Technische Universiteit Delft
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Brownse beweging en de Airy functie.

(Engelse titel: Brownian motion and the Airy function.)

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1 Preface

Random variables can be described as functions from an outcome space to a number, with a probability measure, measuring the size of subsets of the outcome space and therefore the probability of any outcome in that subset happening. Most simple examples are discrete ones which you can use to describe throwing a six-sided dice, which has 6 outcomes each connected to a number 1 to 6, where every outcome has probability $\frac{1}{6}$.

The Brownian Motion can analogously be seen as a function from an outcome space to a space of real valued functions satisfying specific conditions with a specific probability measure, so for every outcome you don't just get a number but an entire function (or in other words uncountably many random variables which are correlated in a specific way).

It's mathematical construction is significantly more complex than the construction of random variables, which will not be treated in this thesis. Same for certain Lemma's which hold for the Brownian Motion (like Doob's submartingale inequality). There are many books written and a course in the Master of Applied Mathematics at the TU Delft which treat the Brownian Motion in detail. While reading you can assume the Brownian Motion exists and is welldefined.

The Brownian Motion describes many things from movements of stockprices to the movement of particles within liquid, firstly described by the botanic Robert Brown around 1827 by examining grains of pollen suspended in water. After which it was mathematically formalized by multiple mathematicians, one of whom is Norbert Wiener (which is why the Brownian Motion is also referred to as the Wiener process). After adding a downward quadratic drift we will prove the maximum exists, is assumed at a unique location almost surely and we will derive an algorithmic method to express it's moments in complex integrals of the Airy function and specific polynomials.

2 Introduction

The location of the maximum of a two-sided Brownian Motion with downward quadratic drift is a random variable V, it's distribution was called Chernoff's distribution by Groeneboom and Wellner [9] since it apparently first appeared in Chernoff [4]. It has been studied by several authors, in particular, Groeneboom [6], [7] gave a description of the distribution and Groeneboom and Wellner [9] give more explicit analytical and numerical formulas; see also Daniels and Skyrme [5]. It has many applications in statistics, see for example Groeneboom and Wellner [9] and the references given there, or, for a more recent example, Anevski and Soulier [3]. The descriptions of the distribution of V in [6], [7] and [9] are rather complicated and do not yield simple formulas for the moments of V, therefore we will prove that V can be expressed in the following form:
For any even $n \in \mathbb{N}$:

$$\mathbb{E}[V^n] = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{p_n(z)}{Ai(z)^2} dz \tag{1}$$

1. p_n is a specific polynomial of at most order $\frac{n}{2}$

2. Ai the Airy function, which is the solution y of:

$$\left\{ \begin{array}{ll} \frac{d^2}{dx^2} y(x) & = xy(x) \\ y(0) & = \frac{3^{-\frac{2}{3}}}{\Gamma(\frac{2}{3})} \\ y'(0) & = \frac{3^{-\frac{1}{3}}}{\Gamma(\frac{1}{3})} \end{array} \right\}$$

For odd n the moments are zero because V is symmetric, which we will show later. The proof of this theorem, which is what the thesis is about, is based around the article by Svante Janson, [10].

3 Moment formula of V

3.1 Defining V

As I've said earlier the Brownian Motion can be seen as a function from an outcome space to the space of real valued functions, analogously it can be seen as a function from the Carthesian product of Ω and $\mathbb{R}$ to $\mathbb{R}$.

Notation:
For $Z : A \times B \to C$ a function by $Z(a, \cdot)$ I mean the implicit function $Z_a : B \to C$ such that $\forall a \in A : \forall b \in B : Z_a(b) = Z(a, b)$.

Firstly the one-sided Brownian Motion:

Definition 3.1.1. Let $(\Omega, \mathcal{A}, \mathbb{P})$ a probability space.
$Z : \Omega \times [0, \infty) \to \mathbb{R}$ is a one-sided Brownian Motion if and only if it has the following properties:
(1): $Z(\cdot, 0) = 0$.
(2): $Z(\omega, \cdot)$ is continuous.
Note: for any $t \in [0, \infty)$ that $Z(\cdot, t)$ is a random variable from Ω to $\mathbb{R}$.
(3): $\forall t, s \in [0, \infty) : Z(\cdot, t) - Z(\cdot, s) \sim N(0, |t - s|)$.
(4): $\Big(Z(\cdot, t_0), ..., Z(\cdot, t_n)\Big)$ has a multivariate normal distribution for any set $\{t_0, ..., t_n\} \subseteq [0, \infty)$.

The multivariate normal distribution is described in the appendix, Definition 5.2.4.
The two sided motion motion is constructed from the one-sided ones the following way:

Definition 3.1.2. Let $(\Omega, \mathcal{A}, \mathbb{P})$ a probability space.
A two-sided Brownian Motion $W : \Omega \times \mathbb{R} \to \mathbb{R}$ is defined by the following property:
$W(\cdot, t)\Big|_{t \geq 0}$ and $W(\cdot, -t)\Big|_{t \geq 0}$ are one sided and independent Brownian Motions.

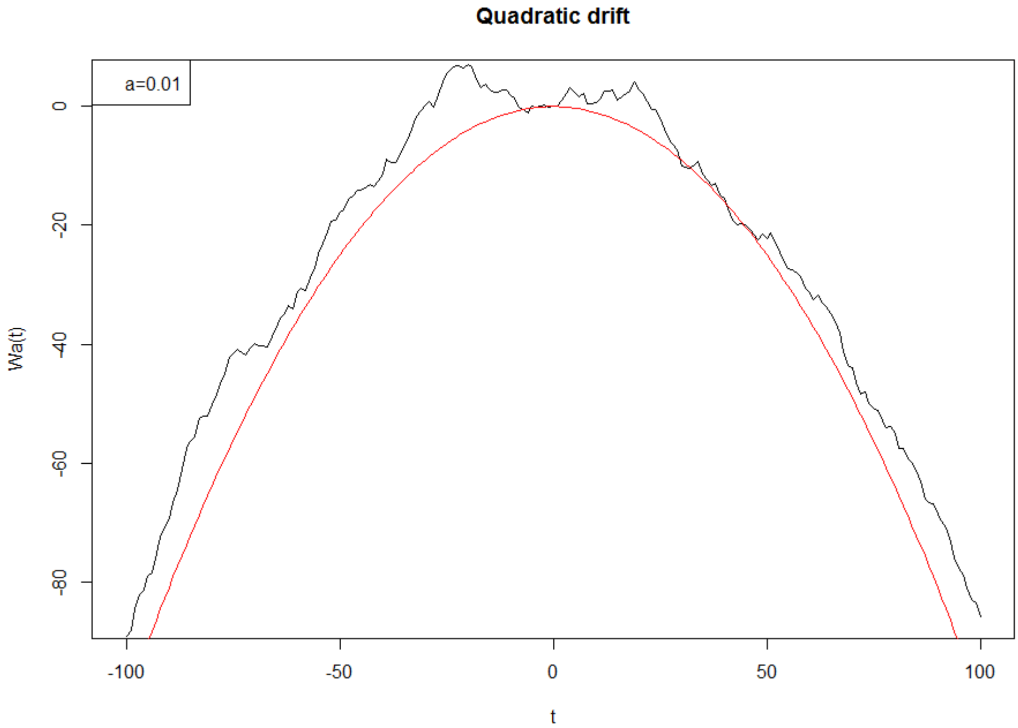
Assume for the rest of the thesis the two-sided Brownian Motion exists and is welldefined (as mentioned in the introduction).

Note from the third condition of the one sided motion follows, for $t \in [0, \infty)$:
$W(\cdot, t) = W(\cdot, t) - W(\cdot, 0) \sim N(0, t)$, therefore $\mathbb{E}[W(\cdot, t)] = 0$.
Also $W(\cdot, -t) = W(\cdot, 0) - W(\cdot, -t) \sim N(0, t)$.

For a some positive number. The Brownian Motion W_a with 'downward' quadratic drift is defined by the property:

$$\forall t \in \mathbb{R} : W_a(\cdot, t) = W(\cdot, t) - at^2. \tag{2}$$

The following is a plot of a realisation of W_a where a is chosen to be 0.01 where t is between -100 and 100. The chosen timesteps are of width 1. The red line represents the drift, defined by the polynomial p such that $p(t) = -at^2$.



Define M_a as:

$$M_a(\cdot) = \max_{t\in\mathbb{R}}(W_a(\cdot, t)). \tag{3}$$

Let $\omega \in \Omega$. It's location V_a (which is a random variable, which we will prove later) is defined implicitly the following way:

$$W_a(\omega, V_a(\omega)) = M_a(\omega) \tag{4}$$

or equivalently:

$$V_a(\omega) = \operatorname*{argmax}_{t\in\mathbb{R}}\{W_a(\omega, t)\}.$$

It can be shown that M_a exists and V_a is unique almost surely (on events of size 1), therefore limitting Ω to the intersection of the events for which this holds makes both M_a and V_a welldefined functions. They are also random variables which follows from Borel-$\mathcal{A}$-measurability, which is another theorem later in the thesis. Note the following Corollary holds for the Brownian Motion:

Corollary 3.1.3. Doob's submartingale inequality
For W a one-sided Brownian Motion and any $n \in \mathbb{N}$ the following holds:

$$\mathbb{P}\Big(\Big\{\omega : \sup_{k\in[n,n+1)} |W(\omega, k)| \geq c\Big\}\Big) = \mathbb{P}\Big(\sup_{k\in[n,n+1)} |W(\cdot, k)| \geq c\Big) \leq \frac{\mathbb{E}[W(\cdot, n+1)^4]}{c^4}. \tag{5}$$

The next Proposition tells us about the almost sure existence of M_a, this can be intuitively justified considering a downward quadratic drift goes faster to $-\infty$ as $|t|$ increases than $W(t)$ can possibly increase.

Proposition 3.1.4. $\mathbb{P}(M_a < \infty) = 1$

Proof. Let $n \in \mathbb{N}$. Let $(\epsilon_n)_{n=1}^{\infty}$ be the sequence defined by:

$$\epsilon_n = \frac{1}{n^{\frac{1}{8}}}.$$

The following inequality holds:

$$\begin{aligned}
\mathbb{P}\Big(\sup_{t\in[n,n+1)} \Big(\frac{W(\cdot,t)}{t}\Big)^4 \geq \epsilon_n^4\Big) &\leq \mathbb{P}\Big(\sup_{t\in[n,n+1)} W(\cdot,t)^4 \geq n^4\epsilon_n^4\Big) \\
&\leq \frac{\mathbb{E}[W(\cdot,n+1)^4]}{n^4\Big(\frac{1}{n^{\frac{1}{8}}}\Big)^4} \\
&= \frac{3(n+1)^2}{n^{3\frac{1}{2}}} \\
&\leq 4\frac{1}{n^{\frac{3}{2}}}.
\end{aligned}$$

The second inequality follows from Corollary 3.1.3. The equality follows from the 4-th moment formula for normal distributions (Lemma 5.2.3 in the appendix), because $W(\cdot, n+1) \sim N(0, n+1)$. From Borel-Cantelli (view appendix) follows:

$$\mathbb{P}\Big(\bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty}\Big\{\sup_{t\in[m,m+1)}\Big(\frac{W(\cdot,t)}{t}\Big)^4 \geq \epsilon_m^4\Big\}\Big) = 0.$$

Define:

$$\Lambda = \bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty}\Big\{\sup_{t\in[m,m+1)}\Big(\frac{W(\cdot,t)}{t}\Big)^4 \geq \epsilon_m^4\Big\}.$$

Note:

$$\begin{aligned}
\Lambda^c &= \bigcup_{n=1}^{\infty}\bigcap_{m=n}^{\infty}\Big\{\sup_{t\in[m,m+1)}\Big(\frac{W(\cdot,t)}{t}\Big)^4 < \epsilon_m^4\Big\} \\
\mathbb{P}(\Lambda^c) &= 1.
\end{aligned}$$

Let $\omega \in \Lambda^c$ and look at $W(\omega,\cdot)$, which we will write as $W(\cdot)$ for the rest of the proof, note ω being in Λ^c implies:

$$\forall n > N : \sup_{t\in[n,n+1)}\Big(\frac{W(t)}{t}\Big)^4 < \epsilon_n^4$$

for some $N \in \mathbb{N}$ depending on ω. This implies:

$$\forall n > N : \sup_{t\in[n,n+1)} W(t) < (n+1)\epsilon_n.$$

This implies:

$$\forall n > N : \sup_{t\in[n,n+1)} W_a(t) < (n+1)\epsilon_n - an^2. \tag{6}$$

Note for some $C \in \mathbb{R}^+$:

$$\forall n \in \mathbb{N} : (n+1)\epsilon_n - an^2 < (n+1) - an^2 < Ca.$$

Therefore W_a is bounded from above on $[N, \infty)$ and from continuity follows it is bounded from above on the closed interval $[0, N]$ therefore also on $[0, \infty)$. Proof is analogous for n negative which proves W_a is bounded from above with probability 1 on $(-\infty, 0]$ for ω from an event of size 1. Taking the intersection of the events tells you W_a is bounded from above with probability 1 on $\mathbb{R}$, therefore $\mathbb{P}(M_a < \infty) = 1$, proving the Proposition.

□

Let $\omega \in \{\omega : M_a(\omega) < \infty\}$. We have proven $W_a(\omega)$ is bounded from above, because $W_a(\omega, \cdot)$ is continuous over the closed set $\mathbb{R}$ it's maximum exists and is taken at one or more points, so $M_a(\omega)$ exists. The next step is proving the uniqueness of this point for any ω from a subset of $\{\omega : M_a(\omega) < \infty\}$ which is also of size 1. Before proving this Proposition we will need a few Lemma's.

Lemma 3.1.5. Let $t_0 \in \mathbb{R}\backslash\{0\}, t_1 \in \mathbb{R} : |t_0| > |t_1|$. Define:

$$H(t_0, t_1) = \mathrm{Cov}(W_a(t_0), W_a(t_1)).$$

Then the following properties hold:
(1):

$$H(t_0, t_0) > H(t_1, t_0). \tag{7}$$

(2): $H(\cdot, t_0)$ is continuous.

Proof. (1): Note:

$$\mathrm{Var}(W(t_0)) \quad = \mathbb{E}[W(t_0)^2] \quad = |t_0|.$$

Now note:

$$H(t_1, t_1) - 2H(t_1, t_0) + H(t_0, t_0) > 0. \tag{8}$$

There are 2 cases to consider in the proof of this inequality.
Case 1: t_1 and t_0 are on the same side of 0:

$$\begin{aligned} |t_1 - t_0| &= \mathrm{Var}(W(t_1) - W(t_0)) \\ &= \mathbb{E}[(W(t_1) - W(t_0) - \mathbb{E}[W(t_1) - W(t_0)])^2] \\ &= \mathbb{E}[(W(t_1) - W(t_0))^2] \\ &= \mathbb{E}[W(t_1)^2] - 2\mathbb{E}[W(t_1)W(t_0)] + \mathbb{E}[W(t_0)^2] \\ &= |t_1| - 2\mathbb{E}[W(t_1)W(t_0)] + |t_0|, \end{aligned}$$

from which follows:

$$\mathbb{E}[W(t_1)W(t_0)] \quad = \frac{|t_1 - t_0| - |t_1| - |t_0|}{-2},$$

from which follows:

$$\begin{aligned}\mathbb{E}[W_a(t_1)W_a(t_0)] &= \mathbb{E}[(W(t_1)-at_1^2)(W(t_0)-at_0^2)]\\ &= \mathbb{E}[W(t_1)W(t_0)]+at_0^2t_1^2\\ &= \frac{|t_1-t_0|-|t_1|-|t_0|}{-2}+at_0^2t_1^2,\end{aligned}$$

from which follows:

$$\begin{aligned}H(t_0,t_1) &= \mathbb{E}[W_a(t_1)W_a(t_0)]-\mathbb{E}[W_a(t_1)]\mathbb{E}[W_a(t_0)]\\ &= \frac{|t_1-t_0|-|t_1|-|t_0|}{-2}+at_0^2t_1^2-at_0^2t_1^2\\ &= \frac{|t_1-t_0|-|t_1|-|t_0|}{-2},\end{aligned} \tag{9}$$

from which follows:

$$\begin{aligned}H(t_0,t_0)-2H(t_1,t_0)+H(t_1,t_1) &= |t_0|+|t_1|+|t_1-t_0|-|t_1|-|t_0|\\ &= |t_1-t_0|,\end{aligned}$$

which proves equation 8 in case 1.
Case 2: t_1 and t_0 are on different sides of 0, clearly $H(t_1,t_0)=0$ since the positive and negative sides are independent Brownian Motions, which implies:

$$H(t_0,t_0)-2H(t_1,t_0)+H(t_1,t_1) \quad = |t_0|+|t_1|. \tag{10}$$

Proving equation (8) in case 2, therefore entirely, note this implies:

$$H(t_0,t_0)+H(t_1,t_1)>2H(t_1,t_0). \tag{11}$$

Note inequality $|t_0|>|t_1|$ implies:

$$H(t_0,t_0)>H(t_1,t_1),$$

combining this inequality with (11) implies:

$$H(t_0,t_0)>H(t_1,t_0)$$

proving (1).

(2): Note again equation (9), it holds if t_1 is on the same side of 0 as t_0, for t_1 is 0 the expression equals 0 and note again that $H(t_0,t_1)$ is 0 if t_1 is on the other side of 0. These conditions imply continuity of $H(\cdot,t_0)$. □

Lemma 3.1.6. Let $t_0\in\mathbb{R}\backslash\{0\}, t_1\in\mathbb{R}:|t_0|>|t_1|$, define:

$$h(t)=\frac{H(t,t_0)}{H(t_0,t_0)}.$$

Where again $H(t_0,t_1)$ is the covariance between $W_a(t_0)$ and $W_a(t_1)$, define:

$$Y(t)=W_a(t)-h(t)W_a(t_0).$$

Then $W_a(t_0)$ and $Y(t)$ are independent for all $t\in\mathbb{R}$.

Proof. Note (follows from Lemma 3.1.5):

$$h(t_0) = \frac{H(t_0, t_0)}{H(t_0, t_0)} > \frac{H(t_1, t_0)}{H(t_0, t_0)} = h(t_1). \tag{12}$$

Define the random variable Y such that:

$$Y(t) = W_a(t) - h(t)W_a(t_0), \tag{13}$$

then both $Y(t)$ and $W_a(t_0)$ are normally distributed for any t, therefore independent because of Lemma 5.2.8, which tells us 2 normally distributed random variables are independent if and only if their covariance is 0:

$$\begin{aligned}
\mathrm{Cov}(Y(t), W_a(t_0)) &= \mathbb{E}[Y(t)W_a(t_0)] - \mathbb{E}[Y(t)]\mathbb{E}[W_a(t_0)] \\
&= \mathbb{E}[(W_a(t) - h(t)W_a(t_0))W_a(t_0)] - \mathbb{E}[W_a(t) - h(t)W_a(t_0)]\mathbb{E}[W_a(t_0)] \\
&= \mathbb{E}[W_a(t)W_a(t_0)] - \mathbb{E}[h(t)W_a(t_0)^2] - \mathbb{E}[W_a(t)]\mathbb{E}[W_a(t_0)] + \mathbb{E}[h(t)W_a(t_0)]\mathbb{E}[W_a(t_0)] \\
&= \mathrm{Cov}(W_a(t), W_a(t_0)) - \frac{H(t, t_0)}{H(t_0, t_0)}\mathrm{Var}(W_a(t_0)) \\
&= H(t, t_0) - \frac{H(t, t_0)}{H(t_0, t_0)}H(t_0, t_0) \\
&= 0.
\end{aligned}$$

□

We're now ready to prove the uniqueness of the location.

Proposition 3.1.7. Let $\{W_a(t) : t \in \mathbb{R}\}$ be a two-sided Brownian Motion with downward quadratic drift depending on $a > 0$, the event of W_a achieving it's supremum at two distinct points of $\mathbb{R}$ is of size 0

Proof. Let $i, j \in \mathbb{Q}$. Define: $K_{i,j} = [i - |j|, i + |j|]$, note $\bigcup_{i,j \in \mathbb{Q}} K_{i,j} = \mathbb{R}$.
It's enough to prove that for 2 disjoint neighbourhoods $K_{i,j}$ and $K_{i',j'}$ of any 2 arbitrary different points in $\mathbb{R}$ the following holds:

$$\mathbb{P}\Big(\sup_{t \in K_{i,j}} W_a(t) = \sup_{t \in K_{i',j'}} W_a(t) = \sup_{t \in \mathbb{R}} W_a(t)\Big) = 0 \tag{14}$$

Since this means the probability of the countable union is also 0, which is the probability of the maximum being attained at 2 distinct points. Note that without loss of generality choosing 2 arbitrary different points in $\mathbb{R}$ is equivalent to choosing $t_0 \in \mathbb{R}\backslash\{0\}$ and $t_1 \in \mathbb{R}$ such that $|t_0| > |t_1|$ which enables us to use Lemma's 3.1.5 and 3.1.6. Define Y and h as in the Lemma 3.1.6. Let $t_0 \in \mathbb{R}\backslash\{0\}, t_1 \in \mathbb{R} : |t_0| > |t_1|$. Now note because of Lemma 3.1.6 and continuity of h, which follows from (2) of Lemma 3.1.5, there exist neighbourhoods K_0 and K_1 of t_0 and t_1 with $K_0, K_1 \in \{K_{1,1}, ...\}$, such that the following holds:

$$\inf_{t \in K_0} h(t) \geq \beta_0 > \beta_1 \geq \sup_{t \in K_1} h(t)$$

Define the following 3 random variables:

$$\Gamma_0(z) = \sup_{t \in K_0} \Big(Y(t) + h(t)z \Big) \tag{15}$$

$$\Gamma_1(z) = \sup_{t \in K_1} \Big(Y(t) + h(t)z \Big) \tag{16}$$

$$\Phi(z) = (\Gamma_1 - \Gamma_0)(z) \tag{17}$$

Assume Y and therefore Φ are known. Let $\Delta \in \mathbb{R}^+$, $z \in \mathbb{R}$:

$$\begin{aligned}
\Gamma_0(z+\Delta) - \Gamma_0(z) &= \sup_{t \in K_0} \Big(Y(t) + h(t)(z+\Delta) \Big) - \sup_{t \in K_0} \Big(Y(t) + h(t)z \Big) \\
&= \sup_{t \in K_0} \Big(Y(t) + h(t)(z+\Delta) \Big) + \inf_{t \in K_0} \Big(-Y(t) - h(t)z) \Big) \\
&= \Big(Y(t_1^*) + h(t_1^*)(z+\Delta) - Y(t_2^*) - h(t_2^*)z \Big) \\
&\geq \Big(Y(t_2^*) + h(t_2^*)(z+\Delta) - Y(t_2^*) - h(t_2^*)z \Big) \\
&\geq \inf_{t \in K_0} \Big(Y(t) + h(t)(z+\Delta) - Y(t) - h(t)z \Big) \\
&= \Delta \inf_{t \in K_0} \Big(h(t) \Big) \\
&\geq \Delta \beta_0.
\end{aligned}$$

Which implies:

$$\Gamma_0(z+\Delta) - \Gamma_0(z) \quad \geq \Delta\beta_0. \tag{18}$$

Analogously:

$$\Gamma_1(z+\Delta) - \Gamma_1(z) \quad \leq \Delta\beta_1 \tag{19}$$

t_1^* and t_2^* can be chosen because K_0 and K_1 are closed, h is continuous and $Y(\omega, \cdot)$ is continuous for any $\omega \in \Omega$. Inequalities (19) and (18) imply:

$$\Phi(z+\Delta) - \Phi(z) = (\Gamma_1(z+\Delta) - \Gamma_1(z)) - (\Gamma_0(z+\Delta) - \Gamma_0(z)) \leq \Delta(\beta_1 - \beta_0) < 0.$$

So because Φ is strictly decreasing if Φ has a root then it's unique.
This implies that Γ_0 and Γ_1 have at most 1 unique intersection point $z^*(Y)$, which depends on Y. Y and $W_a(t_0)$ being independent allows the following interpretation of the probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Let Ω_1, Ω_2 copies of Ω, $\mathcal{A}_1, \mathcal{A}_2$ copies $\mathcal{A}$, $\mathbb{P}_1$,$\mathbb{P}_2$ copies of $\mathbb{P}$ define over the product space $(\Omega_1 \times \Omega_2, \mathcal{A}_1 \times \mathcal{A}_2, \mathbb{P}_1 \times \mathbb{P}_2)$:

$$\widehat{Y} : \Omega_1 \times \Omega_2 \times \mathbb{R} \to \mathbb{R} : \widehat{Y}(\omega_1, \cdot, \cdot) = Y(\omega_1, \cdot)$$

$$\widehat{W_a(t_0)} : \Omega_1 \times \Omega_2 \to \mathbb{R} : \widehat{W_a(t_0)}(\cdot, \omega_2) = W_a(\omega_2, t_0).$$

This gives another definition for random variables Γ_0 and Γ_1:

$$\widehat{\Gamma_0} : \Omega_1 \times \Omega_2 \to \mathbb{R} : \widehat{\Gamma_0} = \sup_{t \in K_0} \Big(\widehat{Y}(\cdot, \cdot, t) + h(t)\widehat{W_a(t_0)}(\cdot, \cdot) \Big)$$

$$\widehat{\Gamma_1} : \Omega_1 \times \Omega_2 \to \mathbb{R} : \widehat{\Gamma_1} = \sup_{t \in K_1} \Big(\widehat{Y}(\cdot, \cdot, t) + h(t)\widehat{W_a(t_0)}(\cdot, \cdot) \Big).$$

This implies:

$$
\begin{aligned}
\mathbb{P}\Big(\sup_{t\in K_0} W_a(t) = \sup_{t\in K_1} W_a(t) = \sup_{t\in\mathbb{R}} W_a(t)\Big) &\leq \mathbb{P}\Big(\sup_{t\in K_0} W_a(t) = \sup_{t\in K_1} W_a(t)\Big) \\
&= \mathbb{P}\Big(\Gamma_0(W_a(t_0)) = \Gamma_1(W_a(t_0))\Big) \\
&= \int_\Omega \mathbf{1}_{\Gamma_0(W_a(t_0))=\Gamma_1(W_a(t_0))} d\mathbb{P} \\
&= \int_\Omega \mathbf{1}_{W_a(t_0)=z^*(Y)} d\mathbb{P} \\
&= \int_{\Omega_1\times\Omega_2} \mathbf{1}_{\widehat{W_a(t_0)}=z^*(\widehat{Y})} d(\mathbb{P}_1 \times \mathbb{P}_2) \\
&= \int_{\Omega_1}\int_{\Omega_2} \mathbf{1}_{\widehat{W_a(t_0)}=z^*(\widehat{Y})} d\mathbb{P}_2 d\mathbb{P}_1 = 0
\end{aligned}
$$

Last equality follows because in the inner integral ω_1 is fixed and noting that $\widehat{W_a(t_0)}(\omega_1, \cdot)$ is a normal random variable with variance $|t_0|$. Also note $z^*(\widehat{Y}(\omega_1, \cdot, \cdot)) = z^*(Y(\omega_1, \cdot))$, which is a fixed number if it exists (if it doesn't exist then the probability of equality is certainly 0) because $Y(\omega_1, \cdot)$ is a fixed real valued function. The probability of a normal random variable with non-trivial variance equalling a fixed number is 0, which proves the Proposition. $\square$

Propositions 3.1.4 and 3.1.7 prove existence and uniqueness of the maximum's location V_a. Limitting Ω to the intersection of the size 1 events from both Propositions makes V_a a welldefined function from Ω to $\mathbb{R}$. Finally V_a is a random variable because it is Borel-$\mathcal{A}$-measurable, which means for any set $O \in \mathcal{B} : \{\omega : V_a(\omega) \in O\} \in \mathcal{A}$.

Let $M_a^*(\cdot, t) = \sup_{s\leq t} W_a(\cdot, s)$. Then M_a^* is a measurable process with continuous paths. This follows from the fact that it suffices to consider rational points s in the supremum (by the continuity of the paths.

Look again at $M_a(\cdot)$, note $M_a(\cdot) = \lim_{t\to\infty} M_a^*(\cdot, t)$. This is also measurable. Now for every $t \in \mathbb{R}$ we have $\{\omega \in \Omega : V_a(\omega) > t\} = \{\omega \in \Omega : M_a^*(t, \omega) < M_a(\omega)\}$. The last set is measurable and hence V_a is Borel-$\mathcal{A}$-measurable, therefore a random variable.

Before going into the formula for even moments, we will prove that odd moments are 0.

Notation:
$\stackrel{d}{=}$ means equal in distribution.

Lemma 3.1.8. odd moments of V_a are 0, in other words:
$\forall n \in 2\mathbb{N} - 1 : \mathbb{E}[V_a^n] = 0$.

Proof. Let $n \in 2\mathbb{N} - 1$. From Definition 3.1.2 $W_a(\cdot, t)\Big|_{t \geq 0}$ and $W_a(\cdot, -t)\Big|_{t \geq 0}$ are independent one-sided Brownian motions. Therefore $\max\limits_{t \in \mathbb{R}^+}\{W_a(\cdot, t)\}$ and $\max\limits_{t \in \mathbb{R}^-}\{W_a(\cdot, t)\}$ are independent and identically distributed, which means V_a is distributed the same as $-V_a$ (in other words V_a is symmetrically distributed). Define:

$$(V_a^+)^n = \max\{0, (V_a)^n\}$$

and

$$\begin{aligned}(V_a^-)^n &= \max\{0, -(V_a)^n\}\\ &= \max\{0, (-V_a)^n\}\end{aligned}$$

because n is odd. Now note:

$$(V_a^+)^n = \max\{0, (V_a)^n\} \stackrel{d}{=} \max\{0, (-V_a)^n\} = (V_a^-)^n.$$

Therefore the following holds:

$$\begin{aligned}\mathbb{E}[(V_a)^n] &= \mathbb{E}[(V_a^+)^n] - \mathbb{E}[(V_a^-)^n]\\ &= \mathbb{E}[(V_a^+)^n] - \mathbb{E}[(V_a^+)^n] = 0.\end{aligned}$$

□

This doesn't hold for even moments, therefore a more complex expression has to be derived, as seen in the introduction. We will start firstly by proving that V_a only differs by a scalar in distribution of V_b for any $a, b > 0$.

Lemma 3.1.9. For any $a > 0 : V_a \stackrel{d}{=} 2^{-\frac{1}{3}} a^{-\frac{2}{3}} V$, where $V = V_{\frac{1}{\sqrt{2}}}$

Proof. Let $a, b > 0$. Note:

$$\begin{aligned}
&2^{\frac{1}{3}} a^{\frac{2}{3}} V_a \\
&= 2^{\frac{1}{3}} a^{\frac{2}{3}} \operatorname*{argmax}_{t_2 \in \mathbb{R}} \left\{ W(t_2) - a t_2^2 \right\} \\
&= \operatorname*{argmax}_{t \in \mathbb{R}} \left\{ W\left(\frac{1}{2^{\frac{1}{3}} a^{\frac{2}{3}}} t\right) - a\left(\frac{1}{2^{\frac{1}{3}} a^{\frac{2}{3}}} t\right)^2 \right\} && \text{substitution } t_2 = \frac{1}{2^{\frac{1}{3}} a^{\frac{2}{3}}} t \\
&\stackrel{d}{=} \operatorname*{argmax}_{t \in \mathbb{R}} \left\{ \frac{1}{\sqrt{2^{\frac{1}{3}} a^{\frac{2}{3}}}} W(t) - \frac{1}{2^{\frac{2}{3}} a^{\frac{4}{3}}} a t^2 \right\} && W(bt) \text{ and } \sqrt{b} W(t) \text{ are Brownian Motions, such that:} \\
& && W(bt) \stackrel{d}{=} \sqrt{b} W(t) \\
&\stackrel{d}{=} \operatorname*{argmax}_{t \in \mathbb{R}} \left\{ W(t) - \frac{\sqrt{2^{\frac{1}{3}} a^{\frac{2}{3}}}}{2^{\frac{2}{3}} a^{\frac{4}{3}}} a t^2 \right\} && \text{multiplication by } \sqrt{2^{\frac{1}{3}} a^{\frac{2}{3}}}. \\
&= \operatorname*{argmax}_{t \in \mathbb{R}} \left\{ W(t) - \frac{1}{2^{\frac{1}{2}}} t^2 \right\} = V.
\end{aligned}$$

□

Therefore knowing the moments of V gives them for V_a for any $a > 0$.
Choosing $a = \frac{1}{\sqrt{2}}$ will provide easier formulas longterm. Similarly it can be proven:

$$M_a \stackrel{d}{=} 2^{\frac{1}{3}} a^{-\frac{1}{3}} M, \text{where } M = M_{\frac{1}{\sqrt{2}}}.$$

Before deriving the the final theorem we will introduce the Airy function in the next section.

3.2 Airy function

The Airy function is the solution of $\frac{d^2y}{dx^2}(x) = xy(x)$ with initial values:

$$y(0) = \frac{3^{-\frac{2}{3}}}{\Gamma(\frac{2}{3})}$$

$$y'(0) = \frac{3^{-\frac{1}{3}}}{\Gamma(\frac{1}{3})}.$$

This gives the following power series expansion:

$$Ai(x) = c_1 f(x) - c_2 g(x) \tag{20}$$

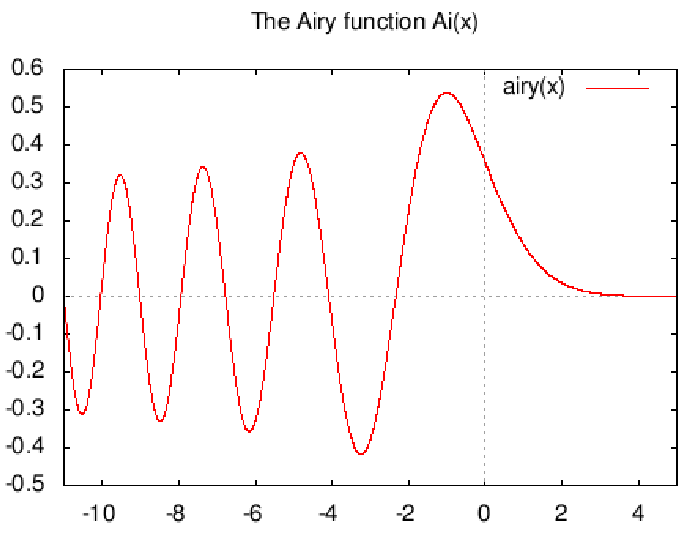
$$f(x) = 1 + \frac{1}{3!}x^3 + \frac{1 \cdot 4}{6!}x^6 + \frac{1 \cdot 4 \cdot 7}{9!}z^9 + ... \qquad c_1 = \frac{3^{-\frac{2}{3}}}{\Gamma(\frac{2}{3})};$$

$$g(x) = z + \frac{2}{4!}z^4 + \frac{2 \cdot 5}{7!}z^7 + \frac{2 \cdot 5 \cdot 8}{10!}z^{10} + ... \qquad c_2 = \frac{3^{-\frac{1}{3}}}{\Gamma(\frac{1}{3})}.$$

Note for any positive z:

$$\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx$$

which enables you to compute $\Gamma(\frac{1}{3})$ and $\Gamma(\frac{2}{3})$ numerically. Limiting domain to $\mathbb{R}$ gives a real valued function looking the following way:



f and g are derived through substituting a power series in the differential equation. Assume y is a solution of $\frac{d^2y}{dx^2}(x) = xy(x)$ and analytic on a disk around 0 (in $\mathbb{C}$), then the following holds:

$$\frac{d^2}{dx^2}\sum_{k=0}^{\infty}\left(a_k x^k\right) = x\sum_{k=0}^{\infty}(a_k x^k)$$

$$\Rightarrow$$

$$\sum_{k=0}^{\infty}\Big(a_k\frac{d^2}{dx^2}x^k\Big)=x\sum_{k=0}^{\infty}(a_kx^k)$$

$$\Rightarrow$$

$$\sum_{k=2}^{\infty}\Big(a_kk(k-1)x^{k-2}\Big)=\sum_{k=0}^{\infty}(a_kx^{k+1})$$

$$\Rightarrow$$

$$\sum_{k=0}^{\infty}\Big(a_{k+2}(k+2)(k+1)x^k\Big)=\sum_{k=1}^{\infty}(a_{k-1}x^k).$$

Differentiation and summation can be exchanged because of Lemma 5.1.4.
Equality holds on the disk iff. coefficients are equal, therefore:
$a_0, a_1 \in \mathbb{R}$, $a_2 = 0$ and for any natural $n \geq 3$:

$$a_n=\frac{a_{n-3}}{n(n-1)}.$$

Note this splits the powerseries of y in 2 parts:

$$\begin{aligned}y(x)&=a_0\Big(1+\frac{1}{2\cdot 3}z^3+\frac{1}{2\cdot 3\cdot 4\cdot 5}z^6+...\Big)\\&+a_1\Big(z+\frac{1}{3\cdot 4}z^4+\frac{1}{3\cdot 4\cdot 6\cdot 7}z^7+...\Big)\\&=a_0f(x)+a_1g(x).\end{aligned}$$

Coefficients a_0 and a_1 determine the function uniquely, which in the case of the Airy function equal $\frac{3^{-\frac{2}{3}}}{\Gamma(\frac{2}{3})}$ and $\frac{3^{-\frac{1}{3}}}{\Gamma(\frac{1}{3})}$ respectively. There are asymptotic formulas for the Airy function (see Handbook of mathematical functions, [1] and the appendix of [10]) and it's derivative, which hold for $|z|$ large, and $|Arg(z)| < \pi$:

$$Ai(z)\sim\frac{1}{2\sqrt{\pi}}z^{-\frac{1}{4}}\exp\Big(-\frac{2}{3}z^{\frac{3}{2}}\Big) \tag{21}$$

$$Ai'(z)\sim-\frac{1}{2\sqrt{\pi}}z^{\frac{1}{4}}\exp\Big(-\frac{2}{3}z^{\frac{3}{2}}\Big). \tag{22}$$

For any complex valued functions $a : \mathbb{C} \to \mathbb{C}$, $b : \mathbb{C} \to \mathbb{C}$
$a \sim b$ means:

$$\forall\theta\in B:\lim_{x\to\infty}\frac{a(x\exp(i\theta))}{b(x\exp(i\theta))}=1.$$

Where the set B depends on the arguments for which the asymptotic expansions hold. The asymptotic expansions for the Airy function hold for z away from the negative real axis, therefore for $\theta \in (-\pi, \pi)$. The formulas imply the following Lemma:

Lemma 3.2.1. The Airy function has the following properties:
(1): $\dfrac{Ai(z)'}{Ai(z)} \sim -z^{\frac{1}{2}}$ for z away from the negative real axis.
(2): $Ai(it)^{-1} = O(e^{-c|t|^{-\frac{3}{2}}})$, for $t \in \mathbb{R}$ and $|t| \geq 1$ for some $c > 0$.
(3): (2) also holds for all derivatives of $Ai(it)^{-1}$ to t.

Proof. (1): Dividing (21) by (22) gives:

$$\frac{Ai(z)'}{Ai(z)} \sim -z^{\frac{1}{2}}. \tag{23}$$

(2): Note for large $|it|$, from formula (21) follows:

$$Ai(it) \sim \frac{1}{2\sqrt{\pi}}(it)^{-\frac{1}{4}} \exp\Big(-\frac{2}{3}(it)^{\frac{3}{2}}\Big). \tag{24}$$

This implies for t real, large and positive:

$$\begin{aligned}
|Ai(it)| &\sim \Big|\frac{1}{2\sqrt{\pi}}(it)^{-\frac{1}{4}} \exp\Big(-\frac{2}{3}(it)^{\frac{3}{2}}\Big)\Big| \\
&\Rightarrow \\
\Big|\frac{1}{Ai(it)}\Big| &\sim \Big|2\sqrt{\pi}(it)^{\frac{1}{4}} \exp\Big(\frac{2}{3}(it)^{\frac{3}{2}}\Big)\Big| \\
&= \Big|2\sqrt{\pi}t^{\frac{1}{4}} \exp\Big(\frac{2}{3}\Big(-\frac{1}{2}\sqrt{2} + i\frac{1}{2}\sqrt{2}\Big)t^{\frac{3}{2}}\Big)\Big| \\
&= \Big|2\sqrt{\pi}t^{\frac{1}{4}} \exp\Big(-\frac{\sqrt{2}}{3}t^{\frac{3}{2}}\Big)\Big| \\
&\leq \Big|2\sqrt{\pi} \exp\Big(\frac{1}{3}t^{\frac{3}{2}}\Big) \exp\Big(-\frac{\sqrt{2}}{3}t^{\frac{3}{2}}\Big)\Big|.
\end{aligned}$$

The first equality follows because powers of i are irrelevant when multiplied within absolute values, having modulus 1. Also because of Euler's formula real multiples of i within the absolute value of the exponential function are irrelevant, again because of modulus 1. Also for large $t : t^{\frac{1}{4}}$ is bounded by $\exp(\frac{1}{3}t^{\frac{3}{2}})$.

Note for t real, large and negative (it has principle argument $-\frac{\pi}{2}$):

$$\begin{aligned}
\Big|\frac{1}{Ai(it)}\Big| &\sim \Big|2\sqrt{\pi}(it)^{\frac{1}{4}} \exp\Big(\frac{2}{3}(it)^{\frac{3}{2}}\Big)\Big| \\
&= \Big|2\sqrt{\pi}a_1|t|^{\frac{1}{4}} \exp\Big(\frac{2}{3}a_2|t|^{\frac{3}{2}}\Big)\Big| : Arg(a_1) = -\frac{1}{8}\pi, Arg(a_2) = -\frac{3}{4}\pi \\
&= \Big|2\sqrt{\pi}|t|^{\frac{1}{4}} \exp\Big(\frac{2}{3}\Big(-\frac{1}{2}\sqrt{2} - i\frac{1}{2}\sqrt{2}\Big)|t|^{\frac{3}{2}}\Big)\Big| \\
&= \Big|2\sqrt{\pi}|t|^{\frac{1}{4}} \exp\Big(-\frac{\sqrt{2}}{3}|t|^{\frac{3}{2}}\Big)\Big| \\
&\leq \Big|2\sqrt{\pi} \exp\Big(\frac{1}{3}|t|^{\frac{3}{2}}\Big) \exp\Big(-\frac{\sqrt{2}}{3}|t|^{\frac{3}{2}}\Big)\Big|.
\end{aligned}$$

Again, for large $|t|$: $|t|^{\frac{1}{4}}$ is bounded by $\exp(\frac{1}{3}|t|^{\frac{3}{2}})$, therefore:

$$\frac{1}{Ai(it)} = O(e^{-\frac{\sqrt{2}-1}{3}|t|^{\frac{3}{2}}}).$$

Therefore (2) holds.

(3): Combining result (1) and the asymptotic formulas gives for t large and real:

$$\begin{aligned}
\Big|\frac{d}{dt}\frac{1}{Ai(it)}\Big| &= \Big|\frac{Ai'(it)}{Ai(it)^2}\Big| \\
&= \Big|\frac{Ai'(it)}{Ai(it)} \cdot \frac{1}{Ai(it)}\Big| \\
&\sim \Big|(it)^{\frac{1}{2}} 2\sqrt{\pi}(it)^{\frac{1}{4}} \exp\Big(\frac{2}{3}(it)^{\frac{3}{2}}\Big)\Big| \\
&= \Big|2\sqrt{\pi}(it)^{\frac{3}{4}} \exp\Big(\frac{2}{3}(it)^{\frac{3}{2}}\Big)\Big| \\
&\leq \Big|2\sqrt{\pi} \exp\Big(\frac{1}{3}|t|^{\frac{3}{2}}\Big) \exp\Big(-\frac{\sqrt{2}}{3}|t|^{\frac{3}{2}}\Big)\Big|,
\end{aligned}$$

which proves (3) for the first derivative.
Note from Lemma 3.3.5 follows $\frac{d^n}{dt^n}\frac{1}{Ai(it)}$ can be expressed as a linear combination of terms $\frac{t^j Ai'(it)^k}{Ai(it)^{k+1}}$. For large $|t|$ because of the earlier results, we can again derive for $|t|$ large enough:

$$\Big|\frac{d^n}{dt^n}\frac{1}{Ai(it)}\Big| \leq \Big|2\sqrt{\pi} \exp\Big(\frac{1}{3}|t|^{\frac{3}{2}}\Big) \exp\Big(-\frac{\sqrt{2}}{3}|t|^{\frac{3}{2}}\Big)\Big|,$$

because $|t^n|$ is bounded by $|\exp\Big(\frac{1}{3}|t|^{\frac{3}{2}}\Big)|$ for large t for any fixed n, which proves (3) for any derivative. $\square$

3.3 Moment formula

We have proven in the previous section that $\frac{1}{Ai(it)}$ and it's derivatives to t have certain properties. More specifically $\frac{1}{Ai(it)}$ and all of it's derivatives are Schwartz functions of t, Definition 5.1.1. The most important thing to know about Schwartz functions in this thesis is that they decrease to 0 very rapidly as $|x|$ goes to ∞, therefore integrals over $\mathbb{R}$ exist:
Let g a Schwartz function, note:

$$\int_{-\infty}^{\infty} g(x)dx < \infty$$

and more specifically:

$$\Big|\int_{-\infty}^{\infty} e^{itx}g(x)dx\Big| < \infty$$

for any real t.
This tells us that the proofs of Lemma's 5.1.6 and 3.3.2 work because the integrals exist. View the appendix for the exact definition of the Schwartz space.

Note the following very important theorem:

Theorem 3.3.1. *For f the density function of V and Ai the Airy function, the following holds:*
(1):$f(x) = \frac{1}{2}g(x)g(-x), \widehat{g}(t) = \dfrac{2^{\frac{1}{2}}}{Ai(it)}$, where $\widehat{g}$ is the Fourier Transform of g, view appendix.
(2):g and $\widehat{g}$ are Schwartz functions.

The proof is too much for this thesis, for it you need a lot of results of the entire book [7], by Groeneboom. Theorem 3.3.1 is very powerful from which combined with some Fourier Analysis we can derive the following Lemma for which the characteristic function is used, Definition 5.2.1.

Lemma 3.3.2. The characteristic function of V satisfies:

$$\forall t \in \mathbb{R} : \phi_V(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{dz}{Ai(it+z))Ai(z)}.$$

Proof. Let $t \in \mathbb{R}$. Note:

$$\begin{aligned}
\phi_V(t) &= \widehat{f}(t) \\
&= \frac{1}{2}\widehat{g\overline{g}}(t) \\
&= \frac{1}{2}\frac{1}{2\pi}(\widehat{g} * \widehat{\overline{g}})(t) \\
&= \frac{1}{4\pi}(\widehat{g} * \overline{\widehat{g}})(t) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{ds}{Ai(i(t-s))Ai(-is)}
\end{aligned}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{ds_2}{Ai(i(t+s_2))Ai(is_2)}$$

$$= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{dz}{Ai(it+z))Ai(z)},$$

where the first two equalities are clear by (1) of Theorem 3.3.1 and the definition of the characteristic function. From (3) of Lemma 5.1.6 in the appendix follows the third equation (with the proof being quite tricky). Fourth equation follows from:

$$\widehat{\overline{g}}(t) = \int_{-\infty}^{\infty} e^{itx}\overline{g}(x)dx = \int_{-\infty}^{\infty} e^{itx}g(-x)dx = \int_{-\infty}^{\infty} e^{i(-t)y}g(y)dy = \overline{\widehat{g}}(t).$$

The following equations also follow from Theorem 3.3.1, since $\widehat{g}(t)$ equals $\frac{\sqrt{2}}{Ai(it)}$ and by substitution $s_2 = -s$ and $z = is_2$. □

This Lemma is also in [8], by Groeneboom.
Differentiating the characteristic function n times, choosing $t = 0$ and scaling by a power of $-i$ gives the n-th moment, we're now ready to derive an expression for the even moments of V, by the following two Lemma's:

Lemma 3.3.3. General formula for the n-th moment:
Let X be a random variable of which all moments exist, then:

$$\mathbb{E}[X^n] = (-i)^n \frac{d^n}{dt^n}\phi_X(0).$$

Proof. Note for any $n \in \mathbb{N}$:

$$\begin{aligned}
(-i)^n \frac{d^n}{dt^n}\phi_X(t) &= (-i)^n \frac{d^n}{dt^n}\mathbb{E}[e^{itX}] \\
&= (-i)^n \frac{d^n}{dt^n}\sum_{k=0}^{\infty} \frac{\mathbb{E}[(itX)^k]}{k!} \\
&= (-i)^n \sum_{k=0}^{\infty} \frac{d^n}{dt^n}(it)^k \frac{\mathbb{E}[X^k]}{k!} \\
&= \sum_{k=n}^{\infty} i^{k-n}t^{k-n}\frac{\mathbb{E}[X^k](k \cdot \ldots \cdot (k-n+1))}{k!},
\end{aligned}$$

where since derivatives of t^0 to t are 0, so the first $n-1$ terms dissappear.

Only the second and third equations require justification, second one follows from dominated convergence and third one from Lemma 5.1.4. Choosing $t = 0$ gives proves the Lemma. □

This result enables us to derive an expression for the n-th moment of V:

Lemma 3.3.4. For any $n \in \mathbb{N}$:

$$\mathbb{E}[V^n] = \frac{1}{2\pi i} \int\limits_{-i\infty}^{i\infty} \frac{d^n}{dz^n}\Big(\frac{1}{Ai(z)}\Big)\frac{dz}{Ai(z)}.$$

Proof. Note:

$$\begin{aligned}
(-i)^n \frac{d^n}{dt^n}\phi_V(t) &= \frac{(-i)^n}{2\pi i}\frac{d^n}{dt^n} \int\limits_{-i\infty}^{i\infty} \frac{dz}{Ai(it+z)Ai(z)} \\
&= \frac{(-i)^n}{2\pi i} \int\limits_{-i\infty}^{i\infty} \frac{d^n}{dt^n}\Big(\frac{1}{Ai(it+z)}\Big)\frac{dz}{Ai(z)} \\
&= \frac{1}{2\pi i} \int\limits_{-i\infty}^{i\infty} \frac{d^n}{dz^n}\Big(\frac{1}{Ai(it+z)}\Big)\frac{dz}{Ai(z)},
\end{aligned}$$

where the first equation follows from Lemma 3.3.2 and the third equation follows from the chain rule, considering differentiating to z is the same as differentiating to t except for some power of i. Lemma follows from combining this result with Lemma 3.3.3. $\square$

Next thing to note is the following relationship following from $(\frac{d^2}{dz^2}A(z) = zAi(z))$.

Lemma 3.3.5. $\frac{d^m}{dz^m}\Big(\frac{1}{Ai(z)}\Big)$ can be expressed as a linear combination of terms of the form:

$$\frac{z^j Ai'(z)^k}{Ai(z)^{k+1}}$$

with j, k non-negative integers and $2j + k \leq m$.

Proof. Proof goes by induction. Taking the first derivative gives:

$$\frac{d}{dz}\Big(\frac{1}{Ai(z)}\Big) = \frac{-\,Ai'(z)}{Ai(z)^2}$$

which is of the correct form with $j = 0, k = 1$, also $2j + k = 1 \leq 1$ (induction basis).
Note the following (induction step):

$$\begin{aligned}
\frac{d}{dz}\frac{z^j Ai'(z)^k}{Ai(z)^{k+1}} &= \frac{Ai(z)^{k+1}(jz^{j-1}Ai'(z)^k + kAi'(z)^{k-1}Ai"(z)z^j)}{Ai(z)^{2k+2}} \\
&- \frac{(k+1)z^j Ai'(z)^k Ai(z)^k Ai'(z)}{Ai(z)^{2k+2}} \\
&= \frac{jz^{j-1}Ai'(z)^k}{Ai(z)^{k+1}} + \frac{kz^{j+1}Ai'(z)^{k-1}}{Ai(z)^k} - \frac{(k+1)z^j Ai'(z)^{k+1}}{Ai(z)^{k+2}},
\end{aligned}$$

which follows from a combination of the product, quotient and chain rules, which is also a linear combination of terms of the correct form because:

(1):if $k = 0$ or $j = 0$ corresponding terms with $j-1$ or $k-1$ dissappear during differentiation
(2):Assuming $2j + k \leq m$ then in the final 3 terms $2j' + k' \leq m+1$ holds, with $(j', k') = (j-1, k)$, $(j+1, k-1)$ or $(j, k+1)$ respectively.
Lemma follows from induction. □

Lemma 3.3.6. Note: for any $j, k \geq 0$:

$$\mathbb{E}[V^{j+k}] = \frac{(-i)^j}{2\pi i} \int_{-i\infty}^{i\infty} \frac{d^j}{dz^j}\Big(\frac{1}{Ai(z)}\Big) \frac{d^k}{dz^k}\Big(\frac{1}{Ai(z)}\Big) dz \tag{25}$$

Proof. Name the right hand side $J(j, k)$. Note firstly:

$$\begin{aligned}
J(j,k) &= \frac{(-1)^j}{2\pi i} \int_{-i\infty}^{i\infty} \frac{d^j}{dz^j}\Big(\frac{1}{Ai(z)}\Big) \frac{d^k}{dz^k}\Big(\frac{1}{Ai(z)}\Big) dz \\
&= \frac{(-1)^j}{2\pi i} \int_{-i\infty}^{i\infty} \frac{d}{dz}\Big(\frac{d^{j-1}}{dz^{j-1}}\Big(\frac{1}{Ai(z)}\Big)\Big) \frac{d^k}{dz^k}\Big(\frac{1}{Ai(z)}\Big) dz \\
&= \frac{(-1)^j}{2\pi i} \Big[\frac{d^{j-1}}{dz^{j-1}}\Big(\frac{1}{Ai(z)}\Big) \frac{d^k}{dz^k}\Big(\frac{1}{Ai(z)}\Big)\Big]_{-i\infty}^{i\infty} \\
&+ \frac{(-1)^{j+1}}{2\pi i} \int_{-i\infty}^{i\infty} \frac{d^{j-1}}{dz^{j-1}}\Big(\frac{1}{Ai(z)}\Big) \frac{d^{k+1}}{dz^{k+1}}\Big(\frac{1}{Ai(z)}\Big) dz \\
&= \frac{(-1)^{j-1}}{2\pi i} \int_{-i\infty}^{i\infty} \frac{d^{j-1}}{dz^{j-1}}\Big(\frac{1}{Ai(z)}\Big) \frac{d^{k+1}}{dz^{k+1}}\Big(\frac{1}{Ai(z)}\Big) dz \\
&= -J(j-1, k+1)),
\end{aligned}$$

where the third equation follows from integration by parts and asymptotic behavior of derivatives of $Ai(z)^{-1}$, (Lemma 3.2.1 and Lemma 3.3.5). This holds for any natural j, k, therefore let j, k and choose $n = k + j$:

$$\begin{aligned}
\mathbb{E}[V^n] &= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{d^n}{dz^n}\Big(\frac{1}{Ai(z)}\Big) \frac{dz}{Ai(z)} \\
&= J(0, n) \\
&= -J(1, n-1) \\
&= (-1)^j J(j, n-j) \\
&= \frac{(-1)^j}{2\pi i} \int_{-i\infty}^{i\infty} \frac{d^j}{dz^j}\Big(\frac{1}{Ai(z)}\Big) \frac{d^{n-j}}{dz^{n-j}}\Big(\frac{1}{Ai(z)}\Big) dz,
\end{aligned}$$

where the third equation follows from repeating the previous computation j times. □

Note that combining Lemma 3.3.5 with Lemma 3.3.4 enables you to write any moment of V as a linear combination of $I(j,k)$, such that:

$$I(j,k) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{z^j Ai'(z)^k}{Ai(z)^{k+2}} dz \tag{26}$$

Lemma 3.3.7. For $j,k \in \mathbb{N} \cup \{0\}$ the following holds:

$$I(j,k) = \left\{ \begin{array}{ll} \frac{j}{k+1} I(j-1,k-1) + \frac{k-1}{k+1} I(j+1,k-2) & k \geq 2 \\ \frac{j}{k+1} I(j-1,0) & k = 1 \end{array} \right\} \tag{27}$$

Proof. Notation:
$[g(z)]_{-i\infty}^{i\infty} = \lim_{a\to\infty} \lim_{b\to\infty} g(ib) - g(-ia)$. Note:

$$\begin{aligned} 0 &= \frac{1}{2\pi i} \Big[\frac{z^j Ai'(z)^k}{Ai(z)^{k+2}} \Big]_{-i\infty}^{i\infty} \\ &= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{d}{dz} \frac{z^j Ai'(z)^k}{Ai(z)^{k+2}} dz \\ &= jI(j-1,k) + kI(j+1,k-1) - (k+2)I(j,k+1). \end{aligned}$$

The first equation follows from the rapidly decreasing behavior of $\frac{1}{Ai(z)^2}$, which goes faster to 0 than $\left| z^j \frac{Ai'(z)^k}{Ai(z)^k} \right|$ goes to ∞ for any $j,k \in \mathbb{N} \cup \{0\}$ (Lemma 3.2.1). The third equation follows analogously to the proof in Lemma 3.3.5. The equation implies:

$$I(j,k+1) = \frac{j}{k+2} I(j-1,k) + \frac{k}{k+2} I(j,k+1) \tag{28}$$

Equation (28) rewrites to (27), proving the Lemma. ☐

Using this Lemma with Lemma 3.3.5 enables you to rewrite every term $I(j,k) : 2j+k \leq n$ into terms $I(j',k')$ with $k' < k$ where $2j' + k' \leq n$ still holds, which follows from combining equation (27) with:

$$[2j+k+1 \leq n] \Rightarrow [(2(j-1)+k \leq n) \wedge (2(j+1)+k-1 \leq n)].$$

Another important result (which will be of use in the next section) is $I(0,1) = 0$ which follows from Lemma 3.3.7. Combining Lemma's 3.3.4, 3.3.5 and 3.3.7 gives the final theorem:

Theorem 3.3.8. *For any even $n \in \mathbb{N}$, $V = V_{\frac{1}{\sqrt{2}}}$ the following holds for the n-th moment:*

$$\mathbb{E}[V^n] = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{p_n(z)}{Ai(z)^2} dz,$$

p_n is a specific polynomial of at most order $\frac{n}{2}$ and
Ai is the Airy function as defined in equation (20).

Proof. Let n even. From Lemma 3.3.3 and 3.3.5 it follows $\mathbb{E}[V^n]$ can be rewritten as an m-fold linear combination of the form:

$$\mathbb{E}[V^n] = \frac{1}{2\pi i}\Big(a_1 \int_{-i\infty}^{i\infty} \frac{z^{j_1} Ai'(z)^{k_1}}{Ai(z)^{k_1+2}} dz + ... + a_m \int_{-i\infty}^{i\infty} \frac{z^{j_m} Ai'(z)^{k_m}}{Ai(z)^{k_m+2}} dz\Big).$$

Note $2j_i + k_i \leq n$, for all i.

After which from Lemma 3.3.7 it follows that every term can be rewritten as a linear combination of terms with lower k_i, which you can repeat until all k_i are 0 resulting in:

$$\mathbb{E}[V^n] = \frac{1}{2\pi i}\Big(a'_1 \int_{-i\infty}^{i\infty} \frac{z^{j'_1}}{Ai(z)^2} dz + ... + a'_{m_2} \int_{-i\infty}^{i\infty} \frac{z^{j'_{m_2}}}{Ai(z)^2} dz\Big).$$

Note again that the relation $2j'_i + k'_i \leq n$ is preserved, therefore the polynomial

$$a'_1 z^{j'_1} + ... + a'_{m_2} z^{j'_{m2}}$$

is at most of order $\frac{n}{2}$.

$\square$

Now to demonstrate the deriviation of the first 2 even moments, by also using Lemma 3.3.6:

$$\begin{aligned}
\mathbb{E}[V^2] &= \frac{-1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{d}{dz}\Big(\frac{1}{Ai(z)}\Big) \frac{d}{dz}\Big(\frac{1}{Ai(z)}\Big) dz \\
&= \frac{-1}{2\pi i} \int_{-i\infty}^{i\infty} \Big(\frac{Ai'(z)^2}{Ai(z)^4}\Big) dz \\
&= -I(0,2) \\
&= -\frac{1}{3} I(1,0) \\
&= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Big(\frac{-\frac{1}{3}z}{Ai(z)^2}\Big) dz
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[V^4] &= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{d^2}{dz^2}\Big(\frac{1}{Ai(z)}\Big) \frac{d^2}{dz^2}\Big(\frac{1}{Ai(z)}\Big) dz \\
&= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Big(\frac{d}{dz}\Big(\frac{-Ai'(z)}{Ai(z)^2}\Big)\Big)^2 dz \\
&= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Big(\frac{d}{dz}\Big(\frac{Ai'(z)}{Ai(z)^2}\Big)\Big)^2 dz
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \Big(\frac{Ai(z)^2Ai''(z) - 2Ai'(z)^2Ai(z)}{Ai(z)^4}\Big)^2 dz \\
&= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \Big(\frac{z}{Ai(z)} - \frac{2Ai'(z)^2}{Ai(z)^3}\Big)^2 dz \\
&= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{z^2}{Ai(z)^2} - \frac{4zAi'(z)^2}{Ai(z)^4} + \frac{4Ai'(z)^4}{Ai(z)^6} dz \\
&= I(2,0) - 4I(1,2) + 4I(0,4) \\
&= I(2,0) - 4\Big(\frac{1}{3}I(0,1) + \frac{1}{3}I(2,0)\Big) + 4\Big(\frac{3}{5}I(1,2)\Big) \\
&= I(2,0) - 4\Big(\frac{1}{3}I(2,0)\Big) + 4\Big(\frac{3}{5}\frac{1}{3}I(2,0)\Big) \\
&= \frac{15}{15}I(2,0) - \frac{20}{15}I(2,0) + \frac{12}{15}I(2,0) = \frac{7}{15}I(2,0) \\
&= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \Big(\frac{\frac{7}{15}z^2}{Ai(z)^2}\Big) dz.
\end{aligned}
$$

Therefore $p_2(z) = -\frac{1}{3}z$ and $p_4(z) = \frac{7}{15}z^2$.

3.4 Python program for computing p_n

I've written a Python program that prints 7 even polynomials from p_2 up to p_{14} (view appendix). Note $p_2(z) = -\frac{1}{3}z$ and $p_4(z) = \frac{7}{15}z^2$ as shown earlier.

```
p_2(z) = -0.333333333333*z^1
p_4(z) = 0.466666666667*z^2
p_6(z) = -1.47619047619*z^3
+1.2380952381*z^0
p_8(z) = 8.46666666667*z^4
+-21.7777777778*z^1
p_10(z) = -77.4242424241*z^5
+398.787878789*z^2
p_12(z) = 1036.24688644*z^6
+-8862.75457919*z^3
+1457.48864467*z^0
p_14(z) = -19112.333334*z^7
+244157.02243*z^4
+-132322.666748*z^1
```

```
p_1(z) = 0.0*z^0
p_3(z) = 0.0*z^0
p_5(z) = 0.0*z^0
+7.1054273576e-15*z^1
p_7(z) = 0.0*z^0
p_9(z) = 0.0*z^0
+-2.91038304567e-11*z^3
+-7.27595761418e-12*z^0
p_11(z) = 0.0*z^0
+-3.72529029846e-09*z^1
p_13(z) = 0.0*z^0
+2.86102294922e-06*z^2
```

Also note that the first 7 odd polynomials are 0 (ignoring the rounding errors) as expected.

4 Conclusion and additional problems/conjectures

After combining results from multiple mathematical fields which may appear very disjoint (Fourier Analysis and Probability Theory) simplifications for difficult formulas can be found. In our case it's formulas for moments of a Brownian Motion with downward quadratic drift. As for the usefulness, one physical interpretation of a Brownian Motion with downward quadratic drift is the movement of a particle through water including the effects of gravity increasing it's downward speed over time, but it's overall movement remaining "random". Knowing the moments then enables you not only to know the place where the particle is expected to be at it's highest point, but also the variance (measure of the likelihood of deviating from the expectation).

The final theorem yields an algorithm for computing the polynomials $p_n(z)$ but no simple formula for them. Therefore the following problems and conjectures can be stated:

Problem 4.0.1. *Is there an explicit formula for the coefficients b_{nj}, and thus for the polynomials $p_n(z)$? Perhaps a recursion formula?*

After computing the polynomials for n between 1 and 100. The following conjectures can be made:

Conjectures 4.0.2. *(1): $p_n(z) = 0$ for every odd n (note that this is stronger than just $\mathbb{E}[V^n] = 0$ for every odd n).*
(2): $p_n(z)$ has degree exactly $n/2$; i.e., the coefficient $b_{n,n/2}$ of $z^{n/2}$ is non zero.
(3): These leading coefficients have exponential generating function:

$$\sum_{n=0}^{\infty} \frac{b_{n,n/2}}{n!} = \frac{x}{\sinh(x)}.$$

Problem 4.0.3. *Is there an explicit formula for the generating function*

$$\sum_{n=0}^{\infty} p_n(z) x^n?$$

These problems and conjectures are from the article by Svante Janson [10].

5 Appendix

Here are definitions and Lemma's, with or without proof, necessary in the thesis, also the code of the program used in the last subsection.

5.1 Fourier and Complex Analysis

Firstly introducing the Schwartz class of rapidly decreasing functions:

Definition 5.1.1. $C^\infty(\mathbb{R})$ is the space of infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{C}$.
Let α, β be non negative integers. A function $f : \mathbb{R} \to \mathbb{C}$ in $C^\infty(\mathbb{R})$ is in the **Schwartz space S** of rapidly decreasing functions if and only if for any $x \in \mathbb{R}$:

$$\sup_{x\in\mathbb{R}} |x^\alpha D^\beta f(x)| < \infty.$$

Where:

$$D^\beta = \Big(\frac{1}{i}\frac{d}{dx}\Big)^\beta.$$

This is the exact definition, note that Schwartz functions and their derivatives multiplied by any polynomial have a bounded absolute value over $\mathbb{R}$.

Definition 5.1.2. The **Fourier Transform** is the function $\widehat{f}$ of the function f defined the followig way, as long as it exists:

$$\widehat{f}(t) = \int_{-\infty}^{\infty} e^{itx} f(x)dx \tag{29}$$

It can be shown Fourier Transforms preserve Schwartz functions.

Definition 5.1.3. The **convolution product** between functions f and g is a function defined as:

$$(f * g)(t) = \int_{-\infty}^{\infty} f(t-x)g(x)dx, \tag{30}$$

as long as the integral exists.

Note:

$$\int_{-\infty}^{\infty} f(t-x)g(x)dx = \int_{\infty}^{-\infty} -f(y)g(t-y)dy = \int_{-\infty}^{\infty} f(y)g(t-y)dy. \tag{31}$$

Therefore:

$$f * g = g * f. \tag{32}$$

Which is visible by substitution $t - x = y$, implying $x = t - y$ and $dy = -dx$, note the borders have to be substituted too.

Lemma 5.1.4. Weierstrass's Theorem for analytic functions
Let $(f_n)_{n=1}^\infty$ be a sequence of analytic functions converging locally uniformly to f on some domain D then f is analytic and f_n' converge locally uniformly to f'.

Note: A domain D is a simply connected open subset of $\mathbb{C}$ on which any interior of a closed curve is within the set.

Lemma 5.1.5. Fubini's theorem
Let $F \in \mathbf{S}^2$, then:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x,y)dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x,y)dydx. \tag{33}$$

Notation:
$\overline{g}(t) = g(-t)$

Lemma 5.1.6. For $f, g \in \mathbf{S}$ the following 3 properties hold:
(1): **Inverse Fourier Transform**

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\widehat{f}(x)dx \tag{34}$$

(2):

$$\frac{1}{2\pi}\widehat{\widehat{f}} = \overline{f} \tag{35}$$

(3):

$$\widehat{fg} = \frac{1}{2\pi}(\widehat{f} * \widehat{g}). \tag{36}$$

Proof. (1): Note firstly from Fubini's Lemma 5.1.5 follows:

$$\begin{aligned}\int_{-\infty}^{\infty} \widehat{f}g(x)dx &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ixy}f(y)dyg(x)dx \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ixy}g(x)dxf(y)dy \\ &= \int_{-\infty}^{\infty} f\widehat{g}(x)dx.\end{aligned}$$

Let $\lambda > 0$, $f^*(x) = f\left(\frac{x}{\lambda}\right)$ (clearly f^* in $\mathbf{S}$) then:

$$\int_{-\infty}^{\infty} f\left(\frac{x}{\lambda}\right)\widehat{g}(x)dx = \int_{-\infty}^{\infty} \widehat{f}(x)g\left(\frac{x}{\lambda}\right)dx.$$

Which follows from:

$$
\begin{aligned}
\int_{-\infty}^{\infty} f\Big(\frac{x}{\lambda}\Big)\widehat{g}(x)dx &= \int_{-\infty}^{\infty} f^*(x)\widehat{g}(x)dx \\
&= \int_{-\infty}^{\infty} \widehat{f^*}(x)g(x)dx \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ixy}f^*(y)dyg(x)dx \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ixy}f\Big(\frac{y}{\lambda}\Big)dyg(x)dx \\
&= \int_{-\infty}^{\infty} \lambda \int_{-\infty}^{\infty} \exp(i\lambda xy_2)f(y_2)dy_2g(x)dx \\
&= \int_{-\infty}^{\infty} \lambda\widehat{f}(\lambda x)g(x)dx \\
&= \int_{-\infty}^{\infty} \widehat{f}(x_2)g\Big(\frac{x_2}{\lambda}\Big)dx_2.
\end{aligned}
$$

Follows from substituting $x_2 = \lambda x$ and $y_2 = \dfrac{y}{\lambda}$.

Note taking the limit $\lambda \to \infty$ (functions are continuous) sends $g\Big(\dfrac{x}{\lambda}\Big)$ to $g(0)$ and $f\Big(\dfrac{x}{\lambda}\Big)$ to $f(0)$. From the dominated convergence Theorem follows

$$
f(0)\int_{-\infty}^{\infty} \widehat{g}(x)dx \quad = g(0)\int_{-\infty}^{\infty} \widehat{f}(x)dx. \tag{37}
$$

This is visible by defining $(f_n)_{n=1}^{\infty}$ such that $f_n(x) = f(\frac{x}{n})$ since both are Schwartz functions the integral of their absolute product is bounded and clearly $\lim\limits_{n\to\infty} f_n(x)\widehat{g}(x) = f(0)\widehat{g}(x)$ for any x therefore: $\lim\limits_{n\to\infty} \int\limits_{\mathbb{R}} f_n(x)\widehat{g}(x)dx = \int\limits_{\mathbb{R}} f(0)\widehat{g}(x)dx$:

This goes analogously for $\widehat{f}(x)g\Big(\dfrac{x}{\lambda}\Big)$. Now note: ($\tau_p g(x) = g(x-p)$ for any $p \in \mathbb{R}$)

$$
g(x) = (\tau_{-x}g)(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{\tau_{-x}g}(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\widehat{g}(t)dt. \tag{38}
$$

Second equation follows from the earlier derived result: (choosing: $\phi(x) = \exp(-\frac{1}{2}x^2)$ and noting $\phi \in \mathbf{S}$)

$$
(\tau_{-x}g)(0)\int_{-\infty}^{\infty} \widehat{\phi}(t)dt = \phi(0)\int_{-\infty}^{\infty} \widehat{\tau_{-x}g}(t)dt. \tag{39}
$$

Combined with: (note again the characteristic function of standard normal variable):

$$\widehat{\phi}(t) = \int_{-\infty}^{\infty} e^{ity} e^{-\frac{1}{2}y^2} dy = \sqrt{2\pi} \int_{-\infty}^{\infty} e^{ity} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy = \sqrt{2\pi} e^{-\frac{1}{2}t^2}.$$

Which implies (integral of a density function over $\mathbb{R}$ equals 1):

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{\phi}(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sqrt{2\pi} e^{-\frac{1}{2}t^2} dt \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt = 1 = e^{-\frac{1}{2}0^2} = \phi(0). \tag{40}$$

After which you can substitute equation (40) in (39) to obtain the second equation of (38). Last equation in (38) follows from:

$$\begin{aligned}
\int_{-\infty}^{\infty} \widehat{\tau_{-x} g}(t) dt &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ity} (\tau_{-x} g)(y) dy dt \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ity} g(y+x) dy dt \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(it(y_2 - x)) g(y_2) dy_2 dt \\
&= \int_{-\infty}^{\infty} e^{-itx} \widehat{g}(t) dt.
\end{aligned}$$

By substituting $y_2 = y + x$.
(1) of the Lemma follows from equation (38).

(2): From (1) follows (substitution $x_2 = -x$):

$$f(-t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \widehat{f}(-x) dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(itx_2) \widehat{f}(x_2) dx_2 = \frac{1}{2\pi} \widehat{\widehat{f}}(t).$$

(3): Note first:

$$\widehat{f}\widehat{g} = \widehat{(f * g)}. \tag{41}$$

This follows from:

$$\begin{aligned}
(\widehat{f}\widehat{g})(t) &= \widehat{f}(t)\widehat{g}(t) \\
&= \int_{-\infty}^{\infty} e^{-itx} f(x) dx \int_{-\infty}^{\infty} e^{-ity} g(y) dy \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-it(x+y)} f(x) g(y) dx dy
\end{aligned}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-it(x_2)} f(x_2 - y) g(y) dy dx_2$$

$$= \int_{-\infty}^{\infty} e^{-it(x_2)} (f * g)(x_2) dx_2$$

$$= \widehat{(f * g)}(t)$$

substitution of $x_2 = x + y$ and Fubini is used here.
Also note:

$$\frac{1}{2\pi}\widehat{(\widehat{f} * \widehat{g})}(t) = \frac{1}{2\pi}\widehat{\widehat{f}}\widehat{\widehat{g}}(t) = 2\pi \overline{f}\overline{g}(t) = 2\pi \overline{fg}(t) = \widehat{\widehat{fg}}(t)$$

first equation follows from equation (41) earlier, second and last equations follow from (2) and note again that $\overline{f}(t) = f(-t)$.
Applying the inverse Fourier Transform gives:

$$\frac{1}{2\pi}(\widehat{f} * \widehat{g})(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \frac{1}{2\pi} \widehat{(\widehat{f} * \widehat{g})}(x) dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \widehat{\widehat{fg}}(x) dx = \widehat{fg}(t).$$

Proving the Lemma. □

5.2 Probability Theory and Real Analysis

Definition 5.2.1. Let X be a random variable, it's **characteristic function** ϕ is defined by:

$$\phi_X(t) = \mathbb{E}[e^{itX}].$$

There are multiple definitions for the Fourier Transform but conveniently we have chosen the Fourier transform equal to the characteristic function of X for which f is the density function. Characteristic functions are useful since they uniquely determine the distribution.

Definition 5.2.2. X is a normally distributed random variable with mean μ and variance σ^2 if and only if the following holds for the cummulative distribution function F_X:

$$F_X(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx, \tag{42}$$

with density function f:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}. \tag{43}$$

This can be written as:
$X \sim N(\mu, \sigma^2)$. For $Z \sim N(0,1)$, Z is standard normally distributed.

Lemma 5.2.3. Fourth moment formula for zero centered normal distributions. Let $\sigma \in [0, \infty)$. Let $X \sim N(0, \sigma^2)$. For X the following holds:

$$\mathbb{E}[X^4] = 3\sigma^4.$$

Proof. By integration by parts and definition 5.2.2:

$$\begin{aligned}
\mathbb{E}[X^4] &= \int_{\infty}^{\infty} x^4 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}x^2} dx \\
&= \frac{1}{\sigma\sqrt{2\pi}}(-\sigma^2) \int_{\infty}^{\infty} x^3 \frac{d}{dx} e^{-\frac{1}{2\sigma^2}x^2} dx \\
&= -\frac{\sigma^2}{\sigma\sqrt{2\pi}} \Big([x^3 e^{-\frac{1}{2}x^2}]_{-\infty}^{\infty} - \int_{\infty}^{\infty} 3x^2 e^{-\frac{1}{2\sigma^2}x^2} dx \Big) \\
&= \frac{3\sigma^2}{\sigma\sqrt{2\pi}} \int_{\infty}^{\infty} x^2 e^{-\frac{1}{2\sigma^2}x^2} dx \\
&= -\frac{3\sigma^4}{\sigma\sqrt{2\pi}} \int_{\infty}^{\infty} x \frac{d}{dx} e^{-\frac{1}{2\sigma^2}x^2} dx \\
&= -\frac{3\sigma^4}{\sigma\sqrt{2\pi}} \Big([x e^{-\frac{1}{2}x^2}]_{-\infty}^{\infty} - \int_{\infty}^{\infty} e^{-\frac{1}{2\sigma^2}x^2} dx \Big)
\end{aligned}$$

$$= 3\sigma^4 \int_{\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}x^2} dx = 3\sigma^4.$$

$\square$

Definition 5.2.4. Multivariate Normal distribution. Let $\left(Z_1, .., Z_n\right)$ be a vector of independent standard normal random variables.
The vector $\left(W_1, .., W_n\right)$ is multinormally distributed iff. there exists some lower triangle positive definite matrix B and a vector μ such that

$$\left(W_1, .., W_n\right)^T \stackrel{d}{=} \mu + B\left(Z_1, .., Z_n\right)^T. \tag{44}$$

It's covariance matrix is

$$BB^T.$$

Lemma 5.2.5. Borel-Cantelli. For any sequence of events $(A_n)_{n=1}^{\infty}$,

$$\left[\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty\right] \Rightarrow \left[\mathbb{P}\left(\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m\right) = 0\right].$$

Lemma 5.2.6. Characteristic function of Z, where $Z \sim N(0, 1)$ is:

$$\phi_Z(t) = e^{-\frac{1}{2}t^2}.$$

Lemma 5.2.7. Dominated convergence theorem.
Let $(f_n)_{n=1}^{\infty}$ be a sequence of lebesque integrable functions converging pointwise to f, so:

$$\lim_{n\to\infty} f_n = f$$

then if there is an integrable g, such that: $\forall n : |f_n| \leq g$ then

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n d\lambda = \int_{\mathbb{R}} f d\lambda. \tag{45}$$

Lemma 5.2.8. Let Z and W be normally distributed from the same probability space, they're independent if and only if $\text{Cov}(Z, W) = 0$.

5.3 Program

Note the earlier result:

$$a\frac{d}{dz}\frac{z^j Ai'(z)^k}{Ai(z)^{k+1}} = a\frac{Ai(z)^{k+1}(jz^{j-1}Ai'(z)^k + kAi'(z)^{k-1}Ai"(z)z^j)}{Ai(z)^{2k+2}}$$
$$- a\frac{(k+1)z^j Ai'(z)^k Ai(z)^k Ai'(z)}{Ai(z)^{2k+2}}$$
$$= \frac{ajz^{j-1}Ai'(z)^k}{Ai(z)^{k+1}} + \frac{akz^{j+1}Ai'(z)^{k-1}}{Ai(z)^k} - \frac{a(k+1)z^j Ai'(z)^{k+1}}{Ai(z)^{k+2}}.$$

Note again that $\frac{d^n}{dz^n}\frac{1}{Ai(z)}$ is a linear combination of terms of the form:

$$a \cdot \frac{z^j Ai'(z)^k}{Ai(z)^{k+2}}.$$

With j, k integers and a the multiplying coefficient.
Logic behind the code is to express these terms in triple vectors (a, j, k), starting with $(1, 0, 0)$, which corresponds with $\frac{1}{Ai(z)}$, then using upper earlier result to reexpress every vector in a matrix of 3 new vectors in every derivation step (leaving out trivial (0,0,0) vectors).
Then by using Lemma 3.3.7 redefine the matrix in terms with lower k until all k are 0, then it's just a matter of summing all coefficients a with corresponding j, which are powers of z.

```
import timeit
import numpy as np
start = timeit.default_timer()
def derivator(Ai):
    object2 = np.zeros((3,3*len(Ai[0,:])))
    for i in range(len(Ai[0,:])):
        a=Ai[0,i]
        j=Ai[1,i]
        k=Ai[2,i]
        if a==0:
            object2[:,3*i] = np.array([0,0,0])
            object2[:,3*i+1] = np.array([0,0,0])
            object2[:,3*i+2] = np.array([0,0,0])
        else:
            if j == 0:
                object2[:,3*i] = np.array([0,0,0])
            else:
                object2[:,3*i] = np.array([a*j,j-1,k])
            if k == 0:
                object2[:,3*i+1] = np.array([0,0,0])
            else:
                object2[:,3*i+1] = np.array([a*k,j+1,k-1])
            if k+1==0:
                object2[:,3*i+2] = np.array([0,0,0])
            else:
                object2[:,3*i+2] = np.array([-a*(k+1),j,k+1])
    count=0
    for i in range(len(object2[0,:])):
        if abs(object2[0,i])>0:
            count = count+1
```

```
    object3 = np.zeros((3,count))
    l=0
    for i in range(len(object2[0,:])):
        if abs(object2[0,i])>0:
            object3[:,l]=object2[:,i]
            l=l+1
    return object3
def Aidiff(Ai,n):
    while n-1>0:
        Ai = derivator(Ai)
        n=n-1
    return Ai
def Keliminator(Ai):
    while sum(abs(Ai[2,:]))>0:
        Ai2 = np.zeros((3,2*len(Ai[0,:])))
        for i in range(len(Ai[0,:])):
            a=Ai[0,i]
            j=Ai[1,i]
            k=Ai[2,i]
            if k >= 2:
                Ai2[0,2*i] = a*j/(k+1)
                if Ai2[0,2*i] == 0:
                    Ai2[1,2*i] = 0
                    Ai2[2,2*i] = 0
                else:
                    Ai2[1,2*i] = j-1
                    Ai2[2,2*i] = k-1
                Ai2[0,2*i+1] = a*(k-1)/(k+1)
                if Ai2[0,2*i+1] == 0:
                    Ai2[1,2*i+1] = 0
                    Ai2[2,2*i+1] = 0
                else:
                    Ai2[1,2*i+1] = j+1
                    Ai2[2,2*i+1] = k-2
            elif k == 1:
                Ai2[0,2*i] = a*j/(k+1)
                if Ai2[0,2*i] == 0:
                    Ai2[1,2*i] = 0
                    Ai2[2,2*i] = 0
                else:
                    Ai2[1,2*i] = j-1
                    Ai2[2,2*i] = 0
                Ai2[0,2*i+1] = 0
                Ai2[1,2*i+1] = 0
                Ai2[2,2*i+1] = 0
            else:
                Ai2[0,2*i] = a
                Ai2[1,2*i] = j
                Ai2[2,2*i] = k
                Ai2[0,2*i+1] = 0
                Ai2[1,2*i+1] = 0
                Ai2[2,2*i+1] = 0
        Ai = Ai2
        count=0
        for i in range(len(Ai[0,:])):
            if abs(Ai[0,i])>0:
                count = count+1
```

```
        object3 = np.zeros((3,count))
        l=0
        for i in range(len(Ai[0,:])):
            if abs(Ai[0,i])>0:
                object3[:,l]=Ai[:,i]
                l=l+1
        Ai = object3
    return Ai
object2 = np.zeros((3,3))
a=1
j=0
k=0
if a==0:
    object2[:,3*0] = np.array([0,0,0])
    object2[:,3*1] = np.array([0,0,0])
    object2[:,3*2] = np.array([0,0,0])
else:
    if j == 0:
        object2[:,0] = np.array([0,0,0])
    else:
        object2[:,0] = np.array([a*j,j-1,k])
    if k == 0:
        object2[:,1] = np.array([0,0,0])
    else:
        object2[:,1] = np.array([a*k,j+1,k-1])
    if k+1==0:
        object2[:,2] = np.array([0,0,0])
    else:
        object2[:,2] = np.array([-a*(k+1),j,k+1])
hmpols = 7
for k in range(hmpols):
    a = Keliminator(Aidiff(object2,2*k+2))
    coefficients=[]
    for z in range(2*hmpols):
        c = np.zeros(2)
        for i in range(len(a[0,:])):
            if a[1,i] == k+1-z:
                c[0]=c[0]+a[0,i]
                c[1]=k+1-z
        coefficients.append(c)
    print('p_'+str(2*k+2)+
          ' = '+str(coefficients[0][0])+'*z^'+str(int(coefficients[0][1])))
    for b in range(len(coefficients)-1):
        if abs(coefficients[b+1][0])>0:
            print("{:11}".format('+'+str(coefficients[b+1][0])+'*z^'+str(int(coefficients[b+1][1]))))
stop = timeit.default_timer()
print('computation time is '+str(stop - start)+' seconds')
```

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