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Output null controllability for linear time-invariant structured discrete-time systems: A graph theoretic condition

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ABSTRACT

In this paper, we consider a linear time-invariant discrete-time system and study the output null controllability problem, i.e., the problem of steering the output to zero in a finite number of steps. We assume that we only know the structure of the system, i.e., the zero/nonzero location in the system matrices. Hence, we consider a structural version of the output null controllability problem. We represent the structure of the system by means of a directed graph and present a graph theoretic sufficient condition for the problem to be generically solvable. Here generically solvable means that the problem is solvable for almost all systems with the same structure. We illustrate the conditions using an example.

1. Introduction

In this paper, we consider a linear time-invariant discrete-time system with a state, an input, and an output. We study the problem of steering the output to zero in a finite number of steps by applying an appropriate sequence of inputs. More specifically, we address the problem from a structural point of view, meaning that we only want to use the structure of the system equations. Hence, we only assume the zero/nonzero structure of the system matrices to be known. Because of this, we can only say something about the possible generic solvability of the problem. Here, generic solvability of the problem means that it is solvable for almost all systems with the same structure, while the set of systems with the same structure for which the problem fails to be solvable forms a set of zero Lebesgue measure.

For a specific numerically specified system, an input that actually steers the output to zero in a finite number of steps, also requires the numerical values of matrix entries, i.e., for such a concrete input actually solving the problem, the structure of the system alone is not enough.

Controllability in the structural context has already been studied for quite some time. The first publication in 1974 is due to Lin (1974). Later other publications on the topic followed, see the introduction of the survey paper (Dion, Commault, & van der Woude, 2003). Originally, the results involved continuous-time systems and full state controllability. Zero state controllability for discrete-time systems was studied in van der Woude (2018). As such, that current paper can be seen as a follow-up and extension of some controllability aspects for linear continuous-time systems. A structural characterisation of output controllability was left as an open problem in Murota and Poljak (1990) and, to the best of our knowledge, no graph characterisation for structural output controllability is available to date. The second difficulty is the intrinsic hardness of the problem: the minimum output controllability problem has recently been proven to be an NP-hard problem (Czeizler, Wu, Gratie, Kanhaiya, & Petre, 2018).

In the overview papers Dion et al. (2003) and Ramos, Aguiar, and Pequito (2022), or in the textbooks Murota (1987) and Reinschke (1988), an extensive motivation for the study of structured systems is given. In general, the study is motivated by the lack of precise knowledge in the description of the systems. For instance, in several applications the nonzero values in the system matrices are obtained via measurements, and thus with certain errors. Or, they appear by using physical laws that are only valid in perfect conditions, thus also with some associated errors in practical situations. In such situations, the structural approach towards the systems may be useful, yielding results that are true generically, i.e., in most practical cases. Also, sometimes certain properties of linear time-invariant systems are hard to compute, such as minimal controllability problems, see Olshevsky (2014) and Pequito, Ramos, Kar, and Aguiar (2017), whereas the structured (practical) versions are easy to solve, see Pequito, Kar, and Aguiar (2016).

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A very important advantage of structured systems is the fact that they are associated in a natural way with a directed graph. This graph is important to visualise the interactions inside the system and also to characterise a lot of properties of the system. This characterisation in graph terms is often very informative in terms of deep structure of the system, but also generally leads to very efficient algorithms to check the properties.

The study of controllability in complex networks/structured systems was given an enormous boost in 2011 by Liu, Slotine, and Barabasi (2011). The paper revived interest in the subject, and many papers on various aspects of controllability have appeared since. See, for instance, Commault and Dion (2013), Pequito et al. (2016) and van der Woude, Boukhobza, and Commault (2018). The paper proved that important network features can be nicely formulated in terms of structured systems properties. This considerably enlarged the range of structured systems applications. Through time also other aspects of linear structured systems have been studied, like structural properties of transfer matrices and various structural (disturbance) decoupling problems. Many of the results were inspired by the geometric and frequency domain approach towards linear system theory, like in Bru, Caccetta, and Rumchev (2005), Commault, van der Woude, and Boukhobza (2017) and Commault, van der Woude, and Frasca (2020).

As mentioned earlier, in the current paper the focus is on discrete-time systems and on steering the output to zero in a finite number of time steps. The problem in this paper has been studied in other works, and nice geometric conditions are known, see Trentelman, Stoorvogel, and Hautus (2001) and Wonham (1985). However, the conditions in these references do not well fit within the structural framework that we adopt in this paper. Therefore, we use conditions that better fit the structural approach. Specially we will use an alternative sufficient condition. The condition will be expressed in terms of the directed graph that easily can be associated with the structured system in this paper. The sufficiency condition is then obtained using a decomposition of the graph of the system that naturally fits the problem under consideration. The main result of this paper, being a sufficient graph theoretic condition for the generic solvability of the problem, can then be obtained easily. We illustrate the condition through an example.

The outline of this paper is as follows. In Section 2, we introduce the type of system studied in this paper. Also, we formulate the state and output null controllability problem and recall a necessary and sufficient condition for their solvability. The presented condition comes from the geometric approach towards linear system theory, see Wonham (1985). For completely known and numerically specified systems, the condition is elegant and also intuitive in a sense. However, the geometric nature of the condition does not fit very well within the structural approach adopted in this paper. This holds in particular for the output null controllability problem, since a structural condition for the generic solvability of the state null controllability problem can be easily given, see van der Woude (2018). Therefore, in Section 3, we present an alternative sufficient condition for the output null controllability problem that better suits our purposes. This paper focuses on finding a solvability condition that matches the adopted structural point of view. In Section 4, several special cases are studied that easily can be dealt with in the structural approach. The special cases will be the foundation of the main result of the paper. In Section 5, the graphs of structured systems will be introduced, together with some elementary notions of graph theory. Also, a decomposition will be described that follows naturally from the problem studied in the paper. In Section 6, parts of the obtained decomposition will be related to existing results in the literature. The combination of these results yields a sufficient condition for the generic solvability of the output null controllability problem in graph terms. The condition is included in Section 6.2, and is illustrated via an example in Section 7. We end the paper with Section 8 with some conclusions and remarks. In particular, the possible necessity of the obtained sufficient condition will be discussed. Also an extension of the obtained condition will be mentioned. The appendix, in Appendix, contains the proof of a statement in a derivation of the alternative sufficient condition.

In this paper, we will frequently use identity matrices I, and zero matrices 0. However, to simplify the notations, we will not precise their dimensions, which will always follow from the context in which they appear.

2. State and output null controllability

We consider the following linear discrete-time system

$$x(k+1) = Ax(k) + Bu(k), \quad y(k) = Cx(k), \tag{1}$$

with $k \geq 0$, the time, and all variables and matrices as usual. More precisely, we have a state $x(k) \in \mathbb{R}^n$, an input $u(k) \in \mathbb{R}^m$, and an output $y(k) \in \mathbb{R}^p$, implying that $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$.

Considering system (1), we denote its state at time $k \geq 0$, given the initial state $x(0) = x_0$ and a control sequence $\mathbf{u} := \{u(0), u(1), u(2), \dots\}$, by $x_{\mathbf{u}}(k, x_0)$. Similarly, $y_{\mathbf{u}}(k, x_0)$ denotes the output at time k, given the initial state x_0 and control sequence $\mathbf{u}$. Note that

$$x_{\mathbf{u}}(k, x_0) = A^k x_0 + \sum_{l=0}^{k-1} A^{k-1-l} Bu(l). \tag{2}$$

Likewise,

$$y_{\mathbf{u}}(k, x_0) = CA^k x_0 + \sum_{l=0}^{k-1} CA^{k-1-l} Bu(l). \tag{3}$$

Considering system (1), we say that for initial state x_0, the *state null controllability* problem is solvable, if there exists a time $K \geq 0$ and a control sequence $\mathbf{u}$ such that $x_{\mathbf{u}}(k, x_0) = 0$ for all $k \geq K$. When the latter holds for any initial state x_0, we say that system (1) is *state null controllable*. We use the abbreviation SNC for 'the *state null controllability* problem', or '*state null controllable*'. Hence, we may say that SNC is solvable for initial state x_0, or that SNC is solvable for system (1), respectively. Or, even simpler, we may refer to it as SNC for x_0, or SNC for system (1), respectively.

We write $\langle A|\text{im } B\rangle$ for the controllable subspace, i.e., the column space of the well-known controllability matrix $\left(B, AB, \cdots, A^{n-1}B \right)$. Recall that $\langle A|\text{im } B\rangle$ is the smallest A-invariant subspace that contains im B. For SNC, necessary and sufficient conditions for A and B are well known, see for instance (Trentelman et al., 2001), Exercise 3.19. Two of such conditions are listed in the next lemma.

Lemma 1. *Let system (1) be given. Then*

(i) SNC is solvable for initial state x_0 if and only if $A^n x_0 \in \langle A|\text{im } B\rangle$.
(ii) SNC is solvable if and only if rank$(A - zI, B) = n$, for all $z \neq 0$.

Now, including the output, we say that for initial state x_0, the *output null controllable* is solvable, if there exists a time $K \geq 0$ and a control sequence $\mathbf{u}$ such that $y_{\mathbf{u}}(k, x_0) = 0$ for all $k \geq K$. When the latter holds for any initial state x_0, we say that system (1) is *output null controllable*. We use the abbreviation ONC for 'the *output null controllability* problem', or '*output null controllable*'. Hence, we may say that ONC is solvable for initial state x_0, or that ONC is solvable for system (1), respectively. Or, even simpler, we may refer to it as ONC for x_0, or ONC for system (1), respectively.

We write $\mathcal{V}^*(\ker C)$ for the largest controlled invariant subspace in $\ker C$, i.e., the largest subspace $\mathcal{V}$ in $\ker C$ such that $A\mathcal{V} \subseteq \mathcal{V} + \text{im } B$. Also, for ONC, necessary and sufficient conditions for A, B, and C can be derived, see Hautus (1979) for some background.

Lemma 2. *Let system (1) be given. Then ONC is solvable for initial state x_0 if and only if $A^n x_0 \in \mathcal{V}^*(\ker C) + \langle A|\text{im } B\rangle$.*

Proof. It can be checked that the geometric condition is a discrete-time analog of the condition in Wonham (1985), Theorem 4.4, when the stability region is the origin in the complex plane. □

Note that both types of null controllability are linear in the initial state. Indeed, assume that SNC is solvable for initial states $x(0) = x_a$ and $x(0) = x_b$ by applying control sequences $\mathbf{u}_a$ and $\mathbf{u}_b$, respectively. Then SNC is also solvable for $x(0) = \alpha_a x_a + \alpha_b x_b$ by applying control sequence $\alpha_a \mathbf{u}_a + \alpha_b \mathbf{u}_b$. A similar statement holds for ONC.

3. Sufficient solvability condition for ONC

From Lemma 2, a (geometric) sufficient condition for the solvability of ONC follows directly. However, this condition does not easily go together with the structural approach that we adopt in this paper. Therefore, to derive a condition that nicely fits the structural approach, we will use an alternative sufficient condition. To introduce this condition, we consider system (1) with initial state $x(0) = x_0$, and we assume that

$$\operatorname{rank}\left(C(zI - A)^{-1}B\right) = \operatorname{rank}\left(C(zI - A)^{-1}\left(B, x_0\right)\right), \tag{4}$$

where $\left(B, x_0\right)$ is the $n \times (m+1)$ matrix obtained by concatenating the matrix B with the column vector x_0, and the rank condition (4) holds for almost all complex z. Then, seen as an equation over the (field of) rational functions, the rank condition in (4) implies that the equation

$$C(zI - A)^{-1}Bu(z) = C(zI - A)^{-1}x_0 \tag{5}$$

has a rational vector $u(z)$ as a solution. It then follows that there exist rational vectors $p(z)$ and $q(z)$ such that

$$(zI - A)p(z) - Bq(z) = x_0 \text{ and } Cp(z) = 0.$$

Indeed, with $u(z)$ as a solution to Eq. (5), take $q(z) = -u(z)$ and $p(z) = (zI - A)^{-1}(x_0 + Bq(z))$.

Next, note that $Cp(z) = 0$ implies that $Cp(z)$ can be seen as a polynomial expression that happens to be the zero polynomial. Hence, we obtain that there exist rational vectors $p(z)$ and $q(z)$ such that

$$(zI - A)p(z) - Bq(z) = x_0 \text{ and } Cp(z) \text{ is polynomial.}$$

Using methods of Schumacher (1983), see also Hautus (1979), it can be proved that the latter implies that (see also a proof in the Appendix)

$$x_0 \in \mathcal{V}^*(\ker C) + \langle A|\mathrm{im}\ B\rangle .$$

Note that the subspace $\mathcal{V}^*(\ker C) + \langle A|\mathrm{im}\ B\rangle$ is A-invariant. Indeed, by the properties mentioned in Section 2, it follows that $A\big(\mathcal{V}^*(\ker C) + \langle A|\mathrm{im}\ B\rangle\big) \subseteq \mathcal{V}^*(\ker C) + \mathrm{im}\ B + \langle A|\mathrm{im}\ B\rangle \subseteq \mathcal{V}^*(\ker C) + \langle A|\mathrm{im}\ B\rangle$. Hence, it follows immediately that

$$A^n x_0 \in \mathcal{V}^*(\ker C) + \langle A|\mathrm{im}\ B\rangle . \tag{6}$$

By Lemma 2, the latter implies the existence of a control sequence $\mathbf{u} = \{u(k)|k \geq 0\}$ for $x(0) = x_0$, such that $y_{\mathbf{u}}(k, x_0) = 0$ for all $k \geq K$ for some appropriate $K \geq 0$. So, we have obtained the following sufficient condition.

Lemma 3. *Consider system (1) with the initial state x_0. If rank condition (4) is satisfied, then ONC is solvable for x_0.*

Proof. If condition (4) is satisfied, condition (6) follows from the above, implying by Lemma 2 that ONC is solvable for x_0. □

Hence, the rank condition in (4) provides a sufficient condition for solving ONC for a specific initial condition. Rank conditions like (4), with x_0 replaced by a known matrix, are useful in the structural approach that we follow in this paper, because they can be implemented in an elegant way.

4. The solvability of ONC in special cases

Before treating the general case, we first look at some special cases in which the solvability of ONC can be treated more easily, and that may be useful for the general case.

(1) Consider the linear discrete time system given by (1).

Proposition 1. *Assume that SNC for system (1) is solvable, then also ONC is solvable for system (1).*

Proof. If for initial state x_0, there is a control sequence $\mathbf{u}$ and an integer $K \geq 0$ such that $x_{\mathbf{u}}(k, x_0) = 0$ for all $k \geq K$, then also $y_{\mathbf{u}}(k, x_0) = Cx_{\mathbf{u}}(k, x_0) = 0$ for all $k \geq K$. Hence, SNC implies ONC. □

(2) Assume that the state $x(k)$, and the matrices A, B and C in (1) are partitioned as

$$x(k) = \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix}, A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}, \tag{7}$$

$$B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}, C = \begin{pmatrix} C_1 & C_2 \end{pmatrix}, \tag{8}$$

with $x_1(k) \in \mathbb{R}^{n_1}, x_2(k) \in \mathbb{R}^{n_2}, A_{11} \in \mathbb{R}^{n_1 \times n_1}, A_{12} \in \mathbb{R}^{n_1 \times n_2}, A_{22} \in \mathbb{R}^{n_2 \times n_2}$, where $n_1 + n_2 = n$, $B_1 \in \mathbb{R}^{n_1 \times m}$, $C_1 \in \mathbb{R}^{p \times n_1}$ and $C_2 \in \mathbb{R}^{p \times n_2}$.

Proposition 2. *Let the partitioning as in (7) and (8) be given, and assume that A_{22} is nilpotent, then ONC is solvable for system (1) if and only if ONC is solvable for the subsystem described by*

$$x_1(k+1) = A_{11}x_1(k) + B_1u(k), y(k) = C_1x_1(k). \tag{9}$$

Proof. The solvability of ONC for system (9) follows from the solvability of ONC for (1), partitioned as in (7) and (8), starting from $x(0) = (x_1^\top(0), 0^\top)^\top$, where $^\top$ denotes transpose, and $0^\top$ denotes a zero row vector of suitable dimension. Conversely, for any $x(0) = (x_1^\top(0), x_2^\top(0))^\top$, and any finite length control sequence $\{u(k)|n_2 > k \geq 0\}$, it follows by the nilpotency of matrix A_{22}, that $x(n_2) = (x_1^\top(n_2), 0^\top)^\top$, i.e., the second component of $x(k)$ goes to zero automatically, and stays there. Next, extending the starting control sequence with a control sequence $\{u(k)|k \geq n_2\}$ such that ONC is solved for system (9) starting from $x_1(n_2)$ at $k = n_2$, it follows directly that ONC is solved for system (1), starting from the original initial state $x(0) = (x_1^\top(0), x_2^\top(0))^\top$, by application of the control sequence $\mathbf{u} = \{u(k)|k \geq 0\}$. □

(3) Next assume that SNC is solvable for the subsystem (9), and therefore, by Proposition 1, also ONC is solvable for any matrix C_1. Then the following equivalence holds.

Proposition 3. *Let the partitioning as in (7) and (8) be given, and assume that rank$\big(A_{11} - zI, B_1\big) = n_1$, for all $z \neq 0$, then ONC is solvable for system (1) if and only if ONC is solvable for system (1) for all $x(0) = (0^\top, x_2^\top(0))^\top$.*

Proof. Indeed, since $\operatorname{rank}\big(A_{11} - zI, B_1\big) = n_1$, for $z \neq 0$, it follows that ONC is solvable for system (1) for all $x(0) = (x_1^\top(0), 0^\top)^\top$. Because of the linearity in the initial state, it then follows that ONC is solvable for system (1) for all $x(0) = (x_1^\top(0), x_2^\top(0))^\top$ if and only if ONC is solvable for system (1) for all $x(0) = (0^\top, x_2^\top(0))^\top$. □

(4) Now assume that the state $x(k)$, and the matrices A, B and C in (1) are partitioned as

$$x(k) = \begin{pmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{pmatrix}, A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{pmatrix}, \tag{10}$$

and

$$B = \begin{pmatrix} B_1 \\ 0 \\ 0 \end{pmatrix}, C = \begin{pmatrix} C_1 & C_2 & C_3 \end{pmatrix}, \tag{11}$$

with $x_1(k) \in \mathbb{R}^{n_1}$, $x_2(k) \in \mathbb{R}^{n_2}$, $x_3(k) \in \mathbb{R}^{n_3}$, $A_{11} \in \mathbb{R}^{n_1\times n_1}$, $A_{12} \in \mathbb{R}^{n_1\times n_2}$, $A_{13} \in \mathbb{R}^{n_1\times n_3}$, $A_{22} \in \mathbb{R}^{n_2\times n_2}$, $A_{23} \in \mathbb{R}^{n_2\times n_3}$, $A_{33} \in \mathbb{R}^{n_3\times n_3}$, where $n_1 + n_2 + n_3 = n$, $B_1 \in \mathbb{R}^{n_1\times m}$, $C_1 \in \mathbb{R}^{p\times n_1}$, $C_2 \in \mathbb{R}^{p\times n_2}$ and $C_3 \in \mathbb{R}^{p\times n_3}$.
The following equivalence is now immediate.

Proposition 4. *Let the partitioning in (10) and (11) be given, and assume that rank$\left(A_{11} - zI, B_1\right) = n_1$, for all $z \neq 0$, and matrix A_{33} is nilpotent. It then follows that ONC is solvable for system (1) if and only if ONC is solvable for the following subsystem of (1) given by*

$$\tilde{x}(k+1) = \tilde{A}\tilde{x}(k) + \tilde{B}u(k), \quad y(k) = \tilde{C}\tilde{x}(k),$$

with

$$\tilde{x}(k) = \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix}, \tilde{A} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix},$$

$$\tilde{B} = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}, \tilde{C} = \begin{pmatrix} C_1 & C_2 \end{pmatrix},$$

for any initial state $\tilde{x}(0) = (0^\top, x_2^\top(0))^\top$, with $x_2(0) \in \mathbb{R}^{n_2}$ arbitrary.

Proof. A proof can be obtained by combining the results of Propositions 2 and 3. □

Hence, because of the two assumptions, for the solvability of ONC for system (1), with a partitioning as in (10) and (11), we can ignore $x_3(k)$, and need only to focus on the evolution of $x_1(k)$ and $x_2(k)$, for $x_1(0) = 0$ and $x_2(0)$ is arbitrary.

(5) Finally, continue with the partitioned system and the assumptions as before, and add a rank condition.

Lemma 4. *Let the partitioning in (10) and (11) be given, and assume that rank$\left(A_{11} - zI, B_1\right) = n_1$, for all $z \neq 0$, and matrix A_{33} is nilpotent. Next, also assume that*

$$\text{rank}\left(C(zI - A)^{-1}B\right) = \text{rank}\left(C(zI - A)^{-1}(B, G)\right), \tag{12}$$

$$\text{with } G^\top = \begin{pmatrix} 0 & I & 0 \end{pmatrix}, \tag{13}$$

where the matrix I in (13) denotes the $n_2 \times n_2$ identity matrix, and the zeros denote zero matrices of suitable dimensions.

Then, for any initial condition x_0 of the form $x_0 = (0^\top, x_2(0)^\top, 0^\top)^\top$, with $x_2(0) \in \mathbb{R}^{n_2}$ the ONC is solvable.

Proof. By the sufficient condition in Lemma 3, it follows that for all initial conditions $x_0 = (0^\top, x_2(0)^\top, 0^\top)^\top$, with $x_2(0) \in \mathbb{R}^{n_2}$ arbitrary, there exists a control sequence $\mathbf{u} = \{u(k)|k \geq 0\}$ such that $y_{\mathbf{u}}(k, x_0) = 0$ for all $k \geq K$ for some appropriate $K \geq 0$, i.e., a control sequence $\mathbf{u}$ that solves ONC for the above x_0. □

Hence, for the partitioned system description as in (10) and (11), the rank assumption (12), and the other two assumptions $\left(\text{rank}(A_{11} - zI, B_1) = n_1 \text{ for all } z \neq 0, \text{ and } A_{33} \text{ nilpotent}\right)$ are sufficient for solving ONC.

Based on the last case, the following sufficient condition can now be given.

Proposition 5. *Consider system (1), partitioned as in (10) and (11). Then ONC is solvable for the system if*

(1) *rank$\left(A_{11} - zI, B_1\right) = n_1$, for all $z \neq 0$,*
(2) *rank condition (12) is satisfied*
(3) *A_{33} is nilpotent.*

Proof. The next observations follow from the various cases. Condition 3 implies that $x_3(k)$, and therefore $C_3x_3(k)$, goes to zero automatically. Condition 2 implies that there is a control that steers $C_2x_2(k)$ to zero, and condition 1 says the same for $x_1(k)$, and therefore for $C_1x_1(k)$. By linearity, it then follows that ONC is solvable for any x_0. □

It turns out that the partitioning, as in (10) and (11), and the checking of the above conditions, can be implemented and performed elegantly for structured systems. Hence, for structured systems, the above yields a sufficient condition for the solvability of ONC in a structural sense.

5. Structured systems

5.1. Graph representation

We assume now that system (1) is structured, i.e., we assume that A, B, and C in (1) are so-called structured matrices, containing free nonzeros and fixed zeros. Let the graph representing the structure of the system be given by $G = (\mathcal{V}, \mathcal{E})$, with node set $\mathcal{V}$ and edge set $\mathcal{E}$. The node set can be written as $\mathcal{V} = \mathcal{X} \cup \mathcal{U} \cup \mathcal{Y}$, with $\mathcal{X} = \{x_1, \ldots, x_n\}$ the set of state nodes, $\mathcal{U} = \{u_1, \ldots, u_m\}$ the set of input nodes, and $\mathcal{Y} = \{y_1, \ldots, y_p\}$ the set of output nodes. The edge set is given by $\mathcal{E} = \{(x_j, x_i)|a_{ij} \neq 0\} \cup \{(u_j, x_i)|b_{ij} \neq 0\} \cup \{(x_j, y_i)|c_{ij} \neq 0\}$, where, for example, (x_j, x_i) denotes an edge from node x_j to node x_i, and $a_{ij} \neq 0$ indicates that the (i, j) element of A is a free nonzero, and similarly for the other edges and nonzero elements.

Given graph $G = (\mathcal{V}, \mathcal{E})$, we say there is a path from node $\tilde{v} \in \mathcal{V}$ to node $\hat{v} \in \mathcal{V}$, if there exist mutually distinct nodes $v_0, v_1, \ldots, v_l \in \mathcal{V}$, with $v_0 = \tilde{v}$, $v_l = \hat{v}$ and $(v_{i-1}, v_i) \in \mathcal{E}$, for $i = 1, 2, \ldots, l$. The path then has length l, and is said to go from node $\tilde{v}(= v_0)$, also called begin node, to node $\hat{v}(= v_l)$, also called end node, and the path is said to consist of the nodes $v_0, v_1, \ldots, v_l$. A cycle is a path with at least one edge, of which the begin node and end node coincide. A path consisting of a single node with no edge to itself, has length 0. Hence, the length of a cycle is always positive.

Given subsets $\tilde{\mathcal{V}}, \hat{\mathcal{V}} \subseteq \mathcal{V}$, we say that $\hat{\mathcal{V}}$ is reachable from $\tilde{\mathcal{V}}$, if there is a path from a node $\tilde{v} \in \tilde{\mathcal{V}}$ to a node $\hat{v} \in \hat{\mathcal{V}}$. We say that a collection of paths from $\hat{\mathcal{V}}$ to $\tilde{\mathcal{V}}$ is disjoint when they mutually have no nodes in common. The size of such a disjoint collection is the number of paths it consists of.

5.2. Graph decomposition

We focus now on the graph $G = (\mathcal{V}, \mathcal{E})$ of the structured system (1) and introduce the following decomposition.

- We let $\mathcal{V}_1$ be the set of nodes of $\mathcal{V}$ that are reachable from $\mathcal{U}$, i.e., that can be reached from a node in $\mathcal{U}$ using a path, possibly of zero length. We write $\mathcal{V}_1 = \mathcal{X}_1 \cup \mathcal{U}_1 \cup \mathcal{Y}_1$, where $\mathcal{U}_1 = \mathcal{U}$ (obviously), $\mathcal{X}_1$ denotes the set of state nodes that are reachable from $\mathcal{U}$, and $\mathcal{Y}_1$ denotes the set of output nodes that are reachable from $\mathcal{U}$.
- Next, consider the complementary set $\mathcal{V}\backslash\mathcal{V}_1 = \{v \in \mathcal{V}|v \notin \mathcal{V}_1\}$. Focusing on the subgraph of G with node set $\mathcal{V}\backslash\mathcal{V}_1$, we let $\mathcal{V}_2 \subseteq \mathcal{V}\backslash\mathcal{V}_1$ denote all nodes that are reachable from a cycle in $\mathcal{V}\backslash\mathcal{V}_1$, i.e., all nodes in $\mathcal{V}\backslash\mathcal{V}_1$ that can be reached using a path from a node in a cycle with nodes in $\mathcal{V}\backslash\mathcal{V}_1$. The cycle has a positive length, the path may be possibly of zero length.
  Let $\mathcal{V}_3$ be all remaining nodes in $\mathcal{V}\backslash\mathcal{V}_1$. Hence, $\mathcal{V}_3 = (\mathcal{V}\backslash\mathcal{V}_1)\backslash\mathcal{V}_2$.

- We note that nodes in $\mathcal{V}\backslash\mathcal{V}_1$ cannot be reached from $\mathcal{U}$, but may be reachable from nodes, and even cycles, in $\mathcal{V}\backslash\mathcal{V}_1$ itself. Further, note that all nodes contained in cycles in $\mathcal{V}\backslash\mathcal{V}_1$ are elements of $\mathcal{V}_2$. Hence, the nodes in $\mathcal{V}_3$ are not contained in any cycle (in $\mathcal{V}\backslash\mathcal{V}_1$). However, the nodes in $\mathcal{V}_3$ may be reached using a path, but such a path cannot start in a cycle.
- We write $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3$, and in particular $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2 \cup \mathcal{X}_3$, $\mathcal{U} = \mathcal{U}_1$, $\mathcal{Y} = \mathcal{Y}_1 \cup \mathcal{Y}_2$, where $\mathcal{X}_1, \mathcal{U}_1, \mathcal{Y}_1$ are as above, $\mathcal{X}_2$ denotes the set of state nodes in $\mathcal{V}\backslash\mathcal{V}_1$ reachable from a cycle in $\mathcal{V}\backslash\mathcal{V}_1$, $\mathcal{X}_3 = \mathcal{X}\backslash(\mathcal{X}_1 \cup \mathcal{X}_2)$ and $\mathcal{Y}_2 = \mathcal{Y}\backslash\mathcal{Y}_1$. Also, observe that $\mathcal{V}\backslash\mathcal{V}_1 = \mathcal{V}_2 \cup \mathcal{V}_3$.

  The matrices A, B, and C can be partitioned as

  $$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}, B = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix},$$
  $$C = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \end{pmatrix},$$

  with the submatrices of suitable dimensions. Note that some of the subsets $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3$ may be empty, in which case the corresponding submatrices are void, i.e., consisting of zero rows and/or columns.
- As a consequence of the definition of $\mathcal{V}_1$, it follows that the submatrices A_{21}, A_{31}, B_2 and B_3, when existing, must be zero matrices. Indeed, a nonzero entry in B_2 or B_3 would mean that there are nodes in $\mathcal{V}\backslash\mathcal{V}_1$ that can be reached from $\mathcal{U}$ directly by an edge starting in $\mathcal{U}$. Similarly, a nonzero entry in A_{21} or A_{31} would mean that there are nodes in $\mathcal{V}\backslash\mathcal{V}_1$ that can be reached from $\mathcal{U}$ via a path that passes through $\mathcal{V}_1$. Both are impossible by the definition of $\mathcal{V}_1$.

  As a consequence of the definition of the sets $\mathcal{V}_2, \mathcal{V}_3 \subseteq \mathcal{V}\backslash\mathcal{V}_1$, it follows that the submatrix A_{32}, when existing, must be a zero matrix. Indeed, a nonzero entry in A_{32} would mean that there are nodes in $\mathcal{V}_3$ that are connected to nodes from $\mathcal{V}_2$, and therefore are connected to a cycle in $\mathcal{V}\backslash\mathcal{V}_1$. Consequently, such a node in $\mathcal{V}_3$ should belong to $\mathcal{V}_2$, which is impossible by the definition of $\mathcal{V}_3$.

  Finally, by definition all edges from $\mathcal{V}_1$ to $\mathcal{Y}$ have the end node in $\mathcal{Y}_1$. Therefore, C_{21}, when existing, must be a zero matrix.
- As a result of these observations, it follows that the matrices A, B and C can be partitioned in more detail as

  $$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{pmatrix}, B = \begin{pmatrix} B_1 \\ 0 \\ 0 \end{pmatrix},$$
  $$C = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ 0 & C_{22} & C_{23} \end{pmatrix},$$

  where some of the submatrices may be void because corresponding node sets that are empty.

6. Existing results and main result

6.1. Incorporating existing results

The following results can be found in literature. Start from the decomposition derived in the previous section.

- Recall that all state nodes in $\mathcal{X}_1$ can be reached from $\mathcal{U}$. The latter can equivalently be expressed by saying that the pair (A_{11}, B_1) is irreducible, cf. Dion et al. (2003). By Theorem 2 of Hosoe and Matsumoto, see Hosoe and Matsumoto (1979), this implies that the generic rank of $(A_{11} - zI, B_1) = n_1$, for all $z \neq 0$, or, by Lemma 1, that SNC is structurally solvable for the structured system given by the pair (A_{11}, B_1).
- The structural version of the rank condition (12) is satisfied if and only if in graph G the maximal number of disjoint paths from $\mathcal{U}$ to $\mathcal{Y}$ equals the maximal number of disjoint paths from $\mathcal{U} \cup \mathcal{X}_2$ to $\mathcal{Y}$. For details, see the survey paper (Dion et al., 2003).
- Recall that, by construction, there are no cycles in $\mathcal{V}_3$. Therefore, the restriction of graph G to the nodes in $\mathcal{X}_3$ does not contain any cycle. The latter implies that any numerical realisation of matrix A_{33} is structurally nilpotent. Indeed, in Theorem 4 of van der Woude (2018), it is shown that $\det(sI - A_{33}) = s^{n_3}$ if and only if the graph of A_{33} contains no cycles.

6.2. Main result

The previous results can be summarised in the following theorem containing sufficient conditions for the structural solvability of ONC. It is the main result of this paper.

Theorem 1. *Consider the structured system (1), and let its graph G be decomposed as described in Section 5.2. Then ONC is generically solvable when the maximal number of disjoint paths from $\mathcal{U}$ to $\mathcal{Y}$ is equal to the maximal number of disjoint paths from $\mathcal{U} \cup \mathcal{X}_2$ to $\mathcal{Y}$.*

Proof. The condition on the equal maximal number of disjoint paths from $\mathcal{U}$ to $\mathcal{Y}$, and from $\mathcal{U} \cup \mathcal{X}_2$ to $\mathcal{Y}$, implies by Theorem 6 in Dion et al. (2003) that the rank condition (12) generically holds. Then condition 2 of Proposition 5 is generically satisfied. Conditions 1 and 3 are generically satisfied by how the partitioning in Section 5.2 is obtained. Hence, in the context of the present theorem, the conditions of Proposition 5 are generically satisfied, and consequently, ONC is generically solvable. □

The graph decomposition in Section 5.2 starts with finding nodes that can be reached from the inputs. The reachable set can be simply obtained by a breadth first algorithm which complexity is linear in the number of edges in the system graph. Note that the first and third bullet in Section 6.1 are satisfied automatically by the decomposition. The second bullet of Section 6.1, i.e., the condition in Theorem 1, can be checked by using ideas based on maximal size linkings, i.e., sets of disjoint paths of maximal size. See Theorem 2 in Section 3.2 of Commault, Dion, and van der Woude (2002) for more details. The computational aspects of the computations can be worked out using bipartite graphs and maximal matchings. See Section 4, and Lemma 4 in Section 5, of Commault et al. (2002) for more details.

To summarise, the conditions of Theorem 1 can be checked using well-known and very efficient (polynomial) algorithms from combinatorial optimisation.

7. Example

In this section, the main result, i.e., Theorem 1, of this paper is illustrated by means of an example.

Consider the structured system (1) given by the structured matrices

$$A = \begin{pmatrix} 0 & * & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & 0 \\ 0 & * & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & 0 & * \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ * \\ 0 \end{pmatrix},$$
$$C = \begin{pmatrix} \times & * & 0 & 0 & 0 & \times \\ * & * & * & 0 & 0 & 0 \end{pmatrix},$$

where the 0's denote fixed zeros and the $*$'s are free nonzeros. The entries $\times$ will be treated below as a fixed zero 0 or as a free nonzero $*$. The graph G of the system is given in Fig. 1. In the graph below the special nature of the entries $\times$ (either a free nonzero or a fixed zero) is indicated by the dotted edge.

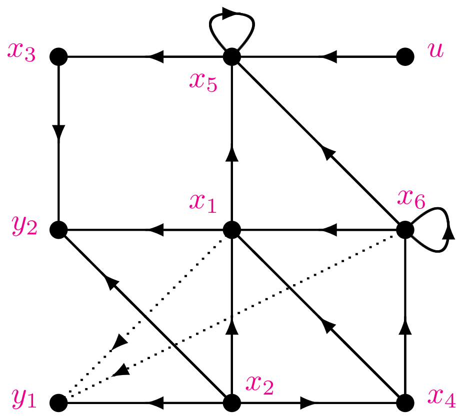

Fig. 1. Graph of example.

From graph G, the decomposition in Section 5.2 easily follows. Indeed, it is straightforward to see that the set of input-connected vertices is $\mathcal{V}_1 = \{u, x_3, x_5, y_2\}$ and then $\mathcal{V}\backslash\mathcal{V}_1 = \{x_1, x_2, x_4, x_6, y_1\}$. If any of the entries $\times$ is a free nonzero, then the set of vertices reachable by a path from a cycle is $\mathcal{V}_2 = \{x_1, x_6, y_1\}$ and then $\mathcal{V}_3 = \{x_2, x_4\}$, else $\mathcal{V}_2 = \{x_1, x_6\}$ and $\mathcal{V}_3 = \{x_2, x_4, y_1\}$. In both cases, it follows that $\mathcal{X}_1 = \{x_3, x_5\}, \mathcal{U}_1 = \{u\}, \mathcal{Y}_1 = \{y_2\}, \mathcal{X}_2 = \{x_1, x_6\}, \mathcal{X}_3 = \{x_2, x_4\}$, and $\mathcal{Y}_2 = \{y_1\}$.

Based on the sets $\mathcal{X}_1$, $\mathcal{X}_2$, $\mathcal{X}_3$, $\mathcal{Y}_1$, and $\mathcal{Y}_2$, the state and output component can be relabelled as follows: $\hat{x}_1 = x_3, \hat{x}_2 = x_5, \hat{x}_3 = x_1, \hat{x}_4 = x_6, \hat{x}_5 = x_2, \hat{x}_6 = x_4$, and $\hat{y}_1 = y_2, \hat{y}_2 = y_1$. Note that u needs no relabelling here.

Then, the associated matrices $\hat{A}$, $\hat{B}$ and $\hat{C}$ can be obtained easily and can be partitioned as described in Section 5.2:

$$\hat{A} = \left(\begin{array}{cc|cc|cc} 0 & * & 0 & 0 & 0 & 0 \\ 0 & * & * & * & 0 & 0 \\ \hline 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & 0 & * \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & 0 \end{array}\right), \quad \hat{B} = \left(\begin{array}{c} 0 \\ * \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \end{array}\right),$$

$$\hat{C} = \left(\begin{array}{cc|cc|cc} * & 0 & * & 0 & * & 0 \\ 0 & 0 & \times & \times & * & 0 \end{array}\right).$$

The main result of this paper (i.e., Theorem 1) is that ONC is structurally solvable for the structured system given by A, B, and C (and by $\hat{A}, \hat{B}$, and $\hat{C}$), if both entries $\times$ are fixed zeros, i.e., if there are no edges from x_1 to y_1 and from x_6 to y_1. Indeed, when $\times = 0$ for both entries, the maximal number of disjoint paths from $\mathcal{U}$ to $\mathcal{Y}$ is one and is equal to the maximal number of disjoint paths from $\mathcal{U} \cup \mathcal{X}_2$ to $\mathcal{Y}$.

If one of the two entries $\times$ is unequal to 0, i.e., $\times \neq 0$, then the maximal number of disjoint paths from $\mathcal{U} \cup \mathcal{X}_2$ to $\mathcal{Y}$ is equal to two. Indeed, then generically $y_1(k) \neq 0$ for all $k \geq 0$, no matter what control sequence $\{u(k)|k \geq 0\}$ is applied.

The above conclusions can be verified numerically by selecting the nonzero entries in A, B, and C randomly, yielding a numerical realisation of the matrices. Next, A^n, $\mathcal{V}^*(\ker C)$, and $\langle A|\text{im } B\rangle$ can be computed and Lemma 2 can be checked numerically. Also, the condition can be checked formally by computing A_λ^n, $\mathcal{V}^*(\ker C_\lambda)$ and $\langle A_\lambda|\text{im } B_\lambda\rangle$, given the matrices A_λ, B_λ, and C_λ parametrised by the vector λ, and checking the condition in Lemma 2.

8. Conclusion and discussion

8.1. Summary

In this paper, we studied the output null controllability problem for a structured linear discrete-time system and studied the generic solvability of the problem. The latter means that the problem is solvable for almost all systems with the same structure. For this, we only needed the zero/nonzero structure of the system matrices. We represented the structure of the system by means of a directed graph and presented a graph theoretic sufficient condition for the generic solvability of the problem.

8.2. Necessity of condition of Theorem 1

The obtained sufficient condition in Theorem 1 is illustrated through an example. In the example, the condition also appeared to be necessary. This phenomenon has shown up in all examples studied thus far. However, the actual necessity of the condition could not be proved yet. The (believed) necessity of the condition of Theorem 1 is a topic for future research.

8.3. Extension

A possible extension of Theorem 1 might be that the set $\mathcal{X}_2$ is restricted to the set of nodes in $\mathcal{X}_2$ that are contained in a cycle in $\mathcal{X}_2$. For the example, this would mean that only node x_4 in $\mathcal{X}_2$ has to be taken into consideration in the application of Theorem 1. A proof of such an extension would require a more detailed investigation of the graph decomposition and all related aspects. To avoid the paper from getting too technical, this possible extension and its proof are omitted.

CRediT authorship contribution statement

Jacob van der Woude: Writing – original draft, Writing – review & editing. **Christian Commault:** Writing – review & editing. **Taha Boukhobza:** Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix

Following the method in Schumacher (1983), write $p(z)$ and $q(z)$ in their Laurent series as follows

$$p(z) = \sum_{k>-\ell} p_k z^{-k} \text{ and } q(z) = \sum_{k\geq-\ell} q_k z^{-k}$$

for some nonnegative integer ℓ, i.e.,

$$\begin{aligned} p(z) = & \qquad p_{1-\ell} z^{\ell-1} + \cdots + p_{-1} z \\ & \qquad\qquad + p_0 + p_1 z^{-1} + p_2 z^{-2} + \cdots, \\ q(z) = & \; q_{-\ell} z^{\ell} + q_{1-\ell} z^{\ell-1} + \cdots + q_{-1} z \\ & \qquad\qquad + q_0 + q_1 z^{-1} + q_2 z^{-2} + \cdots. \end{aligned}$$

Then, by comparing powers of z^{-k} for $k \geq -\ell$, it follows from

$$x_0 = (zI - A)p(z) - Bq(z) \text{ and } Cp(z) \text{ is polynomial,}$$

that

$$\begin{array}{rcll} 0 & = & p_{1-\ell} - Bq_{-\ell} & \\ 0 & = & p_{k+1} - Ap_k - Bq_k & \text{for } -\ell < k < 0 \\ x_0 & = & p_1 - Ap_0 - Bq_0 & \\ 0 & = & p_{k+1} - Ap_k - Bq_k & \text{for } k \geq 1 \end{array}$$

and

$$\begin{array}{rcll} 0 & = & Cp_k & \text{for } k \geq 1 \end{array}$$

Hence, for $k \geq 1$, it follows that

$$p_{k+1} = Ap_k + Bq_k \text{ and } Cp_k = 0.$$

Introducing $\tilde{p}(s) = \sum_{k\geq 1} p_k z^{-k}$ and $\tilde{q}(s) = \sum_{k\geq 1} q_k z^{-k}$, it follows that

$$p_1 = (zI - A)\tilde{p}(z) - B\tilde{q}(z) \text{ and } C\tilde{p}(z) = 0,$$

which implies that $p_1 \in \mathcal{V}^*(\ker C)$, see Hautus (1979).

Further,

$$\begin{array}{lcll} x_0 & = & p_1 \quad - Ap_0 \; - Bq_0 & \\ 0 & = & p_{k+1} - Ap_k \; - Bq_k & \text{for all } -\ell \leq k < 0 \end{array}$$

with $p_{-\ell} = 0$. In particular,

$$\begin{array}{lclll} x_0 & = & p_1 & -Ap_0 & -Bq_0 \\ 0 & = & p_0 & -Ap_{-1} & -Bq_{-1} \\ \vdots & & \vdots & & \vdots \\ 0 & = & p_{2-\ell} & -Ap_{1-\ell} & -Bq_{1-\ell} \\ 0 & = & p_{1-\ell} & & -Bq_{-\ell} \end{array}$$

Multiplying the obtained equations by $I, A, \ldots, A^{\ell-1}$ and A^{ℓ}, respectively, and adding them together, it follows that

$$x_0 = p_1 - Bq_0 - ABq_{-1} - \cdots - A^{\ell} Bq_{-\ell}.$$

Hence, it follows that $x_0 - p_1 \in \langle A|\text{im } B\rangle$. With $p_1 \in \mathcal{V}^*(\ker C)$, it consequently follows that $x_0 \in \mathcal{V}^*(\ker C) + \langle A|\text{im } B\rangle$.

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