Long-term Dynamics

of

Astrophysical Disks

by

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Abstract

context: The long-term evolution of a self-gravitating astrophysical disk can be modeled using secular perturbation theory. Recently, Batygin published a paper[1] where he claims that such a disk with a special density (4.2) can be described by a Schrödinger equation by using this method.
aims: In this thesis, we will study the secular perturbation theory applied to an astrophysical disk with the same density as Batygin, using the Laplace-Lagrange equations. We will take the continuum limit of those equations, and try to find a wave equation like Batygin.
methods: We first apply the Laplace-Lagrange equations to a disk with a large number of planets. Then we take the continuum limit of an infinite number of planets. We then compare the numeric results of the discrete disk to the analytic results of the continuum limit.
results: The eigenmodes of the system are well approximated by damped sinusoids. Mode number n changes sign n times. The eigenvalues are linear in the mode number.
conclusions: The eigenmodes do not satisfy a wave equation.

Contents

Chapter 1

Introduction

Stars are born in molecular clouds[2]. These molecular clouds are giant volumes of cold gas, consisting mostly of hydrogen and helium. They also contain dust particles made from heavier elements in the form of silicates, hydrocarbons and various ices. Parts of such clouds may collapse under their own gravity, and form tens to thousands of stars. At the protostellar stage, there exists a dense envelope of a few 1000 AU around each star. The seed of a new star forms from matter pulled to the center of the envelope by gravity. After about 10^5 years, most of the envelope has collapsed into a disk. The star continues to grow from this disk. After about 10^6 years, the envelope is exhausted, leaving a fully developed star and a massive circumstellar disk of a few Jupiter masses. In this disk, dust particles assemble via low velocity collisions and form larger bodies called planetesimals. These planetesimals grow larger, and the gas slowly disappears. As the planetesimals collide, they grow in size and their orbits become more circular. When the orbits are nearly circular, collisions between them become rare. At this stage, the orbits of the planetesimals are predominantly affected by gravitational forces. The system can be modeled as a debris disk, consisting only of point masses orbiting a central mass. This debris disk model is applicable to this epoch in planetary formation, but can also be applied to other systems, such as planetary ring systems and the stars in the center of a galaxy. We want to study these debris disks. Figure 1.1 shows an example of a protoplanetary disk.

There are three different methods to model such a debris disk[1]. The most obvious method is the N-body simulation, where the trajectory of each individual particle is numerically calculated using Newton's laws of motion. The second method involves solving a partial differential equation for the gravitational potential throughout the disk. The third method uses the Laplace-Lagrange equations to describe how the orbits of the particles change over time scales much larger than the orbital period. The last two methods are less computationally expensive than the N-body simulation.

Recently, Batygin published a paper[1] where he claims that a debris disk with a surface density that scales as the inverse square root of the orbital radius can be described by a Schrödinger equation using the third method. He applies this to the inclination dynamics, which is the evolution of the inclination of the orbits. There, the amplitude of the wave function corresponds to the inclination, and the phase to the argument of the ascending node. In this thesis, we will study Batygin's method. We also use the Laplace-Lagrange equations, but we apply it to the eccentricity dynamics. We introduce a complex function ψ. Here the amplitude is the orbital eccentricity and the phase is the argument of periapsis. As an introduction to the method, we first studied the dynamics for two planets. As an example of the method, we apply this to Jupiter and Saturn. Then we expand the method to an arbitrary number of planets. We apply this to the 8 planets of the solar system.

In order to compare our results to Batygin's, we then considered a system of 101 planets. After that we examined the limit of an infinite number of planets. In chapter 2, we derive Kepler's laws of planetary motion, in order to introduce the elliptic Kepler orbit and the orbital elements. In chapter 3, we introduce the theory of secular perturbations, where we use the Laplace-Lagrange equations to describe how planetary orbits change over large time scales by the gravitational pull of other planets. We apply the theory to the planets of the solar system. In chapter 4, we apply the theory of secular perturbations to a system with 101 planets, as to approximate a continuous disk. We will try to find the eigenmodes of this system. In chapter 5, we take the limit of an infinite number of planets. We will derive the eigenmodes of the system analytically, and compare them to the eigenmodes we found for the discrete disk in chapter 4. We will discuss our results in chapter 6, and compare them to the results of Batygin.

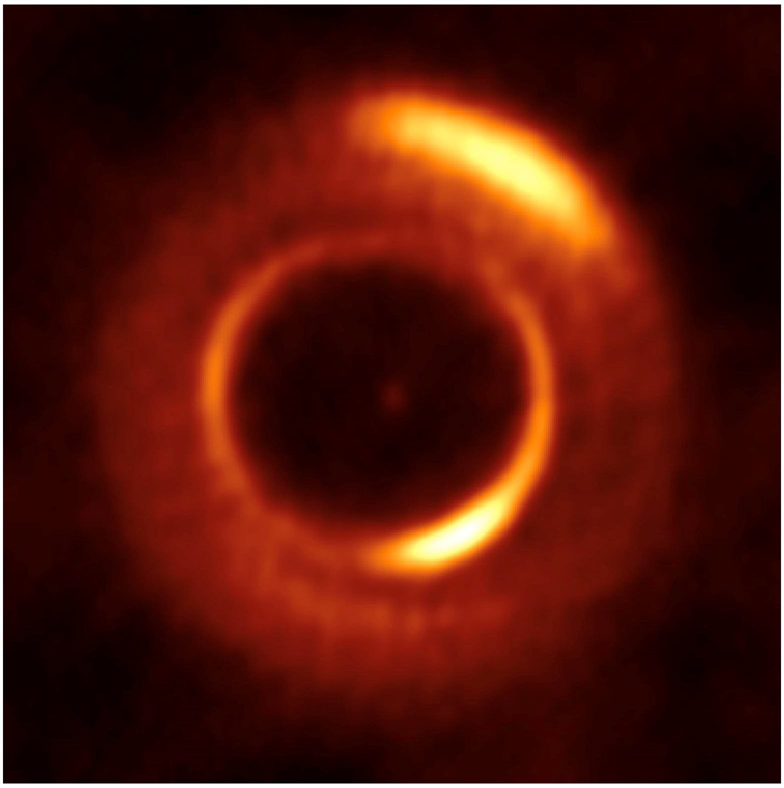


Figure 1.1: An image of a protoplanetary disk taken by the Atacama Large Millimeter/submillimeter Array.[3]

Table 1.1 is a summary of the symbols used in this report.

symbol	meaning
a	semi-major axis
b	semi-minor axis
ϖ	argument of the periapsis
ϵ	eccentricity
G	gravitational constant
m	mass
T	orbital period
Ω	angular orbital frequency
E	eccentric anomaly
j	planet index
k	planet index unequal to j
I	inclination
Ω	argument of the ascending node
A	Hamiltonian matrix for eccentricity dynamics
B	Hamiltonian matrix for inclination dynamics
$\mathcal{R}$	disturbing function
$\vec{\psi}$	complex eccentricity vector $\epsilon \exp(\varpi i)$
$\vec{\phi}$	complex inclination vector $I \exp(\Omega i)$
N	number of planets
v	eigenvector
ω	eigenvalue/frequency
n	modenumber
σ	mass density
λ	damping factor
ϕ	phase offset
κ	wavenumber

Table 1.1: List of symbols

Chapter 2

Planetary Orbits

In this chapter, we will derive Kepler's laws of planetary motion, using the method found in the textbook by Adams[4] and define the orbital parameters needed to describe the orbit of a planet around a star. We need this description in the next chapter, where we will use Lagrange's equations to find how the orbit of a planet evolves under the influence of the gravity of another planet.

2.1 Kepler's Laws of Planetary Motion

First, we will derive Kepler's first law, which states that the orbit of an object around another object with a much greater mass has the shape of an ellipse, with the more massive object at one of the focal points of the ellipse. Figure 2.1 features an ellipse with one of its focal points at the origin O.

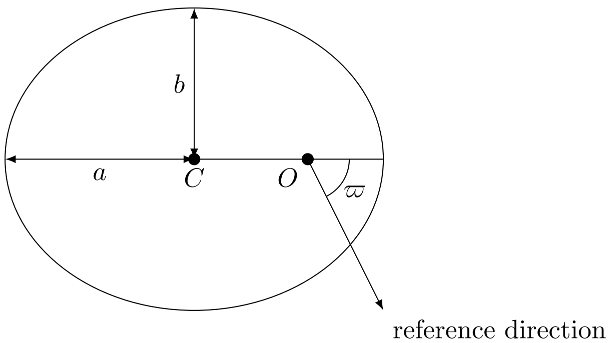


Figure 2.1: An ellipse with semi-major axis a, semi-minor axis b, and center C in the plane, rotated by ϖ with respect to the reference direction.

The point of closest approach of an orbiting object to its parent is known as the periapsis. Thus, the rotation angle ϖ is called the argument of periapsis. The eccentricity ϵ of an ellipse is defined by the ratio between the distance from one of the foci to the center and the semi-major axis. This implies

$$\epsilon = \sqrt{1 - \frac{b^2}{a^2}}. \tag{2.1}$$

An ellipse with one of its focal points at the origin can be expressed in polar coordinates as follows:

$$r(\theta) = \frac{(1-\epsilon^2)a}{1+\epsilon\cos(\theta-\varpi)}. \tag{2.2}$$

We will now derive an orbit to be of this form. We assume that the only force in play is gravity. We place the large mass at the center of the coordinate system. Since the large mass is much larger than the small mass, the gravitational force on the large mass is negligible, and the large mass will be stationary. The gravitational force of the central mass M on the small test mass m is given by

$$\vec{F} = -\frac{GMm}{r^2}\hat{r}, \tag{2.3}$$

where G is the gravitational constant, r is the distance of the test mass from the origin, and $\hat{r}$ is the unit vector in the radial direction. Newton's second law says that

$$\dot{\vec{v}} = \vec{a} = \frac{\vec{F}}{m} = -\frac{GM}{r^2}\hat{r}. \tag{2.4}$$

From this equation, we see that the change in the velocity $\vec{v}$ is in the direction of the position vector $\vec{r}$, and by definition, the change in position is in the direction of the velocity vector. Hence both the position and velocity vectors stay in the plane spanned by the initial position and velocity vectors. We choose this plane to be the xy-plane. We will from here on use cylindrical coordinates (r, θ, z), defined in terms of the Cartesian coordinates (x, y, z) by

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z. \tag{2.5}$$

We also define the corresponding unit vectors

$$\hat{r} = \cos\theta\hat{i} + \sin\theta\hat{j}, \quad \hat{\theta} = -\sin\theta\hat{i} + \cos\theta\hat{j}. \tag{2.6}$$

We can calculate the angular momentum $\vec{L}$ of the test mass, which is constant in time:

$$\vec{L}/m = \vec{r}\times\vec{p}/m = \vec{r}\times\vec{v} = (r\hat{r})\times(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}) = r\dot{r}(\hat{r}\times\hat{r}) + r^2\dot{\theta}(\hat{r}\times\hat{\theta}) = r^2\dot{\theta}\hat{k}, \tag{2.7}$$

which results in the following expression for $\vec{L}$ and its magnitude L:

$$\vec{L} = mr^2\dot{\theta}\hat{k} := L\hat{k}. \tag{2.8}$$

Using (2.4) and (2.8), we find for the change of velocity with respect to angle

$$\frac{d\vec{v}}{d\theta} = \frac{\dot{\vec{v}}}{\dot{\theta}} = \frac{-GM\hat{r}/r^2}{L/mr^2} = -\frac{GMm}{L}\hat{r} = \frac{GMm}{L}\frac{d\hat{\theta}}{d\theta}. \tag{2.9}$$

We can integrate this expression to find $\vec{v}$ in polar form:

$$\vec{v}(\theta) = \int\frac{d\vec{v}}{d\theta}d\theta = \frac{GMm}{L}\hat{\theta} + \vec{c}. \tag{2.10}$$

The integration constant $\vec{c}$ is in the center of a circle. Since we know that $\vec{v}$ will stay in the xy-plane, the z-component of $\vec{c}$ will be 0. We will write $\vec{c}$ in cylindrical coordinates like

$$\vec{c} = \left(\frac{GMm}{L}\epsilon, \varpi + \frac{\pi}{2}, 0\right), \tag{2.11}$$

where ϵ is a dimensionless, positive constant. We will later find ϵ to be the eccentricity of the orbit and ϖ the argument of periapsis. By substituting (2.11) into (2.10), we find:

$$\vec{v}(\theta) = \frac{GMm}{L}\hat{\theta} - \frac{GMm}{L}\epsilon \sin\varpi \hat{i} + \frac{GMm}{L}\epsilon \cos\varpi \hat{j}. \tag{2.12}$$

We will now use the definition for $\vec{L}$ to find $r(\theta)$.

$$\begin{aligned} L\hat{k}/m = \vec{L}/m = \vec{r} \times \vec{v} = (r\hat{r}) \times \left(\frac{GMm}{L}\hat{\theta} - \frac{GMm}{L}\epsilon \sin\varpi \hat{i} + \frac{GMm}{L}\epsilon \cos\varpi \hat{j}\right) &= \\ \frac{GMm}{L} r \cdot \left((\hat{r} \times \hat{\theta}) + \epsilon \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} \times \begin{pmatrix} -\sin\varpi \\ \cos\varpi \\ 0 \end{pmatrix} \right) &= \\ \frac{GMm}{L} r \cdot \left(\hat{k} + \epsilon(\cos\theta\cos\varpi + \sin\theta\sin\varpi)\hat{k}\right) = \frac{GMm}{L} r (1 + \epsilon\cos(\theta - \varpi))\hat{k}, & \end{aligned} \tag{2.13}$$

hence

$$r = \frac{\left(\frac{L^2}{GMm^2}\right)}{1 + \epsilon\cos(\theta - \varpi)}. \tag{2.14}$$

This formula has the same form as the formula of an ellipse (2.2). We now see that ϵ is the eccentricity of the orbit, and ϖ is the argument of periapsis. By equating the numerators from the two formulas, we see that the angular momentum can be expressed by

$$\left(\frac{L^2}{GMm^2}\right) = (1 - \epsilon^2)a. \tag{2.15}$$

Next, we will derive Kepler's second law. Consider a small time interval dt. During this time, a line from the origin to the orbiting object sweeps out a triangle with area $dA = 1/2 \cdot r \cdot rd\theta$. We can rewrite this using (2.8) as

$$2\frac{dA}{dt} = r^2\frac{d\theta}{dt} = \frac{L}{m}. \tag{2.16}$$

Since L is constant, dA/dt is constant. This is Kepler's second law. We can now integrate over one orbital period T, during which the entire area of the ellipse will be swept out. Because the area of the ellipse is πab, we obtain

$$\frac{L}{m}T = \int_0^T \frac{L}{m}dt = 2\int_0^T \frac{dA}{dt}dt = 2A = 2\pi ab. \tag{2.17}$$

We can now express L in terms of the angular orbital frequency $\Omega = 2\pi/T$:

$$L = \Omega mab. \tag{2.18}$$

By combining (2.15) and (2.18), we find Kepler's third law:

$$\Omega = \sqrt{\frac{GM}{a^3}}. \tag{2.19}$$

The square of the orbital period is proportional to the cube of the semi-major axis. A system of one large central mass and many orbiting smaller bodies is called a Keplerian disk. If the bodies have no mutual interaction, they will all orbit in perfect Kepler ellipses.

2.2 Eccentric Anomaly

The eccentric anomaly is a parameter that specifies the position of an object on its orbit. We will need this parameter in section 5.4. The eccentric anomaly is the angle E in figure 2.2.

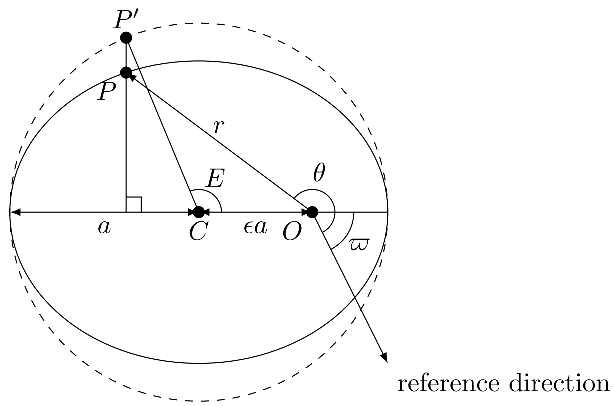


Figure 2.2: The eccentric anomaly E of an object at the point $P = (r, \theta)$ on its orbit. The point P' is obtained by extending a line perpendicular to the major axis through P onto the circle with radius a.

In figure 2.2, we can see by looking at the horizontal component of P, that

$$r\cos(\theta - \varpi) + \epsilon a = a\cos E. \tag{2.20}$$

We can now express $\cos(\theta - \varpi)$ in terms of E:

$$\cos(\theta - \varpi) = \frac{a\cos E - \epsilon a}{r}. \tag{2.21}$$

From figure 2.2, we also find, by looking at the vertical component of P, that

$$r\sin(\theta - \varpi) = \frac{b}{a}a\sin E = b\sin E, \tag{2.22}$$

since the ratio between the vertical component of P and P' is b/a. To find r in terms of E, we use (2.21) and (2.22) and find

$$r^2 = (r\cos(\theta - \varpi))^2 + (r\sin(\theta - \varpi))^2 = (a\cos E - \epsilon a)^2 + (b\sin E)^2 =$$
$$(a\cos E - \epsilon a)^2 + (a\sqrt{1-\epsilon^2})^2(1 - \cos^2 E) = \epsilon^2 a^2\cos^2 E - 2\epsilon a^2\cos E + a^2 = (a - \epsilon a\cos E)^2. \tag{2.23}$$

Now we find that r can be expressed in E by this simple formula:

$$r = a - \epsilon a\cos E. \tag{2.24}$$

Later, we will need an expression for dE/dt. To find one, we will use dr/dt:

$$\frac{dr}{dt} = \hat{r}\cdot\frac{d\vec{r}}{dt} = \hat{r}\cdot\vec{v} = \hat{r}\cdot\left(\frac{GMm}{L}(\hat{\theta} - \epsilon\sin\varpi\hat{i} + \epsilon\cos\varpi\hat{j})\right) = \frac{GMm}{L}\epsilon\sin(\theta - \varpi). \tag{2.25}$$

From (2.24), we know that

$$\frac{dr}{dt} = \frac{d}{dt}(a - \epsilon a \cos E) = \frac{GMm}{L}\epsilon \sin(\theta - \varpi). \tag{2.26}$$

Using the chain rule:

$$\epsilon a \sin E \cdot \frac{dE}{dt} = \frac{GMm}{L}\epsilon \sin(\theta - \varpi). \tag{2.27}$$

We now obtain an expression for dE/dt:

$$\frac{dE}{dt} = \frac{GMm}{L}\frac{\sin(\theta - \varpi)}{a \sin E} = \frac{GMm}{L}\frac{b}{ar}. \tag{2.28}$$

We can rewrite this using (2.18) and (2.19) as

$$\frac{dE}{dt} = \frac{\Omega a}{r}. \tag{2.29}$$

Chapter 3

Secular Perturbations

In chapter 2, we derived the Keplerian planetary orbits. In this chapter, we will use Lagrange's equations to find how these orbits change over time by the gravitational influence of other planets. We will apply the results to the planets of the solar system. We will make three important approximations:

1. The mass of the central body is much larger than the mass of the planets, which means that the changes in a planet's orbit due to the gravity of other planets is much slower than the orbital period. We can therefore average the planet's mass over its orbit. Over the timescale of many orbital periods, the so called secular time scale, the gravity from other planets will change a planet's orbital parameters over time.
2. There are no orbital resonances, which means that the ratio of any two planets orbital periods is never a rational number. If this were the case, those two planets would always be in the same relative positions of each other every few orbits, and the averaging of the planet's mass over the orbit would be incorrect.
3. The eccentricity and inclination of the planets is small. Therefore, we only need to take the lowest order terms in the Lagrange's equations and the disturbing function.

3.1 Two Planets

The approximate Lagrange's equations of the orbital elements are found in Murray [5]:

$$\begin{aligned} \dot{\epsilon}_j &= -\frac{1}{\Omega_j a_j^2 \epsilon_j}\frac{\partial \mathcal{R}_j}{\partial \varpi_j}, \qquad \dot{\varpi}_j = +\frac{1}{\Omega_j a_j^2 \epsilon_j}\frac{\partial \mathcal{R}_j}{\partial \epsilon_j}, \\ \dot{I}_j &= -\frac{1}{\Omega_j a_j^2 I_j}\frac{\partial \mathcal{R}_j}{\partial \Omega_j}, \qquad \dot{\Omega}_j = +\frac{1}{\Omega_j a_j^2 I_j}\frac{\partial \mathcal{R}_j}{\partial I_j}. \end{aligned} \tag{3.1}$$

The dot above a variable indicates a time derivative. In the equation above, $\mathcal{R}_j$ is the so-called disturbing function:

$$\mathcal{R}_j = \Omega_j a_j^2 \left[\frac{1}{2}A_{jj}\epsilon_j^2 + A_{jk}\epsilon_1\epsilon_2\cos(\varpi_1 - \varpi_2) + \frac{1}{2}B_{jj}I_j^2 + B_{jk}I_1 I_2 \cos(\Omega_1 - \Omega_2)\right]. \tag{3.2}$$

Here $j, k \in \{1, 2\}$, and $j \neq k$. The coefficients A and B are given by:

$$
\begin{aligned}
A_{jj} &= +\Omega_j \frac{1}{4} \frac{m_k}{m_c + m_j} \alpha_{12} \bar{\alpha}_{12} b_{3/2}^{(1)} (\alpha_{12}), \\
A_{jk} &= -\Omega_j \frac{1}{4} \frac{m_k}{m_c + m_j} \alpha_{12} \bar{\alpha}_{12} b_{3/2}^{(2)} (\alpha_{12}), \\
B_{jj} &= -\Omega_j \frac{1}{4} \frac{m_k}{m_c + m_j} \alpha_{12} \bar{\alpha}_{12} b_{3/2}^{(1)} (\alpha_{12}), \\
B_{jk} &= +\Omega_j \frac{1}{4} \frac{m_k}{m_c + m_j} \alpha_{12} \bar{\alpha}_{12} b_{3/2}^{(1)} (\alpha_{12}),
\end{aligned} \tag{3.3}
$$

where m_c is the mass of the central body, $\alpha_{12} = a_1/a_2$ and $\bar{\alpha}_{12} = \alpha_{12}$ if $j = 1$, 1 if $j = 2$ and

$$
b_{3/2}^{(n)}(\alpha) = \frac{1}{\pi} \int_0^{2\pi} \frac{\cos n\psi \, \mathrm{d}\psi}{(1 - 2\alpha \cos \psi + \alpha^2)^{\frac{3}{2}}}. \tag{3.4}
$$

It is convenient to continue with the vertical and horizontal components of the eccentricity and inclination vectors:

$$
\begin{aligned}
h_j &= \epsilon_j \sin \varpi_j, \qquad & k_j &= \epsilon_j \cos \varpi_j, \\
p_j &= I_j \sin \Omega_j, \qquad & q_j &= I_j \cos \Omega_j.
\end{aligned} \tag{3.5}
$$

We will now derive the equations of motion for these convenient variables. The disturbing function can be written in terms of these variables:

$$
\mathcal{R}_j = \Omega_j a_j^2 \left[\frac{1}{2} A_{jj} \left(h_j^2 + k_j^2 \right) + A_{jk} \left(h_j h_k + k_j k_k \right) + \frac{1}{2} B_{jj} \left(p_j^2 + q_j^2 \right) + B_{jk} \left(p_j p_k + q_j q_k \right) \right] \tag{3.6}
$$

Using the chain rule, we get the following time derivatives:

$$
\begin{aligned}
\frac{\mathrm{d}h_j}{\mathrm{d}t} &= \frac{\partial h_j}{\partial \epsilon_j} \frac{\mathrm{d}\epsilon_j}{\mathrm{d}t} + \frac{\partial h_j}{\partial \varpi_j} \frac{\mathrm{d}\varpi_j}{\mathrm{d}t}, \qquad & \frac{\mathrm{d}k_j}{\mathrm{d}t} &= \frac{\partial k_j}{\partial \epsilon_j} \frac{\mathrm{d}\epsilon_j}{\mathrm{d}t} + \frac{\partial k_j}{\partial \varpi_j} \frac{\mathrm{d}\varpi_j}{\mathrm{d}t}, \\
\frac{\mathrm{d}p_j}{\mathrm{d}t} &= \frac{\partial p_j}{\partial I_j} \frac{\mathrm{d}I_j}{\mathrm{d}t} + \frac{\partial p_j}{\partial \Omega_j} \frac{\mathrm{d}\Omega_j}{\mathrm{d}t}, \qquad & \frac{\mathrm{d}q_j}{\mathrm{d}t} &= \frac{\partial q_j}{\partial I_j} \frac{\mathrm{d}I_j}{\mathrm{d}t} + \frac{\partial q_j}{\partial \Omega_j} \frac{\mathrm{d}\Omega_j}{\mathrm{d}t},
\end{aligned} \tag{3.7}
$$

where the partial derivatives can be derived from (3.5):

$$
\begin{aligned}
\frac{\partial h_j}{\partial \epsilon_j} &= \frac{h_j}{\epsilon_j}, \qquad & \frac{\partial k_j}{\partial \epsilon_j} &= \frac{k_j}{\epsilon_j}, \qquad & \frac{\partial h_j}{\partial \varpi_j} &= +k_j, \qquad & \frac{\partial k_j}{\partial \varpi_j} &= -h_j, \\
\frac{\partial p_j}{\partial I_j} &= \frac{p_j}{I_j}, \qquad & \frac{\partial q_j}{\partial I_j} &= \frac{q_j}{I_j}, \qquad & \frac{\partial p_j}{\partial \Omega_j} &= +q_j, \qquad & \frac{\partial q_j}{\partial \Omega_j} &= -p_j.
\end{aligned} \tag{3.8}
$$

Combining (3.7) and (3.8), we get the system of equations

$$
\begin{aligned}
\dot{h}_j &= \frac{h_j}{\epsilon_j} \dot{\epsilon}_j + k_j \dot{\varpi}_j, \qquad & \dot{k}_j &= \frac{k_j}{\epsilon_j} \dot{\epsilon}_j - h_j \dot{\varpi}_j, \\
\dot{p}_j &= \frac{p_j}{I_j} \dot{I}_j + q_j \dot{\Omega}_j, \qquad & \dot{q}_j &= \frac{q_j}{I_j} \dot{I}_j - p_j \dot{\Omega}_j
\end{aligned} \tag{3.9}
$$

If we now substitute the time derivatives from the Lagrange equations (3.1), we get the following:

$$\begin{aligned}
\dot{h}_j &= \frac{1}{\Omega_j a_j^2}\left(-\frac{h_j}{\epsilon_j}\frac{1}{\epsilon_j}\frac{\partial \mathcal{R}_j}{\partial \varpi_j} + k_j\frac{1}{\epsilon_j}\frac{\partial \mathcal{R}_j}{\partial \epsilon_j}\right),\\
\dot{k}_j &= -\frac{1}{\Omega_j a_j^2}\left(\frac{k_j}{\epsilon_j}\frac{1}{\epsilon_j}\frac{\partial \mathcal{R}_j}{\partial \varpi_j} + h_j\frac{1}{\epsilon_j}\frac{\partial \mathcal{R}_j}{\partial \epsilon_j}\right),\\
\dot{p}_j &= \frac{1}{\Omega_j a_j^2}\left(-\frac{p_j}{I_j}\frac{1}{I_j}\frac{\partial \mathcal{R}_j}{\partial \Omega_j} + q_j\frac{1}{I_j}\frac{\partial \mathcal{R}_j}{\partial I_j}\right),\\
\dot{q}_j &= -\frac{1}{\Omega_j a_j^2}\left(\frac{q_j}{I_j}\frac{1}{I_j}\frac{\partial \mathcal{R}_j}{\partial \Omega_j} + p_j\frac{1}{I_j}\frac{\partial \mathcal{R}_j}{\partial I_j}\right).
\end{aligned} \tag{3.10}$$

In order to write everything in terms of the components only, we need the Jacobian matrices:

$$\begin{pmatrix} \frac{\partial \epsilon_j}{\partial h_j} & \frac{\partial \epsilon_j}{\partial k_j} \\ \frac{\partial \varpi_j}{\partial h_j} & \frac{\partial \varpi_j}{\partial k_j} \end{pmatrix}\begin{pmatrix} \frac{\partial h_j}{\partial \epsilon_j} & \frac{\partial h_j}{\partial \varpi_j} \\ \frac{\partial k_j}{\partial \epsilon_j} & \frac{\partial k_j}{\partial \varpi_j} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.11}$$

Using the determinant of second matrix:

$$\frac{\partial h_j}{\partial \epsilon_j}\frac{\partial k_j}{\partial \varpi_j} - \frac{\partial h_j}{\partial \varpi_j}\frac{\partial k_j}{\partial \epsilon_j} = -\frac{h_j^2}{\epsilon_j} - \frac{k_j^2}{\epsilon_j} = -\frac{\epsilon_j^2\sin^2(\varpi_j)}{\epsilon_j} - \frac{\epsilon_j^2\cos^2(\varpi_j)}{\epsilon_j} = -\epsilon_j, \tag{3.12}$$

we can invert the second matrix, and obtain

$$\begin{pmatrix} \frac{\partial \epsilon_j}{\partial h_j} & \frac{\partial \epsilon_j}{\partial k_j} \\ \frac{\partial \varpi_j}{\partial h_j} & \frac{\partial \varpi_j}{\partial k_j} \end{pmatrix} = -\frac{1}{\epsilon_j}\begin{pmatrix} \frac{\partial k_j}{\partial \varpi_j} & -\frac{\partial h_j}{\partial \varpi_j} \\ -\frac{\partial k_j}{\partial \epsilon_j} & \frac{\partial h_j}{\partial \epsilon_j} \end{pmatrix}. \tag{3.13}$$

We can rewrite the partial derivatives of the disturbing function, using the chain rule, as follows:

$$\begin{aligned}
\frac{\partial \mathcal{R}_j}{\partial h_j} &= \frac{\partial \varpi_j}{\partial h_j}\frac{\partial \mathcal{R}_j}{\partial \varpi_j} + \frac{\partial \epsilon_j}{\partial p_j}\frac{\partial \mathcal{R}_j}{\partial \epsilon_j}, &\quad \frac{\partial \mathcal{R}_j}{\partial k_j} &= \frac{\partial \varpi_j}{\partial k_j}\frac{\partial \mathcal{R}_j}{\partial \varpi_j} + \frac{\partial \epsilon_j}{\partial p_j}\frac{\partial \mathcal{R}_j}{\partial \epsilon_j},\\
\frac{\partial \mathcal{R}_j}{\partial p_j} &= \frac{\partial \Omega_j}{\partial p_j}\frac{\partial \mathcal{R}_j}{\partial \Omega_j} + \frac{\partial I_j}{\partial p_j}\frac{\partial \mathcal{R}_j}{\partial I_j}, &\quad \frac{\partial \mathcal{R}_j}{\partial q_j} &= \frac{\partial \Omega_j}{\partial q_j}\frac{\partial \mathcal{R}_j}{\partial \Omega_j} + \frac{\partial I_j}{\partial q_j}\frac{\partial \mathcal{R}_j}{\partial I_j}.
\end{aligned} \tag{3.14}$$

If we now use (3.13) to get the unknown partial derivatives, and compare (3.14) with (3.10), we see that

$$\begin{aligned}
\dot{h}_j &= +\frac{1}{\Omega_j a_j^2}\frac{\partial \mathcal{R}_j}{\partial k_j}, &\quad \dot{k}_j &= -\frac{1}{\Omega_j a_j^2}\frac{\partial \mathcal{R}_j}{\partial h_j},\\
\dot{p}_j &= +\frac{1}{\Omega_j a_j^2}\frac{\partial \mathcal{R}_j}{\partial q_j}, &\quad \dot{q}_j &= -\frac{1}{\Omega_j a_j^2}\frac{\partial \mathcal{R}_j}{\partial p_j}.
\end{aligned} \tag{3.15}$$

Taking the partial derivatives of the disturbing function (3.6) is straightforward:

$$\begin{aligned}
\dot{h}_1 &= +A_{11}k_1 + A_{12}k_2, &\quad \dot{k}_1 &= -A_{11}h_1 - A_{12}h_2,\\
\dot{h}_2 &= +A_{21}k_1 + A_{22}k_2, &\quad \dot{k}_2 &= -A_{21}h_1 - A_{22}h_2,\\
\dot{p}_1 &= +B_{11}q_1 + B_{12}q_2, &\quad \dot{q}_1 &= -B_{11}p_1 - B_{12}p_2,\\
\dot{p}_2 &= +B_{21}q_1 + B_{22}q_2, &\quad \dot{q}_2 &= -B_{21}p_1 - B_{22}p_2.
\end{aligned} \tag{3.16}$$

We see that the equations of h and k are decoupled from those of p and q. This can be written in matrix form as follows:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}h_1\\h_2\end{pmatrix}&=\begin{pmatrix}A_{11}&A_{12}\\A_{21}&A_{22}\end{pmatrix}\begin{pmatrix}k_1\\k_2\end{pmatrix},\\ \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}k_1\\k_2\end{pmatrix}&=\begin{pmatrix}-A_{11}&-A_{12}\\-A_{21}&-A_{22}\end{pmatrix}\begin{pmatrix}h_1\\h_2\end{pmatrix},\\ \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}p_1\\p_2\end{pmatrix}&=\begin{pmatrix}B_{11}&B_{12}\\B_{21}&B_{22}\end{pmatrix}\begin{pmatrix}q_1\\q_2\end{pmatrix},\\ \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}q_1\\q_2\end{pmatrix}&=\begin{pmatrix}-B_{11}&-B_{12}\\-B_{21}&-B_{22}\end{pmatrix}\begin{pmatrix}p_1\\p_2\end{pmatrix}.\end{aligned}\tag{3.17}$$

If we now define the following vectors:

$$\vec{h}=\begin{pmatrix}h_1\\h_2\end{pmatrix},\quad \vec{k}=\begin{pmatrix}k_1\\k_2\end{pmatrix},\quad \vec{p}=\begin{pmatrix}p_1\\p_2\end{pmatrix},\quad \vec{q}=\begin{pmatrix}q_1\\q_2\end{pmatrix},\tag{3.18}$$

we can combine h and k, as well as p and q into one complex variable:

$$\vec{\psi}=\vec{k}+i\vec{h},\qquad \vec{\phi}=\vec{q}+i\vec{p},\tag{3.19}$$

which have the following time derivatives:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\vec{\psi}&=\frac{\mathrm{d}}{\mathrm{d}t}\vec{k}+i\frac{\mathrm{d}}{\mathrm{d}t}\vec{h}=-A\vec{h}+iA\vec{k}=iA(i\vec{h}+\vec{k})=iA\vec{\psi},\\ \frac{\mathrm{d}}{\mathrm{d}t}\vec{\phi}&=\frac{\mathrm{d}}{\mathrm{d}t}\vec{q}+i\frac{\mathrm{d}}{\mathrm{d}t}\vec{p}=-B\vec{p}+iB\vec{q}=iB(i\vec{p}+\vec{q})=iB\vec{\phi}.\end{aligned}\tag{3.20}$$

We wil use the equations of motion for the complex vectors:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\vec{\psi}&=iA\vec{\psi},\\ \frac{\mathrm{d}}{\mathrm{d}t}\vec{\phi}&=iB\vec{\phi}.\end{aligned}\tag{3.21}$$

The solutions of (3.21) are given by

$$\vec{\psi}(t)=\mathrm{e}^{iAt}\vec{\psi}(0),\qquad \vec{\phi}(t)=\mathrm{e}^{iBt}\vec{\phi}(0).\tag{3.22}$$

We can get ϵ_j, ϖ_j, I_j and Ω_j from $\vec{\psi}$ and $\vec{\phi}$ as follows:

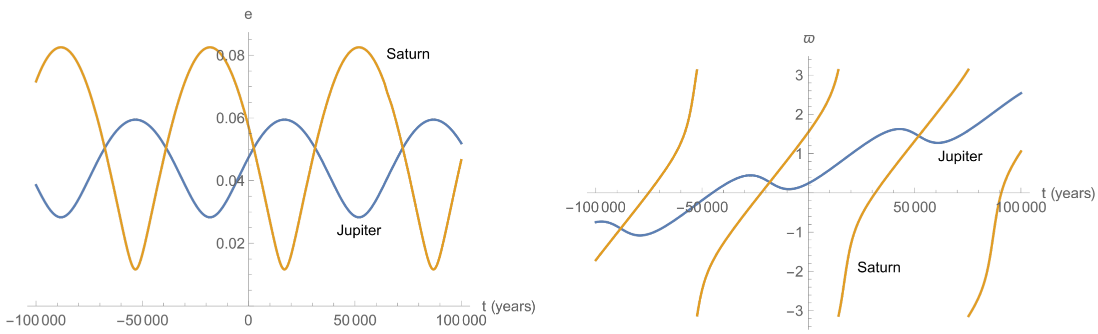
$$\epsilon_j=|\psi_j|,\quad \varpi_j=\operatorname{Arg}\psi_j,\quad I_j=|\phi_j|,\quad \Omega_j=\operatorname{Arg}\phi_j.\tag{3.23}$$

3.2 Jupiter and Saturn

Now we will apply the theory outlined above to the planets Jupiter and Saturn, orbiting the Sun. The parameters of the system in 1983 are given by[5]:

$$
\begin{aligned}
m_1/m_c &= 9.54786 \times 10^{-4}, & m_2/m_c &= 2.85837 \times 10^{-4}, \\
a_1 &= 5.202545 \text{ AU}, & a_2 &= 9.554841 \text{ AU}, \\
\Omega_1 &= 30.3374^\circ \text{y}^{-1}, & \Omega_2 &= 12.1890^\circ \text{y}^{-1}, \\
\epsilon_1 &= 0.0474622, & \epsilon_2 &= 0.0575481, \\
\varpi_1 &= 13.983865^\circ, & \varpi_2 &= 88.719425^\circ, \\
I_1 &= 1.30667^\circ, & I_2 &= 2.48795^\circ, \\
\text{☊}_1 &= 100.0381^\circ, & \text{☊}_2 &= 113.1334^\circ.
\end{aligned}
\tag{3.24}
$$

We will use Wolfram Mathematica to calculate equation 3.22. The code used for the results in this section can be found in appendix A. From the result, we let Mathematica draw the plots in figure 3.1. The plots show the eccentricity and argument of periapsis of both planets. The eccentricity of both planets is a periodic function, with a period of approximately 70,000 years. The eccentricity varies between a minimum and a maximum value. When one planet reaches the maximum value, the other planet reaches the minimum value. The variation of Saturn's eccentricity is larger than that of Jupiter's. This makes sense, because Jupiter is larger, and therefore we would expect Jupiter's influence on Saturn to be greater. The argument of periapsis of both planets precesses. For Saturn, this precession has a period of 70,000 years.



(a) (b)

Figure 3.1: The eccentricity (a) and argument of periapsis (b) of Jupiter and Saturn over a time span of 200,000 years, centered on the year 1983 ($t = 0$).

In figure 3.2, we use Mathematica to draw the orbits of both planets over a period of 20,000 years. Saturn has a larger range of eccentricities, leading to a broader band than Jupiter.

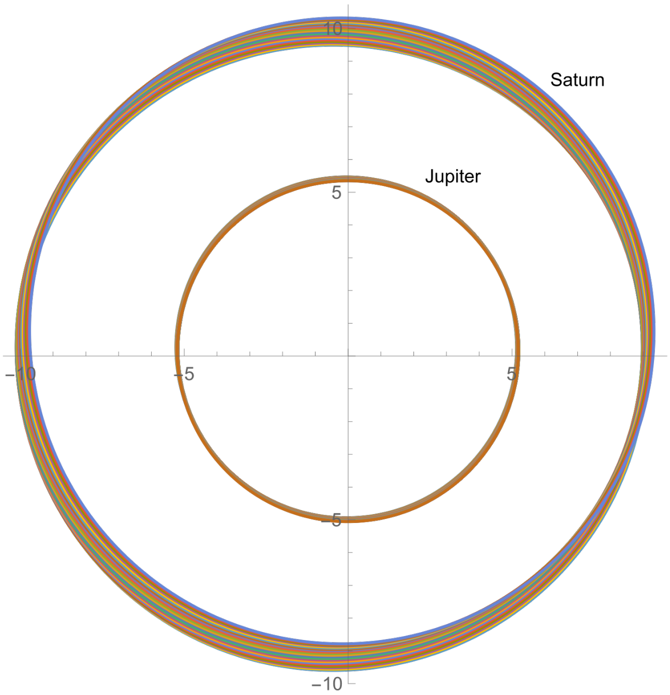
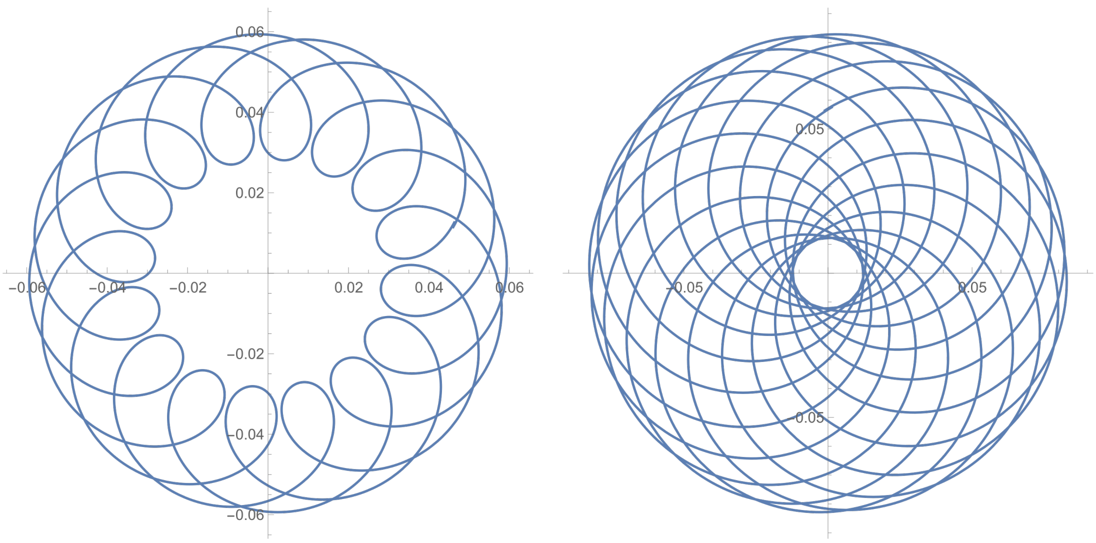


Figure 3.2: The orbits of Jupiter and Saturn in the ecliptic plane over a period of 20,000 years.

Figure 3.3 shows the calculated value of ψ_j in the complex plane. $|\psi_j|$ is the eccentricity of planet j, $\text{Arg}(\psi_j)$ is the argument of periapsis of planet j.



($j = 1$) Jupiter

($j = 2$) Saturn

Figure 3.3: The phase vector $\psi_j = \epsilon_j e^{i\varpi_j}$ in the complex plane over a period of 1.122×10^6 years, starting at the year 1983, for j referring to Jupiter ($j = 1$) and Saturn ($j = 2$).

The curves seem to close after 16 loops. For Jupiter (a), the curve goes around the origin 3 times in this period, while for Saturn (b), the curve goes around the origin 16 times. See figure 3.1.

We can write the solution (3.22) as follows:

$$\psi_j = c_{1j} e^{i\omega_1 t} + c_{2j} e^{i\omega_2 t} \tag{3.25}$$

where $j \in \{1, 2\}$ refers to Jupiter or Saturn, $\omega_1 > \omega_2$ are the eigenvalues of the matrix A, and c_{1j} and c_{2j} are complex coefficients. We can rewrite this equation as

$$\psi_j = c_{2j} e^{i\omega_2 t} \left(1 + \frac{c_{1j}}{c_{2j}} e^{i(\omega_1 - \omega_2)t} \right) \tag{3.26}$$

If c_{2j} is larger than c_{1j}, we see that ψ_j has a frequency of ω_2, with an additional perturbation with a frequency of $\omega_1 - \omega_2$.

For Jupiter this is the case, we have $|c_{12}/c_{22}| = 0.383$. If we compare the frequency of ψ_2 (ω_2) with the frequency of the additional perturbation ($\omega_1 - \omega_2$), we see that $(\omega_1 - \omega_2)/\omega_2 = 5.331 \approx 16/3$. This explains why the curve in figure 3.3 closes on itself after making 16 loops.

3.3 More Than Two Planets

If we have $N > 2$ planets, the disturbing function (3.6) changes to

$$\begin{aligned} \mathcal{R}_j = \Omega_j a_j^2 \Bigg[& \frac{1}{2} A_{jj} \left(h_j^2 + k_j^2 \right) + \sum_{k=1, k \neq j}^{N} A_{jk} \left(h_j h_k + k_j k_k \right) \\ & + \frac{1}{2} B_{jj} \left(p_j^2 + q_j^2 \right) + \sum_{k=1, k \neq j}^{N} B_{jk} \left(p_j p_k + q_j q_k \right) \Bigg]. \end{aligned} \tag{3.27}$$

Here h_j, k_j, p_j, q_j as in equation (3.5). The matrix elements change to

$$\begin{aligned} A_{jj} &= \frac{\Omega_j}{4} \left(\sum_{k=1}^{j-1} \frac{m_k}{m_c + m_j} \frac{a_k}{a_j} b_{3/2}^{(1)} \left(\frac{a_k}{a_j} \right) + \sum_{k=j+1}^{N} \frac{m_k}{m_c + m_j} \left(\frac{a_j}{a_k} \right)^2 b_{3/2}^{(1)} \left(\frac{a_j}{a_k} \right) \right) \\ A_{jk} &= \frac{-\Omega_j}{4} \frac{m_k}{m_c + m_j} \begin{cases} \frac{a_k}{a_j} b_{3/2}^{(2)} \left(\frac{a_k}{a_j} \right), & k < j \\ \left(\frac{a_j}{a_k} \right)^2 b_{3/2}^{(2)} \left(\frac{a_j}{a_k} \right), & k > j \end{cases}. \end{aligned} \tag{3.28}$$

The partial derivatives of h_j, k_j, p_j and q_j are now given by

$$\begin{aligned} \dot{h}_j &= \sum_{k=1}^{N} A_{jk} k_k, \qquad \dot{k}_j = -\sum_{k=1}^{N} A_{jk} h_k, \\ \dot{p}_j &= \sum_{k=1}^{N} B_{jk} q_k, \qquad \dot{q}_j = -\sum_{k=1}^{N} B_{jk} p_k. \end{aligned} \tag{3.29}$$

This can be written in vector form as

$$\frac{\mathrm{d}}{\mathrm{d}t} \vec{h} = A\vec{k}, \quad \frac{\mathrm{d}}{\mathrm{d}t} \vec{k} = -A\vec{h}, \quad \frac{\mathrm{d}}{\mathrm{d}t} \vec{p} = B\vec{q}, \quad \frac{\mathrm{d}}{\mathrm{d}t} \vec{q} = -B\vec{p}. \tag{3.30}$$

Like in section 3.1, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{\psi} = \frac{\mathrm{d}}{\mathrm{d}t}\vec{k} + i\frac{\mathrm{d}}{\mathrm{d}t}\vec{h} = -A\vec{h} + iA\vec{k} = iA(i\vec{h} + \vec{k}) = iA\vec{\psi}, \tag{3.31}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{\phi} = \frac{\mathrm{d}}{\mathrm{d}t}\vec{q} + i\frac{\mathrm{d}}{\mathrm{d}t}\vec{p} = -B\vec{p} + iB\vec{q} = iB(i\vec{p} + \vec{q}) = iB\vec{\phi}. \tag{3.32}$$

3.4 The Planets of the Solar System

We will now apply the theory of secular pertubations to the 8 planets of the solar system. We used Mathematica for the calculations. The code used for the results in this section can be found in appendix B. The parameters of the solar system are taken from the website of the NASA[6]. Solving equation (3.31) for $N = 8$ yields the following:

$$\psi_j = \sum_{n=1}^{8} c_n v_{nj} e^{i\omega_n t}, \tag{3.33}$$

where the v_{nj} are the j^{th} component of the n^{th} eigenvector of the matrix A, and the ω_n are the eigenvalues of the matrix A. The complex coefficients c_n are determined by the initial conditions. If for some n, $c_n v_{nj}$ is much larger in absolute value than for the others, then ψ_n is approximately periodic, with a period determined by the corresponding eigenvalue ω_n. In the table below, this period is given for each planet.

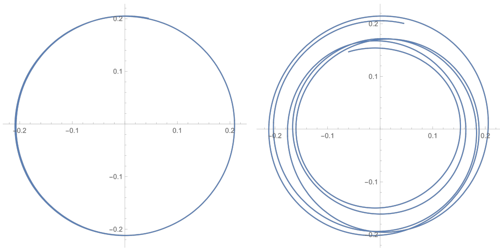
Planet	Period (10,000 years)
Mercury	23.8
Venus	17.7
Earth	17.7
Mars	7.22
Jupiter	34.8
Saturn	5.76
Uranus	34.8
Neptune	204

Since Jupiter and Saturn are much bigger than the other planets, the influences of the other planets might be insignificant. We can ignore the influences of the other planets, by setting their masses to 0. This gives the following values for the period:

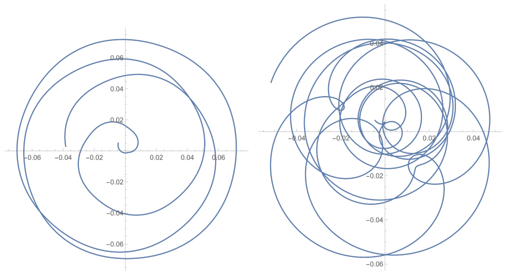
Planet	Period (10,000 years)
Mercury	77.1
Venus	29.4
Earth	17.5
Mars	8.48
Jupiter	37.1
Saturn	5.84
Uranus	55.6
Neptune	336

We can see that these values are very different. Apparently, the influence of the other planets is significant compared to Jupiter's and Saturn's influence. To get a more complete picture, we let

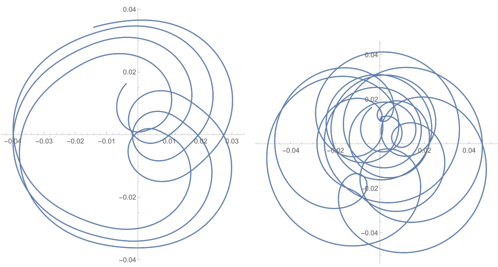
Mathematica draw the ψ_j in the complex plane over a period of 1.2 million years, and compare the results:



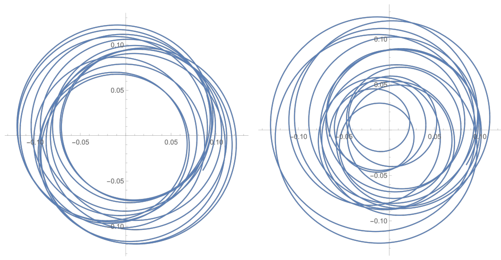
($j = 1$) Mercury



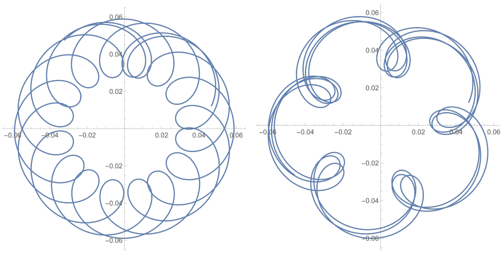
($j = 2$) Venus



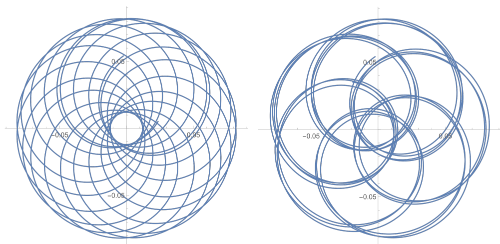
($j = 3$) Earth



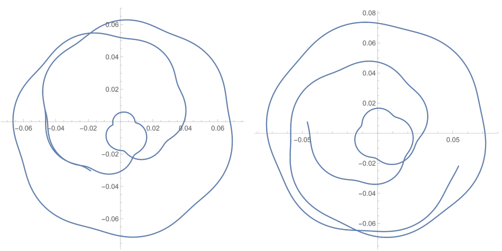
($j = 4$) Mars



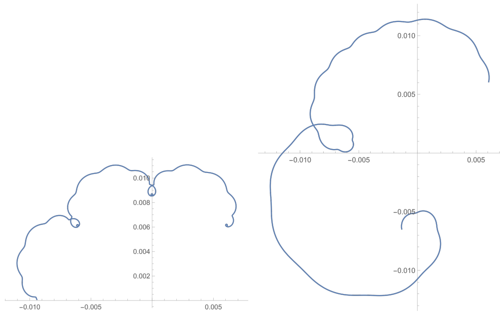
($j = 5$) Jupiter



($j = 6$) Saturn



($j = 7$) Uranus

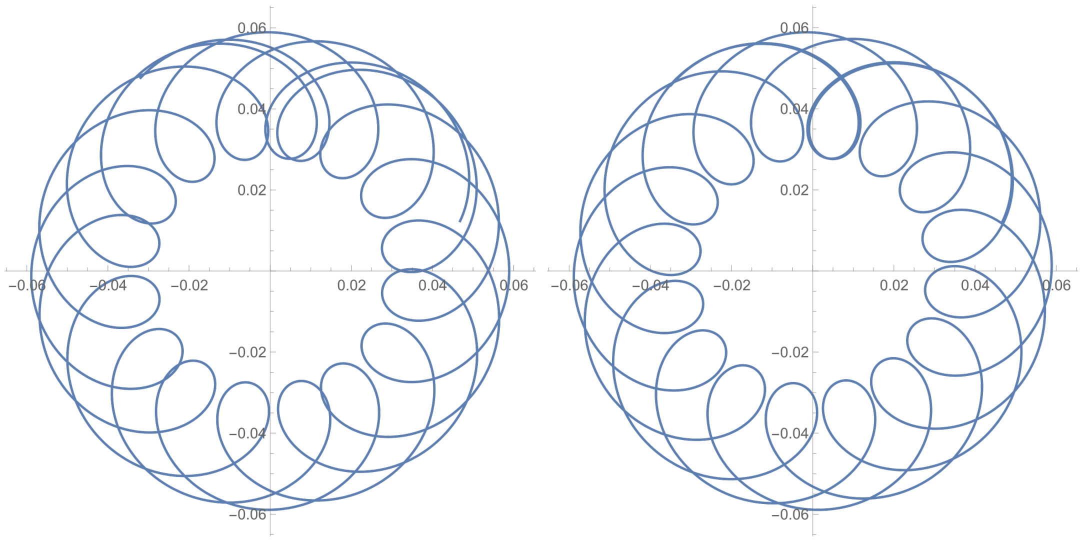


($j = 8$) Neptune

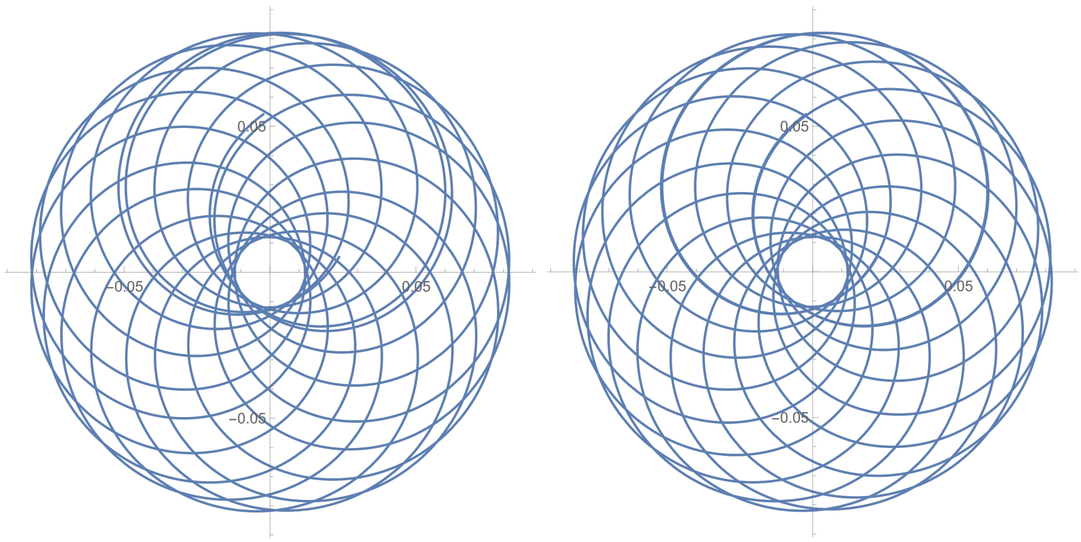
Figure 3.4: The amplitudes $\psi_j = k_j + ih_j$ for the planets of the solar system ($j = 1, ..., 8$) in the complex plane over a period of 1.2 million years. For each planet, the picture on the left shows the results under the influence of only Jupiter and Saturn. The picture on the right shows the results under the influence of all planets.

This picture clearly shows that the results are very different. The resonance of Jupiter and

Saturn observed in section 3.2 is gone. What if we only ignore Uranus and Neptune? Do the inner planets have a significant effect on Jupiter and Saturn? The results of this experiment are shown in figure 3.5:



(a) Jupiter



(b) Saturn

Figure 3.5: ψ_j in the complex plane over a period of 1.12 million years, for Jupiter and Saturn. The picture on the left shows the results under only each other's influence. The picture on the right shows the results under the influence of all planets but Uranus and Neptune.

These results are very similar. The effects of the inner planets on Jupiter and Saturn is small compared to their mutual influence.

Chapter 4

The Disk Model

4.1 Introduction

In this chapter, we apply the secular perturbation theory from chapter 3 to a large number of planets, as to approximate a continuous disk. We want to study the eigenmodes of this system and hope to find a function to describe these.

4.2 The Disk Structure

We take $N + 1 = 101$ planets orbiting around a central body. The central body has a mass of 1 solar mass. The 101 planets have semi-major axes geometrically spaced between 1 and 100 astronomical units. Planet j has semi-major axis

$$a_j = a_{\mathrm{in}} \left(\frac{a_{\mathrm{out}}}{a_{\mathrm{in}}} \right)^{j/N}. \tag{4.1}$$

For the planar density of the system, we take the special density

$$\sigma(a) = \sigma_{\mathrm{in}} \sqrt{\frac{a_{\mathrm{in}}}{a}}, \tag{4.2}$$

where σ_{in} is the density at $a = a_{\mathrm{in}}$. The total mass in the infinitesimal ring element between semi-major axis a and $a + da$ is given by $dm = 2\pi a \sigma(a) da$. Substituting the special density (4.2) gives us

$$\frac{dm}{da} = 2\pi \sigma_{\mathrm{in}} \sqrt{a_{\mathrm{in}} a}. \tag{4.3}$$

The masses of the planets are

$$m_j = \int_{a_j}^{a_{j+1}} dm = \frac{4}{3} \pi \sigma_{\mathrm{in}} \sqrt{a_{\mathrm{in}}} \left(a_{j+1}^{3/2} - a_j^{3/2} \right). \tag{4.4}$$

In the examples, we will take for all planets an initial eccentricity of 0.2, and an initial argument of periapsis 0.

Just like before, the solution has the following form:

$$\vec{\psi} = \sum_{n=0}^{N} c_n \vec{v}_n e^{i\omega_n t}. \tag{4.5}$$

Here the $\vec{v}_n$ are the eigenvectors of the matrix A, and the ω_n are the corresponding eigenvalues. The complex coefficients c_n are determined by the initial conditions.

4.3 The Eigenmodes

We want to study the eigenmodes of the system. The modes are used to solve the equation of motion. Single modes can be exited by external forces. The eigenmodes have the following form:

$$\vec{\psi}_n(t) = \vec{v}_n e^{i\omega_n t} \tag{4.6}$$

for $n = 0, \ldots, N$. The modes have the special property that the eigenvectors $\vec{v}_n$ can be chosen to be real. Therefore the patterns in space stay the same, they only rotate with angular frequency ω_n. Also the orbits of all the planets are aligned. We use Mathematica to calculate the eigenmodes. The code used for the results in this section can be found in appendix C. The first 10 modes are drawn about the horizontal line at their respective eigenvalues in figure 4.1.

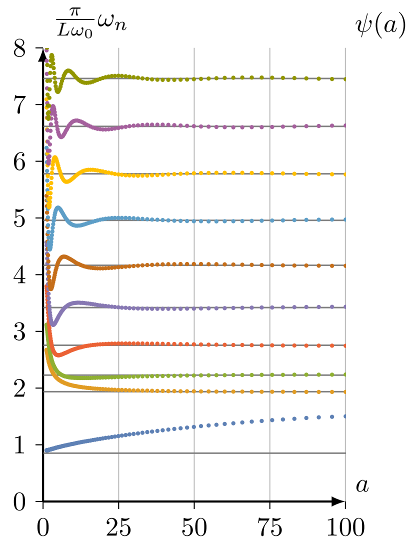


Figure 4.1: The first 10 eigenmodes of the complex vector ψ as a function of semi-major axis, for 100 planets (one dot for each planet), drawn at their respective eigenvalues. Note that mode n changes sign n times.

One remarkable observation is that mode n changes sign n times. We are going to use this property later. Since the semi-major axes are geometrically spaced, it makes sense to draw the eigenmodes as a function of $x = \log(a/a_{\text{in}})$, instead of a itself, see figure 4.2.

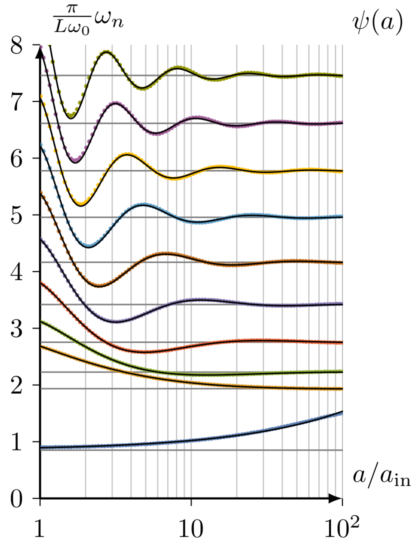


Figure 4.2: The first 10 eigenmodes of the complex vector ψ, for 100 planets, as in figure 4.1. The horizontal axis now has a logarithmic scale. The modes seem to have constant wave lengths. The black lines are damped sinusoids (4.7), fitted to the modes.

In this logarithmic plot, the eigenmodes strongly resemble damped sinusoids. Formula (4.7) gives a possible function describing the eigenmodes:

$$\psi_n = e^{-\lambda_n x} \cos(\kappa_n x + \phi_n). \tag{4.7}$$

Fitting the parameters λ_n, ϕ_n and κ_n to each eigenmode gives the black curves in figure 4.2. The values of the fitted parameters, as well as the eigenvalues, are given in figure 4.3.

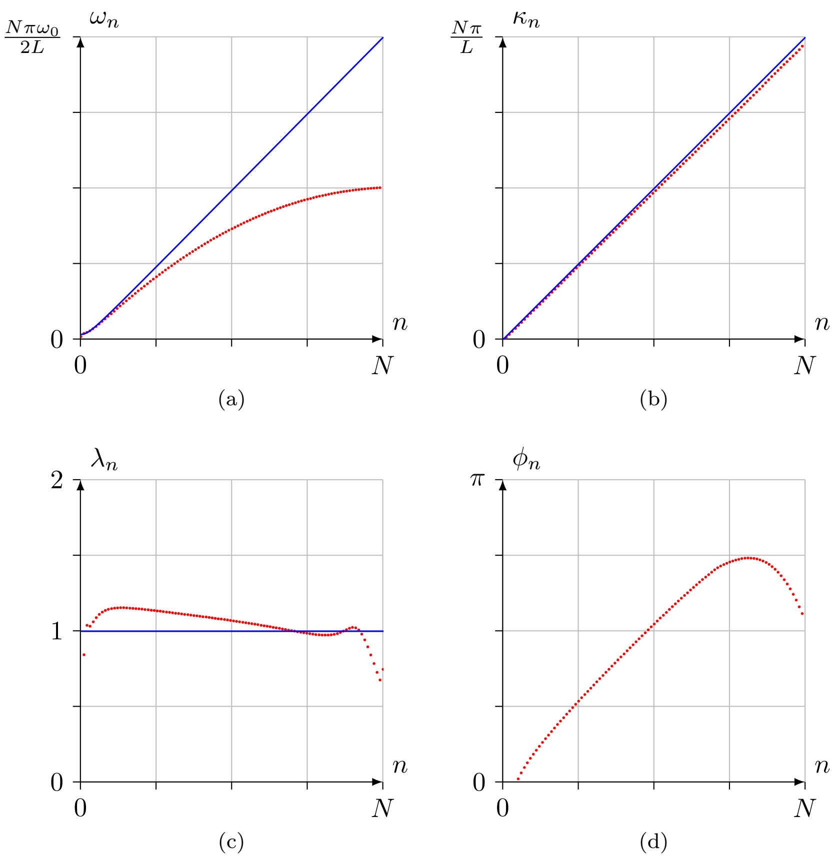


Figure 4.3: The red dots are the eigenvalues (figure (a)) and the values of the fit parameters of the damped sinusoid (4.7), fitted to the eigenmodes, as a function of modenumber n. Figure (b) shows the wavenumber κ_n. Figure (c) shows the exponential factor λ_n. Figure (d) shows the phase ϕ_n. The blue lines in figures (a) and (c) are the corresponding continuum limit.

Chapter 5

The Continuum Limit

In this chapter, we will take the limit $N \to \infty$ of an infinite number of planetesimals. We will derive the modes for the continuous system, and compare them to the modes of the discrete system found in chapter 4.

5.1 Taking the Continuum Limit

We take the limit $N \to \infty$ of an infinite number of planetesimals. The planet index j is replaced by the semi-major axis a.

We have the Schrödinger equation for the discrete system (see (3.31)):

$$i\frac{\partial}{\partial t}\psi_j(t) = \sum_{n=0}^{N} A_{jn}\psi_n(t). \tag{5.1}$$

This equation now transforms into

$$i\frac{\partial}{\partial t}\psi(a,t) = \int_{a_{\text{in}}}^{a_{\text{out}}} \Big(A^{(1)}(a,s)\psi(a,t) - A^{(2)}(a,s)\psi(s,t)\Big)\, ds \tag{5.2}$$

where (3.28) becomes the matrix kernel

$$A^{(n)}(a,s) = \frac{\Omega(a)}{4M}\frac{dm(s)}{ds}\begin{cases} \frac{s}{a} b_{3/2}^{(n)}\left(\frac{s}{a}\right), & s < a \\ \left(\frac{a}{s}\right)^2 b_{3/2}^{(n)}\left(\frac{a}{s}\right), & s > a \end{cases}. \tag{5.3}$$

The integration over $A^{(1)}(a,s)$ corresponds to the summation inside A_{jj}, and the integration over $A^{(2)}(a,s)$ to the summation over A_{jk}. We can make the cases in equation (5.3) symmetrical by factoring out $\sqrt{a/s}$:

$$A^{(n)}(a,s) = \frac{\Omega(a)}{4M}\frac{dm(s)}{ds}\sqrt{\frac{a}{s}}\begin{cases} \left(\frac{s}{a}\right)^{3/2} b_{3/2}^{(n)}\left(\frac{s}{a}\right), & s < a \\ \left(\frac{a}{s}\right)^{3/2} b_{3/2}^{(n)}\left(\frac{a}{s}\right), & s > a \end{cases}. \tag{5.4}$$

If we now bring the fractions within the cases inside of the exponent:

$$A^{(n)}(a,s) = \frac{\Omega(a)}{4M}\frac{dm(s)}{ds}\sqrt{\frac{a}{s}}\begin{cases} \exp\left(\frac{3}{2}\log\frac{s}{a}\right) b_{3/2}^{(n)}\left(\exp\log\frac{s}{a}\right), & s < a \\ \exp\left(\frac{3}{2}\log\frac{a}{s}\right) b_{3/2}^{(n)}\left(\exp\log\frac{a}{s}\right), & s > a \end{cases} \tag{5.5}$$

we can merge the two cases together using the absolute value of the logarithm:

$$A^{(n)}(a,s) = \frac{\Omega(a)}{4M}\frac{dm(s)}{ds}\sqrt{\frac{a}{s}}\left(\exp -\frac{3}{2}\left|\log\frac{a}{s}\right|\right) b_{3/2}^{(n)}\left(\exp -\left|\log\frac{a}{s}\right|\right) \tag{5.6}$$

5.2 Transformation

We now make the following transformation of both the wavefunction and of the position coordinate. The new coordinate is x, the new wavefunction is Ψ:

$$\psi(a,t) = \frac{\Omega(a)}{\sigma(a)}\Psi\left(x(a),t\right), \quad x(a) = \log\frac{a}{a_{\text{in}}}, \quad y(s) = \log\frac{s}{a_{\text{in}}} \tag{5.7}$$

Equation (5.2) now becomes

$$i\frac{\partial}{\partial t}\frac{\Omega(a)}{\sigma(a)}\Psi(x,t) = \int_{a_{\text{in}}}^{a_{\text{out}}}\left(A^{(1)}(a,s)\frac{\Omega(a)}{\sigma(a)}\Psi(x,t) - A^{(2)}(a,s)\frac{\Omega(s)}{\sigma(s)}\Psi(y,t)\right)ds. \tag{5.8}$$

Dividing out $\Omega(a)/\sigma(a)$:

$$i\frac{\partial}{\partial t}\Psi(x,t) = \int_{a_{\text{in}}}^{a_{\text{out}}}\left(A^{(1)}(a,s)\Psi(x,t) - A^{(2)}(a,s)\frac{\Omega(s)\sigma(a)}{\Omega(a)\sigma(s)}\Psi(y,t)\right)ds \tag{5.9}$$

We now take the special density (4.2)

$$\sigma(a) = \sigma_{\text{in}}\sqrt{\frac{a_{\text{in}}}{a}}, \quad \frac{dm}{da} = 2\pi\sigma_{\text{in}}\sqrt{a_{\text{in}}a}. \tag{5.10}$$

Now we can transform the $A^{(n)}$:

$$A^{(n)}(a,s)ds = K^{(n)}(x-y)dy, \quad K^{(n)}(x) = \frac{\omega_0}{4}e^{-x-\frac{3}{2}|x|}b_{3/2}^{(n)}\left(e^{-|x|}\right). \tag{5.11}$$

The resulting $K^{(n)}$ are a function of one variable only. Equation (5.2) now becomes

$$i\frac{\partial}{\partial t}\Psi(x,t) = \int_{-L/2}^{L/2}\left(K^{(1)}(x-y)\Psi(x,t) - K^{(2)}(x-y)e^{x-y}\Psi(y,t)\right)dy. \tag{5.12}$$

For an infinite disk with $a_{\text{in}} \to 0$ and $a_{\text{out}} \to \infty$, the solutions are plane running waves

$$\Psi(x,t) = e^{-i\omega t + ipx} \tag{5.13}$$

with frequency ω and wavenumber p. Substituting the solution in the equation gives the frequency as a function of the wavenumber:

$$\omega(p) = \int_{-\infty}^{\infty}\left(K^{(1)}(-y) - K^{(2)}(-y)e^{ipy-y}\right)dy. \tag{5.14}$$

We can separate out $\omega(0)$ as follows:

$$\begin{aligned}\omega(p) &= \int_{-\infty}^{\infty}\left(K^{(1)}(-y) - K^{(2)}(-y)e^{-y}\right)dy + \\ &\int_{-\infty}^{\infty}\left(K^{(2)}(-y)e^{-y} - K^{(2)}(-y)e^{ipy-y}\right)dy, \\ \omega(p) &= \omega(0) + \int_{-\infty}^{\infty}(1-e^{ipy})K^{(2)}(-y)e^{-y}dy.\end{aligned} \tag{5.15}$$

By working out the integrand, we get

$$\omega(p) = \omega(0) + \int_{-\infty}^{\infty} (1 - \cos(py) - i\sin(py)) \frac{\omega_0}{4} e^{-\frac{3}{2}|y|} b_{3/2}^{(2)} \left(e^{-|y|}\right) dy. \tag{5.16}$$

The sine is odd and all other parts are even, so we can integrate from 0 to get rid of the absolute value:

$$\omega(p) = \omega(0) + \frac{\omega_0}{2} \int_0^{\infty} (1 - \cos(py)) e^{-\frac{3}{2}y} b_{3/2}^{(2)} \left(e^{-y}\right) dy. \tag{5.17}$$

Because the modes with opposite values of p have the same frequency $\omega(p) = \omega(-p)$, we can construct real solutions of the form

$$\Psi(x,t) = \frac{1}{2} e^{-i\omega t} (e^{i(px+\phi)} + e^{-i(px+\phi)}) = e^{-i\omega t} \cos(px + \phi). \tag{5.18}$$

By reversing the transformation (5.7), we obtain

$$\psi(x,t) = e^{-i\omega t - x} \cos(px + \phi). \tag{5.19}$$

5.3 Comparison to the Discrete Disk

In this section, we will compare the continuous wavefunction on the infinite disk (5.19) with the discrete one (4.7). We found both to be damped sinusoids. Since we found for the discrete system that mode n changes sign n times, we may approximate the wavenumber of mode n by

$$p = \kappa_n = \frac{n\pi}{L}. \tag{5.20}$$

We can find the corresponding frequencies $\omega(p)$ by substituting (5.20) into (5.17). The wavenumbers p (blue line) and κ_n (red dots) are drawn in figure (4.3) (b), and the frequencies $\omega(p)$ (blue line) and ω_n (red dots) are drawn in in figure (4.3) (a). We see that the continuous frequencies $\omega(p)$ are a good approximation for the discrete frequencies ω_n for low values of n. The exponential factor in the continuous wavefunction is fixed at the value 1, it is drawn as a blue line in figure (4.3) (c). This is reasonably close to the value of the discrete λ_n.

5.4 Density in the Plane

If the orbits become eccentric, the planar density will not be circularly symmetric like (4.2). In this section we will calculate this density. Consider one mode with modenumber n. The eccentricity is $\epsilon(a,t) = |\psi_n(a,t)| = |\psi_n(a)|$, and the argument of periapsis is $\varpi(a) = \arg(\psi_n(a,t)) = \arg(\psi_n(a,0)) + \omega_n t$. We see that the mode rotates on the secular time scale. Therefore, it is sufficient to calculate the planar density of the mode at $t = 0$ only. For every ring element with mass dm, we average over one orbital period. The contribution to the density is:

$$\frac{\Omega}{2\pi r} \int_0^{2\pi/\Omega} \delta(r - r(t))\delta(\theta - \theta(t)) dt dm. \tag{5.21}$$

The density in the plane is then given by integrating over all ring elements:

$$\sigma(r,\theta) = \frac{\Omega}{2\pi r} \int_{a_{in}}^{a_{out}} \int_0^{2\pi/\Omega} \delta(r - r(t))\delta(\theta - \theta(t)) dt \frac{dm}{da} da. \tag{5.22}$$

We can factor out the polar dependence of r, and use

$$\int_0^{2\pi/\Omega} \delta(r - r(t))\delta(\theta - \theta(t))dt = \delta(r - r(\theta)) \int_0^{2\pi/\Omega} \delta(\theta - \theta(t))dt. \tag{5.23}$$

The integral over the delta function in θ gives $(d\theta/dt)^{-1}$. Therefore, the time average becomes

$$\int_0^{2\pi/\Omega} \delta(r - r(t))\delta(\theta - \theta(t))dt = \delta(r - r(\theta)) \left(\frac{d\theta}{dt}\right)^{-1} \tag{5.24}$$

Since we cannot find $d\theta/dt$ directly, we will be using the eccentric anomaly E from chapter 2. We write

$$\left(\frac{d\theta}{dt}\right)^{-1} = \frac{dt}{dE}\left(\frac{d\theta}{dE}\right)^{-1}. \tag{5.25}$$

By differentiating (2.21), we can find $d\theta/dE$:

$$-\sin(\theta - \varpi)\frac{d\theta}{dE} = \frac{-ra\sin E - (a\cos E - \epsilon a)\epsilon a \sin E}{r^2}, \tag{5.26}$$

$$\frac{d\theta}{dE} = \frac{ra\sin E + (a\cos E - \epsilon a)\epsilon a\sin E}{r^2\sin(\theta - \varpi)}. \tag{5.27}$$

Substituting (2.24) for r in the numerator gives

$$\frac{d\theta}{dE} = \frac{(1-\epsilon^2)a^2\sin E}{r^2\sin(\theta - \varpi)}. \tag{5.28}$$

From figure 2.2, it is clear that $\sin E$ and $\sin(\theta - \varpi)$ always have the same sign. Therefore, their signs cancel out, and we can take the absolute value of both:

$$\frac{d\theta}{dE} = \frac{(1-\epsilon^2)a^2|\sin E|}{r^2|\sin(\theta-\varpi)|} = \frac{(1-\epsilon^2)a^2|\sin E|}{r^2\sqrt{\sin^2(\theta-\varpi)}} = \frac{(1-\epsilon^2)a^2|\sin E|}{r^2\sqrt{1-\cos^2(\theta-\varpi)}}. \tag{5.29}$$

After substituting (2.21) for $\cos(\theta - \varpi)$, the equation simplifies to

$$\frac{d\theta}{dE} = \frac{a}{r}\sqrt{1-\epsilon^2}. \tag{5.30}$$

Using (2.29) and (5.30), we get

$$\sigma(r,\theta) = \frac{r}{2\pi}\int_{a_{\text{in}}}^{a_{\text{out}}} \frac{\delta(r - r(\theta))}{a^2\sqrt{1-\epsilon^2}}\frac{dm}{da}da. \tag{5.31}$$

The integral over the delta function in r gives dr/da, which leads to

$$\sigma(r,\theta) = \frac{r}{2\pi}\int_{a_{\text{in}}}^{a_{\text{out}}} \frac{\delta(r-r(\theta))}{a^2\sqrt{1-\epsilon(a)^2}}\frac{dm}{da}da = \frac{r}{2\pi a(r,\theta)^2\sqrt{1-\epsilon(a(r,\theta))^2}}\frac{dm}{da}\left(\frac{dr(a,\epsilon,\varpi)}{da}\right)^{-1}. \tag{5.32}$$

By the chain rule, we have

$$\frac{dr(a,\epsilon,\varpi)}{da} = \frac{\partial r}{\partial a} + \frac{\partial r}{\partial \epsilon}\frac{d\epsilon}{da} + \frac{\partial r}{\partial \varpi}\frac{d\varpi}{da}. \tag{5.33}$$

By substituting the polar form of r (2.2), we obtain

$$\frac{dr}{da} = \left(1 - \frac{a\epsilon' \cos(\theta - \varpi) + a\epsilon\varpi' \sin(\theta - \varpi)}{1 + \epsilon \cos(\theta - \varpi)} - \frac{2a\epsilon\epsilon'}{1 - \epsilon^2}\right) \frac{r}{a}. \tag{5.34}$$

Here an apostrophe behind a variable indicates its derivative with respect to a. We now replace the orbital parameters in terms of the mode function, by

$$\mathrm{Re}(\psi e^{-i\theta}) = \epsilon \cos(\theta - \varpi), \quad \mathrm{Re}(\psi' e^{-i\theta}) = \epsilon' \cos(\theta - \varpi) + \epsilon\varpi' \sin(\theta - \varpi), \tag{5.35}$$

and obtain

$$\frac{dr}{da} = \left(1 - \frac{r\,\mathrm{Re}(\psi' e^{-i\theta}) + a|\psi^2|'}{1 - |\psi^2|}\right) \frac{r}{a}. \tag{5.36}$$

By substituting this in the formula for the density (5.32), we obtain

$$\sigma(r, \theta, t) = \frac{m'(a)}{2\pi a} \frac{\sqrt{1 - |\psi|^2}}{1 - \mathrm{Re}\left[\left(re^{-i\theta} + 2a\psi^*\right)\psi'\right] - |\psi|^2}. \tag{5.37}$$

For a stationary mode $\psi(a, t) = e^{i\omega_n t}\psi_n(a)$ with real ψ_n, we have

$$\epsilon(a) = |\psi_n(a)|, \quad \varpi(a) = \begin{cases} 0, & \psi_n(a) > 0 \\ \pi, & \psi_n(a) < 0 \end{cases}. \tag{5.38}$$

The density of the mode is then

$$\sigma(r, \theta, t) = \frac{m'(a)}{2\pi a} \frac{\sqrt{1 - \psi^2}}{1 - \psi' r \cos\theta - 2a\psi\psi' - \psi^2}. \tag{5.39}$$

We now need to find a in terms of r and θ. We can find $a(r, \theta)$ from formula (2.2), provided that $a(r, \theta)$ is a proper function. For small values of ϵ, this is indeed the case. $a(r, \theta)$ is drawn in figure 5.1 for the mode $n = 3$ and with a maximum eccentricity of $\epsilon = 0.2$, for $\theta = 0$ and $\theta = \pi$.

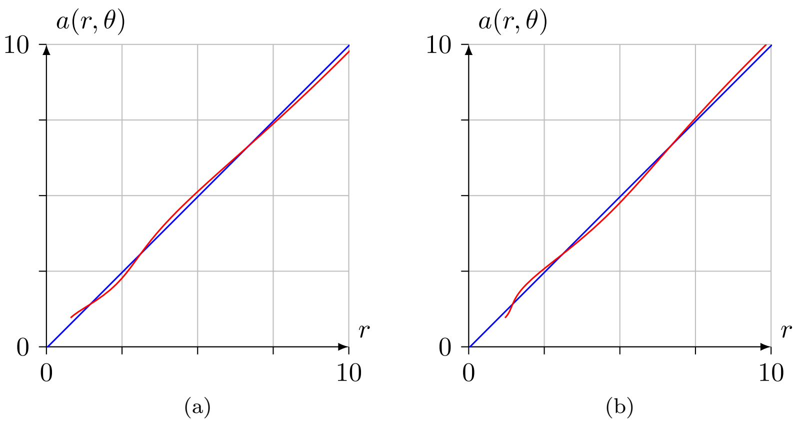


Figure 5.1: $a(r, \theta)$ in red for $\theta = 0$ and $\theta = \pi$. The blue line is the line $a = r$. The axes have units of 1 AU.

We can now plot the density of a mode using Mathematica. The code can be found in appendix D. In figure 5.2, we plot the density for the modes with modenumbers $n = 3$ and $n = 10$.

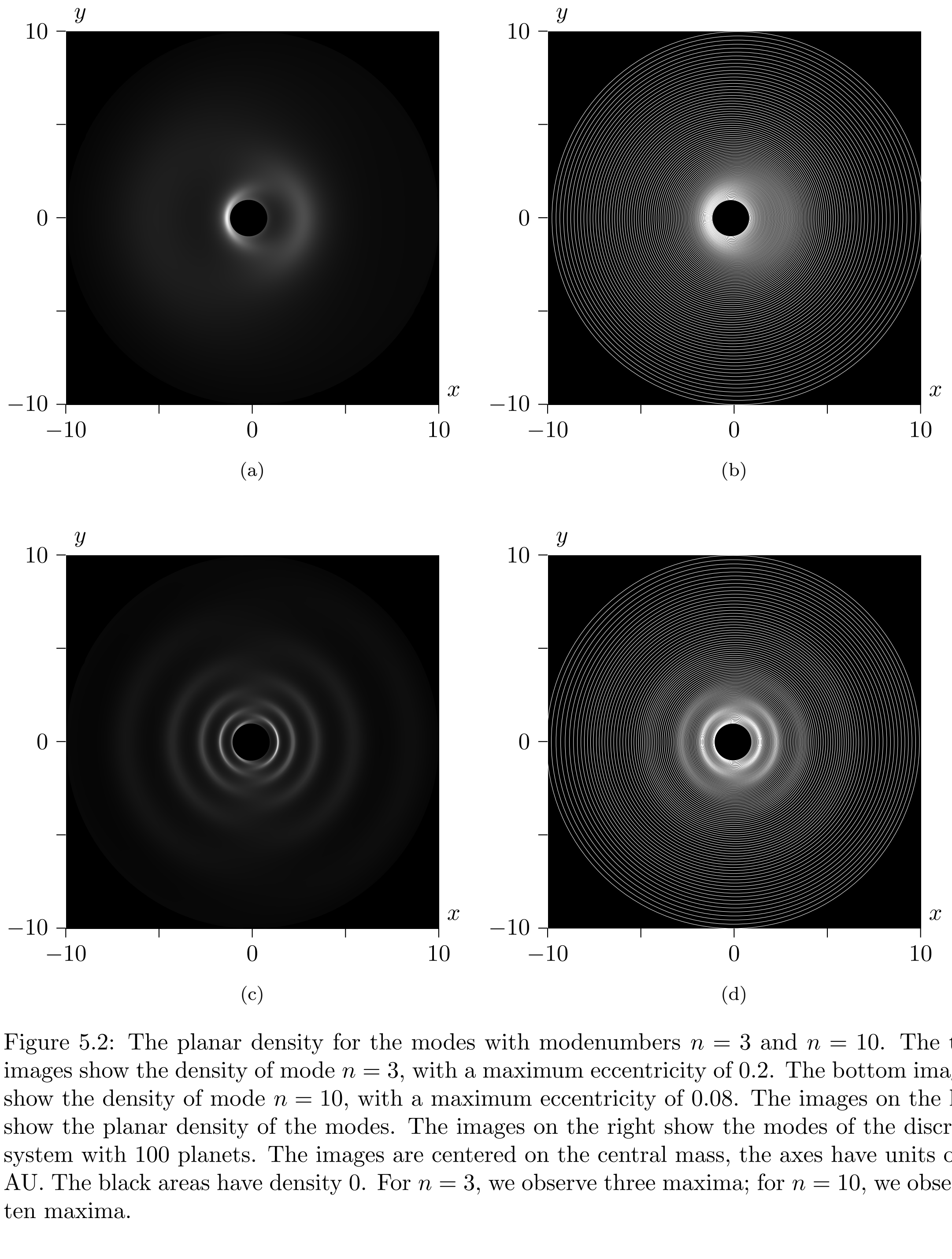


Figure 5.2: The planar density for the modes with modenumbers $n = 3$ and $n = 10$. The top images show the density of mode $n = 3$, with a maximum eccentricity of 0.2. The bottom images show the density of mode $n = 10$, with a maximum eccentricity of 0.08. The images on the left show the planar density of the modes. The images on the right show the modes of the discrete system with 100 planets. The images are centered on the central mass, the axes have units of 1 AU. The black areas have density 0. For $n = 3$, we observe three maxima; for $n = 10$, we observe ten maxima.

Chapter 6

Conclusions

We studied the ecentricity dynamics of a Keplerian disk with a special density (4.2). Recently, Batygin studied the inclination dynamics using this density [1]. He derived a wave equation for the inclination in the complex wave form $\psi = Ie^{i\Omega}$. He found that the eigenvalues of the modes increases quadratically with the mode number. However, in his numerical calculation, the argument of the ascending node increases in time, with a rate (frequency) proportional to the mode number, i.e. $\Omega_n \propto nt$. Thus his numerical results actually indicate that the eigenvalues are linear in the mode number. We studied the dynamics of the eccentricity of the orbits. We introduce the complex number $\psi = \epsilon e^{i\varpi}$ to describe the orbits of the planets. We did our numerical calculations in Wolfram Mathematica. The equation of motion for the system is given by (3.1). The governing matrix is not symmetric, yet we found the eigenvalues to be real and unique. The eigenspectrum is also nearly equidistant. The eigenvector of the lowest state can be chosen to be positive in all elements. The eigenmodes are well approximated by damped sinusoids, given by formula (4.7). If we label the modes with the ground state $n = 0$, then the vector of mode n changes sign n times.

An astrophysical disk consists of many bodies, thus we also studied the continuum limit of the number of planets $N \to \infty$ on a finite disk, then the disk size $L \to \infty$. This limit has an analytical solution. We compared the exact finite solution with the analytical solution. The boundary conditions for the continuous system are taken from the discrete system. For $n << N$, the modes can be approximated by the continuum limit. The eigenmodes for the continuum limit agree with the eigenmodes of the discrete system. In section 5.4, we derived formula (5.39) for the density of the disk. In figure 5.2, we compare the density of the continuum limit to the density of the discrete system. We see that mode number n has n maxima in the density.

In Batygin's paper, he claimed the modes to be cosines. However, he did his calculations using a very thin disk. If we apply our method to a thin disk, the damping factor in the modes becomes negligible. We then also find cosine modes. We also found a linear eigenspectrum, which agrees with Batygin's numerical results. Because the eigenvalues do not increase quadratically with the mode number, his description of the disk with a Schrödinger equation cannot be correct.

Bibliography

[1] K. Batygin, "Schrödinger evolution of self-gravitating disks," *arXiv:1803.01258v1*, 2018.

[2] T. Montmerle et al., *Earth, Moon, and Planets*, ch. 3. Springer, 2006.

[3] ALMA (ESO/NAOJ/NRAO)/Dong et al., "ALMA image of the 0.87 mm continuum emission from the MWC 758 disk." `https://www.almaobservatory.org/wp-content/uploads/2018/06/f4.png`, 2018. Accessed on September 5, 2020.

[4] R. A. Adams and C. Essex, *Calculus: A Complete Course*, ch. 11, pp. 660–661. Pearson, 9 ed., 2018.

[5] C. D. Murray, *Solar System Dynamics*. Cambridge University Press, 2000.

[6] NASA, "Planetary Fact Sheet." `https://nssdc.gsfc.nasa.gov/planetary/factsheet/`, Oct 2019. Accessed in February 2020.

Appendix A

Mathematica Notebook Jupiter and Saturn

```
In[•]:= mu = 1.327124400189 / 149 597 870 700^3 * (31 557 600 * 1.00001742096)^2 * 10^20;
m1 = 9.54786 * 10^ -4;
a1 = 5.202545;
n1 = Sqrt[mu / a1^3];
e10 = 0.0474622;
varpi10 = 13.983865 * Degree;
I10 = 1.30667 * Degree;
Omega10 = 100.0381 * Degree;

m2 = 2.85837 * 10^ -4;
a2 = 9.554841;
n2 = Sqrt[mu / a2^3];
e20 = 0.0575481;
varpi20 = 88.719425 * Degree;
I20 = 2.48795 * Degree;
Omega20 = 113.1334 * Degree;

In[•]:= b[i_, a_] := 1 / Pi * Integrate[Cos[i * x] / (1 - 2 * a * Cos[x] + a^2)^(3 / 2), {x, 0, 2 * Pi}];

In[•]:= A11 = n1 * 1 / 4 * m2 / (1 + m1) * a1 / a2 * a1 / a2 * b[1, a1 / a2];
A12 = -n1 * 1 / 4 * m2 / (1 + m1) * a1 / a2 * a1 / a2 * b[2, a1 / a2];
A21 = -n2 * 1 / 4 * m1 / (1 + m2) * a1 / a2 * 1 * b[2, a1 / a2];
A22 = n2 * 1 / 4 * m1 / (1 + m2) * a1 / a2 * 1 * b[1, a1 / a2];

In[•]:= (AA = {
     {0, 0, A11, A12},
     {0, 0, A21, A22},
     {-A11, -A12, 0, 0},
     {-A21, -A22, 0, 0}} );

In[•]:= lambda = Eigenvalues[AA];

In[•]:= u = Eigenvectors[AA];

In[•]:= A = {
    {A11, A12},
    {A21, A22}};

In[•]:= psi10 = e10 * Exp[I * varpi10];
psi20 = e20 * Exp[I * varpi20];
psi0 = {psi10, psi20};

In[•]:= psi = MatrixExp[I * A * t, psi0];

In[•]:= Plot[{Abs[Part[psi, 1]], Abs[Part[psi, 2]]},
  {t, -100 000, 100 000}, PlotRange → {0, Automatic}]
Plot[{Arg[Part[psi, 1]], Arg[Part[psi, 2]]}, {t, -100 000, 100 000}]
```

In[•]:= $l1 = \frac{A11 + A22}{2} - \sqrt{\left(\frac{A11 - A22}{2}\right)^2 + A21 * A12};$

$l2 = \frac{A11 + A22}{2} + \sqrt{\left(\frac{A11 - A22}{2}\right)^2 + A21 * A12};$

```
In[•]:= c1 = -(e20 * Exp[I * varpi20])/(l2 - l1) + (l2 - A11)/((l2 - l1) A12) e10 * Exp[I * varpi10];
        c2 = (e20 * Exp[I * varpi20])/(l2 - l1) - (l1 - A11)/((l2 - l1) A12) e10 * Exp[I * varpi10];

In[•]:= u1 = {A12, l1 - A11};
        u2 = {A12, l2 - A11};
        f = c1 * u1 * Exp[I * l1 * t] + c2 * u2 * Exp[I * l2 * t];

In[•]:= ParametricPlot[{Re[Part[f, 1]], Im[Part[f, 1]]}, {t, 0, 1122000}]
        ParametricPlot[{Re[Part[f, 2]], Im[Part[f, 2]]}, {t, 0, 1122000}]

In[•]:= t = Range[30000, 50000, 1000];

In[•]:= orbits1 = (a1 (1 - Abs[Part[psi, 1]]^2))/(1 + Re[Exp[I theta] * Part[psi, 1]]);
        orbits2 = (a2 (1 - Abs[Part[psi, 2]]^2))/(1 + Re[Exp[I theta] * Part[psi, 2]]);
        orbits = Join[orbits1, orbits2];
        PolarPlot[orbits, {theta, 0, 2 * Pi}]
```

Appendix B

Mathematica Notebook Solar System

```
In[•]:= mc = 1.9885 * 10^30; (*Mass of sun*)
        G = 6.6740831 * 10^-11; (*Gravitational constant*)
        AU = 149 597 870 700; (*Astronomical unit*)
        year = 31 557 600; (*Year*)

In[•]:= nn = 8; (*Number of planets*)
        names =
          {"Mercury", "Venus", "Earth", "Mars", "Jupiter", "Saturn", "Uranus", "Neptune"};
        m = {.33011, 4.8675, 5.9723, 0.64171, 1898.19, 568.34, 86.813, 102.413} * 10^24;
        (*Planet masses*)
        (*m={0,0,0,0,1898.19,568.34,0,0}*10^24;*) (*Only Jupiter and Saturn*)
        (*m={.33011,4.8675,5.9723,0.64171,1898.19,568.34,0,0}*10^24;*)
        (*All planets except Uranus and Neptune*)
        a = {0.38709893, 0.72333199, 1.00000011, 1.52366231, 5.20336301,
             9.53707032, 19.19126393, 30.06896348} * AU; (*Semi-major axis*)
        e0 = {0.20563069, 0.00677323, 0.01671022, 0.09341233, 0.04839266,
            0.05415060, 0.04716771, 0.00858587}; (*Eccentricity*)
        varpi0 = {77.45645, 131.53298, 102.94719, 336.04084, 14.75385, 92.43194,
             170.96424, 44.97135} * Degree; (*Argument of Periapsis*)
        n = Sqrt[G * mc / a^3];

In[•]:= b[i_, a_] := If[i == 1, 3 a * Hypergeometric2F1[3 / 2, 5 / 2, 2, a^2],
            15 a^2 / 4 * Hypergeometric2F1[3 / 2, 7 / 2, 3, a^2]];

In[•]:= Ajk[j_, k_] := If[k == j,
            Part[n, j] / 4 * (Sum[Part[m, i] / (mc + Part[m, j]) *
                  Part[a, i] / Part[a, j] * b[1, Part[a, i] / Part[a, j]], {i, 1, j - 1}] +
                Sum[Part[m, i] / (mc + Part[m, j]) * (Part[a, j] / Part[a, i])^2 *
                  b[1, Part[a, j] / Part[a, i]], {i, j + 1, nn}]),
            -Part[n, j] / 4 * Part[m, k] / (mc + Part[m, j]) * If[k < j,
              Part[a, k] / Part[a, j] * b[2, Part[a, k] / Part[a, j]],
              (Part[a, j] / Part[a, k])^2 * b[2, Part[a, j] / Part[a, k]]]];

In[•]:= A = Array[Ajk, {nn, nn}];

In[•]:= periods = 1 / Eigenvalues[A] / year / 10 000 * 2 * Pi;
        absev = Abs[Transpose[Eigenvectors[A]]];
        For[j = 1, j ≤ nn, j++, Print[periods[[Ordering[absev[[j]], -1]]]]]

In[•]:= psi0 = e0 * Exp[I * varpi0];
        psi[t_] := MatrixExp[I * A * t, psi0];

In[•]:= For[i = 1, i ≤ nn, i++, ParametricPlot[{Re[Part[psi[t], i]], Im[Part[psi[t], i]]},
            {t, 0, 1 200 000 * year}, PlotLabel → Part[names, i]] // Print]
```

Appendix C

Mathematica Notebook Modes

```
In[•]:= mc = 1.9885 * 10^30;(*Mass of sun*)
        mj = 1898.19 * 10^24;(*Mass of Jupiter*)
        G = 6.6740831 * 10^-11;(*Gravitational constant*)
        AU = 149 597 870 700;(*Astronomical unit*)
        year = 31 557 600;

In[•]:= b[i_, a_] := If[i == 1, 3 a * Hypergeometric2F1[3/2, 5/2, 2, a^2],
           15 a^2/4 * Hypergeometric2F1[3/2, 7/2, 3, a^2]];

In[•]:= nn = 101;
        amax = 100 * AU;
        amin = 1 * AU;
        beta = (amax/amin)^((1)/(nn - 1)) - 1;
        a = Table[amin * (amax/amin)^((j - 1)/(nn - 1)), {j, 1, nn}];
        da = beta * a;
        gamma = 3/4/Pi * mj/(amax^(3/2) - amin^(3/2));
        sigma[a_] := gamma * Sqrt[1/a];
        sigma0 = sigma[a[[1]]];
        m = Table[2 * Pi * a[[i]] * da[[i]] * sigma[a[[i]]], {i, nn}] // N;
        e0 = Table[.2, {i, nn}];
        varpi0 = Table[0, {i, nn}];
        n = Sqrt[G * mc/a^3];

In[•]:= Ajk[j_, k_] := If[k == j, n[[j]]/4 *
             (Sum[m[[i]]/(mc + m[[j]]) * a[[i]]/a[[j]] * b[1, a[[i]]/a[[j]]], {i, 1, j - 1}] +
               Sum[m[[i]]/(mc + m[[j]]) * (a[[j]]/a[[i]])^2 * b[1, a[[j]]/a[[i]]],
                {i, j + 1, nn}]), -n[[j]]/4 *
            m[[k]]/(mc + m[[j]]) * If[k < j, a[[k]]/a[[j]] * b[2, a[[k]]/a[[j]]],
             (a[[j]]/a[[k]])^2 * b[2, a[[j]]/a[[k]]]]];

In[•]:= A = Array[Ajk, {nn, nn}];

In[•]:= psi0 = e0 * Exp[I * varpi0];
        psi[t_] := MatrixExp[I * A * t, psi0];

In[•]:= eigIA = Eigenvalues[I * A];
        eigvIA = Eigenvectors[I * A];
        psii[t_, i_] := Exp[eigIA[[i]] * t] * eigvIA[[i]];

In[•]:= (*Check if mode number n has n sign changes. Print
         the mode numbers for which this is not the case.*)
        For[i = 0, i < nn, i++,
         s = Sign[Re[psii[0, nn - i]]][[1]];
         changes = 0;
         For[j = 2, j ≤ nn, j++,
           If[s != Sign[Re[psii[0, nn - i]]][[j]],
            changes++;
            s *= -1;
           ]
          ] ×
          If[i ≠ changes,
           Print[i]
          ]
        ]

In[•]:= eigA = Eigenvalues[A];
```

```
In[•]:= fits = {};
For[i = nn, i > 0, i--,
 sign = Sign[Re[psii[0, i][[1]]]];
 data = Table[{Log[a[[j]] / AU], sign * Re[psii[0, i]][[j]]}, {j, nn}];
 func =
  alpha * Exp[-lambda * q] * Cos[phi + ((nn - i) * Pi + phi2 - phi) / Log[amax / amin] * q];
 fit = FindFit[data, {func, {-Pi ≤ phi ≤ Pi, -Pi ≤ phi2 ≤ Pi, alpha ≥ 0
      (*lambda==1*) (*mu==(nn-i-.5)*Pi/Log[amax/amin]*)}},
    {{alpha, .25}, {lambda, 1}, {phi, 0}, {phi2, 0}}, q(*,MaxIterations→10000*)];
 table = {alpha, lambda, phi2, phi} /. fit;
 AppendTo[fits, table];
]

In[•]:= alphas = Table[fits[[j]][[1]], {j, nn}];
lambdas = Table[fits[[j]][[2]], {j, nn}];
mus =
  Table[((j - 1) * Pi + fits[[j]][[3]] - fits[[j]][[4]]) / Log[amax / amin], {j, nn}];
phis = Table[fits[[j]][[4]], {j, nn}];

In[•]:= Ke[i_, q_] :=
  Pi * sigma[amin] / 2 * Sqrt[G * amin / mc] * Exp[-q - 3 * Abs[q] / 2] * b[i, Exp[-Abs[q]]];

In[•]:= omega[p_] :=
  NIntegrate[-Ke[1, r] + Ke[2, r] * Exp[I * p * r] - Ke[1, -r] + Ke[2, -r] * Exp[I * p * -r],
   {r, .01, 1}, PrecisionGoal → 10, MaxRecursion → 9, Method → "RiemannRule"];

In[•]:= c0 =
  NIntegrate[Ke[1, -r] - Ke[2, -r] * Exp[I * 0 * r - r], {r, -1, -.005}, PrecisionGoal → 10,
    MaxRecursion → 9] + NIntegrate[Ke[1, -r] - Ke[2, -r] * Exp[I * 0 * r - r],
    {r, .005, 1}, PrecisionGoal → 10, MaxRecursion → 9];

In[•]:= omega[p_] :=
 c0 + 1 / 2 * omega0 * NIntegrate[Exp[-3 * Abs[y] / 2] * b[2, Exp[-Abs[y]]] * (1 - Cos[p * y]),
    {y, 0, Infinity}, Method → "Trapezoidal"]

In[•]:= omega0 = 2 * Pi * sigma[a[[1]]] * Sqrt[G * a[[1]] / mc];
number = 10;
scale = omega0;
scale2 = 1;
size = Large;
aspect = 1.5;
```

```
In[•]:= g1 = {};
g2 = {};
p1 = {};
For[i = nn, i > nn - number, i--,
  AppendTo[g1, Table[{Log[a[[j]] / AU], eigA[[i]] / scale +
      scale2 * Sign[Re[psii[0, i]][[1]]] * Re[psii[0, i]][[j]]}, {j, nn}]]];
For[i = nn, i > nn - number, i--,
  j = 1 + nn - i;
  fit = Plot[eigA[[i]] / scale + scale2 * alphas[[j]] *
      Exp[-lambdas[[j]] * q] * Cos[mus[[j]] * q + phis[[j]]], {q, 0, Log[amax / amin]},
    PlotStyle → {Thickness[0.005], Black}, PlotPoints → 200, PlotRange → All];
  AppendTo[g2, fit]];
For[i = nn, i > nn - number, i--,
  AppendTo[p1, {{0, eigA[[i]] / scale}, {amax / AU, eigA[[i]] / scale}}]];
plotlines = ListLinePlot[p1, PlotStyle → Table[{Gray, Thickness[.005]}, {number}]];
plot1 = ListPlot[g1, PlotMarkers → {Graphics@{Disk[]}, 0.015},
   ImageSize → size, PlotRange → {{0, Automatic}, {0, Automatic}},
   AspectRatio → aspect, AxesLabel → {"q", "E"}, PlotStyle -> Thick];
Show[plotlines, plot1, g2, PlotRange -> {{Log[amin / AU], Log[amax / AU]},
   {0, Pi * 8 / 2 / Log[amax / amin]}}, ImageSize → Large, Axes → False,
 PlotRangePadding → None, PlotRange → Automatic, AspectRatio → aspect]
```

```
In[•]:= scale3 = 1;
```

```
In[•]:= p1 = ListPlot[Table[{k - 1, eigA[[nn - k + 1]] / scale3}, {k, nn}], PlotStyle → {Red, Large},
   PlotRange → {{0, Automatic}, {-6 * 10^-10, Automatic}}, PlotStyle → {Thick, Red},
   PlotMarkers → {Graphics@{Red, Disk[]}, 0.01}];
p2 = Plot[omega[n * Pi / Log[amax / amin]] / scale3,
   {n, 0, nn - 1}, PlotStyle → {Thickness[0.005], Blue}, PlotPoints -> 200];
Show[{p1, p2}, AspectRatio → 1, ImageSize → Large, Axes → False,
 PlotRangePadding → None, PlotRange → Automatic]
```

```
In[•]:= p1 = ListPlot[Table[{k - 1, mus[[k]]}, {k, nn}],
   PlotRange → {{0, Automatic}, {0, Automatic}}, Joined → {False}, PlotStyle → Red];
p2 = ListPlot[{{0, 0}, {nn - 1, (nn - 1) * Pi / Log[amax / amin]}},
   Joined → {True}, PlotStyle → {Thickness[.005], Blue}];
Show[p1, p2, PlotRange -> {{0, nn - 1}, {0, (nn - 1) * Pi / Log[amax / amin]}},
 AspectRatio → 1, Axes → False, PlotRangePadding → None]
```

```
In[•]:= p1 = ListPlot[Table[{k - 1, lambdas[[k]]}, {k, nn}],
   PlotRange → {{0, Automatic}, {0, Automatic}}, Joined → {False}, PlotStyle → Red];
p2 = ListPlot[{{0, 1}, {nn - 1, 1}}, Joined → {True},
   PlotStyle → {Thickness[.005], Blue}];
Show[p1, p2, PlotRange -> {{0, nn - 1}, {0, 2}}, AspectRatio → 1,
 Axes → False, PlotRangePadding → None]
```

```
In[•]:= p1 = ListPlot[Table[{k - 1, Mod[phis[[k]], 2 * Pi]}, {k, nn}], PlotRange →
    {{0, Automatic}, {0, 2 * Pi}}, PlotMarkers → {Graphics@{Red, Disk[]}, 0.01}];
Show[p1, PlotRange → {{0, nn - 1}, {0, Pi}}, AspectRatio → 1,
 Axes → False, PlotRangePadding → None]
```

Appendix D

Mathematica Notebook Density

```
In[•]:= AU = 1149597870700; (*astronomical unit in m*)
nn = 100; (*amount of planets*)
amax = 10 * AU;
amin = 1 * AU;
mj = 1898.19 * 10^24; (*mass of Jupiter in kg*)
gamma = 3 / 4 / Pi * mj / (amax^(3 / 2) - amin^(3 / 2));
sigma0 = gamma * Sqrt[1 / amin]; (*disk surface density at r = amin*)
L = Log[amax / amin];

In[•]:= norm = .08;
psi[n_, a_] := norm amin/a * Cos[n * Pi/L * Log[a/amin] + n * Pi/nn] // N (*mode n*)
psip[n_, a_] :=
 norm -amin/a^2 * (Cos[n * Pi/L Log[a/amin] + n * Pi/nn] + n * Pi/L Sin[n * Pi/L * Log[a/amin] + n * Pi/nn]) //
  N (*d/da of mode n*)

In[•]:= density[a_, theta_, n_] := (sigma0 * Sqrt[amin * a])/(a Sqrt[1 - psi[n, a]^2]) 1/(1 - (a*psip[n,a] Cos[theta])/(1+psi[n,a] Cos[theta]) - (a*2 psi[n,a] psip[n,a])/(1-psi[n,a]^2)) // N

In[•]:= RA[a_, theta_, n_] := ((1 - psi[n, a]^2) a)/(1 + psi[n, a] Cos[theta])

In[•]:= AR[R_, theta_, n_] := Module[{a, ai, x},
  ai = FindRoot[RA[a, theta, n] == R, {a, amin, amax}, MaxIterations → 100];
  a /. ai
 ]

In[•]:= n = 3; (*mode number*)

In[•]:= Do[
 p1 = ParametricPlot[{a, a}, {a, 0, amax / AU},
   (*GridLines→{Range[10]*1, Range[10]*1},*)PlotStyle → {Blue, Thickness[.005]}];
 p2 = ParametricPlot[{RA[a * AU, theta, n] / AU, a},
   {a, amin / AU, amax / AU}, PlotStyle → {Red, Thickness[.005]}];
 (*p3=Plot[AR[AU*r,theta,n]/AU,{r,RA[amin,theta,n]/AU,RA[amax,theta,n]/AU}];*)
 Print@Show[p1, p2, PlotRange → {{0, 10}, {0, 10}},
   AspectRatio → 1, Axes → False, PlotRangePadding → None, ImageSize → Large]
 , {theta, {0, Pi}}]
```

```
In[•]:= rstep = .05 * AU;
        rscale = 1.05;
        thetastep = 2 * Pi / 100;
        points = {};
        points2 = {};
        values = {};
        For[r = 1.01 * amax, r ≥ amin / 2, r /= rscale,
         For[theta = 0, theta < 2 * Pi, theta += thetastep,
          x = r * Cos[theta] / AU;
          y = r * Sin[theta] / AU;
          If[RA[amin, theta, n] < r,
           a = AR[r, theta, n];
           d = density[a, theta, n];
           AppendTo[points, {x, y}];
           AppendTo[values, {x, y, d}];
          ]
         ]
        ]

In[•]:= elmin = Ellipsoid[{-psi[n, amin], 0}, {amin / AU, amin Sqrt[1 - psi[n, amin]^2] / AU}];
        elmax = RegionDifference[Rectangle[{-amax / AU, -amax / AU}, {amax / AU, amax / AU}],
          Disk[{-psi[n, amax], 0}, {amax / AU, amax Sqrt[1 - psi[n, amax]^2] / AU}]];

In[•]:= grayScale = Blend[{GrayLevel[.02], White}, #1] &;

In[•]:= size = 500;
        p1 = Graphics[elmin];
        p0 = DensityPlot[0, {x, y} ∈ elmax, PlotPoints → 100, ColorFunction → GrayLevel];
        p2 = ListDensityPlot[values, ColorFunction → grayScale, PlotRange → All];
        Show[p2, p0, p1, AspectRatio → 1, ImageSize → size, Axes → False,
         PlotRangePadding → None, Frame → None, PlotRange → {{-10, 10}, {-10, 10}}]
```