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BACHELOR THESIS

Applied Mathematics and Applied Physics

Symmetries of the Standard Model

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Abstract

This thesis is divided into five parts (covering chapter 1–5), that together try to give the reader a basic understanding of the symmetries of and the mathematical structure behind the standard model.

The first two chapters cover some Lie and representation theory of the Lie groups $U(1)$, $SU(2)$ and $SU(3)$, which play a central role in the standard model. In the third chapter, the structure of the standard model is given, based purely on observed symmetries and (quantum numbers of) particles. In this chapter, the mathematics of the first two chapters is not used yet.

In the fourth chapter, the Lie theory of chapter 1 is used to treat the gauge theory of $U(1)$, $SU(2)$ and $SU(3)$, where for $SU(2)$ also spontaneous symmetry breaking is incorporated. Finally, the fifth and last theoretical chapter gives examples of how representation theory and gauge theory can together structure the particles of the standard model and can dictate which particles and particle interactions are physically allowed.

The thesis ends with some recommendations for further study of the topic at hand and for further research to symmetries beyond the standard model.

Contents

Introduction

Symmetries can be found in many different and unexpected places in nature: in the spirals of the milky way, in the web of a spider and in the structure of snowflakes. But also, in a way that is both very explicit and beautiful, in the organization and interactions of the elementary particles of which our world is made. Since the first observations of elementary particles, theoretical and mathematical physicists have been trying to mathematically describe the structure and interactions of elementary particles. And to a great extend, they have succeeded: in the form of the *Standard Model* of elementary particles.

The standard model is a theory that describes two different aspects of particle physics that are closely connected. First of all, the standard model classifies all known (observed) particles. Secondly, it describes how three of the four fundamental forces in the universe, namely the electromagnetic, the weak and the strong force, act on these particles. It excludes the gravitational force, which is one of the (increasingly many) reasons that the standard model is not believed to be complete. However, the standard model is still a powerful theory. With it, multiple predictions have been made and verified. The hypothesized and proven existence of the top quark and the Higgs boson are probably the most famous examples of the predictive power of the standard model. Furthermore, the standard model is still used as a basis for many new theories that try to explain observations that the standard model leaves unexplained.

The aim of this bachelor thesis is to provide an introduction to the symmetries of the standard model for undergraduate students in physics and mathematics. The thesis is divided into five parts (covering chapter 1–5), that together give the reader a basic understanding of the mathematical structure behind the standard model.

The first and also the shortest chapter gives an introduction to Lie groups and Lie algebras. Here, we restrict ourselves to matrix Lie groups and focus on the symmetry groups $U(1)$, $SU(2)$ and $SU(3)$. This focus is motivated by the central role that these three groups play in the description of the standard model. In chapter 2, we further expand our knowledge of Lie groups and Lie algebras by looking at the representations of $U(1)$ and $SU(3)$. Then, in chapter 3, the structure of the standard model is described on the basis of the symmetries and elementary particles that are observed in the universe. Hereafter, in chapter 4 we treat the subject of *gauge theory*, motivated by the conservation of certain quantum numbers of particles.

Chapter 5 concludes our treatment of the symmetries of the standard model, by showing how gauge theory can be combined with representation theory, to structure the particles in the standard model and dictate which particle interactions are allowed. In this chapter, we will furthermore see how certain symmetry requirements have led to restrictions on particle masses and to the prediction of new particles.

Finally, in chapter 6, some suggestions for further study are given. For, while this thesis aims to give a basic understanding of the symmetries of the standard model, it is still just an introduction to the subject. Many topics that have been touched upon during the theoretical chapters can be explored further, to get a deeper understanding of the subject.

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I would hereby like to thank my supervisors for all their guidance, their involvement in both my progress and study career and their willingness to accompany me on my journey to the symmetries of the standard model.

Chapter 1

Lie groups and algebras

Together with chapter 2, this chapter aims to give an overview of the definitions and theorems that are necessary to mathematically describe the standard model. While chapter 2 focuses on representation theory, this chapter mainly gives an introduction to basic (matrix) Lie theory. The chapter starts with a short recap of groups and group homomorphisms and from here continues to the subject of Lie groups and Lie algebras.

The last section of this chapter, section 1.3, is the most extensive. It focuses on the Lie algebras of the Lie groups $U(1)$, $SU(2)$ and $SU(3)$, which play a central role in the description of the particles of the standard model.

1.1 Groups and homomorphisms

In this section, a short recap of the basic definitions of group theory will be given, that can often be found in first year algebra courses and that will be used throughout the whole thesis.

Definition 1.1. A *group* is a set G with an operation $G \times G \to G$ (here denoted as $(g_1, g_2) \to g_1 \cdot g_2$), satisfying

- $\forall g_1, g_2, g_3 \in G : g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$
- $\forall g \in G : g \cdot e = e \cdot g = g$
- $\forall g \in G$ there is a $g^{-1} \in G$ such that $g \cdot g^{-1} = g^{-1} \cdot g = e$

here $e \in G$ denotes the *identity* and g^{-1} is called the *inverse* of g. If G additionally satisfies the commutativity property

- $\forall g_1, g_2 \in G : g_1 \circ g_2 = g_2 \circ g_1$

the group is called *abelian*.

As we will only talk about matrix groups whose group operation "$\cdot$" is ordinary matrix multiplication, we will from now on omit the group operator and just write gh instead of $g \cdot h$. For Lie groups consisting of $n \times n$ matrices, the group identity corresponds with the identity matrix. We will denote the $n \times n$ identity matrix by $\mathbb{1}_n$ and write it instead of e.

Later, when we look at particles, the group elements are going to play the role of transformations. The group operation then determines the way in which multiple transformations – leading again to a new transformation – have to be performed. If the group elements are transformations, there has to be something that they transform. Indeed, we will see that particles can be viewed as vectors that are transformed when multiplied by the elements of a matrix group.

When studying a group G, we look at its *subgroups*, defined to be the nonempty subsets H of G such that H is a group. Some types of subgroups are

- a *proper subgroup*: A group H is a proper subgroup of the group G if it is a subgroup containing some but not all elements of G.

- a *normal subgroup*: A group N is a normal subgroup of G if it is a subgroup of G such that $\forall g \in G, n \in N : g^{-1}ng \in N$.

Depending on its type of subgroups, G can be assigned different names.

- a *simple group* is a group that has only itself as a non-trivial normal subgroup, the trivial normal subgroup being $\{e\}$.
- a *semisimple group* is a group that has no abelian, normal subgroups.

1.1.1 Group homomorphism

The relationships between different groups can be expressed by homomorphisms.

Definition 1.2. A *homomorphism* is a map $\varphi : G \to G'$ that preserves products, meaning that for all $g_1, g_2 \in G$ $\varphi(g_1 \circ g_2) = \varphi(g_1) \star \varphi(g_2)$, with "$\circ$" the group operation of G and "$\star$" that of G'.

From this definition, one can easily show that φ also preserves inverse and identity [16] so that the image $\phi[G]$ is itself a group. A homomorphism from G to itself is called an *endomorphism*. The set of endomorphisms is denoted by $\mathrm{End}(G)$. If a homomorphism φ is bijective, it is called an *isomorphism*. In that case the groups can in a way be seen as "equal" because they have a comparable size and structure and we write $G \cong G'$. An endomorphism that is an isomorphism as well, is called an *automorphism*. The set of automorphisms is denoted by $\mathrm{Aut}(G)$.

1.1.2 Direct product

Two or more groups can be combined into another group by taking their direct product.

Definition 1.3. For two groups G_1 and G_2 the *direct product* of G_1 and G_2 is the set

$$G_1 \times G_2 = \{(g_1, g_2) : g_1 \in G_1, g_2 \in G_2\},$$

with the operation "$\cdot$" of $G_1 \times G_2$ given by $(g_1, g_2) \cdot (h_1, h_2) = (g_1 \circ h_1, g_2 \star h_2)$ for all $g_1, h_1 \in G_1$ and $g_2, h_2 \in G_2$. Here "$\circ$" is the group operation of G_1 and "$\star$" the group operation of G_2.

By checking the three group properties of definition 1.1 one can directly show that $G_1 \times G_2$ is itself a group.

1.2 Lie groups

Transformations can take many forms, but usually we see them as the change of position of an object in space (or space-time). Examples of transformations are reflection in a plane, translation along a line or rotation about a certain axis. The concept of transformations is closely related to that of groups, for very often a set of transformations can be described mathematically by a group.

We can distinguish two types of groups of transformations, namely so called *continuous groups* and *discrete groups*. Continuous groups consist of transformations that can be described by one or more continuously varying parameters. For a discrete group, this is not possible for all its elements.

An example of a set of continuous transformations that forms a group is the set of rotations of a 2-dimensional Euclidean space. These rotations are elements of $SO(2)$, the group of orthogonal 2×2 matrices with unit determinant. We can write every element in $SO(2)$ as [2]

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{1.1}$$

for some $\theta \in (0, 2\pi]$, from which we see that indeed elements of $SO(2)$ correspond with rotations in the xy-plane. Furthermore, because the interval $(0, 2\pi] \subseteq \mathbb{R}$ contains infinitely many elements, we see that $SO(2)$ contains infinitely many elements as well.

Clearly, any group that consists of finitely many elements, is a discrete group, as continuity is only possible for infinite sets. The group of rotations of a three dimensional cube, such that after rotation the cube looks the same from all perspectives, is an example of a discrete group, because this group consists of finitely many (namely 24) elements. Intuitively, this is true as well, for the transformations of this discrete group, rotate the cube with discrete “jumps” in space.

We define the *dimension* of a continuous group not to be the number of elements, but the number of independent parameters needed to describe all group elements. These independent parameters are then called the *group parameters.* Because equation 1.1 tells us that each element in $SO(2)$ can be described using only one parameter θ, we conclude that $SO(2)$ has dimension 1.

The same conclusion can be found when deriving the dimension directly from the definition of $SO(2)$. The definition of $SO(2)$

$$SO(2) := \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a, b \in \mathbb{R} \text{ with } a^2 + b^2 = 1 \right\},$$

tells us that each element of SO(2) can be written in terms of the two 1-dimensional variables a and b restricted to $a^2 + b^2 = 1$. The restriction removes one degree of freedom and leaves us with one free parameter. Thus, again it follows that $SO(2)$ has dimension 1.

If both the group multiplication and inverse depend smoothly on the group parameters, the group is a so called *Lie group* [16]. The formal, most general definition of a Lie group is that it is a group that is smooth manifold. However, we will not go into depth of the exact meaning of this definition and restrict ourselves to the informal, but, for our use, tolerable definition given above.

Before we give an overview of the Lie groups that we will encounter, we introduce the general linear group, of which all the groups that we will consider are subsets.

Definition 1.4. The *General Linear group* $GL(V)$ of a finite dimensional vector space V over a *field* $\mathbb{F}$, consists of all $n \times n$ invertible matrices over $\mathbb{F}$. For an n-dimensional vector space $V = \mathbb{F}^n$, we can write $GL(n, \mathbb{F})$ instead of $GL(V)$.

Definition 1.5. A matrix Lie group is a closed[1] subgroup of $GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$, for some $n \in \mathbb{N}$.

In our case, the field $\mathbb{F}$ in definition 1.4 and 1.5 will always be either $\mathbb{R}$ or $\mathbb{C}$[2]. In the sequel, we will restrict our treatment of Lie groups to matrix Lie groups, because these groups are the only Lie groups that play a role in the description of the standard model.

The matrix Lie groups that we will encounter, are given in table 1.1. In this table, U^* denotes the conjugate transpose of the complex matrix U. In the sequel, we will always denote the complex conjugate of a matrix by the superscript *.

Lie Group	Definition
$O(n)$	$\{M \in GL(n, \mathbb{R}) : MM^T = \mathbb{1}\}$
$U(n)$	$\{M \in GL(n, \mathbb{C}) : MM^* = \mathbb{1}\}$
$SO(n)$	$\{M \in O(n) : \det M = 1\}$
$SU(n)$	$\{M \in U(n) : \det M = 1\}$

Table 1.1: *Overview of the matrix Lie groups under consideration.*

1.3 Lie algebras

One of the interesting properties of Lie groups is that they have a Lie algebra, which we will introduce in this section. Lie groups are usually quite complicated, curved objects. However, they have the special property that they can be completely or almost completely captured by a flat vector space, namely by the so called ‘tangent space at the identity’.

[1] Closeness is topological subject that we will not treat in this thesis. Topologically, this subject is treated in much detail in [5]. In [8], it is shown why the Lie groups of our interest (i.e. $U(1)$, $SU(2)$ and $SU(3)$) are closed groups of respectively $GL(\mathbb{C})$, $GL(\mathbb{C}^2$ and $GL(\mathbb{C}^3$.

[2] Because $\mathbb{R}$ and $\mathbb{C}$ are known sets and because we will not encounter any other fields, we will not give the general definition of a field.

1.3.1 The tangent space of a Lie group

Because matrix Lie groups depend in a continuous manner on their group parameters, we can take so called *paths* in the group.

Definition 1.6. A *path* in a group G is a differentiable function[3] A from $[a, b] \subseteq \mathbb{R}$ (with $a < b$) to G.

Hence, the range of a path consists of an infinite set $\{A(t) : t \in [a, b] \subseteq \mathbb{R}\}$ of matrices A that change continuously as a function of the parameter t.

Definition 1.7. A *tangent vector* of G at the identity is equal to $A'(0)$, with $A(t)$ a path in G, whose derivative $A'(t)$ (with respect to t) exists and with initial value $A(0) = \mathbb{1}$.

Definition 1.8. The *tangent space at the identity* $T_1(G)$ of a group G consists of all the tangent vectors in G.

For the matrix Lie groups that we will consider, the tangent space can be determined quite easily. We will show how to obtain the tangent space of $SO(n)$ and state the result for the other relevant groups.

Theorem 1.1. The tangent space of $SO(n)$ is equal to the set of $n \times n$ real matrices X such that $X + X^T = \mathbf{0}$ [16].

Proof. " $\Longrightarrow$ " Suppose that X is a tangent vector of $SO(n)$ at the identity $\mathbb{1}$. We will prove that X satisfies $X + X^T = \mathbf{0}$.

By definition 1.7, every tangent vector X of $SO(n)$ will be of the form $A'(0)$, where $A = A(t)$ is a path of matrices in $SO(n)$ such that $A'(t)$ exists and $A(0) = \mathbb{1}$.

Furthermore, because $A(t) \in SO(n)$ for any t, it holds that $A(t)A(t)^T = \mathbb{1}$ by the definition of $SO(n)$. If we take the derivative to t on both sides of this equality we get, by the product rule

$$A'(t)A(t)^T + A(t)A'(t)^T = \mathbf{0}.$$

The above equation holds for any t and hence for $t = 0$. For $t = 0$ we get

$$A'(0) + A'(0)^T = \mathbf{0},$$

from which follows that any tangent vector $X = A'(0)$ satisfies $X + X^T = \mathbf{0}$.

" $\Longleftarrow$ " Suppose that X is an $n \times n$ real matrix such that $X + X^T = \mathbf{0}$ $(\star)$. We will show that X is an element of the tangent space of $SO(n)$ at the identity.

For $n \times n$ matrices satisfying $AB = BA$ it holds that $e^{A+B} = e^A e^B$. For a proof see e.g. [16]. By our assumption $(\star)$ we have

$$XX^T = X(-X) = (-X)X = X^T X,$$

meaning that

$$e^X e^{X^T} = e^{X+X^T} = e^{\mathbf{0}} = \mathbb{1}.$$

Furthermore, because $(X^T)^m = (X^m)^T$ for any integer m, from the definition of the matrix exponential it follows that $e^{(X^T)} = (e^X)^T$. This means that

$$\mathbb{1} = e^X e^{X^T} = e^X (e^X)^T$$

from which follows that X is an orthogonal matrix. If we can furthermore show that e^X has determinant 1, we can conclude that $e^X \in SO(n)$. And indeed we can, by the following argument.

Because of orthogonality, e^X must have determinant ± 1. If we take a path of matrices tX with $t \in [0, 1]$, varying continuously, then e^{tX} varies continuously from $\mathbb{1} = e^{\mathbf{0}X}$ to $e^{1X} = e^X$. Furthermore, $\det(e^{tX})$ must stay the same, as it cannot jump between only two values when e^{tX} varies continuously. Because $e^{\mathbf{0}}$ has determinant 1, we must therefore have $\det e^X = 1$.

[3]Recall from analysis: a *differentiable function* is a function whose derivate exists and is finite in each point of the domain [11].

Thus we can conclude that if X satisfies $X + X^T = \mathbf{0}$, then $e^X \in SO(n)$. If we now take the path $A(t) = e^{tX}$ in $SO(n)$, we see that for this path X is the tangent vector at $\mathbb{1}$. We have

$$A'(t) = \frac{d}{dt} e^{tX} = X e^{tX},$$

meaning that $A'(0) = X$. Therefore, X must be a tangent vector of $SO(n)$ at the identity. □

Theorem 1.2 (*Lie bracket property* [16])**.** The tangent space $T_1(G)$ of a Lie group G is closed under the Lie bracket, meaning that

$$\forall X, Y \in T_1(G) : [X, Y] := XY - YX \in T_1(G).$$

Definition 1.9. A *matrix Lie algebra* $\mathfrak{g}$ is a vector space of matrices that is closed under the Lie bracket.

Combining definition 1.9 with theorem 1.2, makes us conclude that the tangent space of a Lie group G is a Lie algebra of G. Furthermore, using theorem 1.1, we see that the Lie algebra of $SO(n)$ consists of the $n \times n$ matrices X such that $X + X^T = 0$.

The following table gives the Lie algebras of the groups in table 1.1. The proofs are comparable to the proof of theorem 1.1 and can be found e.g. in [16].

Lie Group	Lie Algebra
$O(n)$	see Lie algebra of $SO(n)$
$U(n)$	$\mathfrak{u}(n) = \{n \times n$ complex matrices $X : X + X^* = \mathbf{0}\}$
$SO(n)$	$\mathfrak{so}(n) = \{n \times n$ real matrices $X : X + X^T = \mathbf{0}\}$
$SU(n)$	$\mathfrak{su}(n) = \{n \times n$ complex matrices $X : X + X^* = \mathbf{0}$ and $\mathrm{Tr}(X) = 0\}$

Table 1.2: *The Lie algebras of the Lie groups in table 1.1.*

1.3.2 From the Lie algebra to the Lie group

In the introduction of this section, we said that Lie groups could often be completely captured by their (flat and therefore less complex) Lie algebras. In this subsection, we will further investigate the relation between the Lie group and its Lie algebra.

For every matrix Lie group $G \subseteq GL(n, \mathbb{F})$, we can write its Lie algebra $\mathfrak{g}$ as [9]

$$\mathfrak{g} = \{X \in GL(n, \mathbb{F}) : \forall a \in \mathbb{R},\ e^{aX} \in G\}.$$

Theorem 1.3. For a compact and connected[4] Lie group G with Lie algebra $\mathfrak{g} = T_1(G)$, the map $\exp : \mathfrak{g} \to G$, given by

$$\exp(X) = e^X,$$

for all $X \in \mathfrak{g}$, is surjective [9].

Definition 1.10. A *Lie algebra homomorphism* is a homomorphism $\phi : \mathfrak{g} \to \mathfrak{g}'$, with $\mathfrak{g}$ the Lie algebra of the Lie group G and $\mathfrak{g}'$ that of G', that respects the Lie bracket, meaning that

$$\phi([X, Y]) = [\phi(x), \phi(y)], \tag{1.2}$$

for all $X, Y \in \mathfrak{g}$.

Theorem 1.4. If $\rho : G_1 \to G_2$ is an isomorphism of the two matrix Lie groups G_1 and G_2, then $\dot{\pi}$ is an isomorphism of the Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$, where $\dot{\pi}$ is defined by

$$\dot{\pi}(X) := \frac{d}{dt}\Big|_{t=0} \rho(e^{tX}) \text{ [9]}.$$

[4]We will not treat the exact meaning of compactness and connectedness. A topological treatment of these subjects can be found in e.g. [5].

Because the homomorphisms in theorem 1.3 and 1.4 are surjective, we can visualize them in the following diagram.

$$\begin{array}{ccc} G_1 & \xrightarrow{\pi} & G_2 \\ {\scriptstyle \exp(G_1)}\uparrow & & \uparrow{\scriptstyle \exp(G_2)} \\ \mathfrak{g}_1 & \xrightarrow{\dot{\pi}} & \mathfrak{g}_2 \end{array}$$

This diagram is said to 'commute' in that it does not matter whether we follow the arrows via the upper or the lower path, when starting from the Lie algebra $\mathfrak{g}_1$: both directions result in the same group G_2.

Suppose that we have a Lie algebra $\mathfrak{g}$ of a Lie group G and that $\mathfrak{g}$ has dimension[5] $d < \infty$. This means that a basis of $\mathfrak{g}$ can be given, in the form of d matrices in the Lie algebra, that span the whole matrix vector space. Such matrices are called *generators* of G. The choice of this name becomes clear when we consider a complete set of generators $\{T_1, ..., T_d\}$ of G. Then every $X \in \mathfrak{g}$ can be written as

$$X = \xi_1 T_1 + ... + \xi_d T_d, \tag{1.3}$$

where the ξ_i $(i = 1, ..., d)$ are elements of $\mathbb{R}$. They are called the *group parameters*. Combining theorem 1.3 and equation 1.3 enables us to write each $g \in G$ as

$$g = e^X = e^{\xi_1 T_1 + ... + \xi_d T_d}.$$

Thus, we see that the generators of G indeed 'generate' the Lie group G. To make our notation more compact, in the following we will use the *Einstein notation*. This is simply a short way of writing a finite summation

$$y = \sum_{i=1}^{n} c_i x^i$$

(where the superscript denotes the ith component of x, not the power) as $y = c_a x^a$, where it should be clear from the context from and till which integer the summation runs. In the above example a runs from 1 to n.

Definition 1.11. A subgroup T of the non-abelian group G is called a *maximal torus*, if it is the largest compact, connected, abelian subgroup of G.

Definition 1.12. The *rank* of a group G is the dimension of the maximal torus of G. If G is abelian, the rank of G is equal to the dimension of G.

1.3.3 The Lie algebras of $U(1), SU(2)$ and $SU(3)$

In the next chapters, we will focus of the Lie groups $U(1)$, $SU(2)$ and $SU(3)$, because of the role that they play in the standard model. In this section, we will, for the three groups separately, determine the dimension of the Lie algebras and give a suitable set of group generators.

The Lie algebras of $U(1)$, $SU(2)$ and $SU(3)$ (which follow directly from the Lie algebras of $U(n)$ and $SU(n)$ as given in table 1.2) are finite dimensional. Their dimension can easily be determined by counting the number of independent real parameters in the matrices of the Lie algebra. We will show this for general n in the the following theorem.

Theorem 1.5. The dimension of the Lie algebra of

- (1): $U(n)$ is equal to n^2
- (2): $SU(n)$ is equal to $n^2 - 1$ [6].

Proof. (1): For $U(n)$, we know that the Lie algebra consists of the $n \times n$ complex matrices such that $X + X^* = \mathbf{0}$. Each matrix in the Lie algebra has $n(n-1)/2$ elements above the diagonal. Those terms can be complex and hence contribute to $2 \cdot n(n-1)/2 = n(n-1)$ independent group parameters. We denote the element of the i-th row and the j-th column of a matrix X in the Lie algebra of $U(1)$ by X_{ij}. Then, to have $X + X^* = \mathbf{0}$, the elements X_{ij} below the diagonal (so

[5] Note that the definition of 'the dimension of a Lie algebra' is the same as that of the dimension of any other Lie group. This definition has been given in section 1.2.

[6] The proof of this theorem is based on a proof in [16].

with $i > j$) must be equal to $-\bar{X}_{ji}$. Hence, these elements depend fully on the already determined group parameters.

To finish our counting, we still have to take into account the n diagonal terms. These must be purely imaginary and hence result in n additional group parameters. Our final result for the dimension of $\mathfrak{u}(n)$ is

$$\dim \mathfrak{u}(1) = n(n-1) + n = n^2.$$

(2): For $SU(n)$, we know that the Lie algebra consists of the $n \times n$ traceless, complex matrices such that $X + X^* = \mathbf{0}$. Without the condition of tracelessness, we have seen in the proof of (1) that the dimension of the Lie algebra is equal to n^2. The condition of $\mathrm{Tr}(X) = 0$ reduces the number of independent group parameters with 1, leaving the dimension of $\mathfrak{su}(n)$ to be $n^2 - 1$. □

Note that the dimensions of the Lie algebras of $U(1)$ and $SU(n)$ is equal to the Lie group dimension, because the Lie algebras generate these Lie groups based on x independent parameters, where x is the dimension of the Lie algebra.

U(1)

By theorem 1.5, $\mathfrak{u}(1)$ has dimension $1^2 = 1$. Therefore $U(1)$ has one generator Y. We can choose

$$Y = i,$$

which clearly spans the Lie algebra $\mathfrak{u}(1) = i\mathbb{R}$, and write, using theorem 1.3, each $z \in U(1)$ as

$$z = e^{aY} = e^{ai} \tag{1.4}$$

with single group parameter $a \in \mathbb{R}$.

SU(2)

By theorem 1.5, $\mathfrak{su}(2)$ has dimension $2^2 - 1 = 3$. Therefore $SU(2)$ has three generators t_a. They are generally written in terms of the *Pauli spin matrices* τ_a $(a = 1, 2, 3)$. We choose the tree generators as $t_a := \frac{1}{2} i \tau_a$, with the Pauli spin matrices given by

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.5}$$

One can check that indeed every element of $\mathfrak{su}(2)$, as given by table 1.2, can be written in terms of the generators t_a and that the generators are linearly independent. Hence, by theorem 1.3, we may write each $U \in SU(2)$ as

$$U = e^{\alpha_a t^a},$$

with $\alpha_a \in \mathbb{R}$, $(a = 1, 2, 3)$. The generators satisfy the following commutation relation

$$[t_a, t_b] = -\epsilon_{abc} t^c, \tag{1.6}$$

where ϵ is the *Levi-Civita* symbol, defined to be

$$\epsilon_{abc} = \begin{cases} 1 & (a,b,c) \text{ is } (1,2,3), (2,3,1) \text{ or } (3,1,2) \\ -1 & (a,b,c) \text{ is } (3,2,1), (2,1,3) \text{ or } (1,3,2) \\ 0 & \text{else.} \end{cases}$$

We see that ϵ_{abc} is *anti-symmetric* in (a, b, c), meaning that symmetric permutations of (a, b, c) result in the same value of ϵ and anti-symmetric permutation in a value of $-\epsilon$.

SU(3)

By theorem 1.5, $\mathfrak{su}(3)$ has dimension $3^2 - 1 = 8$. Therefore $SU(3)$ has eight generators T_a. As a basis, we take the matrices $iT_a := i\lambda_a/2$, where λ_a $(a = 1, ..., 8)$ are the *Gell-Mann matrices*, given by

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$
$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

One can prove that the set of matrices $\{iT_1, iT_2, ..., iT_8\}$ indeed form a basis of $\mathfrak{su}(3)$ by showing that they span $\mathfrak{su}(3)$ and that they are linearly independent. With this basis, again by theorem 1.3, we can write each $g \in SU(3)$ as

$$g = e^{i\xi_a T^a}$$

for some values of the eight real group parameters ξ^a. The following commutation relation holds for the generators

$$[T_a, T_b] = f_{abc} T^c, \tag{1.7}$$

with f_{abc} the structure constants of $SU(3)$. Just as the Levi-Civita symbol, they are anti-symmetric (for a proof see e.g. [10]). They are given by [4]

$$f_{abc} = \begin{cases} 1 & (a,b,c) = (1,2,3) \\ \frac{1}{2} & (a,b,c) = (1,4,7), (1,6,5), (2,4,6), (2,5,7), (3,4,5) \text{ or } (3,7,6) \\ \frac{\sqrt{3}}{2} & (a,b,c) = (4,5,8) \text{ or } (6,7,8), \end{cases}$$

where we have not written down all permutation of (a, b, c) as these can be determined by the fact that f_{abc} is anti-symmetric. The f_{abc} that are not determined by permutations of the indices in equation 1.3.3 have value zero.

Chapter 2

Group representations

Groups and definitely non-matrix groups are often abstract sets. However, many groups can be represented by a group of finite dimensional, invertible matrices: by subsets of the general linear group (definition 1.4), on which ordinary matrix multiplication acts as the group operator. The matrices of the group representation are often easier to categorize than the original group. Hence, a group representation gives a new way to receive information about the original group.

However, in particle physics, representation theory plays another important role. To be able to fully understand this importance, in this chapter, the representation theory of the Lie groups $U(1)$ and $SU(3)$ will be treated. The representation theory of $SU(2)$, which is also relevant for the topic at hand, will not be treated in this chapter. We will encounter it briefly in the beginning of chapter 5.

This chapter is divided into six sections. The first three sections treat respectively unitary, irreducible and Lie algebra representations. Using the theory of weights given in section 2.4, section 2.5 and 2.6 treat the topic of dual and tensor product representations of $SU(3)$.

2.1 Unitary representations

Because the groups $U(1)$, $SU(2)$ and $SU(3)$ are all defined in terms of the field $\mathbb{C}$, in the following, we will assume that the vector spaces under consideration are defined over $\mathbb{C}$ (and not over $\mathbb{R}$ or any other field).

Definition 2.1. A *representation* of a group G on a vector space V is a group homomorphism $\pi : G \to GL(V)$.

The *dimension* of the representation is defined to be equal to the dimension of V. We often denote the representation π of G on the vector space V by (π, V) and call this pair the representation of G. Sometimes, when it is clear about which group representation we are talking, we just call V the representation of G. Besides, we sometimes shorten 'the representation of G' to 'the G-representation'.

Every Lie group G has a so called *trivial representation* (π, V), given by $\pi(g)v = v$ for all $g \in G, v \in V$. Furthermore, each matrix Lie group has a so called *defining representation*, that maps each element to itself. For example, the defining representation of $SU(2)$ acts on the vector space $V = \mathbb{C}^2$ and is the map $\pi : SU(2) \to SU(2)$ given by

$$\pi(U) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = U \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \tag{2.1}$$

for all $U \in SU(2)$. This representation has dimension 2, because it acts on a 2-dimensional vector space, while the group $SU(2)$ itself has dimension 3, as we have seen in section 1.3.

Definition 2.2. The *adjoint representation* (π^{ad}, V) of a matrix Lie group G with Lie algebra $\mathfrak{g}$ is given by the homomorphism $\pi^{\text{ad}} : G \to \text{Aut}(\mathfrak{g})$ as

$$\pi^{\text{ad}}(g)(X) := gXg^{-1}, \tag{2.2}$$

for X any element in $\mathfrak{g}$, $g \in G$[1].

Definition 2.3. A representation (π, V) is called a *unitary representation*, if it respects the inner product on V, meaning that

$$\langle \pi(g)v, \pi(g)w \rangle = \langle v, w \rangle$$

for all $v, w \in V$.

Theorem 2.1. A representation of G is unitary if and only if $\pi(g)^* = \pi(g^{-1})$ for all $g \in G$.

Proof. We have that (π, V) unitary iff

$$\begin{aligned} \langle \pi(g)v, \pi(g)w \rangle &= \langle v, w \rangle \\ &= \langle \pi(g)^{-1}\pi(g)v, w \rangle \\ &= \langle \pi(g)v, (\pi(g)^{-1})^* w \rangle, \end{aligned} \tag{2.3}$$

where the last equality holds because, by the definition of the inner product, for any operator A acting on $v \in V$ we can write

$$\langle Av, w \rangle := (Av)^* w = (v^* A^*) w = v^* (A^* w) = \langle v, A^* w \rangle.$$

From equation 2.3, it follow that $\pi(g)w = (\pi(g)^{-1})^* w$. This can be rewritten as $\pi(g)^* = \pi(g)^{-1} = \pi(g^{-1})$. Because the above argument can be read backwards as well, this completes the proof. $\square$

For a representation $\pi : G \to GL(V)$ we write $\pi|_W : G \to GL(W)$ for a restriction of the mapping to elements of $W \subset V$.

Definition 2.4. A homomorphism $\pi|_W$ is called a *subrepresentation* of the G-representation (π, V) if W is a subspace of V such that W is invariant under G, meaning that

$$\forall w \in W, g \in G : \pi(g)w \in W.$$

Theorem 2.2. If W is a subrepresentation of a unitary representation V, then the orthogonal complement $W^\perp = \{v \in V : \langle v, w \rangle = 0 \; \forall w \in W\}$ is a subrepresentation of V as well.

Proof. Suppose W is a subrepresentation of a unitary representation V. Take $w^\perp \in W^\perp$ arbitrary. Then for any $w \in W$

$$\begin{aligned} \langle \pi(g)w^\perp, w \rangle &= \langle w^\perp, \pi(g)^* w \rangle \\ &= \langle w^\perp, \pi(g)^{-1} w \rangle \qquad \text{theorem 2.1.} \end{aligned}$$

Because W is a subrepresentation, each $w \in W$ can be written as $w = \pi(e)w = \pi(g)\pi(g)^{-1}w = \pi(g)\pi(g^{-1})w = \pi(g)w'$, with $w' \in W$. The last equality is true because W is a subrepresentation, meaning that $\pi(g)w \in W$ for all $g \in G$, so also for g^{-1}. Hence, we are allowed to write

$$\langle w^\perp, \pi(g)^{-1} w \rangle = \langle w^\perp, w' \rangle,$$

for some $w' \in W$. By the definition of $W^\perp$, this means that $\langle w^\perp, w' \rangle = 0$. Combining the the equalities results in $\langle \pi(g)w^\perp, w \rangle = \langle w^\perp, w' \rangle = 0$, meaning that $\pi(g)w^\perp \in W^\perp$, so that $W^\perp$ is indeed a subrepresentation of V. $\square$

Definition 2.5. For two representations (π, V) and (ρ, W) on a group G, the linear map $A : V \to W$ is called an *intertwiner* if

$$A\pi(g) = \rho(g)A$$

for all $g \in G$.

If A is invertible, it is called an *equivalence*. Two representations are called *equivalent* if there exists an equivalence between them. A is a *unitary equivalence* if it is an equivalence that respects the inner product, meaning that

$$\langle v, v' \rangle_V = \langle Av, Av' \rangle_W,$$

where the subscripts denote that the inner product is taken respectively in the vector space V and W. If two representations are unitary equivalent, they can be seen as 'the same'.

[1]This definition is well-defined, as is explained in e.g. [9]

2.1.1 Direct sum of representations

Definition 2.6. For two vector spaces V and W over the same field $\mathbb{F}$, the *direct sum* $V \oplus W$ is defined to be the set

$$V \oplus W = \{(v, w) : v \in V, w \in W\},$$

such that $(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)$ and $\lambda(v, w) = (\lambda v, \lambda w)$ for any $\lambda \in \mathbb{F}$.

Furthermore, a vector space U is the direct sum of two linear subspaces V and W if for any $u \in U$ we can uniquely write $u = v + w$, with $v \in V$ and $w \in W$. In that case we write $U = V \oplus W$.

Definition 2.7. The *direct sum* of two representations (π, V) and (ρ, W) is the vector space $V \oplus W$, with representation ϕ given by $\phi(g)(v \oplus w) := (\pi(g)v, \rho(g)w)$.

We can write ϕ in the above definition equivalently as a representation of G that maps $g \in G, v \oplus w \in V \oplus W$ to

$$\phi(g)(v \oplus w) = \begin{pmatrix} \pi(g) & 0 \\ 0 & \rho(g) \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix},$$

where the zeros denote matrices will all entries zeros of the appropriate dimension (e.g. $\dim(V) \times \dim(W)$ for the upper right zero). We then immediately see that the representation preserves products and hence is well-defined, because for all $g, h \in G$ we have

$$\begin{aligned}
\phi(g)\phi(h) &= \begin{pmatrix} \pi(g) & 0 \\ 0 & \rho(g) \end{pmatrix} \begin{pmatrix} \pi(h) & 0 \\ 0 & \rho(h) \end{pmatrix} \\
&= \begin{pmatrix} \pi(g)\pi(h) & 0 \\ 0 & \rho(g)\rho(h) \end{pmatrix} \\
&= \begin{pmatrix} \pi(gh) & 0 \\ 0 & \rho(gh) \end{pmatrix} && \pi \text{ and } \rho \text{ are representations} \\
&= \phi(gh).
\end{aligned}$$

Theorem 2.3. The direct sum $(\phi, V \oplus W)$ of two representations (π, V) and (ρ, W) is unitary if and only if (π, V) and (ρ, W) are unitary.

Proof. " $\Longrightarrow$ " Suppose ϕ is unitary, so we can write for any $g \in G$ and for all $v, v' \in V$, $w, w' \in W$

$$\begin{aligned}
\langle v, v' \rangle_V + \langle w, w' \rangle_W &= \langle v \oplus w, v' \oplus w' \rangle_{V \oplus W} \\
&= \langle \phi(g)(v \oplus w), \phi(g)(v' \oplus w') \rangle_{V \oplus W} && \phi \text{ is unitary} \\
&= \langle (\pi(g)v, \rho(g)w), (\pi(g)v', \rho(g)w') \rangle_{V \oplus W} && \text{Definition 2.7} \\
&= \langle \pi(g)v, \pi(g)v' \rangle_V + \langle \rho(g)w, \rho(g)w' \rangle_W.
\end{aligned}$$

This equation also holds for $w' = 0$. In that case, we get

$$\begin{aligned}
\langle v, v' \rangle_V &= \langle v, v' \rangle_V + \langle w, 0 \rangle_W && \langle w, 0 \rangle = 0 \\
&= \langle \pi(g)v, \pi(g)v' \rangle_V + \langle \rho(g)w, 0 \rangle_W \\
&= \langle \pi(g)v, \pi(g)v' \rangle_V && \langle \rho(g)w, 0 \rangle_W = 0,
\end{aligned}$$

from which follows that (π, V) is unitary. In the same way we can obtain that (ρ, W) is unitary.

" $\Longleftarrow$ " Suppose that π and ρ are unitary. Then we can write

$$\begin{aligned}
\langle v \oplus w, v' \oplus w' \rangle_{V \oplus W} &= \langle v, v' \rangle_V + \langle w, w' \rangle_W \\
&= \langle \pi(g)v, \pi(g)w' \rangle_V + \langle \rho(g)w, \rho(g)w' \rangle_W && \pi \text{ and } \rho \text{ unitairy} \\
&= \langle (\pi(g)v, \rho(g)w), (\pi(g)v', \rho(g)w') \rangle_{V \oplus W} \\
&= \langle \phi(g)(v \oplus w), \phi(g)(v' \oplus w' \rangle_{V \oplus W},
\end{aligned}$$

from which follows that ϕ is unitairy. □

Theorem 2.4. If W is a subrepresentation of a unitary representation (π, V) (with V a vector space over $\mathbb{C}$), then V is unitary equivalent to $W \oplus W^{\perp}$.

Proof. Suppose that W is a subrepresentation of a unitary representation (π, V). By theorem 2.2, we know that $W^\perp$ is a subrepresentation as well. Because W and $W^\perp$ span V and have an empty intersection, we can write each $v \in V$ as $v = aw + bw^\perp$, for $w \in W, w^\perp \in W^\perp$ and $a, b \in \mathbb{C}$. From definition 2.6, it follows that $V = W \oplus W^\perp$.

Furthermore, by definition 2.7, our representations on W and $W^\perp$ induce a representation on $W \oplus W^\perp$ given by $\phi(g)(w \oplus w^\perp) := (\pi(g)w, \pi(g)w^\perp)$.

Suppose that $\dim(V) = n$. If we then take $A = \mathbb{1}_n$, we see that A is an intertwiner of the representations $(\phi, W \oplus W^\perp)$ and $(\pi \oplus \rho, V)$. This follows directly from definition 2.7. Moreover, for the inner product, we can write

$$\begin{aligned} \langle v, v' \rangle_V &= \langle Av, Av' \rangle_V && A = \mathbb{1}_n \\ &= \langle Av, Av' \rangle_{W \oplus W^\perp} && V = W \oplus W^\perp, \end{aligned} \tag{2.4}$$

from which we conclude that A is an unitary equivalence and hence that V is unitary equivalent to $W \oplus W^\perp$. □

2.2 Irreducible representations

Definition 2.8. An *irreducible representation* is a representation $\pi(g) : G \to GL(V)$ that has only V and $\{0\}$ as a subrepresentation. If a representation is not irreducible, it is called *reducible*.

Theorem 2.5. Every finite-dimensional, unitary representation of G is unitary equivalent to a direct sum of finitely many irreducible, unitary representations.

Proof. We divide the proof into three steps. Let (π, V) be a unitary representation of G with finite dimension n.

(1) If (π, V) is irreducible, then we are done. Otherwise, continue to step 2.

(2) If (π, V) is not irreducible, then it has a non-trivial subrepresentation $W \subset V$, with W of smaller dimension $m < n$. Now, by theorem 2.4, V is unitary equivalent to the direct sum $W \oplus W^\perp$. If W and $W^\perp$ are both irreducible, then we are done. Otherwise, continue to step 3.

(3) We repeat step 1 and 2 for the vector spaces W and/or $W^\perp$ instead of V, and hereafter for their (potential) subrepresentations, and then for the possible subrepresentations of their subrepresentations, etc. The direct sum of the irreducible subrepresentations that we end up with, will be unitary equivalent to (π, V).

However, we still have to show that this repetition will be finite and that the direct sum will consist of finitely many irreducible representations. We can easily argue this, for, because V is finite dimensional and each subrepresentation will be of smaller dimension, we can repeat step 1 and 2 maximally n times, resulting in a direct sum of maximally n irreducible, unitary representations. □

Theorem 2.6 (Schur's Lemma)**.** If (π, V) is an irreducible, unitary representation of G, with V a vector space over a $\mathbb{C}$, then every intertwiner $A : V \to V$ is a multiple of the identity: $A = \lambda\mathbb{1}$ (λ in $\mathbb{C}$, depending on the field over which V is defined).

Proof. Let (π, V) be an irreducible, unitary representation of a group G, with V over $\mathbb{C}$. Let A be an intertwiner from V to V, so that $A\pi(g) = \pi(g)A$ for all $g \in G$. We first show that A has at least one eigenvalue (1), and then that this implies that A must be a multiple of the identity (2).

(1) Denote the dimension of V by n. Take $v \in V, v \neq 0$ and take the set $\{v, Av, A^2v, ..., A^nv\}$. Because this set consists of $n+1$ elements, while a linearly independent set of vectors in V has at most dimension n, the set is linearly dependent. Therefore, there exist complex numbers $a_0, ..., a_n$ that are not all equal to zero and that are such that

$$a_0 v + a_1 Av + ... + a_n A^n v = 0.$$

Furthermore, also the numbers $a_1, ..., a_n$ can not all equal zero, because in that case $a_0 v = 0$, from which unavoidable follows that $a_0 = 0$. We can therefore see the above equation as a polynomial

$p(A)$ with (at least two) non-zero complex coefficients. According to the fundamental theorem of algebra, any polynomial with complex coefficients has at least one root [15]. Hence, we can factorise this polynomial as

$$\begin{aligned} p(A)v &= (a_0\mathbb{1} + a_1 A + ... + a_n A^n)v \\ &= c(A - \lambda_1\mathbb{1})...(A - \lambda_m\mathbb{1})v, \end{aligned}$$

with c and $\lambda_1, ..., \lambda_m$ some complex constants, $c \neq 0$. In the factorization, we write m instead of n ($m < n$), because for $m > 1$ it is possible that the coefficients $a_{m+1}, ..., a_n$ are zero.

Note that the matrices $A - \lambda_i\mathbb{1}$ and $A - \lambda_j\mathbb{1}$ can be interchanged for any $i, j \in \{1, ..., m\}$, because these matrices commute. This means that the following argument holds for any order of the matrices $A - \lambda_i\mathbb{1}$ in $p(A)$.

Because $p(A)v = 0$, we can argue as follows. If the term $(A - \lambda_m\mathbb{1})v$ is equal to zero, we have found an eigenvalue of A. Otherwise, we take the largest $i \in \{1, ..., m\}$ such that

$$0 = (A - \lambda_i\mathbb{1})(A - \lambda_{i+1}\mathbb{1})...(A - \lambda_m\mathbb{1})v = (A - \lambda_i\mathbb{1})w,$$

and $w = (A - \lambda_{i+1}\mathbb{1})...(A - \lambda_m\mathbb{1})v \neq 0$. This results in the eigenvalue λ_i of A. We know that there will always be such an i, because if $w =\neq 0$, for all $i \leq 1$, we still must have

$$0 = \frac{p(A)}{c} = (A - \lambda_1\mathbb{1})w,$$

from which follows that λ_1 is an eigenvalue of A.

(2) Because A has at least one eigenvalue λ, the kernel $\ker(A - \lambda\mathbb{1}) := \{v \in V : Av = \lambda\mathbb{1}v\}$ is nonempty. Now for $v \in \ker(A - \lambda\mathbb{1})$, $g \in G$

$$\begin{aligned} A\pi(g)v &= \pi(g)Av && \text{A is an intertwiner} \\ &= \pi(g)\lambda v && v \in \ker(A - \lambda\mathbb{1}) \\ &= \lambda\pi(g)v, \end{aligned}$$

so that $\pi(g)v$ is an element of the kernel as well. This means that $\ker(A-\lambda\mathbb{1})$ is a subrepresentation of V. Because (π, V) is irreducible and the kernel is non-empty, we must have $\ker(A - \lambda\mathbb{1}) = V$. This means that $\forall v \in V$ we have $Av = \lambda\mathbb{1}v$, from which we conclude that $A = \lambda\mathbb{1}$. □

Theorem 2.7. If T is an abelian group, then every irreducible, unitary representation of T is 1-dimensional.

Proof. Let T be an abelian group and let (π, V) be an irreducible representation of T. Take $t \in T$ arbitrary. Now for each $t \in T$, $\pi(t)$ is an intertwiner from T to T, because for all $t' \in T$

$$\pi(t)\pi(t') = \pi(tt') = \pi(t't) = \pi(t')\pi(t).$$

By theorem 2.6, we know that every intertwiner equals $\lambda\mathbb{1}$ for some $\lambda \in \mathbb{C}$. Thus, we have for any $v, v' \in V, t \in T$

$$\begin{aligned} \langle v, v'\rangle &= \langle \pi(t)v, \pi(t)v'\rangle && \pi \text{ is an intertwiner} \\ &= \langle \lambda v, \lambda v'\rangle && \pi(t) = \lambda\mathbb{1} \\ &= \lambda\bar{\lambda}\langle v, v'\rangle, \end{aligned}$$

so that $\lambda\bar{\lambda} = 1$. Hence, we can write λ as $\lambda = e^{i\mu}$ for some real valued μ. This means that for all t, $\pi(t) = \lambda\mathbb{1}$ is a diagonal matrix, with constant elements on the diagonal. This means that every subspace of V will form a subrepresentation, as for all $v \in V, t \in T$

$$\pi(t)v = e^{i\mu}\mathbb{1}v = e^{i\mu}v. \tag{2.5}$$

However, by assumption, $\pi(t)$ is irreducible and hence has no subrepresentation. Therefore, we must conclude that $\pi(t)$ is a 1×1 matrix, from which it follows that $\pi(t) = e^{i\mu}$ works on the 1-dimensional vector space $\mathbb{C}$ and thus that our representation is 1-dimensional. This completes the proof. □

With use of theorem 2.7, it is relatively easy to classify the irreducible representations of abelian Lie groups. In the next section, we will classify the irreducible representation of the abelian Lie group $U(1)$.

2.3 Lie algebra representations

In section 1.3, we have seen that there is a connection between Lie group and Lie algebra homomorphisms. In this section, we will use this connection to determine the irreducible representations of $U(1)$. As a consequence of theorem 1.4, the following statement about Lie algebra representations holds. For a complete proof, see e.g. [9].

Theorem 2.8. If $\pi : G \to GL(n, \mathbb{F})$ is a continuous representation of the Lie group G, then $\dot{\pi} : \mathfrak{g} \to \mathfrak{gl}(n, \mathbb{F})$ is a Lie algebra representation of the Lie algebra $\mathfrak{g}$ of G. Here $\dot{\pi}$ is defined as

$$\dot{\pi}(X) := \frac{d}{dt}\Big|_{t=0} \pi(e^{tX}), \tag{2.6}$$

for any $X \in \mathfrak{g}$.

From the definition of $\dot{\pi}$, it follows that [9]

$$\pi(e^{tX}) = e^{t\dot{\pi}(X)}. \tag{2.7}$$

We will first show how to find the irreducible representations of the 1-dimensional, abelian Lie group $U(1)$ and then proceed to the non-abelian Lie group $SU(3)$.

Theorem 2.9. The irreducible, unitary representations of the abelian group $U(1) = \{z \in \mathbb{C} : |z| = 1\}$ are labelled by $n \in \mathbb{Z}$ and are given by $\pi_n(z) = z^n$ for each $z \in U(1)$.

Proof. Suppose $\pi : U(1) \to GL(n, \mathbb{C})$ is an irreducible representation of $U(1)$. We will first consider the irreducible representations of its Lie algebra $\mathfrak{u}(1)$ and from it derive an expression for the representation of $U(1)$ itself.

The Lie algebra of $U(1)$ is, as we have seen in section 1.9, equal to $\mathfrak{u}(1) = i\mathbb{R}$. By theorem 2.7, we know that all irreducible representations of an abelian group are 1-dimensional. Therefore, any irreducible representation of $U(1)$ will work on the 1-dimensional vector space $\mathbb{C}$ and will thus be a homomorphism to the set $GL(1, \mathbb{C}) = U(1)$. Hence, by theorem 2.8, the Lie algebra representation $\dot{\pi}$ of $\mathfrak{u}(1)$ will have both a domain and a range $i\mathbb{R}$. This means that we can write $\dot{\pi} : i\mathbb{R} \to i\mathbb{R}$ and get a homomorphism of the following form

$$\dot{\pi}(X) = \mu X \qquad \mu \in \mathbb{R},$$

for any $X \in \mathfrak{u}(1)$. Now, according to equation 2.7,

$$\pi(z) = \pi(e^{tY}) = e^{t\dot{\pi}(Y)} = e^{t\mu Y} = z^{\mu},$$

is the representation of the Lie group for any $z \in U(1)$. In the above, we have used equation 1.4 to be able to write $z = e^{tY}$, with group parameter $t \in \mathbb{R}$ and group generator $Y \in \mathfrak{u}(1) = i\mathbb{R}$.

However, to the above equality cannot be conformed for all $\mu \in \mathbb{R}$. To see this, suppose that $t = 1$ and that X takes the value $i2\pi$ ($\in i\mathbb{R}$) and consider again equation 2.7. We have

$$1 = \pi(1) = \pi(e^0) = \pi(e^{2\pi i}) = e^{\mu 2\pi i},$$

which only holds if $\mu \in \mathbb{Z}$. Thus, the irreducible representations of the abelian group $U(1)$ are given by $\pi_n(z) = z^n$, with $n \in \mathbb{Z}$. □

For the non-abelian Lie group $SU(3)$, our approach will be different. To obtain as much information as possible about the irreducible representations of G, in the next section, we will first find the maximal torus T of $SU(3)$ and then classify the irreducible representations of T using the theory of so called 'weights'.

2.4 Weights

Our next desire is to be able to get as much information as possible about the representations of a general Lie group G and specifically – because of their important role in particle physics – about the the non-abelian Lie groups $SU(2)$ and $SU(3)$. Up and including section 2.6, we will focus on the representation theory of $SU(3)$. For $SU(2)$ a comparable, but easier approach can be used. We will not treat the representation theory of $SU(2)$ in this chapter, but we will encounter and briefly discuss it in chapter 5.

2.4.1 Maximal torus

In section 1.3, we have seen that the Lie algebra of $SU(3)$ is given by the 3×3 complex matrices X, such that $X + X^* = \mathbf{0}$ and $\mathrm{Tr}(X) = 0$ and that this Lie algebra had rank 2. This means that the maximal torus T of $SU(3)$ is 2-dimensional and hence can be generated by two generators that span the Lie algebra $\mathfrak{t}$ of T. We will show that the traceless, unitary diagonal 3×3 matrices form the Lie algebra of the maximal torus of $SU(3)$. Indeed, this group is clearly abelian and a subgroup of $\mathfrak{su}(3)$. Furthermore, it is two dimensional as each element consists of 3 different elements on the diagonal, with one degree of freedom removed by the requirement of zero trace. Hence, we conclude that the Lie algebra $\mathfrak{t}$ of T consists of matrices of the form

$$t = \begin{pmatrix} ix_{11} & 0 & 0 \\ 0 & ix_{22} & 0 \\ 0 & 0 & ix_{33} \end{pmatrix} \tag{2.8}$$

such that $x_{11} + x_{22} + x_{33} = 0$. As a basis of $\mathfrak{t}$, we choose the matrices

$$h_1 = i \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad h_2 = i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

These matrices indeed form a basis as they are linearly independent (there is no $a \in \mathbb{r}$ such that $h_1 = ah_2$) and – by making the substitution $x_{11} \to x_1$ and $x_{33} \to -x_2$ in equation 2.8 – we can write each $t \in \mathfrak{t}$ as

$$t = i \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 - x_1 & 0 \\ 0 & 0 & -x_2 \end{pmatrix} = x_1 i \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + x_2 i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = x_1 h_1 + x_2 h_2.$$

Thus, we see that h_1 and h_2 span $\mathfrak{t}$.

2.4.2 Weight and root lattice

Definition 2.9. A *weight* of the lie algebra representation $(\dot{\pi}, V)$ of the lie algebra $\mathfrak{t}$ of the maximal torus T of a group G is defined to be the linear mapping $\lambda : \mathfrak{t} \to \mathbb{R}$ such that the *weight space* V_λ given by

$$V_\lambda := \{v \in V : \forall X \in \mathfrak{g}, \quad Xv = \lambda(X)v\} \tag{2.9}$$

is nonempty.

So for a weight, there exist vectors v in V such that $Xv = \lambda(X)v$. On the vector space V_λ, λ can be seen as an eigenvalue of X. Because $X = ah_1 + bh_2$ for any $X \in \mathfrak{t}$, the weights of the maximal torus are fully determined by $\lambda(h_1)$ and $\lambda(h_2)$, the eigenvalues of h_1 and h_2.

Theorem 2.10. The weights $\lambda : \mathfrak{t} \to \mathbb{R}$ of the Lie algebra $\mathfrak{t}$ of the group T of unitary diagonal 3×3 matrices integrate into a group representation if and only if $\lambda(h_i) \in \mathbb{Z}$ for $i = 1, 2$.

Proof. " $\Longrightarrow$ " Suppose that the weights $\lambda : \mathfrak{t} \to \mathbb{R}$ integrate into a group representation. For all $t \in T$ we can write

$$\pi(t) = \pi(e^X) = e^{\dot{\pi}(X)} = e^{i\lambda(X)}. \tag{2.10}$$

X is an element of $\mathfrak{t}$ and hence can be written in term of the matrices h_1 and h_2. We write $X = ah_1 + bh_2$, with $a, b \in \mathbb{R}$ to get

$$\begin{aligned} \pi(t) &= \pi(e^{ah_1 + bh_2}) \\ &= \pi(e^{ah_1})\pi(e^{bh_2}) && T \text{ is abelian} \\ &= e^{i\lambda(ah_1)}e^{i\lambda(bh_2)} && \text{equation 2.10} \\ &= e^{ia\lambda(h_1)}e^{ib\lambda(h_2)} && \lambda \text{ is a linear map.} \end{aligned}$$

Now for $t = \mathbb{1}$, we can write

$$\begin{aligned} \mathbb{1} = \pi(\mathbb{1}) &= \pi(e^{\mathbf{0}}) \\ &= \pi(e^{2\pi h_1}) && h_1 \text{ diagonal matrix with entries } 0, \pm i \\ &= e^{2\pi i\lambda(h_1)} && \text{equation 2.10} \end{aligned}$$

to which can be conformed only when $\lambda(h_1)$ is an integer, so we conclude that $\lambda(h_1) \in \mathbb{Z}$. The above equation can also be written for $X = 2\pi h_2$, from which in an equivalent way follows that $\lambda(h_2) \in \mathbb{Z}$.

"$\Longleftarrow$" Suppose that the weights $\lambda(h_i) \in \mathbb{Z}$ for $i = 1, 2$. Equation 2.10 holds in general and can still be used. Hence, by reading the above argument backwards, we can immediately conclude that the weights integrate into a group representation. This completes the proof. □

Inner product

We define an *inner product* κ on $\mathfrak{su}(3)$ by

$$\kappa(X, Y) = -Tr(XY),$$

for all $X, Y \in \mathfrak{su}(3)$. This definition is well-defined, as κ satisfies the three requirements of an inner product: from the definition of the trace, it follows that it is linear in the first argument ($\kappa(aX, Y) = a\kappa(X, Y)$) and furthermore that $\kappa(X, Y) = \overline{\kappa(Y, X)}$. Thirdly, we see that κ is positive definite[2], because

$$\kappa(X, X) = -\operatorname{Tr}(XX) = \operatorname{Tr}((-X)X) = \operatorname{Tr}(X^*X).$$

X^*X is positive semidefinite, meaning that it has nonnegative eigenvalues [3]. The trace is equal to the sum of the eigenvalues of X^*X and hence it follows that $\kappa(X, X) \geq 0$. This also means that the the inner product is equal to zero if and only if all eigenvalues of X^*X are zero. This is the case only when $X = 0$, from which we conclude that $\kappa(X, X) > 0$ for all $X \neq 0$ in $\mathfrak{su}(3)$.

The inner product is $SU(3)$-invariant, as for all $X, Y \in \mathfrak{su}(3)$ and for all $g \in G$

$$\begin{aligned}
\kappa(gXg^{-1}, gYg^{-1}) &= -\operatorname{Tr}(gXg^{-1}, gYg^{-1}) \\
&= -\operatorname{Tr}(gXYg^{-1}) \\
&= -\operatorname{Tr}(XYgg^{-1}) \qquad \text{trace property, cyclic permutation} \\
&= -\operatorname{Tr}(XY) \\
&= \kappa(X, Y).
\end{aligned} \tag{2.11}$$

Using the definition of the inner product, we can calculate the angle between h_1 and h_2. Analogue to the angle between to vectors in an Euclidean vector space, we let the angle θ between two elements X and Y in our Lie algebra be

$$\theta = \arccos\left(\frac{\kappa(X, Y)}{\sqrt{\kappa(X, X)}\sqrt{\kappa(Y, Y)}}\right). \tag{2.12}$$

With this formula, the angle between h_1 and h_2 becomes

$$\theta_{12} = \arccos\left(\frac{1}{\sqrt{2}\sqrt{2}}\right) = \arccos\left(\frac{1}{2}\right) = \pm\frac{\pi}{3}. \tag{2.13}$$

The *root-lattice* of $\mathfrak{t}$ is equal to the set $\mathbb{Z}h_1 + \mathbb{Z}h_2$ and can be drawn on isometric dot paper with a h_1-and h_2-axis that make an angle $\pi/3$ with respect to each other. The root-lattice gives all the possible weights of the Lie algebra, for any representation of $SU(3)$. However, we will see that for each specific representation, only a few weights are 'allowed'.

A representation of the maximal torus T of $SU(3)$ can be decomposed into a direct sum of irreducible representations. As the Lie algebra of T is spanned by h_1 and h_2, we can represent the weights of T as coordinates in the root-lattice[3] of $\mathfrak{t}$. The weights are then given by $\lambda = (\lambda_1(h_1), \lambda_2(h_2))$ for each 1-dimensional, irreducible subrepresentation of T.

Theorem 2.11. The weights of the restriction of the defining representation $(\pi, \mathbb{C}^3)$ of $SU(3)$ to the maximal torus T of $SU(3)$, are given – in the root lattice of the maximal torus of $SU(3)$ – by

$$\lambda_1 = (1, 0), \qquad \lambda_2 = (-1, 1), \qquad \lambda_3 = (0, -1). \tag{2.14}$$

[2]Meaning that for all $X \in \mathfrak{su}(3)$ it holds that $\kappa(X, X) > 0$ if $X \neq \mathbf{0}$.

[3]For a general group, this is the weight-lattice, however, for the representations of $SU(3)$ that we will consider, all weights are elements of the root-lattice. Therefore, we will not consider the weight-lattice here.

Proof. When we restrict the defining representation of $SU(3)$, given by $\pi(U) = U$ for all $U \in SU(3)$, to T, we can argue as follows.

By theorem 2.5, we know that every finite-dimensional, unitary representation of T is unitary equivalent to a direct sum of finitely many irreducible, unitary representations. Furthermore, because T is abelian, from theorem 2.7 it follows that all components of the decomposed representation are 1-dimensional. Hence, the defining representation $\mathbb{C}^3$ is unitary equivalent to the direct sum of three 1-dimensional (irreducible) representations $\mathbb{C}$ and we can write

$$\mathbb{C}^3 = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}.$$

If we now take a basis of $\mathbb{C}^3$ as the set {R,G,B}, with R, G and B given by

$$\mathrm{R} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathrm{G} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathrm{B} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

we see that these vectors are eigenvectors of the elements of the Lie algebra $\mathfrak{t}$ and hence that they are elements of the weight space of T By definition 2.9, for $\mathbf{z} \in \mathbb{C}^3$ the weights of the representation $(\dot{\pi}, \{\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}\}^T)$ can be found by

$$h_1\mathbf{z} = i \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} z \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} iz \\ 0 \\ 0 \end{pmatrix} = i\mathbf{z}, \tag{2.15}$$

$$h_2\mathbf{z} = i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} z \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0\mathbf{z}, \tag{2.16}$$

from which we see that $\lambda(h_1) = 1$ and $\lambda(h_2) = 0$, resulting in the weight $\lambda_1 = (1, 0)$ in the root-lattice of $SU(3)$. In the same way, we can obtain the weight $\lambda_2 = (h_1G, h_2G) = (-1, 1)$ and $\lambda_3 = (h_1B, h_2B) = (0, -1)$. This completes the proof. □

We draw the weights in the root-lattice described above to get a weight diagram of the defining representation of $SU(3)$, as given in figure 2.1.

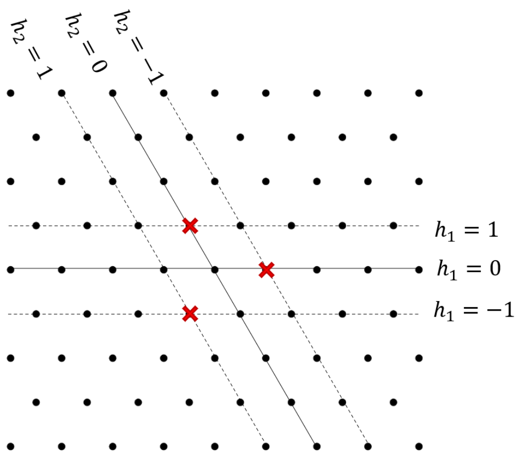

Figure 2.1: *The weight diagram of the defining representation of $SU(3)$. The weights are indicated by the (red) crosses.*

We have now determined the three weights of the defining representation of $SU(3)$. However, to be able to make the step from representation theory to particle physics, we still have need some information. In the next two sections, we will determine the weights of the dual representation $\mathbb{C}^{3^*}$ and the tensor product representations $\mathbb{C}^3 \otimes \mathbb{C}^{3^*}$ and $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ of $SU(3)$.

2.5 Dual representations

Definition 2.10. The *dual vector space* V^* of a vector space V over a field $\mathbb{F}$ is defined to be the set of linear mappings $\alpha : V \to \mathbb{F}$. Here, the linear mappings are equipped with addition and scalar multiplication according to the two calculation rules

$$(\alpha_1 + \alpha_2)(v) = \alpha_1(v) + \alpha_2(v)$$

$$(a\alpha_1)(v) = a(\alpha_1(v))$$

for all $\alpha_1, \alpha_2 \in V^*$, all $v \in V$ and all $a \in \mathbb{F}$.

Definition 2.11. For a representation (π, V) of the matrix group G, the *dual representation* is given by the representation (π', V^*), defined by

$$\pi'(g)v^* := \pi(g)^* v^*,$$

for all $g \in G$ and for all elements v^* of the dual vector space V^* of V. Here, $\pi(g)^*$ corresponds with the conjugate transpose of the representation $\pi(g)$.

Theorem 2.12. If λ is a weight of the unitary representation (π, V) of $SU(3)$, then $-\lambda$ is a weight of the dual representation (π', V^*).

Proof. Suppose that λ is a weight of the unitary representation (π, V) with weight vector v. Then

$$\pi(g)v = \pi(e^X)v = e^{i\lambda(X)}v.$$

Hence, for the dual representation $(\pi'(g), V^*)$ we can write

$$e^{-i\lambda(X)}v^* = (e^{i\lambda(X)}v)^* = (\pi(g)v)^* = \pi(g)^* v^* = \pi'(g)v^*,$$

from which we see that $-\lambda$ is a weight of the dual representation (π', V^*), with weight vector $v^* \in V^*$. □

By theorem 2.12, the weights of the dual representation of $SU(3)$ are given by $\{-\lambda_1, -\lambda_2, -\lambda_3\}$. In figure 2.2, we have draws the weights of the dual representation in the weight diagram of $SU(3)$.

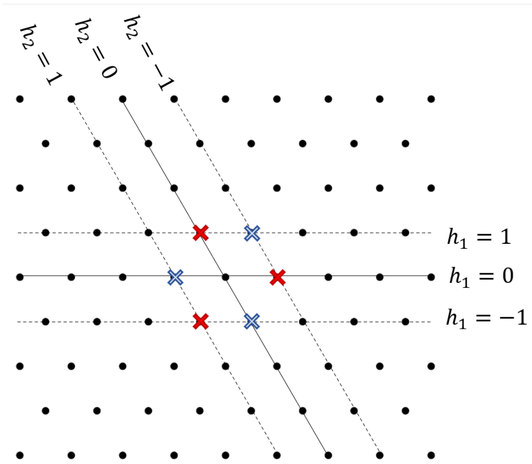

Figure 2.2: *The weight diagram of the defining and dual representation of SU(3). The weights of the defining representation are again indicated by the (red) filled crosses. The weights of the dual representation are indicated by the (blue) unfilled crosses.*

2.6 Tensor product representations

In this section, we will show how to determine the weights of a unitary tensor product representation. We start with some general theory about tensor products and then focus on the $SU(3)$ representations of $\mathbb{C}^3 \otimes \mathbb{C}^{3^*}$ and $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$.

Definition 2.12. Let V and W be two finite dimensional vector spaces V and W over $\mathbb{C}$ with basis e_i, $(i = 1, ..., n)$ and basis f_j, $(j = 1, ..., m)$ respectively. Then the *tensor product* $V \otimes W$ of V and W has $(nm)-$dimensional basis $e_i \otimes f_j$, with elements

$$v \otimes w := \sum_{i=1}^{n} \sum_{j=1}^{m} v_i w_j e_i \otimes f_j, \tag{2.17}$$

for any vectors $v = \Sigma_{i=1}^{n} v_i e_i \in V$ and $w = \Sigma_{j=1}^{m} w_j f_j \in W$.

Theorem 2.13. For two vector spaces V and W, the tensor product $v \otimes w$ of $v \in V$ and $w \in W$ is independent of the choice of basis [17].

Theorem 2.13 tells us that definition 2.12 is well-defined. From the definition of the tensor product it follows that the tensor product is bilinear, meaning that [17]

$$(av_1 + bv_2) \otimes w = a(v_1 \otimes w) + b(v_2 \otimes w)$$

$$v \otimes (aw_1 + bw_2) = a(v \otimes w_1) + b(v \otimes w_2)$$

for all $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$ and $a, b \in \mathbb{C}$.

Note that not every element of $V \otimes W$ can be written in the form $v \otimes w$, where $v \in V, w \in W$. This holds for example for the vector $e_1 \otimes e_1 + e_2 \otimes e_2$ in the tensor product $\mathbb{C}^2 \otimes \mathbb{C}^2$, where we have chosen a standard basis. We can write any two vectors v, w in $\mathbb{C}^2$ as $v = ae_1 + be_2$ and $w = ce_1 + de_2$, for $a, b, c, d \in \mathbb{C}$. Their tensor product $v \otimes w$ should be equal to

$$\begin{aligned} v \otimes w &= (ae_1 + be_2) \otimes (ce_1 + de_2) \\ &= ac(e_1 \otimes e_1) + ad(e_1 \otimes e_2) + bc(e_2 \otimes e_1) + bd(e_2 \otimes e_2) \\ &= e_1 \otimes e_1 + e_2 \otimes e_2. \end{aligned}$$

So we want to solve

$$\begin{pmatrix} ac & ad \\ bc & bd \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

However, because the first matrix has determinant zero, it is not invertible. Hence it has no inverse, which means that equation 2.6 has no solution. Thus we see that we cannot write each vector in $V \otimes W$ in the form $v \otimes w$.

Theorem 2.14. For two linear maps $A : V \to V'$ and $B : W \to W'$, there is a unique linear map $A \otimes B : V \otimes W \to V' \otimes W'$ such that for all $v \in V, w \in W$,

$$(A \otimes B)(v \otimes w) = Av \otimes Bw \text{ [8]}.$$

By theorem 2.14, for two representations (π, V) and (ρ, W) of a group G, there is a unique tensor product representation ϕ of G on $V \otimes W$ defined by

$$\phi(g)(v \otimes w) := (\pi(g) \otimes \rho(g))(v \otimes w) = (\pi(g)v) \otimes (\rho(g)w).$$

Theorem 2.15. If V and W are two inner product spaces, then there exists an unique hermitian inner product on $V \otimes W$ such that for all $v \in V, w \in W$

$$\langle v \otimes w, v' \otimes w' \rangle_{V \otimes W} = \langle v, v' \rangle_V \langle w, w' \rangle_W \text{[12]}.$$

Hence, the inner product between any two vectors $a = \Sigma_{i,j} a_{ij} e_i \otimes f_j$ and $b = \Sigma_{i',j'} b_{i'j'} e_{i'} \otimes f_{j'} \in V \otimes W$ is given by

$$\langle a, b \rangle_{V \otimes W} = \sum_{i,j} \sum_{i',j'} \overline{a_{ij}} b_{i'j'} \langle e_i \otimes f_j, e_{i'} \otimes f_{j'} \rangle_{V \otimes W} = \sum_{i,j} \sum_{i',j'} \overline{a_{ij}} b_{i'j'} \langle e_i, ei' \rangle_V f_j, f_{j'} \rangle_W.$$

If e_i and f_j are orthonormal bases, then $\langle e_i, e_{i'} \rangle = \delta_{ii'}$ and $\langle f_j, f_{j'} \rangle = \delta_{jj'}$ so that the summation reduces to

$$\langle a, b \rangle = \sum_{i,j} \overline{a_{ij}} b_{ij}.$$

Theorem 2.16. If (π, V) and (ρ, W) are unitary representations, then the tensor product representation $(\phi, V \otimes W)$, with $\phi = \pi \otimes \rho$, is unitary as well.

Proof. Suppose that (π, V) and (ρ, W) are unitary representations, so $\langle \pi(g)v, \pi(g)v' \rangle = \langle v, v' \rangle$ and $\langle \rho(g)w, \rho(g)w' \rangle = \langle w, w' \rangle$ for all $v \in V$ and $w \in W$ $(\star)$. Then,

$$\begin{aligned} \langle \phi(g)(v \otimes w), \phi(g)(v' \otimes w') \rangle_{V \otimes W} &= \langle (\pi(g)v) \otimes (\rho(g)w), (\pi(g)v') \otimes (\rho(g)w') \rangle_{V \otimes W} \\ &= \langle \pi(g)v, \pi(g)v' \rangle_V \langle \rho(g)w, \rho(g)w' \rangle_W && \text{theorem 2.15} \\ &= \langle v, v' \rangle_V \langle w, w' \rangle_W && (\text{by } \star), \end{aligned}$$

from which follows that the tensor product representation $(\phi, V \otimes W)$ is unitary as well. □

Theorem 2.17. Let us denote the 1-dimensional representation of the abelian group T with weight λ by $(\pi_\lambda, \mathbb{C})$ and that with weight λ' by $(\pi_{\lambda'}, \mathbb{C})$. Then $(\pi_\lambda \otimes \pi_{\lambda'}, \mathbb{C} \otimes \mathbb{C})$ is the 1-dimensional representation of T with weight $\lambda + \lambda'$.

Proof. Suppose that $(\pi_\lambda, \mathbb{C})$ is a representation with weight λ and $(\pi_{\lambda'}, \mathbb{C})$ with weight λ'. By equation 2.7 we can write for all $t \in T$

$$\pi_\lambda(t) = \pi(e^X) = e^{\dot{\pi}(X)} = e^{i\lambda(X)},$$

$$\pi_{\lambda'}(t) = \pi(e^X) = e^{\dot{\pi}(X)} = e^{i\lambda'(X)},$$

with X an element of the Lie algebra $\mathfrak{t}$ of T. Hence we can write for the tensor product representation

$$(\pi_\lambda \otimes \pi_{\lambda'})(t) = (\pi_\lambda(t) \otimes \pi_{\lambda'}(t)) = e^{i\lambda(X)} \otimes e^{i\lambda'(X)} = e^{i(\lambda+\lambda')}(1 \otimes 1),$$

from which we see that $\lambda + \lambda'$ is a weight of $\pi_\lambda \otimes \pi_{\lambda'}$. The representation is 1-dimensional as it works on the vector space $\mathbb{C} \otimes \mathbb{C}$, which has dimension $1 \cdot 1 = 1$. □

Theorem 2.18. The weights of the $SU(3)$-representation $V \otimes W$ correspond with the elements $\lambda'' = \lambda + \lambda'$, where λ is a weight of V and λ' is a weight of W.

Proof. Let (π, V) and (ρ, W) be two representations of T. By theorem 2.5 and 2.7, T can be decomposed into 1-dimensional irreducible representations of $V = \mathbb{C}^n$ and $W = \mathbb{C}^m$. This results in the 1-dimensional representations $(\pi_{\lambda_i}, \mathbb{C})$ with weight λ_i $(i = 1, ..., n)$ due to the decomposition of V and in the 1-dimensional representations (ρ_{λ_j}, W) with weight λ'_j $(j = 1, ..., m)$ due to the decomposition of W. The tensor product representation $\pi \otimes \rho$ can now be written as

$$(\pi_{\lambda_1} \oplus ... \oplus \pi_{\lambda_n}) \otimes (\rho_{\lambda'_1} \oplus ... \oplus \rho_{\lambda'_m}) = \sum_{i=1}^{n} \sum_{j=1}^{m} \pi_{\lambda_i} \otimes \pi_{\lambda'_j}.$$

The tensor product representations in this sum are all 1-dimensional representations of T, which by theorem 2.17, have weight $\lambda_i + \lambda'_j$. Hence, we see that $V \otimes W$ has weights $\lambda''_{ij} = \lambda_i + \lambda'_j$ for $i = 1, ..., n$ and $j = 1, ..., m$. □

Using the above theorem, we can easily determine the weights for the tensor product representations of $SU(3)$.

2.6.1 The $\mathbb{C}^3 \otimes \mathbb{C}^{3^*}$ representation of $SU(3)$

Because both $(\pi, \mathbb{C}^3)$ and $(\pi', \mathbb{C}^{3^*})$ are unitary representations of $SU(3)$, using theorem 2.12 and 2.18 we can immediately write down the weights of the $SU(3)$ tensor representation $\mathbb{C}^3 \otimes \mathbb{C}^{3^*}$. The weights of $\mathbb{C}^3$ are given by $\lambda = (1,0), (-1,1), (0,-1)$ and the weights of $\mathbb{C}^{3^*}$ by $\lambda' = (-1,0), (1,-1), (0,1)$. By theorem 2.18, the weights λ'' of $\mathbb{C}^3 \otimes \mathbb{C}^{3^*}$ are given by $\lambda + \lambda'$ and form $3 \cdot 3 = 9$ terms

$$\begin{aligned} \lambda'' = (\lambda''(h_1), \lambda''(h_2)) = &(2,-1), (1,1), (-2,1), (-1,2), (-1,-1), (1,-2) && \text{each occurring one time,} \\ &(0,0) && \text{occurring three times.} \end{aligned}$$

These weights are visualised in figure 2.3.

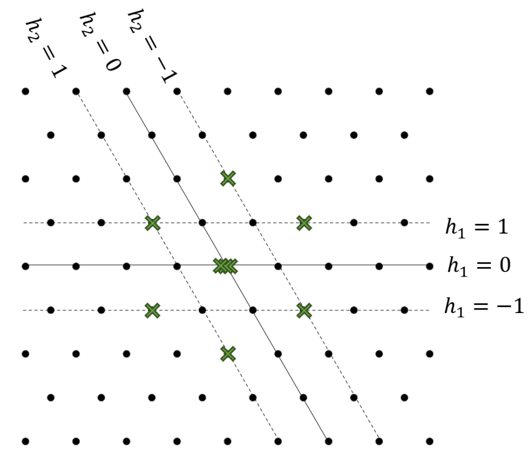

Figure 2.3: *The weight diagram of $SU(3)$ for its $\mathbb{C}^3 \otimes \mathbb{C}^{3^*}$ representation. The weights are indicated by the (green) crosses. Because the weight zero occurs three times, three crosses are placed at $(0,0)$.*

2.6.2 The $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ representation of $SU(3)$

With theorem 2.18, we can immediately write down the weights of the tensor product representation $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ of $SU(3)$, as this is just the sum of the weights of $\mathbb{C}^3$. This results in the following $3 \cdot 3 \cdot 3 = 27$ weights

$$\begin{aligned}(\lambda(h_1), \lambda(h_2)) = &(0,3), (3,-3), (-3,0) && \text{each occurring one time,}\\ &(1,1), (2,-1), (1,-2), (-1,-1), (-2,1), (-1,2) && \text{each occurring three times,}\\ &(0,0) && \text{occurring six times.}\end{aligned}$$

These weights are visualised in figure 2.4.

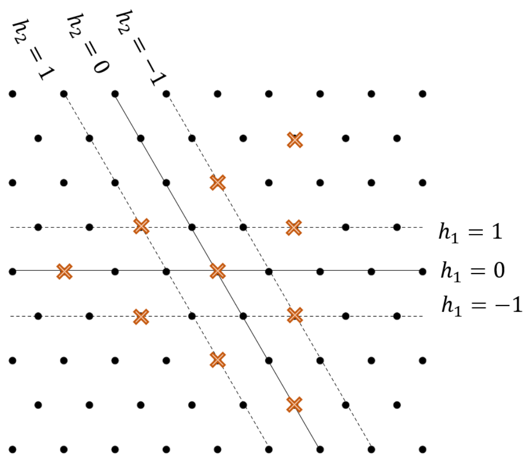

Figure 2.4: *The weight diagram of $SU(3)$ for its $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ representation. The weights are indicated by the (orange) crosses. To keep the picture clear, only one cross has been placed at each weight position, even when the weight occurs multiple times.*

Chapter 3

The structure of the standard model

Over time, physicists have created a theoretical, mathematical framework that corresponds with the symmetries that are observed in the universe, in particle physics. At a certain point in history, this led to the standard model [6]. Meanwhile, new observations have been done, based on which modifications of the standard model have been made (such as the addition of neutrino masses which originally was not included). The framework is still developed further and even extended to theories beyond the standard model, to include recent observations and predict new ones.

We start this chapter with a treatment of the symmetries that are observed in the universe and relevant in particle physics. Because the standard model 'models' the universe, these symmetries should be symmetries of the standard model as well. The symmetries of section 3.1 will motivate us to look at the gauge theory for the groups $U(1)$, $SU(2)$ and $SU(3)$ in chapter 4 and will finally form a theory for the structure and interactions of the particles in the standard model in chapter 5.

In section 3.2, we will structure the particles of the standard model based on their properties in the form of three particle generations and (gauge) bosons. Many parts of this structure, such as the existence of the top quark, have actually been predicted by the standard model (instead of being a motivation for the formulation of the standard model) and have since been experimentally verified. However, we will not follow this historical development and just state the results as they are known now.

Finally, section 3.3 combines the first two sections of this chapter. It makes the symmetries of section 3.1 explicit by giving the quantum numbers that are conserved for the interactions of the particles in section 3.2.

3.1 Symmetries

The standard model is a quantum field theory: a theory that brings together classical field theory, quantum mechanics and special relativity. An important concept in quantum mechanics, that also occurs in the standard model, is that of *quantum numbers*, which tell us that a particle can have only discrete values of certain physical quantities such as energy, momentum and charge. In many cases, certain quantum numbers are conserved during an interaction. A well known example is conservation of charge: in a particle reaction the sum of the charges of the initial state equals the sum of the charges of the final state. The observed conservation of certain properties of a physical system, makes us introduce the following definition.

Definition 3.1. In physics, a physical system is said to have a *symmetry*, if some property of the system is preserved after a certain type of transformation. In that case the property is said to be *invariant* under the transformation.

This definition might still seem to be a bit vague, but will become more concrete in this section, when we will specify both the properties of a system that can be preserved (such as certain quantum numbers) and the types of transformations under which these properties are invariant.

Many physical symmetries can be specified by groups. In section 1.2, we have seen that a group can be discrete or continuous. A physical system is said to have a *discrete* or *continuous symmetry*, when it has a preserved property under transformation by elements of respectively a discrete or continuous group. The standard model has both discrete and continuous symmetries, which we will treat separately in the next two subsections.

3.1.1 Discrete symmetry

The physical world is believed to have one fundamental, discrete symmetry, called *CPT-symmetry*. CPT-symmetry is the invariance of the physical laws under simultaneous transformations of C, P and T, which stand for the following transformations.

- C-*symmetry* (charge symmetry): all particles are replaced by their anti-particle.
- P-*symmetry* (parity symmetry): the position of all objects in space is reflected in the three axis through an arbitrary point.
- T-*symmetry* (time-symmetry): the reversion of the momenta of all objects by the transformation T of time reversal, given as $T : t \to -t$.

Each of the above symmetries is broken when applied individually to the universe. However, the so called CPT-theorem states that any Lorentz invariant quantum field theory (with energy spectra that are bounded from below) must have CPT-symmetry and hence is invariant under the combination of the three symmetries [1]. In the next subsection we will treat the subject of Lorentz-symmetry and see that the standard model is a Lorentz invariant quantum field theory. Hence, CPT-symmetry is a symmetry of the standard model.

3.1.2 Continuous symmetry

Definition 3.1 already tells us that there is a close relation between symmetries and conserved quantities. This relation is made precise by *Noether's theorem*, which states that for each continuous symmetry, a system has a corresponding conservation law[1]. (Continuous) symmetries can be global or local.

Definition 3.2. A system has a *global symmetry* if it has a symmetry whose transformations are not a function of space-time.

An example of a global symmetry is a phase transformation of the electron field

$$\Psi(\mathbf{x}, t) \to e^{i\alpha}\Psi(\mathbf{x}, t), \tag{3.1}$$

where α is the angle by which the field is rotated. By filling in the transformed wave function in the Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi,$$

we see that if $\Psi(\mathbf{x}, t)$ satisfies the Schrödinger equation, then $e^{i\alpha}\Psi(\mathbf{x}, t)$ satisfies it as well. This means that the dynamics of our system is invariant under the transformation in equation 3.1. Furthermore, as already mentioned, only the modulus of the wave function has a physical significance. Hence, not only the dynamics, but also the physical interpretation of the situation is the same after the phase transformation. We can therefore conclude that a phase transformation of the electron field is a global symmetry.

Definition 3.3. A system has a *local symmetry* if it has a symmetry whose transformations are a function of space-time.

Examples of local symmetries in physics are ample. We will introduce them in this subsection and continue their treatment in the whole of chapter 4.

[1]More precisely, Noether's theorem states that every differentiable symmetry of the action of a physical system has a corresponding conservation law. We will not introduce the definition of a differentiable symmetry here. However, as Lie groups are smooth manifolds, it is not strange to think of symmetries under transformations of Lie groups as having a certain differentiability property. The action of a system corresponds with the time-integral over the Lagrangian. We will encounter it in section 4.1, where it is given by equation 4.1.

Lorentz symmetry

A very important global symmetry in physics, is Lorentz symmetry. In the CPT-theorem, we already encountered the term *Lorentz invariance* and stated that the standard model is a Lorentz invariant theory. Lorentz invariance is a necessary requirement for every realistic, physical theory and follows from special relativity. General relativity requires that the laws of physics are the same in every inertial frame[2]. A physical system (for example, a particle field) is said to be Lorentz invariant if it transforms trivially under *Lorentz transformations*, which are elements of the Poincaré group[3].

Furthermore, an equation is Lorentz covariant if the equations of motions are the same after application of a Lorentz transformation to all its elements. Physically, this means that if the equations of motion hold in one, they hold in every inertial frame. It is desired for every physical theory to be Lorentz invariant – meaning that its equations of motion are Lorentz covariant equations – for otherwise the theory is not consistent with general relativity.

Gauge symmetry

Local symmetries play an important role in in the description of particle interactions, in the form of so called 'local gauge symmetries'.

Definition 3.4. A *gauge theory* is a type of field theory in which the Lagrangian of the field is invariant under local symmetries, that correspond with elements of a Lie group.

The local symmetries in definition 3.4 are called *local gauge symmetries* and the Lie group of which they are elements is called the *gauge group.* If the gauge group is not abelian, the gauge theory is called a *non-abelian gauge theory.* When the Lagrangian is invariant under the application of local gauge symmetries to the particle field, we speak of *gauge invariance.*

Probably the most well-known gauge theory is that of quantum electrodynamics (QED), the fundamental theory of electromagnetism that unites quantum mechanics with special relativity [2]. The gauge theory that describes quark interactions, is called quantum chromodynamics (QCD). The theory of weak interaction can be unified with that of electromagnetism in a gauge theory that is called the *unified electroweak model* $U(1) \times SU(2)$. Together with QCD, this theory describes the interactions of all particles of the standard model.

Gauge theory will be the focus of chapter 4. We will see examples of local gauge symmetries in section 4.3 – 4.5, when we look in more detail at the transformations of particle fields. In the remainder of this chapter, we will focus on the quantum numbers of the particles in the standard model, with a main focus on the particles of the first generation of the standard model.

3.2 The elementary particles

The structure of the periodic table led physicists to the discovery of smaller elements that together form the nuclei of the atoms in the periodic table. The nuclei appeared to consist of protons and neutrons, which in their turn consist of quarks that are kept together by the *strong force.* Around the nuclei, electrons move, bound to the nuclei by the *electromagnetic force.* Lastly, the neutron can decay into a proton due to the so called *weak force.* From this decay, together with the proton, also an electron and an anti-neutrino originate.

The above particle interactions can be observed experimentally. In a particle reaction, one can measure which quantities are conserved during that reaction. Examples of such quantities are mass, energy, momentum and spin. Many of these quantities are 'quantized': they are quantum numbers of the observed interacting system. Depending on the type of interaction (e.g. electromagnetic, weak or strong), quantum numbers are found to be conserved or violated. This gives information about the quantum numbers of individual particles, which can be used to subdivide the particles of the standard model into different categories. First of all, all particles are either

[2] An inertial frame is a frame of reference that is not accelerating.

[3] Invariance under the Poincaré group implies e.g. invariance under translation and rotation in space. In the following, we will not pay further attention to Lorentz symmetry. Hence, it is enough to know that invariance under the Poincaré groups correspond with the having same laws of physics in different inertial reference frames. We will not define the Poincaré group or treat its role in the standard model in more detail.

bosons – particles with integer spin ($S = 0, 1, 2, ...$) – or *fermions* – particles with half integer spin ($S = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, ...$).

3.2.1 Bosons

The three forces described above are believed to act by the interchange of a special type of bosons, called *gauge bosons*, which play a central role in the gauge theory that we will treat in chapter 4. Gauge bosons are also called *force carriers*, because by exchanging these bosons, energy, momentum and other quantum numbers which we will specify later are 'carried' from one particle to another. The force carrier of the electromagnetic interaction is the photon γ, that of the strong interaction are eight gluons g_i, ($i = 1, ..., 8$) and that of the weak interaction are three weak bosons $W^{\pm}$ and Z^0. Only the fourth fundamental force, the *gravitational force*, has not yet been explained using particle interactions, although there are many theories on this subject. Some theories suggest the existence of the *graviton*, the hypothesised force carrier of the gravitational force. However, up until now, there has been no evidence for the existence of such a particle.

The standard model includes one other boson, called the *Higgs boson.* The Higgs boson had not yet been observed when it was theoretically predicted by the $SU(2)$ gauge theory that is treated in chapter 4. However, in 2012, physicists have detected the particle in the Large Hadron Collider in CERN, Geneva, which has increased the belief in the correctness of gauge theory.

3.2.2 Fermions

Fermions can be subdivided into three generations. Each generation consists of two *leptons* – particles that undergo electroweak interactions – and two *quarks* – particles that undergo strong interactions – together with their anti-particles. The three generations of fermions are shown in table 3.1. Between different generations, particle masses and flavours differ, but the electroweak and the strong force act in the same way.

	First	**Second**	**Third**
Quarks	u (up)	c (charm)	t (top)
	d (down)	s (strange)	b (bottom)
Leptons	e (electron)	μ (muon)	τ (tau)
	ν_e (electron neutrino)	ν_μ (muon neutrino)	ν_τ (tau neutrino)

Table 3.1: *The three generations of fermions without anti-particles.*

The particles that are formed by quarks and hence feel the strong forces are called *hadrons* and are held together by gluons. The hadrons can be subcategorized into mesons and baryons. *Mesons* are particles that consist of a quark–anti-quark pair. *Baryons* are particles that consist of three quarks. Examples of baryons are the well known proton and neutron.

Particles are either *right-handed* or *left-handed.* Massless particles are right- or left-handed depending on the sign of the projection of their spin component on the axis of their momentum. A massless particle is right-handed if this sign is positive and left-handed if it is negative. For massive particles, this dependence is a bit more complex and we will not fully treat it here. Most important to know is that the right-handed part ψ_R and the left-handed part ψ_L of a particle field ψ can be received mathematically by letting the following projection operators act on ψ [4]

$$P_R := \frac{1-\gamma^5}{2} \text{ to get } \psi_R = P_R\psi \tag{3.2}$$

$$P_L := \frac{1+\gamma^5}{2} \text{ to get } \psi_L = P_L\psi,$$

where γ^5 denotes the fifth *Dirac* (or gamma) *matrix.* The first four Dirac matrices can be written in compact form as

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \tau_i \\ -\tau_i & 0 \end{pmatrix} \qquad (i = 1, 2, 3), \tag{3.3}$$

with τ_i the three Pauli spin matrices (equation 1.5) and $\mathbb{1}_2$ the 2-dimensional identity matrix. The fifth Dirac matrix γ^5 is now defined in terms of these matrices as

$$\gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3.$$

In the next section, we will see that symmetry breaking occurs for left- and right-handed particles in the standard model, as they do not have the same quantum numbers.

3.3 Quantum numbers

Physicists have carried out many experiments to determine the quantum numbers of particles in different reactions. In this section, we will state the conclusions that have been drawn from all these observations: conclusions about the values of quantum numbers of different particles and about which quantum numbers are conserved under which interactions. In chapter 5, we will use these conclusions in our treatment of the mathematical framework of the standard model.

Of all the quantum numbers that a particle can possess, only three are believed to be conserved in all fundamental interactions[4], these are

- B: the baryon number ($\frac{1}{3}$ for each quark, $-\frac{1}{3}$ for each anti-quark, 0 else)
- L: the lepton number (1 for each lepton, -1 for each anti-lepton, 0 else)
- Q: the electric charge, defined by

  $$\mathrm{Q} := \mathrm{I}_3 + \tfrac{1}{2}Y_W,$$

  where I_3 denotes the third component of weak isospin and Y_W is the weak hypercharge.

Weak isospin is a property that is carried by bosons of the weak interaction. The weak bosons $W^{\pm}$ have isospin values ± 1 and the electroweak boson Z^0 has isospin value 0. In the following, because we will only consider the third component of the weak isospin, we will call I_3 just 'weak isospin'. Weak hypercharge is a quantum number that changes under the interaction of particles with the Higgs boson.

An interesting, experimentally observed property of the weak interaction is that it transforms only left-handed fermions and their corresponding right-handed anti-fermions. As a consequence, only left-handed fermions (and their corresponding right-handed anti-fermions) have a nonzero value of weak isospin. For each left-handed fermion with isospin value I_3, its corresponding right-handed antiparticle has isospin value $-\mathrm{I}_3$. The right-handed version of this fermion and its corresponding left-handed anti-fermion, both have zero isospin.

In table 3.2, the quantum numbers that are conserved under both electroweak and strong interactions are given for the first generation of fermions and the gauge bosons of the electromagnetic, weak and strong interactions. Furthermore, the weak isospin and weak hypercharge are given as well. From their values, one can see that in a particle interaction indeed only the weak bosons can change the value of weak isospin and only the Higgs particle that of weak hypercharge. In this table, the anti-particles could be given as well. The quantum numbers are given in so called *natural units*. Natural units are obtained by choosing units such that $\hbar = c = 1$.

Table 3.2 shows the conserved quantum number for left-handed particles. For right handed particles, the table would be looking the same, except from the value of I_3. As already mentioned, this value is zero for all right-handed particles and for their corresponding left-handed anti-particles.

3.3.1 Quark flavour

Furthermore, each quarks can be given a flavour. For example, the flavour quantum number charm (C) is given by

$$C = n_c - n_{\bar{c}} \tag{3.4}$$

[4]There exist grand unification theories in which only the difference between B and L has to be conserved and the individual quantum numbers can vary over an interaction. However, such a violation has not yet been observed [13]

Quarks	Name	S	B	L	Q	I_3	Y_W
u	up	$\frac{1}{2}$	$\frac{1}{3}$	0	$+\frac{2}{3}$	$+\frac{1}{2}$	$\frac{1}{3}$
d	down	$\frac{1}{2}$	$\frac{1}{3}$	0	$-\frac{1}{3}$	$-\frac{1}{2}$	$\frac{1}{3}$
Leptons							
e	electron	$\frac{1}{2}$	0	1	-1	$-\frac{1}{2}$	-1
ν_e	electron neutrino	$\frac{1}{2}$	0	1	0	$+\frac{1}{2}$	-1
Gauge bosons							
γ	photon	1	0	0	0	0	0
$W^{\pm}, Z^0$	weak bosons	1	0	0	±1,0	$\pm 1, 0$	0
$g_i (i = 1, .., 8)$	gluons	1	0	0	0	0	0
Higgs boson							
h	Higgs boson	0	0	0	0	$-\frac{1}{2}$	1

Table 3.2: *The elementary particles under consideration. The last six columns correspond with some quantum numbers of the particles: S denotes the spin, B the baryon number, L the lepton number, Q the charge, I_3 the weak isospin and Y_W the weak hypercharge. The quantum numbers are given in natural units.*

with n_c the number of charm quarks in the system under consideration and $n_{\bar{c}}$ the number of charm anti-quarks. In the same way, flavour quantum numbers can be defined for the strange quark (S), the top quark (T) and the bottom quark (B'). Flavour is conserved by strong interactions, but violated by weak interactions. Therefore, we did not include flavour quantum numbers in table 3.2.

3.3.2 Quark color

Protons are baryons and hence they consist of three quarks. Based on the quantum numbers of u and d, we can identify the proton with the quark combination uud[5]. Then the proton has charge $+\frac{2}{3} + \frac{2}{3} - \frac{1}{3} = +1$, baryon number $+\frac{1}{3} + \frac{1}{3} + \frac{1}{3} = +1$ (so indeed, it is a baryon) and lepton number $0 + 0 + 0 = 0$ (so it is not a lepton). Other particles can be identified with quark combinations as well. For example, the Δ^{++}-baryon corresponds with the quark combination uuu.

However, this identification is puzzling, for u and d are fermions and the Pauli exclusion principle states that two identical fermions are not allowed to be in the same ground state. So why can the proton consist of the quark combination uud, which clearly has two identical quarks u occupying the same state? And if it would, for some yet unknown reason, be allowed to have two identical quarks in the same state, then why have there never been observed any particles that could correspond with a quark combination qq or $\overline{\text{qq}}$ for some quark q with anti-quark $\overline{\text{q}}$? Another interesting observation is that, while the Pauli exclusion principle would allow singe quarks to exists, there have never been observed any particles that could consist of a single quark.

These puzzling observations have led physicists to introduce the quantum number 'color'. Quarks have been assigned a color that is either red (R), green (G) or (B). Their corresponding anti-quarks have the colors cyan ($\overline{\text{R}}$), magenta ($\overline{\text{G}}$) and yellow ($\overline{\text{B}}$). Quark configurations are now required to be colorless, which means that they have to be unchanged by rotations in the R, G, B space.

This requirement has great impact on the allowed particles, for there are only three ways in which quark configurations can be colorless, namely in the combinations

- RGB
- $\overline{\text{RGB}}$
- R$\overline{\text{R}}$, G$\overline{\text{G}}$ or B$\overline{\text{B}}$.

These options are visualised in figure 3.1. From these combinations all physically allowed hadrons can be determined. This will be done in chapter 5, using the representation theory of chapter 2.

[5]In fact it is a superposition of quark combination uud, udu and duu [4]. However, because the proton field is normalized and the combination uud, udu and duu all have the same quantum numbers, the arguments that follow still hold.

By requiring the quark compositions to be colorless, we see that single quarks and many quark combinations that would have been allowed by the Pauli exclusion principle cannot exist.

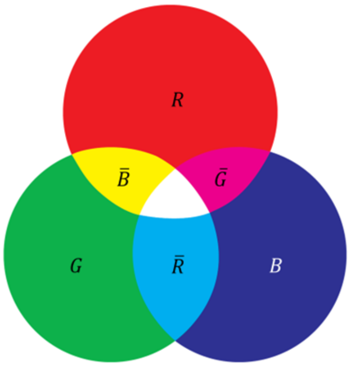

Figure 3.1: *The three circles R, G, B and their overlapping regions show in which ways two our three quarks can can be combined to a colorless (white) particle.*

Evidence for the existence of color is ample. One example is the *R-ratio*: the probability that e^+e^- decays into a hadron (consisting entirely of strongly interacting quarks) divided by the probability that e^+e^- decays into the muon pair $\mu^+\mu^-$. If quarks have color, the decay of e^+e^- into a hadron can happen, with equal probability, in three different ways, as each hadron can be colorless in three different ways, by permutations of (R,G,B). This means that in the decay

$$e^+e^- \rightarrow \text{hadrons}, \tag{3.5}$$

each quark has to be counted three times, one time for each color. The R-ratio can be determined experimentally and indeed is found to be three times larger then would be the case if quarks did not have color, hereby giving evidence for the actual existence of color.

Chapter 4

Gauge theory

The interactions of particles under the electromagnetic, weak and strong force, are believed to be closely connected with certain symmetry principles. More specifically, the interactions are believed to be dictated by so called *local gauge symmetries*. In this chapter, we will describe these symmetries using *gauge theory*, which has been introduced in section 3.1.

This chapter starts in section 4.1 with a general introduction to Lagrange formalism and particle fields, which lie at the basis of gauge theory. In section 4.2, we look at the global invariance of $U(1)$, to prepare ourselves for the treatment of the local gauge invariances of $U(1)$, $SU(3)$ and $SU(2)$ in section 4.3–4.5 respectively. The last of these three sections is the most extensive. It treats the subject of spontaneous symmetry breaking, which is needed to be able to integrate the massive gauge bosons of the weak interaction in our theory.

We will see this necessity in chapter 5, where we will combine the representation theory of chapter 2 and the gauge theories of this chapter to form a mathematical framework that corresponds with the structure of the standard model that has been described in chapter 3. Until chapter 5, we will keep our treatment of the gauge theory of $U(1)$, $SU(2)$ and $SU(3)$ general and give only a few hints to the actual physical situation.

4.1 Lagrange formalism

In this chapter, we will describe particle fields. Particle fields can be seen as functions, that assign some value (that gives information about the particle) to every point in space or space-time. A concrete and well-known example of a particle field that you have probably already seen and that we have encountered in section 3.1, is the wave function $\Psi(\mathbf{x}, t)$ of a non-relativistic electron (with $\mathbf{x} = (x, y, z)$), that satisfies the Schrödinger equation. The physical significance of the wave function lies in its modulus, which defines the probability of an electron to be at a location $\mathbf{x}$ at time t [7].

In the sequel, we will denote the space-time coordinates by the four-vector $x^\mu := (x^0, x^1, x^2, x^3)$. In four-vector notation, x_μ is another four-vector: $x_\mu = (x^0, -x^1, -x^2, -x^3)$. The parameter μ can be seen as a space-time index, running from 0 to 3, with value zero for the time coordinate t and values 1,2,3 for the spatial coordinates x, y and z. When this parameter occurs twice in a multiplication, once as subscript and once as a superscripts (for example: $a_\mu b^\mu$ or $a^\mu b_\mu$), this corresponds to the Einstein summation convention, that has been introduced in section 1.3. In the following, we will always use Greek characters to denote a summation from 0 to 3 for the space-time coordinate x^μ.

The aim of this chapter is to find an equation for the (quantum and relativistic) particle fields of the particles of the standard model. For this, we use the Lagrange density $\mathcal{L}$. It is related to the Lagrangian $\mathscr{L} := T - V$ (where T denotes the kinetic and V the potential energy of the system under consideration), by

$$\mathscr{L} = \int \mathcal{L} \, \mathrm{d}^3\mathbf{x},$$

with $x = (x^1, x^2, x^3)$ in space-time coordinate notation. The action S of a system is defined to be

$$S := \int \mathscr{L} \ \mathrm{d}t = \int \mathcal{L} \ \mathrm{d}^3\mathbf{x}\mathrm{d}t. \tag{4.1}$$

The Euler-Lagrange equations, that describe a system with finitely many generalized coordinates $q_i(t)$, $(i = 1, .., n)$, can be extended to a formalism with continuously varying coordinates $\phi(x_\mu)$. This extension is determined by a transformation of the action S [4]

$$S(q_i, \frac{\partial \dot{q}_i}{\partial t}, t) \to S\left(\phi, \partial_\mu \phi, x_\mu\right), \tag{4.2}$$

where ϕ denotes a particle field and where $\partial^\mu \phi$ is short notation for

$$\begin{aligned} \partial^\mu \phi := \frac{\partial}{\partial x^\mu}\phi &= \left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right)\phi \\ &= \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\phi. \end{aligned} \tag{4.3}$$

The extension from the discrete to the continuous formalism, results in an equation for the Lagrange density [4]

$$\frac{\partial}{\partial x_\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial\phi/\partial x_\mu)}\right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0. \tag{4.4}$$

From now on we will follow the convention to call $\mathcal{L}$ itself the Lagrangian. Equation 4.4 gives a relationship between any particle field and its Lagrangian. In the next sections, we will state the Lagrangian for different particle fields and extend it to a 'complete' Lagrangian using gauge theory[1].

4.2 $U(1)$ global invariance

Before we treat local gauge invariances, we first give an example of a global invariance of the Lagrangian. For this, consider a spin-1/2 particle. This particle can be described by a four-component complex field $\psi(x_\mu)$. The Lagrangian is in this case given by [4]

$$\mathcal{L} = i\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi. \tag{4.5}$$

In this equation, $\bar{\psi} := \psi^* \gamma^0$, m is the mass of the spin-1/2 particle, ∂_μ is again the derivative (equation 4.3) and γ_μ denote the four Dirac matrices that have been introduced in equation 3.3. The Dirac matrices are 4-dimensional. Hence, the multiplication in the first term of equation 4.5 is well-defined, because the wave function is a 4-dimensional column vector, whose conjugate transpose ψ^* is a 4-dimensional row vector.

Suppose that we transform the field as

$$\psi \to e^{iq\alpha}\psi, \tag{4.6}$$

where $\alpha \in \mathbb{R}$ is the phase and where the parameter $q \in \mathbb{R}$ denotes the strength of the phase transformation. By introducing q, we look ahead to chapter 5, where we will look at fields that transform with different strengths and where q will start playing an important role. When ψ transforms according to equation 4.6, its derivative and hermitian conjugate transform as

$$\partial^\mu \psi \to e^{iq\alpha}\partial^\mu \psi, \tag{4.7}$$

$$\bar{\psi} \to e^{-iq\alpha}\bar{\psi}, \tag{4.8}$$

so that by substitution of these transformation in equation 4.5, we can immediately see that the Lagrangian $\mathcal{L}$ is unchanged. We recognise the set of phase transformations of the form $e^{iq\alpha}$, with $q, \alpha \in \mathbb{R}$, as the abelian Lie group $U(1)$.

The invariance of the Lagrangian under the transformation of the field by equation 4.6 is a global symmetry: we can choose α arbitrarily, as for any value of α the Lagrangian is unchanged after transformation.

[1] The explanation of how these initial Lagrangians are derived, goes beyond the scope of this thesis. The derivation is based on the spin and mass of the particle and on the requirement of Lorentz invariance of the Lagrangian. In the following, we thus take these Lagrangians for granted and focus on the development of a gauge theory for the different particle fields.

4.3 $U(1)$ local gauge invariance

The global symmetry of section 4.2 is a very general form of invariance. A local gauge invariance of the Lagrangian would be a stricter requirement and could be found if α is taken to be a function of space-time. In the following, we will often denote a function f, that is a function of the space-time coordinates x^μ, by $f(x)$ and omit the subscript or superscript μ. Hence, we write $\alpha = \alpha(x)$ for the space-time dependent phase.

The requirement of local gauge invariance is motivated by the thought that certain quantum numbers (such as electric charge and color) are conserved not only globally, but locally as well. However, if we make the same substitution as in equation 4.6 - 4.8, but now with α a function of space-time, by the product rule we get

$$\partial^\mu \psi \to e^{iq\alpha}\partial^\mu\psi + iqe^{iq\alpha}\psi\partial^\mu\alpha.$$

Clearly, in the derivative of ψ, an extra term arises that causes the Lagrangian to vary. To get the desired invariance, we search for a modified derivative D_μ, that transforms like ψ itself, namely as

$$D_\mu\psi \to e^{iq\alpha}D_\mu\psi,$$

so that the phase factor cancels the phase factor of $\bar{\psi}$, as it did in the global invariance of section 4.2. Our desire can be fulfilled by defining

$$D_\mu := \partial_\mu - iqA_\mu, \tag{4.9}$$

where A_μ is a new vector field that transforms as

$$A_\mu \to A_\mu + \partial_\mu\alpha. \tag{4.10}$$

The transformation of A_μ is chosen such that D_μ transforms as

$$\begin{aligned} D_\mu\psi &= (\partial_\mu - iqA_\mu)\psi \\ &\to e^{iq\alpha}\partial^\mu\psi + iqe^{iq\alpha}\psi\partial_\mu\alpha - iq(A_\mu + \partial_\mu\alpha)e^{iq\alpha}\psi \\ &= e^{iq\alpha}(\partial^\mu + iq\partial_\mu\alpha - iqA_\mu - iq\partial_\mu\alpha)\psi \\ &= e^{iq\alpha}(\partial^\mu - iqA_\mu)\psi \\ &= e^{iq\alpha}D_\mu\psi, \end{aligned}$$

as desired. A_μ is called the *gauge field*. Its introduction is necessary to be able to meet the demand of local gauge invariance.

With our new derivative D_μ, the Lagrangian of equation 4.5 becomes

$$\mathcal{L}_\psi = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi + q\bar{\psi}\gamma^\mu\psi A_\mu. \tag{4.11}$$

To make the theory complete, we look at whether there can still be added extra, gauge invariant terms to the Lagrangian. The occurrence of A_μ in the 'new' Lagrangian of equation 4.11, motivates us to search for an extra kinetic energy term of the form $\frac{1}{2}\phi^2$ in terms of the field A_μ, that is gauge invariant as well. If such a term indeed exists, we can regard the total Lagrangian $\mathcal{L}$ as the sum of the Lagrangian of the original spin-1/2 particle and the Lagrangian of the new gauge field.

$$\mathcal{L} = \mathcal{L}_\psi + \mathcal{L}_A,$$

with $\mathcal{L}_\phi$ given by equation 4.5 – only with the derivative replaced by the gauge invariant derivative D_μ – and with $\mathcal{L}_A$ given by

$$\mathcal{L}_A = \tfrac{1}{2}\phi^2,$$

for a still unknown field ϕ. However, the addition of $\mathcal{L}_A$ to the total Lagrangian, is only possible if $\mathcal{L}_A$ is gauge invariant. We will show that this is indeed the case, by defining the so called *field strength tensor* $F^{\mu\nu}$ as

$$F^{\mu\nu} := \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{4.12}$$

Now, if our original field ψ transforms with a local phase transformation $e^{iq\alpha}$, $F^{\mu\nu}$ transforms as

$$\begin{aligned} F^{\mu\nu} &:= \partial^\mu A^\nu - \partial^\nu A^\mu \\ &\to \partial^\mu(A^\nu + \partial^\nu\alpha) - \partial^\nu(A^\mu + \partial^\mu\alpha) && \text{equation 4.10} \\ &= \partial^\mu A^\nu + \partial^\mu\partial^\nu\alpha - \partial^\nu\partial^\mu\alpha - \partial^\nu A^\mu \\ &= \partial^\mu A^\nu - \partial^\nu A^\mu && \text{interchanging partial derivatives of third term} \\ &= F^{\mu\nu} \end{aligned}$$

from which it follows that $F^{\mu\nu}$ is gauge invariant. This means that $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ is gauge invariant as well, and hence that we are allowed to add an extra term $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ to the Lagrangian.

There might still be other gauge invariant terms that we can add to the Lagrangian. Could the gauge field possibly induce an extra mass term in the Lagrangian? The answer to this question is no, for this term would be of the form $\frac{1}{2}m_A^2 A_\mu A^\mu$, with m_A denoting the mass of the particle corresponding to the gauge field, and this mass term is not gauge invariant, because A_μ is not gauge-invariant (see equation 4.10). Therefore, we must conclude that the particle of the gauge field is massless.

Thus, the symmetry requirement of local phase invariance of the Lagrangian of an arbitrary spin-1/2 particle, has resulted in an interacting field theory, with the massless gauge field functioning as the interacting field particle. Our final Lagrangian for this particle is

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi + e\bar{\psi}\gamma^\mu\psi A_\mu - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{4.13}$$

4.4 $SU(3)$ local gauge invariance

In the same way as in section 4.3, we can try to obtain an interacting field theory for local gauge invariance under transformation of the Lie group $SU(3)$ (that we will interpret as QCD in section 5.2). The Lagrangian for the field q_j (with $j = R, G, B$) is [4]

$$\mathcal{L} = \bar{q}_j(i\gamma^\mu\partial_\mu - m)q_j.$$

In this equation, $\bar{q}_j := q_j^*\gamma^0$ and γ^μ and ∂_μ are as before. Under $SU(3)$ transformations, the field $\mathbf{q}(x) = (q_R(x), q_G(x), q_B(x))^T$ transforms as

$$\mathbf{q} \to U\mathbf{q} = e^{i\xi_a T^a}\mathbf{q}, \tag{4.14}$$

where U is any element of $SU(3)$, which by theorem 1.3 we are allowed to write as $e^{i\xi_a T^a}$. Here, T^a $(a = 1,..,8)$ are the Gell-Mann matrices, as in section 1.3 and ξ_a the real group parameters. In equation 4.14, again the Einstein notation is used. Because U is hermitian ($UU^* = \mathbb{1}$, from the definition of $SU(3)$) it holds that $U^* = U^{-1}$ and hence that, using the properties of the matrix exponential,

$$(e^{-i\xi_a T^a}) = (e^{i\xi_a T^a})^{-1} = U^{-1} = U^* = (e^{i\xi_a T^a})^* = e^{(i\xi_a T^a)^*} = e^{-i\xi_a^* T^{a^*}}.$$

Because the ξ_a are real, it follows that[2]

$$T^a = T^{a^*}. \tag{4.15}$$

Just as in section 4.3, we look at what happens to $\bar{\mathbf{q}}$ and $\partial_\mu\mathbf{q}$ if $\mathbf{q}$ changes according to equation 4.14, with the group parameters a function of space-time: $\xi_a = \xi_a(x)$. It suffices to look only at infinitesimal phase transformations in $SU(3)$, because by theorem 1.3, we can write each $U \in SU(3)$ as

$$U = e^X = \lim_{n\to\infty}\left(\mathbb{1} + \frac{X}{n}\right), \tag{4.16}$$

for $X \in \mathfrak{su}(3)$. Hence, we see that each element of $SU(3)$ can be built up from infinitesimal transformations around the identity.

[2] This could also be concluded by looking directly at the eight Gell-Mann matrices.

If we take an infinitesimal phase transformation of $\mathbf{q} \to [\mathbb{1} + ig\xi_a T^a]\mathbf{q}$ (with g playing the role of field strength parameter, which was played by q in section 4.3), then we get first of all

$$\mathbf{q}^* \to ([\mathbb{1} + i(g\xi_a T^a)]\mathbf{q})^* = \mathbf{q}^*[\mathbb{1} - i(g\xi_a T^a)^*] = \mathbf{q}^*[\mathbb{1} - ig\xi_a T^a].$$

The last equality holds by equation 4.15. For the derivative we get

$$\partial_\mu \mathbf{q} \to [\mathbb{1} + ig\xi_a T^a]\partial_\mu \mathbf{q} + igT^a \mathbf{q} \partial_\mu \xi_a.$$

The extra term $igT^a \mathbf{q}\partial_\mu \xi_a$ ruins the invariance of the Lagrangian. We therefore again introduce a new derivative D_μ, given by

$$D_\mu := \partial_\mu - igT_a G^a_\mu, \tag{4.17}$$

with this time not one but eight gauge fields G^a_μ. We initially take the gauge fields such that they transform in the same way as A_μ in equation 4.10, namely like

$$G^a_\mu \to G^a_\mu + \partial_\mu \xi_a. \tag{4.18}$$

but then run into a problem. From our definition of D_μ, an extra term arises in the Lagrangian that should be gauge invariant, namely $\mathbf{q}^*\gamma^\mu T_a \mathbf{q}$. We see that

$$\begin{aligned}
\mathbf{q}^*\gamma^\mu T_a \mathbf{q} &\to \mathbf{q}^*[\mathbb{1} - i\xi_a T^a]\gamma^\mu T_a[\mathbb{1} + i\xi_a T^a]\mathbf{q} \\
&= \mathbf{q}^*\gamma^\mu T_a \mathbf{q} + i\xi^b \mathbf{q}^*\gamma^\mu (T_a T_b - T_b T_a)\mathbf{q} + \xi^b \xi^c T_b \mathbf{q}^* \gamma^\mu T_a T_c \mathbf{q} \\
&= \mathbf{q}^*\gamma^\mu T_a \mathbf{q} + i\xi^b \mathbf{q}^*\gamma^\mu [T_a, T_b]\mathbf{q} && \text{(order } \xi^2 \text{ is negligible)} \\
&= \mathbf{q}^*\gamma^\mu T_a \mathbf{q} - f^c_{ab}\xi^b(\bar{\mathbf{q}}\gamma^\mu T_c \mathbf{q}) && \text{(equation 1.7)} \\
&\neq \mathbf{q}^*\gamma^\mu T_a \mathbf{q}.
\end{aligned}$$

In section 4.3, this problem did not occur. For, as $U(1)$ is abelian (and has a 1-dimensional Lie-algebra), its generator always commutes. Indeed, if $SU(3)$ would have been abelian, the structure constants f_{abc} would all be zero, so that the term $\mathbf{q}^*\gamma^\mu T_a \mathbf{q}$ would still be invariant under our $SU(3)$ transformation. This problem can luckily be solved quite easily, by adding an extra term to equation 4.18 and letting G^a_μ transform as [2]

$$G^a_\mu \to G^a_\mu + \partial_\mu \xi^a + g f^a_{bc}\xi^b G^c_\mu. \tag{4.19}$$

Under this transformation of the gauge fields and with our new derivative D_μ, one can check that the Lagrangian $\mathcal{L}$ is invariant under local gauge transformation. It is thus given by

$$\mathcal{L} = \mathbf{q}^*(i\gamma^\mu \partial_\mu - m)\mathbf{q} - g(\mathbf{q}^*\gamma^\mu T_a \mathbf{q})G^a_\mu.$$

Again we look at which additional, invariant terms we can add to the Lagrangian. Specifically, we can look at kinetic and mass terms for each gauge field. However, a term of the form $\frac{1}{2}m^a_G(G_\mu)^2$ is not gauge invariant, unless m_G is zero, from which we conclude that the gauge fields must be massless.

Furthermore, we can search for a kinetic energy term for each gauge field. This is a term of the form $\frac{1}{2}(\partial_\mu \psi^\mu)^2$ for some field ψ that is a function of $\partial_\mu G^\mu_a$. Analogously to our approach for the field strength $F^{\mu\nu}$ in section 4.3, one can show for $G_{\mu\nu}$ that the term $-\frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a$ results in a gauge invariant kinetic energy term for the gauge fields G^a_μ, when $G^a_{\mu\nu}$ is defined by

$$G^a_{\mu\nu} := \partial_\mu G^a_\nu - \partial_\nu G^a_\mu - g f_{abc} G^b_\mu G^c_\nu. \tag{4.20}$$

Adding the kinetic term to the Lagrangian, results in

$$\mathcal{L} = \mathbf{q}^*(i\gamma^\mu \partial_\mu - m)\mathbf{q} - g(\mathbf{q}^*\gamma^\mu T_a \mathbf{q}) - \tfrac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a, \tag{4.21}$$

the final, gauge invariant Lagrangian for the field $\mathbf{q}$.

4.5 $SU(2)$ local gauge invariance

We had a good reason to start with the local gauge invariances of $SU(3)$ instead of the invariances of $SU(2)$, which is of smaller dimension and hence usually easier to examine. In section 5.1, we will see how $SU(2)$ can be used to describe the (electro-)weak interactions. However, the weak interactions have one big difference from the electromagnetic and strong interactions that complicates the determination of the complete Lagrangian for the weak interactions. The difference is that the force carriers of the weak interaction – that are closely connected with the gauge fields of our theory (which we will make explicit in section 5.1) – are massive. If we use the same approach as in section 4.4 for the Lagrangian of the weak interactions, it will once more follow that the gauge fields correspond to massless particles. We do not want our theory to diverge from the physical observations and hence a different approach must be taken.

We could for example let go the demand of invariance of the Lagrangian of the (electro-)weak interaction under local transformations, as this was (initially) just done on "aesthetic grounds". We would then introduce mass terms of the form $\frac{1}{2}m^2 W_\mu W^\mu$ for the gauge field W_μ and ignore the symmetry-breaking that they cause. However, if one does this, one encounters divergences that cannot be restored and make the theory meaningless[3].

We therefore do not give up local gauge invariance and try a second approach in which we do introduce masses and still try not to break the gauge invariance. We will go with small steps to our final goal, by first considering a (nonphysical) real scalar field, then – firstly global and secondly local gauge – invariance of a complex scalar field and finally local gauge invariance of a complex 2-dimensional vector field on which the symmetry group $SU(2)$ is working.

4.5.1 Spontaneous symmetry breaking

In this subsection, we will introduce the subject of spontaneous symmetry breaking on the basis of an nonphysical world that consists only of real, scalar particles. In such a world, a particle field ϕ can be describe by the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \phi)^2 - (\tfrac{1}{2}\mu^2\phi^2 + \tfrac{1}{4}\lambda^2\phi^4), \qquad (4.22)$$

with $\lambda > 0$. Our Lagrangian is invariant under the operation that replaces ϕ by $-\phi$. If $\mu^2 > 0$, we are in a familiar situation. Then the Lagrangian describes a scalar field with mass μ, as equation 4.5 did for the complex field of a spin-1/2 particle, but then with an extra term ϕ^4 that tells us that ϕ is a self-interacting field. The ground state (the state with the lowest energy) is equal to $\phi = 0$.

A new and interesting situation occurs when $\mu^2 < 0$. In that case, the potential energy V,

$$V = \tfrac{1}{2}\mu^2\phi^2 + \tfrac{1}{4}\lambda^2\phi^4,$$

has two minima at $\phi = \pm\nu = \pm\sqrt{-\mu^2/\lambda}$ (and a local maximum at $\phi = 0$). Because the kinetic term vanishes for constant ϕ, the ground states of the Lagrangian for $\mu^2 < 0$ are at $\phi = \pm\nu$. We make the choice to look only at $\phi = \nu$. This is a choice that nature also has to make, for the particle will be in only one of the ground states. The choice causes no loss of generality because of the invariance of the Lagrangian under a change $\phi \to -\phi$. We look at small perturbations around the minimum ν by writing the particle field as

$$\phi(x) = \nu + \eta(x)$$

with $\eta(x)$ representing small fluctuations about the ground state. Substituting this equation for ϕ in the Lagrangian gives a new Lagrangian in terms of $\eta(x)$, namely

$$\begin{aligned}\mathcal{L}' &= \tfrac{1}{2}(\partial_\mu[\nu + \eta(x)])^2 - (\tfrac{1}{2}\mu^2[\nu + \eta(x)]^2 + \tfrac{1}{4}\lambda^2[\nu + \eta(x)]^4) \\ &= \tfrac{1}{2}(\partial_\mu \eta(x))^2 - \lambda\nu^2\eta(x)^2 - \lambda\nu\eta^3 - \tfrac{1}{4}\lambda\eta(x)^4 + \text{ const.} \qquad (\nu \text{ constant}, \mu^2 = -\nu^2\lambda).\end{aligned}$$

[3]'Meaningless' in that it can not predict anything [4]. More specifically, the theory becomes what is called *nonrenormalizable*. A quantum field theory has to be renormalizable to be able to work on all scales. In a way, the place where quantum mechanics and general relavitity clash, is on their renormalizability, as the quantum field theories of all but the gravitational force are renormalizable [14].

In this equation we see that the field η has mass term with sign opposite to the mass term of our original Lagrangian. We namely have a term $-\lambda\nu^2\eta(x)^2$, which is smaller than zero because both λ and ν are assumed to be positive, while equation 4.22 has mass term $-\frac{1}{2}\mu^2\psi^2 > 0$, as $\mu^2 < 0$. We can thus identify a mass in the equation for $\mathcal{L}'$. By writing down the equality

$$-\lambda\nu^2\eta(x)^2 = -\frac{1}{2}m\eta(x)^2,$$

we see that the mass term is equal to

$$m_\eta = \sqrt{2\lambda\nu^2}.$$

Something strange has happened. The two Lagrangians $\mathcal{L}$ and $\mathcal{L}'$ are equivalent and yet they do not translate to the same physical situation. The solution lies in the fact that we have used perturbation theory. One can check that in perturbation theory $\mathcal{L}$ does not converge[4], while $\mathcal{L}'$ does [4]. Therefore, only $\mathcal{L}'$ gives the correct physical result. This means that the scalar particle ϕ described by both $\mathcal{L}$ and $\mathcal{L}'$ has a mass.

We call the way that this mass was 'revealed' *spontaneous symmetry breaking.* One can make the following comparison with a less abstract, physical example. Consider a needle that is compressed with a force F along its axis (which we take to be the z-axis of a 3-dimensional, Euclidean coordinate system). At first sight, the most logical position of the needle, is that it stays with its axis at $x = y = 0$. However, this is only a meta-stable situation. When slightly perturbed in the x-or y-direction, the needle will buckle and stay bend in the direction of the perturbation. In this new ground state, that has been determined arbitrarily, the original symmetry of the system has been broken. The same happens for the Lagrangian of the particle field ϕ: by allowing a slight perturbation $\eta(x)$ (which is indeed present in a realistic world due to quantum fluctuations) of its initial meta-stable state, the reflection symmetry of the Lagrangian has been broken.

4.5.2 Spontaneous breaking of a global symmetry

We now go one step further and look at a complex scalar field $\phi = (\phi_1 + i\phi_2)/\sqrt(2)$. The Lagrangian of this field is given by

$$\begin{aligned}\mathcal{L} &= (\overline{\partial_\mu\phi})(\partial^\mu\phi) - \mu^2\overline{\phi}\phi - \lambda(\overline{\phi}\phi)^2 \\ &= \tfrac{1}{2}(\partial_\mu\phi_1)^2 + \tfrac{1}{2}(\partial_\mu\phi_2)^2 - \tfrac{1}{2}\mu^2(\phi_1^2 + \phi_2^2) - \tfrac{1}{4}\lambda(\phi_1^2 + \phi_2^2)^2,\end{aligned} \tag{4.23}$$

where $\bar{\phi} = \phi^*\gamma^0$. This Lagrangian is invariant under the global transformation $\phi \to e^{i\alpha}\phi$. The first two terms in the last part of the above equation are kinetic energy terms. The last two form the potential energy $-V$. From analysis, we know that a function $V(\phi_1, \phi_2)$ reaches its extremes when

$$\frac{\partial V}{\partial\phi_1} = \frac{\partial V}{\partial\phi_2} = 0,$$

So when

$$\begin{aligned}\frac{\partial V}{\partial\phi_i} &= \mu^2\phi_i + \lambda\phi_i^3 + \lambda\phi_i\phi_j^2 \\ &= \phi_i(\mu^2 + \lambda(\phi_i^2 + \phi_j^2) = 0\end{aligned}$$

for $(i, j) = (1, 2)$ and $(2, 1)$. This corresponds with either $\phi_1 = \phi_2 = 0$ or

$$\phi_1^2 + \phi_2^2 = -\frac{\mu^2}{\lambda} =: \nu^2.$$

The first extreme is local maximum, the second (which is in fact a circle of extremes) is a global minimum. We can – again without loss of generality, this time because or the invariance of the Lagrangian under the global phase transformation $e^{i\alpha}$ – choose one of the minima. We take $\phi_1 = \nu, \phi_2 = 0$. A perturbation around this minimum will be of the form

$$\phi(x) = \sqrt{\tfrac{1}{2}}(\nu + \eta(x) + i\xi(x)). \tag{4.24}$$

[4] For then we would perturb around the unstable 'ground state' $\phi = 0$.

Just as in section 4.5.1, when we substitute the perturbed ground state into the original Lagrangian, we get a Lagrangian of the form

$$\mathcal{L}' = \tfrac{1}{2}(\partial_\mu \xi)^2 + \tfrac{1}{2}(\partial_\mu \eta)^2 + \mu^2\eta^2 + \text{ rest terms.}$$

The rest terms consist of constants and functions in which η or ξ occurs in higher powers than 2. In the new Lagrangian, we can identify a positive mass term $m_\eta = \sqrt{-2\mu^2}$. However, while the field $\xi(x)$ does have a kinetic term, it has no mass term. This means that besides an extra, desired massive gauge field, the spontaneous symmetry breaking of the Lagrangian of a complex scalar particle also gives us an extra, massless scalar particle, which we call the *Goldstone boson*. We will ignore the Goldstone boson for a while and continue to local gauge symmetry in the next subsection.

4.5.3 Spontaneous breaking of a local gauge symmetry

For the gauge group U(1)

We are almost ready to study the spontaneous symmetry breaking of local $SU(2)$ gauge symmetries. In this subsection, we first look at the spontaneous symmetry breaking of the simpler group $U(1)$ and then continue to that of $SU(2)$. In section 4.3, we have seen that to make the Lagrangian invariant under a transformation

$$\phi \to e^{i\alpha(x)}\phi,$$

we must replace ∂_μ by the alternative derivative D_μ given by equation 4.9, where the gauge field A_μ transforms as in equation 4.10. Hence, the gauge invariant Lagrangian for the complex scalar field ϕ becomes

$$\mathcal{L} = (\partial_\mu + iqA_\mu)\bar{\phi}(\partial_\mu - iqA_\mu)\phi - \mu^2\bar{\phi}\phi - \lambda(\bar{\phi}\phi)^2 - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{4.25}$$

by making the substitution $\partial_\mu \to D_\mu$ in the first part of equation 4.23 and adding the gauge invariant kinetic energy term of the gauge field A_μ.

If $\mu^2 > 0$ this equation almost corresponds with the Lagrangian for a charged scalar spin-1/2 particle of mass μ given by equation 4.13. The only difference is the extra 'self interaction term' ϕ^4. However, the situation $\mu^2 > 0$ is not in our interest, for we again want to investigate which masses occur due to spontaneous symmetry breaking. Hence, we look at the case $\mu^2 < 0$.

The potential energy of the Lagrangian in equation 4.25 is the same as that of equation 4.23 and therefore has the same global minima at $\phi_1^2 + \phi_2^2 = \nu^2$. We can again look at the influence of slight perturbations around this minima and substitute 4.24 into equation 4.25 to get

$$\mathcal{L}' = \tfrac{1}{2}(\partial_\mu \xi)^2 + \tfrac{1}{2}(\partial_\mu \eta)^2 - \nu^2\lambda\eta^2 + \tfrac{1}{2}q^2\nu^2 A_\mu A^\mu - q\nu A_\mu \partial^\mu \xi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \text{ rest terms.}$$

The rest terms are again terms in which ξ and η occur in orders higher than two. We do not write them down because we are only interested in the mass terms. In $\mathcal{L}'$, we can identify three particle fields, namely that of the massless Goldstone boson ξ, the massive scalar particle η and, most importantly, a massive gauge field A_μ. This last gauge field could be able to function as the massive, interacting particles of the weak interaction, that we introduced in the beginning of this section and that we desire to merge into our theory. From the Lagrangian, we see that the masses of ξ, η and A_μ are given by

$$m_\xi = 0, \qquad m_\eta = \sqrt{2\lambda\nu^2}, \qquad m_A = q\nu.$$

However, still the unwanted, massless Goldstone boson occurs in our Lagrangian. Luckily, this problem can be solved by noting that the identification of the particles in equation 4.25 cannot be correct. The occurrence of the extra term $A_\mu \partial^\mu \xi$ shows that the particles are not distinct, but have a certain dependency. One can choose a different set of real fields h, θ and A_μ such that the theory is independent of the field θ [5]. This means that our final Lagrangian describes only two interacting particles, namely a massive gauge field A_μ and a massive, real, scalar particle h.

[5] The details on how to choose these fields can be found in e.g. [4].

For the gauge group SU(2)

Finally, we are ready to treat the spontaneous symmetry breaking of local $SU(2)$ gauge symmetries. We again take a Lagrangian comparable to that in equation 4.22, but now with ϕ a complex 2-dimensional vector of the form

$$\phi = \begin{pmatrix} \phi_a \\ \phi_b \end{pmatrix} = \sqrt{\tfrac{1}{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix}.$$

Hence our Lagrangian becomes

$$\mathcal{L} = (\partial_\mu \phi)^* (\partial^\mu \phi) - \mu^2 \phi^* \phi - \lambda (\phi^* \phi)^2.$$

The above Lagrangian is invariant under a global $SU(2)$ transformation

$$\phi \to U\phi = e^{\alpha^a t_a},$$

because $UU^* = \mathbb{1}$, by the definition of $SU(2)$. t_a again denote the group generators and α_a the group parameters of $SU(2)$ (with $a = 1, 2, 3$) as given in section 1.3. However, we require more than global invariance and search for a Lagrangian that is invariant under local $SU(2)$ transformations. This can be done in the same way as we did for $SU(3)$, namely by replacing the derivative ∂_μ by a new derivative D_μ, this time given by

$$D_\mu := \partial_\mu - igt_a W^a_\mu,$$

where W^a_μ are three gauge fields that transform under an infinitesimal $SU(2)$ transformation[6]

$$\phi(x) \to [1 + \alpha_a(x) t^a] \phi(x),$$

as[7] [2]

$$W_\mu \to W_\mu + g\epsilon^a_{bc} W^b_\mu \alpha^c + \partial_\mu \alpha^a.$$

Furthermore, we can add a gauge invariant field strength tensor of the form

$$W^a_{\mu\nu} =:= \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g\epsilon^a_{bc} W^b_\mu W^c_\nu,$$

which is analogue to the field strength tensor in equation 4.20, but with a plus instead of a minus in the last term, which occurs because the structure constants of $SU(2)$ are $-\epsilon_{abc}$ (while those of $SU(3)$ are $+f_{abc}$).

So we get – having passed quickly over the steps that we explained in much more detail for $SU(3)$ – a final Lagrangian for ϕ given by

$$\mathcal{L} = (\partial_\mu - igt_a W^a_\mu \phi)^* (\partial_\mu - igt_a W^a_\mu \phi) - V(\phi) - \frac{1}{4} W^a_{\mu\nu} W^{\mu\nu}_a, \tag{4.26}$$

with $V(\phi)$ the potential energy that can be received directly from the original Lagrangian and is given by

$$V(\phi) = \mu^2 \phi^* \phi + \lambda (\phi^* \phi)^2.$$

We look again at the situation where $\mu^2 < 0$ and $\lambda > 0$. Minima of $V(\phi)$ can be found when

$$\phi^* \phi = \frac{1}{2}(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2) = -\frac{\mu^2}{\lambda}.$$

Again we choose just one of all these minima (without loss of generality because of invariance of the Lagrangian under global $SU(2)$ transformations). We can for example take the minimum

$$\phi_1 = \phi_2 = \phi_4 = 0, \qquad \phi_3^2 = \frac{\mu^2}{\lambda} =: \nu^2.$$

[6]Just as for $SU(3)$, each element $g = e^X \in SU(2)$ can be build up from infinitesimal transformations [2]. See also equation 4.16.

[7]Compare with equation 4.19 to see that the transformation below is exactly as we saw for the gauge fields of $SU(3)$.

At this minimum, the ground state of ϕ becomes

$$\phi = \sqrt{\tfrac{1}{2}} \begin{pmatrix} 0 \\ \nu \end{pmatrix}.$$

One can check that a general perturbation of this ground state (where all four scalar fields are perturbed) corresponds with perturbing the ground state only around $\phi_3 = \nu$ with the amount of perturbation given by the scalar field $h(x)$, so that we get [4]

$$\phi(x) = \sqrt{\tfrac{1}{2}} \begin{pmatrix} 0 \\ \nu + h(x) \end{pmatrix}$$

as a perturbed ground state of ϕ. Substitution of this field in equation 4.26, results in a final, gauge invariant Lagrangian that contains only the scalar field h and the three massive gauge fields W_μ^a.

The masses of the gauge fields can be determined by looking only at the term in which $|t_a W_\mu^a|^2$ occurs. From equation 4.26, we see that this is the term

$$\begin{aligned} \left|igt_a W_\mu^a \phi\right|^2 &= g^2 \left| \frac{1}{2} \frac{1}{\sqrt{2}} \begin{pmatrix} W_\mu^3 & W_\mu^1 - iW_\mu^2 \\ W_\mu^1 + iW_\mu^2 & W_\mu^3 \end{pmatrix} \begin{pmatrix} 0 \\ \nu \end{pmatrix} \right|^2 \qquad \phi = \phi_0, \text{ equation 1.5} \\ &= \frac{g^2\nu^2}{8} \left[(W_\mu^1)^2 + (W_\mu^2)^2 + (W_\mu^3)^2 \right], \end{aligned}$$

where we have written $(\ )^*(\ )$ as $|\ |^2$ to shorten the notation. By comparing this equation to $\frac{1}{2}M^2 B_\mu^2$, the standard form of a mass term of a boson B_μ, we see that

$$M_{W_\mu^a} = \sqrt{\frac{g^2\nu^2}{4}} = \frac{1}{2} g\nu, \tag{4.27}$$

for all three gauge fields W_μ^a.

Chapter 5

Particle representations

In this chapter, we apply the knowledge of Lie groups and representations gained in chapter 1 and 2, to the elementary particles of the standard model that have been described in chapter 3. In chapter 4, we have seen three different types of gauge theories, namely with gauge invariance under

- $U(1)$, a Lie group with one generator Y, in section 4.3 and one gauge field A_μ
- $SU(2)$, a Lie group with three generators $t_a (a = 1, 2, 3)$ and three gauge fields W_μ^a, in section 4.5
- $SU(3)$, a Lie group with eight generators $T_a (a = 1, 2, ..., 8)$ and eight gauge fields G_μ^a, in section 4.4.

In general, particle fields can be represented by functions on $\mathbb{R}^4$ whose domain is some vector space V on which group representations work. For the electroweak model this will be representations of the direct product $U(1) \times SU(2)$, for QCD representations of $SU(3)$. The representations dictate how the particles transform under the group and hence which symmetries the particles have. In this set-up, a system of two particles, can be represented by the tensor product representation $V \otimes V$.

In section 5.1, we will see how the representation and gauge theories of $U(1)$ and $SU(2)$ can be combined to form the electroweak model. Hereafter, in section 5.2, we will explain how the representation and gauge theory of $SU(3)$ can be used to mathematically formulate QCD.

5.1 The electroweak model: $U(1) \times SU(2)$

5.1.1 Representation theory

$SU(2)$ plays a role in the electroweak interactions. The $SU(2)$ transformations can be seen as weak isospin transformations. Under this identification, the third generator of $SU(2)$ can be identified with I_3. We will call these $SU(2)$ transformations weak isospin transformations and denote them by $SU(2)_L$, as only left-handed particles are found to transform under these transformations. Right-handed particles transform trivially, meaning that

$$t_3 \psi_R = 0,$$

for a right-handed particle field ψ_R. Left-handed particles on the other hand, transform under the adjoint representation of $SU(2)$. Physically, this means e.g. that a left-handed electron (with $\mathrm{I}_3 = -\frac{1}{2}$) can be transformed under the weak interaction into a left-handed neutrino (with $\mathrm{I}_3 = +\frac{1}{2}$) with the emission of a weak W^- boson, while on the other hand right-handed particles do not interact under the weak force. Furthermore, the group $U(1)$ can be identified with weak hypercharge transformations. We will denote it by $U(1)_Y$ to distinguish it from the $U(1)$ transformations of the quantum number Q, that we will encounter in the next subsection.

5.1.2 Gauge theory

The theory described in section 4.3, almost completely agrees with QED if we interpret ψ as the electron field; the phase parameter q as the electron charge e; the gauge field A_μ as the photon field; and $F^{\mu\nu}$ as the electromagnetic field strength tensor. In that case, from our conclusion that no gauge invariant mass term of A_μ could be added to the Lagrangian, it follows that the photon is massless, which indeed corresponds with the physical reality [4].

However, it is possible to combine the theory of electromagnetism with that of the weak interaction in the unified electroweak model: a gauge theory whose Lagrangian is invariant under $U(1)\times SU(2)$. As already mentioned in chapter 3, data of experiments suggest that under electroweak interactions, the electric charge Q – which can be seen as a combination of weak isospin (corresponding with the group $SU(2)_L$) and weak hypercharge (corresponding with the group $U(1)_Y$) – is conserved. In this combined theory a different physical interpretation of the gauge fields is made, which we will treat in this section. In the following, we will denote the gauge field of $U(1)$ by B_μ instead of A_μ and use the term A_μ to denote the physical photon field.

We now want to formulate the physical fields $A_\mu, W^\pm$ and Z^0 in terms of the gauge fields of $U(1)$ and $SU(2)$ in such a way that A_μ has zero mass and that the weak bosons are massive. The local gauge theory of $SU(2)$ for massive gauge fields tells us that if we incorporate massive force carriers into the theory, inevitable another scalar field h appears in the Lagrangian of the initial field ϕ. This scalar field corresponds with the field of the Higgs boson. The Higgs boson has been introduced in chapter 3. If we manage to find a $U(1)$ generator which is unbroken by the Higgs field, we know that the corresponding gauge field is massless, as is desired for the photon.

The suitable generator is Q, which we will call the generator of '$U(1)_Q$'. Q is defined by

$$Q := t_3 + \tfrac{1}{2}Y, \tag{5.1}$$

where Y denotes the generator of $U(1)_Y$ and t_3 is the third generator of $SU(2)_L$, given by

$$t_3 = \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

If we compare equation 5.1 with equation 3.3, we can identify t_3 with the weak isospin I_3 and Y with the weak hypercharge $\mathrm{Y_W}$. Then, physically, Q corresponds with the electromagnetic charge. Indeed, we see that it is such that it leaves the Higgs particle invariant. For with Q defined by equation 5.1, we have

$$Q\tfrac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \nu \end{pmatrix} = \tfrac{1}{\sqrt{2}}\left[\tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\begin{pmatrix} 0 \\ \nu \end{pmatrix} = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \nu \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This means that h is left invariant under local $U(1)_Q$ transformations, for then $h \to e^{i\alpha Q}h = h$ for any local phase $\alpha(x)$. So under $U(1)_Q$ transformations, the symmetry remains unbroken and the photon field – the gauge boson A_μ – of the $U(1)_Q$ gauge symmetries is massless.

The gauge field strength q is proportional to the charge of the particle field on which $U(1)_Q$ is working. Because $U(1)$ is abelian, we are free to choose q for different particle fields. This can be seen by the following reasoning.

Consider a compact, connected Lie group G. Let g_1 and g_2 be two elements in G. By theorem 1.3, we can write $g_1 = e^{q_1 X_1}$ and $g_2 = e^{q_1 X_2}$, with X_1 and X_2 elements of the Lie algebra and where we have incorporated the field strength q_1 in our group. Now, for the definition of a group it follows that

$$g_3 = g_1 g_2 \in G.$$

Suppose now that we describe the elements of G with another field with field strength q_2. Take e.g. $q_2 = 2q_1$. Then we can write the squares of our group elements g_1 and g_2 as $g_1^2 = (e^{q_1 X_1})^2 = e^{2q_1 X_1} = e^{q_2 X_1}$ and in the same way $g_1^2 = e^{q_2 X_2}$. We would like to be able to write

$$g_3^2 = g_1^2 \cdot g_2^2,$$

for in that case the new field strength does not change the group multiplication. However, this is only possible if G is abelian, for we only then we are allowed to write

$$g_3^2 = (g_1 \cdot g_2)^2 = g_1 \cdot g_2 \cdot g_1 \cdot g_2 = g_1 \cdot g_1 \cdot g_2 \cdot g_2 = g_1^2 \cdot g_2^2$$

for all $g_1, g_2 \in G$.

The physical fields and the masses of the gauge bosons $W^\pm$, Z^0 and γ of the electroweak interaction can be found as follows. We start with substituting the ground state ϕ that we found in section 4.5 in the $U(1) \times SU(2)$ gauge invariant Lagrangian for the complex vector field ϕ. This causes spontaneous symmetry breaking due to which mass terms arise. These mass terms corresponds with the masses of physical particles. The real, physical, normalized fields that correspond with these masses in terms of the field strengths and gauge fields of $U(1)_Y$ and $SU(2)_L$ are [4]

$$\begin{aligned} W^\pm_\mu &= \frac{W^1_\mu \pm iW^2_\mu}{\sqrt{2}} & M_W &= \tfrac{1}{2}\nu g \text{ (equation 4.27)} \\ Z^0_\mu &= \frac{g'W^3_\mu + gB_\mu}{\sqrt{g^2+g'^2}} & M_Z &= \tfrac{1}{2}\nu\sqrt{g^2+g'^2} \\ A_\mu &= \frac{gW^3_\mu - g'B_\mu}{\sqrt{g^2+g'^2}} & M_A &= 0. \end{aligned}$$

This finishes our treatment of the electroweak interactions.

5.2 Quantum chromodynamics: $SU(3)$

5.2.1 Representation theory

Let $G = SU(3)$, and let $\mathbb{C}^3$ be its defining representation. This representation corresponds with the internal degrees of freedom of a single quark when we interpret the colors red, green and blue as e_1, e_2 and e_3 of the standard basis of $\mathbb{C}^3$ respectively. In section 3.3, we have seen that quarks have to be colorless. This means that they should corresponds with vectors that transform trivially under permutations of {R,B,G}. These permutations can be given by the *Weyl-group* W of $SU(3)$. The Weyl-group is a discrete group of six elements that permutes the color basis of $\mathbb{C}^3$. More precisely, it is the group consisting of transformations $w_\sigma : \mathbb{C}^3 \to \mathbb{C}^3$ such that

$$w_\sigma(e_i) = e_{\sigma(i)}, \tag{5.2}$$

for a permutation σ of $\{e_1, e_2, e_3\}$ = {R,B,G}. An example of an element of W is

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{5.3}$$

which sends R,B and G to G,B and R respectively.

Because the Weyl-group clearly permutes all vectors in both $\mathbb{C}^3$ and $\mathbb{C}^{3^*}$, we already see that no particles consisting of a single quark can exist: any vector in $\mathbb{C}^3$ or $\mathbb{C}^{3^*}$ will change color under transformation by some elements of the Weyl-group and hence is not colorless. Hence, if we want to identify quarks with vectors, we will need to search for vector combinations that are unchanged by transformations of elements in the Weyl-group.

Definition 5.1. The *maximal trivial representation* of the representation (π, V) on a group G is defined to be the vector space

$$V^G = \{v \in V : \pi(g)v = v \ \forall g \in G\}.$$

Because the Weyl-group is a subgroup of $SU(3)$, vectors in V^G will transform trivially under the Weyl-group. This means that all vectors $v \in V^G$ are colorless. Hence, physically allowed particle fields exist only in representations that have a non-empty trivial subrepresentation.

In the introduction of this chapter, we have already seen that if a particle is represented by the unitary representation (π, V), then its anti-particle can be represented by the corresponding dual representation. Hence, mathematically, a meson, which consists of a quark and an anti-quark, would be a basis vector in $\mathbb{C}^3 \otimes \mathbb{C}^{3^*}$, which can be decomposed as [4]

$$\mathbb{C}^3 \otimes \mathbb{C}^{3^*} = \mathbb{C}^8 \oplus \mathbb{C}, \tag{5.4}$$

from which we see that the maximal trivial subrepresentation of $\mathbb{C}^3 \otimes \mathbb{C}^{3^*}$ is 1-dimensional and nonzero. Now, only vectors in the trivial representation of $\mathbb{C}^3 \otimes \mathbb{C}^{3^*}$ are 'colorless' and correspond with physical particles. All other vectors are permuted by the Weyl-group. They can be identified with the coloured gluons of the strong interaction.

We can also, for example, look at the representation of $\mathbb{C}^3 \otimes \mathbb{C}^3$. This representation decomposes as [4]

$$\mathbb{C}^3 \otimes \mathbb{C}^3 = \mathbb{C}^6 \oplus \mathbb{C}^{3^*}, \tag{5.5}$$

from which we see that $\mathbb{C}^3 \otimes \mathbb{C}^3$ has no trivial subrepresentation. Hence, we can conclude that there do not exist any particles that consist of only two quarks!

However, baryons are composed of three quarks, and are thus represented by the color representation $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ of $SU(3)$, which decomposes as [17]

$$\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 = \mathbb{C}^{10} \oplus \mathbb{C}^8 \oplus \mathbb{C}^8 \oplus \mathbb{C}. \tag{5.6}$$

Hence, we see that $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ has one trivial subrepresentation which contains the colorless quarks.

5.2.2 Gauge theory

The theory of QCD can be obtained from the gauge theory of $SU(3)$. We interpret the eight gauge fields G^a_μ as the eight vector gluon fields and g as the coupling constant of the strong force. Contrary to the coupling constant q of the electromagnetic interaction, we have no freedom in choosing g. In section 5.1, we already argued that only because $U(1)$ is abelian, in section 5.1, we were allowed to choose q in multiple ways. g on the other hand is fixed by experimental observations.

From our gauge theory it follows that gluons are massless, because no gauge invariant mass term of the field G^μ_a could be added to the Lagrangian. $G^a_{\mu\nu}$ corresponds with the field strength tensor of QCD.

If we compare the field strength tensor $G_{\mu\nu}$ of QCD with that of QED, we see that $G_{\mu\nu}$ contains an extra term, that arises because $SU(3)$ is non-abelian. Due to this extra term, the gauge bosons are seen to be self-interacting and carry color charge [4], which corresponds with experimental observations.

Now, finally, we can interpret equation 4.21 as the gauge-invariant Lagrangian for QCD, for the eight interacting and self-interacting gluon fields G^a_μ and the colorless quark field $\mathbf{q}$.

Chapter 6

Recommendations

This thesis has given an introduction to the symmetries of the standard model. However, the picture that has been sketched is not complete. The standard model is a very complicated theory, that can be studied at many depths. In this thesis, we have only treated the symmetries till a certain extent and even then the treatment was not always complete. In this chapter, some recommendations for further study (concerning already known topics) and further research (concerning new investigations) will be made.

6.1 Further study

In this section, for each chapter, we name a few topics that can be studied further for a more complete overview and – even more importantly – a deeper understanding of the symmetries of the standard model.

Chapter 1: By the further study of Lie groups, a better intuition can be obtained of how the Lie groups $U(1)$, $SU(2)$ and $SU(3)$ function as symmetry groups and are connected to their Lie algebra.

Chapter 2: We have treated the representation theory of $SU(3)$ (and $U(1)$) only to a certain extent and have omitted the representation theory of $SU(2)$. Further study of the representation theory of $SU(2)$ and $SU(3)$ and specifically of the decomposition of their tensor representations, will enable the reader to further understand how $U(1) \times SU(2)$ and $SU(3)$ dictate the particle interactions under the electroweak and strong force and how they model the conservation and violation of different quantum numbers in these interactions.

Chapter 3: In section 3.1, we have seen that Lorentz invariance is a requirement that follows from special relativity. From general relativity it follows that the speed of light should be the same in different frames of reference. Mathematically, this is a symmetry that can be described by invariance under transformations of the so called Poincaré group. This description, together with the representations of the Poincaré group, can be studied further to get a deeper understanding of the Lorentz symmetry of the standard model.

Moreover, the physical meaning of the phenomenon of left- and right-handed particles and their different treatment by the electroweak interactions, can be investigated further. With this, further understanding of the mathematical description of left- and -right handed particles (as stated by equation 3.2 – 3.3) can be obtained.

Chapter 4: Another reason to further study Lorentz invariance, is to understand how the initial Lagrangians (such as the Dirac equation for spin-1/2 particles) that are posited in chapter 4 can be derived for different particle fields. The derivation of these Lagrangians is also related to the unitary representation theory of the Poincaré group, which ties with and extents the unitary representation theory of chapter 2.

Chapter 5: Chapter 5 has gone briefly over the steps that are to be taken to be able to interpret the representation and gauge theory of the earlier chapters with the particles and particle interaction of the standard model. This interpretation can be made more complete as follows: by extending the representation theory of chapter 2. to be able to treat the electroweak model in more detail; by showing in more detail which vectors in the representations correspond to which particles and how these representations conserve and violate different quantum numbers and by considering the – not fundamental – flavor symmetry of quarks.

6.2 Further research

As already mentioned in the introduction of this thesis, there are still multiple phenomena that the standard model leaves unexplained. Further research has to be done, to better understand these phenomena and to be able to incorporate them in our theory. Physicists are still searching for a so called Grand Unification Theory (GUT), which includes *all* fundamental forces, by developing theories that include symmetries (such as supersymmetry) and particles (such as the graviton) beyond those of the standard model. Both theorists and experimenters try to support these theories, but up until now, no working GUT has been found.

And, as often is the case, these searches for symmetries and particles beyond the standard model, might motivate mathematicians and mathematical physicists to the further investigation of Lie theory, representation theory or other mathematical areas: to mathematics beyond the standard model.

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