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Scaling limit of the odometer of correlated Gaussians on the torus $\mathbb{T}^d$.

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"Scaling limit of the odometer of correlated Gaussians on the torus $\mathbb{T}^d$"

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Abstract

It was shown by Cipriani et al. in [5] that the odometer function for a divisible sandpile with i.i.d. weights on $\mathbb{Z}_n^d$ converges to a continuum bilaplacian field on $\mathbb{T}^d$ after an appropriate scaling. In this thesis, we consider an odometer function associated with correlated Gaussian weights $(\sigma(x))_{x\in\mathbb{Z}_n^d}$, with $\mathbb{E}[\sigma(\cdot)] = 0$ and

$$\mathbb{E}[\sigma(x)\sigma(y)] = \mathcal{K}(x - y),$$

where $\mathcal{K} : \mathbb{Z}^d \to \mathbb{R}$ is a stationary covariance function. We obtain our main result in Theorem 6.1, where we show that after an extra scaling factor depending on $\mathcal{K}$, the odometer still converges to the bilaplacian field on $\mathbb{T}^d$.

Contents

1 Introduction

The world around us is getting more complex every day. Not only is technology advancing rapidly, we are also living with more people than ever, all with their own weird quirks. More than ever now, we need to be able to cope with large, complex systems to make sense of everything. We take as an example the modern financial market: a gigantic, complex system with an enormous amount of participants. At small enough timesteps, a stock price can either go up or down, depending on how much the participants are buying or selling. In a sense, this behaviour resembles a random walk. However, as there are too many players to take into account in the financial market, we approximate our discrete model with a continuous model, the *Brownian motion*, which is much easier to work with.

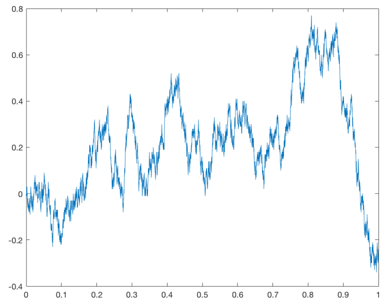

Figure 1: Brownian motion on the interval $[0, 1]$.

This is one of the goals of modern probability theory: taking a complex, random system, that is defined through micro-interactions, and applying abstract theory to say something about the behaviour of the system at large. In the above example, we had our financial market, which was our complex system, our micro-interactions were defined by the buyers and sellers, and using our theoretical tools we obtained the Brownian motion.

Now, modern probability theory has more applications than just modelling the behaviour of stocks in order for you to buy a supercharged Range Rover or a preposterous yacht. We can use the above approach in applications in physics, biology and engineering. The idea stays the same, we take a discrete model with micro-interactions defined on it, and prove a limit theorem to obtain the behaviour for a large-scale system.

One particular example is the *Abelian sandpile model*, which was introduced by Bak et al. in [14]. In this model, we start out with a finite graph V and for each vertex $x \in V$ we assign an integer height $s(x)$, representing a certain amount of particles. In each timestep, we now uniformly at random distribute particles over the vertices $x \in V$ in the graph. We fix a $c \in \mathbb{N}$. If we now have $s(x) > c$ for a certain vertex, we call x unstable, and it *topples* by distributing its particles to its neighbors. Now, if one vertex topples, it can of course cause the toppling of another vertex, which can in turn create a so-called *avalanche* of toppling. This procedure is repeated until all vertices are stable again. In order for this process to become stable, we also have some vertices that act as *sinks*, which absorb the particles. These are usually associated with the boundary when we are working on subsets of $\mathbb{Z}^d$.

This model has applications in for example neuroscience (see for instance [2]), where neurons fire off electrical signals. Neurons are associated with the vertices in the graph, and synapses with the edges. If a neuron reaches a certain amount of electrical charge, it fires off charge to the next neurons through the synapses, and so on.

An extension of the Abelian sandpile, the *divisible sandpile* was introduced by Levine and Peres in ([15],[16]), where they consider a sandpile model with continuous heights, whereas the Abelian sandpile had discrete heights. The divisible sandpile model we will be working with is based on the model in [1] and [5], and is defined in the following way. We start with a collection of i.i.d. standard normals $(\sigma(x))_{x\in\mathbb{Z}_n^d}$. Here we think of $\mathbb{Z}_n^d$ as a discrete, d-dimensional box of side length n. Our initial sandpile configuration now is a function $s : \mathbb{Z}_n^d \to \mathbb{R}$ defined for all $x \in \mathbb{Z}_n^d$ by

$$s(x) = 1 + \sigma(x) - \frac{1}{n^d}\sum_{z\in\mathbb{Z}_n^d}\sigma(z).$$

We think of s as representing the heights at each vertex in the beginning. In the divisible sandpile, a site topples if it has mass > 1. It will then uniformly distribute its excess mass to its neighbors, while keeping mass 1 to itself. Note that this process is completely deterministic, the only randomness is in the initial distribution. Subsequently, for each timestep t, we can look at the amount of mass a vertex has distributed to any of its neighbors. Call this function $e^{(t)} : \mathbb{Z}_n^d \to \mathbb{R}$. As sites can only emit mass, and can't "un-emit" mass, we see that $e^{(t)} \uparrow e$ as $t \to \infty$. We call $e : \mathbb{Z}_n^d \to \mathbb{R}$ the odometer function. If the sandpile stabilizes, we have $e(x) < \infty$ for all $x \in \mathbb{Z}_n^d$. As e depends on the initial configuration, $(e(x))_{x\in\mathbb{Z}_n^d}$ is a collection of random variables, more specifically it was shown in [1] that e is a shifted, discrete bilaplacian field. This roughly means that the odometer e is distributed, up to a constant, like a collection of Gaussians $(\eta(x))_{x\in\mathbb{Z}_n^d}$ such that

$$\Delta^2\mathbb{E}[\eta(x)\eta(y)] = \delta_x(y) - \frac{1}{n^d}.$$

Subsequently, as $n^{-1}\mathbb{Z}_n^d \to \mathbb{T}^d$, the scaling limit of this odometer was considered in [5], where it was shown that the scaling limit of the odometer on $\mathbb{T}^d$ is still a, this time continuous, bilaplacian field. Now this brings us to the main research question of this thesis is:

What is the scaling limit of the odometer if we start with an initial configuration of correlated Gaussians?

In order to answer this question, we first recall the most relevant (for this project) concepts from Probability and Fourier analysis in the Preliminaries section.

In Section 3, we first walk through the basic theory of divisible sandpiles and recall the most important results from [1]. After this we derive a new identity for the odometer of correlated Gaussians in Section 3. This result is similar to Proposition 1.3 in [1].

Section 4 is not integral to answering our main research question, but it is nevertheless another new result related to the divisible sandpile. We consider the speed at which the sandpile stabilizes. In particular, we will see that in the continuous case the divisible sandpile almost surely does not stabilize in finite time, meaning that the properties of the odometer function only hold in the limit.

After this, in Section 5, we go into considerable detail in the proof of the scaling limit for the odometer of i.i.d. Gaussians, Theorem 1 in [5]. We also derive a new identity for the pairwise correlations of the bilaplacian field.

Section 6 contains the main result of this thesis, and we answer our research question. We prove that after an extra scaling factor, depending on the covariance structure, our discrete field still converges to the bilaplacian field, where the convergence holds in the same way as Theorem 1 in [5]. We also compare the maxima of odometers under different covariances.

Now in [5], the result for i.i.d. Gaussians is extended for general i.i.d. weights with mean zero and finite variance. In Section 7 we outline the problems that arise when we try to generalize the result in the correlated, Gaussian case to general correlated weights. We also want to obtain an upper bound as to how fast the sandpile converges to the stable configuration.

We want to stress here that the actual new results are Section 3.3, the entire Section 4, Section 5.6, and the entire Section 6.

2 Preliminaries: a crash course in Probability and Fourier analysis

In this project, we will repeatedly throw around notions from both Probability theory and Fourier analysis, and in fact, we won't do much else. However, as we can not expect from the reader to have followed the 3rd year courses Advanced Probability and Fourier Analysis, we will attempt to give a crash course here. This section is split up in three parts: we will first lay out the basics of measure-theoretic probability, after that we will look at Fourier analysis and analysis on the torus $\mathbb{T}^d$, at last, we will introduce the basics of infinite-dimensional probability theory.

2.1 A crash course in probability theory

This section is set up in the following way: we will first state the basic definitions of probability theory, after that we will have a look at some limit theorems. For a more thorough explanation, we refer the reader to [12].

Definition 2.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a measure space, i.e. $\mathcal{F}$ is a σ-algebra on Ω, and $\mathbb{P} : \mathcal{F} \to \mathbb{R}^+$ is a measure. If $\mathbb{P}(\Omega) = 1$, we call $(\Omega, \mathcal{F}, \mathbb{P})$ a *probability space*.

In otherwords, a probability space is just a measure space with total measure of 1. We say that an event $F \in \mathcal{F}$ happens *almost surely* if $\mathbb{P}(F) = 1$.

Definition 2.2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We call a measurable function $X : \Omega \to \mathbb{R}$ a *random variable*. Equivalently, X is called a random variable if for all $B \in \mathcal{B}(\mathbb{R})$, the Borel σ-algebra, we have

$$X^{-1}(B) \in \mathcal{F}.$$

We are often interested in $\mathbb{P}(X \in B)$, where B is a Borel set. Now the measure $B \mapsto \mathbb{P}(X^{-1}B)$ has name, it's called the *distribution* of X.

Definition 2.3. The measure $\mu : \mathcal{B}(\mathbb{R}) \to \mathbb{R}^+$ defined by

$$\mu(B) := \mathbb{P}(X^{-1}B) = \mathbb{P}(X \in B),$$

is called the *distribution* of X. We call

$$F_X(x) := \mathbb{P}(X \leq x) = \mu((-\infty, x])$$

the *distribution function* of X.

Definition 2.4. For X a random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we define the *expectation* as

$$\mathbb{E}[X] = \int_\Omega X \, \mathrm{d}\mathbb{P},$$

where the integral is the Lebesgue integral over Ω with respect to the measure $\mathbb{P}$.

Definition 2.5. For X a random variable on the space $(\Omega, \mathcal{F}, \mathbb{P})$, we define the *variance*,

$$\mathbb{V}\mathrm{ar}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 .$$

We will also be dealing with a notion of "independence" in this thesis, we see that in some way the outcome of one random variable can influence the outcome of another.

Definition 2.6. For any two random variables X and Y, we say that X and Y are independent if we have for any two Borel sets $B_1, B_2 \in \mathcal{B}(\mathbb{R})$,

$$\mathbb{P}(X \in B_1 \text{ and } Y \in B_2) = \mathbb{P}(X \in B_1)\mathbb{P}(X \in B_2).$$

Furthermore, we define the *covariance* as

$$\mathbb{C}\text{ov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y].$$

In a sense, the covariance measures "how" dependent X and Y are. If X and Y are independent, we have $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$, but be careful here: **zero covariance does not imply independence!**

However, what we will repeatedly do here is look at a sequence of random variables $(X_n)_{n\in\mathbb{N}}$. We write that $(X_n)_{n\in\mathbb{N}}$ is an *i.i.d.* sequence if the X_n's are all (I)ndependent and (I)dentically (D)istributed. The independence is defined pairwise, for all $i, j \in \mathbb{N}$ with $i \neq j$, X_i and X_j are independent. We want to consider a notion of "convergence" for this sequence. There are many forms of convergence in probability theory, but here we will just state *convergence in distribution*, also called *convergence in law*, as this is the main one we will be using in this project.

Definition 2.7. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables with distribution functions F_{X_n}. We say that X_n converges to X in distribution (or in law) if, for all $x \in \mathbb{R}$ such that F_X is continuous in x,

$$F_{X_n}(x) \to F_X(x) \quad \text{as } n \to \infty.$$

If this is the case, then we write

$$X_n \xrightarrow{d} X.$$

As it will turn out, we practically never use the above definition to prove convergence in law. The most common for proving that a sequence $X_n \xrightarrow{d} X$ is by using Levy's Continuity Theorem. We first define the characteristic function

Definition 2.8. Denote with ι the complex unit and let X be a random variable on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We define the *characteristic function* to be

$$\varphi_X(t) := \mathbb{E}\left[\exp(\iota t X)\right] = \int_\Omega \exp(\iota t X) \, \mathrm{d}\mathbb{P}, \quad t \in \mathbb{R}.$$

Now we can state Levy's Continuity Theorem. A more thorough explanation can be found in the lecture notes of my supervisor's course on Advanced Probability.

Theorem 2.1. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of real valued random variables. and φ_{X_n} their corresponding characteristic functions. Let φ_X be the characteristic function of a random variable X. Moreover, assume that*

$$\varphi_{X_n}(t) \to \varphi_X(t),$$

for all $t \in \mathbb{R}$. Then $X_n \xrightarrow{d} X$.

We immediately use this theorem to prove a theorem that will be important later on:

Theorem 2.2. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of Gaussian random variables with $\mathbb{E}[X_n] = m_n$ and $\mathbb{V}\text{ar}(X_n) = \sigma_n^2$, and $\sigma_n^2 \to \sigma^2$, $m_n \to m$. Then*

$$X_n \xrightarrow{d} X,$$

where $X \sim \mathcal{N}(m, \sigma^2)$.

Proof. From Levy's Continuity Theorem we have

$$X_n \xrightarrow{d} X \Leftrightarrow \varphi_{X_n}(t) \to \varphi_X(t),\ \forall t \in \mathbb{R}.$$

Here we write φ_{X_n} and φ_X for the characteristic functions of X_n and X. Now,

$$\varphi_{X_n}(t) = \exp\left(\iota t m_n - \frac{1}{2}\sigma_n^2 t^2\right).$$

Subsequently, since exp is continuous, we have

$$\begin{aligned}\lim_{n\to\infty} \varphi_{X_n}(t) = \lim_{n\to\infty} \exp\left(\iota t m_n - \frac{1}{2}\sigma_n^2 t^2\right) &= \exp\left(\lim_{n\to\infty} \iota t m_n - \frac{1}{2}\sigma_n^2 t^2\right) \\ &= \exp(\left(\iota t m - \frac{1}{2}\sigma_n^2 t^2\right) = \varphi_X(t).\end{aligned}$$

So the claim follows. □

In other words, to show convergence of a sequence of Gaussians, it is enough to consider only the first and second moment. Now we finally state the *Central Limit Theorem*, which forms the basis for most of Statistics and a large part of Probability.

Theorem 2.3. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables with $\mathbb{E}[X_1] = m$ and $\mathbb{V}\mathrm{ar}(X_1) = \sigma^2$. Then*

$$\frac{\sum_{i=1}^n (X_i - m)}{\sqrt{\sigma^2 n}} \xrightarrow{d} Z,$$

with $Z \sim \mathcal{N}(0, 1)$.

This theorem is very powerful, as it says that we can approximate a large sum of i.i.d. random variables of any distribution with a Gaussian random variable, after an appropriate scaling. In fact we can extend this result to more general random variables. We state the Lindeberg-Levy-Feller CLT:

Theorem 2.4. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables with $\mathbb{E}[X_k] = m_k$ and $\mathbb{V}\mathrm{ar}(X_k) = \sigma_k^2 > 0$. Set $s_n^2 = \sum_{k=1}^n \sigma_k^2$. If we have for all $\varepsilon > 0$ that*

$$\lim_{n\to\infty} \frac{1}{s_n^2} \sum_{k=1}^n \mathbb{E}((X_k - m_k)^2 1_{|X_k - m_k| > \varepsilon s_n}) = 0,$$

then

$$\frac{1}{s_n} \sum_{k=1}^n (X_k - m_k) \xrightarrow{d} Z,$$

with $Z \sim \mathcal{N}(0, 1)$.

2.2 Fourier analysis and analysis on the torus $\mathbb{T}^d$

We will do most of our analysis on the torus $\mathbb{T}^d$, which can be viewed as a generalization of a donut. In the case $d = 2$, $\mathbb{T}^2$ is exactly the shape of a regular donut as we, 3-dimensional humans, know it. In some sense, the torus is the perfect space to do analysis on. Because first of all, it is compact, so any smooth function will automatically be integrable. On the other hand, the torus is a group: we can never "escape" the torus just by walking around. This is different than looking at for example a compact interval $[a, b] \subset \mathbb{R}$, which is not a group. In this way it has all of the good features of spaces like $\mathbb{R}$ and $\mathbb{C}$, without the drawbacks of integrability, convergence that come with infinite spaces. We like to view the torus as $(\mathbb{R}/\mathbb{Z})^d$. The intuition behind this quotient group (consider for a moment $d = 1$) is the following: we put each $x \in \mathbb{R}$ in an equivalence class $[x] \in (\mathbb{R}/\mathbb{Z})$, such that $y, x \in [x]$ if and only if there exists an $n \in \mathbb{Z}$ such that $x - y = n$. In a sense now, we can say $[0,1]$" = "$(\mathbb{R}/\mathbb{Z})$, however there is one thing different. If we take any $x \in \mathbb{R}/\mathbb{Z}$, then $x + 1 \equiv x \in \mathbb{R}/\mathbb{Z}$. In this way, we walk out of our interval $[0, 1]$ on one inside, and again walk in on the other side to end up on the same spot after a distance of 1. A visualisation is given in Figure (2).

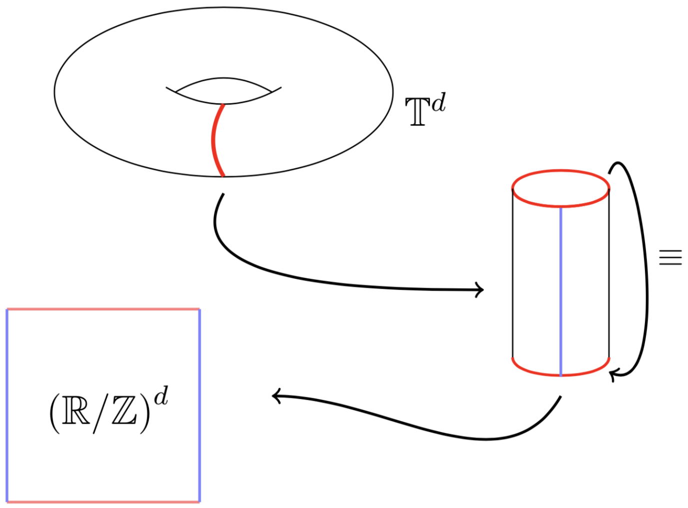

Figure 2: Visualisation of the identification $\mathbb{T}^d \simeq (\mathbb{R}/\mathbb{Z})^d$. First we slice the torus through the red line, and end up with a cylinder. We subsequently cut the cylinder along the blue line and roll it out to obtain a square.

Whenever necessary, we will identify $\mathbb{T}^d$ with $[-\frac{1}{2}, \frac{1}{2}]^d \subset \mathbb{R}^d$, under the equivalence relation as above ("glueing" the ends to each other). Both definitions are equivalent. We define the space $C^\infty(\mathbb{T}^d)$ to be the space of all infinitely differentiable functions on $\mathbb{T}^d$. It is now important to remark that this space is **not** equivalent with the space $C^\infty([-\frac{1}{2}, \frac{1}{2}]^d)$ or $C^\infty([0, 1]^d)$. For the former we require $f^{(n)}(0) = f^{(n)}(1)$, or equivalently $f^{(n)}\left(-\frac{1}{2}\right) = f^{(n)}\left(\frac{1}{2}\right)$, for all $n \in \mathbb{N}_0$, while in the latter case this is not necessary.

Because $\mathbb{T}^d$ is a group, it admits a Fourier transform. Denote with ι the complex unit. We define for any $u \in L^1(\mathbb{T}^d)$,

$$\widehat{u}(\xi) = \int_{\mathbb{T}^d} u(z) \exp(2\pi\iota\xi \cdot z) \, \mathrm{d}z, \quad \xi \in \mathbb{Z}^d.$$

The *Pontryagin dual group* of $\mathbb{T}^d$ is identified with $\mathbb{Z}^d$, what this means is that the Fourier transform of u is a function $\widehat{u} : \mathbb{Z}^d \to \mathbb{C}$. This is the reason why the inverse Fourier transform of

$\mathbb{T}^d$ is a series. For $u \in L^2(\mathbb{T}^d)$, we have

$$u(z) = \sum_{\xi \in \mathbb{Z}^d} \widehat{u}(\xi) \exp(-2\pi\iota\xi \cdot z), \quad z \in \mathbb{T}^d$$

almost everywhere. For more information regarding Pontryagin duality, see [13]. Recall that in the case where u is defined on the real line, we have that both the Fourier transform, and the inverse Fourier transform are integrals over the real line. This is because the Pontryagin dual group of $\mathbb{R}$ is again $\mathbb{R}$. Keeping this idea of Pontryagin duality in the back of our minds, we take a look at another group: $\mathbb{Z}_n^d = (\mathbb{Z}/n\mathbb{Z})^d$, the additive group of integers modulo n. Whenever necessary, we view $\mathbb{Z}_n^d$ as $[-\frac{n}{2}, \frac{n}{2}]^d \cap \mathbb{Z}^d$. Note that $\mathbb{Z}_n^d$ in some sense resembles our torus, as $n \equiv 0$, we again "walk out" on one side, only to enter $\mathbb{Z}_n^d$ again on the other side. This time, $\mathbb{Z}_n^d$ has side length n, whereas the torus had side length 1.

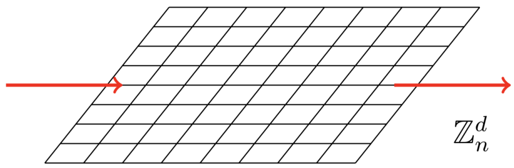

Figure 3: $\mathbb{Z}_n^d$ is just a grid with side length n. If we walk out on one side, we enter the grid again on the opposite side.

Now, as $\mathbb{Z}_n^d$ is a group we can again define a Fourier transform on it. We set for $u \in \ell^1(\mathbb{Z}_n^d)$,

$$\widehat{u}(\xi) = \frac{1}{n^d} \sum_{z \in \mathbb{Z}_n^d} u(z) \exp\left(2\pi\iota\xi \cdot \frac{z}{n}\right).$$

The Pontryagin dual group of $\mathbb{Z}_n^d$ is again identified with $\mathbb{Z}_n^d$, so the Fourier transform is again a function on $\mathbb{Z}_n^d$. We have as inverse transform ($z \in \mathbb{Z}_n^d$):

$$u(z) = \sum_{\xi \in \mathbb{Z}_n^d} \widehat{u}(\xi) \exp\left(2\pi\iota\xi \cdot \frac{z}{n}\right).$$

We can now watch the beauty of this slowly unfold. We set $\mathbb{T}_n^d := \frac{1}{n}\mathbb{Z}_n^d$ for the discretization of $\mathbb{T}^d$ with n^d gridpoints. As we have already seen, $\mathbb{Z}_n^d$ is a torus-like grid of side length n, so $\mathbb{T}_n^d$ will have side length 1. What we can do next is, using the Pontryagin duality of $\mathbb{Z}_n^d$ with itself, "prove" the duality of $\mathbb{T}^d$ with $\mathbb{Z}^d$. To this end, for any C^1-function $u : \mathbb{T}^d \to \mathbb{R}$ we denote $u_n : \mathbb{Z}_n^d \to \mathbb{R}$ for the discretized version $u_n(\cdot) = u\left(\frac{\cdot}{n}\right)$. As u_n is defined on $\mathbb{Z}_n^d$, we can take the Fourier transform to find

$$\widehat{u_n}(\xi) = \frac{1}{n^d} \sum_{z \in \mathbb{Z}_n^d} u_n(z) \exp\left(2\pi\iota\xi \cdot \frac{z}{n}\right) = \frac{1}{n^d} \sum_{z \in \mathbb{Z}_n^d} u\left(\frac{z}{n}\right) \exp\left(2\pi\iota\xi \cdot \frac{z}{n}\right).$$

Here the Fourier transform is defined on $\mathbb{Z}_n^d = [-\frac{n}{2}, \frac{n}{2}] \cap \mathbb{Z}^d$, i.e. $\xi \in [-\frac{n}{2}, \frac{n}{2}] \cap \mathbb{Z}^d$. If we now let $n \to \infty$, the following happens with the Fourier transform,

$$\lim_{n\to\infty} \widehat{u_n}(\xi) = \lim_{n\to\infty} \sum_{z \in \mathbb{Z}_n^d} \frac{1}{n^d} u_n(z) \exp\left(2\pi\iota\xi \cdot \frac{z}{n}\right) = \int_{\mathbb{T}^d} u(z) \exp(2\pi\iota z \cdot \xi) \, \mathrm{d}z.$$

This is because the summation approximates a Riemann integral. But note that this is exactly the Fourier transform as defined on $\mathbb{T}^d$! On the other hand, we see that ξ takes values in $[-\frac{n}{2}, \frac{n}{2}] \cap \mathbb{Z}^d$, so if $n \to \infty$, we obtain that ξ can actually take values in the whole of $\mathbb{Z}^d$. We've come full circle now, and in this way we see that $\mathbb{T}^d$ is the Pontryagin dual of $\mathbb{Z}^d$.

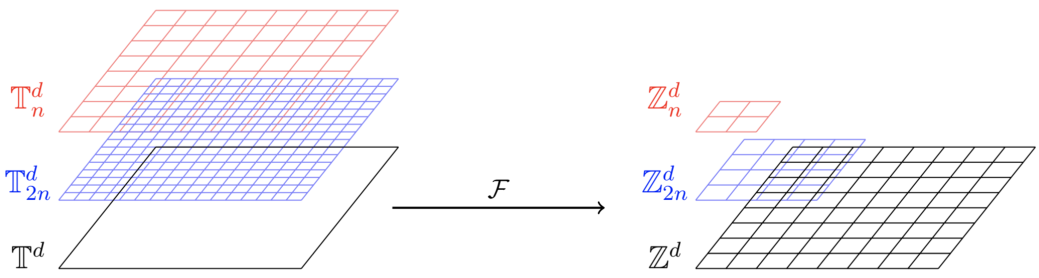

Figure 4: The Pontryagin duality between $\mathbb{Z}^d_n$ with itself, and $\mathbb{T}^d$ and $\mathbb{Z}^d$ visualized. We can see that on the left side, $\mathbb{T}^d_n (= \frac{1}{n}\mathbb{Z}^d_n)$ grows inwards to approximate the torus. On the right side, $\mathbb{Z}^d_n$ expands outwards to form $\mathbb{Z}^d$.

In this project, we will frequently approximate functions u on $\mathbb{T}^d$ with the discretized version u_n, and use the above to find that

$$\lim_{n\to\infty} \widehat{u_n}(\xi) = \widehat{u}(\xi), \quad \xi \in \mathbb{Z}^d.$$

At last, we state and prove the Plancherel theorem for $\mathbb{Z}^d_n$.

Theorem 2.5. *Let $f, g \in L^2(\mathbb{Z}^d_n)$ and $\widehat{f}, \widehat{g}$ be their Fourier transforms. Then*

$$\sum_{\xi \in \mathbb{Z}^d_n} \widehat{f}(\xi)\overline{\widehat{g}(\xi)} = n^{-d} \sum_{z \in \mathbb{Z}^d_n} f(z)\overline{g(z)}.$$

Proof. The proof is a straightforward computation. We have

$$\begin{aligned}
\sum_{\xi \in \mathbb{Z}^d_n} \widehat{f}(\xi)\overline{\widehat{g}(\xi)} &= \sum_{\xi \in \mathbb{Z}^d_n} \left(n^{-d} \sum_{z \in \mathbb{Z}^d_n} f(z) \exp\left(2\pi\iota\xi \cdot \frac{z}{n}\right)\right) \left(n^{-d} \sum_{z' \in \mathbb{Z}^d_n} \overline{g(z')} \exp\left(-2\pi\iota\xi \cdot \frac{z'}{n}\right)\right) \\
&= n^{-d} \sum_{z, z' \in \mathbb{Z}^d_n} f(z)\overline{g(z')} \left(n^{-d} \sum_{\xi \in \mathbb{Z}^d_n} \exp\left(2\pi\iota\xi \cdot \frac{z - z'}{n}\right)\right) \\
&= n^{-d} \sum_{z, z' \in \mathbb{Z}^d_n} f(z)\overline{g(z')} 1_{\{z = z'\}} = n^{-d} \sum_{z \in \mathbb{Z}^d_n} f(z)\overline{g(z)}.
\end{aligned}$$

□

2.3 Abstract Wiener Spaces

This section aims to outline the basics of infinite-dimensional probability theory, for a more in-depth look we refer the reader to [7] or [11]. In this project we will be dealing with a random variable Ξ (our *Gaussian field*) that is infinite-dimensional. We would like to view Ξ as a function $\Xi : \mathbb{T}^d \to \mathbb{R}$, however this is not possible. For our definition of the Gaussian field Ξ to make sense, we need to be able to define a Gaussian random variable on some infinite-dimensional space. Let us start with a finite-dimensional example. For $\mathbb{R}^3$ we can define the following random vector:

$$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix},$$

where the $X_i \sim \mathcal{N}(0,1)$ and are independent for $i = 1,2,3$. For any other vector $\mathbf{a} \in \mathbb{R}^3$ with $\sum_{i=1}^3 a_i = 0$, a simple calculation shows that

$$\mathbb{E}[\langle \mathbf{X}, \mathbf{a} \rangle] = 0,$$

and

$$\mathbb{E}[\langle \mathbf{X}, \mathbf{a} \rangle^2] = \sum_{i=1}^{3} a_i^2 = |\mathbf{a}|^2.$$

The characteristic function of $\mathbf{X}$ in this case is equal to (for $\mathbf{t} \in \mathbb{R}^3$):

$$\mathbb{E}[\exp(\iota \langle \mathbf{X}, \mathbf{t} \rangle)] = \exp\left(-\frac{1}{2}|\mathbf{t}|^2\right).$$

The question now becomes whether, using this characteristic function, we can extend this result to infinite dimensional Hilbert spaces in order to define a Gaussian random variable indexed on $\mathbb{T}^d$, but as it turns out: we can't. Corollary 2.3.2 in [7] tells us for a Hilbert space H with $\dim H = \infty$, that $\varphi(x) = \exp\left(-\frac{1}{2}\|x\|^2\right)$ can NOT be the Fourier transform of any countably additive measure. A different approach then to constructing a Gaussian measure on our infinite dimensional Hilbert space H leads to the *Abstract Wiener Space (AWS)*, the idea is to define the Gaussian measure on a larger Banach space B containing H, but with a different norm $\|\cdot\|_B$ than the norm $\|\cdot\|_H$ on H. In the following part we will use the same construction of our AWS as in [8].

Definition 2.9. An *Abstract Wiener space* is a triple (H, B, μ), where

1. H is a Hilbert space with inner product $(\cdot,\cdot)_H$ and norm $\|\cdot\|_H$.
2. B is the Banach space completion of H under the new *measurable* norm $\|\cdot\|_B$, with the Borel σ-algebra induced by $\|\cdot\|_B$.
3. μ is the unique Borel probability measure on $(B, \mathcal{B})$ such that for all $\phi \in B^*$, we have $\mu \circ \phi^{-1} = \mathcal{N}(0, \|\tilde{\phi}\|_H^2)$, where $\tilde{\phi}$ is the unique element of H such that $\phi(h) = (\tilde{\phi}, h)_H$ for all $h \in H$.

Note that the measure μ here is exactly what we want. Since $H \hookrightarrow B$ densely, we have $B^* \subset H^*$, so note that the functional here is well defined for all $\phi \in B^*$, since we can always find such a $\tilde{\phi} \in H$. The definition of a measurable norm is quite technical, and we will not state it here. Intuitively, a measurable norm on B is a norm such that we can capture most mass of B in finite dimensions. We refer the enthousiastic reader to [8]. We do however, state the following lemma:

Lemma 2.6. *Let H be a Hilbert space with norm $\|\cdot\|_H$ and $\{f_j : j \in \mathbb{N}\}$ an orthonormal basis of H. If $T : H \to H$ is a Hilbert-Schmidt operator, meaning*

$$\sum_{j=1}^{\infty} \|Tf_j\|_H < \infty,$$

then $\|T \cdot\|_H$ is a measurable norm on B.

Observe that this norm is well defined since T is densely defined on B. Intuitively speaking, the Hilbert-Schmidt operator squeezes our orthonormal basis down to something that is summable, this again ties in with the concept of a measurable norm, which tries to capture most mass in finite dimensions.
Recall that we originally wanted to define a Gaussian measure on $H^{-1}(\mathbb{T}^d)$. We are going to do this by finding a suitable Hilbert-Schmidt operator and using the construction above, in exactly the same way as in [5]. To this end, let $\sim$ be the equivalence relation on $C^\infty(\mathbb{T}^d)$ defined by $f \sim g$ if and only if f and g differ by a constant. We define the inner product

$$(f, g)_a = \sum_{\nu \in \mathbb{Z}^d \setminus \{0\}} \|\nu\|^{4a} \widehat{f}(\nu)\widehat{g}(\nu) = ((-\Delta)^a f, (-\Delta)^a g)_{L^2(\mathbb{T}^d)},$$

and write $H^a(\mathbb{T}^d)$ for the completion of $C^\infty(\mathbb{T}^d)/\sim$ with respect to this inner product. An important remark to make here is that the space $H^a(\mathbb{T}^d)$ does *not* agree with the usual definition of the Sobolev space $H^a(U)$ for $U \subset \mathbb{R}^d$ and $a \in \mathbb{N}$. Next, define the Hilbert space

$$\mathcal{H}_a := \{u \in L^2(\mathbb{T}^d) : (-\Delta)^a u \in L^2(\mathbb{T}^d)\}/\sim,$$

equipped with the norm

$$\|u\|^2_{\mathcal{H}_a} := ((-\Delta)^a u, (-\Delta)^a u)_{L^2(\mathbb{T}^d)}.$$

We want $\mathcal{H}_a$ to be our Banach space completion of $H^a(\mathbb{T}^d)$, to this end, note that $\{(-\Delta)^{-a}\mathbf{e}_\nu\}_{\nu \neq 0}$ (we can take $\mathbf{e}_0 \equiv 0$, since we look at the space over the equivalence relation $\sim$, so set the first coefficient to 0) forms an orthonormal basis for $H^a(\mathbb{T}^d)$. Indeed,

$$\begin{aligned}
((-\Delta)^{-a}\mathbf{e}_\nu(\vartheta), (-\Delta)^{-a}\mathbf{e}_\kappa(\vartheta))_{H^a(\mathbb{T}^d)} &= \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \|k\|^{4a} \widehat{(-\Delta)^{-a}\mathbf{e}_\nu}(k) \widehat{(-\Delta)^{-a}\mathbf{e}_\kappa}(k) \\
&= \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \|k\|^{4a} \left(\|k\|^{-4a} \widehat{\mathbf{e}_\nu}(k) \widehat{\mathbf{e}_\kappa}(k)\right) \\
&= \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \|k\|^{4a} \|k\|^{-4a} \delta_{\nu=k}\delta_{\kappa=k} = \delta_{\nu=\kappa}.
\end{aligned}$$

Since $H^a(\mathbb{T}^d)$ is a subspace of $L^2(\mathbb{T}^d)$ and $(-\Delta)^{-a}\mathbf{e}_\nu$ is just a rescaling of the Fourier coefficients, this is also a basis for $H^a(\mathbb{T}^d)$. We will now see that $T := (-\Delta)^{b-a}$ is a Hilbert-Schmidt operator on $H^a(\mathbb{T}^d)$, whenever $b < a - \frac{d}{4}$. Subsequently we find, in the same way as in Section 6.2 of [8], the following for all $\nu \neq 0$:

$$\begin{aligned}
\|T(-\Delta)^{-a}\mathbf{e}_\nu\|_{H^a(\mathbb{T}^d)} &= ((-\Delta)^a(-\Delta)^{b-a}(-\Delta)^{-a}\mathbf{e}_\nu, (-\Delta)^a(-\Delta)^{b-a}(-\Delta)^{-a}\mathbf{e}_\nu)_{L^2(\mathbb{T}^d)} \\
&= ((-\Delta)^{b-a}\mathbf{e}_\nu, (-\Delta)^{b-a}\mathbf{e}_\nu)_{L^2(\mathbb{T}^d)} \\
&= \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \|k\|^{4(b-a)} \delta_{\nu=k} = \|\nu\|^{4(b-a)} \leq \|\nu\|^{-(d+\delta)},
\end{aligned}$$

for some $\delta > 0$. Now by the Euler-Maclaurin formulas,

$$\begin{aligned}\sum_{\nu \in \mathbb{Z}^d \setminus \{0\}} \|T(-\Delta)^{-a} \mathbf{e}_\nu\|_{H^a(\mathbb{T}^d)} \leq \sum_{\nu \in \mathbb{Z}^d \setminus \{0\}} \|\nu\|^{-(d+\delta)} \quad &\sim \quad \int_1^\infty \rho^{d-1} \rho^{-(d+\delta)} \, \mathrm{d}\rho \\ &= \quad \int_1^\infty \rho^{-(1+\delta)} \, \mathrm{d}\rho < \infty.\end{aligned}$$

So in fact T is a Hilbert-Schmidt operator and $\|T \cdot \|_{H^a(\mathbb{T}^d)}$ is a measurable norm on our Banach space $\mathcal{H}_a$. We now set $a := -1$, and $-\varepsilon := b < 0$ such that $\varepsilon > 1 + \frac{d}{4}$. In conclusion, we now have a unique Gaussian measure $\mu_{-\varepsilon}$ on our space $\mathcal{H}_{-\varepsilon}$, and $(H^{-1}(\mathbb{T}^d), \mathcal{H}_{-\varepsilon}, \mu_{-\varepsilon})$ is our AWS, where $\mu_{-\varepsilon}$ has characteristic function

$$\Phi(u) := \exp\left(-\frac{\|u\|_{-1}^2}{2}\right).$$

Now the operator Δ^{-2} is our infinite-dimensional analog of the covariance matrix.

3 Divisible Sandpiles

3.1 Introduction

Before we dive into the scaling limit, we first build up some theory around divisible sandpile models. The presentation here is based on [1]. Consider a connected, undirected graph $G = (V, E)$. Write $x \sim y$ if $(x, y) \in E$ and $\deg(x) = \#\{y \in V : y \sim x\}$. We want to start with some initial configuration $s : V \to \mathbb{R}$, think of this as a mass assigned to each vertex. If $s(x) > 1$, then the site x is called *unstable*, otherwise x is called *stable*. Whenever x is an unstable site, it *topples*, meaning that it distributes its excess fat evenly among its neighbors, while keeping mass 1 to itself. At each discrete time step, all unstable sites topple at the same time. Consider the following example of a graph with 5 vertices, the number in each vertex denotes the mass:

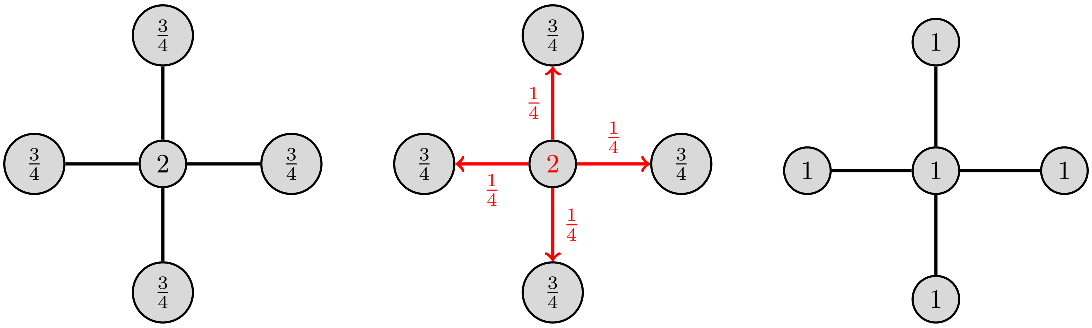

Figure 5: Left: the initial configuration, Middle: the unstable site (red) topples, Right: stabilized.

Once every site has mass ≤ 1, we call the sandpile *stabilized*. We see in the example above, that each vertex will have total mass of 1, and thus this sandpile has stabilized after one discrete time step. A natural question that arises is whether the sandpile stabilizes or not. In our case it is enough to consider only finite, connected graphs, and as it turns out, we are in luck: Lemma 7.1 in [1] states that for an initial configuration $s : V \to \mathbb{R}$ with $\sum_{x \in V} s(x) = |V|$, the sandpile stabilizes to the all-one configuration. We will come back to this later. Now consider for any site x the amount of mass emitted to any neighbor, up to and including the discrete time step t, denoted by $e^{(t)}(x)$. We see that $e^{(t)}$ is an increasing sequence (a site can not "un-emit" mass), and if the sandpile stabilizes, converges to some function e,

$$e^{(t)} \uparrow e < \infty.$$

We call this $e : V \to \mathbb{R}$ the *odometer* function. Note that if s does not stabilize, the odometer blows up. Consider the following sandpile:

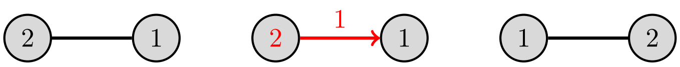

Figure 6: The odometer function $e(x)$ explodes.

In the above example, the t-th odometer satisifes

$$e^{(t)}(x) = \begin{cases} 1 + \lfloor \frac{t}{2} \rfloor, & \text{if } x \text{ is the left vertex} \\ \lfloor \frac{t}{2} \rfloor, & \text{if } x \text{ is the right vertex} \end{cases}.$$

In this case $e^{(t)}(x) \uparrow \infty$ for all $x \in V$, this is because V is finite and $\sum_{x \in V} s(x) > |V|$.

Given an initial configuration $s : V \to \mathbb{R}$, the sandpile evolves deterministically, so the odometer function e is dependent only on the initial configuration s. Before we make all of this rigorous, we need some definitions.

3.2 Theoretical Background

In the following section (based on Section 2 in [1]), $G = (V, E)$ is a finite, connected, undirected graph, and we write $\mathcal{X} = \mathbb{R}^V$ for the set of all divisible sandpile configurations on G. We define for $u : V \to \mathbb{R}$ the *graph Laplacian* to be

$$\Delta e(x) := \sum_{y \sim x} (e(y) - e(x)) .$$

What this intuitively does is taking the unweighted average of $e(x)$ with its neighbors, and this is no surprise: the continuous Laplace operator in fact does something similar (this can be seen by taking for example a finite difference approximation of continuous Δ). Now, as we have already seen before, in each timestep a site can "topple", meaning that it distributes its extra mass among his neighboring sites.

Definition 3.1. Let $T \subset [0, \infty)$ be a well-ordered set of toppling times such that $0 \in T$ and T is a closed subset of $[0, \infty)$. A *toppling procedure* is a function $T \times V \to [0, \infty)$ defined by

$$(t, x) \mapsto e^{(t)}(x),$$

such that for all $x \in V$

1. $e^{(0)}(x) = 0$.
2. $e^{(t_1)}(x) \leq e^{(t_2)}(x)$ for all $t_1 \leq t_2$.
3. If $t_n \to t$ then $e^{(t_n)}(x) \uparrow u_t(x)$.

A toppling procedure is essentially the amount of mass emitted from site x, up to and including time t. We can then quantify the amount of mass $s_t(x)$ for any site $x \in V$ and $t \in T$ by the following equation

$$s_t(x) = s(x) + \Delta e^{(t)}(x).$$

Here $\Delta e^{(t)}(x)$ acts as the net gain of site x in the time $[0, t] \subset T$, and $s \in \mathcal{X}$ is the initial mass. Now write $a^+ := \max(a, 0)$ and $t^- := \sup\{r \in T : r < t\}$.

Definition 3.2. We call a toppling procedure *legal* for initial configuration s if for all $x \in V$ and $t \in T \setminus \{0\}$,

$$e^{(t)}(x) - e^{(t^-)}(x) \leq \frac{(s_{t^-}(x) - 1)^+}{\deg(x)}.$$

This is saying a few things at once. Consider first the case $s_{t^-}(x) > 1$. Then the mass emitted at timestep t is less than the excess mass, divided by the degree of x. In other words: it has to distribute its mass equally over its neighbors. Second, if $s_{t^-}(x) \leq 1$, then $u_t(x) - e^{(t^-)}(x) \leq 0$, so it can not distribute any mass at all.

Definition 3.3. A toppling procedure e is called *finite* if for all $x \in V$ we have

$$e^{(\infty)}(x) := \lim_{t \to \sup T} e^{(t)}(x) < \infty.$$

Note that since $e^{(t)}(x)$ is non-decreasing, this limit exists in $[0, \infty]$. If $e^{(\infty)}(x) = \infty$, then we call the procedure *infinite*.

Definition 3.4. Let $s \in \mathcal{X}$. A toppling procedure e is called *stabilizing* if e is finite and $s_\infty \leq 1$ pointwise. We say that s *stabilizes* if there exists a stabilizing toppling procedure for s.

Now we are finally getting to the most important definition of this section.

Definition 3.5. Let $s \in \mathcal{X}$. The function $e^{(\infty)} : V \to [0, \infty]$ is called the *odometer* of s. If s stabilizes, then its *stabilization* is the configuration

$$s_\infty = s + \Delta e^{(\infty)}.$$

To sum up all of the above: for our initial configuration s, our toppling procedure $e^{(t)}(x)$ describes the mass distributed to any neighbor of x, in the time $[0, t]$. This toppling procedure can either be finite (whenever it stabilizes), or infinite (if it does not stabilize). The limit of the toppling procedure is called the *odometer*.

We may now prove a very useful lemma (Lemma 7.1 in [1]):

Lemma 3.1. *Let $G = (V, E)$ be a finite connected graph with $|V| = n$ and let $s : V \to \mathbb{R}$ be a divisible sandpile with $\sum_{x\in V} s(x) = n$. Then s stabilizes to the all 1 configuration and the odometer of s is the unique function e satisfying*

$$\begin{cases} s + \Delta e = 1 \\ \min u \quad = 0 \end{cases} .$$

Proof. We begin the proof by showing some fundamental properties of the discrete Laplacian. First note that all harmonic functions $f : V \to \mathbb{R}$ (i.e. $\Delta f = 0$) are constant. Indeed, assume that $\Delta f = 0$ and f is not constant on V. Since V is finite we can find $x \in V$ such that $f(x) > f(y)$ for all $y \in V$. For this x we have

$$\Delta f(x) = \sum_{z\sim x} (f(z) - f(x)) < 0,$$

a contradiction, so f must be constant on V. As we can view Δ as a linear operator acting on vectors in $\mathbb{R}^{|V|}$, we can say that Δ has a 1-dimensional kernel spanned by the constant function. In this way we see that Δ must have rank $n - 1$. Next, we have $\sum_{x\in V} \Delta f(x) = 0$. Indeed,

$$\begin{aligned} \sum_{x\in V} \Delta f(x) = \sum_{x\in V} \sum_{y\sim x} (f(y) - f(x)) &= \sum_{x\in V} \left[-\deg(x) f(x) + \sum_{y\sim x} f(y) \right] \\ &= -\sum_{x\in V} \deg(x) f(x) + \sum_{x\in V} \sum_{y\in V} 1_{y\sim x} f(y) \\ &= -\sum_{x\in V} \deg(x) f(x) + \sum_{y\in V} \sum_{x\in V} 1_{y\sim x} f(y) \\ &= -\sum_{x\in V} \deg(x) f(x) + \sum_{y\in V} \deg(y) f(y) = 0 \end{aligned}$$

In our case we want to solve $\Delta v = 1 - s$, and since $\sum_{x\in V} (1 - s(x)) = 0$, the problem has a solution. Let $w = v - \min v$, then $w \geq 0$ and $s + \Delta w = 1$, so s stabilizes. Now for any function u that satisfies $s + \Delta u \leq 1$,

$$\sum_{x\in V} (s + \Delta u)(x) = \sum_{x\in V} s(x) + \sum_{x\in V} \Delta u(x) = n.$$

We see that we need to have $s + \Delta e = 1$, so in fact the sandpile stabilizes to the all one configuration, and any two functions e that satisfy $s + \Delta e \leq 1$ differ by only a constant. Proposition 2.5 in [1] then tells us that the odometer is the smallest non-negative one, in other words, we have $\min e = 0$. □

We are interested in the case where the initial configuration is Gaussian. Let $(\sigma(x))_{x \in V}$ be a collection of i.i.d. standard normals, and set $s(x) = 1 + \sigma(x) - \frac{1}{|V|} \sum_{z \in V} \sigma(z)$. If our graph is a 50×50 grid (more precisely, $\mathbb{Z}_{50}^2$) our sandpile will look something like in Figure (7).

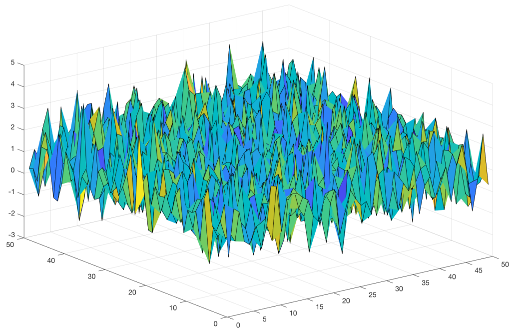

Figure 7: Sandpile with Gaussian heights.

Now if we let each site x topple as described above (by keeping mass 1 to itself, and distributing the rest to its neighbors), then we may a run simulation to see what our odometer will look like as $n \to \infty$ by taking sufficiently many timesteps. A result of such a simulation is shown in Figure (8). We stress that this is **not** the configuration s_t, but instead a surface plot of an approximation of the odometer $e^{(\infty)}$. Note that this is a much nicer, smoother surface than the initial configuration.

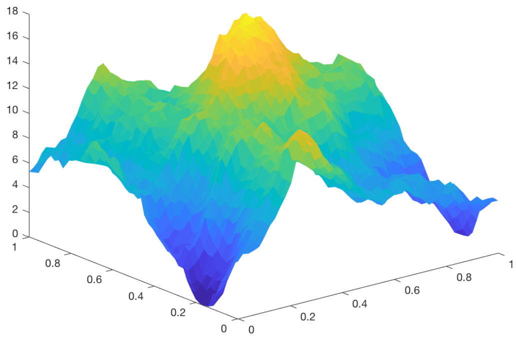

Figure 8: Odometer after 10000 timesteps.

Since the odometer e satisfies $s + \Delta e = 1$, and s is a random variable, then e is also a random variable. Proposition 1.3 in [1] tells us that e is in fact Gaussian as well. We will state both the proposition and its proof (as found in [1]) here, as the proof contains many valuable techniques we will also use in the next section.

Theorem 3.2. *(Proposition 1.3 in [1]) Let $G = (V, E)$ be a finite connected graph and $(\sigma(x))_{x\in V}$ a collection of i.i.d. standard normals. Consider the divisible sandpile*

$$s(x) = 1 + \sigma(x) - \frac{1}{|V|}\sum_{y\in V}\sigma(y).$$

Then s stabilizes to the all 1 configuration and the distribution of its odometer $e : V \to [0,\infty)$ is

$$(e(x))_{x\in V} \overset{d}{=} (\eta(x) - \min\eta)_{x\in V},$$

where the $\eta(x)$ are again Gaussian with mean zero and covariance

$$\mathbb{E}[\eta(x)\eta(y)] = \frac{1}{\deg(x)\deg(y)}\sum_{z\in V} g(z,x)g(z,y).$$

Here g is defined by $g(x,y) = \frac{1}{|V|}\sum_{z\in V} g^z(x,y)$ and $g^z(x,y)$ is the expected number of visits to y by a simple random walk started at x before hitting z.

Proof. Define for $x, y, z \in V$,

$$g^z(x,y) = \mathbb{E}\left[\text{number of visits to } y \text{ from a random walk starting in } x \text{ before hitting } z\right].$$

For $x, z \in V$ fixed we find that $\Delta\frac{g^z(x,y)}{\deg(y)} = \delta_z - \delta_x$ and $g^z(x,z) = 0$ (see for a more thorough discussion of this fact Appendix A.1, where we perform this calculation in the case $V = \mathbb{Z}_n^d$). We subsequently set $g(x,y) := g^z(x,y)$.

As we have already seen above in Lemma 3.1, the divisible sandpile stabilizes since $\sum_{x\in V}\sigma(x) = n$, and the odometer u satisfies $s + \Delta e = 1$, with $\min e = 0$. Now set

$$v^z(y) := \frac{1}{\deg(y)}\sum_{x\in V} g^z(x,y)(s(x)-1).$$

Since $\Delta\frac{g^z(x,\cdot)}{\deg(\cdot)} = \delta_z - \delta_x$, we have the following for $y \neq z$ (note $\delta_z(y) = 0$ in this case),

$$\begin{aligned}\Delta v^z(y) = \sum_{x\in V}\Delta\frac{g^z(x,y)}{\deg(y)}(s(x)-1) &= \sum_{x\in V}(\delta_z(y) - \delta_x(y))(s(x)-1)\\ &= 1 - s(y).\end{aligned}$$

On the other hand, if $y = z$, then

$$\Delta\frac{g^z(x,z)}{\deg(z)} = \delta_z(z) - \delta_x(z) = 1 - \delta_x(z).$$

So in fact

$$\begin{aligned}\Delta v^z(z) &= \sum_{x\in V}(1-\delta_x(z))(s(x)-1)\\ &= \sum_{x\in V}(s(x)-1) - \sum_{x\in V}\delta_x(z)(s(x)-1) = 1 - s(z).\end{aligned}$$

We find then that $\Delta(e - v^z) = \Delta e - \Delta v^z = 0$, so $e - v^z$ is constant. Now let $v = \frac{1}{n}\sum_{z\in V} v^z$. Because $e - v^z$ is constant, we also have that $e - v$ is constant, furthermore

$$\begin{aligned} v(y) = \frac{1}{n}\sum_{z\in V} v^z(y) &= \frac{1}{n}\sum_{z\in V}\frac{1}{\deg(y)}\sum_{x\in V} g^z(x,y)(s(x)-1) \\ &= \frac{1}{\deg(y)}\sum_{x\in V}\left(\frac{1}{n}\sum_{z\in V} g^z(x,y)\right)(s(x)-1) \\ &= \frac{1}{\deg(y)}\sum_{x\in V} g(x,y)(s(x)-1). \end{aligned}$$

We now explicitly calculate the covariance of v, note first that

$$\begin{aligned} \mathbb{E}[(s(z)-1)(s(w)-1)] &= \mathbb{E}\left[\left(\sigma(z) - \frac{1}{n}\sum_{y\in V}\sigma(y)\right)\left(\sigma(w) - \frac{1}{n}\sum_{y\in V}\sigma(y)\right)\right] \\ &= \mathbb{E}[\sigma(z)\sigma(w)] - \frac{1}{n}\mathbb{E}\left(\sigma(w)\sum_{y\in V}\sigma(y)\right) \\ &- \frac{1}{n}\mathbb{E}\left(\sigma(z)\sum_{y\in V}\sigma(y)\right) + \frac{1}{n^2}\mathbb{E}\left(\sum_{y\in V}\sigma(y)\right)^2 \\ &= 1_{z=w} - \frac{1}{n} - \frac{1}{n} + \frac{1}{n} = 1_{z=w} - \frac{1}{n}. \end{aligned}$$

Then,

$$\begin{aligned} \mathbb{E}[v(x)v(y)] &= \frac{1}{\deg(x)\deg(y)}\sum_{z,w\in V} g(z,x)g(w,y)\mathbb{E}[(s(z)-1)(s(w)-1)] \\ &= \frac{1}{\deg(x)\deg(y)}\left(\sum_{z\in V} g(z,x)g(z,y) - \frac{1}{n}\left(\sum_{z\in V} g(z,x)\right)\left(\sum_{w\in V} g(w,y)\right)\right). \end{aligned}$$

Define $K(y) := \sum_{w\in V}\frac{g(w,y)}{\deg(y)}$. Then $\Delta K = \sum_{z,w\in V}\frac{1}{n}(\delta_z - \delta_w) = 0$, so K is constant. The second term on the right then equals $\frac{K^2}{n}$, so define $Y \sim \mathcal{N}\left(0, \frac{K^2}{n}\right)$, independent of v. Now if η is as given in Theorem 3.2, then

$$\eta \overset{d}{=} v + C.$$

Indeed, since v and C are both centered Gaussians,

$$\mathbb{E}[\eta(x)] = \mathbb{E}[v(x) + C] = 0.$$

Furthermore,

$$\begin{aligned} \mathbb{E}[(v(x)+C)(v(y)+C)] &= \mathbb{E}[v(x)v(y) + Cv(x) + Cv(y) + C^2] \\ &= \mathbb{E}[v(x)v(y)] + \mathbb{E}[C^2] = \frac{1}{\deg(x)\deg(y)}\sum_{z\in V} g(z,x)g(z,y) = \mathbb{E}[\eta(x)\eta(y)]. \end{aligned}$$

So the result is indeed consistent. Now, $e - v$ is constant and $\min e = 0$, so we need to have

$$e = v - \min v \overset{d}{=} \eta - \min \eta.$$

□

3.3 Sandpiles with correlated initial distribution

In this section, we will use the same techniques as in the previous section and in [1] to derive similar results for the odometer under a correlated initial distribution. Recall that in last section we were working with the sandpile given by

$$s(x) = \sigma(x) + 1 - \frac{1}{|V|}\sum_{z\in V}\sigma(z),$$

where $(\sigma(z))_{z\in V}$ was a collection of **i.i.d.** standard normal random variables. The question now is what happens if we correlate $(\sigma(z))_{z\in V}$ according to a covariance function,

$$\mathbb{E}[\sigma(x)\sigma(y)] = K(x,y).$$

Here we will only consider the case where $V = \mathbb{Z}_n^d := [-\frac{n}{2}, \frac{n}{2}] \cap \mathbb{Z}^d$, because this is our main interest in the next section where we calculate the scaling limit. Note that we have $|\mathbb{Z}_n^d| = n^d$. In this section we will state and prove the correlated analog to Proposition 1.3 in [1], we only consider the case where $K(x,y) = \mathcal{K}(x-y)$. Covariance functions of this form are called *stationary covariance functions.*

Theorem 3.3. *Let $(\sigma(x))_{x\in\mathbb{Z}_n^d}$ be a collection of centered Gaussian random variables with covariance $\mathbb{E}[\sigma(x)\sigma(y)] = \mathcal{K}(x-y)$ and consider the divisible sandpile on $\mathbb{Z}_n^d$ given by*

$$s(x) = 1 + \sigma(x) - \frac{1}{n^d}\sum_{z\in\mathbb{Z}_n^d}\sigma(z)$$

for each $z \in \mathbb{Z}_n^d$. Then the sandpile stabilizes to the all one configuration and the distribution of the odometer $e(x)$ is given by

$$e(x) \overset{d}{=} \eta(x) - \min_{z\in\mathbb{Z}_n^d}\eta(z).$$

Here $(\eta(x))_{x\in\mathbb{Z}_n^d}$ is a collection of centered Gaussian random variables with covariance

$$\mathbb{E}[\eta(x)\eta(y)] = \frac{1}{(2d)^2}\sum_{z,z'\in\mathbb{Z}_n^d}\mathcal{K}(z-z')g(z,x)g(z',y).$$

Proof. The proof follows the same strategy as the proof of Proposition 1.3 in [1], and in fact the first part of the proof is exactly the same. A simple calculation shows us that

$$\sum_{z\in\mathbb{Z}_n^d} s(z) = n^d,$$

so by Lemma 7.1 in [1] the sandpile stabilizes and the odometer e satisfies $\min e = 0$ and

$$\Delta e = 1 - s.$$

Now, since $\Delta\frac{g^z(x,y)}{2d} = \delta_z - \delta_x$, we get that

$$v^z(y) := \frac{1}{2d}\sum_{x\in\mathbb{Z}_n^d} g^z(x,y)(s(x)-1))$$

satisfies the equation $\Delta v^z(y) = 1 - s(y)$ for all $y \in \mathbb{Z}_n^d$. But then $\Delta(e - v^z) = 0$ for all $z \in \mathbb{Z}_n^d$, and thus it must hold that $e - v^z$ is constant on $\mathbb{Z}_n^d$. Setting $v = \frac{1}{n^d}\sum_{z\in\mathbb{Z}_n^d} v^z =$

$\frac{1}{2d}\sum_{x\in\mathbb{Z}_n^d} g(x,y)(s(x)-1)$, we can see in a similar way that $e-v$ is constant. So $e=v+c$ for some constant $c\in\mathbb{R}$, but because $\min u=0$, it must hold that

$$u = v - \min v.$$

Now, since each $v(x)$ is a linear combination of (possibly) correlated Gaussian random variables, v is again Gaussian with covariance

$$\mathbb{E}[v(x)v(y)] = \frac{1}{(2d)^2}\sum_{z,z'\in\mathbb{Z}_n^d} g(z,x)g(z',y)\mathbb{E}[(s(z)-1)(s(z')-1)]. \tag{1}$$

We first observe that

$$\sum_{w\in\mathbb{Z}_n^d}\mathcal{K}(z-w) = \sum_{w'\in\{z-w|w\in\mathbb{Z}_n^d\}}\mathcal{K}(w') = \sum_{w\in\mathbb{Z}_n^d}\mathcal{K}(w),$$

so $\sum_{w\in\mathbb{Z}_n^d}\mathcal{K}(z-w)$ does not depend on z anymore, so say $\sum_{w\in\mathbb{Z}_n^d}\mathcal{K}(z-w)=C$. The expectation in the summation can be calculated in the following way:

$$\begin{aligned}
\mathbb{E}[(s(z)-1)(s(z')-1)] &= \mathbb{E}\left(\sigma(z)-\frac{1}{n^d}\sum_{w\in\mathbb{Z}_n^d}\sigma(w)\right)\left(\sigma(z')-\frac{1}{n^d}\sum_{w\in\mathbb{Z}_n^d}\sigma(w)\right)\\
&= \mathbb{E}[\sigma(z)\sigma(z')] - \mathbb{E}\left[\sigma(z')\frac{1}{n^d}\sum_{w\in\mathbb{Z}_n^d}\sigma(w)\right]\\
&- \mathbb{E}\left[\sigma(z)\frac{1}{n^d}\sum_{w\in\mathbb{Z}_n^d}\sigma(w)\right] + \frac{1}{n^{2d}}\mathbb{E}\left[\sum_{w,w'\in\mathbb{Z}_n^d}\sigma(w)\sigma(w')\right]\\
&= \mathcal{K}(z-z') - \frac{1}{n^d}\sum_{w\in\mathbb{Z}_n^d}[\mathcal{K}(z'-w)+\mathcal{K}(z-w)] + \frac{1}{n^{2d}}\sum_{w,w'\in\mathbb{Z}_n^d}K(w-w')\\
&= \mathcal{K}(z-z') - \frac{2C}{n^d} + \frac{n^dC}{n^{2d}} = \mathcal{K}(z-z') - \frac{C}{n^d}.
\end{aligned}$$

If we now plug this into 1, we obtain

$$\begin{aligned}
\mathbb{E}[v(x)v(y)] &= \frac{1}{(2d)^2}\sum_{z,z'\in\mathbb{Z}_n^d}\mathcal{K}(z-z')g(z,x)g(z',y) - \frac{C}{n^d(2d)^2}\left(\sum_{z,z'\in\mathbb{Z}_n^d} g(z,x)g(z',y)\right)\\
&= \frac{1}{(2d)^2}\sum_{z,z'\in\mathbb{Z}_n^d}\mathcal{K}(z-z')g(z,x)g(z',y) - \frac{C}{n^d(2d)^2}\left(\sum_{z\in\mathbb{Z}_n^d} g(z,x)\right)^2.
\end{aligned}$$

Here we used the fact that $\sum_{z\in\mathbb{Z}_n^d} g(z,x)$ does not depend on x anymore. Call $R=\frac{C}{n^d(2d)^2}\left(\sum_{z\in\mathbb{Z}_n^d} g(z,x)\right)^2$, and define $Y\sim\mathcal{N}(0,R)$ independent of v. Then

$$(v+Y)_{x\in\mathbb{Z}_n^d} \overset{d}{=} (\eta(x))_{x\in\mathbb{Z}_n^d}.$$

Where $(\eta(x))_{x\in\mathbb{Z}_n^d}$ is a collection of centered Gaussians with

$$\mathbb{E}[\eta(x)\eta(y)] = \frac{1}{(2d)^2}\sum_{z,z'\in\mathbb{Z}_n^d}\mathcal{K}(z-z')g(z,x)g(z',y),$$

Now since $e - v$ is constant and $\min e = 0$, we find

$$e \overset{d}{=} \eta - \min_{z \in \mathbb{Z}_n^d} \eta(z).$$

□

3.4 Brief summary

In this section, we have first defined our *divisible sandpile*. On a finite, connected graph $G = (V, E)$ we defined $s : V \to \mathbb{R}$, where for each discrete time step, a site $x \in V$ *topples* if $s(x) > 1$. If x topples, it distributes mass $s(x) - 1$ equally to its neighbors while keeping mass 1 to itself. The sandpile *stabilizes* if each site x has mass $s(x) \leq 1$.

Consider $e^{(t)} : V \to \mathbb{R}$ to be the mass emitted by a site up to and including discrete time-step t. If the sandpile stabilizes, we call $e = \lim_{t\to\infty} e^{(t)}$ the *odometer* of the sandpile. We then saw (Lemma 3.1, Lemma 7.1 in [1]) that if $\sum_{x \in V} s(x) = |V|$, then the sandpile stabilizes and the odometer e satisfies

$$s + \Delta e = 1,$$

with $\min e = 0$. As the sandpile evolves deterministically, our main interest was the probabilistic distribution of the odometer e when the sandpile is defined by

$$s(x) = \sigma(x) + 1 - \frac{1}{|V|} \sum_{z \in V} \sigma(z),$$

where $(\sigma(z))_{z \in V}$ is a collection of i.i.d. standard normals. Now Theorem 3.2 (Proposition 1.3 in [1]) tells us that e is again Gaussian,

$$e \overset{d}{=} \eta - \min_{x \in V} \eta_x,$$

where $(\eta_x)_{x \in V}$ is a collection of centered Gaussians with covariance

$$\mathbb{E}[\eta(x)\eta(y) = \frac{1}{\deg(x)\deg(y)} \sum_{z \in V} g(z, x) g(z, y).$$

The question we have attempted to answer in this section is what happens when we take the $(\sigma(z))_{z \in V}$ to be correlated according to some correlation function $\mathcal{K}$,

$$\mathbb{E}[\sigma(x)\sigma(y)] = \mathcal{K}(x - y).$$

We have taken $V = \mathbb{Z}_n^d$, and using the same proof techniques as [1], our key result is Theorem 3.3, which tells us that for $\mathcal{K}$ a stationary covariance function, our odometer is again distributed as

$$e = \eta - \min_{x \in \mathbb{Z}_n^d} \eta_x,$$

with this time

$$\mathbb{E}[\eta(x)\eta(y)] = \frac{1}{(2d)^2} \sum_{z, z' \in \mathbb{Z}_n^d} \mathcal{K}(z - z') g(z, x) g(z', y).$$

4 Stabilization speed for the divisible sandpile

4.1 Introduction

We consider again the divisible sandpile on $\mathbb{Z}_n^d$ for $n \geq 3$ with initial configuration given by

$$s(x) = 1 + \sigma(x) - \frac{1}{n^d} \sum_{z \in \mathbb{Z}_n^d} \sigma(z),$$

where $(\sigma(z))_{z \in \mathbb{Z}_n^d}$ is a collection of i.i.d. $\mathcal{N}(0,1)$ random variables. We write $s_t(x) = s(x) + \Delta e^{(t)}(x)$ for the mass present at site $x \in \mathbb{Z}_n^d$ at discrete timestep $t \geq 1$ and we set $s_0(x) := s(x)$. Now one might think that after enough timesteps, the divisible sandpile stabilizes to the all-one configuration. At the end of the day, there is only a finite amount of mass we are shuffling around on a finite grid, so as Jeremy Clarkson would say: "how hard can it be?!" As it turns out, very hard. In this section we will show that this intuition is wrong and the divisible sandpile does NOT stabilize in finite time, and we have the relation

$$s + \Delta e = 1,$$

only in the limit. We first consider a simulation of the divisible sandpile on $\mathbb{Z}_{50}^2$, we have a "before" and an "after" picture

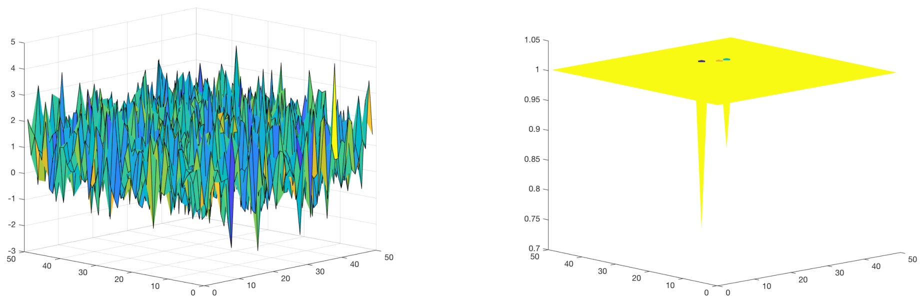

Figure 9: Sandpile: initial configuration and after 10.000 timesteps.

At first look, it may seem like we have made a programming error, as the sinks look unusual: it looks like we have "lost" mass in our graph. This is, however, not the case: analysing the image reveals that the rest of the graph actually does not have mass 1, but is in fact higher than 1 by a small amount, making all these sites unstable. In each iteration, the heights get shuffled around, and only a very small amount will end up in the sink.

4.2 Main proof for $n \geq 3$

Our main theorem of this section then is:

Theorem 4.1. *The divisible sandpile $(s(x))_{x \in \mathbb{Z}_n^d}$ for $n \geq 3$, where s is as defined before, does not stabilize in finite time almost surely.*

We will prove this theorem in a few steps, but the idea is very basic: once we find two nodes x, y that are adjacent to each other, where at least one has mass > 1, and the other mass ≥ 1, we are done. Indeed, the unstable site x will distribute some mass to its neighbor y, making y unstable in the next iteration. In the next step then, y will topple and give some mass to x. Again now x is unstable, and will give some mass to y again, and we can continue like this. This mass will decrease in each iteration, but it will only be zero in the limit.

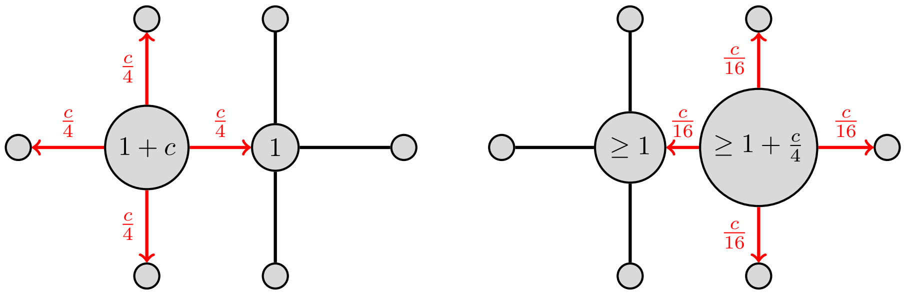

Figure 10: The idea behind our proof: there will always be at least one unstable node.

Lemma 4.2. *Let s be a divisible sandpile as given above. If at any timestep t we find $x \sim y$ such that $s_t(x) > s_t(y) \geq 1$, then we have that $s_{t+1}(y) > 1$. As a consequence, the sandpile does not stabilize in finite time.*

Proof. Assume that we find $x, y \in \mathbb{Z}_n^d$ and $t \geq 0$ such that $x \sim y$ and $s_t(x) > s_t(y) \geq 1$. We write $s_t(x) = 1 + c$ for some $c > 0$. As $s_t(x) > 1$, vertex x is unstable and topples in this timestep, distributing its mass equally over its neighboring vertices. Now $y \sim x$ and we consider two cases, in the first case $s_t(y) = 1$. Then $s_{t+1}(y) \geq 1 + \frac{c}{2d} > 1$, as y receives $\frac{c}{2d}$ mass from x. If $s_t(y) > 1$, then y topples and keeps mass 1 to itself. Because x also topples in the same timestep, we again find $s_{t+1}(y) \geq 1 + \frac{c}{2d} > 1$, thus proving the claim. In the same way, we see that in timestep $t+2$, we have $s_{t+2}(x) > 1$. Proceeding inductively, the result follows. □

The problem then reduces to finding two sites $x, y \in \mathbb{Z}_n^d$ with $x \sim y$, of which at least one has mass > 1. Indeed, if we find this we can use the above lemma to show that the sandpile does not stabilize in finite time. Our next claim is that we can always find such $x \sim y$ after just one iteration.

Theorem 4.3. *At timestep $t = 1$, there are $x, y \in \mathbb{Z}_n^d$ with $x \sim y$ such that $s_t(x) > s_t(y) \geq 1$.*

If we have proven this theorem, we can utilize Lemma 4.2 to obtain that the sandpile does not stabilize in finite time. The idea behind the proof and the theorem is that there is "too much mass" in the graph for this not to happen.

Proof. We write

$$V^+ := \{z \in \mathbb{Z}_n^d : s(z) > 1\} \subset \mathbb{Z}_n^d,$$

for the collection of all sites that topple in the first iteration. V^+ is non-empty, this is easy to see: if V^+ is empty, then $s(z) < 1$ for all $z \in \mathbb{Z}_n^d$, because we have $\mathbb{P}(s(x) = 1) = 0$ for all $x \in \mathbb{Z}_n^d$. If $s(z) < 1$ for all $z \in \mathbb{Z}_n^d$, then $\sum_{z \in \mathbb{Z}_n^d} s(z) < n^d$, and this is a contradiction. Now, if there exist two $x, y \in V^+$ such that $x \sim y$, then we are done, as we will have $s_1(x), s_1(y) > 1$ by Lemma 4.2. Therefore it is enough to consider only the case where each $x \in V^+$ is isolated in the sense that we have for no $x, y \in V^+$ that $x \sim y$. In this case, we have for all $x \in V^+$ that $s(y) < 1$ for all $y \sim x$. Choose $x \in V^+$. We can either find y with $y \sim x$ such that $s(y) + \frac{1}{2d} \sum_{z \sim y} (s(z) - 1)^+ > 1$, or we have for all $y \sim x$ that $s(y) + \frac{1}{2d} \sum_{z \sim y} (s(z) - 1)^+ \leq 1$. We will show that the first case leads to the desired result, while the second case gives a contradiction almost surely for $n \geq 3$.

In the first case, we find that since for all $z \in V^+$, we have $s(z) > 1$, these sites topple. Now as there is a y such that

$$s_1(y) = s(y) + \frac{1}{2d}\sum_{z\sim y}(s(z)-1)^+ > 1,$$

with $y \sim x$, and $x \in V^+$, the result follows because $s_1(x) \geq 1$.

Now consider the second case: Fix $x \in V^+$. Then for all $y \sim x$ we have that $s(y) + \frac{1}{2d}\sum_{z\sim y}(s(z)-1)^+ \leq 1$. In words: after the first toppling, all sites adjacent to any site in V^+ still have mass of at most 1. In this case, we either find a $y \sim x$ for some $x \in V^+$ such that

$$s(y) + \frac{1}{2d}\left(\sum_{x\sim y}(s(x)-1)^+\right) < 1,$$

or we have for all $x \in V^+$, and $y \sim x$ that

$$s(y) + \frac{1}{2d}\left(\sum_{x\sim y}(s(x)-1)^+\right) = 1.$$

We show now that the first case leads to a contradiction. We write

$$\widetilde{V}^+ := V^+ \cup \{z \in \mathbb{Z}_n^d : z \sim x \text{ for some } x \in V^+\} = V^+ \cup \{\text{ all neighbors of } V^+\}$$

and look at

$$\sum_{z\in\mathbb{Z}_n^d} s(z) = \sum_{z\in\widetilde{V}^+} s(z) + \sum_{z\in\mathbb{Z}_n^d\setminus\widetilde{V}^+} s(z). \tag{2}$$

As we have for all $z \in \mathbb{Z}_n^d \setminus \widetilde{V}^+$ by definition that $s(z) < 1$,

$$\sum_{z\in\mathbb{Z}_n^d\setminus\widetilde{V}^+} s(z) \leq |\mathbb{Z}_n^d \setminus \widetilde{V}^+| \leq n^d - |\widetilde{V}^+|. \tag{3}$$

Note that equality in the above occurs in the case where $\mathbb{Z}_n^d \setminus \widetilde{V}^+$ is empty. Now, if $z \in V^+$ topples, it distributes its mass over the neighbors, and by definition we have that these are in $\widetilde{V}^+$, so in fact

$$\sum_{z\in\widetilde{V}^+} s(z) = \sum_{z\in\widetilde{V}^+} s_1(z).$$

We have already seen that each $z \in V^+$ is isolated, so that it does not receive any mass from toppling neighbors, then

$$\begin{aligned}
\sum_{z\in\widetilde{V}^+} s_1(z) &= \sum_{z\in V^+} s_1(z) + \sum_{z\in\widetilde{V}^+\setminus V^+} s_1(z) \\
&= |V^+| + \sum_{z\in\widetilde{V}^+\setminus V^+}\left[s(z) + \frac{1}{2d}\sum_{x\sim y}(s(x)-1)^+\right] \\
&< |V^+| + |\widetilde{V}^+| - |V^+| = |\widetilde{V}^+|.
\end{aligned}$$

The last inequality is due to the fact that we could find at least one $y \in \widetilde{V}^+$ with $s(z) + \frac{1}{2d}\sum_{x\sim y}(s(x)-1)^+ < 1$. However, taking the above result together with Equation (3) and

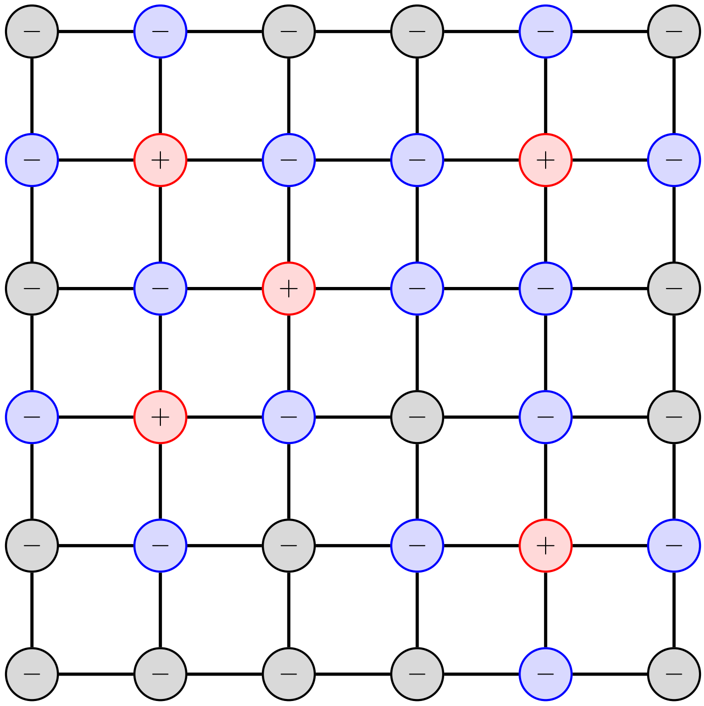

Figure 11: Visualisation of the proof on $\mathbb{Z}_6^2$. The blue nodes are the nodes in $\widetilde{V}^+ \setminus V^+$, while the red nodes are the nodes in V^+. A $+$ denotes nodes with mass > 1, a $-$ denotes nodes with mass < 1.

plugging it back into Equation (2), we obtain

$$\begin{aligned}\sum_{z\in\mathbb{Z}_n^d} s(z) &= \sum_{z\in\widetilde{V}^+} s(z) + \sum_{z\in\mathbb{Z}_n^d\setminus\widetilde{V}^+} s(z) \\ &< |\widetilde{V}^+| + n^d - |\widetilde{V}^+| = n^d,\end{aligned}$$

and this is a contradiction as $\sum_{z\in\mathbb{Z}_n^d} s(z) = n^d$ by definition of s. Now we still have to show that for all $x \in V^+$ and $y \sim x$

$$s(y) + \frac{1}{2d}\sum_{x\sim y}(s(x)-1)^+ = 1$$

almost surely does not happen when $n \geq 3$. We prove this in a separate theorem. □

Theorem 4.4. *In the notation of above, we have for fixed $x \in V^+$ and $y \sim x$, that*

$$\mathbb{P}\left(s(y) + \frac{1}{2d}\sum_{z\sim y}(s(z)-1)^+ = 1\right) = 0,$$

for $n \geq 3$.

Proof. For more convenient notation, we write

$$\Lambda(x) := \sigma(x) - \frac{1}{n^d}\sum_{w\in\mathbb{Z}_n^d}\sigma(w).$$

Note first that

$$\mathbb{P}\left(s(y)+\frac{1}{2d}\sum_{z\sim y}(s(z)-1)^{+}=1\right)=\mathbb{P}\left(\Lambda(y)+\frac{1}{2d}\sum_{z\sim y}1_{\Lambda(z)>0}\Lambda(z)=0\right).$$

Next, enumerate the neighbors of y from 1 to $2d$. It is then clear that proving the above probability is zero is equivalent to proving that for any combination of neighbors $x_1,\dots,x_k$ of y, $1\le k\le 2d$, we have

$$\mathbb{P}\left(\Lambda(y)+\frac{1}{2d}\sum_{j=1}^{k}\Lambda(x_j)=0\right)=0.$$

Explicitly calculating the above gives

$$\begin{aligned}\Lambda(y)+\frac{1}{2d}\sum_{j=1}^{k}\Lambda(x_j)&=\sigma(y)-\frac{1}{n^d}\sum_{w\in\mathbb{Z}_n^d}\sigma(w)+\frac{1}{2d}\sum_{j=1}^{k}\left(\sigma(x_j)-\frac{1}{n^d}\sum_{w\in\mathbb{Z}_n^d}\sigma(w)\right)\\&=\sigma(y)+\frac{1}{2d}\sum_{j=1}^{k}\sigma(x_j)-\left(1+\frac{k}{2d}\right)\frac{1}{n^d}\sum_{w\in\mathbb{Z}_n^d}\sigma(w).\end{aligned}$$

As we are only interested in the case where the above equals zero, we multiply everything with n^d and find that the above is equal to

$$\left(n^d-\left(1+\frac{k}{2d}\right)\right)\sigma(y)+\left(\frac{n^d}{2d}-\left(1+\frac{k}{2d}\right)\right)\sum_{j=1}^{k}\sigma(x_j)-\left(1+\frac{k}{2d}\right)\sum_{w\in\mathbb{Z}_n^d\setminus\{x_1,\dots,x_k,y\}}\sigma(w).$$

This is a linear combination of the i.i.d. Gaussians $(\sigma(x))_{x\in\mathbb{Z}_n^d}$, which is never equal to zero unless all coefficients are equal to zero. However, as we have taken $n\ge 3$, and $k\le 2d$, the first coefficient is always >0. This proves the claim. □

Now putting the above theorems together, gives the claim that the divisible sandpile on $\mathbb{Z}_n^d$ almost surely does not stabilize in finite time.

Remark: Note that the above proof also works for more general variables. We can repeat the proof for any sandpile configuration with continuous, i.i.d. weights $(\sigma(x))_{x\in\mathbb{Z}_n^d}$.

4.3 The case $n=2, d=1$

Note that in the case $\mathbb{Z}_n^d=\mathbb{Z}_2$, the sandpile stabilizes after the first iteration. There are a few reasons why the above proof fails in this case. As we have taken $n\ge 3$, we find that each node has $2d$ neighbors, however this does not generalize to the case $n=2$, where there are only two nodes, each has only the other as neighboring node. Here we indeed have

$$\mathbb{P}\left(s(y)+\frac{1}{2d}\sum_{z\sim y}(s(z)-1)^{+}=1\right)=1. \tag{4}$$

This is seen in the following way: as $\deg(x)=1$ for $x\in\mathbb{Z}_2$, we constantly replace the factor $2d$ by 1 in the proof of Theorem 4.4. Note that the only neighbor of y, has to have $s>1$. Looking at the last line, we see that this becomes for $k=0,1$

$$(2-(1+k))\,\sigma(y)+(2-(1+k))\,\sigma(x)-(1+k)\sum_{w\in\mathbb{Z}_n^d\setminus\{x,y\}}\sigma(w). \tag{5}$$

For $k = 1$, the first and second coefficient are equal to zero, and as $\mathbb{Z}_2 \setminus \{x, y\} = \emptyset$, the last term vanishes as well. As a consequence, we see that Equation (5) simplifies to 0, so that indeed (4) holds.

Brief Summary

We have seen that the divisible sandpile, defined as before, almost surely does not stabilize in finite time. We have first observed that, whenever we find two neighboring sites x, y, of which one has mass ≥ 1, and one > 1, then the process only stabilizes in the limit. In order to find two of these sites, we partitioned our graph in V^+, the sites with initial mass > 1, $\widetilde{V}^+ \setminus V^+$, the sites which receive mass in the first toppling, and $\mathbb{Z}_n^d \setminus \widetilde{V}^+$, the sites which don't receive any mass in the first toppling, and have initial mass < 1. We have seen that it is enough to only consider the case where each $x \in V^+$ is isolated, in other words, no two $x, y \in V^+$ are neighbors. In this way, no $x \in V^+$ receives mass after the first toppling. In this way we have seen that

$$\sum_{z \in \mathbb{Z}_n^d \setminus \widetilde{V}^+} s_1(z) \leq |\mathbb{Z}_n^d \setminus \widetilde{V}^+| \text{ and } \sum_{z \in V^+} s_1(z) = 1.$$

Now if we can not find a $y \in \widetilde{V}^+ \setminus V^+$ with mass > 1 after the first toppling, then either all $y \in \widetilde{V}^+ \setminus V^+$ have mass 1, and we have shown that this almost surely does not happen. In the other case, we can find $y \in \widetilde{V}^+ \setminus V^+$ with $s_1(y) < 1$. In this case $\sum_{z \in \widetilde{V}^+ \setminus V^+} s_1(z) < |\widetilde{V}^+ \setminus V^+|$. This leads to a contradiction, as

$$\begin{aligned} n^d = \sum_{z \in \mathbb{Z}_n^d} s(z) = \sum_{z \in \mathbb{Z}_n^d} s_1(z) &= \sum_{z \in \mathbb{Z}_n^d \setminus \widetilde{V}^+} s_1(z) + \sum_{z \in V^+} s_1(z) + \sum_{z \in \widetilde{V}^+ \setminus V^+} s_1(z) \\ &< |\mathbb{Z}_n^d \setminus \widetilde{V}^+| + |V^+| + |\widetilde{V}^+ \setminus V^+| = n^d. \end{aligned}$$

So there must be at least one $y \in \widetilde{V}^+ \setminus V^+$ with mass > 1. As $y \sim x$ for some $x \in V^+$, and x has mass 1, we are done.

5 Scaling Limit for i.i.d. Gaussians

5.1 Introduction

Consider a simple random walk on $\mathbb{Z}$. That is, define $X_0 = 0$, and for all $n \in \mathbb{N}$

$$X_n = \begin{cases} 1, & \text{w.p. } \frac{1}{2} \\ -1, & \text{w.p. } \frac{1}{2} \end{cases}.$$

Now set $S_n := \sum_{k=0}^{n} X_k$. We can make a plot now of the random walk, as it will look something like in Figure 12. In this picture we see that we first jump down, then make two jumps upwards, a jump down again, and so on.

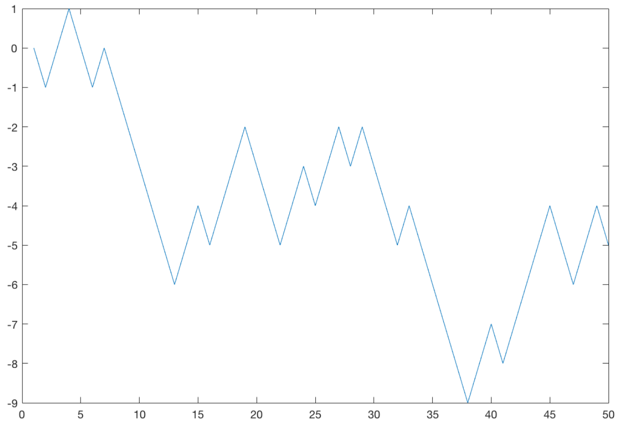

Figure 12: A plot of $(S_n)_{n\geq 0}$ for $n \leq 50$.

In general, we want to be able to quantify our random walk. For example, we might want to know the probability of $S_n \geq 10$ for n very large. As S_n takes values in $[-\frac{n}{2}, \frac{n}{2}] \cap \mathbb{Z}$, we can explicitly calculate this probability. Consider $\mathbb{P}(S_n = k)$. If n is even, then S_n takes only even values, and if n is odd, S_n takes only odd values. To this end, assume that both n and k are even. As we require $S_n = k$, we need to have k jumps up, $\frac{n-k}{2}$ jumps down, and again $\frac{n-k}{2}$ jumps down, not necessarily in that order. The problem is then equivalent to computing how many combinations there are where we have $\frac{n+k}{2}$ jumps up, and $\frac{n-k}{2}$ jumps down, and this follows essentially a binomial distribution. We have, for k even,

$$\mathbb{P}(S_n = k) = \binom{n}{\frac{n+k}{2}} \left(\frac{1}{2}\right)^n.$$

Subsequently,

$$\mathbb{P}(S_n \geq 10) = \sum_{\substack{10 \leq k \leq n \\ k \text{ even}}} \binom{n}{\frac{n+k}{2}} \left(\frac{1}{2}\right)^n.$$

Now, even though it is possible in theory to explicitly compute this probability, it is usually not very practical as the binomial coefficients are computationally expensive to calculate. Another approach then is to use a limit theorem, the *Central Limit Theorem* roughly states that for a sequence of reasonable enough i.i.d. random variables $(X_n)_{n\geq 1}$, the following convergence holds

$$\frac{\sum_{k=1}^n X_k - \mathbb{E}[X_1]}{\sigma\sqrt{n}} \xrightarrow{d} Z,$$

with $Z \sim \mathcal{N}(0,1)$. In our case we have $\mathbb{E}[X_1] = 0$ and $\sigma = 1$. Now, by the Central Limit Theorem $S_n/\sqrt{n} \to Z$ in distribution, and for large n we have $10/\sqrt{n} \to 0$, so we obtain

$$\mathbb{P}(S_n \geq 10) = \mathbb{P}\left(\frac{S_n}{\sqrt{n}} \geq \frac{10}{\sqrt{n}}\right) \approx \mathbb{P}(Z \geq 0) = \frac{1}{2}.$$

Note that this is a much quicker and easier way to approximate the probability we could not explicitly calculate before. This is exactly the reason why studying scaling limits is so useful, because instead of working on a gigantic discrete space, we can circumvent the combinatorics and approximate our probability with a continuous model. In a lot of cases this simplifies our computation significantly.

This Section, with the exception of subsection 5.6, is entirely based on Cipriani et al. [5], and the purpose of this section is to explain the techniques they used to prove Theorem 1 in [5]. It is perhaps important to remark now that we will not go into full detail with their proof, as we believe it is more important to explain "the big picture" of what is happening.

5.2 Scaling limit of the odometer in the i.i.d. case: the limiting field

We consider a divisible sandpile s on $\mathbb{Z}_n^d$ associated with the initial i.i.d. weights $(\sigma(z))_{z\in\mathbb{Z}_n^d}$, as described in Theorem 3.2. Write $e_n : \mathbb{Z}_n^d \to \mathbb{R}$ for the odometer in this case, as we have seen

$$e_n \stackrel{d}{=} \left(\eta - \min_{z\in\mathbb{Z}_n^d} \eta(z)\right),$$

with $(\eta(z))_{z\in\mathbb{Z}_n^d}$ centered Gaussians with

$$\mathbb{E}[\eta(x)\eta(y)] = \frac{1}{\deg(x)\deg(y)} \sum_{w\in\mathbb{Z}_n^d} g(x,w)g(w,y).$$

As we want to calculate the scaling limit, we are interested in the behaviour of e_n as $n \to \infty$. Now, e_n is defined on $\mathbb{Z}_n^d$, and in a sense we can view $\frac{1}{n}\mathbb{Z}_n^d$ as a discretization of the torus $\mathbb{T}^d$. Our limiting field will then also be some sort of function on $\mathbb{T}^d$, however it will turn out to be a bit more subtle than that. First of all, note that we have, for Δ the graph Laplacian,

$$\begin{aligned}
\Delta_y^2 \mathbb{E}[\eta(x)\eta(y)] &= \Delta_y^2 \left(\frac{1}{\deg(x)\deg(y)} \sum_{w\in\mathbb{Z}_n^d} g(x,w)g(w,y)\right) \\
&= \Delta_y \left(\sum_{w\in\mathbb{Z}_n^d} \frac{g(x,w)}{\deg(x)} \Delta_y \frac{g(w,y)}{\deg(y)}\right).
\end{aligned}$$

Now recall our previous calculations with the graph Laplacian (see for example the calculations in the proof of Theorem 3.2). We find

$$\begin{aligned}\Delta_y \frac{g(w,y)}{\deg(y)} = \sum_{z\in\mathbb{Z}_n^d} \frac{1}{n^d}\Delta_y g^z(w,y) &= \sum_{z\in\mathbb{Z}_n^d} \frac{1}{n^d}\left(\delta_z(y)-\delta_w(y)\right) \\ &= \frac{1}{n^d}\sum_{z\in\mathbb{Z}_n^d}\delta_z(y) - \frac{1}{n^d}\sum_{z\in\mathbb{Z}_n^d}\delta_w(y) = \frac{1}{n^d} - \delta_w(y).\end{aligned}$$

Plugging this into the above yields

$$\begin{aligned}\Delta_y\left(\sum_{w\in\mathbb{Z}_n^d}\frac{g(x,w)}{\deg(x)}\left(\frac{1}{n^d}-\delta_w(y)\right)\right) &= \Delta_y\left(\frac{1}{n^d}\sum_{w\in\mathbb{Z}_n^d}\frac{g(x,w)}{\deg(x)} - \frac{g(x,y)}{\deg(x)}\right) \\ &= \delta_x(y) - \frac{1}{n^d}.\end{aligned}$$

This is because $\frac{1}{n^d}\sum_{w\in\mathbb{Z}_n^d}\frac{g(x,w)}{\deg(x)}$ is constant, and thus vanishes under the graph Laplacian. For the second term, we can repeat the same computation as above. All of this together gives us that

$$\Delta^2\mathbb{E}[\eta(x)\eta(y)] = \delta_x(y) - \frac{1}{n^d}.$$

So in a certain sense, our covariance matrix approximates the inverse of a discrete bi-Laplacian. We will call the limiting field Ξ. It might now be tempting to simply take the limit $n\to\infty$ in the above and to conclude that the continuous limit is exactly the inverse bi-Laplacian, but there are a few problems with this. First of all, the limiting field Ξ is infinite-dimensional, so how do we quantify a covariance matrix in this case? Second, how does one go about rigorously proving convergence for random variables taking values in infinite dimensional spaces?

Before answering these questions, we first need to build up some theory around Fourier analysis and Sobolev spaces on the torus $\mathbb{T}^d$.

5.3 Sobolev spaces on the torus $\mathbb{T}^d$

For any smooth, rapidly decaying function φ on $\mathbb{R}$, we define the *Fourier transform* on $\mathbb{R}$ to be

$$\widehat{\varphi}(\xi) = \int_{\mathbb{R}} \varphi(x)\mathrm{e}^{\iota\xi x}\,\mathrm{d}x.$$

Now note that, by integration by parts,

$$\widehat{\varphi^{(n)}}(\xi) = \int_{\mathbb{R}}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\varphi(x)\mathrm{e}^{\iota\xi x}\,\mathrm{d}x = (-1)^n\int_{\mathbb{R}}\varphi(x)\frac{\mathrm{d}^n}{\mathrm{d}x^n}\mathrm{e}^{\iota\xi x}\mathrm{d}x = (-\iota\xi)^n\widehat{\varphi}(\xi).$$

Funny thing really, since we found that $\widehat{\varphi^{(n)}}(\xi) = (-\iota\xi)^n\widehat{\varphi}(\xi)$, we can in this way define fractional and negative order derivatives in the following way, for $a\in\mathbb{R}$ we can say

$$\varphi^{(a)}(x) := \left((-\iota\xi)^a\widehat{\varphi^{(a)}}(\xi)\right)^{\vee}.$$

Now, since the Fourier transform is also defined on $\mathbb{T}^d$, we can do something similar to what we have seen above. We define for any function $f\in L^2(\mathbb{T}^d)$ and $\nu\in\mathbb{Z}^d$,

$$\widehat{f}(\nu) = \int_{\mathbb{T}^d} f(x)\mathrm{e}^{2\pi\iota\nu\cdot x}\,\mathrm{d}x.$$

Subsequently for all $x \in \mathbb{T}^d$,

$$f(x) = \sum_{\nu \in \mathbb{Z}^d} \widehat{f}(\nu) \mathrm{e}^{-2\pi\iota\nu\cdot x}.$$

We can define for any $a \in \mathbb{R}$ the operator $(-\Delta)^a$ by the action of its Fourier transform, let $(\mathbf{e}_\nu)_{\nu\in\mathbb{Z}^d}$ denote the Fourier basis of $L^2(\mathbb{Z}^d)$ (i.e., $\mathbf{e}_\nu := \exp(-2\pi\iota\nu)$) and $u(\vartheta) = \sum_{\nu\in\mathbb{Z}^d} \widehat{u}(\nu)\mathbf{e}_\nu(\vartheta)$. Similar to what we have seen on $\mathbb{R}^d$, we define

$$(-\Delta)^a \sum_{\nu\in\mathbb{Z}^d} \widehat{u}(\nu)\mathbf{e}_\nu(\vartheta) = \sum_{\nu\in\mathbb{Z}^d\setminus\{0\}} \|\nu\|^{2a}\widehat{u}(\nu)\mathbf{e}_\nu(\vartheta).$$

Note that $\widehat{u}(0) = 0$ since u is a mean zero function. We define the *Sobolev space of order* -1 $H^{-1}(\mathbb{T}^d)$ to be the subset of $L^2(\mathbb{T}^d)$ such that the norm

$$\|u\|_{-1}^2 := (u, \Delta^{-2}u)_{L^2(\mathbb{T}^d)}.$$

is finite. Using the Fourier transform, we can now see what this norm actually means:

$$\|u\|_{-1}^2 = \sum_{\nu\in\mathbb{Z}^d\setminus\{0\}} \|\nu\|^{-4}|\widehat{u}(\nu)|^2.$$

We have essentially just rescaled the Fourier coefficients. This norm will turn out to be very important in the construction of our field Ξ, as we will see in the next section!

5.4 Construction of the limiting field Ξ

The presentation here is based on [5] and is a quick rundown of the "Abstract Wiener Spaces" section in the preliminaries section. We write $e_n(\cdot)$ for the odometer on $\mathbb{Z}_n^d$ associated with the i.i.d. weights $(\sigma(z))_{z\in\mathbb{Z}_n^d}$. Set

$$\Xi_n(x) := 4\pi^2 \sum_{z\in\mathbb{T}_n^d} n^{\frac{d-4}{2}} e_n(nz) 1_{B\left(z,\frac{1}{2n}\right)}(x),$$

for $x \in \mathbb{T}_n^d$. Here we have taken $B\left(z, \frac{1}{2n}\right)$ in the ℓ^∞ norm, so instead of a ball, this will look like a box. Theorem 1 in [5] tells us that, as $n \to \infty$, $\Xi_n \xrightarrow{d} \Xi$ in the Sobolev space $\mathcal{H}_{-\varepsilon}(\mathbb{T}^d)$ for $\varepsilon > \max\{1 + \frac{d}{4}, \frac{d}{2}\}$ (don't worry about this last part just yet). However, up until now we have not really explained what Ξ actually is. In this section we will explain and quantify what Ξ is. As Ξ_n is a function on $\mathbb{T}^d$, we might expect our field Ξ to be that as well. However, as it turns out Ξ can be seen as a *random distribution* on $\mathbb{T}^d$. In this way, we consider Ξ as a collection of random variables $\{\langle \Xi, u\rangle : u \in H^{-1}(\mathbb{T}^d)\}$, where $\langle\cdot,\cdot\rangle$ is the usual L^2-inner product. Each $\langle \Xi, u\rangle$ now is a Gaussian random variable with $\mathbb{E}[\langle \Xi, u\rangle] = 0$ and

$$\mathbb{E}\left[\langle \Xi, u\rangle^2\right] = \|u\|_{-1}^2 := (u, \Delta^{-2}u)_{L^2(\mathbb{T}^d)}.$$

Now for a more rigorous construction of Ξ, we refer the reader to the Preliminaries section. Here we will state just the essentials to understand the proof of Theorem 1 as given in [5]. Define the equivalence relation $\sim$ such that $f \sim g$ if they differ only by a constant. We set for any $a \in \mathbb{R}$:

$$\mathcal{H}_a(\mathbb{T}^d) := \{u \in L^2(\mathbb{T}^d) : (-\Delta)^a u \in L^2(\mathbb{T}^d)\}/\sim,$$

and supply $\mathcal{H}_a(\mathbb{T}^d)$ with the norm

$$\|u\|_{\mathcal{H}_a}^2 = ((-\Delta)^a u, (-\Delta)^a u)_{L^2(\mathbb{T}^d)}.$$

Then $\Xi \in \mathcal{H}_{-\varepsilon}(\mathbb{T}^d)$ for $\varepsilon > 1 + \frac{d}{4}$. Note that actually Ξ is the random distribution associated with the Gaussian probability measure $\mu_{-\varepsilon}$ on $\mathcal{H}_{-\varepsilon}(\mathbb{T}^d)$. Here $\mu_{-\varepsilon}$ has characteristic functional

$$\Phi(u) := \exp\left(-\frac{\|u\|_{-1}^2}{2}\right).$$

However strange this definition may seem, it is in fact the infinite-dimensional version of a random variable X taking values in the 1-dimensional space $\mathbb{R}$, with characteristic function

$$\mathbb{E}\left[\mathrm{e}^{\iota t X}\right] = \mathrm{e}^{-\frac{t^2}{2}}.$$

For now, just think of Ξ as a random surface, that we can only quantify through inner products.

5.5 Proving convergence of Ξ_n

We repeat Theorem 1 in [5],

Theorem 5.1. *Let $d \geq 1$ and $(\sigma(x))_{x \in \mathbb{Z}_n^d}$ be a collection of i.i.d. standard Gaussians. Let $e_n(\cdot) := e_{\mathbb{Z}_n^d}(\cdot)$ be the odometer on $\mathbb{Z}_n^d$ associated to these weights. The formal field*

$$\Xi_n(x) := 4\pi^2 \sum_{z \in \mathbb{T}_n^d} n^{\frac{d-4}{2}} e_n(nz) 1_{B(z, \frac{1}{2n})}(x), \; x \in \mathbb{T}^d$$

converges in law as $n \to \infty$ to the bilaplacian field Ξ on $\mathbb{T}^d$. The convergence holds in the Sobolev space $\mathcal{H}_{-\varepsilon}(\mathbb{T}^d)$ with the topology induced by the norm $\|\cdot\|_{\mathcal{H}_{-\varepsilon}(\mathbb{T}^d)}$ for any $\varepsilon > \max\{1 + \frac{d}{4}, \frac{d}{2}\}$.

In this section we will dissect the proof of this theorem as given in [5], and explain the motivation behind the steps taken.

For random variables $(X_n)_{n \in \mathbb{N}}$ taking values in the 1-dimensional space $\mathbb{R}$, proving convergence $X_n \xrightarrow{d} X$ is relatively simple. A common technique is to look at the convergence of their characteristic functions. Write $\varphi_n(t) := \mathbb{E}[\exp(\iota t X_n)]$ for the characteristic function of X_n and $\varphi(t) := \mathbb{E}[\exp(\iota t X)]$ for the characteristic function of X. If

$$\varphi_n(t) \to \varphi(t), \; \forall t \in \mathbb{R},$$

then $X_n \xrightarrow{d} X$. As we have already hinted at before, a similar approach is necessary in the infinite dimensional case: in our case we will be dealing with the sequence $(\Xi_n)_{n \in \mathbb{N}}$ in the space $\mathcal{H}_{-\varepsilon}(\mathbb{T}^d)$, with limit $\Xi \in \mathcal{H}_{-\varepsilon}(\mathbb{T}^d)$. Now, a remark under Lemma 2.2 in Section 2.1 in [11] states that for random variables X_n taking values in a Banach space B, $X_n \xrightarrow{d} X$ as soon as for all $f \in B'$ we have $f(X_n) \xrightarrow{d} f(X)$ as a sequence of real-valued random variables, *and* the sequence $(X_n)_{n \in \mathbb{N}}$ is tight, meaning for all $\varepsilon > 0$ there exists a compact $K \subset B$ such that for all $n \in \mathbb{N}$,

$$\mathbb{P}(X_n \in K) \geq 1 - \varepsilon.$$

Subsequently, to show that the convergence $\Xi_n \xrightarrow{d} \Xi$ holds in $\mathcal{H}_{-\varepsilon}(\mathbb{T}^d)$, we need to show two things:

1. The sequence $(\Xi_n)_{n \in \mathbb{N}}$ is tight in the space $\mathcal{H}_{-\varepsilon}(\mathbb{T}^d)$.

2. For all mean zero $u \in C^\infty(\mathbb{T}^d)$, we have

$$\langle \Xi_n, u\rangle \xrightarrow{d} \langle \Xi, u\rangle \quad \text{as } n \to \infty.$$

The proof of 5.1 is then split up in two parts, we first show the convergence of the real valued random variables $(\langle \Xi_n, u\rangle)_{n\in\mathbb{N}}$ for any mean zero $u \in C^\infty(\mathbb{T}^d)$, so

$$\langle \Xi_n, u\rangle \xrightarrow{d} \langle \Xi, u\rangle, \ \forall u \in C^\infty(\mathbb{T}^d).$$

The authors of [5] call this (P2). Now by Theorem 2.2 it is enough to show $\mathbb{E}\left[\langle \Xi_n, u\rangle\right] \to \mathbb{E}\left[\langle \Xi, u\rangle\right] = 0$ and

$$\mathbb{E}\left[\langle \Xi_n, u\rangle^2\right] \to \mathbb{E}\left[\langle \Xi, u\rangle^2\right] = \|u\|_{-1}^2.$$

The authors of [5] prove this in a few steps. They first show $\langle \Xi_n, u\rangle$ is a centered Gaussian random variable for all $n \in \mathbb{N}$ through a straightforward argument:

$$\langle \Xi_n, u\rangle = 4\pi^2 \sum_{z\in\mathbb{T}_n^d} n^{\frac{d-4}{2}} e_n(nz) \int_{\mathbb{T}^d} u(x) 1_{B(z,\frac{1}{2n})}(x)\ \mathrm{d}x = 4\pi^2 \sum_{z\in\mathbb{T}_n^d} n^{\frac{d-4}{2}} e_n(nz) \int_{B(z,\frac{1}{2n})} u(x)\ \mathrm{d}x.$$

Now, recall that $e_n(nz) \stackrel{d}{=} \left(\chi_{nz} - \min_{z'\in\mathbb{Z}_n^d} \chi_{z'}\right)$. If we then plug this into the above equation, we obtain

$$\begin{aligned}4\pi^2 \sum_{z\in\mathbb{T}_n^d} n^{\frac{d-4}{2}} e_n(nz) \int_{B(z,\frac{1}{2n})} u(x)\ \mathrm{d}x &\stackrel{d}{=} 4\pi^2 \sum_{z\in\mathbb{T}_n^d} n^{\frac{d-4}{2}} \left(\chi_{nz} - \min_{z'\in\mathbb{Z}_n^d} \chi_{z'}\right) \int_{B(z,\frac{1}{2n})} u(x)\ \mathrm{d}x \\ &= 4\pi^2 \sum_{z\in\mathbb{T}_n^d} n^{\frac{d-4}{2}} \chi_{nz} \int_{B(z,\frac{1}{2n})} u(x)\ \mathrm{d}x - \left(\min_{z'\in\mathbb{Z}_n^d} \chi_{z'}\right) 4\pi^2 n^{\frac{d-4}{2}} \sum_{z\in\mathbb{T}_n^d} \int_{B(z,\frac{1}{2n})} u(x)\ \mathrm{d}x\end{aligned}$$

As we have taken $u \in C^\infty(\mathbb{T}^d)$ to have integral 0, we see that the last term vanishes, so in fact

$$\langle \Xi_n, u\rangle \stackrel{d}{=} 4\pi^2 \sum_{z\in\mathbb{T}_n^d} n^{\frac{d-4}{2}} \chi_{nz} \int_{B(z,\frac{1}{2n})} u(x)\ \mathrm{d}x.$$

Now we recall from 3.2 that χ_{nz} are centered Gaussians, so $\mathbb{E}[\chi_{nz}] = 0$ for all $z \in \mathbb{T}_n^d$. Furthermore, note that $\langle \Xi_n, u\rangle$ is a Gaussian random variable again, as it is a linear combination of a finite amount of Gaussians. The claim then follows, as $\mathbb{E}\left[\langle \Xi_n, u\rangle\right] = 0$ for all $n \in \mathbb{N}$, so the limit agrees with $\mathbb{E}\left[\langle \Xi, u\rangle\right] = 0$. The proof for the second moment is way more technical, and to this end the authors of [5] start off by proving the following Lemma (Proposition 4 in [5]):

Lemma 5.2. *The odometer $e_n(\cdot)$ on $\mathbb{Z}_n^d$ admits the representation*

$$(e_n(x))_{x\in\mathbb{Z}_n^d} \stackrel{d}{=} \left(\chi_x - \min_{z\in\mathbb{Z}_n^d} \chi_z\right),$$

where the χ_x are centered Gaussians with correlation

$$\mathbb{E}[\chi_x\chi_y] = \frac{n^{-d}}{16} \sum_{z\in\mathbb{Z}_n^d\setminus\{0\}} \frac{\exp\left(2\pi\iota(y-x)\cdot\frac{z}{n}\right)}{\left(\sum_{i=1}^d \sin^2\left(\pi\frac{z_i}{n}\right)\right)^2}. \tag{6}$$

The proof relies on the Fourier transform of $g_x(a) := g(x,a) = g(a,x)$, where g is as defined before. We have for all $a \neq 0$,

$$\lambda_a \widehat{g_x}(a) = -2dn^{-d} \exp\left(-2\pi\iota x \cdot \frac{a}{n}\right).$$

Here we set

$$\lambda_w = -4\sum_{i=1}^{d} \sin^2\left(\frac{\pi w_i}{n}\right),\ w \in \mathbb{Z}_n^d.$$

For $a = 0$ we find

$$\widehat{g_x}(0) = n^{-d} \sum_{y\in\mathbb{Z}_n^d} g_x(y) =: L.$$

Now the proof of the above assertion follows from applying the Plancherel Theorem to the covariance function in 3.2, and using the above facts for the Fourier transform of g_x. For the second moment, we approximate the integrals by their value at the centre,

$$\int_{B(z,\frac{1}{2n})} u(x)\ \mathrm{d}x = n^{-d}u(z) + K_n(u).$$

We obtain

$$\langle \Xi_n, u\rangle = 4\pi^2 \sum_{z\in\mathbb{T}_n^d} n^{\frac{d-4}{2}} \chi_{nz} \int_{B(z,\frac{1}{2n})} u(x)\ \mathrm{d}x = 4\pi^2 \sum_{z\in\mathbb{T}_n^d} n^{\frac{-(d+4)}{2}} \chi_{nz} u(z) + R_n(u).$$

For now, we will disregard the $R_n(u)$ term, but it can be shown that $\mathbb{E}[(R_n(u))^2] \to 0$. Our focus here is on the first term, by neglecting $R_n(u)$, we find the following:

$$\begin{aligned}\langle \Xi_n, u\rangle^2 &= \left(4\pi^2 \sum_{z\in\mathbb{T}_n^d} n^{\frac{-(d+4)}{2}} \chi_{nz} u(z)\right)\left(4\pi^2 \sum_{z'\in\mathbb{T}_n^d} n^{\frac{-(d+4)}{2}} \chi_{nz'} u(z')\right)\\ &= 16\pi^4 \sum_{z,z'\in\mathbb{T}_n^d} n^{-(d+4)} \chi_{nz}\chi_{nz'} u(z)u(z').\end{aligned}$$

The beauty now comes when we take the expectation and plug in the covariance from (6). We see that

$$\mathbb{E}\left[\langle \Xi_n, u\rangle^2\right] = \pi^4 \sum_{z,z'\in\mathbb{T}_n^d} n^{-(2d+4)} u(z)u(z') \sum_{w\in\mathbb{Z}_n^d\setminus\{0\}} \frac{\exp\left(2\pi\iota(z-z')\cdot w\right)}{\left(\sum_{i=1}^d \sin^2\left(\pi\frac{w_i}{n}\right)\right)^2}. \tag{7}$$

Note that the above sums are finite, so we can exchange their order:

$$(7) = \sum_{w\in\mathbb{Z}_n^d\setminus\{0\}} \frac{\pi^4 n^{-4}}{\left(\sum_{i=1}^d \sin^2\left(\pi\frac{w_i}{n}\right)\right)^2} \left(n^{-d}\sum_{z\in\mathbb{T}_n^d} u(z)\exp\left(2\pi\iota z\cdot w\right)\right)\left(n^{-d}\sum_{z'\in\mathbb{T}_n^d} u(z')\exp\left(-2\pi\iota z'\cdot w\right)\right).$$

We denote by $u_n : \mathbb{Z}_n^d \to \mathbb{R}$ the function defined by $u_n(\cdot) := u\left(\frac{\cdot}{n}\right)$. Recall the Fourier transform on $\mathbb{Z}_n^d$, and one might now notice that

$$n^{-d}\sum_{z\in\mathbb{T}_n^d} u(z)\exp(2\pi\iota z\cdot w) = n^{-d}\sum_{z\in\mathbb{Z}_n^d} u_n(z)\exp\left(2\pi\iota z\cdot\frac{w}{n}\right) = \widehat{u_n}(w).$$

Notice as well how the above approximates a Riemann integral, and since u is sufficiently smooth we have the convergence $\widehat{u_n}(w) \to \widehat{u}(w)$, where $\widehat{u}$ is just the regular Fourier transform on $\mathbb{T}^d$. Similarly, we see that the third term is equal to $\widehat{u_n}(w)$, so plugging this into the above, we find that it simplifies to

$$\sum_{w \in \mathbb{Z}_n^d \setminus \{0\}} \frac{\pi^4 n^{-4}}{\left(\sum_{i=1}^d \sin^2\left(\pi \frac{w_i}{n}\right)\right)^2} |\widehat{u_n}(w)|^2.$$

To deal with the fraction term in the summation, Cipriani et al. have made a sharp observation: Lemma 7 in [5] says

Lemma 5.3. *There exists $c > 0$ such that for all $n \in \mathbb{N}$ and $w \in \mathbb{Z}_n^d \setminus \{0\}$ we have*

$$\frac{1}{\|\pi w\|^4} \leq n^{-4} \left(\sum_{i=1}^d \sin^2\left(\frac{\pi w_i}{n}\right) \right)^{-2} \leq \left(\frac{1}{\|\pi w\|^2} + \frac{c}{n^2} \right)^2.$$

Multiplying by π^4 and subsequently taking the limit of $n \to \infty$ in the above, we see what it's really about. We find

$$\frac{\pi^4 n^{-4}}{\left(\sum_{i=1}^d \sin^2\left(\pi \frac{w_i}{n}\right)\right)^2} \to \frac{1}{\|w\|^4}.$$

The proof is based on the approximation $\sin(x) \approx x$ for small x. Now, as $n \to \infty$, the argument in our sin's also tends to 0, so heuristically speaking,

$$\sum_{i=1}^d \sin^2\left(\frac{\pi w_i}{n}\right) \approx \sum_{i=1}^d \left(\frac{\pi w_i}{n}\right)^2 = n^{-2}\|\pi w\|^2.$$

In this way we can see,

$$\left(\sum_{i=1}^d \sin^2\left(\frac{\pi w_i}{n}\right) \right)^{-2} \approx \frac{n^4}{\|\pi w\|^4}.$$

Now Lemma 5.3 follows by scaling appropriately. For a more rigorous proof, we refer the reader to [5]. Using this approximation then, it is enough to show the convergence of

$$\sum_{w \in \mathbb{Z}_n^d \setminus \{0\}} \|w\|^{-4} |\widehat{u_n}(w)|^2 = \sum_{w \in \mathbb{Z}^d \setminus \{0\}} 1_{w \in \mathbb{Z}_n^d} \|w\|^{-4} |\widehat{u_n}(w)|^2. \tag{8}$$

Ultimately, we would like to switch limit and integral in the above, to obtain (note $1_{w \in \mathbb{Z}_n^d} \to 1_{w \in \mathbb{Z}^d}$ and $\widehat{u_n} \to \widehat{u}$):

$$\lim_{n \to \infty} \sum_{w \in \mathbb{Z}^d \setminus \{0\}} 1_{w \in \mathbb{Z}_n^d} \|w\|^{-4} |\widehat{u_n}(w)|^2 = \sum_{w \in \mathbb{Z}^d \setminus \{0\}} \|w\|^{-4} |\widehat{u}(w)|^2 = (u, \Delta^{-2} u)_{L^2(\mathbb{T}^d)}. \tag{9}$$

In the case $d \leq 3$, we see that for some $C > 0$ and n large enough,

$$\begin{aligned} |\widehat{u_n}(w)| = \left| \frac{1}{n^d} \sum_{z \in \mathbb{Z}_n^d} u\left(\frac{z}{n}\right) \exp\left(2\pi \iota w \cdot \frac{z}{n}\right) \right| &\leq \frac{1}{n^d} \sum_{z \in \mathbb{Z}_n^d} \left| u\left(\frac{z}{n}\right) \right| \\ &\leq C \int_{\mathbb{T}^d} |u(z)| \, \mathrm{d}z \leq C \|u\|_{L^1(\mathbb{T}^d)} < \infty. \end{aligned}$$

So $1_{w\in\mathbb{Z}_n^d}\|w\|^{-4}|\widehat{u_n}(w)|^2 \leq C^2\|w\|^{-4}\|u\|^2_{L^1(\mathbb{T}^d)}$. As this function is integrable over $\mathbb{Z}^d$ when $d \leq 3$, we can use the dominated convergence theorem to switch limit and integral as in 9, and the required convergence follows. The case $d \geq 4$ is more complicated, as we can not easily bound our function by an integrable function anymore. The strategy used in [5] (and we will repeatedly use in the next section as well) is to use a mollifier. For now, we will only heuristically explain what is happening in their proof. Take any $\phi \in \mathcal{S}(\mathbb{R}^d)$ with compact support on $[-\frac{1}{2}, \frac{1}{2}]^d$ and $\int_{\mathbb{R}^d}\phi(x)\,\mathrm{d}x = 1$. Write $\phi_\kappa(x) := \kappa^{-d}\phi\left(\frac{x}{\kappa}\right)$. We now know a few things about the Fourier transform $\widehat{\phi_\kappa}$. First of all, as ϕ is smooth, the rescaled ϕ_κ is smooth as well, so the Fourier transform $\widehat{\phi_\kappa}(\xi)$ decays rapidly as $|\xi| \to \infty$. Second, ϕ_κ will resemble a delta peak, so we would expect the Fourier transform to $\widehat{\phi_\kappa}(\xi) \to 1$ pointwise as $\kappa \to 0$. The fact that $\widehat{\phi_\kappa}(\xi) \to 1$ pointwise as $\kappa \to 0$, and that $\widehat{\phi_\kappa}(\xi)$ decays rapidly in ξ for all $\kappa > 0$ makes this a very useful tool in proving convergence. Cipriani et al. use this to split up (9) in two parts. We have

$$\sum_{w\in\mathbb{Z}_n^d\setminus\{0\}} \left(1-\widehat{\varphi_\kappa}(w)\right)\|w\|^{-4}|\widehat{u_n}(w)|^2 + \sum_{w\in\mathbb{Z}_n^d\setminus\{0\}} \widehat{\varphi_\kappa}(w)\|w\|^{-4}|\widehat{u_n}(w)|^2. \tag{10}$$

We will be taking the limit $n \to \infty$ and $\kappa \to 0$. First, they derive a clever bound. We have

$$|\widehat{\varphi_\kappa}(w) - 1| \leq C\kappa\|w\|,$$

for some $C > 0$ and all $w \in \mathbb{Z}^d$. We then use this to swiftly bound the first term in (10),

$$\begin{aligned}\sum_{w\in\mathbb{Z}_n^d\setminus\{0\}} \left(1-\widehat{\varphi_\kappa}(w)\right)\|w\|^{-4}|\widehat{u_n}(w)|^2 \leq C\kappa \sum_{w\in\mathbb{Z}_n^d\setminus\{0\}} \|w\|^{-3}|\widehat{u_n}(w)|^2 &\leq C\kappa \sum_{w\in\mathbb{Z}_n^d\setminus\{0\}} |\widehat{u_n}(w)|^2 \\ &\leq C\kappa\|u\|^2_{L^2(\mathbb{T}^d)} \to 0,\end{aligned}$$

as $\kappa \to 0$. For the second term now, we will first be taking the limit $n \to \infty$, and then the limit $\kappa \to 0$. Note that we now have our Fourier transformed mollifier in the summation, which goes to 0 rapidly, thus it justified to use the dominated convergence theorem to obtain

$$\lim_{n\to\infty} \sum_{w\in\mathbb{Z}_n^d\setminus\{0\}} \widehat{\varphi_\kappa}(w)\|w\|^{-4}|\widehat{u_n}(w)|^2 = \sum_{w\in\mathbb{Z}^d\setminus\{0\}} \widehat{\varphi_\kappa}(w)\|w\|^{-4}|\widehat{u}(w)|^2.$$

Now we have done away with the $\widehat{u_n}(w)$ terms. Moreover, we have the bound $|\widehat{\varphi_\kappa}(w)| \leq 1$ for all $w \in \mathbb{Z}^d$, so again we can bound the function in the summation by an integrable function. Using the dominated convergence theorem again, we find

$$\lim_{\kappa\to 0} \sum_{w\in\mathbb{Z}^d\setminus\{0\}} \widehat{\varphi_\kappa}(w)\|w\|^{-4}|\widehat{u}(w)|^2 = (u, \Delta^{-2}u)_{L^2(\mathbb{T}^d)}.$$

So by Theorem 2.2, we have now proven for any $u \in C^\infty(\mathbb{T}^d)$ with integral 0, that

$$\langle \Xi_n, u\rangle \xrightarrow{d} \langle \Xi, u\rangle,$$

with $\langle \Xi, u\rangle \sim \mathcal{N}(0, \|u\|^2_{-1})$. Assertion (P2) as in [5] is proven now, however we are not done yet, as we still have to show tightness of the sequence $(\Xi_n)_{n\in\mathbb{N}}$ in the space $\mathcal{H}_{-\varepsilon}(\mathbb{T}^d)$. We closely follow the proof from Section 4.2 in [5]. By Section 2.1 in [11], we have to find a compact $K \subset \mathcal{H}_{-\varepsilon}(\mathbb{T}^d)$ such that for all $n \in \mathbb{N}$ we have

$$\mathbb{P}(\Xi_n \in K) \geq 1 - \varepsilon.$$

Note that proving this is equivalent with showing that for all $\delta > 0$, we can find K compact such that

$$\sup_{n\in\mathbb{N}} \mathbb{P}(\Xi_n \notin K) \leq \delta. \tag{11}$$

Before we proceed, we state Rellich's Theorem (as can be found in [5] or [6]), which will turn out to be very useful in the next step of our proof:

Theorem 5.4. *For $k_1 < k_2$, the inclusion operator $H^{k_2}(\mathbb{T}^d) \hookrightarrow H^{k_1}(\mathbb{T}^d)$ is a compact operator. Equivalently, if $V \subset H^{k_2}(\mathbb{T}^d)$ is bounded, then $\overline{V} \subset H^{k_1}(\mathbb{T}^d)$ is compact.*

From Section 2 in [5] (or see Appendix 2.3), we know that for any a there is an isomorphism between $H^a(\mathbb{T}^d)$ and $\mathcal{H}_a(\mathbb{T}^d)$, so the above theorem will also apply for the spaces we are working with. The strategy for showing tightness then is the following: as $\mathcal{H}_{-\frac{\varepsilon}{2}}(\mathbb{T}^d) \hookrightarrow \mathcal{H}_{-\varepsilon}(\mathbb{T}^d)$ is a compact embedding, it suffices to find a bounded $K \subset \mathcal{H}_{-\frac{\varepsilon}{2}}(\mathbb{T}^d)$ such that $\sup_{n\in\mathbb{N}} \mathbb{P}(\Xi_n \notin K) \leq \delta$. Because if we have found such a K, then $\overline{K}$ is compact in $\mathcal{H}_{-\varepsilon}(\mathbb{T}^d)$, and the required condition (Equation 11) holds for $\overline{K}$.

We will show, as in the proof of (P1) in [5], that for any $\delta > 0$, there exists R_δ such that $\sup_{n\in\mathbb{N}} \mathbb{P}(\Xi_n \notin B_{\mathcal{H}_{-\frac{\varepsilon}{2}}}(0, R_\delta)) \leq \delta$. By Theorem 5.4, this is sufficient. Note that this is just equivalent to showing

$$\sup_{n\in\mathbb{N}} \mathbb{P}\left(\|\Xi_n\|_{\mathcal{H}_{-\frac{\varepsilon}{2}}} \geq R_\delta\right) \leq \delta.$$

We now use Markov's inequality to obtain,

$$\mathbb{P}\left(\|\Xi_n\|_{\mathcal{H}_{-\frac{\varepsilon}{2}}} \geq R_\delta\right) \leq \frac{\mathbb{E}\left[\|\Xi_n\|^2_{\mathcal{H}_{-\frac{\varepsilon}{2}}}\right]}{R_\delta^2}.$$

Putting sup's in front of everything above, we see that it is enough to show that $\mathbb{E}\left[\|\Xi_n\|^2_{\mathcal{H}_{-\frac{\varepsilon}{2}}}\right]$ is uniformly bounded in n. First of all,

$$\|\Xi_n\|^2_{L^2(\mathbb{T}^d)} = 16\pi^4 n^{d-4} \sum_{x,y\in\mathbb{T}^d} \left(\chi_{nx} - \min_{w\in\mathbb{Z}_n^d} \chi_w\right)\left(\chi_{ny} - \min_{w\in\mathbb{Z}_n^d} \chi_w\right).$$

So for n fixed, we have $\Xi_n \in L^2(\mathbb{T}^d)$ almost surely, as it is just a finite combination of Gaussians. Now, since $\Xi_n \in L^2(\mathbb{T}^d)$ it admits a Fourier representation, so we have

$$\|\Xi_n\|^2_{\mathcal{H}_{-\frac{\varepsilon}{2}}} = \sum_{\nu\in\mathbb{Z}^d\setminus\{0\}} \|\nu\|^{-2\varepsilon} \left|\widehat{\Xi_n}(\nu)\right|^2,$$

with

$$\widehat{\Xi_n}(\vartheta) = \int_{\mathbb{T}^d} \Xi_n(\vartheta)\mathbf{e}_\nu(\vartheta)\,\mathrm{d}\vartheta = 4\pi^2 \sum_{x\in\mathbb{T}_n^d} n^{\frac{d-4}{2}} \chi_{nx} \int_{B(x,\frac{1}{2n})} \mathbf{e}_\nu(\vartheta)\,\mathrm{d}\vartheta.$$

Now explicitly calculating the expectation,

$$\begin{aligned}\mathbb{E}\left[\|\Xi_n\|^2_{\mathcal{H}_{-\frac{\varepsilon}{2}}}\right] &= 16\pi^4 \sum_{\nu\in\mathbb{Z}^d\setminus\{0\}} \sum_{x,y\in\mathbb{T}_n^d} \|\nu\|^{-2\varepsilon} n^{d-4} \mathbb{E}[\chi_{nx}\chi_{ny}] \int_{B(x,\frac{1}{2n})} \mathbf{e}_\nu(\vartheta)\,\mathrm{d}\vartheta \int_{B(y,\frac{1}{2n})} \overline{\mathbf{e}_\nu(\vartheta)}\,\mathrm{d}\vartheta \\ &= 16\pi^4 \sum_{\nu\in\mathbb{Z}^d\setminus\{0\}} \sum_{x,y\in\mathbb{T}_n^d} \|\nu\|^{-2\varepsilon} n^{d-4} H(nx,ny) \int_{B(x,\frac{1}{2n})} \mathbf{e}_\nu(\vartheta)\,\mathrm{d}\vartheta \int_{B(y,\frac{1}{2n})} \overline{\mathbf{e}_\nu(\vartheta)}\,\mathrm{d}\vartheta.\end{aligned}$$

We write $F_{n,\nu} : \mathbb{T}_n^d \to \mathbb{R}$ for $F_{n,\nu}(x) := \int_{B(x,\frac{1}{2n})} \mathbf{e}_\nu(\vartheta)\ \mathrm{d}\vartheta$. Now $\mathbf{e}_\nu \in L^2(\mathbb{T}^d)$, so by Cauchy-Schwarz we have that $F_{n,\nu}$ is bounded, and therefore integrable on $\mathbb{T}_n^d$. We make the following claim: there exists $C' > 0$ such that

$$\sup_{\nu\in\mathbb{Z}^d}\sup_{n\in\mathbb{N}} \sum_{x,y\in\mathbb{T}_n^d} n^{d-4} H(nx,ny) F_{n,\nu}(x)\overline{F_{n,\nu}(y)} \le C'. \tag{12}$$

If this holds true, we find that (because $-\varepsilon < -\frac{d}{2}$),

$$\begin{aligned}\mathbb{E}\left[\|\Xi_n\|^2_{\mathcal{H}_{-\frac{\varepsilon}{2}}}\right] &= 16\pi^4 \sum_{\nu\in\mathbb{Z}^d\setminus\{0\}} \|\nu\|^{-2\varepsilon} \sum_{x,y\in\mathbb{T}_n^d} n^{d-4} H(nx,ny) F_{n,\nu}(x)\overline{F_{n,\nu}(y)} \\ &\le C' \sum_{k=1}^{\infty} k^{d-1-2\varepsilon} \le C < \infty.\end{aligned}$$

Since the last constant C does not depend on n, we have that the expectation is uniformly bounded in n, which is exactly what we needed. Now in proving (12), we will stray off the beaten path, as we believe we have found a shorter proof of this claim. First we denote $G_{n,\nu} : \mathbb{Z}_n^d \to \mathbb{R}$ for the rescaled version of $F_{n,\nu}$, set $G_{n,\nu}(\cdot) := F_{n,\nu}\left(\frac{\cdot}{n}\right)$. We have

$$\sum_{x,y\in\mathbb{T}_n^d} n^{d-4} H(nx,ny) F_{n,\nu}(x)\overline{F_{n,\nu}(y)} = C \sum_{x,y\in\mathbb{Z}_n^d} \sum_{w\in\mathbb{Z}_n^d\setminus\{0\}} n^{-4} \frac{\exp\left(2\pi\iota(x-y)\cdot\frac{w}{n}\right)}{\left(\sum_{i=1}^d \sin^2\left(\frac{\pi w_i}{n}\right)\right)^2} G_{n,\nu}(x)\overline{G_{n,\nu}(y)}. \tag{13}$$

From [5], and 5.3 before, we can bound $n^{-4}\left(\sum_{i=1}^d \sin^2\left(\frac{\pi w_i}{n}\right)\right)^{-2} \le C\|w\|^{-4}$ for some $C > 0$. Furthermore,

$$\begin{aligned}(13) &\le C \sum_{x,y\in\mathbb{Z}_n^d} \sum_{w\in\mathbb{Z}_n^d\setminus\{0\}} \frac{\exp\left(2\pi\iota(x-y)\cdot\frac{w}{n}\right)}{\|w\|^4} G_{n,\nu}(x)\overline{G_{n,\nu}(y)} \\ &= Cn^{2d} \sum_{w\in\mathbb{Z}_n^d\setminus\{0\}} \|w\|^{-4}\left|\widehat{G_{n,\nu}}(w)\right|^2 \le Cn^{2d} \sum_{w\in\mathbb{Z}_n^d} \left|\widehat{G_{n,\nu}}(w)\right|^2.\end{aligned}$$

Now using the Plancherel Theorem, we obtain the following bound for some $C' > 0$ (this is calculation (4.18) in [5]),

$$\begin{aligned}\sum_{w\in\mathbb{Z}_n^d} \left|\widehat{G_{n,\nu}}(w)\right|^2 &= n^{-d} \sum_{w\in\mathbb{Z}_n^d} G_{n,\nu}(w)\overline{G_{n,\nu}(w)} = n^{-d} \sum_{w\in\mathbb{T}_n^d} F_{n,\nu}(w)\overline{F_{n,\nu}(w)} \\ &\le n^{-2d} \sum_{w\in\mathbb{T}_n^d} \int_{B(w,\frac{1}{2n})} |\mathbf{e}_\nu(\vartheta)|\ \mathrm{d}\vartheta = n^{-2d} \int_{\mathbb{T}^d} |\mathbf{e}_\nu(\vartheta)|\ \mathrm{d}\vartheta \\ &\le n^{-2d} \|\mathbf{e}_\nu\|_{L^1(\mathbb{T}^d)} \le C' n^{-2d}.\end{aligned}$$

Here we have used, in the second to last line, that $|F_{n,\nu}(w)| \le n^{-d}$: this fact is easily proven as we are integrating a function bounded in modulus by 1, over an area of measure n^{-d}. Now putting all of this together, we have

$$(13) \le Cn^{2d} \sum_{w\in\mathbb{Z}_n^d} \left|\widehat{G_{n,\nu}}(w)\right|^2 \le Cn^{2d} C' n^{-2d} \le C.$$

This proves the tightness, and we are done.

5.6 Calculating the covariance of the random variables $\langle \Xi, f\rangle$

In the previous parts, we have shown the convergence of the real-valued random variables $\langle \Xi, u\rangle$. We obtained that for each $u \in C^\infty(\mathbb{T}^d)$, $\langle \Xi, u\rangle \sim \mathcal{N}(0, \|u\|_{-1}^2)$. Considering now $\{\langle \Xi, u\rangle : u \in H^{-1}(\mathbb{T}^d)\}$ as a collection of real-valued random variables, we might be interested in the covariance of any two $\langle \Xi, f\rangle$, $\langle \Xi, f'\rangle$. We can do this in the same way as the proof of 5.1, however here and there we will tweak the argument to make everything work out. We state and prove the following:

Theorem 5.5. *For any pair $f, f' \in C^\infty(\mathbb{T}^d)$ of mean zero smooth test functions, we have the following identity:*

$$\mathbb{E}[\langle \Xi, f\rangle\langle \Xi, f'\rangle] = (\Delta^{-1} f, \Delta^{-1} f')_{L^2(\mathbb{T}^d)}.$$

Proof. We find from [5], that

$$\langle \Xi_n, u\rangle = 4\pi^2 n^{-\frac{d+4}{2}} \sum_{z\in\mathbb{T}_n^d} \chi_{nz} u(z) + R_n(u).$$

Now,

$$\mathbb{E}[\langle \Xi_n, f\rangle\langle \Xi_n, f'\rangle] = 16\pi^4 n^{-(d+4)} \sum_{z,z'\in\mathbb{T}_n^d} f(z) f'(z')\mathbb{E}[\chi_{nz}\chi_{nz'}] + \mathcal{R}_n(f, f'). \tag{14}$$

We first show that the first term converges to the required result, in the end we will specify $\mathcal{R}_n(f, f')$ and show that it goes to 0 as $n \to \infty$. Recall the covariance from equation (3.3) in [5]. Plugging this into the above equation yields

$$\begin{aligned}(14) - \mathcal{R}_n(f, f') &= \pi^4 n^{-2d} n^{-4} \sum_{z,z'\in\mathbb{T}_n^d} f(z) f'(z') \sum_{w\in\mathbb{Z}_n^d\setminus\{0\}} \frac{\mathrm{e}^{2\pi\iota(z-z')\cdot w}}{\left(\sum_{i=1}^d \sin^2\left(\frac{\pi w_i}{n}\right)\right)^2} \\ &= \pi^4 n^{-4} \sum_{w\in\mathbb{Z}_n^d\setminus\{0\}} \frac{\widehat{f_n}(w)\overline{\widehat{f'_n}(w)}}{\left(\sum_{i=1}^d \sin^2\left(\frac{\pi w_i}{n}\right)\right)^2}. \end{aligned}\tag{15}$$

Here we have defined $f_n : \mathbb{Z}_n^d \to \mathbb{R}$ by $f_n(z) = f\left(\frac{z}{n}\right)$ for more convenient notation. Subsequently, we want to say something about the behaviour of the denominator as $n \to \infty$. We claim that it gets arbitrarily close to $\|w\|^{-4}$. Consider the following:

$$\left|(15) - \sum_{w\in\mathbb{Z}_n^d\setminus\{0\}} \frac{\widehat{f_n}(w)\overline{\widehat{f'_n}(w)}}{\|w\|^4}\right| \le \sum_{w\in\mathbb{Z}_n^d\setminus\{0\}} |\widehat{f_n}(w)\overline{\widehat{f'_n}(w)}| \left|\frac{\pi^4 n^{-4}}{\left(\sum_{i=1}^d \sin^2\left(\frac{\pi w_i}{n}\right)\right)^2} - \frac{1}{\|w\|^4}\right|.$$

We know from Lemma 7 in [5] that for some $C > 0$ and all $n \in \mathbb{N}$, we have

$$\left|\frac{\pi^4 n^{-4}}{\left(\sum_{i=1}^d \sin^2\left(\frac{\pi w_i}{n}\right)\right)^2} - \frac{1}{\|w\|^4}\right| \le \frac{C}{n^2}.$$

On the other hand, we claim that $\sum_{w\in\mathbb{Z}_n^d\setminus\{0\}} |\widehat{f_n}(w)\overline{\widehat{f'_n}(w)}|$ is uniformly bounded in n. First of all, by the Cauchy-Schwarz inequality

$$\begin{aligned}\left(\sum_{w\in\mathbb{Z}_n^d\setminus\{0\}}|\widehat{f_n}(w)\overline{\widehat{f_n'}(w)}|\right)^2 &\leq \left(\sum_{w\in\mathbb{Z}_n^d\setminus\{0\}}|\widehat{f_n}(w)|^2\right)\left(\sum_{w'\in\mathbb{Z}_n^d\setminus\{0\}}|\widehat{f_n'}(w')|^2\right)\\ &= \left(n^{-d}\sum_{w\in\mathbb{Z}_n^d}\left|f\left(\frac{w}{n}\right)\right|^2\right)\left(n^{-d}\sum_{w'\in\mathbb{Z}_n^d}\left|f'\left(\frac{w'}{n}\right)\right|^2\right).\end{aligned}$$

Now, $n^{-d}\sum_{w\in\mathbb{Z}_n^d}|f(w/n)|^2 \to \int_{\mathbb{T}^d} f(x)^2\ \mathrm{d}x < \infty$, so indeed the previous term is uniformly bounded in n. Subsequently,

$$\left|(15) - \sum_{w\in\mathbb{Z}_n^d\setminus\{0\}}\frac{\widehat{f_n}(w)\overline{\widehat{f_n'}(w)}}{\|w\|^4}\right| \leq Cn^{-2}\to 0.$$

It is now enough to consider only the convergence of

$$\sum_{w\in\mathbb{Z}_n^d\setminus\{0\}}\frac{\widehat{f_n}(w)\overline{\widehat{f_n'}(w)}}{\|w\|^4} = n^{-2d}\sum_{z,z'\in\mathbb{T}_n^d}f(z)f'(z')\sum_{w\in\mathbb{Z}_n^d\setminus\{0\}}\frac{\mathrm{e}^{2\pi\iota(z-z')\cdot w}}{\|w\|^4}. \tag{16}$$

Next we split our proof. The case $d \leq 3$ is more straightforward, since $\|w\|^{-4}$ is integrable on $\mathbb{Z}^d$. First of all, note that

$$\widehat{f_n}(w) = n^{-d}\sum_{z\in\mathbb{Z}_n^d}f(z)\mathrm{e}^{2\pi\iota w\cdot\frac{z}{n}} \to \int_{\mathbb{T}^d}f(x)\mathrm{e}^{2\pi\iota w\cdot x}\ \mathrm{d}x = \widehat{f}(w).$$

Using this fact, we can use the dominated convergence theorem in the same way as [5]. We find

$$(16) \to \sum_{w\in\mathbb{Z}_n^d\setminus\{0\}}\frac{\widehat{f}(w)\overline{\widehat{f'}(w)}}{\|w\|^4} = (\Delta^{-1}f,\Delta^{-1}f')_{L^2(\mathbb{T}^d)}.$$

Next consider $d \geq 4$. In this case another approach is necessary since $\|w\|^{-4}$ is not integrable on $\mathbb{Z}^d$. To this end, take any $\phi\in\mathcal{S}(\mathbb{R}^d)$ with support in $[-\frac{1}{2},\frac{1}{2})^d$ and $\int_{\mathbb{R}^d}\phi(x)\ \mathrm{d}x = 1$. For $\kappa > 0$ define $\phi_\kappa(x) := \kappa^{-1}\phi\left(\frac{x}{\kappa}\right)$. It is now sufficient to consider

$$\lim_{\kappa\to 0}\lim_{n\to\infty}\sum_{z,z'\in\mathbb{T}_n^d}f(z)f'(z')\sum_{w\in\mathbb{Z}_n^d\setminus\{0\}}\widehat{\phi_\kappa}(w)\frac{\mathrm{e}^{2\pi\iota(z-z')\cdot w}}{\|w\|^4},$$

as we claim

$$\lim_{\kappa\to 0}\limsup_{n\to\infty}n^{-2d}\sum_{z,z'\in\mathbb{T}_n^d}f(z)f'(z')\sum_{w\in\mathbb{Z}_n^d\setminus\{0\}}\left(\widehat{\phi_k}(w)-1\right)\frac{\mathrm{e}^{2\pi\iota(z-z')\cdot w}}{\|w\|^4} = 0.$$

To prove this, we recall from [5] that

$$\left|\widehat{\phi_\kappa}(w)-1\right| \leq C\kappa\|w\|.$$

Subsequently we have

$$\begin{aligned}\left| n^{-2d} \sum_{z,z' \in \mathbb{T}_n^d} f(z) f'(z') \sum_{w \in \mathbb{Z}_n^d \setminus \{0\}} \left(\widehat{\phi_k}(w) - 1 \right) \frac{\mathrm{e}^{2\pi\iota(z-z')\cdot w}}{\|w\|^4} \right| &= \left| \sum_{w \in \mathbb{Z}_n^d \setminus \{0\}} \frac{\widehat{\phi_\kappa}(w) - 1}{\|w\|^4} \widehat{f_n}(w) \overline{\widehat{f_n'}(w)} \right| \\ &\leq C\kappa \sum_{w \in \mathbb{Z}_n^d \setminus \{0\}} \|w\|^{-3} |\widehat{f_n}(w) \overline{\widehat{f_n'}(w)}| \\ &\leq C\kappa \sum_{w \in \mathbb{Z}_n^d \setminus \{0\}} |\widehat{f_n}(w) \overline{\widehat{f_n'}(w)}|.\end{aligned}$$

As we have already seen before, $\sum_{w \in \mathbb{Z}_n^d \setminus \{0\}} |\widehat{f_n}(w) \overline{\widehat{f_n'}(w)}| < \infty$ uniformly in n. Now letting $\kappa \to 0$ in the above expression gives the required result. We now look at the convergence of the other expression,

$$\lim_{\kappa \to 0} \lim_{n \to \infty} \sum_{z,z' \in \mathbb{T}_n^d} f(z) f'(z') \sum_{w \in \mathbb{Z}_n^d \setminus \{0\}} \widehat{\phi_\kappa}(w) \frac{\mathrm{e}^{2\pi\iota(z-z')\cdot w}}{\|w\|^4}.$$

Since ϕ is a smooth function, transformed $\widehat{\phi_\kappa}(w)$ decays fast at infinity, resolving the issue with the integrability of $\|w\|^{-4}$. We can now apply the dominated convergence theorem:

$$\lim_{n \to \infty} n^{-2d} \sum_{z,z' \in \mathbb{T}_n^d} f(z) f'(z') \sum_{w \in \mathbb{Z}_n^d \setminus \{0\}} \widehat{\phi_\kappa}(w) \frac{\mathrm{e}^{2\pi\iota(z-z')\cdot w}}{\|w\|^4} = \sum_{w \in \mathbb{Z}^d \setminus \{0\}} \widehat{\phi_\kappa}(w) \frac{\widehat{f}(w) \overline{\widehat{f'}(w)}}{\|w\|^4}.$$

Now since $|\widehat{\phi_\kappa}(\cdot)| \leq 1$ uniformly in κ, we again utilize the dominated convergence theorem to obtain

$$\lim_{\kappa \to 0} \sum_{w \in \mathbb{Z}^d \setminus \{0\}} \widehat{\phi_\kappa}(w) \frac{\widehat{f}(w) \overline{\widehat{f'}(w)}}{\|w\|^4} = \sum_{w \in \mathbb{Z}^d \setminus \{0\}} \frac{\widehat{f}(w) \overline{\widehat{f'}(w)}}{\|w\|^4} = (\Delta^{-1} f, \Delta^{-1} f')_{L^2(\mathbb{T}^d)}.$$

It now rests us to prove that $\mathcal{R}_n(f, f') \to 0$ as $n \to \infty$. The rest term is given by

$$\mathcal{R}_n(f, f') = \mathbb{E}\left[R_n(f') 4\pi^2 n^{-\frac{(d+4)}{2}} \sum_{z \in \mathbb{T}_n^d} f(z) \chi_{nz} \right] + \mathbb{E}\left[R_n(f) 4\pi^2 n^{-\frac{(d+4)}{2}} \sum_{z' \in \mathbb{T}_n^d} f'(z') \chi_{nz'} \right] + \mathbb{E}[R_n(f) R_n(f')].$$

For the first and second term, we apply Cauchy-Schwarz and see:

$$\begin{aligned}\left(\mathbb{E}\left[R_n(f') 4\pi^2 n^{-\frac{(d+4)}{2}} \sum_{z \in \mathbb{T}_n^d} f(z) \chi_{nz} \right] \right)^2 &\leq \left(\mathbb{E}[R_n^2(f)] \right) \mathbb{E}\left(4\pi^2 n^{-\frac{(d+4)}{2}} \sum_{z \in \mathbb{T}_n^d} f(z) \chi_{nz} \right)^2 \\ &\leq C \cdot \mathbb{E}[R_n^2(f)] \|f\|_{-1}^2 \to 0.\end{aligned}$$

The convergence to 0 follows from Proposition 6 in [5], which states that $\mathbb{E}[R_n^2(f)] \to 0$. For the last term, we use Cauchy-Schwarz again:

$$\mathbb{E}[R_n(f) R_n(f)]^2 \leq \mathbb{E}[R_n^2(f)] \mathbb{E}[R_n^2(f')] \to 0.$$

Lastly, note that for large $n \in \mathbb{N}$ there exists $C > 0$ such that

$$\mathbb{E}[\langle \Xi_n, f \rangle \langle \Xi_n, f' \rangle]^2 \leq \mathbb{E}[\langle \Xi_n, f \rangle^2] \mathbb{E}[\langle \Xi_n, f' \rangle^2] \leq C \|f\|_{-1}^2 \|f'\|_{-1}^2.$$

Now by the dominated convergence theorem (again), we have

$$\mathbb{E}[\langle \Xi, f\rangle\langle \Xi, f'\rangle] = \lim_{n\to\infty} \mathbb{E}[\langle \Xi_n, f\rangle\langle \Xi_n, f'\rangle] = (\Delta^{-1}f, \Delta^{-1}f')_{L^2(\mathbb{T}^d)}.$$

□

5.7 Brief summary

We will now give a brief overview of what we have seen so far. Recall our definition of the sandpile on $\mathbb{Z}_n^d$. We had initial configuration given by,

$$s(x) = 1 + \sigma(x) - \frac{1}{n^d}\sum_{z\in\mathbb{Z}_n^d}\sigma(z),$$

where the $(\sigma(x))_{x\in\mathbb{Z}_n^d}$ are i.i.d. standard normals. In the previous section, we have already seen that the odometer $e_n : \mathbb{Z}_n^d \to \mathbb{R}$ is distributed as

$$e_n(\cdot) \overset{d}{=} \left(\eta - \min_{z\in\mathbb{Z}_n^d}\eta(z)\right),$$

with η correlated, centered Gaussians. An interesting observation now comes looking at the covariance of $(\eta(x))_{x\in\mathbb{Z}_n^d}$, we find (here Δ_g denotes the discrete Laplacian, as can be found in the previous section),

$$\Delta_g^2\mathbb{E}[\eta(x)\eta(y)] = \delta_x(y) - \frac{1}{n^d}.$$

In some sense, our covariance structure approximates an inverse bilaplacian. It can also be thought of as a Green's function for the discrete bilaplacian. The question now is whether our field is still bilaplacian if we take the limit $n \to \infty$ and consider our sandpile on $\frac{1}{n}\mathbb{Z}_n^d$. To answer this question, we walked through the proof of Theorem 1 in [5], step by step. We have defined a rescaled version of the odometer, Ξ_n, and we have seen that our limiting field is defined as a distribution, $\Xi = \{\langle \Xi, u\rangle : u \in H^{-1}(\mathbb{T}^d)\}$, with

$$\mathbb{E}\left[\langle \Xi, u\rangle^2\right] = (u, \Delta^{-2}u)_{L^2(\mathbb{T}^d)}.$$

The inverse bilaplacian Δ^{-2} was defined through the Fourier coefficients, as $\Delta \sim \|\xi\|^2$, we have $\Delta^{-2} \sim \|\xi\|^{-4}$. To rigorously prove the convergence $\Xi_n \overset{d}{\to} \Xi$, we had to check two things:

1. $\langle \Xi_n, u\rangle \overset{d}{\to} \langle \Xi, u\rangle$ for all $u \in C^\infty(\mathbb{T}^d)$, as a sequence of real-valued random variables.
2. The sequence $(\Xi_n)_{n\in\mathbb{N}}$ is tight in the space $\mathcal{H}_{-\varepsilon}(\mathbb{T}^d)$.

The first statement was proven by first observing that each $\langle \Xi_n, u\rangle$ is Gaussian, since it is a finite linear combination of Gaussian random variables. Moreover, we have seen that $\mathbb{E}[\langle \Xi_n, u\rangle] = 0$. We have seen that for a sequence of Gaussians $(X_n)_{n\in\mathbb{N}}$ with mean $\mathbb{E}[X_n] = m_n$ such that $m_n \to m < \infty$, and $\mathbb{V}\mathrm{ar}(X_n) = \sigma_n^2 \to \sigma^2 < \infty$, the following convergence holds

$$X_n \overset{d}{\to} X,$$

with $X \sim \mathcal{N}(m, \sigma^2)$. Thus to show the first assertion we have proven that

$$\mathbb{E}\left[\langle \Xi_n, u\rangle^2\right] \to (u, \Delta^{-2}u)_{L^2(\mathbb{T}^d)}.$$

Because the first moments are always 0, this was sufficient. Now, to prove the tightness, we had to find, for any $\delta > 0$, a compact $K \subset \mathcal{H}_{-\varepsilon}(\mathbb{T}^d)$ such that for all $n \in \mathbb{N}$,

$$\mathbb{P}(\Xi_n \notin K) \leq \delta.$$

We have found this K with a bit of a detour, we have used Rellich's Theorem to first find a bounded set $V \in \mathcal{H}_{-\frac{\varepsilon}{2}}(\mathbb{T}^d)$, such that $\mathbb{P}(\Xi_n \notin V) \leq \delta$ holds. Since $\mathcal{H}_{-\frac{\varepsilon}{2}}(\mathbb{T}^d) \hookrightarrow \mathcal{H}_{-\varepsilon}(\mathbb{T}^d)$ is a compact embedding, we have that $\overline{V} \subset \mathcal{H}_{-\varepsilon}(\mathbb{T}^d)$. So now we have a compact $K := \overline{V} \subset \mathcal{H}_{-\varepsilon}(\mathbb{T}^d)$, with $\mathbb{P}(\Xi_n \notin K) \leq \delta$. This proves the second statement, so we are done.

After this, we used the same techniques as described above in proving that for any two $f, f' \in C^\infty(\mathbb{T}^d)$, we have

$$\mathbb{E}\left[\langle \Xi, f \rangle \langle \Xi, f' \rangle\right] = (\Delta^{-1} f, \Delta^{-1} f')_{L^2(\mathbb{T}^d)}.$$

In this result we can see on the left hand side the $L^2(\Omega)$ inner product between the Gaussians $\langle \Xi, f \rangle, \langle \Xi, f' \rangle$, and on the right hand side the $H^{-1}(\mathbb{T}^d)$ inner product between the test functions $f, f' \in C^\infty(\mathbb{T}^d)$.

6 Scaling Limit for Correlated Gaussians

6.1 Introduction

In the previous section, we started with a divisible sandpile with Gaussian i.i.d. weights $(\sigma(x))_{x\in\mathbb{Z}_n^d}$. We saw that a rescaling of the odometer e_n, called Ξ_n, converges in law to the bilaplacian field Ξ. We characterized this field in the following way: for any $u \in C^\infty(\mathbb{T}^d)$, we have that $\langle \Xi, u\rangle$ is a Gaussian with $\langle \Xi, u\rangle \sim \mathcal{N}(0, \|u\|_{-1}^2)$. In this section, we consider the odometer e_n for a divisible sandpile given by weights $(\sigma(x))_{x\in\mathbb{Z}_n^d}$, where the weights are correlated by a certain covariance function $K(x, y)$,

$$\mathbb{E}[\sigma(x)\sigma(y)] = K(x, y).$$

The most natural assumption to make on such a covariance function K is that it depends on the distance between x and y. We thus set $K(x, y) = \mathcal{K}(x - y)$ for some even function $\mathcal{K} : \mathbb{Z}_n^d \to \mathbb{R}$. This is called a **stationary covariance function**. We will then determine the convergence of the rescaled odometer by using the same techniques as [5].

6.2 Covariance functions on $\mathbb{Z}_n^d$

There are, as ever, some challenges in defining a proper covariance function in our case. The first problem is that our graph is changing for each n, so we will be dealing with a sequence of covariance functions $(\mathcal{K}_n)_{n\in\mathbb{N}}$, where for all n, we have $\mathcal{K}_n : \mathbb{Z}_n^d \to \mathbb{R}$. In some way, these covariance functions need to be related to each other. One way to define our sequence $(\mathcal{K}_n)_{n\in\mathbb{N}}$ is by first considering $\mathcal{K} : \mathbb{T}^d \to \mathbb{R}$ with $\mathcal{K}(0)$. We will subsequently define for every $n \in \mathbb{N}$ and $x, y \in \mathbb{Z}_n^d$,

$$\mathbb{E}[\sigma(x)\sigma(y)] = \mathcal{K}_n(x - y) := \mathcal{K}\left(\frac{x}{n} - \frac{y}{n}\right).$$

In this way, we are discretizing the torus again, and considering our $\mathcal{K}_n$ on discretizations of $\mathbb{T}^d$ that get more refined as n grows. However, the problem with this definition is that we can not generate an i.i.d. collection of Gaussians. To do this, we would like $\mathcal{K}(x - y) = 1$ if $x = y$ and $\mathcal{K}(x - y) = 0$ if $x \neq y$. The only function that satisfies this is $\mathcal{K} : \mathbb{T}^d \to \mathbb{R}$ defined by $\mathcal{K}(z) = 1_{\{z=0\}}$ for all $z \in \mathbb{T}^d$. Seeing as we do most of our analysis in $L^1(\mathbb{T}^d)$ and $L^2(\mathbb{T}^d)$ the function $1_{\{z=0\}}$ is identified exactly with 0, and will thus disappear under Fourier transforms or convolutions.

Another way to construct a covariance function that is defined on $\mathbb{Z}_n^d$ for all $n \in \mathbb{N}$ is to define $\mathcal{K} : \mathbb{Z}^d \to \mathbb{R}$ such that $(x, y) \mapsto \mathcal{K}(x - y)$ is positive definite on $\mathbb{Z}_n^d \times \mathbb{Z}_n^d$ for every $n \in \mathbb{N}$. We also require $\|\mathcal{K}\|_{\ell^1(\mathbb{Z}^d)} < \infty$. We subsequently define $\mathcal{K}_n : \mathbb{Z}_n^d \to \mathbb{R}$ by $\mathcal{K}_n = \mathcal{K}|_{\mathbb{Z}_n^d}$. An example of such a $\mathcal{K}$ is given by

$$\mathcal{K}(x - y) = \begin{cases} \|x - y\|^{-\alpha}, & \text{for } x \neq y \\ C, & \text{for } x = y. \end{cases},$$

where $C > \sum_{z\in\mathbb{Z}_n^d} \|z\|^{-\alpha}$. First of all note that $(z, z') \mapsto \mathcal{K}_n(x - y)$ is positive definite for all $n \in \mathbb{Z}_n^d$. Choose any $n \in \mathbb{N}$. We have $\mathcal{K}_n(z - z') = \|z - z'\|^{-\alpha} = \|z' - z\|^{-\alpha} = \mathcal{K}_n(z' - z)$, so $\mathcal{K}_n$ is symmetric. Second, the matrix defined by $\mathcal{K}_n$ is strictly diagonally dominant, as we have $\mathcal{K}_n(z - z) \geq \sum_{z'\in\mathbb{Z}_n^d} \mathcal{K}_n(z - z')$ by the way $C = \mathcal{K}_n(0)$ was defined. Putting all this together proves that $\mathcal{K}$ is positive definite for any restriction on $\mathbb{Z}_n^d$. This time, we *can* define an i.i.d. collection, by simply setting $\mathcal{K}(z) = 1_{\{z=0\}}$, as this is indeed a well-defined function on $\mathbb{Z}^d$.

6.3 Main results

In this section we will prove the following theorem:

Theorem 6.1. *Let $e_n : \mathbb{Z}_n^d \to \mathbb{R}$ be the odometer associated with the weights $(\sigma(x))_{x\in\mathbb{Z}_n^d}$, which have covariance given $\mathcal{K} : \mathbb{Z}^d \to \mathbb{R}$ as described in Section 6.2. Set $\mathcal{C}_\mathcal{K} = \sum_{w\in\mathbb{Z}^d} \mathcal{K}(w)$ and define for $\mathcal{C}_\mathcal{K} \neq 0$,*

$$\Xi_n^\mathcal{K}(x) = \mathcal{C}_\mathcal{K}^{-1/2} 4\pi^2 \sum_{z\in\mathbb{T}_n^d} n^{\frac{d-4}{2}} e_n(nz) 1_{B(z,\frac{1}{2n})}(x), \quad x \in \mathbb{T}^d.$$

Then $\Xi_n^\mathcal{K} \xrightarrow{d} \Xi$ in $\mathcal{H}_{-\varepsilon}(\mathbb{T}^d)$ for $\varepsilon > \max\{1 + \frac{d}{4}, \frac{d}{2}\}$, where Ξ is the bilaplacian field: we have for all mean zero $u \in C^\infty(\mathbb{T}^d)$ that $\langle \Xi, u\rangle \sim \mathcal{N}(0, \|u\|_{-1}^2)$.

Note that we are using the same definition of Ξ_n as in [5], but that we now added a scaling factor $\mathcal{C}_\mathcal{K}^{-1/2}$. This constant is necessary as we will see in the next few examples. We consider the field Ξ_n without the appropriate scaling factor (so without the extra $\mathcal{C}_\mathcal{K}^{-1/2}$), under different kinds of covariances. We will indeed see that the larger $\mathcal{C}_\mathcal{K}$, the larger the peaks of our limiting field.

Example 1: We consider a divisible sandpile on $\mathbb{Z}_n^2$ with correlation given by

$$\mathcal{K}(x-y) = \begin{cases} 7, & \text{for } x = y \\ -\|x-y\|^{-3}, & \text{for } x \neq y. \end{cases}.$$

In this case, the factor 7 is chosen such that $\mathcal{C}_\mathcal{K}$ is very small. In our case we have

$$\mathcal{C}_\mathcal{K} \approx 0.48,$$

when the summation is considered on $\mathbb{Z}_{100}^2$. We can now run a simulation of the scaling limit, which will look as in Figure 13.

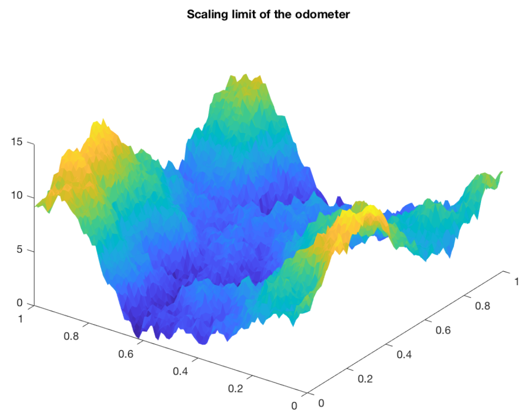

Figure 13: Scaling limit of the odometer under negative covariances, $\mathcal{K}$ is as specified above.

We remark that in the i.i.d. case, the surface frequently fluctuates at a height of around 20, with the minimum still being 0. In our case now, our peaks have a height of about 15. Our idea of a “rescaled” bilaplacian field seems to make sense then.

Example 2: We now consider a sandpile with covariance function

$$\mathcal{K}(x-y) = \begin{cases} 7, & \text{for } x = y \\ \|x-y\|^{-3}, & \text{for } x \neq y. \end{cases} .$$

Now we have

$$\mathcal{C}_{\mathcal{K}} \approx 13.5,$$

so we expect our surface to have higher peaks than in the i.i.d. case. Simulating the sandpile with this covariance, indeed confirms our expectations. See Figure 14. Note that the peaks of our surface almost hit a height of 50.

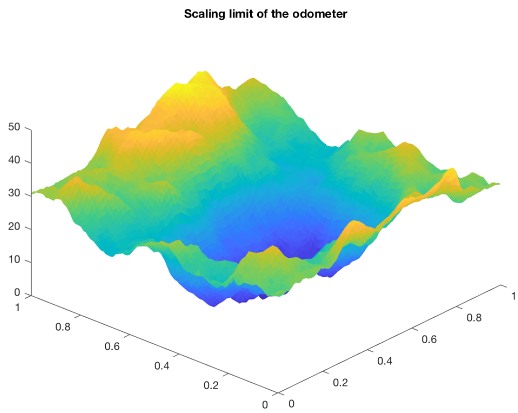

Figure 14: Scaling limit of the odometer under positive covariance $\mathcal{K}$ as above.

6.4 Proof of Theorem 6.1

We will prove Theorem 6.1 in a few steps. We will first derive a useful formula for the covariance of the odometer $e_n(\cdot)$ on $\mathbb{Z}_n^d$, similar to Equation (3.3) in [5]. After this we will use this to determine the scaling limit, and at last we will show the tightness of our sequence $(\Xi_n)_{n\in\mathbb{N}}$ in the space $\mathcal{H}_{-\varepsilon}(\mathbb{T}^d)$. Recall from Theorem 3.3 that the odometer had distribution

$$e_n(\cdot) = \left(\eta - \min_{z\in\mathbb{Z}_n^d} \eta(z)\right),$$

with (for $\mathcal{K}_n$ our covariance on $\mathbb{Z}_n^d$),

$$\mathbb{E}[\eta(x)\eta(y)] = \frac{1}{(2d)^2} \sum_{z,z'\in\mathbb{Z}_n^d} \mathcal{K}_n(z-z')g(z,x)g(z',y). \tag{17}$$

We recall that the eigenvalues of the Laplacian Δ are given by ($\xi \neq 0$):

$$\lambda_\xi = -4\sum_{i=1}^{d} \sin^2\left(\pi\frac{\xi_i}{n}\right).$$

Then:

Theorem 6.2. *Let $e_n : \mathbb{Z}_n^d \to \mathbb{R}$ be the odometer associated with the stationary covariance functions $(\mathcal{K}_n)_{n\in\mathbb{N}}$ as above. Then*

$$e_n \stackrel{d}{=} \left(\chi - \min_{z\in\mathbb{Z}_n^d} \chi_z\right).$$

Here $(\chi_z)_{z\in\mathbb{Z}_n^d}$ is a collection of centered Gaussians with covariance

$$\mathbb{E}[\chi_x\chi_y] = \sum_{\xi\in\mathbb{Z}_n^d\setminus\{0\}} \widehat{\mathcal{K}_n}(\xi)\frac{\exp\left(2\pi\iota(x-y)\cdot\frac{\xi}{n}\right)}{\lambda_\xi^2} =: \mathcal{H}_n(x,y).$$

Proof. As we have remarked above, the covariance of the field $(\eta(z))_{z\in\mathbb{Z}_n^d}$ associated with the odometer is given by

$$\mathbb{E}[\eta(x)\eta(x)] = \frac{1}{(2d)^2}\sum_{z,z'\in\mathbb{Z}_n^d} \mathcal{K}_n(z-z')g(z,x)g(z',y).$$

We denote $g_x(\cdot) = g(\cdot, x)$. Subsequently we utilize the Plancherel theorem to find

$$\begin{aligned}\frac{1}{(2d)^2}\sum_{z,z'\in\mathbb{Z}_n^d} \mathcal{K}_n(z-z')g(z,y)g(z',x) &= \frac{1}{(2d)^2}\sum_{z\in\mathbb{Z}_n^d} g(z,y)\sum_{z'\in\mathbb{Z}_n^d}\mathcal{K}_n(z-z')g(z',x)\\ &= \frac{n^d}{(2d)^2}\sum_{z\in\mathbb{Z}_n^d} g(z,y)\sum_{\xi\in\mathbb{Z}_n^d}\exp\left(2\pi\iota z\cdot\frac{\xi}{n}\right)\widehat{\mathcal{K}_n}(\xi)\overline{\widehat{g_x}(\xi)}.\end{aligned}$$

Here we have used the fact that $\mathcal{K}_n(z-z') = \mathcal{K}_n(z'-z)$, and then used the shift theorem. Next, swapping the order of summation,

$$\begin{aligned}(\ldots) &= \frac{n^d}{(2d)^2}\sum_{\xi\in\mathbb{Z}_n^d}\left(\sum_{z\in\mathbb{Z}_n^d} g(z,y)\exp\left(2\pi\iota z\cdot\frac{\xi}{n}\right)\right)\widehat{\mathcal{K}_n}(\xi)\overline{\widehat{g_x}(\xi)}\\ &= \frac{n^{2d}}{(2d)^2}\sum_{\xi\in\mathbb{Z}_n^d}\widehat{\mathcal{K}_n}(\xi)\widehat{g_y}(\xi)\overline{\widehat{g_x}(\xi)}\\ &= \frac{n^{2d}}{(2d)^2}\widehat{\mathcal{K}_n}(0)\widehat{g_y}(0)\overline{\widehat{g_x}(0)} + \frac{n^{2d}}{(2d)^2}\sum_{\xi\in\mathbb{Z}_n^d\setminus\{0\}}\widehat{\mathcal{K}_n}(\xi)\widehat{g_y}(\xi)\overline{\widehat{g_x}(\xi)}.\end{aligned}$$

We find $\widehat{g_x}(0) = \sum_{z\in\mathbb{Z}_n^d} g(z,x)$, which does not depend on x. In this way, we see that the first term is constant, and thus vanishes in a similar way as in the proof of Proposition 1.3 of [1] and the proof of Proposition 4 in [5]. Considering now the second term above, we recall Equation (20) in [1], for $\xi \neq 0$,

$$\widehat{g_x}(\xi) = -2dn^{-d}\lambda_\xi^{-1}\exp\left(-2\pi\iota\xi\cdot\frac{x}{n}\right).$$

Plugging this into the above, we obtain

$$\mathbb{E}[\chi_x\chi_y] = \sum_{\xi\in\mathbb{Z}_n^d\setminus\{0\}}\widehat{\mathcal{K}_n}(\xi)\frac{\exp\left(2\pi\iota(x-y)\cdot\frac{\xi}{n}\right)}{\lambda_\xi^2}.$$

Now we still need to show the positive-definiteness of $\mathcal{H}_n(x,y)$. For any function $c : \mathbb{Z}_n^d \to \mathbb{C}$, such that c is not the zero function, we show that $\sum_{x,y\in\mathbb{Z}_n^d} \mathcal{H}_n(x,y)c(x)\overline{c(y)} > 0$. First of all, since $\mathcal{K}_n(z-z')$ is positive definite, we have

$$0 < \sum_{z,z'\in\mathbb{Z}_n^d} \mathcal{K}_n(z-z')c(z)\overline{c(z')} = \sum_{z\in\mathbb{Z}_n^d} c(z) \sum_{z'\in\mathbb{Z}_n^d} \mathcal{K}_n(z-z')\overline{c(z')}. \tag{18}$$

In the same way as above, we obtain

$$0 < [18] = n^{2d} \sum_{\xi\in\mathbb{Z}_n^d} \widehat{\mathcal{K}_n}(\xi)\widehat{c}(\xi)\overline{\widehat{c}(\xi)} = n^{2d} \sum_{\xi\in\mathbb{Z}_n^d} \widehat{\mathcal{K}_n}(\xi)|\widehat{c}(\xi)|^2.$$

Since this needs to hold for all such functions $c(x)$, we conclude that $\widehat{\mathcal{K}_n}(\xi) > 0$ for all $\xi \in \mathbb{Z}_n^d$. Next we find

$$\begin{aligned}
\sum_{x,y\in\mathbb{Z}_n^d} \mathcal{H}_n(x,y)c(x)\overline{c(y)} &= \sum_{x,y\in\mathbb{Z}_n^d} c(x)\overline{c(y)} \sum_{\xi\in\mathbb{Z}_n^d\setminus\{0\}} \widehat{\mathcal{K}_n}(\xi)\frac{\mathrm{e}^{2\pi\iota(x-y)\cdot\frac{\xi}{n}}}{\lambda_\xi^2} \\
&= n^{2d} \sum_{\xi\in\mathbb{Z}_n^d\setminus\{0\}} \frac{\widehat{\mathcal{K}_n}(\xi)}{\lambda_\xi^2}\widehat{c}(\xi)\overline{\widehat{c}(\xi)} \\
&= n^{2d} \sum_{\xi\in\mathbb{Z}_n^d\setminus\{0\}} \frac{\widehat{\mathcal{K}_n}(\xi)}{\lambda_\xi^2}|\widehat{c}(\xi)|^2 > 0.
\end{aligned}$$

Since $\lambda_\xi^2 > 0$ the positive definiteness of $\mathcal{H}_n(x,y)$ follows, as we are just summing positive, real terms. □

We are now ready to prove Theorem 6.1. We will use the same techniques as the proof of Theorem 1 in [5], and as described in Section 5. We remark first that

$$\begin{aligned}
\widehat{\mathcal{K}_n}(\xi) &= n^{-d} \sum_{w\in\mathbb{Z}_n^d} \mathcal{K}_n(w) \exp\left(2\pi\iota w \cdot \frac{\xi}{n}\right) \\
&= n^{-d} \sum_{w\in\mathbb{Z}_n^d} \mathcal{K}_n(w) 1_{w\in\mathbb{Z}_n^d} \exp\left(2\pi\iota w \cdot \frac{\xi}{n}\right).
\end{aligned}$$

The proof is again split up in two parts, we first prove the convergence of moments, i.e.

$$\mathbb{E}\left[\langle \Xi_n^{\mathcal{K}}, u\rangle^2\right] \to \|u\|_{-1}^2.$$

In the next section, we prove tightness of $(\Xi_n^{\mathcal{K}})_{n\in\mathbb{N}}$ in $\mathcal{H}_{-\varepsilon}(\mathbb{T}^d)$. This then concludes the proof of Theorem 6.1.

Proof. As we have $\Xi_n^{\mathcal{K}} = \mathcal{C}_{\mathcal{K}}^{-1/2}\Xi_n$, we can recycle the first calculations in the proof of Theorem 1 in [5] to obtain

$$\langle \Xi_n^{\mathcal{K}}, u\rangle = \mathcal{C}_{\mathcal{K}}^{-1/2} 4\pi^2 n^{-\frac{d+4}{2}} \sum_{z\in\mathbb{T}_n^d} \chi_{nz} u(z) + \mathcal{C}_{\mathcal{K}}^{-1/2} R_n(u).$$

In the same way, we can split the variance

$$\mathbb{E}[\langle \Xi_n^{\mathcal{K}}, u\rangle^2] = 16\pi^4 \mathcal{C}_{\mathcal{K}}^{-1} n^{-(d+4)} \sum_{z,z'\in\mathbb{T}_n^d} u(z)u(z')\mathbb{E}[\chi_{nz}\chi_{nz'}] + \mathcal{C}_{\mathcal{K}}^{-1}\mathbb{E}[R_n^2(u)]$$
$$+4\pi^2 \mathcal{C}_{\mathcal{K}}^{-1}\mathbb{E}\left[n^{-\frac{d+4}{2}} \sum_{z\in\mathbb{T}_n^d} u(z)\chi_{nz} R_n(u)\right].$$

For the first term, we find

$$\begin{aligned}
&\mathcal{C}_{\mathcal{K}}^{-1} 16\pi^4 n^{-(d+4)} \sum_{z,z'\in\mathbb{T}_n^d} u(z)u(z')\mathbb{E}[\chi_{nz}\chi_{nz'}] \\
&= \mathcal{C}_{\mathcal{K}}^{-1} 16\pi^4 n^{-(d+4)} \sum_{z,z'\in\mathbb{T}_n^d} u(z)u(z')n^{-d} \sum_{\xi\in\mathbb{Z}_n^d\setminus\{0\}} \left(\sum_{w\in\mathbb{Z}^d} \mathcal{K}(w)1_{w\in\mathbb{Z}_n^d} \exp\left(2\pi\iota w\cdot \frac{\xi}{n}\right)\right) \frac{\exp\left(2\pi\iota(z-z')\cdot\frac{\xi}{n}\right)}{\lambda_\xi^2} \\
&= \mathcal{C}_{\mathcal{K}}^{-1} 16\pi^4 n^{-4} \sum_{\xi\in\mathbb{Z}_n^d\setminus\{0\}} \lambda_\xi^{-2} \left(\sum_{w\in\mathbb{Z}^d} \mathcal{K}(w)1_{w\in\mathbb{Z}_n^d} \exp\left(2\pi\iota w\cdot \frac{\xi}{n}\right)\right) \widehat{u_n}(\xi)\overline{\widehat{u_n}(\xi)}.
\end{aligned}$$

Where we have used the same notation as before, with $u_n : \mathbb{Z}_n^d \to \mathbb{R}$ defined by $u_n(z) := u\left(\frac{z}{n}\right)$. We will now show convergence of the above in three steps; first we will show that the summation in w is uniformly bounded and converges to $\mathcal{C}_{\mathcal{K}}$, then we will show that the other terms are also uniformly bounded, at last we will split the proof for dimensions $d \leq 3$ and $d \geq 4$. First of all, we have by assumption that $\mathcal{K} \in \ell^1(\mathbb{Z}^d)$, so

$$\left|\sum_{w\in\mathbb{Z}^d} \mathcal{K}(w)1_{w\in\mathbb{Z}_n^d} \exp\left(2\pi\iota w\cdot\frac{\xi}{n}\right)\right| \leq \sum_{w\in\mathbb{Z}^d} |\mathcal{K}(w)|1_{w\in\mathbb{Z}_n^d} \leq \sum_{w\in\mathbb{Z}^d} |\mathcal{K}(w)| < \infty.$$

Furthermore for each $w \in \mathbb{Z}^d$ we have $\left|\mathcal{K}(w)1_{w\in\mathbb{Z}_n^d}\exp\left(2\pi\iota w\cdot\frac{\xi}{n}\right)\right| \leq |\mathcal{K}(w)|$. We introduce the notation $\Sigma_n(\mathcal{K}) := \sum_{w\in\mathbb{Z}^d} \mathcal{K}(w)1_{w\in\mathbb{Z}_n^d}\exp\left(2\pi\iota w\cdot\frac{\xi}{n}\right)$, and use the dominated convergence theorem to find

$$\begin{aligned}
\lim_{n\to\infty}\Sigma_n(\mathcal{K}) = \lim_{n\to\infty}\sum_{w\in\mathbb{Z}^d} \mathcal{K}(w)1_{w\in\mathbb{Z}_n^d}\exp\left(2\pi\iota w\cdot\frac{\xi}{n}\right) &= \sum_{w\in\mathbb{Z}^d}\lim_{n\to\infty}\mathcal{K}(w)1_{w\in\mathbb{Z}_n^d}\exp\left(2\pi\iota w\cdot\frac{\xi}{n}\right) \\
&= \sum_{w\in\mathbb{Z}^d}\mathcal{K}(w) = \mathcal{C}_{\mathcal{K}}.
\end{aligned}$$

We remark here that since $\mathcal{K}_n : \mathbb{Z}_n^d \to \mathbb{R}$ is even and positive-definite, all Fourier coefficients are real-valued and ≥ 0. It then follows that, since $\Sigma_n(\mathcal{K}) = n^d\widehat{\mathcal{K}_n}(\xi)$ for some ξ, it is real valued and ≥ 0 as well. Now, as we have already seen, there exists some $C > 0$ such that

$$\left|\frac{16\pi^4 n^{-4}}{\lambda_\xi^2} - \frac{1}{\|\xi\|^4}\right| \leq Cn^{-2}.$$

Again, we find

$$\left|\mathcal{C}_{\mathcal{K}}^{-1}16\pi^4 n^{-4}\sum_{\xi\in\mathbb{Z}_n^d\setminus\{0\}}\lambda_\xi^{-2}\Sigma_n(\mathcal{K})\widehat{u_n}(\xi)\overline{\widehat{u_n}(\xi)}-\mathcal{C}_{\mathcal{K}}^{-1}\sum_{\xi\in\mathbb{Z}_n^d\setminus\{0\}}\|\xi\|^{-4}\Sigma_n(\mathcal{K})\widehat{u_n}(\xi)\overline{\widehat{u_n}(\xi)}\right|$$
$$\leq \mathcal{C}_{\mathcal{K}}^{-1}Cn^{-2}\sum_{\xi\in\mathbb{Z}_n^d\setminus\{0\}}|\Sigma_n(\mathcal{K})||\widehat{u_n}(\xi)|^2$$
$$\leq \mathcal{C}_{\mathcal{K}}^{-1}Cn^{-2}\|\mathcal{K}\|_{\ell^1(\mathbb{Z}^d)}\sum_{\xi\in\mathbb{Z}_n^d\setminus\{0\}}|\widehat{u_n}(\xi)|^2\to 0.$$

Here we have used the fact from [5] that $\sum_{\xi\in\mathbb{Z}_n^d\setminus\{0\}}|\widehat{u_n}(\xi)|^2$ is uniformly bounded in n. The above computation shows that it is now enough to compute the limit of

$$\mathcal{C}_{\mathcal{K}}^{-1}\sum_{\xi\in\mathbb{Z}_n^d\setminus\{0\}}\|\xi\|^{-4}\Sigma_n(\mathcal{K})\widehat{u_n}(\xi)\overline{\widehat{u_n}(\xi)}.$$

In dimension $d\leq 3$, we have that $\|\xi\|^{-4}$ is integrable on $\mathbb{Z}^d$, and both $|\widehat{u_n}(\xi)|^2$ and $\Sigma_n(\mathcal{K})$ are uniformly bounded above, we can interchange limit and integral to find

$$\lim_{n\to\infty}\mathcal{C}_{\mathcal{K}}^{-1}\sum_{\xi\in\mathbb{Z}_n^d\setminus\{0\}}\|\xi\|^{-4}\Sigma_n(\mathcal{K})\widehat{u_n}(\xi)\overline{\widehat{u_n}(\xi)}=\lim_{n\to\infty}\mathcal{C}_{\mathcal{K}}^{-1}\sum_{\xi\in\mathbb{Z}^d\setminus\{0\}}1_{\xi\in\mathbb{Z}_n^d}\|\xi\|^{-4}\Sigma_n(\mathcal{K})\widehat{u_n}(\xi)\overline{\widehat{u_n}(\xi)}$$
$$=\mathcal{C}_{\mathcal{K}}^{-1}\sum_{\xi\in\mathbb{Z}^d}\|\xi\|^{-4}\mathcal{C}_{\mathcal{K}}|\widehat{u}(\xi)|^2=\sum_{\xi\in\mathbb{Z}^d}\|\xi\|^{-4}|\widehat{u}(\xi)|^2.$$

In the case $d\geq 4$ we can again use a mollifying procedure. For $\phi\in\mathcal{S}(\mathbb{R}^d)$ supported on $[-\frac{1}{2},\frac{1}{2}]^d$ with integral 1, we set $\phi_\kappa(x):=\kappa^{-d}\phi\left(\frac{x}{\kappa}\right)$ for $\kappa>0$. Recall our previous technique, we show first that

$$\lim_{\kappa\to 0}\limsup_{n\to\infty}\mathcal{C}_{\mathcal{K}}^{-1}\sum_{\xi\in\mathbb{Z}_n^d\setminus\{0\}}\|\xi\|^{-4}\left(1-\widehat{\phi_\kappa}(\xi)\right)\Sigma_n(\mathcal{K})|\widehat{u_n}(\xi)|^2=0. \tag{19}$$

As in Equation (4.11) from [5], we have

$$\left|\widehat{\phi_\kappa}(\xi)-1\right|\leq C\kappa\|\xi\|.$$

We then plug this into Equation [19], to find

$$\left|\mathcal{C}_{\mathcal{K}}^{-1}\sum_{\xi\in\mathbb{Z}_n^d\setminus\{0\}}\|\xi\|^{-4}\left(1-\widehat{\phi_\kappa}(\xi)\right)\Sigma_n(\mathcal{K})|\widehat{u_n}(\xi)|^2\right| \leq \mathcal{C}_{\mathcal{K}}^{-1}C\kappa\sum_{\xi\in\mathbb{Z}_n^d\setminus\{0\}}\|\xi\|^{-3}|\Sigma_n(\mathcal{K})||\widehat{u_n}(\xi)|^2$$
$$\leq \mathcal{C}_{\mathcal{K}}^{-1}C\kappa\|\mathcal{K}\|_{\ell^1(\mathbb{Z}^d)}\|u\|_{\mathbb{T}^d}^2.$$

Letting $\kappa\to 0$ in the above expression gives the convergence to 0. For the other term, note that $\widehat{\phi_\kappa}$ has fast decay, $\widehat{u_n}(\xi)\to\widehat{u}(\xi)$ for all $\xi\in\mathbb{Z}^d$ and $|\Sigma_n(\mathcal{K})|\leq\|\mathcal{K}\|_{\ell^1(\mathbb{Z}^d)}<\infty$. Applying now the dominated convergence theorem in n,

$$\lim_{n\to\infty}\mathcal{C}_{\mathcal{K}}^{-1}\sum_{\xi\in\mathbb{Z}^d\setminus\{0\}}1_{\xi\in\mathbb{Z}_n^d}\widehat{\phi_\kappa}(\xi)\|\xi\|^{-4}\Sigma_n(\mathcal{K})|\widehat{u_n}(\xi)|^2 = \mathcal{C}_{\mathcal{K}}^{-1}\mathcal{C}_{\mathcal{K}}\sum_{\xi\in\mathbb{Z}^d\setminus\{0\}}\widehat{\phi_\kappa}(\xi)\|\xi\|^{-4}|\widehat{u}(\xi)|^2$$
$$=\sum_{\xi\in\mathbb{Z}^d\setminus\{0\}}\widehat{\phi_\kappa}(\xi)\|\xi\|^{-4}|\widehat{u}(\xi)|^2.$$

From [5] we know that taking the limit $\kappa \to 0$ in the above gives that

$$\lim_{\kappa\to 0} \sum_{\xi\in\mathbb{Z}^d\setminus\{0\}} \widehat{\phi_\kappa}(\xi)\|\xi\|^{-4}|\widehat{u}(\xi)|^2 = \|u\|_{-1}^2.$$

At last we compute $\mathbb{E}[R_n^2(u)]$, so

$$\begin{aligned}
\mathbb{E}[R_n^2(u)] &= 16\pi^4 n^{-2d} \sum_{z,z'\in\mathbb{T}_n^d} n^{d-4}\mathcal{H}_n(nz,nz')K_n(z)K_n(z') \\
&\le Cn^{-2d} \sum_{z,z'\in\mathbb{T}_n^d} \sum_{\xi\in\mathbb{Z}_n^d\setminus\{0\}} \Sigma_n(\mathcal{K})\frac{\exp(2\pi\iota(z-z')\cdot\xi)}{\|\xi\|^4}K_n(z)K_n(z') \\
&\le Cn^{-2d} \sum_{z,z'\in\mathbb{T}_n^d} \sum_{\xi\in\mathbb{Z}_n^d\setminus\{0\}} \Sigma_n(\mathcal{K})\exp(2\pi\iota(z-z')\cdot\xi)K_n(z)K_n(z'),
\end{aligned}$$

since $\|\xi\| \ge 1$. Next, set $K_n'(x) := K_n\left(\frac{x}{n}\right)$. Now we have, thanks to calculations on page 14 of [5],

$$\sum_{\xi\in\mathbb{Z}_n^d\setminus\{0\}} \Sigma_n(\mathcal{K})\widehat{K_n'}(\xi)\overline{\widehat{K_n'}(\xi)} \le \mathcal{C}_\mathcal{K} C n^{-2} \to 0.$$

Then $R_n(u) \to 0$ in L^2. □

6.5 Tightness in $\mathcal{H}_{-\varepsilon}$

To complete the proof, we show that the convergence in law $\Xi_n^\mathcal{K} \xrightarrow{d} \Xi$ holds in the Sobolev space $\mathcal{H}_{-\varepsilon}(\mathbb{T}^d)$ for any $\varepsilon > \max\{1+\frac{d}{4}, \frac{d}{2}\}$. We state the following theorem:

Theorem 6.3. *Define $\Xi_n^\mathcal{K}$ as above. Then the sequence $(\Xi_n^\mathcal{K})_{n\in\mathbb{N}}$ is tight in $\mathcal{H}_{-\varepsilon}(\mathbb{T}^d)$, in other words, for all $\delta > 0$ there exists $R_\delta > 0$ such that*

$$\sup_{n\in\mathbb{N}} \mathbb{P}\left(\|\Xi_n^\mathcal{K}\|_{\mathcal{H}_{-\frac{\varepsilon}{2}}} \ge R_\delta\right) \le \delta.$$

Proof. The proof of this theorem is analogous to the proof of tightness in [5]. We first apply Markov's inequality and see

$$\mathbb{P}\left(\|\Xi_n^\mathcal{K}\|_{\mathcal{H}_{-\frac{\varepsilon}{2}}} \ge R_\delta\right) \le \frac{\mathbb{E}\left[\|\Xi_n^\mathcal{K}\|_{\mathcal{H}_{-\frac{\varepsilon}{2}}}^2\right]}{R_\delta^2}.$$

Now whenever we have

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\|\Xi_n^\mathcal{K}\|_{\mathcal{H}_{-\frac{\varepsilon}{2}}}^2\right] \le C,$$

the assertion follows as we can choose R_δ such that

$$\mathbb{P}\left(\|\Xi_n^\mathcal{K}\|_{\mathcal{H}_{-\frac{\varepsilon}{2}}} \ge R_\delta\right) \le \frac{C}{R_\delta^2} < \delta.$$

Next, observe that for $n \in \mathbb{N}$ fixed

$$\|\Xi_n^\mathcal{K}\|_{L^2(\mathbb{T}^d)}^2 = \mathcal{C}_\mathcal{K}^{-1} 16\pi^4 n^{d-4} \sum_{x,y\in\mathbb{T}^d} \left(\chi_{nx} - \min_{w\in\mathbb{Z}_n^d}\chi_w\right)\left(\chi_{ny} - \min_{w\in\mathbb{Z}_n^d}\chi_w\right) < \infty \text{ a.s.}$$

Now since $\Xi_n^{\mathcal{K}} \in L^2(\mathbb{T}^d)$ it has Fourier expansion, and we write $\Xi_n^{\mathcal{K}}(\vartheta) = \sum_{\nu\in\mathbb{Z}^d} \widehat{\Xi_n^{\mathcal{K}}}(\nu)\mathbf{e}_\nu(\vartheta)$, with

$$\widehat{\Xi_n^{\mathcal{K}}}(\nu) = \int_{\mathbb{T}^d} \Xi_n^{\mathcal{K}}(\vartheta)\mathbf{e}_\nu(\vartheta)\,\mathrm{d}\vartheta = \mathcal{C}_{\mathcal{K}}^{-1/2}4\pi^2 \sum_{x\in\mathbb{T}_n^d} n^{\frac{d-4}{2}}\chi_{nx}\int_{B(x,\frac{1}{2n})}\mathbf{e}_\nu(\vartheta)\,\mathrm{d}\vartheta.$$

We can then write

$$\|\Xi_n^{\mathcal{K}}\|^2_{\mathcal{H}_{-\frac{\varepsilon}{2}}} = \sum_{\nu\in\mathbb{Z}^d\setminus\{0\}} \|\nu\|^{-2\varepsilon}|\widehat{\Xi_n^{\mathcal{K}}}(\nu)|^2.$$

Furthermore we explicitly calculate the expectation, we obtain

$$\begin{aligned}\mathbb{E}\left[\|\Xi_n^{\mathcal{K}}\|^2_{\mathcal{H}_{-\frac{\varepsilon}{2}}}\right] &= \mathcal{C}_{\mathcal{K}}^{-1}16\pi^4 \sum_{\nu\in\mathbb{Z}_n^d\setminus\{0\}}\sum_{x,y\in\mathbb{T}_n^d}\|\nu\|^{-2\varepsilon}n^{d-4}\mathbb{E}[\chi_{nx}\chi_{ny}]\int_{B(x,\frac{1}{2n})}\mathbf{e}_\nu(\vartheta)\,\mathrm{d}\vartheta\int_{B(y,\frac{1}{2n})}\overline{\mathbf{e}_\nu(\vartheta)}\,\mathrm{d}\vartheta\\ &= \mathcal{C}_{\mathcal{K}}^{-1}16\pi^4 \sum_{\nu\in\mathbb{Z}_n^d\setminus\{0\}}\sum_{x,y\in\mathbb{T}_n^d}\|\nu\|^{-2\varepsilon}n^{d-4}H(nx,ny)\int_{B(x,\frac{1}{2n})}\mathbf{e}_\nu(\vartheta)\,\mathrm{d}\vartheta\int_{B(y,\frac{1}{2n})}\overline{\mathbf{e}_\nu(\vartheta)}\,\mathrm{d}\vartheta.\end{aligned}$$

Now define $F_{n,\nu} : \mathbb{T}_n^d \to \mathbb{R}$ as $F_{n,\nu}(x) := \int_{B(x,\frac{1}{2n})}\mathbf{e}_\nu(\vartheta)\,\mathrm{d}\vartheta$. We have that both $1_{B(x,\frac{1}{2n})}, \mathbf{e}_\nu \in L^2(\mathbb{T}^d)$ so by Cauchy-Schwarz $F_{n,\nu} \in L^1(\mathbb{T}^d)$. Next we claim that for some $C' > 0$,

$$\sup_{\nu\in\mathbb{Z}^d}\sup_{n\in\mathbb{N}}\left|\sum_{x,y\in\mathbb{T}_n^d} n^{d-4}H(nx,ny)F_{n,\nu}(x)\overline{F_{n,\nu}(y)}\right| \le C'. \tag{20}$$

Remark that in the same way as bound (4.5) in [5], we have $|n^{-4}\lambda_\xi^{-2}| \le C\|\xi\|^{-4}$ for some $C > 0$. We write $G_{n,\nu} : \mathbb{Z}_n^d \to \mathbb{R}$ for $G_{n,\nu}(z) := F_{n,\nu}\left(\frac{\cdot}{n}\right)$. Using this, we find

$$\begin{aligned}\left|\sum_{x,y\in\mathbb{T}_n^d} n^{d-4}H(nx,ny)F_{n,\nu}(x)\overline{F_{n,\nu}(y)}\right| &= \left|\sum_{x,y\in\mathbb{T}_n^d} n^{-4}\sum_{\xi\in\mathbb{Z}_n^d\setminus\{0\}}\Sigma_n(\mathcal{K})\frac{\exp(2\pi\iota(x-y)\cdot\xi)}{\lambda_\xi^2}F_{n,\nu}(x)\overline{F_{n,\nu}(y)}\right|\\ &= \left|\sum_{\xi\in\mathbb{Z}_n^d\setminus\{0\}} n^{-4}\lambda_\xi^{-2}\Sigma_n(\mathcal{K})n^{2d}|\widehat{G_{n,\nu}}(\xi)|^2\right|\\ &\le Cn^{2d}\|\mathcal{K}\|_{\ell^1(\mathbb{Z}^d)}\sum_{\xi\in\mathbb{Z}_n^d\setminus\{0\}}\|\xi\|^{-4}|\widehat{G_{n,\nu}}(\xi)|^2.\end{aligned}$$

Here we have again exploited the fact that $|\Sigma_n(\mathcal{K})| \le \|\mathcal{K}\|_{\ell^1(\mathbb{Z}^d)}$. Now by the triangle inequality,

$$|F_{n,\nu}(w)| = \left|\int_{B(w,\frac{1}{2n})}\mathbf{e}_\nu(\vartheta)\,\mathrm{d}\vartheta\right| \le \int_{B(w,\frac{1}{2n})}\mathrm{d}\vartheta = n^{-d}.$$

Now,

$$\begin{aligned}\sum_{\xi\in\mathbb{Z}_n^d\setminus\{0\}}\|\xi\|^{-4}|\widehat{G_{n,\nu}}(\xi)|^2 &\le \sum_{\xi\in\mathbb{Z}_n^d}|\widehat{G_{n,\nu}}(\xi)|^2 = n^{-d}\sum_{z\in\mathbb{Z}_n^d}G_{n,\nu}(z)\overline{G_{n,\nu}(z)}\\ &= n^{-d}\sum_{z\in\mathbb{T}_n^d}F_{n,\nu}(z)\overline{F_{n,\nu}(z)} \le n^{-2d}\sum_{z\in\mathbb{T}_n^d}\int_{B(z,\frac{1}{2n})}|\mathbf{e}_\nu(\vartheta)|\,\mathrm{d}\vartheta\\ &\le n^{-2d}\|\mathbf{e}_\nu\|_{L^1(\mathbb{T}^d)} = Cn^{-2d}.\end{aligned}$$

We then use this bound to obtain

$$Cn^{2d}\|\mathcal{K}\|_{\ell^1(\mathbb{Z}^d)}\sum_{\xi\in\mathbb{Z}_n^d\setminus\{0\}}\|\xi\|^{-4}|\widehat{G_{n,\nu}}(\xi)|^2\le Cn^{2d}\|\mathcal{K}\|_{\ell^1(\mathbb{Z}^d)}n^{-2d}=C\|\mathcal{K}\|_{\ell^1(\mathbb{Z}^d)}.$$

This is a constant that does not depend on n or ν, so the claim (20) is proven. Using the claim now, we have by the Euler-Maclaurin formulas

$$\begin{aligned}\mathbb{E}\left[\left\|\Xi_n^{\mathcal{K}}\right\|^2_{\mathcal{H}_{-\frac{\varepsilon}{2}}}\right]&=\mathcal{C}_{\mathcal{K}}^{-1}16\pi^4\sum_{\nu\in\mathbb{Z}^d\setminus\{0\}}\|\nu\|^{-2\varepsilon}\sum_{x,y\in\mathbb{T}_n^d}n^{d-4}H(nx,ny)F_{n,\nu}(x)\overline{F_{n,\nu}(y)}\\&\le\mathcal{C}_{\mathcal{K}}^{-1}C'\sum_{k\ge1}k^{d-1-2\varepsilon}\le C.\end{aligned}$$

The last estimate here is due to the fact that $-\varepsilon<-\frac{d}{2}$. □

6.6 Comparing the maxima of correlated odometers

As we have seen in the simulations earlier, and as we would expect from Theorem 6.1, the maximum of the Gaussian field with large $\mathcal{C}_{\mathcal{K}}$, is larger than the maximum of the Gaussian field with small $\mathcal{C}_{\mathcal{K}}$. In this subsection we will investigate whether we can say something about the maximum of the odometer under different kinds of covariance functions. To this end, let $(\sigma(x))_{x\in\mathbb{Z}_n^d}$ be a collection of correlated Gaussians. We consider a divisible sandpile $s:\mathbb{Z}_n^d\to\mathbb{R}$ given by

$$s(x)=1+\sigma(x)-\frac{1}{n^d}\sum_{z\in\mathbb{Z}_n^d}\sigma(z).$$

We are concerned with the maximum of the odometer $e:\mathbb{Z}_n^d\to\mathbb{R}$ under different covariance functions. To this end, we state and prove the following theorem:

Theorem 6.4. *Let $\mathcal{K}^+$ and $\mathcal{K}^-$ be two stationary covariance functions on $\mathbb{Z}_n^d$ such that $\mathcal{K}^+-\mathcal{K}^-$ is still positive definite. Define e_+ to be the odometer associated with $\mathcal{K}^+$ and e_- the odometer associated with $\mathcal{K}^-$. Then*

$$\mathbb{E}\left[(e_-)^*\right]\le\mathbb{E}\left[(e_+)^*\right].$$

The proof of this theorem is based on the Sudakov-Fernique inequality:

Theorem 6.5. *Let X,Y be $n\times1$ Gaussian vectors with $\mathbb{E}[X_i]=\mathbb{E}[Y_i]$ for all $1\le i\le n$, and assume that $\mathbb{E}\left[(X_i-X_j)^2\right]\le\mathbb{E}\left[(Y_i-Y_j)^2\right]$ for all $i\neq j$. Then*

$$\mathbb{E}[X^*]\le\mathbb{E}[Y^*].$$

Using this theorem, we will now prove 6.4:

Proof. As we have already seen,

$$e_\pm\overset{d}{=}\left(\eta_\pm-\min_{z\in\mathbb{Z}_n^d}\eta_\pm(z)\right),$$

with

$$\mathbb{E}[\eta_\pm(x)\eta_\pm(y)]=\frac{1}{(2d)^2}\sum_{z,z'\in\mathbb{Z}_n^d}\mathcal{K}^\pm(z-z')g(z,x)g(z',y).\tag{21}$$

Now, since η is a centered random variable, we have

$$\mathbb{E}\left[(e_\pm)^*\right] = \mathbb{E}\left[\eta_\pm^* - \min_{z\in\mathbb{Z}_n^d} \eta_\pm(z)\right] = 2\mathbb{E}\left[\eta_\pm^*\right].$$

In this way we see that in order to prove Theorem 6.4, it is enough to only consider the maxima of the Gaussian fields $(\eta_+(x))_{x\in\mathbb{Z}_n^d}$ and $(\eta_-(x))_{x\in\mathbb{Z}_n^d}$, where we have again used the convenient notation of associating η_+ with $\mathcal{K}^+$ and η_- with $\mathcal{K}^-$. In other words, proving Theorem 6.4 is equivalent with proving

$$\mathbb{E}\left[(\eta_-)^*\right] \le \mathbb{E}\left[(\eta_+)^*\right].$$

By Theorem 6.5 it is sufficient to show now that for all $x, y \in \mathbb{Z}_n^d$, $x \ne y$, the following holds:

$$\mathbb{E}\left[(\eta_-(x) - \eta_-(y))^2\right] \le \mathbb{E}\left[(\eta_+(x) - \eta_+(y))^2\right]. \tag{22}$$

We can explicitly calculate these expectations using 21. We obtain

$$\begin{aligned}\mathbb{E}\left[(\eta_-(x) - \eta_-(y))^2\right] &= \mathbb{E}[\eta_-(x)^2] - 2\mathbb{E}[\eta_-(x)\eta_-(y)] + \mathbb{E}[\eta_-(y)^2] \\ &= 2\left(\mathbb{E}[\eta_-(x)^2] - \mathbb{E}[\eta_-(x)\eta_-(y)]\right) \\ &= \frac{2}{(2d)^2} \sum_{z,z'\in\mathbb{Z}_n^d} \mathcal{K}^-(z-z')g(z,x)\left(g(z',x) - g(z',y)\right).\end{aligned}$$

We find a similar result for η_+. Next, we show 22 by showing

$$\sum_{z,z'\in\mathbb{Z}_n^d} \mathcal{K}^-(z-z')g(z,x)(g(z',x) - g(z',y)) \le \sum_{z,z'\in\mathbb{Z}_n^d} \mathcal{K}^+(z-z')g(z,x)(g(z',x) - g(z',y)).$$

To this end, note that this is equivalent with

$$0 \le \sum_{z,z'\in\mathbb{Z}_n^d} \left(\mathcal{K}^+ - \mathcal{K}^-\right)(z-z')g(z,x)(g(z',x) - g(z',y)),$$

which is in turn equivalent to showing

$$\sum_{z,z'\in\mathbb{Z}_n^d} \left(\mathcal{K}^+ - \mathcal{K}^-\right)(z-z')g(z,x)g(z',y) \le \sum_{z,z'\in\mathbb{Z}_n^d} \left(\mathcal{K}^+ - \mathcal{K}^-\right)(z-z')g(z,x)g(z',x). \tag{23}$$

Now define $\mathcal{K} = \mathcal{K}^+ - \mathcal{K}^-$. By assumption, this is positive definite and thus a well-defined covariance function. Now let $(\chi(x))_{x\in\mathbb{Z}_n^d}$ be the field associated with the odometer under covariance $\mathcal{K}$. As we have already seen before, then

$$\mathbb{E}[\chi(x)\chi(y)] = \frac{1}{(2d)^2} \sum_{z,z'\in\mathbb{Z}_n^d} \mathcal{K}(z-z')g(z,x)g(z',y).$$

In this case, we see that 23 simplifies to

$$\mathbb{E}[\chi(x)\chi(y)] \le \mathbb{E}[\chi(x)^2],$$

which holds by the Cauchy-Schwarz inequality and because $\mathbb{E}[\chi(x)^2] = \mathbb{E}[\chi(y)^2]$, and so the proof is finished. $\square$

Example: We consider $\mathcal{K}^- = I$, in other words, an i.i.d. initial distribution, and define

$$\mathcal{K}^+(x-y) = \begin{cases} C, & \text{for } x-y=0 \\ \|x-y\|^{-(d+1)} & \text{for } x-y \neq 0 \end{cases},$$

where we have chosen C large enough such that $\mathcal{K}^+ - I$ is positive definite. Now we have that

$$\mathcal{K}^+ - \mathcal{K}^- = \begin{cases} C-1, & \text{for } x-y=0 \\ \|x-y\|^{-(d+1)} & \text{for } x-y \neq 0 \end{cases},$$

and this is positive definite. So here we have

$$\mathbb{E}\left[(e_-)^*\right] \leq \mathbb{E}\left[(e_+)^*\right].$$

Remark 6.6(i): This theorem is really not as powerful as we would like it to be. The difficulty here is that we would like to have $\mathcal{K}^+ \geq \mathcal{K}^-$ in some sense, however this is not as easy it seems on first sight. A common ordering on the collection of positive definite matrices is the Löwner ordering, which is defined as

$$A \succeq B \text{ if and only if } A - B \text{ is positive-definite.}$$

We have used this ordering here as well. One might think on first sight that if $A \geq B$ pointwise, and A, B are both positive definite, that this would imply that $A - B$ is positive definite as well, but in general this is not the case, and in fact rules out a lot of covariance structures that we would like to compare.
Remark 6.6(ii): Another problem is that in the Sudakov-Fernique inequality, we are not dealing with variables that have equal second moment. If we look at the above example again, we have found that the odometer in the positively correlated case has a larger expected maximum than in the i.i.d. case, but is this really a consequence of the random variables $(\sigma(x))_{x\in\mathbb{Z}_n^d}$ being correlated? To make $\mathcal{K}^+$ a positive definite matrix, the diagional elements (i.e., $\mathbb{E}[(\sigma(x)^2])$ need to be sufficiently large compared to the covariances. If we compare now (for $C > 1$)

$$\mathcal{K}^+(x-y) = C1_{x=y}$$

with the i.i.d. case, we see that the expected maximum is still larger than the expected maximum for the i.i.d. case. We see now that the expected maximum might have more to do with the variances themselves, than with the pairwise correlation. An interesting problem to study in the future then would be whether Theorem 6.4 also holds under weaker assumptions.

6.7 Brief summary

Recall that in previous section, we started out with a sandpile configuration given by the i.i.d. weights $(\sigma(x))_{x\in\mathbb{Z}_n^d}$. We viewed $\mathbb{Z}_n^d$ as a discretization of the torus $\mathbb{T}^d$, and saw that after sufficient scaling, the odometer $e_n : \mathbb{Z}_n^d \to \mathbb{R}$ converges to the bilaplacian field Ξ. For all $u \in C^\infty(\mathbb{T}^d)$ we have that $\langle \Xi, u\rangle$ is a centered Gaussian with

$$\mathbb{E}\left[\langle \Xi, u\rangle^2\right] = (u, \Delta^{-2}u)_{L^2(\mathbb{T}^d)} =: \|u\|_{-1}^2.$$

In this section, we were concerned with the scaling limit of the odometer given an initial distribution of correlated Gaussians, for each $n \in \mathbb{N}$,

$$\mathbb{E}[\sigma(x)\sigma(y)] = \mathcal{K}_n(x-y).$$

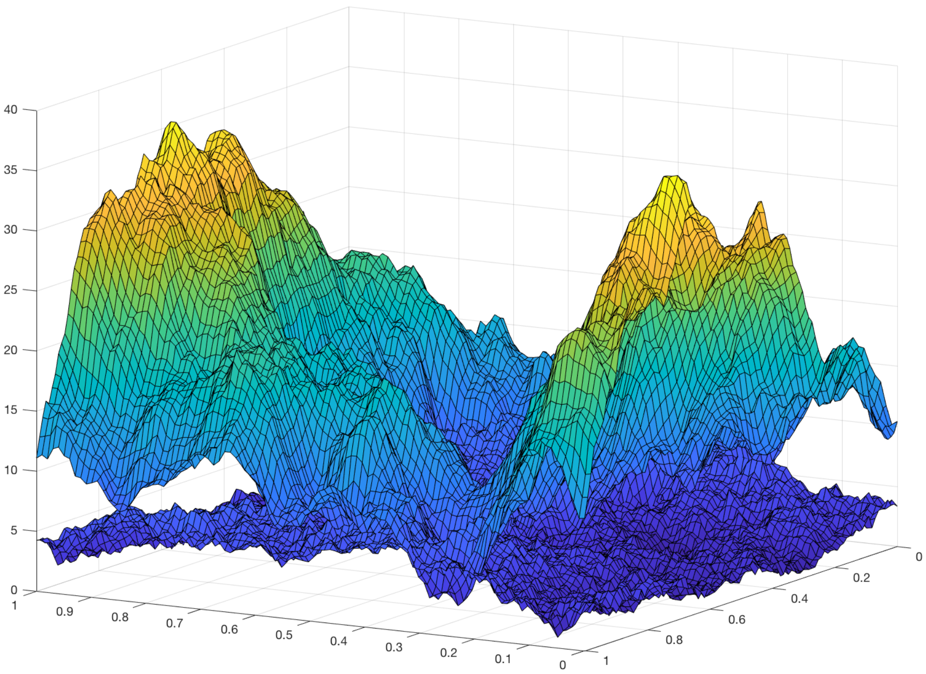

Figure 15: Comparison of the odometers of correlations $\mathcal{K}^{\pm} = \pm\|x-y\|^{-3}$ for $x \neq y$ and $\mathcal{K}^{\pm} = C$ for $x = y$. We observe that the $\mathcal{K}^{+}$ surface indeed lies a lot higher than the $\mathcal{K}^{-}$ surface.

We defined $\mathcal{K} : \mathbb{Z}^d \to \mathbb{R}$ as an even function, and for each $n \in \mathbb{N}$ we set $\mathcal{K}_n := \mathcal{K}|_{\mathbb{Z}_n^d}$. We first derived a useful identity for distribution of the odometer for fixed n. We had

$$e_n(\cdot) \overset{d}{=} \left(\chi - \min_{z \in \mathbb{Z}_n^d}\right),$$

with $(\chi_z)_{z\in\mathbb{Z}_n^d}$ centered, correlated Gaussian with covariance

$$\mathbb{E}[\chi_x \chi_y] = \sum_{\xi \in \mathbb{Z}_n^d \setminus \{0\}} \widehat{\mathcal{K}_n}(\xi) \frac{\exp\left(2\pi\iota(x-y)\cdot\frac{\xi}{n}\right)}{\lambda_\xi^2}.$$

Subsequently, we took the scaling limit with an extra factor $\mathcal{C}_{\mathcal{K}}^{-1/2}$ and found that in some sense, $\widehat{\mathcal{K}_n}(\xi)$ "behaves" like

$$n^{-d} \sum_{w\in\mathbb{Z}^d} \mathcal{K}(w) \sim n^{-d}\mathcal{C}_{\mathcal{K}}.$$

Now the n^{-d} factor is the same factor as in 5.2, and the $\mathcal{C}_{\mathcal{K}}$ comes in front of the scaling limit, as it does not depend on ξ anymore as $n \to \infty$. This then cancels out the $\mathcal{C}_{\mathcal{K}}^{-1}$. After this, we have used the same techniques as [5] to prove that our rescaled field $\Xi_n^{\mathcal{K}}$ converges in distribution to

Ξ in $\mathcal{H}_{-\varepsilon}(\mathbb{T}^d)$, here Ξ is again the bilaplacian field, we have for all $u \in C^\infty(\mathbb{T}^d)$ that $\langle \Xi, u \rangle$ is a centered Gaussian with

$$\mathbb{E}\left[\langle \Xi, u \rangle^2\right] = \|u\|_{-1}^2.$$

Finally, we have compared the maxima of odometers under different kinds of covariance structures.

7 Further research

For further research, two main questions arise. First of all, we have seen that for correlated Gaussians, after an extra scaling factor, the odometer converges again to the bilaplacian field $\langle \Xi, u \rangle \sim \mathcal{N}(0, \|u\|_{-1}^2)$. The question now is to what other types of random variables we can extend this result. In this section we will look at the techniques used in [5], and see if we can use these in proving a scaling limit result for a more general setting. Not surprisingly, there are a few issues, otherwise this would not be in the "Further research" section. Second, we are very interested in the stabilization speed of the sandpile with Gaussian weights.

7.1 Scaling limit for bounded, correlated random variables

We recall Theorem 11 in [5]:

Theorem 7.1. *Assume $(\sigma(x))_{x\in\mathbb{Z}_n^d}$ is a collection of i.i.d. variables with $\mathbb{E}[\sigma] = 0$ and $\mathbb{E}[\sigma^2] = 1$. Moreover, assume there exists $K < \infty$ such that $|\sigma| < K$ almost surely. Let $d \geq 1$ and $e_n(\cdot)$ the corresponding odometer. Then the formal field Ξ_n, defined as in the Gaussian case, converges in law to Ξ on $\mathbb{T}^d$. The convergence holds on the same fashion as the convergence for the i.i.d. Gaussian case.*

The strategy used to prove this theorem is the method of moments. We will attempt to do the same, and outline where the problems are in correlated case. In fact we will be able to show the convergence of the first and second moment, however in proving the convergence of moments ≥ 3, the proof of [5] fails in the correlated case and we need to think of another approach.

We consider a sandpile given by weights $(\sigma(x))_{x\in\mathbb{Z}_n^d}$, such that $\mathbb{E}[\sigma(x)\sigma(y)] = \mathcal{K}_n(x-y)$, as before. We also assume that for some K, we have $|\sigma| < K$ almost surely. The odometer e_n in this case still satisfies $\Delta e_n + s = 1$, with $\min_{z\in\mathbb{Z}_n^d} e_n(z) = 0$. To this end, we define

$$v_n(y) = \frac{1}{2d} \sum_{x\in\mathbb{Z}_n^d} g(x,y)(s(x)-1).$$

As we have already seen, $\Delta(e_n - v_n)(z) = 0$ for all z, implying that $e_n = v_n + C$, for some constant C. As we need $\min_{z\in\mathbb{Z}_n^d} e_n(z)$, we find

$$e_n(x) = v_n(x) - \min_{z\in\mathbb{Z}_n^d} v_n(z).$$

Like before, we are not interested in the minimum term, as it is constant and will vanish when taking the inner product with a mean zero $u \in C^\infty(\mathbb{T}^d)$. Now, as $s(x) - 1 = \sigma(x) - n^{-d}\sum_{y\in\mathbb{Z}_n^d}\sigma(y)$, we have

$$v_n(y) = \frac{1}{2d}\sum_{x\in\mathbb{Z}_n^d} g(x,y)\sigma(x) - \frac{1}{2dn^d}\sum_{x\in\mathbb{Z}_n^d} g(x,y)\sum_{z\in\mathbb{Z}_n^d}\sigma(z).$$

Note that in the above, the last term is again constant, as both $\sum_{x\in\mathbb{Z}_n^d} g(x,y)$ and $\sum_{z\in\mathbb{Z}_n^d}\sigma(z)$ do not depend on y anymore, and thus vanishes in a similar fashion as the minimum term. Define then

$$w_n(y) = \frac{1}{2d}\sum_{x\in\mathbb{Z}_n^d} g(x,y)\sigma(x),$$

and set for $h_n : \mathbb{Z}_n^d \to \mathbb{R}$

$$\Xi_{h_n}(x) := 4\pi^2 n^{\frac{d-4}{2}} \sum_{z\in\mathbb{T}_n^d} h_n(nz) 1_{B(z,\frac{1}{2n})}(x), \quad x \in \mathbb{T}^d.$$

We see then that

$$\langle \Xi_{e_n}, u\rangle = \langle \Xi_{w_n}, u\rangle.$$

Next, by assumption, we had $\mathbb{E}[\sigma(x)] = 0$ for all $x \in \mathbb{Z}_n^d$. Subsequently, we find

$$\begin{aligned}\mathbb{E}\left[\langle \Xi_{w_n}, u\rangle\right] &= 4\pi^2 n^{\frac{d-4}{2}} \sum_{z\in\mathbb{T}_n^d} \mathbb{E}[w_n(nz)] \int_{B(z,\frac{1}{2n})} u(y)\ \mathrm{d}y \\ &= 4\pi^2 n^{\frac{d-4}{2}} \sum_{z\in\mathbb{T}_n^d} \left(\frac{1}{2d}\sum_{x\in\mathbb{Z}_n^d} g(x,nz)\mathbb{E}[\sigma(x)]\right) \int_{B(z,\frac{1}{2n})} u(y)\ \mathrm{d}y = 0.\end{aligned}$$

For the second moment then, we find

$$\begin{aligned}\langle \Xi_{w_n}, u\rangle^2 &= 16\pi^4 n^{d-4} \sum_{z,z'\in\mathbb{T}_n^d} \left(\sum_{x,x'\in\mathbb{Z}_n^d} \mathbb{E}[\sigma(x)\sigma(x')] g(x,nz)g(x',nz')\right) \int_{B(z,\frac{1}{2n})} u(y)\ \mathrm{d}y \int_{B(z',\frac{1}{2n})} u(y')\ \mathrm{d}y' \\ &= 16\pi^4 n^{d-4} \sum_{z,z'\in\mathbb{T}_n^d} \left(\frac{1}{(2d)^2}\sum_{x,x'\in\mathbb{Z}_n^d} \mathcal{K}_n(x-x') g(x,nz)g(x',nz')\right) \int_{B(z,\frac{1}{2n})} u(y)\ \mathrm{d}y \int_{B(z',\frac{1}{2n})} u(y')\ \mathrm{d}y' \\ &= 16\pi^4 n^{d-4} \sum_{z,z'\in\mathbb{T}_n^d} \mathcal{H}_n(nz,nz') \int_{B(z,\frac{1}{2n})} u(y)\ \mathrm{d}y \int_{B(z',\frac{1}{2n})} u(y')\ \mathrm{d}y'.\end{aligned}$$

However, this is the exact same term we were working with in Theorem 6.1, but then without the factor $\mathcal{C}_{\mathcal{K}}^{-1}$. In this way, we obtain that

$$\mathbb{E}\left[\langle \Xi_{e_n}, u\rangle^2\right] = \mathbb{E}\left[\langle \Xi_{w_n}, u\rangle^2\right] = \mathcal{C}_{\mathcal{K}}\|u\|_{-1}^2,$$

which is equivalent to what we have shown in Theorem 6.1. The proof in [5] now proceeds by calculating the higher moments, however, here the problem immediately becomes clear. Fix $u \in C^\infty(\mathbb{T}^d)$ and define $T_n : \mathbb{T}^d \to \mathbb{R}$ by

$$T_n(z) = \int_{B(z,\frac{1}{2n})} u(y)\ \mathrm{d}y, \quad z \in \mathbb{T}^d.$$

Using this notation, we obtain for the third moment,

$$\begin{aligned}\mathbb{E}\left[\langle \Xi_{w_n}, u\rangle^3\right] &= \left(\frac{4\pi^2 n^{\frac{d-4}{2}}}{2d}\right)^3 \sum_{z_1,z_2,z_3\in\mathbb{T}_n^d} \mathbb{E}[w_n(nz_1)w_n(nz_2)w_n(nz_3)]T_n(z_1)T_n(z_2)T_n(z_3) \\ &= \left(\frac{4\pi^2 n^{\frac{d-4}{2}}}{2d}\right)^3 \sum_{z_1,z_2,z_3\in\mathbb{T}_n^d}\ \sum_{x_1,x_2,x_3\in\mathbb{Z}_n^d} \mathbb{E}\left[\prod_{j=1}^3 \sigma(x_j)\right] \prod_{j=1}^3 g(x_j,nz_j)T_n(z_j).\end{aligned}$$

In the i.i.d. case, we have

$$\mathbb{E}\left[\prod_{j=1}^3 \sigma(x_j)\right] = 1_{x_1=x_2=x_3}\mathbb{E}[\sigma^3].$$

This simplifies the proof significantly, as in the correlated case, we have no way of expressing $\mathbb{E}[\sigma(x_1)\sigma(x_2)\sigma(x_3)]$ in their pairwise covariances. We could restrict ourselves to the case where

$(\sigma(x))_{x\in\mathbb{Z}_n^d}$ is collection of *symmetric* random variables. In this case we have that the odd moments vanish. Indeed, as $\sigma(x) \overset{d}{=} -\sigma(x)$ for all $x \in \mathbb{Z}_n^d$, we find

$$\mathbb{E}[\sigma(x_1)\sigma(x_2)\sigma(x_3)] = (-1)^3\mathbb{E}[\sigma(x_1)\sigma(x_2)\sigma(x_3)] = -\mathbb{E}[\sigma(x_1)\sigma(x_2)\sigma(x_3)],$$

implying that $\mathbb{E}[\sigma(x_1)\sigma(x_2)\sigma(x_3)] = 0$ for all combinations $x_1, x_2, x_3 \in \mathbb{Z}_n^d$. This fact also generalizes to higher odd moments. However, this still leaves us with the problem of calculating the even moments.

7.2 Speed of convergence for the divisible sandpile

We have considered a sandpile given by

$$s(x) = 1 + \sigma(x) - \frac{1}{n^d}\sum_{z\in\mathbb{Z}_n^d}\sigma(z),$$

for $(\sigma(z))_{z\in\mathbb{Z}_n^d}$ i.i.d. standard normals. We have seen that in this case, our sandpile s_t stabilizes to the all one configuration, in other words $s_t(x) \to 1$ as $t \to \infty$. We were particularly interested in the speed of stabilization. One of the things we have shown is that the sandpile almost surely does not stabilize in finite time, furthermore we obtained the following lower bound:

$$\frac{c}{(2d)^t} \leq \frac{1}{n^d}\sum_{x\in\mathbb{Z}_n^d}|s_t - 1|,$$

for some $c \in \mathbb{R}$. However, we have not managed to show an upper bound to the above quantity. One of the approaches we could try to find this upper bound is looking at mixing times of Markov chains, however this idea is just crawling out of the primordial ooze at the time of writing.

8 Conclusion

In this thesis, we have used the techniques from Levine et al. [1] and Cipriani et al. [5] to generalize Theorem 1 from [5] to a sandpile configuration with correlated Gaussians. More specifically, we obtained the following theorem as our main result:

Theorem 6.1: *Let $e_n : \mathbb{Z}_n^d \to \mathbb{R}$ be the odometer associated with the weights $(\sigma(x))_{x\in\mathbb{Z}_n^d}$, which have covariance given $\mathcal{K} : \mathbb{Z}^d \to \mathbb{R}$ as described in Section 6.2. Set $\mathcal{C}_\mathcal{K} = \sum_{w\in\mathbb{Z}^d} \mathcal{K}(w)$ and define for $\mathcal{C}_\mathcal{K} \neq 0$,*

$$\Xi_n^\mathcal{K}(x) = \mathcal{C}_\mathcal{K}^{-1/2} 4\pi^2 \sum_{z\in\mathbb{T}_n^d} n^{\frac{d-4}{2}} e_n(nz) 1_{B(z,\frac{1}{2n})}(x), \quad x \in \mathbb{T}^d.$$

Then $\Xi_n^\mathcal{K} \xrightarrow{d} \Xi$ in $\mathcal{H}_{-\varepsilon}(\mathbb{T}^d)$ for $\varepsilon > \max\{1 + \frac{d}{4}, \frac{d}{2}\}$, where Ξ is the bilaplacian field: we have for all mean zero $u \in C^\infty(\mathbb{T}^d)$ that $\langle \Xi, u\rangle \sim \mathcal{N}(0, \|u\|_{-1}^2)$.

To prove this theorem, we walked through a few steps, the first was proving the correlated analogue to Proposition 1.3 in [1]. We did this in Section 3: we defined a covariance function $\mathcal{K} : \mathbb{Z}^d \to \mathbb{R}$ and a collection of Gaussians $(\sigma(z))_{z\in\mathbb{Z}_n^d}$ such that for any $x, y \in \mathbb{Z}_n^d$,

$$\mathbb{E}[\sigma(x)\sigma(y)] = \mathcal{K}(x - y).$$

After this we defined a sandpile $s : \mathbb{Z}_n^d \to \mathbb{R}$ by

$$s(x) = \sigma(x) + 1 - n^{-d} \sum_{z\in\mathbb{Z}_n^d} \sigma(z),$$

and we considered the distribution of the odometer $e_n : \mathbb{Z}_n^d \to \mathbb{R}$. We saw that

$$e_n(\cdot) \stackrel{d}{=} \eta - \min_{z\in\mathbb{Z}_n^d} \eta(z),$$

where $(\eta(z))_{z\in\mathbb{Z}_n^d}$ is a collection of correlated, centered Gaussians with

$$\mathbb{E}[\eta(x)\eta(y)] = \frac{1}{(2d)^2} \sum_{z,z'\in\mathbb{Z}_n^d} \mathcal{K}(z - z') g(z, x) g(z', y).$$

After this, we have used the same approach as in [5] to show that the odometer e_n, after an extra scaling, again converges to the bilaplacian field. We first showed the convergence of moments, and for a Gaussian it is enough to compute only the first and second moment. We observed that $\mathbb{E}\left\langle \Xi_n^\mathcal{K}, u\right\rangle = 0$, and then explicitly computed $\mathbb{E}\left[\langle \Xi_n^\mathcal{K}, u\rangle^2\right]$ to show that it indeed converges to $(u, \Delta^{-2}u)_{L^2(\mathbb{T}^d)}$. In proving the tightness of the sequence $(\Xi_n^\mathcal{K})_{n\in\mathbb{N}}$ we have used a quicker proof than the one given in [5]. We have seen that the mollifier approach they are using is not necessary, and we have shortened their proof of tightness as well in Section 5.5.

At the end of Section 6 we have compared the maxima of odometers under different covariances, if $\mathcal{K}^+$ and $\mathcal{K}^-$ are both covariances, and we have that $\mathcal{K}^+ - \mathcal{K}^-$ is still positive-definite, then we can say (for e_+ the odometer associated with $\mathcal{K}^+$ and e_- the odometer associated with $\mathcal{K}^-$)

$$\mathbb{E}\left[\max_{z\in\mathbb{Z}_n^d} e_-(z)\right] \leq \mathbb{E}\left[\max_{z\in\mathbb{Z}_n^d} e_+(z)\right].$$

This theorem however was not as powerful as we would like it to be, and we could investigate whether this theorem also holds under weaker assumptions.

On the other hand, in Section 4 we were concerned with the stabilization speed of the sandpile to the all-one configuration. In this section we have proven that the divisible sandpile almost surely does not stabilize in finite time:

Theorem 4.1: *The divisible sandpile* $(s(x))_{x\in\mathbb{Z}_n^d}$ *for* $n \geq 3$*, where* s *is as defined before, does not stabilize in finite time almost surely.*

At first this result seemed counter-intuitive, but running a few simulations gives a better insight into the statement. We have therefore used a very hands-on approach in proving this theorem. We have seen that if, at any timestep, we find two sites of which one has mass > 1, and the other mass ≥ 1, then we can't have stabilization in finite time. Now we showed that this in fact already happens after the first timestep, and to show this we have used a proof by contradiction: there is too much mass in the graph for this not to happen.

In Section 5, we have also calculated an identity for the pairwise covariances of the collection of real-valued random variables $\langle \Xi, u \rangle$, for $u \in C^\infty(\mathbb{T}^d)$. For $f, f' \in C^\infty(\mathbb{T}^d)$, we saw that

$$\mathbb{E}[\langle \Xi, f\rangle\langle \Xi, f'\rangle] = (\Delta^{-1}f, \Delta^{-1}f')_{L^2(\mathbb{T}^d)}.$$

Note that the inner product (covariance) on the left between the variables $\langle \Xi, f\rangle$ and $\langle \Xi, f'\rangle$ corresponds to the $H^{-1}(\mathbb{T}^d)$ inner product on the right.

However, after all these results, a lot of work still needs to be done if we want to extend Theorem 6.1 to general correlated random variables. We would also like to find an upper bound to the speed of stabilization of the divisible sandpile. So in the end, this is just the beginning.

A Green's function on a finite graph

Define on $\mathbb{Z}^d$ the Green's function $g(x,y)$ as the expected number of visits from x to y. Since g is translation-invariant, we find $g(x,y) = g(o, y-x)$, and therefore it is enough to consider just $g(o,z)$ for $z \in \mathbb{Z}^d$ and $o = (0,0,\dots,0) \in \mathbb{Z}^d$. Let $(S_n)_{n\geq 0}$ be a simple random walk on $\mathbb{Z}^d$, then

$$\begin{aligned} g(o,z) = g(z,o) = \mathbb{E}^z\left[\sum_{n=0}^{\infty} 1_{S_n=o}\right] &= 1_{S_0=o} + \mathbb{P}^z(S_1=y)\mathbb{E}^y\left[\sum_{n=1}^{\infty} 1_{S_n=o}\right] \\ &= 1_{S_0=o} + \sum_{y\sim z} \mathbb{P}^z(S_1=y) g(y,o). \end{aligned}$$

We have essentially just taken out the first step here, if $z = o$, then we get an extra visit, otherwise we just consider the random walk from any of the adjacent tiles. Re-arranging the order of the above gives

$$\sum_{y\sim z} \mathbb{P}^z(S_1=y) g(o,y) - g(o,z) = -1_{S_0=o}.$$

Recall the definition of the graph Laplacian. Since $\mathbb{P}^z(S_1=y) = (2d)^{-1} 1_{y\sim z}$, we find

$$\Delta g(o,\cdot) = -1_{\cdot=o} = -\delta_o.$$

Now we have created a Green's function on the infinite graph $\mathbb{Z}^d$. On finite graphs, things become a bit different as

$$\mathbb{E}[\text{number of visits from } x \text{ to } y] = \infty,$$

for all pairs $x, y \in V$. Intuitively, we have only finite amount of space to walk around for an infinite time. It is necessary to think of another strategy in this case, and as it turns out the function

$$g^z(x,y) := \mathbb{E}[\text{number of visits to } y \text{ starting from } x \text{ before hitting } z],$$

has useful properties that are very much like our Green's function above. We will try to use the same proof techniques as [3] to show this.

Lemma A.1. *Define the function $g^z(x,y)$ on $\mathbb{Z}_n^d$ by*

$$g^z(x,y) := \mathbb{E}[\textit{number of visits to } y \textit{ starting from } x \textit{ before hitting } z].$$

Then g^z satisfies the equation

$$\Delta \frac{g^z(x,y)}{2d} = \delta_z - \delta_x,$$

where Δ is the graph Laplacian.

Proof. We first consider the case where $y \neq z$. Note that we can use the same approach as in Section 4 in [3] here. We use the symmetry of the function $g^z(x,y)$ to find:

$$\begin{aligned} g^z(x,y) = g^z(y,x) &= \delta_x(y) + \sum_{w\in\mathbb{Z}_n^d} p(y,w) g^z(w,x) \\ &= \delta_x(y) + \sum_{w\sim y} \frac{g^z(x,w)}{2d}. \end{aligned}$$

What we've essentially done here is "take out the first step". If $x = y$, then we already have one visit, if not, we just consider the random walk starting at any node connected to y with equal probability. Rewriting the above, gives us

$$\sum_{w \sim y} \left(\frac{g^z(x,y)}{2d} - \frac{g^z(x,w)}{2d} \right) = \delta_x(y).$$

But this is exactly $-\Delta \frac{g^z(x,y)}{2d}$, so the formula holds in the case $y \neq z$. Now consider the case where $y = z$. From Proposition 7.1 in [4] we derive the following formula for finite sets A, C in our state space with stationary probability measure π:

$$\pi(A) = \int_C \mathbb{E}^y \left[\sum_{k=0}^{\tau_C - 1} \mathbf{1}_{\{S_k \in A\}} \right] \pi(dy).$$

Consider now $C = \{y\}$ and $A = \{w \in \mathbb{Z}_n^d : w \sim y\}$. Then the integral reduces to:

$$\mathbb{E}^y \left[\sum_{k=0}^{\tau_y - 1} \mathbf{1}_{\{S_k \in A\}} \right] \pi(\{y\}) = \pi(A).$$

Note that since S_n is just a simple random walk, π is the uniform measure on $\mathbb{Z}_n^d$ after a rescaling, so $\pi(A) = 2d$. Next, we have:

$$2d = \mathbb{E}^y \left[\sum_{k=0}^{\tau_y - 1} \mathbf{1}_{\{S_k \in A\}} \right] = \sum_{w \in A} \frac{1}{2d} g^y(w, A).$$

Here we are starting from y, but not actually counting $\tau_y = 0$ as a stopping time. Essentially we now have $2d$ choices to move from, each with the same probability. By symmetry, it now holds that $g^y(w, A) = g^y(w', A)$ for all $w, w' \in A$. So in fact $g^y(w, A) = 2d$ for all $w \in A$. Now, starting the random walk from $x \neq y$, set W as the first point where the walk enters A. We find

$$g^y(x, A) = \sum_{w \sim y} g^y(x, w) = \sum_{w \sim y} \mathbb{P}_x(W = w) g^y(w, A) = 2d \sum_{w \sim y} \mathbb{P}_x(W = w) = 2d.$$

So whenever $z = y$,

$$\sum_{w \sim y} \frac{g^y(x,w)}{2d} = 1.$$

Note that since $g^y(y, x) = 0$ for all $x \in \mathbb{Z}_n^d \setminus \{y\}$ in this case, the following formula holds

$$g^z(x,y) = \delta_x(y) - \delta_z(y) + \sum_{w \sim y} \frac{g^y(x,w)}{2d}. \tag{24}$$

Whenever $x = y = z$, everything in the above formula reduces to 0, so we can say that 24 holds in general, so

$$\Delta \frac{g^z(x,y)}{2d} = \delta_x(y) - \delta_z(y).$$

$\square$

B Matlab code: i.i.d. scaling limit

```matlab

%% create initial distribution of the sandpile

n = 100;

S = zeros(n,n);

for i = 1:n
    for j = 1:n

        S(i,j) = randn;

    end
end

total = sum(sum(S));

S = S - (1/(n*n))*total + ones(n,n);

%surf(S)
%hold on;

%% iterate the timesteps

timesteps = 20000;

odometer = zeros(n,n);

figure
hold on

for t = 1:timesteps

    A = zeros(n,n);

    for i = 1:n
        for j = 1:n

            if S(i,j) > 1

                extra = S(i,j) - 1;
                distr = 0.25 * extra;

                id_N = i-1;

                if id_N == 0
                    id_N = n;
```

```
                end

                id_S = i+1;

                if id_S == n+1
                    id_S = 1;
                end

                id_W = j-1;

                if id_W == 0
                    id_W = n;
                end

                id_E = j+1;

                if id_E == n+1
                    id_E = 1;
                end

                odometer(i,j) = odometer(i,j)+ distr;

                A(i,id_W) = A(i,id_W)+distr;
                A(i,id_E) = A(i,id_E)+distr;
                A(id_N,j) = A(id_N,j)+distr;
                A(id_S,j) = A(id_S,j)+distr;

                A(i,j) = A(i,j) -extra;

            end
        end
    end

    S = S + A;

end

odometer = (4*pi^2)*(1/n)*odometer;

h = surf(linspace(0,1,n),linspace(0,1,n),odometer);
title('Scaling limit of the odometer')
set(h,'LineStyle','none');
disp(h)
```

C Matlab code: correlated case

```
%% create initial distribution of the sandpile

n = 100;

S = zeros(n,n);

Sigma = zeros(n^2,n^2);
MU = zeros(n^2,1);

% We evaluate the norm from each point in our grid to any other point in
% the grid, we start with 1 in the upper left corner, and work down to the
% lower right corner

K = zeros(n^2,n^2);
index_x = zeros(n^2,n^2);
index_y = zeros(n^2,n^2);

for i = 1:n^2


    i_x = mod(i,n);
    if i_x == 0
        i_x = n;
    end
    i_y = ((i-i_x)/n)+1;



    for j = 1:n^2


        j_x = mod(j,n);
        if j_x == 0
            j_x = n;
        end
        j_y = ((j-j_x)/n)+1;
        dist = zeros(9,1);
        dist(1) = abs(j_x-i_x) + abs(j_y-i_y);
        dist(2) = (abs(i_x-n)+j_x)+abs(j_y-i_y);
        dist(3) = (abs(j_x-n)+i_x)+abs(j_y-i_y);

        dist(4) = (abs(j_x-n)+i_x) + (abs(i_y-n)+ j_y);
        dist(5) = (abs(i_x-n)+j_x) + (abs(i_y-n) + j_y);
        dist(6) = abs(j_x-i_x) + (abs(i_y-n)+j_y);

        dist(7) = abs(j_x-i_x) + (abs(j_y-n)+i_y);
```

```
        dist(8) = (abs(j_x-n)+i_x) + (abs(j_y-n)+i_y);
        dist(9) = (abs(i_x-n)+j_x) + (abs(j_y-n)+i_y);

        K(i,j) = min(dist);
        index_x(i,j) = i_x;
        index_y(i,j) = i_y;
    end

end

for i = 1:n^2
    for j = 1:n^2
        if i == j
            Sigma(i,j) = 6;
        else
            Sigma(i,j) = -(K(i,j))^(-4);
        end
    end
end

R = mvnrnd(zeros(n^2,1),Sigma);

sum(Sigma(1))

S = zeros(n,n);

for i = 1:n^2
    i_x = mod(i,n);
    if i_x == 0
        i_x = n;
    end
    i_y = ((i-i_x)/n)+1;

    S(i_x,i_y) = R(i);

end

total = sum(sum(S));

S = S - (1/(n*n))*total + ones(n,n);

%surf(S)


%% iterate the timesteps

timesteps = 10000;

odometer = zeros(n,n);
```

```
figure
hold on

for t = 1:timesteps

    A = zeros(n,n);

    for i = 1:n
        for j = 1:n

            if S(i,j) > 1

                extra = S(i,j) - 1;
                distr = 0.25 * extra;

                id_N = i-1;

                if id_N == 0
                    id_N = n;
                end

                id_S = i+1;

                if id_S == n+1
                    id_S = 1;
                end

                id_W = j-1;

                if id_W == 0
                    id_W = n;
                end

                id_E = j+1;

                if id_E == n+1
                    id_E = 1;
                end

                odometer(i,j) = odometer(i,j)+ distr;

                A(i,id_W) = A(i,id_W)+distr;
                A(i,id_E) = A(i,id_E)+distr;
                A(id_N,j) = A(id_N,j)+distr;
                A(id_S,j) = A(id_S,j)+distr;

                A(i,j) = A(i,j) -extra;

```

```
            end
        end
    end

    S = S + A;

%     g = surf(S);
%     set(g,'LineStyle','none');
%     disp(g);
%     hold on

end

odometer = (4*pi^2)*(1/n)*odometer;

h = surf(linspace(0,1,n),linspace(0,1,n),odometer);
title('Scaling limit of the odometer')
set(h,'LineStyle','none');
disp(h)
hold on
C_K = 0;

for i = 1:n^2
    C_K = C_K + Sigma(1,i);
end
```

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