e^{-t\sqrt{-a}}$ if $a < 0$ and $y(t) = c_1 \cos(t\sqrt{a}) + c_2 \sin(t\sqrt{a})$ if $a > 0$ ($c_1, c_2 \in \mathbb{R}$). This case is less interesting, because there

would be no delay term $y(t-1)$ and the equation can (easily) be solved analytically. Letting $a, b \neq 0$ results in a whole different system in which the solution and qualitative properties are not so obvious. This thesis mainly focuses on the stability properties of (1).

What is stability exactly? The stability of general delay differential equations will be studied in Section 2. Formal definitions of stability, instability and asymptotic stability will be stated. A key tool for determining the behaviour of solutions of delay differential equations is the *characteristic equation*. The main result of this section concerns the relationship between characteristic equations and stability. In Section 3 the stability of (1) will be discussed. Cahlon and Schmidt have already considered the stability of second order delay differential equations in [3], but only for $a > 0$ and $b < 0$. Our goal is to give statements for all $a, b \neq 0$. Subsequently, in Section 4 some numerical solutions will be shown to confirm the results in Section 3.

2 Stability

A naturally arising question is how the solution $y(t)$ of

$$y''(t) + ay(t) + by(t-1) = 0, \quad a, b \neq 0, \tag{2}$$

behaves after a long period of time. Will the solution grow over time? Or will it converge to a certain value? Logically, the answers to these questions will depend on a and b. A bit more work is required in order to give qualitative properties of (2).

The definitions in this section follow the ones in Hale and Lunel [1] and in Smith [2].

2.1 General delay differential equations

Before an in-depth analysis of the qualitative properties of (2), some remarks on general delay differential equations need to be made.

Let $n \in \mathbb{N}$. The space $\mathbb{R}^n$ is equipped with the Euclidean norm $\| \cdot \|$. Let $r > 0$ be the delay and define $C := C([-r, 0], \mathbb{R}^n)$ as the space of continuous functions $[-r, 0] \to \mathbb{R}^n$ endowed with the norm $\| \cdot \|_\infty$. The norm $\| \cdot \|_\infty : C \to \mathbb{R}$ is defined as $\|\phi\|_\infty = \sup\{\|\phi(\theta)\| : -r \leq \theta \leq 0\}$.

Define $y_t \in C$ by $y_t(\theta) = y(t+\theta)$ for $\theta \in [-r, 0]$ (for convenience).
Given a continuous function $f : \mathbb{R} \times C \to \mathbb{R}^n$ and an initial continuous function $\phi \in C$, consider the system of delay differential equations

$$y'(t) = f(t, y_t),\ t \geq \sigma, \tag{3}$$
$$y_\sigma = \phi,$$

where $\sigma \in \mathbb{R}$ is the initial time. This means that $y(\sigma + \theta) = y_\sigma(\theta) = \phi(\theta)$ for $\theta \in [-r, 0]$ or equivalently, that $y(t) = \phi(t - \sigma)$ for $t \in [\sigma - r, \sigma]$.

This notation might seem unnecessary, but it will turn out that it will be helpful for giving a formal definition of (asymptotic) stability for (3). For clarification on how this notation works, two (simple) examples will be presented.

Example 2.1. $(n = 1)$
Let $r > 0$ be the delay and $\phi : [-r, 0] \to \mathbb{R}$ a known continuous function. Consider the initial value problem

$$y'(t) = -y(t - r),\ t \geq 0, \tag{4}$$
$$y(t) = \phi(t),\ t \in [-r, 0].$$

Now define $g : [0, \infty] \times C \to \mathbb{R}$ by $g(t, \psi) = -\psi(-r)$ and define $y_t \in C$ by $y_t(\theta) = y(t + \theta)$ for $\theta \in [-r, 0]$. So that $g(t, y_t) = -y_t(-r) = -y(t-r)$ for $t \geq 0$. But then $y'(t) = -y(t-r) = g(t, y_t)$ for $t \geq 0$ and $y_0 = \phi$ on $[-r, 0]$. Therefore problem (4) is equivalent to

$$y'(t) = g(t, y_t),\ t \geq 0,$$
$$y_0(t) = \phi(t),\ t \in [-r, 0].$$

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Example 2.2. $(n = 2)$

Let $r > 0$ be the delay and $\phi : [-r, 0] \to \mathbb{R}$ a known differentiable function. Consider the initial value problem

$$y''(t) + ay(t) + by(t - r) = 0, \ t \geq 0, \tag{5}$$
$$y(t) = \phi(t), \ t \in [-r, 0],$$

with $a, b \neq 0$. Let $x_1(t) = y(t)$ and $x_2(t) = y'(t)$. Then $x_1'(t) = x_2(t)$ and $x_2'(t) = -ay(t) - by(t - r) = -ax_1(t) - bx_1(t - r)$. Equivalently, this results in the equation

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -b & 0 \end{pmatrix} \begin{pmatrix} x_1(t - r) \\ x_2(t - r) \end{pmatrix}. \tag{6}$$

Now define $h : [0, \infty] \times C \to \mathbb{R}^2$ by $h(t, \psi) = A\psi(0) + B\psi(-r)$ with $A = \begin{pmatrix} 0 & 1 \\ -a & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ -b & 0 \end{pmatrix}$. Let $\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ and define $\mathbf{x}_t \in C$ by $\mathbf{x}_t(\theta) = \mathbf{x}(t + \theta)$ for $\theta \in [-r, 0]$. Then $h(t, \mathbf{x}_t) = A\mathbf{x}_t(0) + B\mathbf{x}_t(-r) = A\mathbf{x}(t) + B\mathbf{x}(t - r) = \mathbf{x}'(t)$. Moreover, $\mathbf{x}_0(\theta) = \begin{pmatrix} y(\theta) \\ y'(\theta) \end{pmatrix} = \begin{pmatrix} \phi(\theta) \\ \phi'(\theta) \end{pmatrix} =: \boldsymbol{\Phi}(\theta)$ for $\theta \in [-r, 0]$. Therefore, problem (5) is equivalent to

$$\mathbf{x}'(t) = h(t, \mathbf{x}_t), \ t \geq 0,$$
$$\mathbf{x}_0 = \boldsymbol{\Phi} \ \text{ on } \ [-r, 0].$$

2.2 Stability definitions

It is now possible to formulate formal definitions for (asymptotic) stability of the zero solution. Consider the following system of delay differential equations

$$y'(t) = f(t, y_t).$$

Suppose that $f(t, 0) = 0 \in \mathbb{R}^n$ for all $t \in \mathbb{R}$ so that $y = 0 \in \mathbb{R}^n$ is a solution. Moreover, let $y(t, \sigma, \phi)$ be the notation for the solution of (3).

Definition 2.1. The solution $y = 0$ is *stable* if for any $\sigma > 0$ and $\epsilon > 0$, there exists $\delta = \delta(\sigma, \epsilon) > 0$ such that $\phi \in C$ and $\|\phi\|_\infty < \delta$ implies $\|y(t, \sigma, \phi)\| < \epsilon$ for $t \geq \sigma$.
We say that $y = 0$ *unstable* if it is not stable.

Definition 2.2. The solution $y = 0$ is *asymptotically stable* if it is stable and if for all $\sigma > 0$ there exists a $d = d(\sigma) > 0$ such that whenever $\phi \in C$ and $\|\phi\|_\infty < d$, then $y(t, \sigma, \phi) \to 0$ as $t \to \infty$.

2.3 Linear delay systems and characteristic equations

The next definitions follow the ones in Rynne and Youngson [5].

Definition 2.3. Let X and Y be normed vector spaces and let $T : X \to Y$ be a linear transformation. T is said to be *bounded* if there exists a positive real number $k > 0$ such that $\|T(x)\|_Y \leq k\|x\|_X$ for all $x \in X$.

Let X and Y be normed linear spaces. The set of all bounded operators from X to Y is denoted by $B(X, Y)$.

$B(X, Y)$ is a normed vector space with (operator) norm $\| \cdot \|_{op} : B(X, Y) \to \mathbb{R}$ defined by $\|T\|_{op} = \sup\{\|T(x)\|_Y : \|x\|_X \le 1\}$. For more information on bounded operators see Chapter 4 in Rynne and Youngson [5].

Now let $T \in B(X, Y)$ and $x \in X$. The following is an important consequence of this definition:

$$\|T(x)\|_Y = \left\| T\left(\frac{x}{\|x\|_X} \right) \right\|_Y \cdot \|x\|_X$$
$$\le \|T\|_{op} \|x\|_X. \tag{$*$}$$

The subsequent theorem shows that a huge class of functions are bounded.

Theorem 2.1. Let X be a finite-dimensional normed space, let Y be any normed linear space and let $T : X \to Y$ be a linear transformation. Then T is bounded.

Proof. See Theorem 4.9 in [5]. $\square$

Consequently, every linear transformation $T : \mathbb{C}^n \to \mathbb{C}^m$ is bounded, i.e. $T \in B(\mathbb{C}^n, \mathbb{C}^m)$. And therefore $\|T\|_{op} < \infty$ is well-defined. In addition, $T(x) = Ax$ for some $n \times m$ matrix A (as the underlying field is $\mathbb{C}$) and the norm of the matrix A is defined by $\|A\|_{op} = \|T\|_{op}$.

The goal in this subsection is to state an important theorem about the stability of the system of linear delay differential equations

$$y'(t) = L(y_t), \tag{7}$$

where $L \in B(C, \mathbb{C}^n)$ and $C := C\big([-r, 0], \mathbb{C}^n\big)$ with $r > 0$. Note that complex-valued solutions are allowed even though one is only interested in real valued solutions. It will make it easier studying system (7). The norm $\| \cdot \|$ on $\mathbb{C}^n$ will be defined by $\|\mathbf{z}\| = \sqrt{|z_1|^2 + \ldots + |z_n|^2}$. An important example is the discrete-delay case.

Example 2.3. Let $C := C\big([-r, 0], \mathbb{C}^n\big)$ for $r > 0$ as usual and let A and B be $n \times n$ matrices. Define $L : C \to \mathbb{C}^n$ by $L(\phi) = A\phi(0) + B\phi(-r)$. Then L is linear and for all $\phi \in C$ we have

$$
\begin{aligned}
\|L(\phi)\| &= \|A\phi(0) + B\phi(-r)\| \\
&\le \|A\phi(0)\| + \|B\phi(-r)\| &&\text{(by triangle inequality)} \\
&\le \|A\|_{op} \cdot \|\phi(0)\| + \|B\|_{op} \cdot \|\phi(-r)\| &&\text{(by ($*$) and Theorem 2.1)} \\
&\le \|A\|_{op} \cdot \|\phi\|_\infty + \|B\|_{op} \cdot \|\phi\|_\infty &&\text{(by definition of $\| \cdot \|_\infty$)} \\
&= K\|\phi\|_\infty,
\end{aligned}
$$

with $K = \|A\|_{op} + \|B\|_{op}$. Consequently, L is bounded and (7) takes the form

$$y'(t) = Ay(t) + By(t - r).$$

A key tool for determining the stability of delay differential equations are the corresponding characteristic equations. For that, consider (7). We will seek exponential solutions of the form $y(t) = e^{\lambda t} v$ with $\lambda \in \mathbb{C}$, $t \in \mathbb{R}$ and $v \in \mathbb{C}^n \setminus \{0\}$. It is useful defining the function

$\exp_\lambda : [-r, 0] \to \mathbb{C}$ by $\exp_\lambda(\theta) = e^{\lambda\theta}$, because then y_t (corresponding to y) can be expressed by $y_t(\theta) = y(t + \theta) = e^{\lambda t} \exp_\lambda(\theta) v$. This implies $y_t = e^{\lambda t}(\exp_\lambda) v$. Consequently, if $y(t) = e^{\lambda t} v$ is a solution of (7), then $\lambda e^{\lambda t} v = y'(t) = L(y_t) = e^{\lambda t} L(\exp_\lambda v)$ since L is linear. Equivalently,

$$L(\exp_\lambda v) = \lambda v. \tag{8}$$

This equation almost looks like an eigenvalue problem. To make it a proper eigenvalue problem, define the bounded operator $L_\lambda : \mathbb{C}^n \to \mathbb{C}^n$ by $L_\lambda(v) = L(\exp_\lambda v)$. Equation (8) is now equivalent to

$$L_\lambda(v) = \lambda v, \tag{9}$$

which is a proper eigenvalue problem. Hence, $y(t) = e^{\hat\lambda t} v$ is a nontrivial solution of (7) if $\hat\lambda$ satisfies the *characteristic equation* $\det(L_\lambda - \lambda I) = 0$ and $v \in \ker\left(L_{\hat\lambda} - \hat\lambda I\right) \setminus \{0\}$.

Since L is bounded and in particular linear, the *superposition* principle holds for (7); a linear combination of solutions is a solution again. Thus, a finite sum $y(t) = \sum_n c_n e^{\lambda_n t} v_n$ is a solution of (7) with λ_n *characteristic roots* of $\det(L_\lambda - \lambda I) = 0$ and v_n the corresponding *eigenvectors*. In this case $c_n \in \mathbb{C}$ are arbitrary constants. Infinite sums are also solutions under suitable conditions to ensure convergence.

One is of course interested in the general solution of (7). We will discuss this later in Section 3. For now, the attention still lies on the characteristic equation and the following definitions and theorems are given.

Lemma 2.2. The function $h : \mathbb{C} \to \mathbb{C}$ defined by $h(\lambda) = \det(L_\lambda - \lambda I)$ is an entire function.

Proof. See Lemma 4.1 in [2] $\qquad\qquad\square$

Recall that an entire function is an analytic function defined on the entire complex plane $\mathbb{C}$. Hence, $h(\lambda) = \det(L_\lambda - \lambda I)$ is arbitrarily often complex differentiable on $\mathbb{C}$. We have the following definition.

Definition 2.4. Let $h(\lambda) = \det(L_\lambda - \lambda I)$. We call $\lambda \in \mathbb{C}$ a root of *multiplicity* $k \geq 1$ if $h(\lambda) = h'(\lambda) = h''(\lambda) = \cdots = h^{(k-1)}(\lambda) = 0$ and $h^{(k)}(\lambda) \neq 0$.

The following theorem shows the huge importance of characteristic equations which basically determines the asymptotic stability and instability of the solution $y = 0$ of (7).

Theorem 2.3. Consider equation (7). Suppose that $\Re(\lambda) < \mu$ for every characteristic root λ. Then there exists $K > 0$ such that

$$\|y(t, \phi)\| \leq K e^{\mu t} \|\phi\|_\infty, \ t \geq 0, \ \phi \in C, \tag{10}$$

where $y(t, \phi)$ is the solution of (7) satisfying $y_0 = \phi$.

In particular, $y = 0$ is a asymptotically stable for (7) if $\Re(\lambda) < 0$ for every characteristic root; it is unstable if there is a root satisfying $\Re(\lambda) > 0$.

Proof. See Theorem 4.3 in [2] and Theorem 5.2 in [1]. $\qquad\qquad\square$

3   Stability of a harmonic oscillator with delay

As far as we know the results in this section have not been obtained before, unless otherwise stated.

We now turn to the harmonic oscillator. A harmonic oscillator with a certain delay system is described by the equation

$$y''(t) + ay(t) + by(t-1) = 0, \ t \geq 0, \tag{11}$$
$$y(t) = \psi(t), \ -1 \leq t \leq 0,$$

with an initial continuous function $\psi \in C([-1,0], \mathbb{R})$. In Example 2.2 we have already seen that (11) can be written in the form of (7). For additional clarity, this will be restated.

Equation (11) is equivalent to

$$\mathbf{x}'(t) = L(\mathbf{x}_t), \tag{12}$$

with $\mathbf{x}(t) = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}$ and $L : C \to \mathbb{R}^2$ defined by $L(\phi) = \begin{pmatrix} 0 & 1 \\ -a & 0 \end{pmatrix} \phi(0) + \begin{pmatrix} 0 & 0 \\ -b & 0 \end{pmatrix} \phi(-1)$ and $C = C([-1,0], \mathbb{R}^2)$. Moreover, in Example 2.3 it was shown that L is bounded, i.e. $L \in B(C, \mathbb{R}^2)$. Hence, Theorem 2.3 can be applied to (11) to determine the stability of $y = 0$ for various $a, b \in \mathbb{R}$.

Thus, we seek exponential solutions of the form $y(t) = \exp(\lambda t)$ with $\lambda \in \mathbb{C}$ and $t \in \mathbb{R}$. Substituting $y(t) = \exp(\lambda t)$ in (11) gives

$$\lambda^2 \exp(\lambda t) + a \exp(\lambda t) + b \exp(\lambda(t-1)) = 0 \iff \lambda^2 + a + b \exp(-\lambda) = 0.$$

The equation $\lambda^2 + a + b \exp(-\lambda) = 0$ is called the *characteristic equation* of (11).

Let $A = \begin{pmatrix} 0 & 1 \\ -a & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ -b & 0 \end{pmatrix}$. Note that, if $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ with $\mathbf{v} \in \mathbb{R}^2$ is a solution of (12), then $\lambda e^{\lambda t}\mathbf{v} = \mathbf{x}'(t) = L(\mathbf{x}_t) = A\mathbf{x}(t) + B\mathbf{x}(t-1) = Ae^{\lambda t}\mathbf{v} + Be^{\lambda t}e^{-\lambda}\mathbf{v}$ which is equivalent to $A\mathbf{v} + e^{-\lambda}B\mathbf{v} = \lambda\mathbf{v}$. The corresponding characteristic equation is

$$\det(A + e^{-\lambda}B - \lambda I) = 0 \iff \begin{vmatrix} -\lambda & 1 \\ -a - be^{-\lambda} & -\lambda \end{vmatrix} = 0$$
$$\iff \lambda^2 + a + be^{-\lambda} = 0.$$

Thus, the characteristic equations coincide which should be expected.

The question now is: for what choices of a and b will the equilibrium solution $y = 0$ be (asymptotically) stable or unstable for (11)? The focus will be first on asymptotic stability of $y = 0$. It is possible to reformulate this question in terms of the characteristic equation $\lambda^2 + a + be^{-\lambda} = 0$ using Theorem 2.3:

Determine all a and b such that all roots of $\lambda^2 + a + be^{-\lambda} = 0$ have negative real part.

3.1 Pontryagin's Theorem

Multiplying e^λ on both sides of $\lambda^2 + a + be^{-\lambda} = 0$ results, equivalently, in

$$H(\lambda) := \lambda^2 e^\lambda + ae^\lambda + b = 0. \tag{13}$$

H is a special function, usually called an *exponential polynomial* or a *quasi-polynomial*. The problem analyzing the distribution of the zeros in the complex plane has received a great deal of attention.

Definition 3.1. Let $h(z, w)$ be a polynomial in two (complex) variables z and w (with complex coefficients); $h(z, w) = \sum_{m,n} a_{mn} z^m w^n$ in which the sum is finite. We call the term $a_{rs} z^r w^s$ the *principal term* of $h(z, w)$ if $a_{rs} \neq 0$, and for each term $a_{mn} z^m w^n$ with $a_{mn} \neq 0$, we have $r \geq m$ and $s \geq n$.

Note that not every polynomial with two variables has a principal term. However, $H(\lambda) = h(\lambda, e^\lambda)$ with $h(z, w) = z^2 w + aw + b$. Hence, it is clear from Definition 3.1 that $h(z, w)$ has principal term $z^2 w$.

Before stating an important theorem, the definition of interlacing polynomials will be given.

Definition 3.2. Let f be a monic real-rooted polynomial of degree n and g a monic real-rooted polynomial of degree n or $n-1$, with roots $\alpha_n \leq \ldots \leq \alpha_1$, $\beta_n \leq \ldots \leq \beta_1$, respectively (ignoring β_n if g is of degree $n-1$.) We say that g *interlaces* f if $\beta_n \leq \alpha_n \leq \beta_{n-1} \leq \ldots \leq \beta_1 \leq \alpha_1$.

This definition will not be used any further. It is stated for the completeness of the following theorem that is very important and helpful for studying asymptotic stability of (11). It is a theorem of Pontryagin. See [3] and [7].

Theorem 3.1. Let $H(\lambda) = h(\lambda, e^\lambda)$, where $h(z, w)$ is a polynomial with a principal term. The function $H(iy)$ is now separated into real and imaginary parts; that is, we set $H(iy) = F(y) + iG(y)$. If all the zeros of the function $H(\lambda)$ lie in the open left half plane, then the zeros of the functions $F(y)$ and $G(y)$ are real, interlacing, and

$$\Delta(y) := G'(y)F(y) - G(y)F'(y) > 0 \tag{14}$$

for all $y \in \mathbb{R}$. Moreover, in order that all the zeros of the function $H(\lambda)$ lie in the open left half plane, it is sufficient that one of the following conditions be satisfied:

(a) All the zeros of the functions $F(y)$ and $G(y)$ are real interlace, and the inequality (14) is satisfied for at least one value of y.
(b) All the zeros of the function $F(y)$ are real and for each of these zeros $y = y_0$ condition (14) is satisfied; that is, $F'(y_0)G(y_0) < 0$.
(c) All the zeros of the function $G(y)$ are real and for each of these zeros $y = y_0$ condition (14) is satisfied; that is, $G'(y_0)F(y_0) > 0$.

Proof. See [7]. □

So in our case, we have

$$
\begin{aligned}
H(iy) &= (iy)^2 e^{iy} + ae^{iy} + b \\
&= -y^2 \cos(y) - iy^2 \sin(y) + a\cos(\lambda) + ia\sin(y) + b \\
&= -y^2 \cos(y) + a\cos(y) + b + i(-y^2 \sin(y) + a\sin(y)) \\
&= F(y) + iG(y)
\end{aligned}
$$

with $F(y) = -y^2 \cos(y) + a\cos(y) + b$ and $G(y) = -y^2 \sin(y) + a\sin(y)$.

3.2 Consequences of Pontryagin's Theorem

The following lemma gives a necessary condition for asymptotic stability of the zero solution of (11). This lemma is basically Lemma 3.1 in [3], but we have included the (short) proof whereas the proof was omitted in [3].

Lemma 3.2. If the zero solution of (11) is asymptotically stable, then $\Delta(0) = a(a+b) > 0$.

Proof. We have

$$G(y) = (a - y^2)\sin(y),$$
$$G'(y) = -2y\sin(y) + (a - y^2)\cos(y)$$
$$F(y) = -y^2\cos(y) + a\cos(y) + b,$$
$$F'(y) = -2y\cos(y) + y^2\sin(y) - a\sin(y).$$

All the zeros of H lie in the left open half plane, because $y = 0$ is asymptotically stable. So we must have $\Delta(y) > 0$ for all $y \in \mathbb{R}$ by Theorem 3.1. In particular,

$$\Delta(0) = G'(0)F(0) - G(0)F'(0) = a(a+b) > 0.$$

$\square$

The following theorem is Lemma 3.2 in [3], but we have translated it in our parameters a and b. The proof of Lemma 3.2 in [3] has therefore also slightly been adjusted to fit our context.

Theorem 3.3. If $a < 0$, then the zero solution of (10) is not asymptotically stable.

Proof. $G(y) = (a - y^2)\sin(y)$, therefore $\pm i\sqrt{-a}$ is a complex root of G as $a < 0$. This means that there is a root $\hat{\lambda}$ of H with $\Re(\hat{\lambda}) \geq 0$ by Theorem 3.1. This implies that $y = 0$ is not asymptotically stable by Theorem 2.3. $\square$

The following two theorems determine precisely all $a, b \neq 0$ such that $y = 0$ is asymptotically stable for (11). The case $b < 0$ will be considered first, then $b > 0$.

Theorem 3.4. Assume $b < 0$. Then the zero solution of (11) is asymptotically stable if and only if $a > 0$, and there exists a $k \in \mathbb{N} \cup \{0\}$ such that

$$2k\pi < \sqrt{a} < (2k+1)\pi$$

and

$$b > \max\{(2k)^2\pi^2 - a, a - (2k+1)^2\pi^2\}.$$

Proof. We have

$$G(y) = (a - y^2)\sin(y),$$
$$G'(y) = -2y\sin(y) + (a - y^2)\cos(y)$$
$$F(y) = -y^2\cos(y) + a\cos(y) + b.$$

First, assume that $y = 0$ is asymptotically stable. If $a = 0$, then $\Delta(0) = 0$ and by Lemma 3.2 the zero solution of (11) is not asymptotically stable. If $a < 0$, Theorem 3.3 yields that the zero solution of (11) is not asymptotically stable. Thus, it is necessary that $a > 0$.

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The zeros of G are $y = \pm\sqrt{a}$ and $y = n\pi$ with $n \in \mathbb{Z}$. If y_0 is a zero of G, then

$$\Delta(y_0) = F(y_0)G'(y_0) = \Big(-y_0^2\cos(y_0) + a\cos(y_0) + b\Big)\Big(-2y_0\sin(y_0) + (a - y_0^2)\cos(y_0)\Big).$$

All the zeros of F and G are real and $\Delta(y) > 0$ for all $y \in \mathbb{R}$ by Theorem 3.1. So as $b < 0$ and $y = \pm\sqrt{a}$ are zeros of G, the following holds

$$\begin{aligned}\Delta(-\sqrt{a}) = \Delta(\sqrt{a}) > 0 &\iff -2b\sqrt{a}\sin(\sqrt{a}) > 0 \\ &\iff \sin(\sqrt{a}) > 0 \\ &\iff 2k\pi < \sqrt{a} < (2k+1)\pi \text{ for some } k = 0, 1, 2, \ldots\end{aligned}$$

So there exists a $\overline{k} \in \mathbb{N} \cup \{0\}$ such that $2\overline{k}\pi < \sqrt{a} < (2\overline{k}+1)\pi$.

At the points $y = n\pi$ $(n \in \mathbb{Z})$ we have

$$\begin{aligned}\Delta(n\pi) &= \Big((-n^2\pi^2 + a)(-1)^n + b\Big)\Big((a - n^2\pi^2)(-1)^n\Big) \\ &= (a - n^2\pi^2)^2 + b(a - n^2\pi^2)(-1)^n.\end{aligned}$$

Thus,

$$\begin{aligned}\Delta(n\pi) > 0 &\iff (a - n^2\pi^2)^2 + b(a - n^2\pi^2)(-1)^n > 0 \\ &\iff (n^2\pi^2 - a)^2 > b(n^2\pi^2 - a)(-1)^n. \end{aligned} \tag{15}$$

Inequality (15) must hold by Theorem 2.3. We distinguish two cases for $n \in \mathbb{Z}$.

Case 1: Let $n > 2\overline{k}$.
Thus $n\pi > 2\overline{k}\pi$ and $0 < 2\overline{k}\pi < \sqrt{a}$, so that $(2\overline{k})^2\pi^2 < a < (2\overline{k}+1)^2\pi^2 \leq n^2\pi^2$ and therefore $n^2\pi^2 - a > 0$. For n even, the right hand side of (15) is negative, and inequality (15) is satisfied. For n odd, (15) is equivalent to $a - n^2\pi^2 < b$. Observe that $a - n^2\pi^2 < b$ holds for all odd n with $n > 2\overline{k}$ if and only if it holds for $n = 2\overline{k} + 1$.

Case 2: Let $0 \leq n \leq 2\overline{k}$
Thus $0 \leq n\pi \leq 2\overline{k}\pi < \sqrt{a}$ and $n^2\pi^2 - a < 0$. For n odd, the right hand side of (15) is negative, and inequality (15) is satisfied. For n even, (15) is equivalent to $n^2\pi^2 - a < b$. Observe that $n^2\pi^2 - a < b$ holds for all even n with $0 \leq n \leq 2\overline{k}$ if and only if it holds for $n = 2\overline{k}$.

Combining these two cases results in $b > \max\{(2\overline{k})^2\pi^2 - a, a - (2\overline{k}+1)^2\pi^2\}$.

Now suppose that $a > 0$ and there exists a $k \in \mathbb{N} \cup \{0\}$ such that $2k\pi < \sqrt{a} < (2k+1)\pi$ and $b > \max\{(2k)^2\pi^2 - a, a - (2k+1)^2\pi^2\}$. Then all the zeros of G are real and for each zero $y_0 \in \{\sqrt{a}, -\sqrt{a}, n\pi : n \in \mathbb{Z}\}$ of G we have $\Delta(y_0) = F(y_0)G(y_0) > 0$ by reversing the arguments above. We only have to note that Δ is an even function. So no additional analysis is needed for negative $n \in \mathbb{Z}$. Therefore, every zero of H has negative real part by Theorem 3.1.(c). Asymptotic stability now follows from Theorem 2.3. $\qquad\square$

Theorem 3.4 is basically Theorem 3.2 in [3] from Cahlon and Schmidt, but it has been translated in our parameters a and b. The proof of Theorem 3.2 in [3] has therefore also slightly been adjusted to fit our context. Cahlon and Schmidt only considered 'mixed coefficients' in [3], i.e. $b < 0$ and $a > 0$. Of course, we are not only interested in mixed coefficients. So the next theorem includes the case $b > 0$ which as far as we know has not been studied before.

Theorem 3.5. Assume $b > 0$. Then the zero solution of (11) is asymptotically stable if and only if $a > 0$ and there exists a $k \in \mathbb{N} \cup \{0\}$ such that

$$(2k + 1)\pi < \sqrt{a} < (2k + 2)\pi$$

and

$$b < \min\{a - (2k + 1)^2\pi^2, (2k + 2)^2\pi^2 - a\}.$$

The proof is analogue to the proof of Theorem 3.4.

Proof. We have

$$G(y) = (a - y^2)\sin(y),$$
$$G'(y) = -2y\sin(y) + (a - y^2)\cos(y)$$
$$F(y) = -y^2\cos(y) + a\cos(y) + b.$$

First, assume that $y = 0$ is asymptotically stable. If $a = 0$, then $\Delta(0) = 0$ and by Lemma 3.2 the zero solution of (11) is not asymptotically stable. If $a < 0$, Theorem 3.3 yields that the zero solution of (11) is not asymptotically stable. Thus, it is necessary that $a > 0$.

The zeros of G are $y = \pm\sqrt{a}$ and $y = n\pi$ with $n \in \mathbb{Z}$. If y_0 is a zero of G, then

$$\Delta(y_0) = F(y_0)G'(y_0) = \left(-y_0^2\cos(y_0) + a\cos(y_0) + b\right)\left(-2y_0\sin(y_0) + (a - y_0^2)\cos(y_0)\right).$$

All the zeros of F and G are real and $\Delta(y) > 0$ for all $y \in \mathbb{R}$ by Theorem 3.1. So as $b > 0$ and $y = \pm\sqrt{a}$ are zeros of G, the following holds

$$\Delta(-\sqrt{a}) = \Delta(\sqrt{a}) > 0 \iff -2b\sqrt{a}\sin(\sqrt{a}) > 0$$
$$\iff \sin(\sqrt{a}) < 0$$
$$\iff (2k + 1)\pi < \sqrt{a} < (2k + 2)\pi \text{ for some } k = 0, 1, 2, \ldots$$

So there exists a $\bar{k} \in \mathbb{N} \cup \{0\}$ such that $(2\bar{k} + 1)\pi < \sqrt{a} < (2\bar{k} + 2)\pi$.

At the points $y = n\pi$ ($n \in \mathbb{Z}$) we have

$$\Delta(n\pi) = \left((-n^2\pi^2 + a)(-1)^n + b\right)\left((a - n^2\pi^2)(-1)^n\right)$$
$$= (a - n^2\pi^2)^2 + b(a - n^2\pi^2)(-1)^n.$$

Thus,

$$\Delta(n\pi) > 0 \iff (a - n^2\pi^2)^2 + b(a - n^2\pi^2)(-1)^n > 0$$
$$\iff (n^2\pi^2 - a)^2 > b(n^2\pi^2 - a)(-1)^n. \tag{16}$$

Inequality (16) must hold by Theorem 2.3. We distinguish two cases for $n \in \mathbb{Z}$.

Case 1: Let $n > 2(\bar{k} + 1)$.
Thus $n\pi > 2(\bar{k} + 1)\pi$ and $0 < 2(\bar{k} + 1)\pi < \sqrt{a}$, so that $(2\bar{k} + 1)^2\pi^2 < a < (2\bar{k} + 2)^2\pi^2 \leq n^2\pi^2$ and therefore $n^2\pi^2 - a > 0$. For n odd, the right hand side of (16) is negative, and inequality

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(16) is satisfied. For n even, (16) is equivalent to $n^2\pi^2 - a > b$. Observe that $n^2\pi^2 - a > b$ holds for all even n with $n > 2(\overline{k}+1)$ if and only if it holds for $n = 2\overline{k}+2$.

Case 2: Let $0 \le n \le 2(\overline{k}+1)$
Thus $0 \le n\pi \le 2(\overline{k}+1)\pi < \sqrt{a}$ and $n^2\pi^2 - a < 0$. For n even, the right hand side of (16) is negative, and inequality (16) is satisfied. For n odd, (16) is equivalent to $a - n^2\pi^2 > b$. Observe that $a - n^2\pi^2 > b$ holds for all odd n with $0 \le n \le 2(\overline{k}+1)$ if and only if it holds for $n = 2\overline{k}+1$.

Combining these two cases results in $b < \min\{a - (2\overline{k}+1)^2\pi^2, (2\overline{k}+2)^2\pi^2 - a\}$.

Now suppose that $a > 0$ and there exists a $k \in \mathbb{N} \cup \{0\}$ such that $2k\pi < \sqrt{a} < (2k+1)\pi$ and $b < \min\{a - (2k+1)^2\pi^2, (2k+2)^2\pi^2 - a\}$. Then all the zeros of G are real and for each zero $y_0 \in \{\sqrt{a}, -\sqrt{a}, n\pi : n \in \mathbb{Z}\}$ of G we have $\Delta(y_0) = F(y_0)G(y_0) > 0$ by reversing the arguments above. We only have to note that Δ is an even function. So no additional analysis is needed for negative $n \in \mathbb{Z}$. Therefore, every zero of H has negative real part by Theorem 3.1.(c). Asymptotic stability now follows from Theorem 2.3. $\qquad\square$

3.3 (Asymptotic) stability region

Now combining Theorem 3.3, 3.4 and 3.5, all values of $b \ne 0$ and $a \in \mathbb{R}$ are determined such that $y = 0$ is asymptotically stable for (11). See figure 2 for a part of this region. Every $k \in \mathbb{N} \cup \{0\}$ defines a set $A_k \subseteq \mathbb{R}^2$ such that $y = 0$ is asymptotically stable for (11) if $(a, b) \in A_k$ by using Theorem 3.4 and 3.5. In this case A_k is a triangle. All values of $a \in \mathbb{R}$ and $b \ne 0$ for which $y = 0$ is asymptotically stable for (11) consists of infinitely many triangles A_k that alternate around $b = 0$ and they 'grow bigger' as $a \to \infty$. We denote $S = \bigcup_{k=0}^{\infty} A_k$ as the *asymptotic stability region*. So what does Figure 2 exactly say? It tells us that a point (a, b) is in the interior of one of those triangles (i.e. $(a, b) \in S$) if and only if $y = 0$ is asymptotically stable for (11).
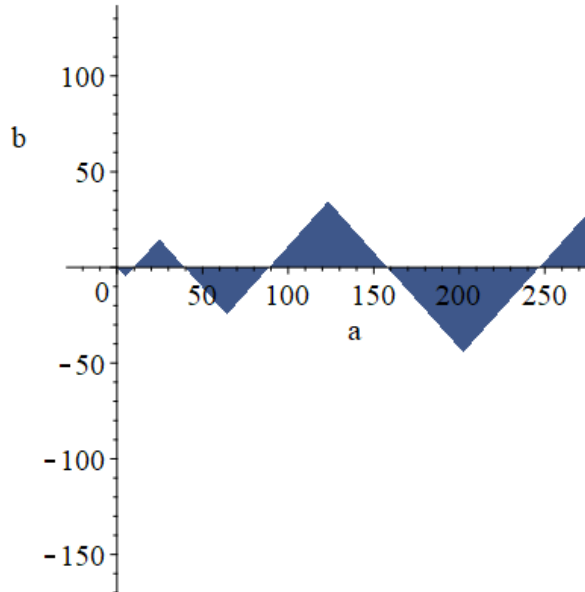

Figure 2: Asymptotic stability region.

An interesting matter to examine is how the solution of (11) behaves on the boundaries of the triangles. The lines that overlap the boundaries are explicitly given by $b_n(a) = (-1)^n(n^2\pi^2 - a)$ ($n = 0, 1, 2, ...$) by Theorem 3.4 and/or 3.5. A few lines are plotted in Figure 3. The yellow line corresponds with $n = 0$, the red line corresponds with $n = 1$, the blue line corresponds with $n = 2$, the green line corresponds with $n = 3$, the purple line corresponds with $n = 4$ and the brown line corresponds with $n = 5$. Figure 4 shows the asymptotic stability region that is enclosed by $b_n(a) = (-1)^n(n^2\pi^2 - a)$. Interestingly, these lines coincide with the following:

Substituting $\lambda = i\omega$ with $\omega \in \mathbb{R}$ in $\lambda^2 + a + be^{-\lambda} = 0$ results in

$$\begin{cases} -\omega^2 + a + b\cos(\omega) = 0 \\ b\sin(\omega) = 0 \end{cases} \tag{17}$$

by separating the real and imaginary part. Now, $\sin(\omega) = 0$ if and only if $\omega = k\pi$ with $k \in \mathbb{Z}$ and therefore $a = \omega^2 - b\cos(\omega) = k^2\pi^2 - b(-1)^k$ or $b_k(a) = (-1)^k(k^2\pi^2 - a)$. Note that $b_k(a) = b_{-k}(a)$ for $k \in \mathbb{Z}$. So there are infinitely many branches/lines $b_n(a) = (-1)^n(n^2\pi^2 - a)$ with $n \in \{0, 1, 2, ...\}$ such that the characteristic equation has (at least) two purely imaginary solutions, namely $\lambda = \pm in\pi$. Note that the complex conjugate is also a solution;

$$\left(\overline{\mu}\right)^2 + a + be^{-\overline{\mu}} = \overline{\mu^2} + a + b\overline{e^{-\mu}} = \overline{\mu^2 + a + be^{-\mu}} = \overline{0} = 0$$

if $\mu^2 + a + be^{-\mu} = 0$. We have the following lemma that makes the preceding more explicit.

Lemma 3.6. There exists a purely imaginary solution $\lambda = i\omega$ of $\lambda^2 + a + be^{-\lambda} = 0$ if and only if $(a, b) \in \{(x, y) \in \mathbb{R}^2 : y = (-1)^k(k^2\pi^2 - x)\}$ for some $k \in \mathbb{Z}$.

Proof. First, suppose that $\lambda = i\omega$ ($\omega \in \mathbb{R}$) is a purely imaginary solution. Then $\omega = k\pi$ for some $k \in \mathbb{Z}$ by the second equation of (17). This implies $a = k^2\pi^2 - b(-1)^k$ or $b = (-1)^k(k^2\pi^2 - a)$ by the first equation of (17), so that $(a, b) \in \{(x, y) \in \mathbb{R}^2 : y = (-1)^k(k^2\pi^2 - x)\}$ for some $k \in \mathbb{Z}$.

Now assume that $(a, b) \in \{(x, y) \in \mathbb{R}^2 : y = (-1)^k(k^2\pi^2 - x)\}$ for some $k \in \mathbb{Z}$. Set $\lambda = ik\pi$ and substitute it in $\lambda^2 + a + be^{-\lambda} = 0$. Then $\sin(k\pi) = 0$ and $-k^2\pi^2 + a + (-1)^{2k}(k^2\pi^2 - a) = 0$. So $\lambda = ik\pi$ satisfies both the equations of (17) and therefore $\lambda = ik\pi$ satisfies the characteristic equation. Hence, there exists a purely imaginary solution of $\lambda^2 + a + be^{-\lambda} = 0$. $\square$
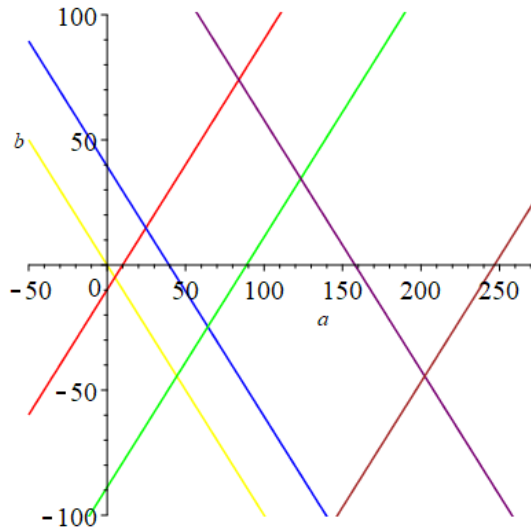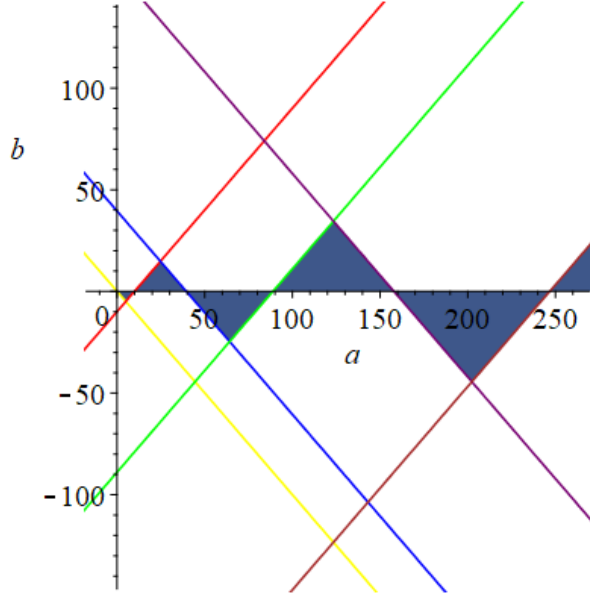

Figure 3: Critical lines.

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Figure 4: Asymptotic stability region enclosed by critical lines.

With this being said, there are therefore (at least) two purely imaginary solutions of $\lambda^2 + a + be^{-\lambda} = 0$ on the critical lines shown in Figure 3. Let $C_k = \{(x,y) \in \mathbb{R}^2 : y = (-1)^k(k^2\pi^2 - x)\}$. Then purely imaginary solutions of the characteristic equation exist if and only if $(a,b) \in C_k$ for some $k \in \mathbb{Z}$ by just rephrasing Lemma 3.6. In this case purely imaginary solutions are $\lambda = \pm ik\pi$. Keep in mind that the complex conjugate also is a solution. Also note that $C_k = C_{-k}$ for all $k \in \mathbb{Z}$. So we can restrict ourselves to C_n for $n = 0, 1, 2, \ldots$

The asymptotic stability region was already denoted by S. For clarification: $(a,b) \in S$ if and only if $y = 0$ is asymptotically stable for (11).

Theorem 3.7. If $(a,b) \notin \bigcup_{n=0}^{\infty} C_n \cup S$, then $y = 0$ is unstable for (11).

Proof. We prove by contradiction. Assume that $(a,b) \notin \bigcup_{n=0}^{\infty} C_n \cup S$ and suppose $y = 0$ is stable for (11). Then $(a,b) \in \bigcap_{n=0}^{\infty} (C_n)^c \cap S^c$. Let $\hat{\lambda}$ be an arbitrary solution of $\lambda^2 + a + be^{-\lambda} = 0$. Then $\Re(\hat{\lambda}) \leq 0$, otherwise $y = 0$ would have been unstable by Theorem 2.3. However, $(a,b) \notin C_n$ for all $n \in \mathbb{N} \cup \{0\}$. So we must have $\Re(\hat{\lambda}) < 0$ by Lemma 3.6. This implies that $y = 0$ is asymptotically stable by Theorem 2.3 and that $(a,b) \in S$. This is a contradiction. Therefore, $y = 0$ is unstable for (11). $\square$

Theorem 3.7 indicates that instability of $y = 0$ for (11) occurs when $(a,b) \notin \bigcup_{n=0}^{\infty} C_n \cup S$. However, there may be more values for a and b such that $y = 0$ is unstable. The remaining candidates for instability or stability of $y = 0$ are the points $(a,b) \in \bigcup_{n=0}^{\infty} C_n$. The lines C_n are tricky, because on these lines the characteristic equation $\lambda^2 + a + be^{-\lambda} = 0$ has (at least) two purely imaginary solutions $\lambda = \pm in\pi$. We already know for a fact that asymptotic stability of $y = 0$ can not occur if $(a,b) \in \bigcup_{n=0}^{\infty} C_n$. Repeating the results up until now will probably give more clarity:

So let $S_{1,k} = \{(a,b) \in \mathbb{R}^2 : (2k)^2\pi^2 < a < (2k+1)^2\pi^2, \ 0 > b > \max\{(2k)^2\pi^2 - a, a - (2k+1)^2\pi^2\}\}$ and $S_{2,k} = \{(a,b) \in \mathbb{R}^2 : (2k+1)^2\pi^2 < a < (2k+2)^2\pi^2, \ 0 < b < \min\{a - (2k+1)^2, (2k+2)^2 - a\}\}$. Then $S = \bigcup_{k=0}^{\infty} S_{1,k} \cup S_{2,k}$ is the asymptotic stability region. The points $(a,b) \in S$ are precisely the values for a and b such that $y = 0$ is asymptotically stable for (11). This was proven by

Theorem 3.4 and 3.5. Moreover, let $C_n = \{(x, y) \in \mathbb{R}^2 : y = (-1)^n(n^2\pi^2 - x)\}$ for $n \in \mathbb{N} \cup \{0\}$. If $(a, b) \notin \bigcup_{n=0}^{\infty} C_n \cup S$, then $y = 0$ is unstable for (11). Therefore, the stability type of $y = 0$ has so far been determined for all $(a, b) \in \mathbb{R}^2 \setminus \bigcup_{n=0}^{\infty} C_n$.

The remaining question is: Is $y = 0$ stable or unstable for (11) if $(a, b) \in \bigcup_{n=0}^{\infty} C_n$? In this case the characteristic equations become

$$\lambda^2 + a + (-1)^n(n^2\pi^2 - a)e^{-\lambda} = 0, \qquad (n = 0, 1, 2, ...) \tag{18}$$

because $b = (-1)^n(n^2\pi^2 - a)$. It has already been shown that $\lambda = \pm in\pi$ are purely imaginary solutions of (18). But what about the other zeros? The zeros determine the (in)stability of $y = 0$ of (11). However, in this case there are two parameters a and n that can change the distribution of the zeros. Unfortunately, Theorem 3.1 can not be used because it gives necessary and sufficient conditions about the zeros lying in the *open* left half plane. And equation (18) has purely imaginary solutions for all $a \in \mathbb{R}$ and $n \in \mathbb{N}$. So Theorem 3.1 needs to be modified to give a statement about the *closed* left half plane or something else needs to be found that potentially can help. It turns out that the stability can be determined for a part of $\bigcup_{n=0}^{\infty} C_n$ without the Theorem of Pontryagin.

Let the boundary of the asymptotic stability region S be denoted by ∂S. So ∂S consists of the boundaries of the triangles in Figure 2. Then it turns out that $y = 0$ is stable for (11) if $(a, b) \in \partial S \setminus \{(0, 0)\}$ as it will soon be shown why. We already know that $y = 0$ is stable for (11) if $b = 0$ and $a > 0$, because then $y(t) = c_1 \cos(t\sqrt{a}) + c_2 \sin(t\sqrt{a})$ is the general solution with $c_1, c_2 \in \mathbb{R}$. So the case $b \neq 0$ is left to be considered. Before we can show stability of $y = 0$ on ∂S, we need a few results.

Theorem 3.8. The functions $t^k e^{\lambda_0 t}$, $k = 0, 1, 2..., m - 1$, are solutions of equation (11) if and only if λ_0 is a root of multiplicity at least m of the characteristic equation $\lambda^2 + a + be^{-\lambda} = 0$.

This theorem was already proven in [2] and in [1] for the equations $y'(t) + ay(t - 1) = 0$ and $y'(t) + ay(t) + by(t - 1) = 0$, respectively. The theorem also holds for our equation $y''(t) + ay(t) + by(t - 1) = 0$ and the proof is analogue to the proofs in [2] and in [1].

Proof. Introduce the linear operator, defined on the (complex) differentiable functions, by $L(y) = y''(t) + ay(t) + by(t - 1)$ and let $H(\lambda) = \lambda^2 + a + be^{-\lambda}$. Then

$$L(t^k e^{\lambda t}) = L\left(\frac{\partial^k}{\partial \lambda^k}(e^{\lambda t})\right)$$

$$= \frac{\partial^k}{\partial \lambda^k} L(e^{\lambda t})$$

$$= \frac{\partial^k}{\partial \lambda^k}\left(e^{\lambda t} H(\lambda)\right)$$

$$= \sum_{j=0}^{k}\left[\binom{k}{j} H^{(j)}(\lambda)\frac{\partial^{(k-j)}}{\partial \lambda^{(k-j)}}(e^{\lambda t})\right]$$

$$= e^{\lambda t}\sum_{j=0}^{k}\binom{k}{j}t^{k-j}H^{(j)}(\lambda),$$

by using the General Leibniz Rule and noticing that $\frac{\partial^k}{\partial \lambda^k}$ commutes with L.

Assume λ_0 is a root of multiplicity at least m. Then $H^{(k)}(\lambda_0) = 0$ for all $k \in \{0, 1, ..., m - 1\}$

by Definition 2.4. Therefore, $L(t^k e^{\lambda_0 t}) = 0$ for all $k \in \{0, 1, ..., m-1\}$ by the equation above. Consequently, $y(t) = t^k e^{\lambda_0 t}$ is a solution of equation (11) for $k = 0, 1, 2..., m-1$.

If the functions $t^k e^{\lambda_0 t}$, $k = 0, 1, 2..., m-1$, are solutions of equation (11), then $L(t^k e^{\lambda_0 t}) = 0$ for all $k \in \{0, 1, ..., m-1\}$. This implies that $H^{(k)}(\lambda_0) = 0$ for all $k \in \{0, 1, ..., m-1\}$ by the equation above. Thus, λ_0 is a characteristic root of multiplicity at least m. $\qquad\square$

The general solution of a linear delay differential equation was mentioned in Section 2.3, but we chose not to go in detail until now. It turns out that the general solution of (11) is an infinite linear combination of such solutions in Theorem 3.8 under suitable conditions to ensure convergence. To make it more precise, consider problem (11) with initial function $y(t) = \psi(t)$ on $[-1, 0]$. Then

$$y(t) = \sum_{k=1}^{\infty} a_k t^{n_k} e^{\lambda_k t}$$

is the solution of problem (11) where $a_k \in \mathbb{C}$ are coefficients depending on $\psi(t)$ and $n_k \in \mathbb{N} \cup \{0\}$ are non-negative integers depending on the multiplicity of the characteristic root λ_k. The roots λ_k do not have to be distinct in the sum! The coefficients a_k are determined using Laplace transforms. Some of these results can be found in Hale and Lunel [1] and in Bellman and Cooke [8].

Theorem 3.9. If $(a, b) \in \partial S \setminus \{(0, 0)\}$, then $y = 0$ is stable for (11).

Proof. We only have to consider the case $b \neq 0$.
Suppose $(a, b) \in \partial S \setminus \{(0, 0)\} \subseteq \bigcup_{n=0}^{\infty} C_n$, then the purely imaginary characteristic roots are necessarily of the form $\lambda = ik\pi$ for some $k \in \mathbb{Z}$. This is immediate clear from (17). Furthermore, let $H(\lambda) = \lambda^2 + a + be^{-\lambda}$ then $H'(\lambda) = 2\lambda - be^{-\lambda}$. Suppose $H'(ik\pi) = 0$, then it follows that $2ik\pi = \pm b$. This is a contradiction as the left hand side is purely imaginary and b is a real number unequal to zero. Hence, $\lambda = ik\pi$ is a root of multiplicity 1 of the characteristic equation.

It is well known that the roots $\lambda = \lambda(a, b)$ of $\lambda^2 + a + be^{-\lambda} = 0$ depend continuously on $a \in \mathbb{R}$ and $b \neq 0$. Now assume that there is a root of the characteristic equation with positive real part if $(a, b) \in \partial S \setminus \{(0, 0)\}$. Then there is a purely imaginary solution $\mu = \mu(\hat{a}, \hat{b}) \in \mathbb{C}$ of the characteristic equation for some $(\hat{a}, \hat{b}) \in S$ by continuity of the roots and by the fact that in S all characteristic roots $\lambda \in \mathbb{C}$ satisfy $\Re(\lambda) < 0$. This is now forms a contradiction because $\Re(\mu(\hat{a}, \hat{b})) = 0$ with $(\hat{a}, \hat{b}) \in S$. Therefore, all roots of $\lambda^2 + a + be^{-\lambda} = 0$ have real part less or equal to zero if $(a, b) \in \partial S \setminus \{(0, 0)\}$.

We know that the general solution of (11) has the form $y(t) = \sum_{j=1}^{\infty} a_j t^{n_j} e^{\lambda_j t}$. Since all purely imaginary roots $\lambda = ik\pi$ have multiplicity 1, $t^m e^{ik\pi t}$ are no solutions of (11) with $m \geq 1$. And the corresponding solutions of the characteristic root $\lambda = ik\pi$ are $\cos(k\pi t)$ and $\sin(k\pi t)$ which are periodic functions. This follows from Theorem 3.8. Therefore, the general solution of (11) can be written as

$$y(t) = p(t) + \sum_{l=1}^{\infty} a_l t^{n_l} e^{\lambda_l t},$$

with $p(t)$ a periodic function and characteristic roots $\lambda_l \in \mathbb{C}$ satisfying $\Re(\lambda_l) < 0$ for all $l \in \mathbb{N}$. We have $\lim_{t \to \infty} a_l t^{n_l} e^{\lambda_l t} = 0$ for all $l \in \mathbb{N}$. Moreover, $\sum_{l=1}^{\infty} a_l t^{n_l} e^{\lambda_l t}$ is uniformly convergent for all $t \geq -1$ as $\Re(\lambda_l) < 0$ for all $l \in \mathbb{N}$ (see Theorem 6.5 in [8]). Therefore, $y(t) \to p(t)$ as $t \to \infty$. Hence, the solution $y(t)$ converges to a periodic solution and therefore $y = 0$ is stable for (11). $\qquad\square$

The *stability region* has now been expanded a bit. We have the following statement.

Corollary 3.10. Let S be the asymptotic stability region defined implicitly by Theorem 3.4 and 3.5. If $(a, b) \in (S \cup \partial S) \setminus \{(0,0)\} = \overline{S} \setminus \{(0,0)\}$, then $y = 0$ is stable for (11) where $\overline{S}$ denotes the closure of S.

What still remains to be studied is the stability of $y = 0$ of (11) when $(a, b) \in \bigcup_{n=0}^{\infty} C_n \setminus \overline{S}$. Unfortunately, until this day this is still an open problem. This is not the end, because a numerical scheme for (11) will be presented in Section 4. In particular, numerical solutions of (11) will be shown for $(a, b) \in \bigcup_{n=0}^{\infty} C_n \setminus \overline{S}$. And then presumptions can be made based on these solutions. Although there is not a mathematical proof for (in)stability if $(a, b) \in \bigcup_{n=0}^{\infty} C_n \setminus \overline{S}$, the numerical solutions might give a good indication whether $y = 0$ is stable or unstable. See Section 4 for more information. (For now, our guess is that $y = 0$ is unstable for (11) because, visually by Figure 4, all points $(a, b) \in \bigcup_{n=0}^{\infty} C_n \setminus \overline{S}$ are surrounded by points $(\hat{a}, \hat{b}) \notin \bigcup_{n=0}^{\infty} C_n \cup S$ such that $y = 0$ is unstable for $y''(t) + \hat{a}y(t) + \hat{b}y(t - 1) = 0$ by Theorem 3.7.)

Thus far, we have only restricted ourselves to the case of stability of $y = 0$. Now what about the stability of other equilibrium points of (11) other than zero? Equilibrium points satisfy the equation $y'(t) = 0$. This implies that $y(t) = c$ for some $c \in \mathbb{R}$ and that c is an equilibrium point. The following lemma gives a necessary and sufficient condition for the existence of nontrivial equilibrium solutions.

Lemma 3.11. Let $a, b \neq 0$. There exists an equilibrium solution $\hat{y}(t) = c$ of (11) with $c \in \mathbb{R}$ arbitrary if and only if $a + b = 0$

Proof. Let $c \in \mathbb{R}$ be arbitrary. If $\hat{y}(t) = c \in \mathbb{R}$ is equilibrium solution of (11), then $\hat{y}''(t) = 0$. So that (11) becomes $a\hat{y}(t) + b\hat{y}(t - 1) = 0$. This implies $\hat{y}(t) = -\frac{b}{a}\hat{y}(t - 1)$. This is a difference equation. The corresponding auxiliary equation is $\lambda + \frac{b}{a} = 0$. Hence, $\hat{y}(t) = d(-\frac{b}{a})^t$ with $d, t \in \mathbb{R}$ is the general solution of $\hat{y}(t) = -\frac{b}{a}\hat{y}(t - 1)$. Therefore, $c = \hat{y}(t) = d(-\frac{b}{a})^t$ for all $t \in \mathbb{R}$ which implies that $-\frac{b}{a} = 1$ (and $c = d$). And of course, $-\frac{b}{a} = 1$ if and only if $a + b = 0$.

Suppose $a + b = 0$, then (11) becomes $y''(t) + a(y(t) - y(t - 1)) = 0$. It is directly seen that $\hat{y}(t) = c \in \mathbb{R}$ is an equilibrium solution for all $c \in \mathbb{R}$. $\square$

So nontrivial equilibrium solutions exist if and only if $a + b = 0$, and then the equilibrium points are $y(t) = c$ with $c \in \mathbb{R}$ arbitrary.

Theorem 3.12. If $(a, b) \in (\partial S \cap C_0) \setminus \{(0,0)\}$, then $y(t) = c$ is a stable equilibrium solution of (11) for all $c \in \mathbb{R}$.

Proof. Let $c \in \mathbb{R}$ be arbitrary and assume that $(a, b) \in (\partial S \cap C_0) \setminus \{(0,0)\}$. Then $a + b = 0$ as $(a, b) \in C_0 = \{(x, y) \in \mathbb{R}^2 : y = -x\}$. So $y(t) = c$ is an equilibrium solution of (19) by Lemma 3.11.

Now set $w(t) = y(t) - c$. Then

$$
\begin{aligned}
w''(t) &= y''(t) \\
&= -ay(t) + ay(t - 1) & \text{(because } b = -a) \\
&= -a(w(t) + c) + a(w(t - 1) + c) \\
&= -aw(t) + aw(t - 1) \\
&= -aw(t) - bw(t - 1).
\end{aligned}
$$

Hence, the stability of $y(t) = c$ of (11) is equivalent to the stability of $w(t) = 0$ of $w''(t) + aw(t) + bw(t-1) = 0$. Since $(a, b) \in \partial S \setminus \{(0,0)\}$, we have that $w(t) = 0$ is stable for $w''(t) + aw(t) + bw(t-1) = 0$ by Theorem 3.9. Therefore, $y(t) = c$ is a stable equilibrium solution of (11). $\qquad \square$

The proof in Theorem 3.12 actually tells us that the stability of nontrivial equilibrium solutions of (11) are equivalent to the stability of $y(t) = 0$ of (11) if $(a, b) \in C_0$.

All in all, Figure 5 shows visually the results that we obtained. The gray area contains the values for a and b such that $y = 0$ is unstable for (11). The blue area without boundary contains precisely all values for a and b such that $y = 0$ is asymptotically stable for (11). The red lines contain the values for a and b such that $y = 0$ is stable for (11). The stability of $y = 0$ is still unknown on the black lines (that are not the coordinate axes). Note that this is not the whole (a, b) plane, but the structure in Figure 5 is the same for larger a.
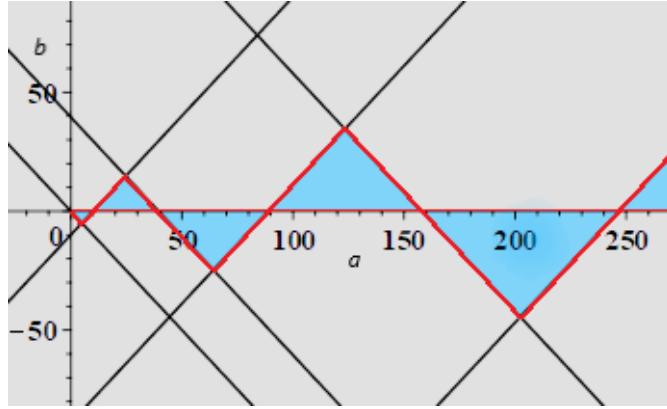
Figure 5: Grey region indicates instability, red lines indicate stability, blue region indicates asymptotic stability and the stability is unknown on the black lines.

3.4   Stability of a harmonic oscillator with arbitrary delay

So far, the qualitative properties of $y''(t) + ay(t) + by(t-1) = 0$ has been thoroughly studied. In particular, the stability type of $y = 0$ is determined for various $(a, b) \in \mathbb{R}^2$. In this case the delay $r > 0$ was chosen to be equal to 1. Now what about other delays $r > 0$? What can be said about the stability of

$$y''(t) + ay(t) + by(t-r) = 0 \tag{19}$$

with an arbitrary delay $r > 0$?

Now let $r > 0$ be arbitrary and fixed. To say something about the (asymptotic) stability and instability of (19), it is useful to use Theorem 2.3. Substituting $y(t) = e^{\lambda t}$ in (19) results in the characteristic equation

$$\lambda^2 + a + be^{-\lambda r} = 0 \tag{20}$$

with $\lambda \in \mathbb{C}$ and $r > 0$.

So the question again is: for what choices of a and b will the equilibrium solution $y = 0$ be (asymptotically) stable or unstable for (19)? This question can be directly translated in relation to the characteristic equation $\lambda^2 + a\lambda + be^{-\lambda r} = 0$ by Theorem 2.3. Studying the distribution of the zeros of (20) can be troublesome as it was seen in previous sections. It actually looks more difficult now as we now have an extra parameter $r > 0$. However, this will not entirely be the case. The results in previous subsections will be used to answer stability questions around equation (19).

The first step is writing (19) in the form of a delay differential equation with delay equal to 1. This can be done by a coordinate transformation $\tau = \frac{t}{r}$. Now define $y(t)$ by $y(t) = \hat{y}(\tau(t))$. Then

$$
y'(t) = \frac{d}{d\tau}\hat{y}(\tau(t)) \cdot \frac{d\tau}{dt}(t) = \hat{y}'(\tau)\frac{1}{r},
$$
$$
y''(t) = \frac{1}{r}\frac{d}{d\tau}\hat{y}'(\tau(t)) \cdot \frac{d\tau}{dt}(t) = \hat{y}''(\tau)\frac{1}{r^2},
$$
$$
y(t - r) = \hat{y}\left(\frac{t-r}{r}\right) = \hat{y}(\tau - 1).
$$

Therefore, (19) is equivalent to

$$
\hat{y}''(\tau) + ar^2\hat{y}(\tau) + br^2\hat{y}(\tau - 1) = 0, \ \tau \geq 1 \tag{21}
$$

The following remark is important for studying stability properties of (19).

Remark 3.1. We have $\lim\limits_{t\to\infty} y(t) = 0$ if and only if $\lim\limits_{\tau\to\infty} \hat{y}(\tau) = 0$, because $t \to \infty$ if and only if $\tau \to \infty$. Hence, $\hat{y} = 0$ is asymptotically stable for (21) if and only if $y = 0$ is asymptotically stable for (19).

With this being said, it is now possible to obtain the asymptotic stability region for (19). The following two theorems determine this region for a fixed delay $r > 0$.

Theorem 3.13. Let $r > 0$ be the delay and assume $b < 0$. Then the zero solution of (19) and (21) is asymptotically stable if and only if $a > 0$, and there exists a $k \in \mathbb{N} \cup \{0\}$ such that

$$2k\pi < \sqrt{ar^2} < (2k+1)\pi$$

and

$$br^2 > \max\{(2k)^2\pi^2 - ar^2, ar^2 - (2k+1)^2\pi^2\}.$$

Proof. Apply Theorem 3.4 to equation (21) (and use Remark 3.1) $\qquad\square$

Theorem 3.14. Let $r > 0$ be the delay and assume $b > 0$. Then the zero solution of (19) and (21) is asymptotically stable if and only if $a > 0$ and there exists a $k \in \mathbb{N} \cup \{0\}$ such that

$$(2k+1)\pi < \sqrt{ar^2} < (2k+2)\pi$$

and

$$br^2 < \min\{ar^2 - (2k+1)^2\pi^2, (2k+2)^2\pi^2 - ar^2\}.$$

Proof. Apply Theorem 3.5 to equation (21) (and use Remark 3.1) □

The asymptotic stability regions for fixed delays $r > 0$ look the same as the asymptotic stability region when $r = 1$. See Figure 6 and 7 for a part of the asymptotic stability regions with delay $r = 2$ and $r = 4$, respectively. Note that the triangles keep alternating around $b = 0$, but they get substantially 'smaller' when r gets bigger. But that doesn't mean that asymptotic stability should occur with $r = 4$ when asymptotic stability occurs when $r = 2$ or exactly the opposite. For example, if $(a, b) = (16, -1)$, then $y = 0$ is asymptotically stably for (19) with $r = 2$. One can check this visually in Figure 6 or choose $k = 1$ in Theorem 3.13 and check the conditions. However, $y = 0$ is not asymptotically stable for (19) with $r = 4$ when $(a, b) = (16, -1)$. One can check this visually in Figure 7 or check that the conditions do not hold in Theorem 3.13. The zero solution is actually unstable in the latter case as it will be shown later in Theorem 3.16. So changing the delay $r > 0$ might change the stability of $y = 0$ for fixed $(a, b) \in \mathbb{R}^2$.
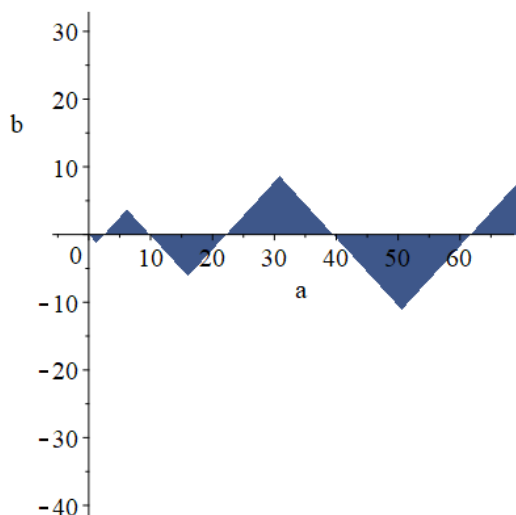
Figure 6: Asymptotic stability region with $r = 2$.
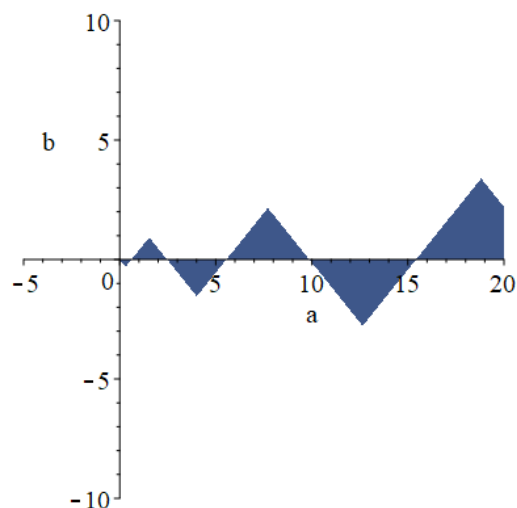
Figure 7: Asymptotic stability region with $r = 4$.

It should be no surprise that these stability regions are enclosed by 'critical' lines as it was the same case with $r = 1$. The lines are given by $b_n(a) = (-1)^n \left(\frac{n^2 \pi^2}{r^2} - a \right)$ for $n = 0, 1, 2, ...$ This can be derived from Theorem 3.13 and/or 3.14 or by substituting $\lambda = i\omega$ in the characteristic equation (20) with $\omega \in \mathbb{R}$. Substituting $\lambda = i\omega$ in $\lambda^2 + a + be^{-\lambda r} = 0$ results in

$$\begin{cases} -\omega^2 + a + b\cos(\omega r) = 0 \\ b\sin(\omega r) = 0 \end{cases} \tag{22}$$

by separating real and imaginary parts. We have $b\sin(\omega r) = 0$ if and only if $\omega = \frac{n\pi}{r}$ for $n \in \mathbb{N} \cup \{0\}$. So, indeed, $b_n(a) = (-1)^n \left(\frac{n^2 \pi^2}{r^2} - a \right)$ for $n = 0, 1, 2, ...$ by the first equation of (22). The asymptotic stability regions that are enclosed by these lines are partially shown in Figure 8 and 9 with $r = 2$ and $r = 4$, respectively. The yellow line corresponds with $n = 0$, the red line corresponds with $n = 1$, the blue line corresponds with $n = 2$, the green line corresponds with $n = 3$, the purple line corresponds with $n = 4$ and the brown line corresponds with $n = 5$.
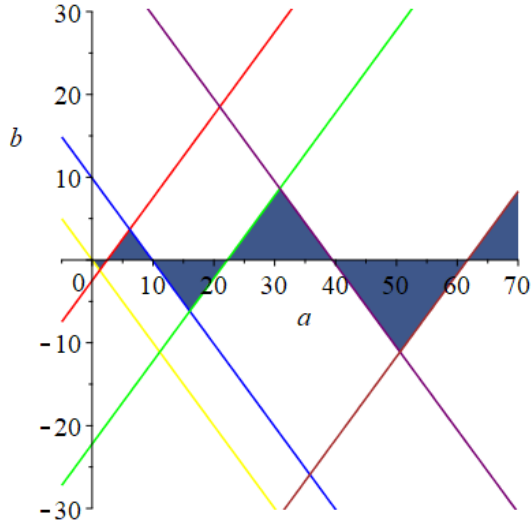
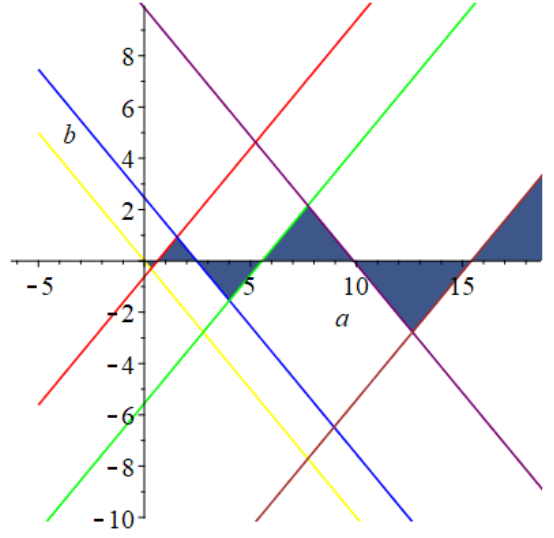Figure 8: Asymptotic stability region enclosed by critical lines with $r = 2$.

Figure 9: Asymptotic stability region enclosed by critical lines with $r = 4$.

Let $C_{n,r} = \left\{ (x,y) \in \mathbb{R}^2 : y = (-1)^n \left(\frac{n^2 \pi^2}{r^2} - a \right) \right\}$ with $n \in \mathbb{N} \cup \{0\}$ and $r > 0$. Then the following lemma is a generalization of Lemma 3.6, but now for arbitrary delays $r > 0$.

Lemma 3.15. Let $r > 0$. There exist a purely imaginary solution $\lambda = i\omega$ of $\lambda^2 + a + be^{-\lambda r} = 0$ if and only if $(a,b) \in C_{n,r}$ for some $n \in \mathbb{N} \cup \{0\}$.

The proof is analogous to the proof of Lemma 3.6.

Proof. First, suppose that $\lambda = i\omega$ ($\omega \in \mathbb{R}$) is a purely imaginary solution. Then $\omega = \frac{k\pi}{r}$ for some $k \in \mathbb{Z}$ by the second equation of (22). This implies $b = (-1)^k \left(\frac{k^2 \pi^2}{r^2} - a \right)$ by the first equation of (22). Note that $C_{-j,r} = C_{j,r}$ for all $j \in \mathbb{Z}$, so that $(a,b) \in C_{n,r}$ for some $n \in \mathbb{N} \cup \{0\}$.

Now assume that $(a,b) \in C_{n,r}$ for some $n \in \mathbb{N} \cup \{0\}$. Now set $\lambda = iy$ ($y \in \mathbb{R}$) and substitute it in $\lambda^2 + a + be^{-\lambda} = 0$. Choose $y = \frac{n\pi}{r}$, then $\sin(yr) = 0$ and $-\frac{n^2 \pi^2}{r^2} + a + (-1)^{2n} \left(\frac{n^2 \pi^2}{r^2} - a \right) = 0$. So $\lambda = i\frac{n\pi}{r}$ satisfies both the equations of (22) and therefore $\lambda = i\frac{n\pi}{r}$ satisfies the characteristic equation. Hence, there exists a purely imaginary solution of $\lambda^2 + a + be^{-\lambda} = 0$. $\square$

Denote S_r by the asymptotic stability region that depends on the delay $r > 0$. So $(a,b) \in S_r$ if and only if $y = 0$ is asymptotically stable for (19). S_r contains all the values for $a, b \neq 0$ such that $y = 0$ is asymptotically stable for (19). Theorem 3.13 and 3.14 give the explicit region S_r. One can expect that Theorem 3.7 also holds for arbitrary delays $r > 0$. That is indeed the case.

Theorem 3.16. Let $r > 0$. If $(a,b) \notin \bigcup_{n=0}^{\infty} C_{n,r} \cup S_r$, then $y = 0$ is unstable for (19).

The proof is analogous to the proof of Theorem 3.7.

Proof. We prove by contradiction. Assume that $(a,b) \notin \bigcup_{n=0}^{\infty} C_{n,r} \cup S_r$ and suppose $y = 0$ is stable for (19). Then $(a,b) \in \bigcap_{n=0}^{\infty} (C_{n,r})^c \cap (S_r)^c$. Let $\hat{\lambda}$ be an arbitrary solution of $\lambda^2 + a + be^{-\lambda r} = 0$. Then $\Re(\hat{\lambda}) \leq 0$, otherwise $y = 0$ would have been unstable by Theorem 2.3. However, $(a,b) \notin C_{n,r}$ for all $n \in \mathbb{N} \cup \{0\}$. So we must have $\Re(\hat{\lambda}) < 0$ by Lemma 3.15. This implies that $y = 0$ is asymptotically stable by Theorem 2.3 and that $(a,b) \in S_r$. This is a contradiction. Therefore, $y = 0$ is unstable for (19). $\square$

As we can see, the proofs of Lemma 3.15 and Theorem 3.16 are essentially identical to the proofs of Lemma 3.6 and Theorem 3.7, respectively. Hence, we will now only state the remaining stability results of equation (19) without proofs as the proofs are also almost identical to the proofs of the corresponding theorems in the previous section. The detailed proofs are given in Appendix B with some additional results.

Theorem 3.17. Let $r > 0$ be the delay. If $(a, b) \in \partial S_r \setminus \{(0, 0)\}$, then $y = 0$ is stable for (19).

Theorem 3.18. Let $r > 0$ be the delay. If $(a, b) \in (\partial S_r \cap C_{0,r}) \setminus \{(0, 0)\}$, then $y(t) = e$ is a stable equilibrium solution of (19) for all $e \in \mathbb{R}$.

Hence, the stability structure of the $y''(t) + ay(t) + by(t - r) = 0$ is similar to the stability structure of $y''(t) + ay(t) + by(t - 1) = 0$. To be more specific, everything in Figure 5 also holds for $y''(t) + ay(t) + by(t - r) = 0$ but scaled appropriately according to $r > 0$.

3.5 Stability of a harmonic oscillator with delay and damping

With all the results that are obtained so far, it is also possible to give a statement about the stability of a harmonic oscillator with a damping. Let $\delta > 0$ be the damping coefficient and consider the delay differential equation

$$y''(t) + \delta y'(t) + ay(t) + by(t - r) = 0, \tag{23}$$

with delay $r > 0$. Intuitively, there is now a damping in a mass springs system with delayed feedback and the influence of this damping is $\delta y'(t)$. Equation (23) is of course different than all the other delay differential equations that have been considered until now, because there is a new term $\delta y'(t)$. However, the following transformation makes it possible to give a statement about the stability of $y = 0$ of (23).

Let $y(t) = e^{-\frac{\delta}{2}t} z(t)$, with $z(t)$ a smooth function, be a solution of (23). The goal is to derive a delay differential equation for $z(t)$ that has the form of (11), so that Theorem 3.4 and 3.5 are applicable. We have

$$y'(t) = -\frac{\delta}{2} e^{-\frac{\delta}{2}t} z(t) + e^{-\frac{\delta}{2}t} z'(t),$$

$$y''(t) = \frac{\delta^2}{4} e^{-\frac{\delta}{2}t} z(t) - \delta e^{-\frac{\delta}{2}t} z'(t) + e^{-\frac{\delta}{2}t} z''(t),$$

and therefore

$$\begin{aligned}
0 &= y''(t) + \delta y'(t) + ay(t) + by(t - r) \\
&= e^{-\frac{\delta}{2}t} z''(t) + \frac{\delta^2}{4} e^{-\frac{\delta}{2}t} z(t) - \frac{\delta^2}{2} e^{-\frac{\delta}{2}t} z(t) + ae^{-\frac{\delta}{2}t} z(t) + be^{-\frac{\delta}{2}(t-r)} z(t - r) \\
&= e^{-\frac{\delta}{2}t} \left[z''(t) + \left(a - \frac{\delta^2}{4} \right) z(t) + be^{\frac{\delta}{2}r} z(t - r) \right].
\end{aligned}$$

Hence, we obtain

$$z''(t) + \left(a - \frac{\delta^2}{4} \right) z(t) + be^{\frac{\delta}{2}r} z(t - r) = 0. \tag{24}$$

Now let $z(t) = \hat{z}(\tau)$ with $\tau = \frac{t}{r}$. Then (24) becomes

$$\hat{z}''(t) + \left(a - \frac{\delta^2}{4} \right) r^2 \hat{z}(t) + br^2 e^{\frac{\delta}{2}r} \hat{z}(\tau - 1) = 0. \tag{25}$$

Equation (25) is derived similarly as equation (21).

Now $\lim_{t\to\infty} z(t) = 0$ if and only if $\lim_{\tau\to\infty} \hat{z}(\tau) = 0$. Moreover, $\lim_{t\to\infty} y(t) = 0$ if $\lim_{t\to\infty} z(t) = 0$. So $\lim_{t\to\infty} y(t) = 0$ if $\lim_{t\to\infty} \hat{z}(t) = 0$. But then asymptotic stability of $\hat{z} = 0$ of (25) implies asymptotic stability of $y = 0$ of (23). Note that asymptotic stability of $y = 0$ of (23) does **not** imply that $\hat{z} = 0$ is asymptotically stable for (25).

Theorem 3.4 and 3.5 can be applied to equation (25) to find values for $a, b \in \mathbb{R}$ such that $y = 0$ is asymptotically stable for (23). Let $\hat{a} = \left(a - \frac{\delta^2}{4}\right)r^2$ and $\hat{b} = br^2 e^{\frac{\delta}{2}r}$ such that (25) is equivalent to

$$\hat{z}''(t) + \hat{a}\hat{z}(t) + \hat{b}\hat{z}(\tau - 1) = 0.$$

This will give the following sufficient (but not necessary) conditions for asymptotic stability of $y = 0$ for (23) with four parameters $a, b \in \mathbb{R}$ and $r, \delta > 0$:

There exists a $k \in \mathbb{N} \cup \{0\}$ such that

$$\hat{b} < 0,$$
$$\hat{a} > 0,$$
$$2k\pi < \sqrt{\hat{a}} < (2k+1)\pi \quad \text{and}$$
$$\hat{b} > \max\{(2k)^2\pi^2 - \hat{a}, \hat{a} - (2k+1)^2\pi^2\},$$

or

$$\hat{b} > 0,$$
$$\hat{a} > 0,$$
$$(2k+1)\pi < \sqrt{\hat{a}} < (2k+2)\pi \quad \text{and}$$
$$\hat{b} < \min\{\hat{a} - (2k+1)^2\pi^2, (2k+2)^2\pi^2 - \hat{a}\}.$$

If $r = \delta = 1$, then Figure 9 is obtained that indicates some values for a and b such that $y = 0$ is asymptotically stable for (23). Note that these values are not all values for a and b for which $y = 0$ is asymptotically stable.
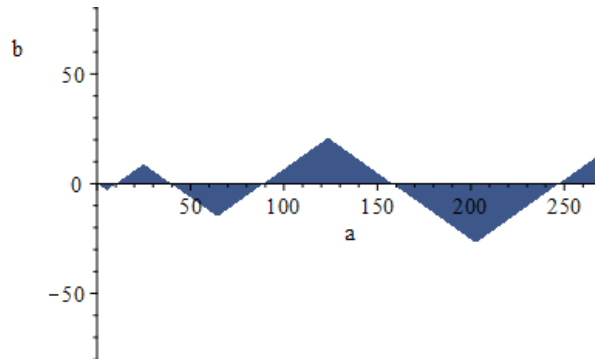

Figure 10: $(r = \delta = 1)$ Blue region indicates asymptotic stability of $y = 0$ for (23).

4 Numerical Solution

Now that the stability of $y''(t) + ay(t) + by(t-r) = 0$ has been studied, it will be satisfying if the solutions coincide with the results in Chapter 3. Comparing the results of Chapter 3 with solutions requires the solutions to be defined for longer periods of time. However, it will soon be made clear that exact solutions are not easily found for large periods of time. Thus, a numerical method needs to introduced in order to obtain a solution defined on a large interval.

4.1 An attempt solving DDE of a harmonic oscillator analytically

Solving the delay differential equation

$$y''(t) + ay(t) + by(t-1) = 0, \ 0 \leq t \leq M, \tag{26}$$
$$y(t) = \phi(t), \ -1 \leq t \leq 0,$$

on an interval $[-1, M]$ can be done by the *Method of Steps* with $M \in \mathbb{N}$. Nevertheless, this method will make clear that numerical methods are needed when $M \gg 0$.

Let $a > 0$ and suppose $\phi = 1$, so that $y(t) = 1$ if $-1 \leq t \leq 0$.
Now assume that $0 \leq t \leq 1$, then $-1 \leq t - 1 \leq 0$ so that $y(t-1) = 1$ for all $t \in [0,1]$. Consequently, we obtain the initial value problem

$$y''(t) + ay(t) + b = 0, \ \ t \in [0,1], \tag{27}$$
$$y(0) = 1,$$
$$y'(0) = 0,$$

on $[0,1]$. Setting $y_p(t) = c \in \mathbb{R}$ and substituting it in (27) gives $c = -\frac{b}{a}$. So a particular solution of (27) is $y_p(t) = -\frac{b}{a}$. Then the general solution of (27) is $y(t) = c_1 \cos(\sqrt{a}t) + c_2 \sin(\sqrt{a}t) - \frac{b}{a}$, since $y(t) = c_1 \cos(\sqrt{a}t) + c_2 \sin(\sqrt{a}t)$ is the general solution of the homogeneous equation $y'' + ay = 0$. We also have $1 = y(0) = c_1 - \frac{b}{a}$, so $c_1 = 1 + \frac{b}{a}$. And $0 = y'(0) = c_2\sqrt{a}$, so $c_2 = 0$. Therefore, the solution of the initial value problem (27) is $\tilde{y}(t) = (1 + \frac{b}{a})\cos(\sqrt{a}t) - \frac{b}{a}$ and $\tilde{y}(t)$ satisfies (26) on $[0,1]$

Now assume $1 \leq t \leq 2$. Then $0 \leq t - 1 \leq 1$ so that $y(t-1) = \tilde{y}(t-1)$ for all $t \in [1,2]$. Consequently, we obtain the initial value problem

$$y''(t) + ay(t) + b \cdot \frac{(a+b)\cos(\sqrt{a}(t-1)) - b}{a} = 0, \ \ t \in [1,2], \tag{28}$$
$$y(1) = \hat{y}(1) = \frac{(a+b)\cos(\sqrt{a}) - b}{a},$$
$$y'(1) = \hat{y}'(1) = -\frac{\sqrt{a}(a+b)\sin(\sqrt{a})}{a},$$

on $[1,2]$. It is directly seen that problem (28) is not easily solved, so moving over to numerical computations would for now be a better idea. Nevertheless, these steps can be repeated for $t \in [(n-1), n]$ with $n \in \mathbb{N}$.

This procedure above is called the Method of Steps for obvious reasons. The whole interval is divided into sub-intervals and then the original problem (26) is solved on those sub-intervals. In this case it was not worth it solving it any further after one step, let alone the next steps. The (obvious) alternative to obtain a solution of (26) for $M \gg 0$ is approximating it by numerical computations. A numerical scheme needs to be introduced for this.

4.2 Numerical method

The Method of Steps has shown that the sub-problems of (26) contain a ordinary differential equation with a complicated nonhomogeneous term. One can solve all these sub-problems numerically to obtain a approximation for (26) by 'concatenating' the approximated solutions defined on the sub-intervals. That might take some work, so introducing a numerical method that relates directly to (26) might be a better idea.

Let $h > 0$ be the step size. Then in order to approximate the solution of (26), the time interval is divided into a set of discrete points $t_n = nh$, $n = 0, 1, 2, ...$ Let y_n be the approximation of $y(t_n)$ with $y(t_n)$ the solution of (26) at time t_n. The equation $y''(t) + ay(t) + by(t-1) = 0$ is a second order delay differential equation. So there needs to be a formula $Q(h)$ that approximates the second derivative $y''(t)$. The central-difference formula

$$Q(h) = \frac{y(t+h) - 2y(t) + y(t-h)}{h^2} \tag{29}$$

can be used to approximate $y''(t)$. The error in this formula is $\mathcal{O}(h^2)$. See Chapter 3 in [6] for more information about numerical differentiation and see Section 3.6 in [6] for the derivation of (29).

Then $y(t_n) \simeq y_n$ and $y''(t_n) \simeq \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2}$ by (29). Furthermore, $t_n - 1 = nh - 1 = \left(n - \frac{1}{h}\right)h = t_{n-\frac{1}{h}}$. Therefore, $y(t_n - 1) = y\left(t_{n-\frac{1}{h}}\right) \simeq y_{n-\frac{1}{h}}$. Now by substituting these approximations in (26), the following numerical method is obtained

$$y_{n+1} = 2y_n - y_{n-1} - h^2\left(ay_n + by_{n-\frac{1}{h}}\right) \tag{30}$$

to approximate solutions of (26).

4.3 Numerical solutions

It is now possible approximating the solution of (26) by using method (30). Let for example $\phi(t) = \sin(t)$ be the initial function. Choosing another initial function is also possible obviously. However, the problem being considered now is

$$y''(t) + ay(t) + by(t-1) = 0, \ t \geq 0, \tag{31}$$
$$y(t) = \sin(t), \ -1 \leq t \leq 0.$$

See Appendix A for the implementation of method (30) for problem (31).

The solution will of course depend on a and b. Recall the asymptotic stability region S of $y = 0$ and the critical lines C_n in Section 3.3. The results in that section should coincide with the numerical solutions of (26). Specifically, $\lim_{t \to \infty} \tilde{y}(t) = 0$ if $(a, b) \in S$, $\hat{y}(t)$ converges to a periodic function if $(a, b) \in \partial S \setminus \{(0, 0)\}$ and $\tilde{y}(t)$ diverges as $t \to \infty$ if $(a, b) \notin \bigcup_{n=0}^{\infty} C_n \cup S$ where $\tilde{y}(t)$ is the numerical solution. A part of the asymptotic stability region with the critical lines will be shown again for the sake of convenience. See Figure 11.
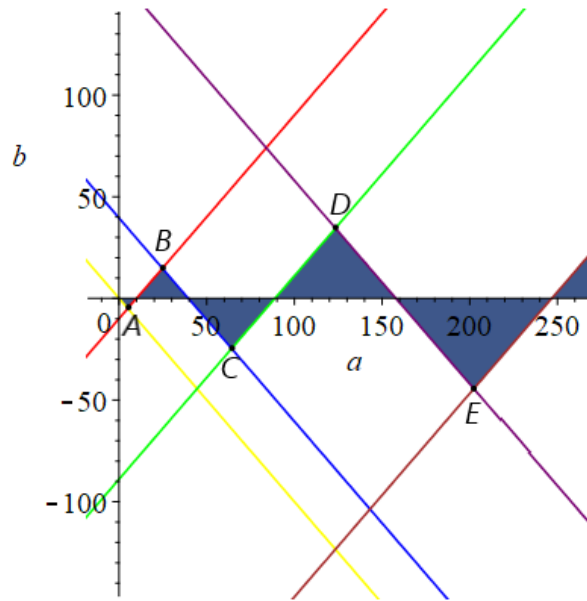
Figure 11: Asymptotic stability region enclosed by critical lines ($r = 1$).

The following figures show some numerical solutions on $[-1, 20]$. It will be directly clear whether the solution is periodic, converges to zero or diverges.
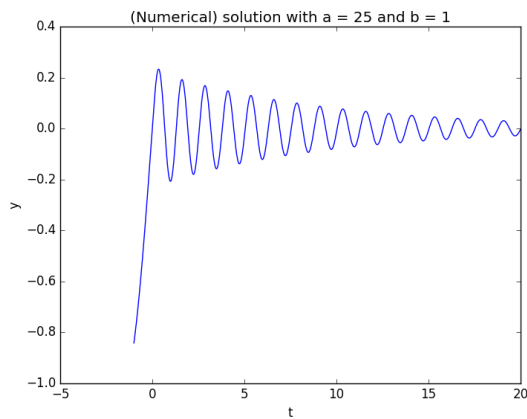

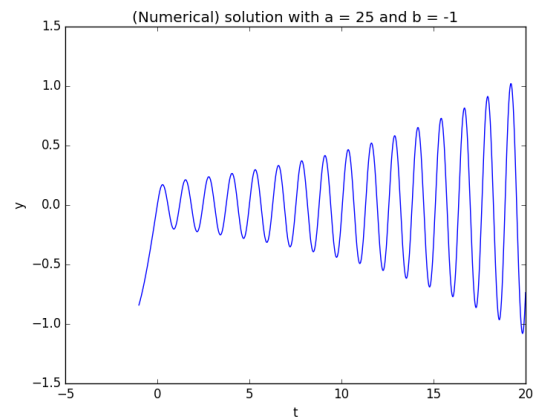Figure 12: Numerical solution with $a = 25$ and $b = 1$.

Figure 13: Numerical solution with $a = 25$ and $b = -1$.

Figure 12 shows the numerical solution with $a = 25$ and $b = 1$. In this case, $(25, 1) \in S$ and the solution does indeed converge to zero. As opposed to this, Figure 13 shows divergence for $a = 25$ and $b = -1$. This also coincides with the results in Section 3.3 (Theorem 3.7).

Figure 14 and 15 show the numerical solution for $(a, b) = (5, -1)$ and $(a, b) = (5, -10)$, respectively. The behaviour of the solutions agrees with the results in Chapter 3.3.
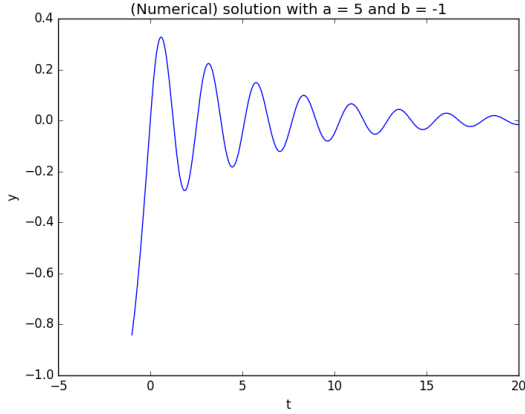
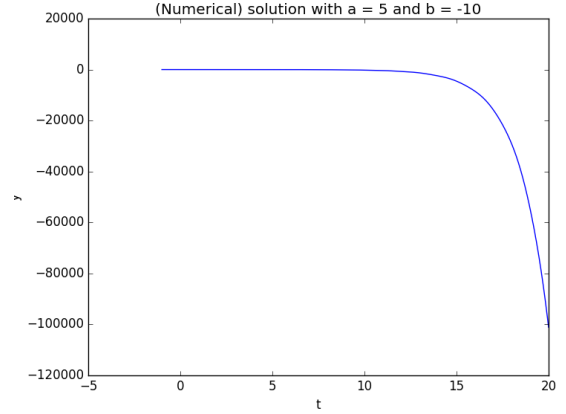Figure 14: Numerical solution with $a = 5$ and $b = -1$.

Figure 15: Numerical solution with $a = 5$ and $b = -10$.

We now turn to the the boundary of the asymptotic stability region ∂S and the lines C_n. Recall that the lines C_n were defined by $C_n = \{(x, y) \in \mathbb{R}^2 : y = (-1)^n(n^2\pi^2 - x)\}$ and that $\partial S \subseteq \bigcup_{n=0}^{\infty} C_n$. We have for example $(100, -9\pi^2 + 100) \in C_3 \cap \partial S$ and $(150, -9\pi^2 + 150) \in C_3 \setminus \partial S$. See Figure 11 where C_3 is the green line. Hence, $y = 0$ is stable with $a = 100$ and $b = -9\pi^2 + 100$ by Theorem 3.9. Figure 16 shows indeed the stability of $y = 0$ whereas Figure 17 shows instability of $y = 0$.
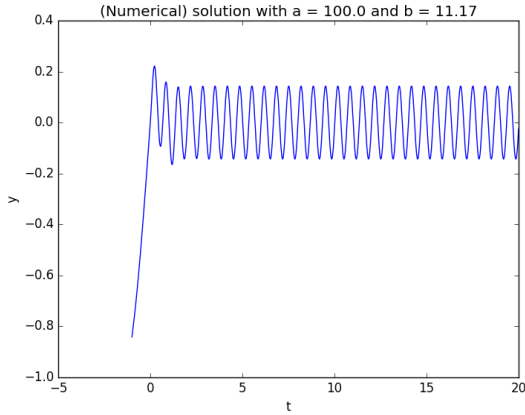

Figure 16: Numerical solution with $a = 100$ and $b = -9\pi^2 + 100$.
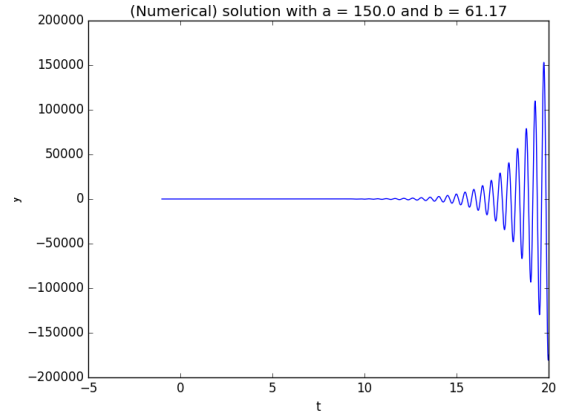

Figure 17: Numerical solution with $a = 150$ and $b = -9\pi^2 + 150$.

Remember that the stability is still unknown for $(a, b) \in \bigcup_{n=0}^{\infty} C_n \setminus \partial S$. Trying two more values for $(a, b) \in \bigcup_{n=0}^{\infty} C_n \setminus \partial S$ might give an indication whether $y = 0$ is stable or unstable. Two more solutions are shown in Figure 18 and 19 and they diverge for $(50, -\pi^2 + 50) \in C_1 \setminus \partial S$ and $(150, -25\pi^2 + 150) \in C_5 \setminus \partial S$, respectively. See Figure 11 where C_1 is the red line and C_5 is the brown line.

Now, one can guess that $y = 0$ is unstable for (11) if $(a, b) \in \bigcup_{n=0}^{\infty} C_n \setminus \partial S$. However, we still have to be critical as there are numerical errors we have to take into account. And, of course, a mathematical proof is necessary before one can say that $y = 0$ is (un)stable on $\bigcup_{n=0}^{\infty} C_n \setminus \partial S$.
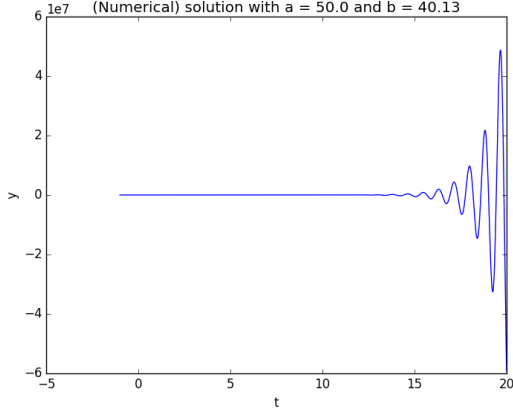
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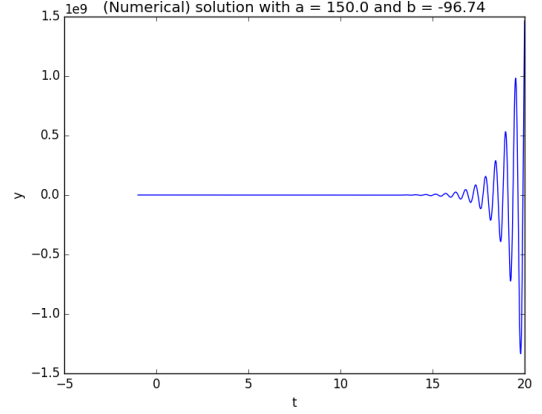Figure 18: Numerical solution with $a = 50$ and $b = -\pi^2 + 50$.

Figure 19: Numerical solution with $a = 150$ and $b = -25\pi^2 + 150$.

Finally, some solutions are shown that are periodic and have a double period for large enough $t > 0$. These solutions occur, for example, on the points B, C, D and E (Figure 11). This happens because there are four purely imaginary roots of the characteristic equation $\lambda^2 + a + be^{-\lambda} = 0$, namely $\lambda \pm ik\pi$ and $\lambda = \pm i(k+1)\pi$ for some $k \in \mathbb{N}$ if $(a, b) \in \{B, C, D, E\}$. The corresponding solutions are $\cos(k\pi t), \sin(k\pi t), \cos((k+1)\pi)$ and $\sin((k+1)\pi t)$. If $(a, b) = A$, then there are only two non-zero purely imaginary roots $\lambda = \pm i\pi$, because $\lambda = 0$ is the other 'imaginary' solution. Generally speaking, double periodic solutions arise on the vertices of the triangles where two critical lines intersect, except for point A as shown in Figure 11. We have the following coordinates for A, B, C, D and E and the solution is shown in Figure 20 and 21 for $(a, b) = C$ and $(a, b) = E$, respectively.

$$A = \left(\frac{\pi^2}{2}, -\frac{\pi^2}{2}\right),\ B = \left(\frac{5\pi^2}{2}, \frac{3\pi^2}{2}\right),\ C = \left(\frac{13\pi^2}{2}, -\frac{5\pi^2}{2}\right),$$
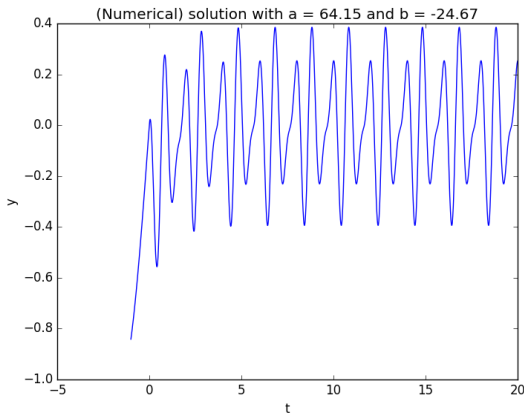$$D = \left(\frac{25\pi^2}{2}, \frac{7\pi^2}{2}\right),\ E = \left(\frac{41\pi^2}{2}, -\frac{9\pi^2}{2}\right).$$
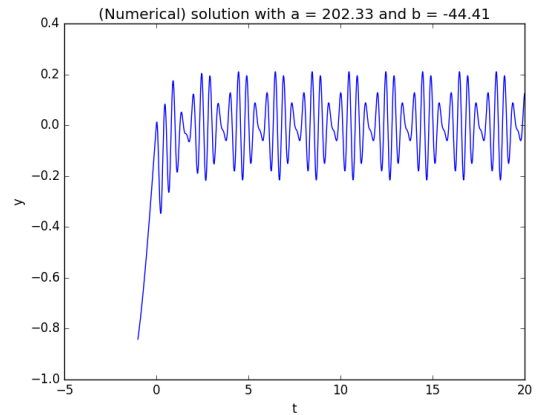

Figure 20: Numerical solution with $(a, b) = C$.

Figure 21: Numerical solution with $(a, b) = E$.

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5   Conclusion and discussion

In this thesis we have looked at the stability type of $y = 0$ for the delay differential equation $y''(t) + ay(t) + by(t-1) = 0$ with $a, b \in \mathbb{R}$. The solution of $y''(t) + ay(t) + by(t-1) = 0$ is of course dependent on a and b, and so is the stability. There are three types of stability that have been considered: stable, unstable and asymptotically stable. It was, unfortunately, not possible to determine the stability for all $a, b \in \mathbb{R}$, but for the most part we were successful. Figure 5 in Section 3.3 summarizes visually all the results that have been obtained.

Following this, we were able to generalize the stability problem for $y''(t) + ay(t) + by(t-r) = 0$ for all delays $r > 0$. This was done by a coordinate transformation for t such that all the results for $y''(t) + ay(t) + by(t-1) = 0$ were indirectly applicable to $y''(t) + ay(t) + by(t-r) = 0$. It turned out that the stability structure for arbitrary delays is similar to the stability structure shown in Figure 5.
It was also possible to make statement about the stability for $y''(t) + \delta y'(t) + ay(t) + by(t-r) = 0$ with $\delta, r > 0$. Some values for a and b were found such that $y = 0$ is asymptotically stable for $y''(t) + \delta y'(t) + ay(t) + by(t-r) = 0$ with δ and r fixed.
Subsequently, these results could be verified by numerical solutions. First, a numerical method was derived for $y''(t) + ay(t) + by(t-1) = 0$. Then numerical solutions were shown that all agreed with the mathematical results.

All in all, the qualitative properties of $y''(t) + ay(t) + by(t-1) = 0$ have been discussed. However, the whole stability problem is not solved yet. So we invite the reader to make an attempt solving the remaining open problem. Moreover, throughout the thesis the focus was on one delay differential equation. So the stability of other delay differential equations can be studied for future research. Or studying the stability of a perturbed equation $y''(t) + ay(t) + by(t-1) = f(t, y)$ can also be done by adding a nonlinear term $f(t, y)$. An exponential polynomial also came across that arose from studying the stability. One can look into general exponential polynomials and study their roots. There are enough (sub)topics that can be studied concerning delay differential equations.

A Python code

```python
import matplotlib.pyplot as plt
import numpy as np

def phi(t):
    return np.sin(t)

a = 1
b = -1
r = 1 #delay
teind = 20
h = 0.001 #step size
nsteps = int(teind/h)
t = -r
t_lst = [t]
y_lst = [phi(t)]

for i in range(1,int(r/h)+1):
    y = phi(-r + i*h)
    t = t + h

    y_lst.append(y)
    t_lst.append(t)

for i in range(1,nsteps):
    y = 2*y_lst[-1] - y_lst[-2] - h**2*(a*y_lst[-1] +
        b*y_lst[-int(r/h)-1])
    t = t + h

    y_lst.append(y)
    t_lst.append(t)

plt.plot(t_lst, y_lst)
plt.xlabel('t')
plt.ylabel('y')
plt.title('(Numerical) solution with a = ' + str(round(a,2)) +
          ' and b = ' + str(round(b,2)))
plt.show()
```

B   Additional theorems with proofs

Theorem B.1. The functions $t^k e^{\lambda_0 t}$, $k = 0, 1, 2..., m - 1$, are solutions of equation (19) if and only if λ_0 is a root of multiplicity at least m of the characteristic equation $\lambda^2 + a + be^{-\lambda r} = 0$.

Proof. Introduce the linear operator, defined on the (complex) differentiable functions, by $L(y) = y''(t) + ay(t) + by(t - r)$ and let $H(\lambda) = \lambda^2 + a + be^{-\lambda r}$. Then

$$
\begin{aligned}
L\big(t^k e^{\lambda t}\big) &= L\left(\frac{\partial^k}{\partial \lambda^k} (e^{\lambda t}) \right) \\
&= \frac{\partial^k}{\partial \lambda^k} L\big(e^{\lambda t}\big) \\
&= \frac{\partial^k}{\partial \lambda^k} \big(e^{\lambda t} H(\lambda)\big) \\
&= \sum_{j=0}^{k} \left[\binom{k}{j} H^{(j)}(\lambda) \frac{\partial^{(k-j)}}{\partial \lambda^{(k-j)}} (e^{\lambda t}) \right] \\
&= e^{\lambda t} \sum_{j=0}^{k} \binom{k}{j} t^{k-j} H^{(j)}(\lambda),
\end{aligned}
$$

by using the General Leibniz Rule and noticing that $\frac{\partial^k}{\partial \lambda^k}$ commutes with L.

Assume λ_0 is a root of multiplicity at least m. Then $H^{(k)}(\lambda_0) = 0$ for all $k \in \{0, 1, ..., m - 1\}$ by Definition 2.4. Therefore, $L(t^k e^{\lambda_0 t}) = 0$ for all $k \in \{0, 1, ..., m - 1\}$ by the equation above. Consequently, $y(t) = t^k e^{\lambda_0 t}$ is a solution of equation (11) for $k = 0, 1, 2..., m - 1$.

If the functions $t^k e^{\lambda_0 t}$, $k = 0, 1, 2..., m - 1$, are solutions of equation (11), then $L\big(t^k e^{\lambda_0 t}\big) = 0$ for all $k \in \{0, 1, ..., m - 1\}$. This implies that $H^{(k)}(\lambda_0) = 0$ for all $k \in \{0, 1, ..., m - 1\}$ by the equation above. Thus, λ_0 is a characteristic root of multiplicity at least m. $\qquad \square$

Theorem B.2. Le $r > 0$. If $(a, b) \in \partial S_r \setminus \{(0, 0)\}$, then $y = 0$ is stable for (19).

Proof. We only have to consider the case $b \neq 0$.

Suppose $(a, b) \in \partial S_r \setminus \{(0, 0)\} \subseteq \bigcup_{n=0}^{\infty} C_{n,r}$, then the purely imaginary characteristic roots are necessarily of the form $\lambda = \frac{ik\pi}{r}$ for some $k \in \mathbb{Z}$. This is immediate clear from (22). Furthermore, let $H(\lambda) = \lambda^2 + a + be^{-\lambda r}$ then $H'(\lambda) = 2\lambda - be^{-\lambda r}$. Suppose $H'(ik\pi) = 0$, then it follows that $2ik\pi = \pm b$. This is a contradiction as the left hand side is purely imaginary and b is a real number unequal to zero. Hence, $\lambda = \frac{ik\pi}{r}$ is a root of multiplicity 1 of the characteristic equation.

It is well known that the roots $\lambda = \lambda(a, b)$ of $\lambda^2 + a + be^{-\lambda r} = 0$ depend continuously on $a \in \mathbb{R}$ and $b \neq 0$. Now assume that there is a root of the characteristic equation with positive real part if $(a, b) \in \partial S \setminus \{(0, 0)\}$. Then there is a purely imaginary solution $\mu = \mu(\hat{a}, \hat{b}) \in \mathbb{C}$ of the characteristic equation for some $(\hat{a}, \hat{b}) \in S_r$ by continuity of the roots and by the fact that in S_r all characteristic roots $\lambda \in \mathbb{C}$ satisfy $\Re(\lambda) < 0$. This is now forms a contradiction because $\Re(\mu(\hat{a}, \hat{b})) = 0$ with $(\hat{a}, \hat{b}) \in S_r$. Therefore, all roots of $\lambda^2 + a + be^{-\lambda r} = 0$ have real part less or equal to zero if $(a, b) \in \partial S_r \setminus \{(0, 0)\}$.

We know that the general solution of (19) has the form $y(t) = \sum_{j=1}^{\infty} a_j t^{n_j} e^{\lambda_j t}$. Since all purely imaginary roots $\lambda = \frac{ik\pi}{r}$ have multiplicity 1, $t^m e^{\frac{ik\pi t}{r}}$ are no solutions of (11) with $m \geq 1$. And

the corresponding solutions of the characteristic root $\lambda = \frac{ik\pi}{r}$ are $\cos\left(\frac{k\pi t}{r}\right)$ and $\sin\left(\frac{k\pi t}{r}\right)$ which are periodic functions. This follows from Theorem B.1. Therefore, the general solution of (19) can be written as

$$y(t) = p(t) + \sum_{l=1}^{\infty} a_l t^{n_l} e^{\lambda_l t},$$

with $p(t)$ a periodic function and characteristic roots $\lambda_l \in \mathbb{C}$ satisfying $\Re(\lambda_l) < 0$ for all $l \in \mathbb{N}$. We have $\lim_{t\to\infty} a_l t^{n_l} e^{\lambda_l t} = 0$ for all $l \in \mathbb{N}$. Moreover, $\sum_{l=1}^{\infty} a_l t^{n_l} e^{\lambda_l t}$ is uniformly convergent for all $t \geq -r$ as $\Re(\lambda_l) < 0$ for all $l \in \mathbb{N}$ (see Theorem 6.5 in [8]). Therefore, $y(t) \to p(t)$ as $t \to \infty$. Hence, the solution $y(t)$ converges to a periodic solution and therefore $y = 0$ is stable for (19). $\qquad\square$

Lemma B.3. Let $r > 0$ and $a, b \neq 0$. There exists an equilibrium solution $\hat{y}(t) = e$ of (19) with $e \in \mathbb{R}$ arbitrary if and only if $a + b = 0$

Proof. Let $e \in \mathbb{R}$ be arbitrary. If $\hat{y}(t) = e \in \mathbb{R}$ is equilibrium solution of (19), then $\hat{y}''(t) = 0$. So that (19) becomes $a\hat{y}(t) + b\hat{y}(t-r) = 0$. This implies $\hat{y}(t) = -\frac{b}{a}\hat{y}(t-r)$. This is a difference equation. The corresponding auxiliary equation is $\lambda^r + \frac{b}{a} = 0$. Hence, $\hat{y}(t) = c(-\frac{b}{a})^{\frac{t}{r}}$ with $c, t \in \mathbb{R}$ is the general solution of $\hat{y}(t) = -\frac{b}{a}\hat{y}(t-r)$. Therefore, $e = \hat{y}(t) = c(-\frac{b}{a})^{\frac{t}{r}}$ for all $t \in \mathbb{R}$ which implies that $-\frac{b}{a} = 1$ (and $e = c$). And of course, $-\frac{b}{a} = 1$ if and only if $a + b = 0$.

Suppose $a + b = 0$, then (11) becomes $y''(t) + a(y(t) - y(t-r)) = 0$. It is directly seen that $\hat{y} = e \in \mathbb{R}$ is an equilibrium solution for all $e \in \mathbb{R}$. $\qquad\square$

Theorem B.4. Let $r > 0$. If $(a, b) \in (\partial S_r \cap C_{0,r}) \setminus \{(0,0)\}$, then $y(t) = e$ is a stable equilibrium solution of (19) for all $e \in \mathbb{R}$.

Proof. Let $e \in \mathbb{R}$ be arbitrary and assume that $(a, b) \in (\partial S_r \cap C_{0,r}) \setminus \{(0,0)\}$. Then $a + b = 0$ as $(a, b) \in C_{0,r} = \{(x, y) \in \mathbb{R}^2 : y = -x\}$. So $y(t) = e$ is an equilibrium solution of (19) by Lemma B.3.

Now set $w(t) = y(t) - e$. Then

$$\begin{aligned}
w''(t) &= y''(t) \\
&= -ay(t) + ay(t-r) && \text{(because } b = -a) \\
&= -a(w(t) + e) + a(w(t-r) + e) \\
&= -aw(t) + aw(t-r) \\
&= -aw(t) - bw(t-r).
\end{aligned}$$

Hence, the stability of $y(t) = e$ of (11) is equivalent to the stability of $w(t) = 0$ of $w''(t) + aw(t) + bw(t-r) = 0$. Since $(a, b) \in \partial S_r \setminus \{(0.0)\}$, we have that $w(t) = 0$ is stable for $w''(t) + aw(t) + bw(t-r) = 0$ by Theorem B.2. Therefore, $y(t) = e$ is a stable equilibrium solution of (19). $\qquad\square$

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