The Category of Sets in Homotopy Type Theory

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Abstract

This paper is a literature survey on homotopy type theory, analyzing the formalization of sets within homotopy type theory. Set theory is embedded in homotopy type theory via h-sets, with all h-sets forming the type **Set**. This paper presents the properties of the type **Set** from a categorical perspective, comparing it with its set-theoretic counterpart. We will also compare homotopy type theory to standard axiomatic set theory from the point of view of mathematical foundations, discussing the axiom of the empty set, the power set axiom and the axiom of choice and their equivalents in homotopy type theory.

Keywords: Homotopy Type Theory, Sets, h-sets, ∞-groupoid, ΠW-pretopos, ZFC

1. Introduction

Homotopy Type Theory (HoTT) is a novel research area at the intersection of multiple fields, primarily mathematics and computer science (The Univalent Foundations Program, 2013)[1]. Within mathematics, HoTT has two main applications. Firstly, it is a new foundational program, referred to as *Univalent Foundations*, aiming to replace the standard ZFC[2] set theory as a constructive[3] foundation of mathematics (Awodey, 2014). Secondly, it can be regarded as an alternative language for category theory, more suitable for capturing the structural invariance of isomorphic constructs (Ahrens et al., 2015). Within computer science, HoTT presents novel applications for computer proof-assistants, extending the inherently computational nature of type theory to higher-categorical mathematics, lending itself to novel formalizations in computer proof assistants, such as `Coq` and `Agda` (Bauer et al., 2017) .

HoTT is an extension of intuitionistic type theory, or Martin-Löf Type-Theory (Martin-Löf, 1975), with Voevodsky's Univalence Axiom (Awodey et al., 2013) and higher inductive types. It gives a homotopical interpretation to the regular type-theoretical constructs. Types are regarded as spaces, with identity types of objects conveying paths between objects in a certain space, and identity types between paths conveying paths between paths, and so on. This gives rise to HoTT as a natural language for the categorical notion of ∞-groupoids, i.e., mathematical structures that capture this infinite upwards connection between higher paths. The novel addition consists of the Univalence Axiom, which says that equivalence is equivalent to identity. In other words, if two structures are equivalent, then they are identical.

Due to the novelty of the field of research, there are many current open problems, e.g., extending the semantics and computer formalizations of this theory (Awodey et al., 2015). Within the foundations of mathematics, type theory, especially extended to homotopy type theory, is preferred to the standard ZFC due to its computational and constructive properties (Tsementzis, 2017). Further, for certain branches of mathematics, most notably category theory, for which weaker notions of identity than equality suffice, homotopy type theory provides a more adequate language, as category theory *fits the least comfortably in set theoretic foundations* (The Univalent Foundations Program, 2013, p. 307). As a foundational effort, univalent foundations should also lend themselves useful to other branches of

1. Page references are to the on-screen view version of the book
2. The axiomatic set theory of Zermelo–Fraenkel with the axiom of choice
3. To oversimplify, constructive mathematics reject the law of excluded middle ($P \vee \neg P$) and proofs by contradiction, i.e., the equivalence ($P \equiv \neg P \to \bot$)

mathematics. As such, it is interesting to see how sets, which form the foundation in ZFC, fit within homotopy type theory, and to what extent they can be used by the practising mathematician.

This paper aims to investigate the type of sets in homotopy type theory, based on the work done by Rijke and Spitters (2015). The paper tries to give an answer to two questions. Firstly, why do we want sets in HoTT, if one of the main aims of HoTT, the foundational one, is to replace ZFC as the default foundation of mathematics. Secondly, it aims to investigate how the formalization of sets in HoTT differs behaviourally from the standard ZFC. In other words, do sets in HoTT exhibit the same properties as sets in ZFC, and if not, where do they differ.

This paper will be structured as follows: Section 2 will give an overview of the notation, main definitions and basic theorems of homotopy type theory used throughout this paper. Section 3 will present the formalization of sets in homotopy type theory as done by Rijke and Spitters (2015), presenting their main results in a manner that aims to convert the category theory definitions to the case of sets. Section 4 will then analyse the behaviour of sets in homotopy type theory, contrasting it with sets in the classical ZFC, and present the similarities, as well as the main differences.

2. Homotopy type theory preliminaries

This section will present the necessary theorems, definitions and results needed to understand the category of sets in homotopy type theory. It is assumed that the reader is familiar with dependent type theory as can be found in, e.g., The Univalent Foundations Program (2013, Chapter 1), along with some basic notions of category theory such as those in e.g. Awodey (2010). We will mostly follow the notation present in The Univalent Foundations Program (2013).

Definition 2.1 (-2-type) *A type $P : \mathcal{U}$ is called **contractible** if there is a point $a : P$, such that $a =_P x$ for any $x : P$.*

$$\mathsf{isContr}(P) :\equiv \sum_{(a:P)} \prod_{(x:P)} (a = x)$$

In a topological sense, a contractible space is a single point, while the logical reading says that P is inhabited by an object a, and any object is equal to a. As a visual guide, we can think of contractible spaces of those that **can** be continuously deformed into a single point.

Definition 2.2 (-1-type) *A type $P : \mathcal{U}$ is a **mere proposition** if for all $x, y : P$, $x =_P y$.*

$$\mathsf{isProp}(P) :\equiv \prod_{x,y:P} (x = y)$$

In other words, mere propositions are types with at most one inhabitant. While the $\mathsf{isContr}(P)$ and $\mathsf{isProp}(P)$ types seems similar, the important distinction is that a contractible type P is always inhabited, while a mere proposition might not be inhabited. For example, the contradiction type $\mathbf{0}$ is a mere proposition, but it is not contractible, since, intuitively, nothing can prove a contradiction, and thus, under the propositions as types paradigm, the contradiction type $\mathbf{0}$ is uninhabited.

Definition 2.3 (0-type) *A type $A : \mathcal{U}$ is a set (0-type) if for all $x, y : A$, and all $p, q : x =_A y$, then $p =_{x=_A y} q$*

$$\mathsf{isSet}(A) :\equiv \prod_{(x,y:A)} \prod_{(p,q:x=y)} (p = q)$$

At this point, the term *set* has been used with at least three different meanings. To avoid notational confusion, throughout this paper we will use **Set** to denote the category of ZFC-sets. Types that are sets in the meaning of Definition 2.3 will be called *h-set*. To refer to the standard ZFC sets, we will simply call them sets.

A type is an h-set, in other words, if there are no non-trivial paths between paths. Another way to think of this is by means of the following counterexample. We will show a type that is not an h-set. The classic example is the universe type $\mathcal{U}$, which, historically, was the main motivation for type theory as a way to overcome Russell's Paradox. For this, consider the boolean type $\mathbf{2} : \mathcal{U}$. While $\mathbf{2}$ is a type, when considering the universe of types $\mathcal{U}$, then types become objects of type $\mathcal{U}$. $\mathcal{U}$ can be thought of as a *type of types*. The boolean type contains two inhabitants, $0_{\mathbf{2}} : \mathbf{2}$ and $1_{\mathbf{2}} : \mathbf{2}$, which can be thought of as the usual booleans $0, 1$. They are indexed to clarify that they are different from the objects of type $\mathbb{N}$, since types are disjoint. We can define two functions $id, neg : \mathbf{2} \to \mathbf{2}$, setting $id(0_{\mathbf{2}}) = 0_{\mathbf{2}}$, $id(1_{\mathbf{2}}) = 1_{\mathbf{2}}$ and $neg(0_{\mathbf{2}}) = 1_{\mathbf{2}}$, $neg(1_{\mathbf{2}}) = 0_{\mathbf{2}}$. If the universe were a set, then $id = neg$. Then, if that were the case, then we would reach the conclusion that $id(0_{\mathbf{2}}) = neg(0_{\mathbf{2}})$, meaning $0_{\mathbf{2}} = 1_{\mathbf{2}}$, which is a contradiction. See Figure 1 for a visual representation of the non-identical loops. Thus, the universe is a non-example of a set.

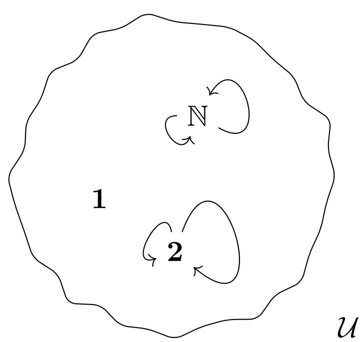


Figure 1: Universe $\mathcal{U}$ and its objects, which are types. The existence of two non-identical loops shows that $\mathcal{U}$ is not an h-set

The category (or type) **Set** of sets consists of all types $A : \mathcal{U}$ for which $\mathsf{isSet}(A)$ is inhabited. In formal notation:

$$\mathbf{Set} :\equiv \sum_{A:\mathcal{U}} \mathsf{isSet(A)}$$

Intuitively, the type of sets does not form a set. To prove that, it suffices to find a set P : **Set** and a path $p : P = P$ such that $p \neq \mathsf{refl}_P$. Taking the boolean type **2**, we can reuse the same proof that shows the universe $\mathcal{U}$ is not a set.

The hierarchy of homotopical structures is upwards closed. This means that any contractible type is also a proposition, any proposition is a set, and so on. However, there also exists an operation that *squashes* down the homotopical space to a lower level, in a sense discarding information about the objects above that certain level. The main one we will be using is the propositional truncation, which can reduce any n-type to a mere proposition type.

Definition 2.4 (Propositional Truncation) *For any type P, we can define the* ***propositional truncation*** *type* $||P||$ *such that for any* $x : P$, $|x| : ||P||$, *and for any* $x, y : ||P||$, *then* $x = y$.

One of the interesting aspects of HoTT is the equivalence between identity and equivalence. The *identity-to-equivalence* direction of the univalence axiom is trivial, since two identical objects are equivalent as well. Voevodsky's novelty comes in the other direction. Before presenting the formal statement of Voevodsky's axiom, we will sketch the definition and main intuition of the equivalence type.

Definition 2.5 (Homotopy) *Given two functions* $f, g : A \to B$, *we call the* ***homotopy*** *of f and g the type*

$$(f \sim g) :\equiv \prod_{x:A} (f(x) = g(x))$$

Definition 2.6 (Equivalence function) *A function* $f : A \to B$ *is called an equivalence if there exist functions* $g, h : B \to A$ *such that for all* $x : A,\ y : B$, $g \circ f(x) = id_A$ *and* $f \circ h(x) = id_B$, *where "∘" represents function composition.*

$$\mathsf{isEquiv}(f) :\equiv (\sum_{g:B \to A} (g \circ f) \sim id_A) \times (\sum_{h:B \to A} (f \circ h) \sim id_B)$$

Definition 2.7 (Type Equivalence) *Two types* $A, B : \mathcal{U}$ *are equivalent if there exists a function* $f : A \to B$ *that is an equivalence.*

$$(A \simeq B) :\equiv \sum_{f:A \to B} \mathsf{isEquiv}(f)$$

Axiom 2.1 (Univalence) *For any types* $A, B : \mathcal{U}$

$$(A =_{\mathcal{U}} B) \simeq (A \simeq_{\mathcal{U}} B)$$

What the univalence axiom says is that equivalent types can be identified, i.e., they are identical. Equivalences can be thought of as isomorphisms in ZFC. That is to say, abusing concepts, that a bijection can be established between the two types, and any relationships between objects of the type are maintained.

As Ahrens and North (2019) point out, the notion of sameness in ZFC is a very strong notion, and mathematicians seek *weaker notions of sameness and those properties that are invariant under such notions* (p. 138). A more familiar example, in theoretical computer science and modal logic, is the notion of bisimilar transition systems. Two transition systems are called bisimilar if they satisfy the same formulas. Thus, regardless of the internal configuration of the systems, the expressiveness of the two systems would be identical, and the two systems would be considered equivalent. This is one such *weaker notion of sameness* that mathematicians are interested in.

3. What does it mean that Set is a ΠW-pretopos?

This section constitutes a presentation of the results given by Rijke and Spitters (2015). Their main claim is that **Set** is a ΠW-pretopos. We will analyse this concept and try to explain in simpler terms how this result was obtained. To that extent, we will reconstruct their proof in a way that is more approachable for readers without a strong background in category theory, doing so in an iterative, bottom-up fashion, focusing on intuitive aspects rather than mathematical rigour. This section will be more involved in terms of category theory, but we will aim to explain everything as simply as possible, with a focus on what the result means for **Set**. To that extent, for each category-theory concept, we will first restrict it to the specific case of the category **Set**. This is, in a sense, the inverse aim of category theory, whose goal is to generalize set-theoretic notions. Therefore, most of the category theory generalizations do stem from set-theoretic constructs, and as such, the inverse direction should be more familiar.

Isomorphisms between sets are called bijections. Naturally, a bijective function is a function that is both surjective and injective. These two terms have their own meaning in homotopy type theory, being types as well. We will first define the fibre type, which we will then use to define injective and surjective function types

Definition 3.1 *The fibre of a function $f : A \to B$ over a point $y : B$ is a point $x : A$ such that $f(x) = y$*

$$\mathsf{fib}_f(y) :\equiv \sum_{x:A}(f(x) = y)$$

We are now ready to define the familiar concepts of injective and surjective functions, together with the image of a function.

Definition 3.2 *A function $f : A \to B$ is **injective** if there is an inhabitant of the type*

$$\mathsf{inj}(f) :\equiv \prod_{x:A} \mathsf{isContr}(\mathsf{fib}_f(f(x))).$$

or, equivalently, if A and B are h-sets,

$$\mathsf{inj}(f) :\equiv \prod_{x,y:A} (f(x) = f(y)) \to (x = y)$$

Definition 3.3 *A function $f : A \to B$ is **surjective** if the following type is inhabited*

$$\mathsf{surj}(f) :\equiv \prod_{y:B} ||\mathsf{fib}_f(y)||$$

Definition 3.4 *For any function $f : A \to B$, we define the* ***image*** *of f as the type*

$$\mathsf{im}(f) :\equiv \sum_{b:B} ||\mathsf{fib}_f(b)||$$

We are now ready to present the first categorical concept, the pullback. The pullback, although a categorical concept, can be given, as most concepts in category theory, an interpretation in HoTT. In set theoretical foundations, restricted to **Set**, given two functions $f : A \to C$, $g : B \to C$, we call pullback the subset of the Cartesian product $A \times B$ defined as $A \times_C B = \{(a, b)|f(a) = g(b)\}$. In HoTT, pullbacks are defined similarly.

Definition 3.5 *Given functions $f : A \to C$ and $g : B \to C$, the* ***pullback*** *of f and g is the type*

$$A \times_C B :\equiv \sum_{a:A} \sum_{b:B} (f(a) =_C g(b))$$

together with the projections $\pi_1 : A \times_C B \to A$ and $\pi_2 : A \times_C B \to B$

We are now going to reconstruct, in an iterative bottom-up fashion, the results of Rijke and Spitters (2015). We will be focusing on conveying the intuition behind the results, rather than the full, formal proofs.

3.1 Set is regular

To show that **Set** is a regular category, it is sufficient to show, using Gran (2021, Theorem 1.14), that **Set** is a finitely complete category, that any function can be written as a combination of a surjective[4] function followed by an injective function, and that factorization is stable under pullbacks. This is similar to Theorem 7.6.6 in The Univalent Foundations Program (2013), restricted to (-1)-types.

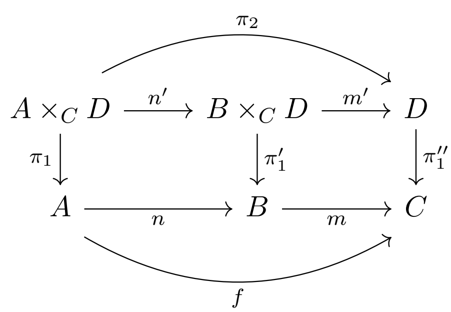
Thus, let us first show that **Set** is finitely complete. We will be making use of Borceux (1994a, Proposition 2.8.2). Thus, it is enough to show that a category admits pullbacks and terminal objects in order to show it is finitely complete. We know trivially that **Set** admits terminal objects, since any singleton is a terminal object in **Set**. Now, the easiest way to show that **Set** has pullbacks is using Awodey (2010, Corollary 5.6.), and show that **Set** has equalizers and products. Trivially, it has (Cartesian) products. Equalizers, for some *parallel* functions, i.e., functions with the same domain and codomain, try to capture the concept of *sameness* for the two functions. In a sense, an equalizer wants to see where the two functions *agree*. Thus, for any two functions $f, g : A \to B$, we can simply define, restricted to **Set**, the equalizer of f, g as the subset of A on which the two functions agree. We can define the equalizer type as $Eq(f, g) :\equiv \sum_{(a:A)}(f(a) = g(a))$, together with the projection $\pi : Eq(f, g) \to A$.

We can call $f = m \circ n$ the *factorization* of f. Now, we will show, following The Univalent Foundations Program (2013, Lemma 7.6.4.), that any function f admits such a factorization, where m is injective and n is surjective. To that extent, assume there is a function $f : A \to B$. Define $n : A \to \mathsf{im}(f)$, and $m : \mathsf{im}(f) \to B$, such that $n(a) :\equiv (f(a), |(a, \mathsf{refl}_{f(a)})|)$ and

4. Technically, the definition says that it must be a regular epimorphism, but it can be shown that, restricted to **Set**, surjective functions are always regular epimorphisms, see Rijke and Spitters (2015, Theorem 3.10)

$m :\equiv \pi_1(\mathsf{im}(f))$. We first show that n is surjective. $A \simeq \sum_{b:B} \mathsf{fib}_f(b)$, so n becomes akin to $\prod_{b:B}(\mathsf{fib}_f(b) \to ||\mathsf{fib}_f(b)||)$, which is surjective by The Univalent Foundations Program (2013, Corollary 7.5.8, Lemma 7.5.13). To show m is injective, given $x, y : \mathsf{im}_f$ such that $m(x) = m(y)$, we show that $x = y$. Supposing $x : (b, |\alpha|)$ and $y : (b', |\beta|)$, then $m(x) = m(y)$ amounts to $b = b'$. Since in the propositional truncation any two objects are equal, it follows that $|\alpha| = |\beta|$. Thus, $x = y$, and therefore m is injective, completing the proof.

What does it mean that the *factorization is stable under pullbacks*? To simplify, given the following diagram, where $f = m \circ n$, then $\pi_2 = m' \circ n'$ is the factorization of $\pi_2 : A \times_C D \to D$ such that n' is also surjective and m' is injective, given that the squares are pullback squares.



Monomorphisms (i.e., injective functions) are always stable under pullbacks (Johnstone, 2002, p. 18). Therefore, since m is injective, then so is m'. Supposing that n is surjective, we just need to show n' is also surjective. We base our proof on Gran (2021, Section 1.3). Take some $(x, y) : B \times_C D$. Since $B \times_C D$ is a pullback, and π'_1 is its projection, then there is $b : B$ such that $\pi'_1((x, y)) = b$. Further, n is surjective. Then there is some $a : A$ such that $n(a) = b$. By The Univalent Foundations Program (2013, Lemma 2.1.2), it follows that $n(a) = \pi'_1((x, y))$. But then, since all the squares are pullbacks, the left square is also a pullback, and thus there is an element $(a, y) : A \times_C D$ such that $n'((a, y)) = (x, y)$, and thus n' is surjective.

3.2 Set is exact

To form an exact category, a category must be regular, and equivalence relations must be effective. Having shown that **Set** is regular, we only need to show the second condition.

Just as in classical mathematics, an equivalence relation is a relation that is reflexive, symmetric and transitive. That is, given a relation $R : A \to A \to \mathcal{U}$, there is an inhabitant of the type $\mathsf{isEqRel}(R) :\equiv (\prod_{x:A} R(x, x)) \times (\prod_{x,y:A} R(x, y) \to R(y, x)) \times (\prod_{x,y,z:A} R(y, z) \to R(x, y) \to R(x, z))$, and showing that, for some $a, b : A$, aRb holds means giving an inhabitant of $\sum_{a,b:A} R(a, b)$.

What does it mean that equivalence relations are *effective*? We will base our answer on the presentation done by Birkedal et al. (1998). Take some equivalence relation, customary denoted by $\sim$[5], and some set S on which this relation is defined. For example, some set of integers being divided into equivalence classes with respect to modular arithmetic. In category theory, the object $S/\sim$ is called the quotient of S by $\sim$, analogous to how we think

5. Since the homotopy type is not used in this subsection, $\sim$ will only denote equivalence relations for this subsection

of quotients in division[6]. More intuitively, this can be thought of as the equivalence class of S with respect to $\sim$, or as a partition of S defined by $\sim$, where each equivalence class is disjoint. Being effective means that, according to Definition 2.5.3 in Borceux (1994b), in the following diagram

$$\sim \underset{r_2}{\overset{r_1}{\rightrightarrows}} S \xrightarrow{\;q\;} S/\sim$$

q exists as the coequalizer of (r_1, r_2) and (r_1, r_2) is the kernel pair of q. We have seen the equalizer before. The coequalizer is its dual. We can think of it as the arrow that *collapses* S into its quotient set. In other words, equivalent elements of S are being placed into an equivalence class, which becomes then an element of the quotient set. For (r_1, r_2) to form the kernel pair of q, it suffices for the following to be a pullback square

$$\begin{array}{ccc} \sim & \xrightarrow{\;r_2\;} & S \\ {\scriptstyle r_1}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ S & \xrightarrow[\;q\;]{} & S/\sim \end{array}$$

Combining the two requirements and applying the definition of a pullback, it means showing that $\sim \simeq S \times_{S/\sim} S \equiv \sum_{(a,b:S)}(q(a) = q(b))$. In other words, that $a \sim b \simeq (q(a) = q(b))$, i.e., that equivalent objects in S belong to the same equivalence class in $S/\sim$.

Rijke and Spitters (2015, Theorem 3.22), show that the equivalence holds in the following way. They extend $\sim$ from a relation on S to a relation on $S/\sim$. Call this relation $\sim_q\colon S/\sim\to S/\sim\to \mathcal{U}$, defined inductively as $\sim_q (q(x), q(y)) :\equiv \sim (x, y)$. The gist of the proof comes in showing that $w \sim_q w' \simeq w =_{S/\sim} w'$. This has been proven in two ways. In Rijke and Spitters (2015), it is done by showing there is an inhabitant of $\prod_{w:S/\sim} \mathsf{isContr}(\sum_{w':S/\sim} w \sim_q w')$. In The Univalent Foundations Program (2013, Lemma 10.1.8), this is proven in the usual way, by giving the two functions that result in the equivalence. As an alternative, more intuitive proof, see Borceux and Bourn (2004, Example A.5.14).

3.3 Set is a pretopos

A category is a pretopos when it is exact and extensive. We have already discussed why **Set** is exact, so all that is left to show is that it is extensive. We will be using the definition of extensive, distributive and lextensive categories as stated in Carboni et al. (1993), and Theorem 2.23 from Rijke and Spitters (2015). This amounts to showing, using Carboni et al. (1993, Proposition 2.2), the equivalence

$$(A \times_D C) + (B \times_D C) \simeq (A + B) \times_D C$$

holds, for functions $f : A \to D$, $g : B \to D$ and $h : C \to D$, or, in other words, that coproducts commute with pullbacks. This equivalence can be thought of as akin to distributivity of summation and multiplication in arithmetic, or distributivity of conjunction and disjunction in propositional logic, generalized to categorical level, over pullbacks. Reduced

6. I.e., what is *left* after computing the division

to the level of **Set**, this means that disjoint union distributes over pullbacks (which are, in **Set**, just special subsets of the Cartesian product).[7]

The way this is proven in Rijke and Spitters (2015, Theorem 2.23) is using Σ-types. From a set-theoretic perspective, Cartesian products and disjoint union produce an identical structure: a set of pairs, but of course with a different formation rule. Therefore, it suffices to show that

$$\sum_{a:A}(P(a) \times_D B) \simeq \sum_{a:A} P(a) \times_D B$$

for $P : A \to \mathcal{U}$, $f : \sum_{a:A} P(a) \to D$ and $g : B \to D$. The equivalence is straightforward, using the rules for the Σ-type. The equivalence used is $\sum_{(a:A)} \sum_{(p:P(a))}(a, p) \simeq \sum_{(x:\sum_{a:A} P(a))} x$. As such, unravelling the definition of the pullback and applying the above equivalence, $\sum_{a:A}(P(a)\times_D B) \equiv \sum_{a:A} \sum_{p:P(a)} \sum_{b:B}(f(a, p) =_D g(b) \simeq \sum_{(x:\sum_{a:A} P(a))} \sum_{b:B} (f(x) =_D g(b) \equiv \sum_{a:A} P(a) \times_D B$.

What does **Set** lack for it to be a topos, instead of a pretopos? In short, it lacks power objects, which, for sets, are power sets. An additional axiom, called propositional resizing, would allow **Set** to exhibit power sets, but it would also make the theory impredicative. We will discuss more on that in section 4.3.

3.4 Set is a ΠW-pretopos

A **Π**W-pretopos is a pretopos which is also a locally Cartesian closed category and exhibits W-types. We will first sketch the importance of locally cartesian closed categories, and then give some ideas about the proof that **Set** is indeed locally cartesian closed. We will skip the categorical definitions in this section, and instead focus on the connection between (dependent) type theory and (locally) cartesian closed categories

Cartesian closed categories, as pointed out by Johnstone (2002, p. 44), are categories that correspond to typed λ-calculus, having the proper properties needed to model the rules of the system. Categories can be used to model different formal systems, such as first or second order logic, as well as λ-calculus and dependent type theory. Different systems require different properties, hence the importance of the classifications. For example, cartesian closed categories model λ-calculus, while locally cartesian closed categories model Martin-Löf type theory (Awodey, 2010, p. 237).

To show that **Set** is locally cartesian closed, we can rely on the interpretation of Martin-Löf type theory in locally cartesian closed categories. The rules of type theory are follows by **Set**, as a type of the theory. Since type theory is interpreted in cartesian closed categories, and dependent type theory is interpreted in locally cartesian closed category, then the rules of dependent type theory model additional properties. Those come in the form of the dependent types, i.e., the Σ-type and the Π-type. To oversimplify, these types model certain categorical constructs which allow **Set**, as a part of Martin-Löf type theory, to be locally cartesian closed. The proof of the equivalence between dependent type theory and locally cartesian closed categories is given in Hofmann (1995) and Seely (1984).

W-types are a kind of inductive type, whose purpose is to generalize many inductive types in order to facilitate an easier formalization of their properties (The Univalent Founda-

7. This is a special case of what MacLane (1971, p. 210) calls the *interchange of limits*, or the commutativity of limits, the treatment of which is outside the scope of this paper.

tions Program, 2013, p. 154). Many familiar recursive structures are captured by W-types, such as the natural numbers, or, to give a data structure more familiar to computer scientists, lists. For simplicity, the type theory of Martin-Löf includes W-types as shown in The Univalent Foundations Program (2013, Section 5.3), and so **Set** does have W-types. Since it is also locally cartesian closed, it means that **Set** is a ΠW-pretopos.

4. Contrasting HoTT and ZFC

The motivation behind h-sets is a categorical one, under which the main focus is structural invariance. However, a foundational effort in mathematics should model a wide range of mathematical domains, or at least the most basic fields upon which the others are built. The Univalent Foundations Program (2013) gives a presentation of how HoTT can be used to model Homotopy Theory, Category Theory, Real Analysis and, most importantly, Set Theory. In this section, we will compare some aspect of HoTT and ZFC, focusing first on the foundational aspects of each theory, and then how some axioms of ZFC are modelled in HoTT.

4.1 Foundational differences

The first foundational difference between HoTT and ZFC is in terms of primitives. Both theories confer a basic status to certain elements, from which the rest of the formal system is then expanded. In ZFC, the two primitives are sets and the membership relation "$\in$". In ZFC, everything is a set. This is where the first contrast between the formal theory and regular use comes to light. To most practitioners, statements such as $\{\emptyset\} \in 4$ seem nonsense, but are true in ZFC due to the way natural numbers are defined.

In HoTT, types form, just like in Martin-Löf Type-Theory, the primitive elements, being given the homotopical interpretation of *spaces*. In other words, types and elements are the primitive, being seen through the homotopical lense as spaces, points, and paths.

When speaking of HoTT as a foundational effort, the focus falls upon Axiom 2.1. The idea of replacing ZFC with a type theoretical foundation of mathematics dates back to Russell's type theory and Church's λ-calculus (Coquand, 2018). Therefore, one may wonder if the novelty comes mainly in the form of the univalence axiom, what prevents us from enriching ZFC with the univalence axiom? The answer is straightforward, and is proven in Proposition 4.1.

Proposition 4.1 *(ZFC + Univalence) is inconsistent*

Proof Take two sets, $S_0 = \{0\}$ and $S_1 = \{1\}$. The two sets belong to the same hierarchical universe $\mathcal{U}$. In categorical terms, isomorphism in set theory consists of a bijective function. Thus, take $f : S_0 \to S_1$ defined as $f(0) = 1$. Trivially, f is bijective, and thus $S_0 \simeq S_1$. According to ZFC's axiom of extensionality, $S_0 \neq S_1$. Since the axiom holds in ZFC, it also holds in ZFC + Univalence. Then, according to univalence, since $S_0 \simeq S_1$, then $S_0 = S_1$. Thus, $(ZFC + Univalence) \vdash \bot$. ■

Thus, as a result, in standard ZFC, the univalence axiom would identify all sets of the same cardinality. However, adding the univalence axiom to Martin-Löf type theory does not lead to an inconsistency, which is what made it a suitable starting point for HoTT. In addition, univalence leads to another axiom, more familiar to its equivalent in ZFC, the extensionality axiom (The Univalent Foundations Program, 2013, Section 4.9).

In ZFC, extensionality means that two sets are deemed equal iff they have the same elements. In HoTT, extensionality applies similarly, but only to functions, i.e., two functions[8] are equal iff they are equal for all elements of the domain. While in ZFC extensionality applies to the primitive elements of the theory, in HoTT this is not the case. In type theory, types are disjunct. This means that no two types share an object. If, by some means, it is shown that two types share a common object, then automatically those types are deemed equal. As Angere (2021) points out, this can imply that type theory has a weaker notion of extensionality than ZFC.

We will now analyse a few of the axioms of ZFC. We will base our analysis on the following goal. We will present a simple theorem in HoTT, for which an axiom is needed in ZFC (i.e., the axiom of the empty set). Then, we will cover an axiom from ZFC which does not hold in HoTT without an additional, separate axiom (i.e., the power set axiom). Finally, we will discuss the well known case of the axiom of choice, and its different implications in HoTT.

4.2 The Axiom of Empty Set

In ZFC, the existence of the empty set must be stated as an axiom. This is because, in ZFC, anything must behave as a set, and as such the existence of the empty set is stated to prevent further inconsistencies. For example, if two sets A, B are disjoint, then $A \cap B$ would not be defined without the existence of an empty set. Thus, in ZFC, the axiom of the empty set is defined as

$$\exists x \forall y \neg (y \in x)$$

Or, informally, that there exists at least one set such that no set is a member of that set. Using the axiom of extensionality, it is then proven that there is exactly one such set, called the empty set, usually denoted as $\emptyset$ or $\{\}$.

In Martin-Löf Type-Theory, and by extension in HoTT as well, the existence of an empty type is a theorem of the system. In other words, no axiom is needed to state that such a type exists, unlike in ZFC. Formally, these are the rules for the introduction and the elimination of the empty type, as given in The Univalent Foundations Program (2013, A.2):

$$\frac{\Gamma \ \mathrm{ctx}}{\Gamma \vdash \mathbf{0} : \mathcal{U}_i}\ \mathbf{0}\text{-form} \qquad\qquad \frac{\Gamma, x : \mathbf{0} \vdash C : \mathcal{U}_i \qquad \Gamma \vdash a : \mathbf{0}}{\Gamma \vdash \mathsf{ind}_{\mathbf{0}}(x.C, a) : C[a/x]}\ \mathbf{0}\text{-elim}$$

The formation rule simply states that the empty type $\mathbf{0}$ exists as part of some universe $\mathcal{U}_i$. The elimination rule is the equivalent of the *ex falso* principle in logic, which states that a contradiction implies anything. In HoTT, whenever we can construct a proof $a : \mathbf{0}$, then we are allowed to imply anything we want by virtue of its elimination rule, giving us its induction principle.

The empty type is also a set since, according to the induction principle of the empty type, for any two inhabitants $x, y : \mathbf{0}$, we may deduce anything, and in particular, we can

8. With the same domain and codomain

deduce an inhabitant $p : \mathsf{isSet}(\mathbf{0})$. Therefore, the empty type also behaves like a set, and therefore it is a member of **Set**.

The empty type is thus a simple example of a construction in HoTT that arises as a simple consequence of the established rule of the formal theory, and no axiom is needed to state the existence of this type. This is a simple example of the different characteristic of each theory.

4.3 The Axiom of Power Set

In ZFC, the power set axiom states, roughly, that given any set W, there exists a set called the power set of W, usually denoted as $\mathcal{P}(W)$, which contains all the subsets of W. In formal notation, the axiom is stated as:

$$\forall x \exists y \forall z (z \in y \iff \forall w (w \in z \to w \in x))$$

Unlike the empty set axiom discussed previously, Martin-Löf type theory does not include as part of the formal theory an equivalent formulation. Adding the univalence axiom does not lead to anything similar to the power set axiom in HoTT either. However, there is a method to obtain such a construct in HoTT, by adding another axiom to the theory, which will be discussed below.

The power set axiom is an example of an existence axiom which, contrary to the empty set axiom, is not obtained *for free* as a consequence of the rules and primitives of type theory. To obtain something similar power sets in HoTT, an additional axiom is needed:

Axiom 4.1 (Propositional resizing) *Given the type of all propositions in a universe $\mathcal{U}_i$, denoted as $\mathsf{Prop}_{\mathcal{U}_i} :\equiv \sum_{A:\mathcal{U}_i} \mathsf{isProp}(A)$, we have that $\mathsf{Prop}_{\mathcal{U}_i} \simeq \mathsf{Prop}_{\mathcal{U}_{i+1}}$*

Using this axiom, power sets can then be defined in HoTT as follows: for a set A, define $\mathcal{P}(A) :\equiv A \to \mathsf{Prop}_{\mathcal{U}_0}$, where $\mathsf{Prop}_{\mathcal{U}_0}$ is the type of all mere propositions.

As The Univalent Foundations Program (2013, p. 116) point out, the addition of Axiom 4.1 would make HoTT an *impredicative* theory. Roughly speaking, a definition is impredicative if it makes reference to something which contains the defined object. Thus, the power set axiom, and the existence of power objects in general, makes a theory impredicative (Crosilla, 2020, Section 1.3.2).

Within set-theoretical foundations, the category of sets forms a topos. In other words, in addition to the already described properties of a ΠW-pretopos, it also has, among other characteristics, power sets. Therefore, adding propositional resizing to **Set** would make **Set** a topos, but it would also make the theory impredicative.

However, as it will become clear in the next subsection, there are also advantages to adding such axioms. The final result of The Univalent Foundations Program (2013, Chapter 10) is that the entirety of ZFC *can* be modelled within HoTT. In other words, ZFC can be embedded within HoTT, allowing the practice of set theory within univalent foundations. This provides, to use the terminology employed by Maddy (2019), the *generous arena* and *shared standard* of ZFC, i.e., the possibility of formalizing a multitude of mathematical fields, that share a common base for what counts as good mathematical practice. In addition, HoTT provides an excellent field for proof checking, and thus ZFC embedded within HoTT would benefit from the good computational properties of HoTT.

4.4 The Axiom of Choice

The axiom of choice has, within ZFC, many equivalent statements (Herrlich, 2006, Chapter 2). One of those equivalent statements is the following: For $S = (A_i)$ a collection (or a set) of pairwise disjoint non-empty sets, there exists a set $C = \{x_i | x_i \in A_i\}$ (Jech, 1977). In other words, for a collection of non-empty sets, there is a set which contains (or chooses) exactly one element from each set. Alternatively, one can formulate it in terms of a choice function: For every set $S = (A_i)$ of non-empty sets, there is a function f such that $f(A_i) \in A_i$.

For constructivists, at a first glance, the axiom of choice poses serious problems. As Jech (1977) points out, it states the existence of a set or a function, without specifying a way to construct such a set or function. Therefore, it is surprising that, given the non-constructive nature of this axiom, that there is a formulation of the axiom of choice in HoTT. The logical statement of the axiom of choice used for the type-theoretical formalization is the following (The Univalent Foundations Program, 2013, p. 119):

$$\forall (x : X).(\exists (a : A(x)).P(x,a)) \to \exists (g : \prod_{x:X} A(x)).\forall (x : X).P(x, g(a))$$

where it is important that $X, A(x)$ are sets and $P(x,a)$ a mere proposition, for otherwise the axiom fails (The Univalent Foundations Program, 2013, Lemma 3.8.5.). In other words, if for any object from the h-set X, there is an object a from the h-set $A(x)$ such that $P(x,a)$, then there is a dependent function g from $x : X$ to $A(x)$, such that for all x in X, $P(x, g(a))$ holds. In terms of the set-theoretical formulations given at the beginning of the subsection, to abuse concepts, X can be thought of as the family of sets, $A(x)$ is similar to each A_i smaller non-empty set, while P, as a mere proposition, can be thought of as akin to the membership relation.

The formulation in HoTT of the above logical statement is the following (The Univalent Foundations Program, 2013, p. 119):

$$\left(\prod_{x:X}\left(\left\| \sum_{a:A(x)} P(x,a) \right\|\right)\right) \to \left\| \sum_{(g:\prod_{(x:X)} A(x))} \prod_{(x:X)} P(x, g(a)) \right\|$$

We could also add the conditions that $X, A(x), P$ are h-sets and h-propositions, respectively, in the antecedent. The interesting aspect comes in the presence of the propositional truncation in the type of the axiom of choice. The role of the propositional truncation is to allow us to assume the existence of some object of a certain type, without having to provide the exact construction, and use that object in our proof as long as the result does not depend on the particular value of the object. However, this works solely when the codomain is a mere proposition.

Furthermore, it was proven that the axiom of choice implies the law of excluded middle (Diaconescu, 1975), one of the main logical principles rejected by constructivist mathematics. Therefore, from the perspective of foundational efforts, HoTT can, indeed, be used to model most of mathematics depending on which axiom one chooses to incorporate into the theory. The axiom of choice can be generalized to higher levels by replacing propositional truncation with n-truncation, or even eliminating it entirely for ∞-groupoids (The Univalent Foundations Program, 2013, Chapter 7). Therefore, the axiom of choice has multiple non-trivial variants within HoTT, as opposed to the unified account in ZFC.

5. Conclusion

In this paper, we provided a review of the type of sets in homotopy type theory. We first presented the preliminaries of homotopy type theory, the concepts of h-sets, as well as the importance of the univalence axiom from the perspective of structuralist approaches to mathematics. We have then presented the formalization of the type **Set** as done by Rijke and Spitters (2015) and The Univalent Foundations Program (2013, Chapter 10.1), reconstructing their results and giving more intuition to the reader without a strong background in category theory. Finally, we compared homotopy type theory and ZFC from a foundational perspective, as well as from the point of view of three set-theoretic axioms, showing whether they can be formalized in homotopy type theory.

Various additional axioms can be added to HoTT, such as propositional resizing, in order to obtain differing models and results. As Shulman (2017) points out, HoTT is a young field, and as such there is still active research involving the core theory. But importantly, this proves that HoTT can be enhanced to suit one's needs, be them modeling ZFC or keeping the constructive purity of the theory.

Possible areas of future research into the connections between ZFC and HoTT include the formalization of all the set-theoretic axioms in ZFC, besides the ones already mentioned here. For example, an interesting case would be the axiom of infinity, which states the existence of at least one infinite set in ZFC. Such a set cannot be constructed from finite sets, and as such could be thought of as an inductive type, similarly to the natural numbers. However, as with most axioms of existence, care has to be taken to follow the constructive principles which motivate HoTT.

Responsible Research

The work conducted in this paper falls within the broad field of pure mathematics. While HoTT's most obvious application is computer-assisted proof checking, that was not the focus of our research. As such, problems of experiment reproducibility, code availability, or data bias are not applicable.

We have aimed to follow established guidelines within mathematical research, such as those published by established mathematical societies, such as the American Mathematical Society[9] and the European Mathematical Society[10]. As such, all theorems and lemmas used in our proofs are properly referenced, and mathematical results are properly attributed to their respective authors.

A commonly held ethical objection against pure mathematics is that researchers should focus on more pressing, concrete problems, instead of abstract research with no immediate applications (Franklin, 1991). However, as Kachapova (2014) points out, many purely mathematical results found, in time, concrete applications, e.g., number theory in cryptography, or Fourier analysis in computer graphics and signal processing. Further, most constructions in pure mathematics, due to their high degree of abstraction, can be readily applied in multiple other domains, such as groups and fields in cryptography, or sets being

9. https://www.ams.org/about-us/governance/policy-statements/sec-ethics
10. https://euro-math-soc.eu/system/files/uploads/COP-approved.pdf

used in almost all scientific fields. As HoTT is a young field, the practical applications might still be unclear, but progress is being made, see, e.g., Kunii and Hilaga (2015).

References

Ahrens, B., Kapulkin, K., & Shulman, M. (2015). Univalent categories and the Rezk completion. *Mathematical Structures in Computer Science*, *25*(5), 1010–1039. https://doi.org/10.1017/s0960129514000486

Ahrens, B., & North, P. R. (2019). Univalent foundations and the equivalence principle. In S. Centrone, D. Kant, & D. Sarikaya (Eds.), *Reflections on the foundations of mathematics: Univalent foundations, set theory and general thoughts* (pp. 137–150). Springer International Publishing. https://doi.org/10.1007/978-3-030-15655-8_6

Angere, S. (2021). Identity and intensionality in Univalent Foundations and philosophy. *Synthese*, *198*(5), 1177–1217. https://doi.org/10.1007/s11229-016-1301-z

Awodey, S. (2010). *Category theory* (Second Edition). Oxford University Press.

Awodey, S. (2014). Structuralism, Invariance, and Univalence. *Philosophia Mathematica*, *22*(1), 1–11. https://doi.org/10.1093/philmat/nkt030

Awodey, S., Gambino, N., & Palmgren, E. (2015). Introduction – from type theory and homotopy theory to univalent foundations. *Mathematical Structures in Computer Science*, *25*(5), 1005–1009. https://doi.org/10.1017/S0960129514000474

Awodey, S., Pelayo, Á., & Warren, M. (2013). Voevodsky's univalence axiom in homotopy type theory. *Notices of the American Mathematical Society*, *60*(9), 1164–1167. https://doi.org/10.1090/noti1043

Bauer, A., Gross, J., Lumsdaine, P. L., Shulman, M., Sozeau, M., & Spitters, B. (2017). The HoTT Library: A Formalization of Homotopy Type Theory in Coq. *Proceedings of the 6th ACM SIGPLAN Conference on Certified Programs and Proofs*, 164–172. https://doi.org/10.1145/3018610.3018615

Birkedal, L., Carboni, A., Rosolini, G., & Scott, D. (1998). Type theory via exact categories. *Proceedings. Thirteenth Annual IEEE Symposium on Logic in Computer Science (Cat. No.98CB36226)*, 188–198. https://doi.org/10.1109/LICS.1998.705655

Borceux, F. (1994a). *Handbook of categorical algebra* (Vol. 1). Cambridge University Press.

Borceux, F. (1994b). *Handbook of categorical algebra* (Vol. 2). Cambridge University Press.

Borceux, F., & Bourn, D. (2004). *Mal'cev, Protomodular, Homological and Semi-Abelian Categories*. Springer Dordrecht. https://doi.org/10.1007/978-1-4020-1962-3

Carboni, A., Lack, S., & Walters, R. (1993). Introduction to extensive and distributive categories. *Journal of Pure and Applied Algebra*, *84*(2), 145–158. https://doi.org/https://doi.org/10.1016/0022-4049(93)90035-R

Coquand, T. (2018). Type Theory. In E. N. Zalta (Ed.), *The Stanford encyclopedia of philosophy* (Fall 2018). Metaphysics Research Lab, Stanford University.

Crosilla, L. (2020). Set Theory: Constructive and Intuitionistic ZF. In E. N. Zalta (Ed.), *The Stanford encyclopedia of philosophy* (Summer 2020). Metaphysics Research Lab, Stanford University.

Diaconescu, R. (1975). Axiom of choice and complementation. *Proceedings of the American Mathematical Society*, *51*(1), 176–178. https://doi.org/10.2307/2039868

Franklin, J. (1991). Ethics of mathematics. *The Mathematical Intelligencer*, *13*(1), 4. https://doi.org/10.1007/BF03024064

Gran, M. (2021). An Introduction to Regular Categories. In M. M. Clementino, A. Facchini, & M. Gran (Eds.), *New Perspectives in Algebra, Topology and Categories: Summer School, Louvain-la-Neuve, Belgium, September 12-15, 2018 and September 11-14, 2019* (pp. 113–145). Springer International Publishing. https://doi.org/10.1007/978-3-030-84319-9_4

Herrlich, H. (2006). *Axiom of choice*. Springer Berlin, Heidelberg. https://doi.org/10.1007/11601562

Hofmann, M. (1995). On the interpretation of type theory in locally cartesian closed categories. In L. Pacholski & J. Tiuryn (Eds.), *Computer science logic* (pp. 427–441). Springer Berlin Heidelberg.

Jech, T. J. (1977). About the Axiom of Choice. In J. Barwise (Ed.), *Handbook of Mathematical Logic* (pp. 345–370). Elsevier. https://doi.org/10.1016/S0049-237X(08)71107-8

Johnstone, P. T. (2002). *Sketches of an Elephant: a Topos theory Compendium*. Oxford University Press.

Kachapova, F. (2014). On the importance of pure mathematics. *Journal of Mathematics and Statistics*, *10*, 421–422. https://doi.org/10.3844/jmssp.2014.421.422

Kunii, T. L., & Hilaga, M. (2015). Homotopy type theory for big data. *2015 International Conference on Cyberworlds (CW)*, 204–209. https://doi.org/10.1109/CW.2015.9

MacLane, S. (1971). *Categories for the Working Mathematician*. Springer-Verlag.

Maddy, P. (2019). What Do We Want a Foundation to Do? In S. Centrone, D. Kant, & D. Sarikaya (Eds.), *Reflections on the foundations of mathematics: Univalent foundations, set theory and general thoughts* (pp. 293–311). Springer International Publishing. https://doi.org/10.1007/978-3-030-15655-8_13

Martin-Löf, P. (1975). An intuitionistic theory of types: Predicative part. In H. Rose & J. Shepherdson (Eds.), *Logic colloquium '73* (pp. 73–118). Elsevier. https://doi.org/10.1016/S0049-237X(08)71945-1

Rijke, E., & Spitters, B. (2015). Sets in homotopy type theory. *Mathematical Structures in Computer Science*, *25*(5), 1172–1202. https://doi.org/10.1017/S0960129514000553

Seely, R. A. G. (1984). Locally cartesian closed categories and type theory. *Mathematical Proceedings of the Cambridge Philosophical Society*, *95*(1), 33–48. https://doi.org/10.1017/S0305004100061284

Shulman, M. (2017). Homotopy type theory: A synthetic approach to higher equalities. In E. Landry (Ed.), *Categories for the Working Philosopher*. Oxford University Press. https://doi.org/10.1093/oso/9780198748991.003.0003

The Univalent Foundations Program. (2013). *Homotopy type theory: Univalent foundations of mathematics*. https://homotopytypetheory.org/book.

Tsementzis, D. (2017). Univalent foundations as structuralist foundations. *Synthese*, *194*(9), 3583–3617. https://doi.org/10.1007/s11229-016-1109-x