Nonlinear Fourier transform algorithm using a higher order exponential integrator

Chimmalgi, Shrinivas; Prins, Peter; Wahls, Sander

DOI
10.1364/SPPCom.2018.SpM4G.5

Publication date
2018

Document Version
Accepted author manuscript

Published in
Advanced Photonics 2018

Nonlinear Fourier Transform Algorithm Using a Higher Order Exponential Integrator

Shrinivas Chimmalgi **Peter J. Prins** **Sander Wahls**

Delft Center for Systems and Control, Delft University of Technology, 2628 CD Delft, The Netherlands

s.chimmalgi@tudelft.nl

Abstract: We present a nonlinear Fourier transform algorithm whose accuracy, at a comparable runtime and for moderate step sizes, is orders of magnitude better than that of the classical Boffetta-Osborne method.

OCIS codes: 060.4370, 060.2330.

1. Introduction and Theory

The exploitation of nonlinear Fourier transforms (NFTs) in optical communication has recently gained significant attention [1]. Many of the communication schemes being developed rely on numerical NFT algorithms to compute the reflection coefficient (one part of the nonlinear Fourier spectrum) accurately and efficiently[1]. Although several NFT algorithms have been proposed for this purpose, the classical algorithm based on a one-step exponential integrator (BO) in [2] seems to remain the standard method. BO provides a good trade-off between accuracy, computation time and ease of implementation. In this work, we present a new algorithm based on a commutator-free fourth-order integrator (CF4) [3]. At similar runtimes, the new algorithm is significantly more accurate than BO, which e.g. would allow us to decrease the sampling rates in NFT-based communication systems or the computational costs of simulations.

The reflection coefficient $\rho(\lambda)$ of any signal $q(t)$ that vanishes sufficiently fast for $t \to \pm\infty$ is found by solving the Zakharov-Shabat scattering problem [4]. Here $\lambda \in \mathbb{R}$ is a parameter. For numerical schemes, the computation domain is fixed to $t \in [T_-, T_+]$, which results in a step-size $h = (T_+ - T_-)/(D-1)$. The given samples are $q_k := q(t_k)$, where $t_k = T_- + kh$, $k = 0, \ldots, D-1$. The scattering problem is then solved in [2] by multiplying individual scattering matrices ($\mathbf{U}_k$), one for each sample. We follow exactly the same approach as [2], but instead apply CF4 [3] to determine the per-sample scattering matrices. That is, we replace [2, Eq. 4.2] with

$$\mathbf{U}_k = \exp\left(\begin{bmatrix} -i\lambda/2 & r_k^{(1)} \\ -r_k^{(1)*} & i\lambda/2 \end{bmatrix} h\right) \exp\left(\begin{bmatrix} -i\lambda/2 & r_k^{(2)} \\ -r_k^{(2)*} & i\lambda/2 \end{bmatrix} h\right), \tag{1}$$

where $r_k^{(1)} := (q_k^{(1)} + q_k^{(2)})/4 + \sqrt{3}(q_k^{(2)} - q_k^{(1)})/6$, $r_k^{(2)} := (q_k^{(1)} + q_k^{(2)})/4 - \sqrt{3}(q_k^{(2)} - q_k^{(1)})/6$, $q_k^{(1)} := q(t_k - (\sqrt{3}/6)h)$ and $q_k^{(2)} := q(t_k + (\sqrt{3}/6)h)$, $k = 0, 1, \ldots, D-1$. The new scattering matrices therefore require nonequispaced samples. In many cases it is not practical to obtain such nonequispaced samples. Hence, we use local cubic interpolation to obtain the nonequispaced samples at the required time instants. Specifically, we obtain the nonequispaced samples $q_k^{(1)}$ and $q_k^{(2)}$ by interpolation of the given equispaced samples q_k.

2. Numerical Example

We choose the signal $q(t) = 4.4i\,\mathrm{sech}(t)$ for the numerical example. We let the domain of computation be $[-34.2, 34.2]$ such that $|q(t)| < 100\varepsilon \;\forall\; t \notin [-34.2, 34.2]$, where $\varepsilon = 2.2204\mathrm{e}{-16}$ (machine epsilon), in order to avoid truncation errors. As a result more samples than usual are required to reach acceptable step sizes. The closed form of the reflection coefficient is given by $\rho_a(\lambda) = \frac{i\sin(4.4\pi)}{\cosh(\pi\lambda)} \cdot \frac{\Gamma(0.5-i\lambda+4.4)\Gamma(0.5-i\lambda-4.4)}{\Gamma^2(0.5-i\lambda)}$, where Γ is the gamma function [6, Eq. 27]. Let the numerically computed reflection coefficient be $\rho_n(\lambda)$. The relative L^2-error $\|\rho_a(\lambda) - \rho_n(\lambda)\|_2 / \|\rho_a(\lambda)\|_2$, $|\lambda| < \lambda_{max}$, is used as the measure of accuracy. The CF4 algorithm is compared against BO and the Implicit-Adams method (IA2), which is another higher order method; its use for NFTs has been recently proposed in [7, IA_2]. As with the linear

[1] The method presented in this paper can also be applied to the so-called b-coefficient recently considered in [8].

This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 716669).

Fourier transform, the sampling process limits the range $|\lambda| < \lambda_{max}$ that we can resolve. Since we are not aware of a theoretical quantification of this effect, we plot $|\rho_a(\lambda)|$ and the error $|\rho_a(\lambda) - \rho_n(\lambda)|$ for $D = 2^8$ for the three algorithms in Fig. 1. Based on this plot, we choose $|\lambda| < 3$ to compute the relative L^2-error for increasing number of samples. The results are shown in Fig. 2.

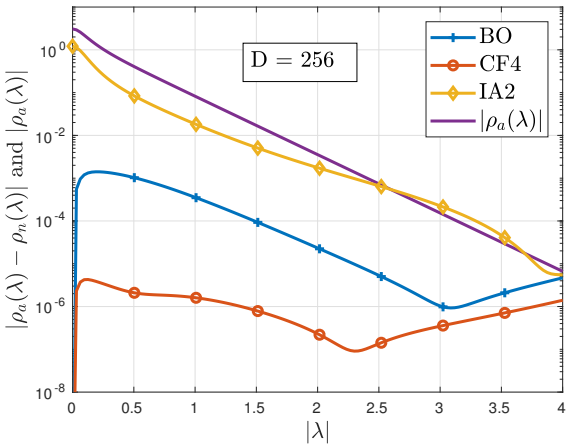

Fig. 1: Absolute error in reflection coefficient.

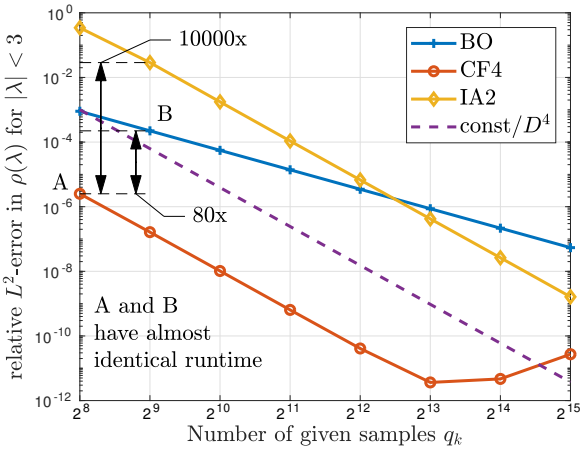

Fig. 2: Error in reflection coefficient vs number of samples.

From Fig. 1 it is clear that the CF4 integrator can compute the spectrum more accurately than BO and IA2. Fig. 2 gives a better visualization of the scale of improvement. As the interpolation step effectively doubles the number of values used by the CF4 method, we compare the error at $D = 2^8$ given samples for CF4 with the error at $D = 2^9$ for BO and IA2. The relative error for CF4 is roughly 80 times smaller than BO and 10000 times smaller than IA2. Both IA2 and CF4 converge with order 4, but IA2 becomes better than BO only for $D > 2^{12}$. We remark that the runtimes of CF4 with $D = 2^8$ and BO with $D = 2^9$ are almost identical. We have verified that the ideas used in [5, 6] to make BO fast also apply to CF4, leading to a fast and highly accurate NFT algorithm. The details will be published elsewhere.

3. Conclusion and Discussion

An NFT algorithm which uses a fourth order exponential integrator was proposed and shown to compute the reflection coefficient better than the current standard algorithm [2]. The algorithm is fourth order convergent and can be made fast. In Fig. 2, as the number of samples is increased, the error of CF4 reaches a minimum and starts rising again. The underlying cause of the error rise is under investigation, but it is not a result of the cubic interpolation used to obtain the nonequispaced samples. We however point out that the minimal error achieved by CF4 is very low.

References

1. S. K. Turitsyn et al., "Nonlinear Fourier Transform for Optical Data Processing and Transmission: Advances and Perspectives," in *Optica*, vol. 4, 307–322 (2017).
2. G. Boffetta and A. R. Osborne, "Computation of the Direct Scattering Transform for the Nonlinear Schrödinger Equation," in *Journal of Computational Physics*, vol. 102, no. 2, 252–264 (1992).
3. M. Thalhammer, "A fourth-order commutator-free exponential integrator for nonautonomous differential equations," in *SIAM Journal on Numerical Analysis*, vol. 44, no. 2, 851–864 (2006).
4. V. Zakharov and A. Shabat, "Exact Theory of Two-Dimensional Self-Focusing and One-Dimensional Self-Modulation of Waves in Nonlinear Media," in *Soviet Physics JETP*, vol. 34, p. 62 (1972).
5. S. Wahls and H.V. Poor, "Introducing the Fast Nonlinear Fourier Transform," in *Proc. of IEEE ICASSP*, Vancouver, Canada, May 2013.
6. P. J. Prins and S. Wahls, "Higher Order Exponential Splitttings for the Fast Non-Linear Fourier Transform of the Korteweg-de Vries Equation," to be published in *Proc. of IEEE ICASSP*, Calgary, Canada, April 2018.
7. V. Vaibhav, "Higher Order Convergent Fast Nonlinear Fourier Transform," Preprint, Dec. 2017, Link: `https://arxiv.org/pdf/1712.06647.pdf`.
8. S. Wahls, "Generation of Time-Limited Signals in the Nonlinear Fourier Domain via b-Modulation," in *Proc. European Conference on Optical Communcation (ECOC)*, Gothenburg, Sweden, September 2017.