BSc Thesis Applied Mathematics

Large deviations analysis for the log-normal distribution

Grote afwijkingen analyse voor lognormaal verdeelde toevalsvariabelen

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Abstract

We inspect the behavior of the probability that a weighted sum of random variables with log-normal tails is greater than its expected value. Under the right conditions for the weights and the variance being set to 1; we were able to bound a suitable transformation of this probability with the upper bound being a fixed factor of $\sqrt{e}$ above the lower bound. Beyond this, we analyse the conditions on the weights and determine a method for letting the weights be random and give an example.

We end off by extending our result to general variance, where we see that the deviation between the lower and upper bound as well as the domain for the result are dependant on the variance.

Contents

1 Introduction

Large Deviations Theory is a topic in mathematics that deals with the likelihood of extreme events. The popularity of this topic has steadily increased in the past century with the rise of the financial market and along with that, insurance companies. These companies would like to receive more premiums than claims. Premiums are the income, which are set. However, the claims are randomly distributed, possibly leading to a situation where an insurance company could become bankrupt as a result of too many incoming claims or claims that are too large. Insurance companies want enough clients so that the chance of bankruptcy is small. Consequently, the interest in the study of Large Deviations Theory grew.

The problem sketched above is inherently different from the Law of Large Numbers [4, p. 234-235] and the Central Limit Theorem [4, p. 172]. To elaborate upon this, the mathematical representation of these two theorems is given by

$$\lim_{n\to\infty} \mathbb{P}\left(S_n \geq \mu + a\right) = 0, \tag{1}$$

$$\mathbb{P}\left(S_n \geq \mu + \frac{\sigma a}{\sqrt{n}}\right) \approx 1 - \Phi\left(a\right). \tag{2}$$

where $\{X_j\}_{j=1}^n$ is an i.i.d. sequence of random variables and $S_n = \frac{\sum_{j=1}^n X_j}{n}$; $\mu = \mathbb{E}[X_1]$; $\sigma^2 = var(X_1)$; $a > 0$; and $\Phi(x)$ is the CDF for the normal distribution.

We see that the Law of Large Numbers implies that the probability of any deviation of the sample mean from the mean tends to 0. On the other hand, the Central Limit Theorem tells us that the standardized sample mean approaches the standard normal distribution and we can approximate the probability of deviations in this manner. We see that neither of these two methods can be used for the above insurance company problem.

As we shall see, our problem corresponds to studying the decay of $\mathbb{P}\left(S_n \geq \mu + a\right)$. In the standard case of Cramér [3], [6]; the following solution was found:

$$\mathbb{P}\left(S_n \geq \mu + a\right) \approx \exp\left(-nI(\mu + a)\right); \tag{3}$$

where I is the rate function and we have that $I \geq 0$ and $I(\mu) = 0$. We see that that the probability goes to 0, for $a > 0$. We can deduce that Large Deviations Theory shows the development towards the result of the Law of Large Numbers; in the sense that the probability of a deviation goes to 0 as $n \to \infty$. We will now move on to the origin of this solution.

At first, a solution for the Large Deviations problem was given by Harald Cramér [3]; however, we shall be working with the modern formulation of his work [6]. Let $\{X_j\}_{j=1}^n$ be a sequence of i.i.d. random variables with an existing expected value; $m := \mathbb{E}[X_1] < \infty$ and let $S_n = \frac{\sum_{j=1}^n X_j}{n}$ be the sample mean. We define light-tailed distributions here as distributions that have a finite moment generating function; i.e. $M(t) = \mathbb{E}[\exp(X_1 t)]$ exists for some $t > 0$. Cramér's Theorem tells us that light-tailed distributions have the property that deviations from the mean occur with exponentially decreasing probability. That is,

$$\lim_{n\to\infty} \frac{\log\left(\mathbb{P}\left(S_n \geq x\right)\right)}{n} = -\sup_{t>0}\left(tx - \log(M(t))\right), \tag{4}$$

where the right-hand side corresponds to the rate function. Taking the inverse of the transformation on the probability shows that the probability tends to 0 exponentially.

$$\lim_{n\to\infty} \frac{\log\left(\mathbb{P}\left(S_n \geq x\right)\right)}{n} = -\sup_{t>0}\left(tx - \log(M(t))\right); \tag{5}$$

$$\Rightarrow$$

$$\frac{\log\left(\mathbb{P}\left(S_n \geq x\right)\right)}{n} \approx -\sup_{t>0}\left(tx - \log(M(t))\right); \tag{6}$$

$$\mathbb{P}\left(S_n \geq x\right) \approx \exp\left(-n\sup_{t>0}\left(tx - \log(M(t))\right)\right). \tag{7}$$

However, as stated above, this theorem only provides a solution for light-tailed distribution; whilst many problems revolve around heavy-tailed distributions.
The problem for heavy-tailed distributions is that the moment generating function is infinite for all positive t. But because all tail events can occur with more probability, we can consider the possibility that just one observation, X_j, causes the sample mean to be greater than the expected value. The methods developed in this thesis are based on this idea and gave rise to the first steps of both the lower and upper bound for the probability.

In a paper by Gantert et al [5], a Large Deviations Theorem was set up for weighted sums of stretched exponential variables, where the weighted sum corresponds to $\bar{S}_n = \sum_{j=1}^{n} a_j(n)X_j$ and $a_j(n)$ is an array of positive weights satisfying Assumptions 3.1. Previously, we had that $a_j(n) = \frac{1}{n}$ for $j \leq n$, in the case of Cramér. The stretched exponential distribution is a heavy-tailed distribution; thus the moment generating function is infinite for all $t > 0$. Therefore, this is a solution to the large deviations problem for heavy-tailed distributions. However, the knowledge of Large Deviations Theory for this category of distributions is still quite limited.
We are going to analyse the conditions of the theorem and proof by Gantert et al [5]. After, we will create a Large Deviations Theorem for log-normal random variables. We would like to do this to further advance Large Deviations Theory, but also because the log-normal distribution satisfies certain conditions that the stretched exponential distribution does as well. Besides this, the log-normal distribution arises in many areas; including log-returns on stocks.

2 Preliminaries

Throughout this paper, we work with slowly-varying functions and their properties. For this reason we dedicated this section to explaining what they entail; to avoid confusion. We will start by giving the definition of a slowly-varying function.

Definition 2.1. *(Slowly-varying function)*
We say a function $g : (0, \infty) \to (0, \infty)$ *is slowly-varying at infinity if it satisfies the following property:*

$$\lim_{x\to\infty} \frac{g(\alpha x)}{g(x)} = 1, \ \forall \alpha > 0. \tag{8}$$

Remark 2.1. *If* $f(x)$ *is a polynomial of finite order, then* $\ln(f(x))$ *is a slowly-varying function*

Next, we shall introduce properties of these functions that we shall use later on.

Proposition 2.1. *Given a slowly-varying function* $g(x) : (0, \infty) \to (0, \infty)$*, we have that* g *satisfies the following:*

1. $\lim_{x\to\infty} \frac{\ln(g(x))}{\ln(x)} = 0$,
2. $\forall \alpha > 0, \ x^\alpha g(x) \to \infty$ *and* $x^{-\alpha} g(x) \to 0$,
3. *for* $\gamma \in \mathbb{R}$, $g(x)^\gamma$ *is also slowly-varying,*
4. $\lim_{x\to\infty} \frac{\ln(g(x)x)}{\ln(x)} = 1$.

Proof. The first three properties were already used in the paper by Gantert et al [5] and proven in [1]. The last property was not given, but is a direct consequence of the property of logarithms:

$$\log(ab) = \log(a) + \log(b). \tag{9}$$

The last property can be proven as follows:

$$\begin{aligned} \lim_{n\to\infty} \frac{\ln(g(x)x)}{\ln(x)} &= \lim_{n\to\infty} \frac{\ln(g(x)) + \ln(x)}{\ln(x)}, &(10)\\ &= \lim_{n\to\infty} \frac{\ln(g(x))}{\ln(x)} + \frac{\ln(x)}{\ln(x)}, &(11)\\ &= 1. &(12) \end{aligned}$$

□

Remark 2.2. *Note that in the last step of the proof of the fourth property, the first property of Proposition 2.1 was applied.*

3 Stretched exponential random variables

The stretched exponential distribution is found by stretching out the exponential distribution. This can be seen by comparing the density functions. Where normally the exponential density function is given by $\lambda \exp(-\lambda t)$, the density function of the stretched exponential distribution is given by $\lambda \exp(-\lambda t^r)$, for $r \in (0, 1)$. This slight change in the power of t causes the density function of the stretched exponential to go to 0 slower and as a consequence the tails thicken. As a consequence of the thicker tails, the moment generating function, $M(t)$, is no longer finite for any $t > 0$. As stated before, a Large Deviations Theorem for weighted sums of stretched exponential random variables was set up by Gantert et al [5]. Since this is an extension of Cramér's theorem for a heavy-tailed distribution, we will study this theorem thoroughly.
In this section, we are going to explore their theorem. This will be done through analysing the conditions: we will look into the assumptions on the weights and the tail probability bounds. After this, we inspect the different elements of the statement and find out more about their significance with the help of their proof for the lower bound of their statement. We start with an analysis of the weights.

3.1 Weights

For the weights used in the paper for the theorem, only two assumptions were made.

Assumptions 3.1. *Let $a_j(n)$ be an array of non-negative weights and let $a_{max}(n) = \max_{j=1,..,n}(a_j(n))$ such that:*

1. $\exists\ s_1 > 0$ *such that:*
$$\lim_{n\to\infty} \sum_{j=1}^{n} a_j(n) = s_1, \tag{13}$$

2. $\exists\ s > 0$, *such that*
$$\lim_{n\to\infty} a_{max}(n)n = s. \tag{14}$$

Remark 3.1. *In the classical case of Cramér's Theorem [6], equal weights are taken; that is $a_j(n) = \frac{1}{n}$, $\forall j = 1, .., n$. With these equal weights, we get that:*

$$s_1 = \lim_{n\to\infty} \sum_{j=1}^{n} \frac{1}{n} = \lim_{n\to\infty} n\frac{1}{n} = 1,$$
$$s = \lim_{n\to\infty} a_{max}(n)n = \lim_{n\to\infty} \frac{1}{n}n = 1.$$

3.1.1 Random weights

We shall first introduce the conditions that we require for the distribution. After, we shall show for Example 1 that setting

$$a_j(n) = \begin{cases} \frac{Z_j}{n}, & \text{for } j \le n, \\ 0, & \text{for } j > n; \end{cases} \tag{15}$$

where $\{Z_j\}$ is a sequence of i.i.d. random variables such that $Z_j \sim U(0,1)$; works as a distribution for the weights and we will calculate s and s_1 for this example.
In order to take a sample from a distribution for the sequence of weights, the first thing we note is the existence of $\lim_{n\to\infty} a_{max}(n)n$. To ensure this converges to the positive constant s, we require that $a_{max}(n)$ grows with a factor n^{-1}. From this it follows that we need a distribution for which the upper bound of the domain decreases at a rate of n^{-1}. This condition alone suffices for the distribution for the sequence of weights. This follows from the fact that $s_1 \leq s$:

$$s_1 = \lim_{n\to\infty} \sum_{j=1}^{n} a_j(n) \leq \lim_{n\to\infty} \sum_{j=1}^{n} a_{max}(n) = \lim_{n\to\infty} a_{max}(n)n = s. \tag{16}$$

In other words, the existence of s implies the existence of s_1.
Since we multiplied realizations of $U(0,1)$ with the factor $\frac{1}{n}$, we have that the example satisfies the required growth condition. We will now calculate the corresponding constants, s and s_1.

Example 1. *As an example, one could use weights as defined in* (15)*, with Z_j a sequence of i.i.d random variables with $Z_j \sim U(0,1)$; as the distribution for the weights. Since we have already shown that this distribution works, we shall calculate the corresponding constants s_1 and s. Since we are working with a distribution, we can try to interpret the different assumptions to calculate s and s_1. We start with s_1.*
Let the sequence $\{Z_j\}_{j=1}^n$ be a sequence of independent observations from $U(0,1)$. We see that we thus have that the weights equal $a_j(n) = \frac{Z_j}{n}$ and thus condition one becomes:

$$s_1 = \sum_{j=1}^{n} a_j(n) = \frac{1}{n} \sum_{j=1}^{n} Z_j. \tag{17}$$

From the equation above, we see that s_1 is equal to the sample mean of observations from $U(0,1)$, therefore we have that $s_1 = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} Z_j = \frac{1}{2}$, by the Law of Large Numbers [4, p. 234-235].
Next, we calculate s. We want to find out how we can represent $a_{max}(n)$. We know that every weight can be written as $a_j(n) = \frac{Z_j}{n}$, for $j = 1,..,n$. From this, it follows that the greatest weight corresponds to the greatest of the n observations from $U(0,1)$. Thus, we have that $a_{max}(n) = \frac{\max_{j=1,..,n}\{Z_j\}}{n}$. Filling this out in (14) *gives*

$$s = \lim_{n\to\infty} n a_{max}(n) = \lim_{n\to\infty} \frac{n \max_{j=1,..,n}\{Z_j\}}{n} = \lim_{n\to\infty} \max_{j=1,..,n}\{Z_j\}. \tag{18}$$

Furthermore, $\max_{j=1,..,n}\{Z_j\}$ will tend to the essential supremum of $U(0,1)$. That is, the greatest observation will tend towards the greatest possible realization; and therefore $s = \lim_{n\to\infty} \max_{j=1,..,n}\{Z_j\} \to 1$.

3.2 Main result

The following theorem was the key result found by Gantert et al [5].

Theorem 3.1. *(Large Deviations for Weighted Sums, Stretched Exponential Tails). Let $\{X_j\}_{j\in\mathbb{N}}$ be a sequence of i.i.d. random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with*

$$\mathbb{E}[|X_1|^k] < \infty, \ \forall k \in \mathbb{N}, \tag{19}$$

and let $m := \mathbb{E}[X_1]$*. Suppose that there exists a constant* $r \in (0,1)$*, slowly-varying functions* $b,\ c_1,\ c_2 : (0,\infty) \to (0,\infty)$ *and a constant* $t^* > 0$ *such that for* $t \geq t^*$*,*

$$c_1(t) \exp\left(-b(t)t^r\right) \leq \mathbb{P}(X_1 \geq t) \leq c_2(t) \exp\left(-b(t)t^r\right). \tag{20}$$

Let $\{a_j(n)\}_{j\in\mathbb{N}},\ n \in \mathbb{N}$*, be an infinite array of non-negative real numbers that satisfy Assumptions 3.1; let* s *and* s_1 *be the associated constants, respectively; and let* $\{\bar{S}_n\}_{n\in\mathbb{N}}$ *be the sequence of weighted sums*

$$\bar{S}_n := \sum_{j=1}^{n} a_j(n) X_j. \tag{21}$$

Then

$$\lim_{n\to\infty} \frac{1}{b(n)n^r} \log \mathbb{P}\left(\bar{S}_n \geq x\right) = -\left(\frac{x}{s} - \frac{s_1}{s}m\right)^r,\ \forall x > s_1 m. \tag{22}$$

Remark 3.2. *Comparing* (22) *to the Cramér case from* (4)*, we see that we had to take* n^r *with* $r \in (0,1)$ *as opposed to just multiplying by* n *in the Cramér case. This leads to a decay of* $\exp\left(-(nc)^r\right)$ *in the case of Gantert et al, which decreases slower than* $\exp(-nc)$ *from Cramér.*

3.3 Conditions

Keeping our goal in mind of adapting this theorem so that it works for log-normally distributed random variables, we would like to alter the conditions as little as we can. The only condition we will truly have to change is (20). This particular inequality needs to be changed as it describes the way the tail probability of the distribution behaves; which, for this case, is stretched exponentially. Therefore, we will need to adapt this condition so that the tail bound behaves log-normally. However, we will see that we also have to adapt the statement.

3.4 Statement

With regards to the statement, (22), of Theorem 3.1; we are interested in the transformations applied to $\mathbb{P}\left(\bar{S}_n \geq x\right)$. From the statement, we can see that first the logarithm was taken, which we shall call f^{-1}; then a factor was introduced, in this case $\frac{1}{b(n)n^r}$, which we shall call $\psi(n)$; and finally the limit was taken. Next, we want to know why these steps were taken and what their significance was. We will do this by walking through their proof of the lower bound of their statement.

3.4.1 Lower Bound

Let $\varepsilon > 0$, $m = \mathbb{E}[X_1]$, $x > m$ and $j^* = \inf\{j \in \{1,..,n\} : a_j(n) = a_{max}(n)\}$; in other words, let $j^*(n)$ be the lowest index such that the weight $a_{j^*}(n)$ is equal to the greatest weight. As stated in the introduction, large deviations of the type $\bar{S}_n \geq x$ are caused by one extremely large observation for thick-tailed distributions. The beginning of this proof is based on this idea and only uses the i.i.d.-property of the sequence of random variables, thus:

$$\mathbb{P}\left(\bar{S}_n \geq x\right) = \mathbb{P}\left(\sum_{j=1,..,n} a_j(n)(X_j - m) \geq x - \sum_{j=1,..,n} a_j(n)m\right), \tag{23}$$

$$\geq \mathbb{P}\left(\left\{a_{max}(n)(X_{j^*(n)} - m) \geq x - \sum_{j=1,..,n} a_j(n)m + \varepsilon\right\},\ R_n\right), \tag{24}$$

where

$$R_n := \left\{\sum_{j=1,..,n;j\neq j^*} a_j(n)(X_j - m) \geq -\varepsilon\right\}. \tag{25}$$

We can see that the idea of having one great realization is incorporated in the inequality above; in (24). We will come back to this later in (39). All of the other X_j's can be found in R_n. We can interpret $\mathbb{P}(R_n)$ as the probability that the weighted sample mean of $n-1$ random variables minus the mean is bigger than any non-positive constant. Intuitively and by the Law of Large Numbers, we have that $\mathbb{P}(R_n) \to 1$ as $n \to \infty$. We shall prove this now first.

Lemma 3.2. *Let $X_1, X_2, ...$ be a a sequence of i.i.d. log-normal random variables and let $a_j(n)$ be an array of weights that satisfy Assumptions 3.1. Then, $\forall \varepsilon > 0$, we have*

$$\lim_{n\to\infty} \mathbb{P}\left(R_n\right) = 1, \tag{26}$$

where R_n is defined in (25).

Proof. We would like to work with the Chebyshev Inequality, see Theorem A.3; so we are going to show that the complement goes to 0, i.e.

$$\lim_{n\to\infty} \mathbb{P}\left(\sum_{j=1,..,n,\ j\neq j^*} a_j(n)\left(X_j - \mathbb{E}[X_1]\right) \leq -\varepsilon\right) = 0. \tag{27}$$

The Chebyshev Inequality gives us:

$$\mathbb{P}\left(\sum_{j=1,..,n,\ j\neq j^*} a_j(n)\left(X_j - \mathbb{E}[X_1]\right) \leq -\varepsilon\right) \tag{28}$$

$$\leq \mathbb{P}\left(\sum_{j=1,..,n,\ j\neq j^*} a_j(n)\left(X_j - \mathbb{E}[X_1]\right) \leq -\varepsilon\right) + \mathbb{P}\left(\sum_{j=1,..,n,\ j\neq j^*} a_j(n)\left(X_j - \mathbb{E}[X_1]\right) \geq \varepsilon\right), \tag{29}$$

$$= \mathbb{P}\left(a_{max}(n) \sum_{j=1,..,n,\ j\neq j^*} |X_j - \mathbb{E}[X_1]| \geq \varepsilon\right), \tag{30}$$

$$\leq \frac{\mathbb{E}\left[\left(a_{max}(n)\sum_{j=1,..,n,\ j\neq j^*} |X_j - \mathbb{E}[X_1]|\right)^2\right]}{\varepsilon^2}. \tag{31}$$

Next, we will write out the expectation in the numerator of (31). We see that the terms equal the variance and covariance. Recall the independence of our X_j. This implies that $cov(X_i, X_j) = 0, i \neq j$. We have that

$$\frac{\mathbb{E}\left[\left(a_{max}(n)\sum_{j=1,..,n,\ j\neq j^*}|X_j - \mathbb{E}[X_1]|\right)^2\right]}{\varepsilon^2} \tag{32}$$

$$= \frac{a_{max}^2(n)\left((n-1)\ var(X_1) + (n-1)(n-2)cov\left(X_1, X_2\right)\right)}{\varepsilon^2}, \tag{33}$$

$$\leq \frac{n\ a_{max}^2(n)\ var(X_1)}{\varepsilon^2}, \tag{34}$$

$$= \frac{\left(n\ a_{max}(n)\right)^2\ \ var(X_1)}{n\ \varepsilon^2}. \tag{35}$$

Next, using (14) and (19), the overall limit becomes:

$$\lim_{n\to\infty} \mathbb{P}\left(\sum_{j=1,..,n,\ j\neq j^*} a_j(n)\left(X_j - \mathbb{E}[X_1]\right) \leq -\varepsilon\right) \leq \lim_{n\to\infty} n^{-1}\frac{\left(a_{max}(n)\ n\right)^2\ \ var(X_1)}{\varepsilon^2}, \tag{36}$$

$$= 0; \tag{37}$$

since $na_{max}(n) \to s$ by Assumption 3.1.
Therefore, we have that

$$\lim_{n\to\infty} \mathbb{P}\left(\sum_{j=1,..,n,\ j\neq j^*} a_j(n)\left(X_j - \mathbb{E}[X_1]\right) \leq -\varepsilon\right) = 0. \tag{38}$$

□

Moving on, we use the independence of the X_j on (24):

$$\mathbb{P}\left(\left\{a_{max}(n)(X_{j^*(n)} - m) \geq x - \sum_{j=1,..,n} a_j(n)m + \varepsilon\right\},\ R_n\right) \geq \mathbb{P}\left(X_1 \geq t_1(n)\right)\mathbb{P}\left(R_n\right), \tag{39}$$

where

$$t_1(n) = \frac{n}{na_{max}(n)}\left(x - \sum_{j=1,..,n} a_j(n)m + a_{max}(n)m + \varepsilon\right). \tag{40}$$

It might not be immediately clear, but the constant in the statement (22), $-\left(\frac{x}{s} - \frac{s_1 m}{s}\right)^r$; is processed into $t_1(n)$ and is isolated after dividing by n, taking the limit $n \to \infty$ and letting $\varepsilon \downarrow 0$. We shall show the constant isolation first.

Lemma 3.3. *We have that*

$$\lim_{n\to\infty} \frac{t_1(n)}{n} = \left(\frac{x}{s} - \frac{s_1 m}{s}\right) + \frac{\varepsilon}{s}. \tag{41}$$

Proof. We begin by writing out $t_1(n)$:

$$\frac{t_1(n)}{n} = \frac{x - \sum_{j=1,..,n} a_j(n)m + a_{max}(n)m + \varepsilon}{na_{max}(n)}. \tag{42}$$

From the assumptions, we have the following limits:

1. $\lim_{n\to\infty} \sum_{j=1,..,n} a_j(n) = s_1$,
2. $\lim_{n\to\infty} a_{max}(n)n = s$,
3. $\lim_{n\to\infty} a_{max}(n) = 0$.

Using these limits, we see that:

$$\lim_{n\to\infty} \frac{t_1(n)}{n} = \lim_{n\to\infty} \frac{\left[x - \sum_{j=1,..,n} a_j(n)m + a_{max}(n)m + \varepsilon\right]}{na_{max}(n)}, \tag{43}$$

$$= \frac{x - s_1 m + \varepsilon}{s}. \tag{44}$$

□

Remark 3.3. *After letting* $\varepsilon \downarrow 0$ *on the result of Lemma 3.3, the constant* $\left(\frac{x}{s} - \frac{s_1 m}{s}\right)$ *remains.*

Next, they used the tail bound from (20) on (39):

$$\mathbb{P}(X_1 \geq t_1(n))\,\mathbb{P}(R_n) \geq c_1(t_1(n)) \exp\left(-t_1(n)^r b(t_1(n))\right) \mathbb{P}(R_n). \tag{45}$$

Now we have everything to isolate the constant. The log was taken so that the b-term and the $\mathbb{P}(R_n)$-term can be easily split off from the $t_1(n)^r$ and functions as an inverse function for the density function of the stretched exponential distribution. Besides this, the continuity of the log and Lemma 3.2 together give us that that $\log(\mathbb{P}(R_n))$-term goes to 0. The other factor, $\frac{1}{b(n)n^r}$, is included so that the constant can get completely isolated from $b(t)t^r$ and the other terms go to 0. Once again, we shall refer to the inverse as f^{-1} and the other introduced factor as $\psi(n)$.
Therefore, in order to adapt the theorem so that it works for log-normally distributed random variables; there are three things we must do. We must adapt the tail probability bounds; find the inverse density function for log-normal distribution, f^{-1}; and find a suitable $\psi(n)$ so that the final limit converges. We will start by adapting the tail bounds and finding f^{-1}.

Remark 3.4. *Even though* f^{-1} *is not necessarily the inverse of the entire density function; we shall refer to* f^{-1} *as the inverse density function since it concerns the density function of the distribution at hand.*

4 Log-normal distribution

Let $X \sim N(\mu,\ \sigma^2)$, i.e. let X be normally distributed with mean μ and variance σ^2. A log-normally distributed random variable is defined as the variable Y in $X = \ln(Y)$, [8]. In this section, we are going to find the tail bounds for the log-normal distribution as well as the inverse density function. In addition, we will also give our main result and we will use the rest of the thesis to prove the result. We start with the tail bounds.

4.1 Tail probability bounds

Similarly to the bounds for the stretched exponential random variables, our bounds will work starting from a certain t^*. We will bound $\mathbb{P}(X_1 \geq x)$. Note that the previous probability is equal to the cumulative distribution function for the log-normal distribution, therefore, we can represent the CDF with the error function as

$$\mathbb{P}(X_1 \geq x) = 1 - \left(\frac{1}{2} + \frac{1}{2}\mathrm{erf}\left(\frac{\ln(x)-\mu}{\sigma}\right)\right) = \frac{1}{2} - \frac{1}{2}\mathrm{erf}\left(\frac{\ln(x)-\mu}{\sigma}\right). \tag{46}$$

The error function is defined as

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp\left(-t^2\right) dt. \tag{47}$$

The error function is used to calculate the CDF of the normal distribution, hence filling in $\ln(x)$ in the input for the error function, [7], in (46) allows us to use the error function for the CDF of the log-normal distribution.

Lemma 4.1. *Let X be log-normally distributed and let $\sigma = 1$. Then there exists slowly-varying functions b, c_1 and c_2; and a constant $t^* > 0$ such that $\forall t > t^*$ we have that*

$$c_1(t)\exp\left(-\ln^2(tb(t))\right) \leq \mathbb{P}(X \geq t) \leq c_2(t)\exp\left(-\ln^2(tb(t))\right). \tag{48}$$

Proof. We will use the Q-function to bound (46), where the Q-function is given by:

$$Q(x) := \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-\frac{u^2}{2}}\mathrm{d}u. \tag{49}$$

Using the fact that $\lim_{x\to\infty}\mathrm{erf}(x) = 1$ and setting $I(t) = \frac{\ln(t)-\mu}{\sigma\sqrt{2}}$, we see that:

$$\frac{1}{2} - \frac{1}{2}\mathrm{erf}\left[I(t)\right] = \frac{1}{\sqrt{\pi}}\int_0^\infty e^{-t^2}\mathrm{d}t - \frac{1}{\sqrt{\pi}}\int_0^{I(t)} e^{-t^2}\mathrm{d}t, \tag{50}$$

$$= \frac{1}{\sqrt{\pi}}\int_{I(t)}^\infty e^{-t^2}\mathrm{d}t. \tag{51}$$

Next, using substitution with $u(x) = \frac{x}{\sqrt{2}}$, we get that:

$$\frac{1}{\sqrt{\pi}}\int_{I(n)}^\infty e^{-t^2}\mathrm{d}t = \frac{1}{\sqrt{\pi}}\int_{u^{-1}(I(n))}^\infty \frac{\mathrm{d}u}{\mathrm{d}x}e^{u(t)^2}\mathrm{d}t, \tag{52}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{\sqrt{2}I(n)}^\infty e^{\frac{t^2}{2}}\mathrm{d}t, \tag{53}$$

$$= Q(\sqrt{2}I(n)). \tag{54}$$

Thus, $\mathbb{P}(X_1 \geq t) = Q\left(\frac{\ln(t)-\mu}{\sigma}\right)$. Filling in $\sigma = 1$ gives

$$\mathbb{P}(X_1 \geq t) = Q(\ln(t) - \mu). \tag{55}$$

The reason why we wanted to represent the CDF with the Q-function is because there exists bounds for the Q-function. Let $\phi(x)$ denote the density function for the standard normal distribution. We then have the following inequality [2]:

$$\frac{x}{1+x^2}\phi(x) \leq Q(x) \leq \frac{1}{x}\phi(x). \tag{56}$$

We would like to extend (56) to a slowly-varying functions inequality, where we also take into account the input for the functions. We note that $\ln(t) - \mu = \ln(t) - \ln(\exp(\mu)) = \ln(t\exp(-\mu))$, so we set $\exp(-\mu) = b(t)$; so $x = \ln(t) - \mu = \ln(tb(t))$. Since all converging functions are slowly-varying at infinity and the two prefactors, $\frac{x}{1+x^2}$ and $\frac{1}{x}$, are convergent, we represent them with the slowly-varying functions $c_1(t)$ and $c_2(t)$, respectively. That is,

$$c_1(t) = \frac{\ln(tb(t))}{1+\ln^2(tb(t))}, \tag{57}$$

$$c_2(t) = \frac{1}{\ln(tb(t))}. \tag{58}$$

This leads to:

$$c_1(t)\exp\left(-\ln^2(tb(t))\right) \leq Q(\ln(tb(t))) \leq c_2(t)\exp\left(-\ln^2(tb(t))\right). \tag{59}$$

This gives rise to the following tail bounds.

$$c_1(t)\exp\left(-\ln^2(tb(t))\right) \leq \mathbb{P}(X_1 \geq t) \leq c_2(t)\exp\left(-\ln^2(tb(t))\right). \tag{60}$$

□

Remark 4.1. *Note that for $\sigma \neq 1$, $\frac{\ln(t)-\mu}{\sigma}$ cannot be expressed as $\ln(tb(t))$ for some slowly-varying function b. It would rather be $\ln(t^\alpha b(t))$, where we can set $b(t) = b^\alpha(t)$ by property 3 of Proposition 2.1; for $\alpha = \frac{1}{\sigma}$. However, we shall work with $\sigma = 1$ in order to extract the main ideas. See (228) for the extension of the tail bounds for general variance.*

We, now, have adapted the tail bounds so that it is suited for log-normally distributed random variables. The next step is to find the inverse of the log-normal density function.

4.2 Inverse density function

Next, we would like to find the inverse of the density function for log-normal random variables. That is, we want to find the inverse of $f(x) = \phi(\ln(x))$.

Corollary 4.2. *Given the density function $f(x) = \phi(\ln(x))$, where $\phi(x)$ is the standard normal density function, i.e.*

$$\phi(x) = \exp\left(-x^2\right). \tag{61}$$

The inverse for f(x) is, accordingly, given by

$$f^{-1}(x) = \exp\left(-\sqrt{-\ln(x)}\right). \tag{62}$$

Remark 4.2. *We overlook the constant part of the density function, the factor* $\frac{1}{\sqrt{2\pi}}$. *This can be done as the constant can be processed into the slowly-varying functions* c_1 *and* c_2 *of the tail bounds.*

Proof.

$$f(x) = \exp\left(-\ln^2(x)\right), \tag{63}$$

$$\ln(f(x)) = -\ln^2(x), \tag{64}$$

$$\pm\sqrt{-\ln(f(x))} = \ln(x), \tag{65}$$

$$\exp\left(\pm\sqrt{-\ln(f(x))}\right) = x, \tag{66}$$

$$f^{-1}(x) = \exp\left(-\sqrt{-\ln(x)}\right). \tag{67}$$

□

Remark 4.3. *The reason why we chose for this inverse function instead of* $\exp\left(\sqrt{-\ln(x)}\right)$, *is because the inverse function, as defined in* (62), *maintains order relations.*

Having found the inverse function allows us to start working with the bounds for $\mathbb{P}\left(\bar{S}_n \geq x\right)$ for log-normally distributed random variables.

4.3 Main Result

Now that we have found two of the main components, we will give the main result. In contrast to the Large Deviations Theorem for stretched exponential random variables, we do not have that the upper bound and lower bound converge to the same constant. In our case, they differ from each other with a factor $\exp\left(\frac{1}{2}\right)$. This is not completely optimal, however, it is a good first step towards a Large Deviations Theorem for the log-normal distribution. We will now present our result for log-normally distributed random variables.

Theorem 4.3. *(Large Deviations for Weighted Sums, Log-Normal Tails).*
Let $\{X_j\}_{j\in\mathbb{N}}$ *be a sequence of i.i.d. random variables on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *with*

$$\mathbb{E}[|X_1|^k] < \infty, \ \forall k \in \mathbb{N}; \tag{68}$$

$m := \mathbb{E}[X_1]$ *and* $var(X_1) = 1$. *Suppose that there exists slowly-varying functions* $b,\ c_1,\ c_2 : (0,\infty) \to (0,\infty)$ *and a constant* $t^* > 0$ *such that for* $t \geq t^*$,

$$c_1(t)\exp\left(-\ln^2(b(t)t)\right) \leq \mathbb{P}(X_1 \geq t) \leq c_2(t)\exp\left(-\ln^2(b(t)t)\right). \tag{69}$$

Let $\{a_j(n)\}_{j\in\mathbb{N}}$, $n \in \mathbb{N}$, *be an infinite array of non-negative real numbers that satisfy Assumptions 3.1; let* s *and* s_1 *be the associated constants, respectively; and let* $\{\bar{S}_n\}_{n\in\mathbb{N}}$ *be the sequence of weighted sums*

$$\bar{S}_n := \sum_{j=1}^{n} a_j(n)X_j. \tag{70}$$

Let f^{-1} *be the function defined in Corollary 4.2, then*

$$\left(\frac{x}{s} - \frac{s_1}{s}m\right)^{-1} \leq \lim_{n\to\infty} b(n)n\ f^{-1}\left(\mathbb{P}\left(\bar{S}_n \geq x\right)\right) \leq \exp\left(\frac{1}{2}\right)\left(\frac{x}{s} - \frac{s_1}{s}m\right)^{-1},\ \forall x > s + s_1 m. \tag{71}$$

Remark 4.4. *Note that, just like stretched exponential random variables, log-normal random variables satisfy the finiteness of the moments assumption* (68)*. Furthermore, see Theorem 8.1 for the extension to* $\sigma^2 \in \mathbb{R}_{>0}$*; where* $\sigma^2 = var(X_1)$.

The rest of the thesis is the proof of Theorem 4.3. Within this proof, we split the upper bound into two terms and show that the lower bound as well as both parts of the upper bound can be represented by one equation. Then we calculate the limit of this equation for the three situations. Lastly, we use the Laplace Principle to combine the two parts of the upper bound.

5 A general form

In this section, we are going to work out the bounds for the log-normal distribution until the part where we multiply the bounds with $\psi(n)$. We are going to show that each part of the bound can be written in the form

$$\psi(n)\left(tb(t)\right)^{\Gamma(n)}. \tag{72}$$

In Section 6.2, we will calculate the limits of this equation, where we assume that $\psi(n) = (nb(n))^{\alpha}$, for $\alpha \in \mathbb{R}$.
We will start with the lower bound.

5.1 Lower Bound

As we saw before in (39), the first few steps followed only from the i.i.d. property of the sequence of X_j's; therefore we can continue from here and utilize our tail bounds from Lemma 4.1:

$$\mathbb{P}\left(\bar{S}_n \geq x\right) \geq \mathbb{P}\left(X_1 \geq t_1(n)\right)\mathbb{P}(R_n), \tag{73}$$

$$\mathbb{P}\left(X_1 \geq t_1(n)\right)\mathbb{P}(R_n) \geq c_1(t_1(n))\exp\left(-\ln^2(t_1(n)b(t_1(n)))\right)\mathbb{P}(R_n). \tag{74}$$

Next, we apply our inverse function from Corollary 4.2. Overall, we then get:

$$f^{-1}\left(\mathbb{P}\left(\bar{S}_n \geq x\right)\right) \geq f^{-1}\left(c_1(t_1(n))\exp\left(-\ln^2(t_1(n)b(t_1(n)))\right)\mathbb{P}(R_n)\right), \tag{75}$$

$$\geq \exp\left(-\sqrt{\ln^2(t_1(n)b(t_1(n))) - \ln(c_1(t_1(n))\mathbb{P}(R_n))}\right), \tag{76}$$

$$\geq \exp\left(-\ln(t_1(n)b(t_1(n)))\sqrt{1-\frac{\ln(c_1(t_1(n))\mathbb{P}(R_n))}{\ln(t_1(n)b(t_1(n)))^2}}\right), \tag{77}$$

$$\geq \left(t_1(n)b(t_1(n))\right)^{\Gamma_1(n)}, \tag{78}$$

where

$$\Gamma_1(n) := -\sqrt{1-\frac{\ln(c_1(t_1(n))\mathbb{P}(R_n))}{\ln(t_1(n)b(t_1(n)))^2}}. \tag{79}$$

The next step is when we introduce our factor $\psi(n)$. This gives

$$\psi(n)f^{-1}\left(\mathbb{P}\left(\bar{S}_n \geq x\right)\right) \geq \psi(n)\left(t_1(n)b(t_1(n))\right)^{\Gamma_1(n)}. \tag{80}$$

Next, we are going to work out the upper bound up until this step.

5.2 Upper Bound

Analogously to the lower bound, the first few steps of the upper bound are done using conditions that our theorem also satisfies. It starts by splitting the upper bound into two parts.

$$\mathbb{P}\left(\bar{S}_n \geq x\right) \leq A_1^n + A_2^n; \tag{81}$$

where

$$A_1^n := \mathbb{P}\left(\max_{j=1,..,n}\{X_j\} \geq t_2(n)\right), \tag{82}$$

$$A_2^n := \mathbb{P}\left(\bar{S}_n \geq x, \max_{j=1,..,n}\{X_j\} < t_2(n)\right), \tag{83}$$

$$t_2(n) = n\left(\frac{x}{s} - \frac{s_1 m}{s}\right). \tag{84}$$

We can tell from the above definitions that A_1^n represents the situation where at least one of the observations takes on such a high value that could cause the deviation; whereas A_2^n describes a situation in which all observations are slightly higher in value than one would expect, causing the weighted mean to be greater than the expected value.
We are going to show that these two probabilities individually can also be written in the general form (72). The limits of the individual parts will be calculated and brought back together in Proposition 7.5.
We will start with A_1^n

5.2.1 A_1^n

Analogous to the proof of Gantert et al [5], we can start by performing the union bound. This, accordingly, makes way for the tail bound, which results in the following for the log-normal case:

$$A_1^n \leq n\mathbb{P}(X_1 \geq t_2(n)) \leq nc_2(t_2(n)\exp(-\ln(t_2(n)b(t_2(n))))). \tag{85}$$

Now, we apply f^{-1} and then multiply by $\psi(n)$. This gives us:

$$\psi(n)f^{-1}(A_1^n) \leq \psi(n)f^{-1}(nc_2(t_2(n)\exp(-\ln(t_2(n)b(t_2(n)))))), \tag{86}$$

$$\leq \psi(n)\exp\left(-\sqrt{\ln^2(t_2(n)b(t_2(n))) - \ln(c_2(t_2(n))n)}\right), \tag{87}$$

$$\leq \psi(n)\exp\left(-\ln(t_2(n)b(t_2(n)))\sqrt{1 - \frac{\ln(c_2(t_2(n))n)}{\ln^2(t_2(n)b(t_2(n)))}}\right), \tag{88}$$

$$\leq \psi(n)(t_2(n)b(t_2(n)))^{\Gamma_2(n)}; \tag{89}$$

where

$$\Gamma_2(n) := -\sqrt{1 - \frac{\ln(c_2(t_2(n))n)}{\ln^2(t_2(n)b(t_2(n)))}}. \tag{90}$$

5.2.2 A_2^n

Lastly, we would like to do the same for A_2^n.
This starts by applying the Chernoff Bound, see Theorem A.2; with the positive real parameter $\frac{\beta(n)}{s}$, where

$$\beta(n) := \ln^2(t_2(n)b(t_2(n))). \tag{91}$$

Remark 5.1. *Note that $\beta(n)$ is a slowly-varying function by Remark 2.1*

Applying this bound to A_2^n leads to

$$A_2^n \leq \exp\left(-\beta(n)\frac{x}{s}\right) \prod_{j=1,.,,n} \mathbb{E}\left[\exp\left(\beta(n)\frac{a_j(n)}{s}X_j\right) \mathbb{1}_{\{X_j<t_2(n)\}}\right]. \tag{92}$$

Now, we can once again apply our inverse function and consequently multiply by $\psi(n)$. This gives:

$$\psi(n)f^{-1}(A_2^n) \leq \psi(n)f^{-1}\left(\exp\left(-\beta(n)\frac{x}{s}\right) \prod_{j=1,.,,n} \mathbb{E}\left[\exp\left(\beta(n)\frac{a_j(n)}{s}X_j\right) \mathbb{1}_{\{X_j<t_2(n)\}}\right]\right), \tag{93}$$

$$\leq \psi(n)\exp\left(-\sqrt{\frac{\ln^2(t_2(n)b(t_2(n)))x}{s} - \sum_{j=1,..,n}\Lambda_j}\right), \tag{94}$$

$$\leq \psi(n)\left(t_2(n)b(t_2(n))\right)^{\Gamma_3(n)}; \tag{95}$$

where

$$\Gamma_3(n) := -\sqrt{\frac{x}{s} - \frac{\sum_{j=1,..,n}\Lambda_j}{\ln^2(t_2(n)b(t_2(n)))}}, \tag{96}$$

$$\Lambda_j := \ln\left(\mathbb{E}\left[\exp\left(\beta(n)\frac{a_j(n)}{s}X_j^{(n)}\right)\right]\right), \tag{97}$$

$$X_j^{(n)} := X_j \mathbb{1}_{\{X_j<t_2(n)\}}. \tag{98}$$

5.3 Summary

All in all, we were successfully able to write each separate part of the bounds; the lower bound, A_1^n, and A_2^n; in the given form (72). We will now state each different t and $\Gamma(n)$, respectively. We will put all the results together for clarity.
For the lower bound, we found

$$\psi(n)f^{-1}\left(\mathbb{P}\left(\bar{S}_n \geq x\right)\right) \geq \psi(n)\left(t_1(n)b(t_1(n))\right)^{\Gamma_1(n)}; \tag{99}$$

where

$$t_1(n) = \frac{n}{na_{max}(n)}\left(x - \sum_{j=1,..,n} a_j(n)m + a_{max}(n)m + \varepsilon\right), \tag{100}$$

$$\Gamma_1(n) = -\sqrt{1 - \frac{\ln(c_1(t_1(n))\mathbb{P}(R_n))}{\ln(t_1(n)b(t_1(n)))^2}}; \tag{101}$$

see Remark 5.2 below for the form of $t_1(n)$.
Next, for A_1^n, we found

$$\psi(n)f^{-1}\left(A_1^n\right) \leq \psi(n)\left(t_2(n)b(t_2(n))\right)^{\Gamma_2(n)}; \tag{102}$$

where

$$t_2(n) = n\left(\frac{x}{s} - \frac{s_1 m}{s}\right), \tag{103}$$

$$\Gamma_2(n) = -\sqrt{1 - \frac{\ln(c_2(t_2(n))n)}{\ln^2(t_2(n)b(t_2(n)))}}. \tag{104}$$

Lastly, we were also able to represent A_2^n in the form (72) as follows:

$$\psi(n)f^{-1}\left(A_2^n\right) \leq \psi(n)\left(t_2(n)b(t_2(n))\right)^{\Gamma_3(n)}; \tag{105}$$

where

$$t_2(n) = n\left(\frac{x}{s} - \frac{s_1 m}{s}\right), \tag{106}$$

$$\Gamma_3(n) = -\sqrt{\frac{x}{s} - \frac{\sum_{j=1,..,n}\Lambda_j}{\ln^2(t_2(n)b(t_2(n)))}}. \tag{107}$$

Remark 5.2. *For both $t_1(n)$ and $t_2(n)$, we have a function of the form:*

$$t(n) \sim c\, n, \tag{108}$$

where $c = \frac{x-s_1m}{s} + \frac{\varepsilon}{s}$ for $t_1(n)$ and $c = \frac{x-s_1m}{s}$ for $t_2(n)$. This follows from Lemma 3.3 and the definition of $t_2(n)$ from (84).

Next, we are going to give $\psi(n)$ such that the above cases converge.

6 Convergence of the bounds

Now that we have worked out all the bounds until the introduction of $\psi(n)$, we can have a further look into this factor. Based on the statement of the theorem by Gantert et al [5], (22), we assume that

$$\psi(n) = (nb(n))^{\alpha}, \ \alpha \in \mathbb{R}. \tag{109}$$

In order to get a better indication of this α, we will calculate the limits of the different Γ's first, where we find that $\Gamma_1(n)$ and $\Gamma_2(n)$ tend to -1. Accordingly, we will show that the lower bound and A_1^n converge for $\alpha = 1$. For A_2^n, we will see that the $-\ln(\psi(n))\Gamma_3^2(n)$-term does not converge when calculating the full limit; but rather diverges to $\pm\infty$ for certain values of x. However, the divergence occurs in the exponent and we will be able to work our way around it when $-\ln(\psi(n))\Gamma_3^2(n) \to -\infty$. We start with the calculation of the limits of $\Gamma_i(n), \ i \in \{1, 2, 3\}$.

Remark 6.1. *We could have chosen to already set $\alpha = 1$ and we will do so in the following sections. We would like to note that this is only the case because we set $\sigma^2 = 1$ in Lemma 4.1. This will be elaborated upon in the discussion; see Section 8.*

6.1 Limit of Γ_i

In this section, we are going to look into the limit of $\Gamma_i(n), \ i \in \{1, 2, 3\}$. We will look at the different limits individually. We begin with $\Gamma_1(n)$.

6.1.1 Limit of $\Gamma_1(n)$

Lemma 6.1. *We find the following result for the limit for $\Gamma_1(n)$, where $\Gamma_1(n)$ is defined in (79). We then have that the limit is*

$$\lim_{n\to\infty} \Gamma_1(n) = -1. \tag{110}$$

Proof. For $\Gamma_1(n)$, we have:

$$\lim_{n\to\infty} \Gamma_1(n) = \lim_{n\to\infty} -\sqrt{1 - \frac{\ln(c_1(t_1(n))\mathbb{P}(R_n))}{\ln(t_1(n)b(t_1(n)))^2}}, \tag{111}$$

$$= \lim_{n\to\infty} -\sqrt{1 - \frac{\ln(c_1(t_1(n))\mathbb{P}(R_n))}{\ln(t_1(n)b(t_1(n)))}\frac{1}{\ln(t_1(n)b(t_1(n)))}}. \tag{112}$$

From Lemma 3.2, we know that $\mathbb{P}(R_n) \to 1$ and since the log-function is continuous around 1, we have that $\ln(\mathbb{P}(R_n)) \to 0$. Furthermore, we can split $\mathbb{P}(R_n)$ off of the product by (9). For this reason, we only need to look at $\lim_{n\to\infty} \frac{\ln(c_1(t_1(n)))}{\ln(t_1(n)b(t_1(n)))}\frac{1}{\ln(t_1(n)b(t_1(n)))}$. This also tends to 0 as $n \to \infty$, by property 1 of Proposition 2.1, since $t_1(n)b(t_1(n)) \to \infty$ by property 2 of Proposition 2.1. Putting all of this together, we get that:

$$\lim_{n\to\infty} \Gamma_1(n) = \lim_{n\to\infty} -\sqrt{1 - \frac{\ln(c_1(t_1(n))\mathbb{P}(R_n))}{\ln(t_1(n)b(t_1(n)))}\frac{1}{\ln(t_1(n)b(t_1(n)))}} = -1. \tag{113}$$

□

6.1.2 Limit of $\Gamma_2(n)$

Lemma 6.2. *We find the following result for the limit for* $\Gamma_2(n)$*, where* $\Gamma_2(n)$ *is defined in* (90)*. We then have that the limit is*

$$\lim_{n\to\infty} \Gamma_2(n) = -1. \tag{114}$$

Proof. We consider $\Gamma_2(n)$:

$$\lim_{n\to\infty} \Gamma_2(n) = \lim_{n\to\infty} \sqrt{1 - \frac{\ln(c_2(t_2(n))n)}{\ln(t_2(n)b(t_2(n)))} \frac{1}{\ln(t_2(n)b(t_2(n)))}}. \tag{115}$$

Here, we have that $\lim_{n\to\infty} \frac{\ln(c_2(t_2(n))n)}{\ln(t_2(n)b(t_2(n)))} = 1$ by property 4 of Proposition 2.1. It follows that $\lim_{n\to\infty} \frac{\ln(c_2(t_2(n))n)}{\ln(t_2(n)b(t_2(n)))} \frac{1}{\ln(t_2(n)b(t_2(n)))} = 0$ and the overall limit becomes:

$$\lim_{n\to\infty} \Gamma_2(n) = \lim_{n\to\infty} \sqrt{1 - \frac{\ln(c_2(t_2(n))n)}{\ln(t_2(n)b(t_2(n)))} \frac{1}{\ln(t_2(n)b(t_2(n)))}} = -1. \tag{116}$$

□

6.1.3 Limit of $\Gamma_3(n)$

Lastly, we direct our attention to the limit of $\Gamma_3(n)$. Unlike the other two cases, we are not able to directly calculate the limit. We are going to bound it instead using Lemma 6.4. We will first give the result.

Lemma 6.3. *We find the following results for the limits for* $\Gamma_3(n)$*, where* $\Gamma_3(n)$ *is defined in* (96)*. We then have that the limit is*

$$\lim_{n\to\infty} \Gamma_3(n) \leq -\sqrt{\frac{x}{s} - \frac{s_1 m}{s}}. \tag{117}$$

Proof. Using the continuity of the square root function and taking the limit of $\Gamma_3(n)$ from (96) gives:

$$\lim_{n\to\infty} -\sqrt{\frac{x}{s} - \frac{\sum_{j=1,..,n} \Lambda_j}{\ln^2(t_2(n)b(t_2(n)))}} \leq -\sqrt{\frac{x}{s} - \frac{s_1 m}{s}}, \tag{118}$$

by Lemma 6.4. □

This proof was based on the following lemma.

Lemma 6.4. *With* $\beta(n)$ *as defined in* (91) *and* Λ_j *as defined in* (97)*, we have that*

$$\lim_{n\to\infty} \frac{\sum_{j=1,..,n} \Lambda_j}{\beta(n)} \leq \frac{s_1 m}{s}. \tag{119}$$

Proof. We use the same estimates as in the paper by Gantert et al [5]: $\ln(x) \leq x - 1$ and $e^x - 1 \leq \sum_{j=1}^{k} \frac{x^j}{j!} + \frac{x^{k+1}}{(k+1)!}e^x$; for some $k \in \mathbb{N}$, which we shall define later. Using these estimates, we have:

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{n} \Lambda_j}{\beta(n)} \leq \lim_{n\to\infty} \left(\sum_{i=1}^{k} \sum_{j=1}^{n} \frac{\mathbb{E}\left[\left(\beta(n)\frac{a_j(n)}{s}X_j^{(n)}\right)^i\right]}{\beta(n)i!} \right) + \frac{B_0}{(k+1)!}. \tag{120}$$

Here, $B_0 := \lim_{n\to\infty} \beta(n)^{-1} \sum_{j=1}^{n} \mathbb{E}\left[\left(X_j^{(n)}\right)^{k+1} \exp\left(\beta(n)\frac{a_j(n)}{s}X_j^{(n)}\right)\right]$.

Now, we will show that $\lim_{n\to\infty} \sum_{i=1}^{k} \sum_{j=1}^{n} \frac{\mathbb{E}\left[\left(\beta(n)\frac{a_j(n)}{s}X_j^{(n)}\right)^i\right]}{\beta(n)\ i!} \leq \frac{s_1 m}{s}$.

Fix i and look at $\lim_{n\to\infty} \sum_{j=1}^{n} \frac{\mathbb{E}\left[\left(\beta(n)\frac{a_j(n)}{s}X_j^{(n)}\right)^i\right]}{\beta(n)\ i!}$. We have that

$$\lim_{n\to\infty} \sum_{j=1}^{n} \frac{\mathbb{E}\left[\left(\beta(n)\frac{a_j(n)}{s}X_j^{(n)}\right)^i\right]}{\beta(n)\ i!} = \lim_{n\to\infty} \mathbb{E}\left[\left(X_1^{(n)}\right)^i\right] \frac{\beta(n)^{i-1}}{s^i} \frac{\sum_{j=1}^{n} a_j(n)^i}{i!}. \tag{121}$$

For i=1, we have that $E\left[X_1^{(n)}\right] \leq m$ and by (13) of Assumptions 3.1, we have that $\lim_{n\to\infty} \sum_{j=1}^{n} a_j(n) \to s_1$. Filling this in gives that $\lim_{n\to\infty} \mathbb{E}\left[\left(X_1^{(n)}\right)^i\right] \frac{\beta(n)^{i-1}}{s^i} \frac{\sum_{j=1}^{n} a_j(n)^i}{i!} \leq \frac{s_1}{s}m$.

Next, for the other $i > 1$, we use the fact that $\sum_{j=1}^{n} a_j(n)^i \leq n\ a_{max}(n)^i$ and that $\lim_{n\to\infty} n a_{max}(n) = s$; the latter of which corresponds to (14) of Assumptions 3.1. This gives:

$$\begin{aligned} \lim_{n\to\infty} \mathbb{E}\left[\left(X_1^{(n)}\right)^i\right] \frac{\beta(n)^{i-1}}{s^i} \frac{\sum_{j=1}^{n} a_j(n)^i}{i!} &\leq \lim_{n\to\infty} \mathbb{E}\left[\left(X_1^{(n)}\right)^i\right] \frac{\beta(n)^{i-1}}{s^i} \frac{n a_{max}(n)^i}{i!}, \\ &\leq \lim_{n\to\infty} \mathbb{E}\left[\left(X_1^{(n)}\right)^i\right] n^{-i+1} \frac{\beta(n)^{i-1}}{s^i} \frac{(n a_{max}(n))^i}{i!}. \end{aligned} \tag{122}$$

Here we have that $\mathbb{E}\left[\left(X_1^{(n)}\right)^i\right] < \infty$; $\lim_{n\to\infty} n^{-i+1} \frac{\beta(n)^{i-1}}{s^i} \to 0$, by (68) and property 2 of Proposition 2.1, since $\beta(n)$ is a slowly-varying function; and $\lim_{n\to\infty} \frac{(n a_{max}(n))^i}{i!} \to \frac{s^i}{i!} < \infty$. Therefore, we have $\lim_{n\to\infty} \mathbb{E}\left[\left(X_1^{(n)}\right)^i\right] \frac{\beta(n)^{i-1}}{s^i} \frac{\sum_{j=1}^{n} a_j(n)^i}{i!} = 0$.

All this together shows that:

$$\lim_{n\to\infty} \left(\sum_{i=1}^{k} \sum_{j=1}^{n} \frac{\mathbb{E}\left[\left(\beta(n)\frac{a_j(n)}{s}X_j^{(n)}\right)^i\right]}{\beta(n)\ i!} \right) \leq \frac{s_1}{s}m. \tag{123}$$

In order to complete the proof, we need to show that $B_0 = 0$:

$$\lim_{n\to\infty} \beta(n)^{-1} \sum_{j=1}^{n} \mathbb{E}\left[\left(X_j^{(n)}\right)^{k+1} \exp\left(\beta(n)\frac{a_j(n)}{s}X_j^{(n)}\right)\right] = 0. \tag{124}$$

We will follow the same steps as in the paper. We begin by bounding B_0 by $\limsup_{n\to\infty}(B_1(n)+B_2(n))$, where

$$B_1(n) := \beta(n)^{-1}\sum_{j=1}^{n}\left(\beta(n)\frac{a_j(n)}{s}\right)^{k+1}(t^*)^{k+1}\exp\left(\beta(n)\frac{a_j(n)}{s}t^*\right), \tag{125}$$

$$\leq \beta(n)^{-1}n^{-k}\left(\beta(n)\frac{na_{max}(n)}{s}\right)^{k+1}(t^*)^{k+1}\exp\left(n^{-1}\beta(n)\frac{na_{max}(n)}{s}t^*\right); \tag{126}$$

and

$$B_2(n) := \beta(n)^{-1}\sum_{j=1}^{n}\left(\beta(n)\frac{a_j(n)}{s}\right)^{k+1}\mathbb{E}\left[\left(X_j^{(n)}\right)^{k+1}\exp\left(\beta(n)\frac{a_j(n)}{s}X_j^{(n)}\right)\mathbb{1}_{\{t^*\leq X_j^{(n)}\}}\right], \tag{127}$$

$$\leq \beta(n)^{-1}n^{-k}\left(\beta(n)\frac{na_{max}(n)}{s}\right)^{k+1}\mathbb{E}\left[\left(X_1^{(n)}\right)^{k+1}\exp\left(\beta(n)\frac{a_{max}(n)}{s}X_j^{(n)}\right)\mathbb{1}_{\{t^*\leq X_1^{(n)}\}}\right]. \tag{128}$$

Since $\beta(n)$ is slowly-varying, we have that $\lim_{n\to\infty}n^{-1}\beta(n)^p = 0$, for any $p\in\mathbb{N}$ by property 2 and 3 of Proposition 2.1. Along with the fact that $t^*<\infty$ and (14) from Assumptions 3.1, we have that $\lim_{n\to\infty}B_1(n)=0$.
Now all that is left to prove is that $\lim_{n\to\infty}B_2(n)=0$. Since we only need one factor of n^{-1} in (128), we can rewrite (128) as follows:

$$\beta(n)^{-1}n^{-k}\left(\beta(n)\frac{na_{max}(n)}{s}\right)^{k+1}\mathbb{E}\left[\left(X_1^{(n)}\right)^{k+1}\exp\left(\beta(n)\frac{a_{max}(n)}{s}X_j^{(n)}\right)\mathbb{1}_{\{t^*\leq X_1^{(n)}\}}\right], \tag{129}$$

$$= n^{-1}\left(\beta(n)\frac{na_{max}(n)}{s}\right)^{k+1}n^{-k+1}\beta(n)^{-1}\mathbb{E}\left[\left(X_1^{(n)}\right)^{k+1}\exp\left(\beta(n)\frac{a_{max}(n)}{s}X_j^{(n)}\right)\mathbb{1}_{\{t^*\leq X_1^{(n)}\}}\right]. \tag{130}$$

Using the previous limits, we see that what we need to show is:

$$\lim_{n\to\infty}n^{-k+1}\beta(n)^{-1}\mathbb{E}\left[\left(X_1^{(n)}\right)^{k+1}\exp\left(\beta(n)\frac{a_{max}(n)}{s}X_j^{(n)}\right)\mathbb{1}_{\{t^*\leq X_1^{(n)}\}}\right]<\infty. \tag{131}$$

Next, we bound the expectation in (131) using Hölder's inequality. That is, $\forall\varepsilon>0$:

$$\mathbb{E}\left[\left(X_1^{(n)}\right)^{k+1}\exp\left(\beta(n)\frac{a_{max}(n)}{s}X_j^{(n)}\right)\mathbb{1}_{\{t^*\leq X_1^{(n)}\}}\right] \tag{132}$$

$$\leq \mathbb{E}\left[\left(X_1^{(n)}\right)^{(k+1)\frac{1+\varepsilon}{\varepsilon}}\mathbb{1}_{\{t^*\leq X_1^{(n)}\}}\right]^{\frac{\varepsilon}{1+\varepsilon}}\mathbb{E}\left[\exp\left((1+\varepsilon)\beta(n)\frac{a_{max}(n)}{s}X_j^{(n)}\right)\mathbb{1}_{\{t^*\leq X_1^{(n)}\}}\right]^{\frac{1}{1+\varepsilon}}. \tag{133}$$

Using the fact that the log-normal distribution has finite moments, we have that

$$\mathbb{E}\left[\left(X_1^{(n)}\right)^{(k+1)\frac{1+\varepsilon}{\varepsilon}}\mathbb{1}_{\{t^*\leq X_1^{(n)}\}}\right]^{\frac{\varepsilon}{1+\varepsilon}}<\infty, \tag{134}$$

which leaves to prove that:

$$\lim_{n\to\infty} n^{-k+1}\beta(n)^{-1}\mathbb{E}\left[\exp\left((1+\varepsilon)\beta(n)\frac{a_{max}(n)}{s}X_j^{(n)}\right)\mathbb{1}_{\{t^*\leq X_1^{(n)}\}}\right]^{\frac{1}{1+\varepsilon}} < \infty. \tag{135}$$

Bear in mind that we have not yet set k. We will prove (135) with the same Integration by Parts Theorem as in the paper, which can be found in the appendix, Theorem B.1. Applying this theorem, we can bound (135) as follows:

$$n^{-k+1}\beta(n)^{-1}\mathbb{E}\left[\exp\left((1+\varepsilon)\beta(n)\frac{a_{max}(n)}{s}X_j^{(n)}\right)\mathbb{1}_{\{t^*\leq X_1^{(n)}\}}\right] \tag{136}$$

$$\leq n^{-k+1}(1+\varepsilon)\beta(n)^{-1}\beta(n)\frac{a_{max}(n)}{s}\int_{t^*}^{t_2(n)}\exp\left((1+\varepsilon)\beta(n)\frac{a_{max}(n)}{s}z\right)\mathbb{P}\left(X_1\geq z\right)dz \tag{137}$$

$$+\,n^{-k+1}\beta(n)^{-1}\exp\left((1+\varepsilon)\beta(n)\frac{a_{max}(n)}{s}t^*\right). \tag{138}$$

We have that for $k \geq 2$, that the term in (138) will go to 0 in its limit, as a result of Assumptions 3.1 and due to the fact that $\beta(n)$ is slowly-varying. To clarify, we can rewrite the term in (138) as follows:

$$n^{-k+1}\beta(n)^{-1}\exp\left((1+\varepsilon)\beta(n)\frac{a_{max}(n)}{s}t^*\right) = n^{-k+1}\beta(n)^{-1}\exp\left((1+\varepsilon)\frac{\beta(n)}{n}\frac{a_{max}(n)n}{s}t^*\right). \tag{139}$$

Therefore, we shift our focus to (137). To start off, we are going to bound $\mathbb{P}\left(X_1\geq z\right)$ using the tail bounds from Lemma 4.1:

$$n^{-k+1}(1+\varepsilon)\frac{a_{max}(n)}{s}\int_{t^*}^{t_2(n)}\exp\left((1+\varepsilon)\beta(n)\frac{a_{max}(n)}{s}z\right)\mathbb{P}\left(X_1\geq z\right)dz \leq$$
$$n^{-k+1}(1+\varepsilon)\frac{a_{max}(n)}{s}\int_{t^*}^{t_2(n)}c_2(z)\exp\left((1+\varepsilon)\beta(n)\frac{a_{max}(n)}{s}z-\ln^2\left(zb(z)\right)\right)dz. \tag{140}$$

Lastly, we apply substitution with $y := \frac{z}{t_2(n)}$. Then, we have that $\frac{dy}{dz} = \frac{1}{t_2(n)} \Rightarrow t_2(n)dy = dz$. Substituting into (140) gives:

$$n^{-k+1}(1+\varepsilon)\frac{a_{max}(n)}{s}\int_{t^*}^{t_2(n)}c_2(z)\exp\left((1+\varepsilon)\beta(n)\frac{a_{max}(n)}{s}z-\ln^2\left(zb(z)\right)\right)dz$$
$$= n^{-k+1}t_2(n)(1+\varepsilon)\frac{a_{max}(n)}{s}\int_{\frac{t^*}{t_2(n)}}^{1}c_2(z)\exp\left((1+\varepsilon)\beta(n)\frac{a_{max}(n)}{s}t_2(n)y-\ln^2\left(t_2(n)yb(t_2(n)y)\right)\right)\mathrm{d}y, \tag{141}$$

$$\leq (1-\frac{t^*}{t_2(n)})n^{-k}t_2(n)(1+\varepsilon)\frac{a_{max}(n)n}{s}c_2(t^*)\exp\left((1+\varepsilon)\beta(n)\frac{a_{max}(n)}{s}t^*-\ln^2\left(t^*b(t^*)\right)\right); \tag{142}$$

where we have that the following limits:

$$\lim_{n\to\infty}(1-\frac{t^*}{t_2(n)})=1,$$
$$\lim_{n\to\infty}\frac{t_2(n)}{n^k}=0,\ k\geq 2,\ \text{using the definition of } t_2(n) \text{ from (84)},$$
$$\lim_{n\to\infty}a_{max}(n)n=s,$$
$$\lim_{n\to\infty}\beta(n)a_{max}(n)=0.$$

All the terms in (142) that are not found in the list above are constants. We thus have that

$$\lim_{n\to\infty}(1-\frac{t^*}{t_2(n)})n^{-k}t_2(n)(1+\varepsilon)\frac{a_{max}(n)n}{s}c_2(t^*)\exp\left((1+\varepsilon)\beta(n)\frac{a_{max}(n)}{s}t^*-\ln^2\left(t^*b(t^*)\right)\right)=0. \tag{143}$$

This concludes the proof of this lemma. □

Knowing the limits of the different $\Gamma(n)$'s gives us more insight in the calculations of the overall limits, which we shall do next. For notational convenience, we will henceforth write t instead of either $t_1(n)$ or $t_2(n)$, where the definition of t will be specified. In the context of general equations, t will be used to represent both $t_1(n)$ and $t_2(n)$.

6.2 The overall limit

Using (108), Definition 2.1, Remark 5.2 and (109); we can rewrite (72):

$$\psi(n)\left(t(n)b(t(n))\right)^{\Gamma(n)}=(nb(n))^{\alpha}\left(cnb(t(n))\right)^{\Gamma(n)}, \tag{144}$$
$$=\left(\frac{b(n)}{b(t)}\right)^{\alpha}(b(t)n)^{\alpha+\Gamma(n)}c^{\Gamma(n)}. \tag{145}$$

Remark 6.2. *The definition of slowly-varying functions was used here with the introduction of* $\left(\frac{b(t)}{b(t)}\right)^{\alpha}$. *We saw in Remark 5.2 that both* $t_1(n)$ *and* $t_2(n)$ *can be seen as a constant times* n. *This implies that* $\frac{b(t)}{b(n)}\to 1$ *as* $n\to\infty$.

As a result of the above remark and the convergence of the $\Gamma(n)$'s; the convergence of (145) only requires the convergence of the last term:

$$(nb(t))^{\alpha+\Gamma(n)}. \tag{146}$$

In order to calculate the limit of this term, we need to work $nb(t)$ into the exponent. The identity transformation $x\to\exp(\ln(x))$ allows this:

$$\lim_{n\to\infty}(nb(t))^{\alpha+\Gamma(n)}=\lim_{n\to\infty}\exp\left(\ln\left((nb(t))^{\alpha+\Gamma(n)}\right)\right), \tag{147}$$
$$=\lim_{n\to\infty}\exp\left((\alpha+\Gamma(n))\ln\left(nb(t)\right)\right). \tag{148}$$

When we look back at the definition of the $\Gamma(n)$'s, we see that all of them include a square root. With the goal of getting rid of the square root in the numerator, we raise (148) to the power $\frac{\alpha-\Gamma(n)}{\alpha-\Gamma(n)}$, giving us the following

$$\lim_{n\to\infty} (nb(t))^{\alpha+\Gamma(n)} = \lim_{n\to\infty} \exp\left((\alpha+\Gamma(n))\frac{\alpha-\Gamma(n)}{\alpha-\Gamma(n)}\ln(nb(t))\right), \tag{149}$$

$$= \lim_{n\to\infty} \exp\left(\frac{\alpha^2-\Gamma(n)^2}{\alpha-\Gamma(n)}\ln(nb(t))\right). \tag{150}$$

Therefore, we need (150) to converge so that the bounds converge. We will check this equation for all three of our cases, starting with the lower bound. We will prove that this is the case for $\alpha = 1$. However, as we shall see, A_2^n will require us to make additional conditions for convergence.

6.2.1 Lower Bound

Proposition 6.5. *Under the assumptions of Theorem 4.3, we have*

$$\left(\frac{x}{s} - \frac{s_1}{s}m\right)^{-1} \leq \lim_{n\to\infty} b(n)n\ f^{-1}\left(\mathbb{P}\left(\bar{S}_n \geq x\right)\right),\ if\ x > s_1 m. \tag{151}$$

Proof. Let $t = t_1(n)$. From (151) we can see that this $\psi(n) = nb(n)$ corresponds to the case when $\alpha = 1$.
For the lower bound, we found $\Gamma(n) = -\sqrt{1 - \frac{\ln(c_1(t)\mathbb{P}(R_n))}{\ln(tb(t))^2}}$, in (79). Filling this out in (150) gives us:

$$\lim_{n\to\infty} \exp\left(\frac{\alpha^2-\Gamma(n)^2}{\alpha-\Gamma(n)}\ln(nb(t))\right) = \lim_{n\to\infty} \exp\left(\frac{\alpha^2 - (1 - \frac{\ln(c_1(t)\mathbb{P}(R_n))}{\ln(tb(t))^2})}{\alpha-\Gamma(n)}\ln(nb(t))\right), \tag{152}$$

$$= \lim_{n\to\infty} \exp\left(\frac{\alpha^2\ln(nb(t)) - \ln(nb(t)) + \frac{\ln(nb(t))\ln(c_1(t)\mathbb{P}(R_n))}{\ln(tb(t))^2})}{\alpha-\Gamma(n)}\right). \tag{153}$$

Take a closer look at $\frac{\ln(nb(t))\ln(c_1(t)\mathbb{P}(R_n))}{\ln(tb(t))^2})$. Rewriting this as follows; $\frac{\ln(nb(t))}{\ln(tb(t))}\frac{\ln(c_1(t)\mathbb{P}(R_n))}{\ln(tb(t))}$, makes it easier to see that this term goes to 0 as a result of Lemma 3.2 and by property 3 of Proposition 2.1. This also implies that if we set $\alpha = 1$, then the limit of the factor inside the exponent in (153) goes to 0, since $\Gamma(n)$ converges to -1 from Lemma 6.1. That is,

$$\lim_{n\to\infty} \exp\left(\frac{\ln(nb(t)) - \ln(nb(t)) + \frac{\ln(nb(t))\ln(c_1(t)\mathbb{P}(R_n))}{\ln(tb(t))^2})}{1-\Gamma(n)}\right) \tag{154}$$

$$= \lim_{n\to\infty} \exp\left(\frac{\frac{\ln(nb(t))}{\ln(tb(t))}\frac{\ln(c_1(t)\mathbb{P}(R_n))}{\ln(tb(t))})}{1-\Gamma(n)}\right) = \exp\left(\frac{1\cdot 0}{2}\right) = 1. \tag{155}$$

This means, that for the overall limit for the lower bound, we get from (145):

$$\lim_{n\to\infty}\left(\frac{b(n)}{b(t)}\right)^1 (b(t_1(n))n)^{1+\Gamma(n)}\, c^{\Gamma(n)} = 1\cdot 1\cdot\left(\frac{x}{s} - \frac{s_1 m}{s} - \frac{\varepsilon}{s}\right)^{-1}, \tag{156}$$

$$= \left(\frac{x-s_1 m}{s}\right)^{-1}, \tag{157}$$

where c is from Lemma 3.3 and we let $\varepsilon \downarrow 0$ in (157), for $\alpha = 1$. □

We would like to note that we calculated the constant in (157) by immediately working with the constant, instead of the function that converges to the constant, from Lemma 3.3. With the following result, we show that the calculation still works.

Lemma 6.6. *Let $a, f(n) > 0$, and let $g(n)$ and b such that*

$$\lim_{n\to\infty} f(n) = a, \tag{158}$$

$$\lim_{n\to\infty} g(n) = b. \tag{159}$$

Then

$$\lim_{n\to\infty} f(n)^{g(n)} = a^b. \tag{160}$$

Proof. We apply the identity transformation $x \to \exp(\log(x))$:

$$f(n)^{g(n)} = \exp\left(\log\left(f(n)^g(n)\right)\right), \tag{161}$$

$$= \exp\left(g(n)\log(f(n))\right). \tag{162}$$

Now, we have by the conditions that $f(n)$ is positive, therefore, $\log(f(n))$ is well-defined and by continuity of the log we have that $\lim_{n\to\infty}\log(f(n)) = \log(a)$. Then, by the multiplication rule of limits, we have that the following equation holds

$$\lim_{n\to\infty} g(n)\log(f(n)) = b\log(a). \tag{163}$$

Also using the continuity of the exponential function, this gives us

$$\lim_{n\to\infty} f(n)^{g(n)} = \lim_{n\to\infty} \exp\left(\log\left(f(n)^{g(n)}\right)\right), \tag{164}$$

$$= \lim_{n\to\infty} \exp(g(n)\log(f(n))), \tag{165}$$

$$= \exp(b\log(a)), \tag{166}$$

$$= a^b. \tag{167}$$

□

We applied Lemma 6.6 for

$$f(n) = \frac{t_1(n)}{n} = \frac{1}{na_{max}(n)}\left(x - \sum_{j=1,..,n} a_j(n)m + a_{max}(n)m + \varepsilon\right), \tag{168}$$

$$g(n) = \Gamma_1(n) = -\sqrt{1 - \frac{\ln(c_1(t)\mathbb{P}(R_n))}{\ln(tb(t))^2}}. \tag{169}$$

To give us that

$$\lim_{n\to\infty} f(n)^{g(n)} = \left(\frac{x - s_1 m}{s} + \frac{\varepsilon}{s}\right)^{-1}, \tag{170}$$

for (157).

Remark 6.3. *The limits of $f(n)$ and $g(n)$ were found in Lemma 3.3 and Lemma 6.1, respectively.*

6.2.2 A_1^n

Next, we are going to show that the A_1^n-bound also converges for $\psi(n) = nb(n)$.

Lemma 6.7. *Let A_1^n be the probability defined in* (82)*. Under the same conditions as Theorem 4.3, we have that*

$$\lim_{n\to\infty} b(n)n\, f^{-1}\left(A_1^n\right) \leq \exp\left(\frac{1}{2}\right)\left(\frac{x}{s} - \frac{s_1 m}{s}\right)^{-1}, \ \textit{if } x > s_1 m. \tag{171}$$

Proof. Let $t = t_2(n)$. We once again start by noting that this for this particular case we found $\Gamma(n) = -\sqrt{1 - \frac{\ln(c_2(t)n)}{\ln^2(tb(t))}}$ in (90). We insert this into (150), giving

$$\lim_{n\to\infty} \exp\left(\frac{\alpha^2 - \Gamma(n)^2}{\alpha - \Gamma(n)} \ln\left(nb(t)\right)\right) = \lim_{n\to\infty} \exp\left(\frac{\alpha^2 - (1 - \frac{\ln(c_2(t)n)}{\ln^2(tb(t))})}{\alpha - \Gamma(n)} \ln(nb(t))\right), \tag{172}$$

$$= \lim_{n\to\infty} \exp\left(\frac{\alpha^2 \ln(nb(t)) - \ln(nb(t)) + \frac{\ln(nb(t))\ln(c_2(t)n)}{\ln^2(tb(t))})}{\alpha - \Gamma(n)}\right). \tag{173}$$

Analogously to the lower bound, we need to inspect the term $\frac{\ln(nb(t))\ln(c_2(t)n)}{\ln^2(tb(t))}$; and we can rewrite as follows: $\frac{\ln(nb(t))}{\ln(tb(t))}\frac{\ln(c_2(t)n)}{\ln(tb(t))}$. By property 4 of Proposition 2.1, we have that the fraction converges to 1:

$$\lim_{n\to\infty} \frac{\ln(nb(t))}{\ln(tb(t))}\frac{\ln(c_2(t)n)}{\ln(tb(t))} = 1 \cdot 1 = 1. \tag{174}$$

Applying this along with $\alpha = 1$ and Lemma 6.2 in (173), we get:

$$\lim_{n\to\infty} \exp\left(\frac{\ln(nb(t)) - \ln(nb(t)) + \frac{\ln(nb(t))\ln(c_2(t)n)}{\ln^2(tb(t))}}{1 - \Gamma(n)}\right) \tag{175}$$

$$= \lim_{n\to\infty} \exp\left(\frac{\frac{\ln(nb(t))}{\ln(tb(t))}}{1 - \Gamma(n)}\right) = \exp\left(\frac{1 \cdot 1}{1 + 1}\right) = \exp\left(\frac{1}{2}\right). \tag{176}$$

This previous limit implies that we get

$$\lim_{n\to\infty} \left(\frac{b(n)}{b(t)}\right)^1 (b(t)n)^{1+\Gamma(n)}\, c^{\Gamma(n)} = 1 \cdot \exp\left(\frac{1}{2}\right) \cdot c^{-1}, \tag{177}$$

$$= \exp\left(\frac{1}{2}\right)\left(\frac{x - s_1 m}{s}\right)^{-1}; \tag{178}$$

for the overall limit of A_1^n with $\alpha = 1$, where c is the constant from $t_2(n)$ as defined in (84). □

The constant found in (178) is not exactly the same constant as the one we found for the lower bound in (157) but it did converge. The next case is where we run into a bit of trouble.

6.2.3 A_2^n

We ran into trouble for this case as a result of the fact that we were only able to find a bound for $\Gamma_3(n)$ in Lemma 6.3. Using the bound, we are only able to definitely state the following the lemma.

Lemma 6.8. *Let A_2^n be the probability defined in* (83)*, using the same assumptions as in Theorem 4.3; but for general $var(X_1) = \sigma^2$ and $\alpha = \frac{1}{\sigma}$, gives us*

$$\lim_{n\to\infty} (b(n)n)^\alpha \ f^{-1}\left(A_2^n\right) = 0, \ \text{if } x > s\alpha^2 + s_1 m. \tag{179}$$

Proof. Let $t = t_2(n)$. Using the $\Gamma(n)$ we found for this case in (96), $\Gamma(n) = -\sqrt{\frac{x}{s} - \frac{\sum_{j=1,..,n}\Lambda_j}{\ln^2(tb(t))}}$, (150) then becomes:

$$\lim_{n\to\infty} \exp\left(\frac{\alpha^2 - \Gamma(n)^2}{\alpha - \Gamma(n)}\ln\left(nb(t)\right)\right) = \lim_{n\to\infty} \exp\left(\frac{\alpha^2 - \left(\frac{x}{s} - \frac{\sum_{j=1,..,n}\Lambda_j}{\ln^2(tb(t))}\right)}{\alpha - \Gamma(n)}\ln(nb(t))\right). \tag{180}$$

Looking at (180), we see that we need to calculate the limit of $\ln(nb(t))\Gamma^2(n)$. Calculating this limit directly will not work with the method we used in the proof Lemma 6.4, but rather it would diverge. What we can look at is when $\ln(nb(t))\frac{\alpha^2-\Gamma^2(n)}{\alpha-\Gamma(n)} \to -\infty$.
We need to check the sign of the above term. We begin by noting that both $\ln(nb(t))$ and $\alpha - \Gamma(n)$ are both greater than 0, see (96). Then, only $\alpha^2 - \Gamma^2(n)$ remains. We want to find out if and when $\lim_{n\to\infty} \alpha^2 - \Gamma^2(n) < 0$.
Using our bound from Lemma 6.3, we have that

$$\lim_{n\to\infty} \alpha^2 - \Gamma^2(n) \leq \alpha^2 - \frac{x}{s} + \frac{s_1 m}{s}. \tag{181}$$

We want this to be less than 0. This gives:

$$\alpha^2 - \frac{x}{s} + \frac{s_1 m}{s} < 0, \tag{182}$$

$$x > s_1 m + s\alpha^2. \tag{183}$$

□

Remark 6.4. *We only considered the case where $\ln(nb(t))\frac{\alpha^2-\Gamma^2(n)}{\alpha-\Gamma(n)} \to -\infty$; since the divergence of the above limit to infinity does not produce any significant results. This is a consequence from the fact that we were only able to bound $\Gamma_3(n)$ because we cannot make any absolute statements about the the limit of $\ln(nb(t))\frac{\alpha^2-\Gamma^2(n)}{\alpha-\Gamma(n)}$ for $s_1 m \leq x \leq s_1 m + s$.*

However, since we work with $\alpha = 1$ for the lower bound and A_1^n, we are particularly interested in Lemma 6.8 for this case. Giving us the following corollary.

Corollary 6.9. *Under the same conditions as Theorem 4.3 and with A_2^n as defined in* (83)*, we have that*

$$\lim_{n\to\infty} nb(n)f^{-1}(A_2^n) = 0, \ \text{if } x > s + s_1 m, \tag{184}$$

All that is left is to use the Laplace Principle to combine the upper bound.

7 Laplace Principle

When computing the limit of the bound for A_2^n, a problem arose. Besides not converging to the either of the two constants found in Proposition 6.5 or Lemma 6.7, A_2^n only converged for certain values of x. However, the method for reuniting A_1^n and A_2^n for the upper bound also provides a solution for this problem.
In the paper by Gantert et al [5], the Laplace principle was used. We start by giving the statement and its proof. Accordingly, we look at what it accomplished for the stretched exponential case and see what it can do for us.

Theorem 7.1. *(Laplace principle)*
Let $a_n,\ b_n \geq 0$ *be two sequences such that*

$$\lim_{n\to\infty} \frac{1}{n} \log(a_n) = a, \tag{185}$$

$$\lim_{n\to\infty} \frac{1}{n} \log(b_n) = b. \tag{186}$$

Then

$$\lim_{n\to\infty} \frac{1}{n} \log\left(a_n + b_n\right) = \max(a,\ b). \tag{187}$$

Proof. The proof of this result is based on the following inequality, for $x, y > 0$:

$$\max(x,\ y) \leq x + y \leq 2\max(x,\ y). \tag{188}$$

We will use this bound for the sum in $\frac{1}{n}\log(a_n + b_n)$:

$$\frac{1}{n}\log(\max(a_n,\ b_n)) \leq \frac{1}{n}\log(a_n + b_n) \leq \frac{1}{n}\log(2\max(a_n,\ b_n)), \tag{189}$$

$$\leq \frac{\log(2)}{n} + \frac{1}{n}\log(\max(a_n,\ b_n)). \tag{190}$$

Since the log-function preserves order relations, we can interchange max and log.

$$\max\left(\frac{1}{n}\log(a_n),\ \frac{1}{n}\log(b_n)\right) \leq \frac{1}{n}\log(a_n + b_n) \leq \frac{\log(2)}{n} + \max\left(\frac{1}{n}\log(a_n),\ \frac{1}{n}\log(b_n)\right). \tag{191}$$

We were able to put the factor $\frac{1}{n}$ inside of the max on account of this term being non-negative.
Lastly, the result follows when taking the limit of (191), since $\frac{\log(2)}{n} \to 0$. □

We will now demonstrate the Laplace Principle application in the paper by Gantert et al [5].

7.1 Stretched exponential case

Gantert et al [5] used the Laplace Principle for the upper bound. This was necessary due to the fact that the upper bound was split into two parts. They found that both parts converged to the same constant. That is, for the upper bound they found the following two limits

$$\lim_{n\to\infty} \frac{1}{b(n)n^r} \log\left(A_1^n\right) \leq \left(\frac{x}{s} - \frac{s_1 m}{s}\right)^r, \tag{192}$$

$$\lim_{n\to\infty} \frac{1}{b(n)n^r} \log\left(A_2^n\right) \leq \left(\frac{x}{s} - \frac{s_1 m}{s}\right)^r. \tag{193}$$

Given these limits, the Laplace Principle could be applied and gave in their case:

$$\lim_{n\to\infty} \frac{1}{b(n)n^r} \log\left(\mathbb{P}\left(\bar{S}_n \geq x\right)\right) \leq \lim_{n\to\infty} \frac{1}{b(n)n^r} \log\left(A_1^n + A_2^n\right), \tag{194}$$

$$= -\left(\frac{x}{s} - \frac{s_1 m}{s}\right)^r. \tag{195}$$

Now, let us apply this principle for log-normally distributed random variables. We will try to work it out alongside of the proof.

7.2 Laplace extension

Gantert et al [5] were able to apply the Laplace principle for their upper bound as the log-function was their f^{-1}. However, our inverse density function, as found in Corollary 4.2, is not just the log-function, but a composition of functions including the log. In this section, we will show that the Laplace Principle still holds for our f^{-1}.

Theorem 7.2. *(Laplace Extension).*
Let $0 < a_n,\ b_n < 1$. If $\lim_{n\to\infty} \psi(n) f^{-1}(a_n) = a$; $\max\{a_n,\ b_n\} = a_n$ for large n; and $\psi(n) \to \infty$, then

$$\lim_{n\to\infty} \psi(n) f^{-1}(a_n + b_n) = a. \tag{196}$$

The proof of Theorem 7.2 is based on the following lemma.

Lemma 7.3. *Let $a_n,\ a,\ b_n,\ c_n > 0$ and let $\ln^2(x_n) - \ln(c_n) = -\ln(a_n)$. If $\lim_{n\to\infty} x_n f^{-1}(a_n) = a$ and*

$$\lim_{n\to\infty} \frac{\ln(b_n)}{\ln(x_n)} = 0, \tag{197}$$

then

$$\lim_{n\to\infty} x_n f^{-1}(b_n a_n) = a. \tag{198}$$

Proof. Let us begin by writing out the inverse function in the limit of (198)

$$\lim_{n\to\infty} x_n f^{-1}(b_n a_n) = \lim_{n\to\infty} x_n \exp\left(-\sqrt{-\ln(b_n) - \ln(a_n)}\right), \tag{199}$$

$$= \lim_{n\to\infty} x_n \exp\left(-\sqrt{\ln^2(x_n) - \ln(b_n) - \ln(c_n)}\right), \tag{200}$$

$$= \lim_{n\to\infty} x_n x_n^{\Gamma(n)}, \tag{201}$$

where

$$\Gamma(n) = -\sqrt{1 - \frac{\ln(b_n) + \ln(c_n)}{\ln^2(x_n)}}. \tag{202}$$

Using the identity transformation $x \to \exp(\ln(x))$ on (201) gives

$$\lim_{n\to\infty} x_n^{1+\Gamma(n)} = \lim_{n\to\infty} \exp\left((1+\Gamma(n))\ln(x_n)\right), \tag{203}$$

$$= \lim_{n\to\infty} \exp\left(\frac{1-\Gamma^2(n)}{1-\Gamma(n)}\ln(x_n)\right); \tag{204}$$

where we raised (203) to the power $\frac{1-\Gamma(n)}{1-\Gamma(n)}$ in the last step. Writing out the $\Gamma^2(n)$-term results in the following

$$\lim_{n\to\infty} \exp\left(\frac{1-\Gamma^2(n)}{1-\Gamma(n)}\ln(x_n)\right) = \lim_{n\to\infty} \exp\left(\frac{1-1+\frac{\ln(b_n)+\ln(c_n)}{\ln^2(x_n)}}{1-\Gamma(n)}\ln(x_n)\right), \tag{205}$$

$$= \lim_{n\to\infty} \exp\left(\frac{\frac{\ln(b_n)}{\ln(x_n)}}{1-\Gamma(n)}\right)\exp\left(\frac{\frac{\ln(c_n)}{\ln(x_n)}}{1-\Gamma(n)}\right). \tag{206}$$

By our assumptions, we have that $\lim_{n\to\infty} \exp\left(\frac{\frac{\ln(b_n)}{\ln(x_n)}}{1-\Gamma(n)}\right) = 1$; and $\lim_{n\to\infty} \exp\left(\frac{\frac{\ln(c_n)}{\ln(x_n)}}{1-\Gamma(n)}\right) = a$. The latter limit follows by writing out the assumption concerning $\lim_{n\to\infty} x_n f^{-1}(a_n)$ analogously to the method above; gives the following result:

$$\lim_{n\to\infty} x_n f^{-1}(a_n) = \lim_{n\to\infty} \exp\left(\frac{1-1+\frac{\ln(c_n)}{\ln^2(x_n)}}{1+\sqrt{1-\frac{\ln(c_n)}{\ln^2(x_n)}}}\ln(x_n)\right) = a. \tag{207}$$

We see that this assumption implies the convergence of $\sqrt{1-\frac{\ln(c_n)}{\ln^2(x_n)}}$. Alongside the assumption that $\lim_{n\to\infty} \frac{\ln(b_n)}{\ln(x_n)} = 0$ and the continuity of the square root; we have that $1-\Gamma(n)$ converges. Altogether, we have that

$$\lim_{n\to\infty} x_n f^{-1}(b_n a_n) = a. \tag{208}$$

□

We now move on to the proof of Theorem 7.2.

Proof. Since $0 < a_n,\ b_n < 1$, we have that

$$\max(a_n,\ b_n) < a_n + b_n < 2\max(a_n,\ b_n). \tag{209}$$

Since our inverse density function preservers order relations, this inequality still holds after applying the inverse function as well as after multiplying by the positive factor $\psi(n)$ as follows

$$f^{-1}(\max(a_n,\ b_n)) < f^{-1}(a_n+b_n) < f^{-1}(2\max(a_n,\ b_n)), \tag{210}$$

$$\psi(n)f^{-1}(\max(a_n,\ b_n)) < \psi(n)f^{-1}(a_n+b_n) < \psi(n)f^{-1}(2\max(a_n,\ b_n)), \tag{211}$$

$$\Rightarrow$$

$$\psi(n)f^{-1}(a_n) < \psi(n)f^{-1}(a_n+b_n) < \psi(n)f^{-1}(2a_n). \tag{212}$$

This last step is the result of $\max(a_n,\ b_n) = a_n$ for large n. The last step is to take the limit of (212) giving

$$\lim_{n\to\infty} \psi(n) f^{-1}(a_n) < \lim_{n\to\infty} \psi(n) f^{-1}(a_n + b_n) < \lim_{n\to\infty} \psi(n) f^{-1}(2a_n), \tag{213}$$

$$\Rightarrow$$

$$a < \lim_{n\to\infty} \psi(n) f^{-1}(a_n + b_n) < a. \tag{214}$$

Note that we applied Lemma 7.3 for the right-hand side of the inequality. This was possible as a result of $\psi(n) \to \infty$, since x_n grows at the same rate as $\psi(n)$. As a consequence, we have that $\lim_{n\to\infty} \frac{2}{x_n} = 0$. □

By Theorem 7.2, we have that the following equation holds

$$\lim_{n\to\infty} b(n) n f^{-1}\left(\mathbb{P}\left(A_1^n + A_2^n\right)\right) \leq \exp\left(\frac{1}{2}\right)\left(\frac{x - s_1 m}{s}\right)^{-1}; \tag{215}$$

if $A_1^n \geq A_2^n$. We dedicate the next subsection to finding any new conditions so that this inequality holds.

7.3 Final conditions

In this section, we will find out if any more conditions must be taken in order for the upper bound to converge. That is, we need to find out if more conditions are required in order to have that the inequality $A_1^n \geq A_2^n$ is satisfied. However, we worked out the limits for the bounds of A_1^n and A_2^n, rather than A_1^n and A_2^n. We shall refer to the bounds as C_1^n and C_2^n from (85) and (92), respectively. We will first show that it suffices to compare the bounds. Accordingly, we will show that this is the case for $x > s_1 m + s$.
As stated, it suffices to compare C_1^n and C_2^n to find out which of the two is greater, as a result of

$$\mathbb{P}\left(\bar{S}_n \geq x\right) \leq A_1^n + A_2^n \leq C_1^n + C_2^n, \tag{216}$$

$$C_1^n := n c_2(t_2(n)) \exp\left(-\ln^2(t_2(n) b(t_2(n)))\right), \tag{217}$$

$$C_2^n := \exp\left(-\beta(n)\frac{x}{s}\right) \prod_{j=1}^{n} \mathbb{E}\left[\exp\left(\beta(n)\frac{a_j(n)}{s} X_j\right) \mathbb{1}_{\{X_j \leq t_2(n)\}}\right]. \tag{218}$$

Lemma 7.4. *Under the same conditions as Theorem 4.3 and with C_1^n and C_2^n as defined above in* (217) *and* (218)*, respectively; we have that for large n:*

$$C_1^n \geq C_2^n,\ \forall x > s_1 m + s. \tag{219}$$

Proof. We would like to begin by restating the fact that our inverse density function f^{-1} preserves order relations; i.e.

$$x \geq y \Longleftrightarrow f^{-1}(x) \geq f^{-1}(y). \tag{220}$$

Next, applying our inverse density function and the inequality described at (188) gives:

$$f^{-1}\left(\mathbb{P}\left(\bar{S}_n \geq x\right)\right) \leq f^{-1}\left(C_1^n + C_2^n\right); \tag{221}$$

$$f^{-1}\left(\max\{C_1^n,\ C_2^n\}\right) \leq f^{-1}\left(C_1^n + C_2^n\right) \leq f^{-1}\left(2\max\{C_1^n,\ C_2^n\}\right). \tag{222}$$

Since our inverse function preserves order relations, we can interchange the max and the inverse function as follows

$$f^{-1}\left(\max\{C_1^n,\ C_2^n\}\right) \leq f^{-1}\left(C_1^n + C_2^n\right) \leq f^{-1}\left(2\max\{C_1^n,\ C_2^n\}\right), \tag{223}$$

$$\Longleftrightarrow$$

$$\max\{f^{-1}(C_1^n), f^{-1}(C_2^n)\} \leq f^{-1}\left(C_1^n + C_2^n\right) \leq \max\{f^{-1}(2C_1^n), f^{-1}(2C_2^n)\}. \tag{224}$$

This means that we can compare the limits of $\psi(n)f^{-1}(C_1^n)$ and $\psi(n)f^{-1}(C_2^n)$. We already calculated these limits in Lemma 6.7 and Lemma 6.8. We see from these two lemmas that

$$\lim_{n\to\infty} nb(n)f^{-1}\left(C_1^n\right) \geq \lim_{n\to\infty} nb(n)f^{-1}\left(C_2^n\right),\ \forall x > s_1 m + s; \tag{225}$$

where Lemma 6.8 requires us to restrict the domain to $x > s_1 m + s$. □

Now that we have proved Lemma 7.4, the following lemma holds.

Proposition 7.5. *Under the same conditions as Theorem 4.3 and with both A_1^n and A_2^n as the probabilities defined in* (82) *and* (83)*, respectively. Then*

$$\lim_{n\to\infty} b(n)n\ f^{-1}\left(\mathbb{P}\left(\bar{S}_n \geq x\right)\right) \leq \exp\left(\frac{1}{2}\right)\left(\frac{x}{s} - \frac{s_1 m}{s}\right)^{-1}. \tag{226}$$

Proof. The result directly follows from applying Theorem 7.2 on Lemma 7.4 along with the limits found in Lemma 6.7 and Lemma 6.8. □

8 Discussion

In this discussion, we will elaborate upon a few of the ideas given in the thesis. Most significantly, we will work out Theorem 4.3 for $\sigma^2 \in \mathbb{R}_{>0}$; where we will also compare the rate of decrease found for log-normal random variables with those found by Cramér and Gantert et al.
Furthermore, we would like to elaborate upon the limits of the upper bound.

8.1 $\sigma^2 \in \mathbb{R}_{>0}$

We begin by examining the consequences of remark right after Lemma 4.1. In this remark, we stated that for general σ, we would get that (48) becomes:

$$\exists \alpha \in \mathbb{R}_{>0}, \text{ namely } \alpha := \frac{1}{\sigma} : \tag{227}$$

$$c_1(t) \exp\left(-\ln^2(t^\alpha b(t))\right) \leq \mathbb{P}\left(X_1 \geq t\right) \leq c_2(t) \exp\left(-\ln^2(t^\alpha b(t))\right). \tag{228}$$

We are going to check how this changes the general equations.
This change in the tail bounds becomes clear once we apply (228) for the bounds of $\mathbb{P}(\bar{S}_n \geq x)$. For example, we can look at the lower bound from (73) and see that applying (228) we get

$$\mathbb{P}\left(X_1 \geq t_1^\alpha(n)\right) \mathbb{P}(R_n) \geq c_1(t) \exp\left(-\ln^2(t^\alpha b(t))\right) \mathbb{P}(R_n). \tag{229}$$

Then, applying f^{-1} and working it out we get that

$$f^{-1}\left(\mathbb{P}\left(\bar{S}_n \geq x\right)\right) \geq \left(t_1(n) b(t_1(n))\right)^{\alpha \Gamma_1(n)} \tag{230}$$

$$\Gamma_1(n) := -\sqrt{1 - \frac{\ln(c_1(t_1^\alpha(n))\mathbb{P}(R_n))}{\ln(t_1^\alpha(n) b(t_1(n)))^2}}. \tag{231}$$

Note that since $b(t)$ is a slowly-varying function, we can represent all powers of $b(t)$ with $b(t)$ by property 3 of Proposition 2.1. Multiplying (230) with $\psi(n)$ gives

$$\psi(n) f^{-1}\left(\mathbb{P}\left(\bar{S}_n \geq x\right)\right) \geq \psi(n) \left(t_1(n) b(t_1(n))\right)^{\alpha \Gamma_1(n)}. \tag{232}$$

When working out the different parts of the upper bound, we see that we can also work it out to get a the same general equation as in (232). That is, (72) becomes

$$\psi(n)(tb(t))^{\alpha \Gamma(n)}. \tag{233}$$

Since the exponent only differs by a constant from what we had before, we have that the limits of $\Gamma_i(n)$ for $i \in \{1,\ 2,\ 3\}$ all converge to α times their limit. From here, we can tell that this changes the final limit of the constant and $\psi(n)$. We shall show the work for the lower bound. We will show that the main statement of Theorem 4.3 becomes

$$\left(\frac{x}{s} - \frac{s_1 m}{s}\right)^{-\alpha} \leq \lim_{n \to \infty} n^\alpha b(n) f^{-1}\left(\mathbb{P}\left(\bar{S}_n \geq x\right)\right), \tag{234}$$

for the lower bound and changing the work for the upper bound gives the analogous result.
We had that $\lim_{n \to \infty} \Gamma_1(n) = -1$, therefore $\lim_{n \to \infty} \alpha \Gamma_1(n) = -\alpha$. Since we required

that $\psi(n) = nb(n)$ to counter $\Gamma_1(n)$ previously it is clear that in the new case we require that $\psi(n) = n^\alpha b(n)$, the $b(n)$ term once again does not require the α in the exponent as a consequence of $b(n)$ being slowly varying. When altering the calculations of the proof of Proposition 6.5 this becomes clear. The constant, $\left(\frac{x}{s} - \frac{s_1 m}{s}\right)^{-1}$, changes as a result of the change of the general equation. When we want to split off the constant from this equation, analogously to (145), we see that the only difference is the exponent; that is

$$\psi(n)\left(t(n)b(t(n))\right)^{\alpha\Gamma(n)} = (nb(n))^\alpha \left(cnb(t(n))\right)^{\alpha\Gamma(n)}, \tag{235}$$

$$= \left(\frac{b(n)}{b(t)}\right)^\alpha (b(t)n)^{\alpha+\alpha\Gamma(n)} c^{\alpha\Gamma(n)}, \tag{236}$$

where we now have that $c^{\alpha\Gamma_1(n)} \to c^{-\alpha}$.
The analog of this bound for the upper bound can be shown in the same fashion. We will now present the main theorem of this thesis for any $\sigma \in \mathbb{R}_{>0}$.

Theorem 8.1. *(Large Deviations for Weighted Sums, Log-Normal Tails).*
Let $\{X_j\}_{j\in\mathbb{N}}$ be a sequence of i.i.d. random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with

$$\mathbb{E}[|X_1|^k] < \infty, \ \forall k \in \mathbb{N}, \tag{237}$$

and let $m := \mathbb{E}[X_1]$. Suppose that there exists slowly-varying functions b, c_1, c_2 : $(0,\infty) \to (0,\infty)$ and constants $\alpha > 0$ and $t^* > 0$ such that for $t \geq t^*$,

$$c_1(t)\exp\left(-\ln^2(b(t)t^\alpha)\right) \leq \mathbb{P}(X_1 \geq t) \leq c_2(t)\exp\left(-\ln^2(b(t)t^\alpha)\right). \tag{238}$$

Let $\{a_j(n)\}_{j\in\mathbb{N}}$, $n \in \mathbb{N}$, be an infinite array of non-negative real numbers that satisfy conditions (3.1); let s and s_1 be the associated constants, respectively; and let $\{\bar{S}_n\}_{n\in\mathbb{N}}$ be the sequence of weighted sums

$$\bar{S}_n := \sum_{j=1}^n a_j(n)X_j. \tag{239}$$

Let f^{-1} be the function defined in (62), then

$$\left(\frac{x}{s} - \frac{s_1}{s}m\right)^{-\alpha} \leq \lim_{n\to\infty} b(n)n^\alpha\ f^{-1}\left(\mathbb{P}\left(\bar{S}_n \geq x\right)\right) \leq \exp\left(\frac{\alpha}{2}\right)\left(\frac{x}{s} - \frac{s_1}{s}m\right)^{-\alpha}, \ \forall x > s\alpha^2 + s_1 m. \tag{240}$$

Remark 8.1. *Note that the domain is also dependant on the variance, the cause of this can be traced back to Lemma 6.8.*

Now that we have found the bounds for general variance in Theorem 8.1, we can move on to comparing the rate of decreases for $\mathbb{P}(\bar{S}_n \geq x)$.

8.1.1 Speed of decay

We can tell that the probability for log-normally distributed random variables approaches 0 slower than the Cramér case [6] and the stretched exponential case from Gantert et al [5]. This can be done by comparing the main statements of each of the theorems.
We already compared the rate of decay between Cramér and Gantert et al in a remark

below Theorem 3.1. In order to see what the speed of decrease is for our result, we calculate the inverse analogously to calculating the inverse for Cramér in the Introduction, Section 1 of this thesis. That is;

$$\left(\frac{x}{s}-\frac{s_1}{s}m\right)^{-\alpha} \lessapprox b(n)n^\alpha \; f^{-1}\left(\mathbb{P}\left(\bar{S}_n \geq x\right)\right) \lessapprox \exp\left(\frac{\alpha}{2}\right)\left(\frac{x}{s}-\frac{s_1}{s}m\right)^{-\alpha}; \tag{241}$$

$$f\left((b(n)n^\alpha)^{-1}\left(\frac{x}{s}-\frac{s_1}{s}m\right)^{-\alpha}\right) \lessapprox \mathbb{P}\left(\bar{S}_n \geq x\right) \lessapprox f\left((b(n)n^\alpha)^{-1}\exp\left(\frac{\alpha}{2}\right)\left(\frac{x}{s}-\frac{s_1}{s}m\right)^{-\alpha}\right); \tag{242}$$

where $f(x) = \phi(\ln(x))$ from Corollary 4.2. Filling this in in the equation above, (242); gives

$$\begin{aligned}&\exp\left(-\ln^2\left((b(n)n^\alpha)^{-1}\left(\frac{x}{s}-\frac{s_1}{s}m\right)^{-\alpha}\right)\right)\\ &\quad\lessapprox \mathbb{P}\left(\bar{S}_n \geq x\right)\\ &\qquad\lessapprox \exp\left(-\ln^2\left((b(n)n^\alpha)^{-1}\exp\left(\frac{1}{2}\right)\left(\frac{x}{s}-\frac{s_1}{s}m\right)^{-\alpha}\right)\right).\end{aligned} \tag{243}$$

Using the following property for logarithms,

$$\ln^2\left(x^{-r}\right) = \left(-r\ln(x)\right)\left(-r\ln(x)\right) = \left(r\ln(x)\right)^2; \tag{244}$$

we have that (243) becomes

$$\begin{aligned}&\exp\left(-\ln^2\left(b(n)n^\alpha\left(\frac{x}{s}-\frac{s_1}{s}m\right)^{\alpha}\right)\right)\\ &\quad\lessapprox \mathbb{P}\left(\bar{S}_n \geq x\right)\\ &\qquad\lessapprox \exp\left(-\ln^2\left(b(n)n^\alpha\exp\left(-\frac{1}{2}\right)\left(\frac{x}{s}-\frac{s_1}{s}m\right)^{\alpha}\right)\right).\end{aligned} \tag{245}$$

Now, we can compare the different exponents for the three cases. We are particularly interest in the speed function; that is

$$\mathbb{P}\left(\bar{S}_n \geq x\right) \approx \exp(-R(n)c), \tag{246}$$

where $R(n)$ is the speed function given by

$$R(n) = \begin{cases} n, & \text{for Cramér;} \\ n^r, & r \in (0,1), \text{ for Gantert et al;} \\ \ln^2\left(b(n)n^\alpha\right), & \alpha \in \mathbb{R}_{>0} \text{ for log-normal random variables.} \end{cases} \tag{247}$$

We shall call the speed functions $R_1(n)$, $R_2(n)$, and $R_3(n)$; respectively.

Remark 8.2. *We used c to represent the constant parts of the speed functions. For this reason, we were able to summarize the lower bound as well as the upper bound of the decay speed with* $-\ln^2\left(b(n)n^\alpha c\right)$. *We are mainly interested in how fast the equations go to* $-\infty$ *in (247).*
Also note that in the Cramér case we consider the probability $S_n \geq x$*, rather than* $\bar{S}_n \geq x$. *This corresponds to setting* $a_j(n)\frac{1}{n}$, $j \leq n$.

We are now able to compare the decay speeds by comparing the speed at which $R_i(n) \to \infty$, for $i \in \{1,\ 2,\ 3\}$. Doing so shows that Cramér's speed of decrease is the fastest.
We have that

$$\lim_{n\to\infty} \frac{R_2(n)}{R_1(n)} = \lim_{n\to\infty} \frac{n^r}{n} = 0; \tag{248}$$

and

$$\lim_{n\to\infty} \frac{R_3(n)}{R_2(n)} = \frac{\ln^2\left(b(n)n^\alpha\right)}{n^r} = 0. \tag{249}$$

The last limit follows from property 2 of Proposition 2.1.
As a result of Remark 2.1, the speed function, $R_3(n)$, is a slowly-varying function. Therefore, it tends to infinity much slower than the other two functions. Beyond this, Cramér's Theorem considers light-tailed distributions, so deviations are bound to tend to 0 much more quickly than heavy-tailed distributions and therefore, $R_1(n)$ grows much quicker than the other functions. Concerning the stretched exponential distribution and the log-normal distribution; especially for greater variance, the log-normal distribution has a lot more mass in its tails than the stretched exponential distribution. This is also a consequence of the logarithm being a slowly-varying function. For this reason, you would also expect the rate function for the log-normal case to grow slower than the rate function for stretched exponential case; which we found in (249).

8.2 Limits of the upper bound

In the result of Theorem 4.3, we found that the upper bound was a factor $\exp\left(\frac{1}{2}\right)$ off from the lower bound. When considering general variance, we even found that this factor was dependant on the variance, that is, for $\sigma^2 = var(X)$ and $\alpha = \frac{1}{\sigma}$, we have that the factor became $\exp\left(\frac{\alpha}{2}\right)$ in Theorem 8.1.
When comparing this result to that the findings of Cramér and Gantert et al, we see that their upper bound and lower bound converged to the same constant instead of finding that they deviated from each other like our result did.
Besides this, our A_2^n did not converge. We had to put an extra constraint on the domain as a consequence.
This leaves the question of whether or not further research could get rid of these two problems.

8.2.1 A_1^n

We start with A_1^n. By the application of the Laplace Principle and the lack of convergence of A_2^n, we know that the constant for our upper bound is solely derived from A_1^n. Therefore, in order to get rid of the $\exp\left(\frac{\alpha}{2}\right)$-term, we need to trace back to the root of this deviation.
This can be viewed from two different perspectives. In (174), we saw that we had

$$\lim_{n\to\infty} \frac{\ln(c_2(n)n)}{\ln(t_2(n)b(n))} = 1. \tag{250}$$

In order to get rid of the deviation, we must have that this limit goes to 0. For this reason, we could either look at getting rid of the n in $\ln(c_2(n)n)$, which traces back to applying the union bound in (85); or we need to choose $t_2(n)$ differently. The latter of which results in defining a $t_2(n)$ such that the above limit goes to 0.

8.2.2 A_2^n

Next, we take a look at A_2^n. The problem here, was that $\Gamma_3(n)\ln(\psi(n))$ no longer converged when calculating the final limit in Lemma 6.8. Inspecting the limit of $\Gamma_3(n)$ in Lemma 6.3, we see that (123) does not go to a constant anymore.
We saw in the proof above (123) that changing the $\beta(n)$ will not necessarily tackle the convergence problem, as the $\beta(n)$ from the numerator canceled the $\beta(n)$ from the denominator for $i = 1$. However, we needed to use a well-chosen $\beta(n)$ in order to optimize the Chernoff Bound, see Remark A.1. It could still be the case that there exists a $\beta(n)$ that would work better.
It does not make sense to look for a different $t_2(n)$ here because we would need it to be decreasing, which is a definite contradiction to the A_1^n-case.

8.2.3 Summary

Altogether, we have that we could correct the deviation from the lower bound by either looking for a stronger growing $t_2(n)$ or a tighter upper bound than the union bound for A_1^n. Looking for a different $t_2(n)$ has all sorts of implications for all the limits of the paper, since we would have to change the lower bound and might cause $\Gamma_3(n) \to \infty$. This would result in a lot of research and will not necessarily ensure the riddance of the deviation $\exp\left(\frac{\alpha}{2}\right)$.
Concerning the convergence of $\Gamma_3(n)\ln(\psi(n))$; this might follow from using tighter estimates for the terms in the expectation of the Λ_j. Besides this, only a decreasing $\psi(n)$ would help otherwise but this would only cause further problems for the lower bound as well as A_1^n.

8.3 Conclusion

All in all, we were able to bound the probability of the weighted mean of log-normally distributed random variables up to a deviation between bounds dependant on the variance. For log-normal random variables this probability tends to 0 much slower than for stretched exponential random variables or light-tailed distributions on account of the log-normal random variables having a lot of mass in their tails.
Perhaps with the help of further research we could get rid of this mismatch; but we are pleased with how far we have gotten with setting up a Large Deviations Theorem for log-normal random variables.

A Probabilistic Inequalities

In this appendix, we will give the statements and the proofs of the Markov Inequality, the Chernoff Bound and the Chebyshev Inequality.

Theorem A.1. *(Markov Inequality)*
Let X be a non-negative random variable with finite mean and let $\alpha > 0$. Then

$$\mathbb{P}\left(X \geq \alpha\right) \leq \frac{\mathbb{E}[X]}{\alpha}. \tag{251}$$

Proof. By the definition of expectation, we have that

$$\mathbb{E}[X] = \int_0^\infty t f_X(t) dt. \tag{252}$$

We split this integral into two parts, one part up until α and the other part is the rest.

$$\int_0^\infty t f_X(t) dt = \int_0^\alpha t f_X(t) dt + \int_\alpha^\infty t f_X(t) dt, \tag{253}$$

$$\geq \int_\alpha^\infty t f_X(t) dt, \tag{254}$$

$$\geq \alpha \int_\alpha^\infty f_X(t) dt, \tag{255}$$

$$= \alpha \mathbb{P}\left(X \geq \alpha\right). \tag{256}$$

It follows that

$$\mathbb{E}[X] \geq \alpha \mathbb{P}\left(X \geq \alpha\right). \tag{257}$$

The desired result follows by dividing both sides of (257) by α. □

Theorem A.2. *(Chernoff Bound)*
Let X be a non-negative random variable with finite mean and let $\alpha > 0$ and $t > 0$. Then

$$\mathbb{P}\left(X \geq \alpha\right) \leq \frac{\mathbb{E}[e^{tX}]}{e^{t\alpha}}. \tag{258}$$

Proof. The proof is fairly straightforward. We apply two transformations to $\mathbb{P}(X \geq \alpha)$. First, we multiply the inequality within the probability by t, then we apply it on the exponential function. Both of these transformations preserve order relations, so that the set of values for X for which $X \geq \alpha$ stays the same; i.e.

$$\{X \in [0, \infty) : X \geq \alpha\} \tag{259}$$

$$= \{X \in [0, \infty) : tX \geq t\alpha\}, \tag{260}$$

$$= \left\{X \in [0, \infty) : e^{tX} \geq e^{t\alpha}\right\}. \tag{261}$$

Since X has the same distribution for all cases, all three situations have the same density function. All of the above together shows that

$$\mathbb{P}\left(X \geq \alpha\right) = \mathbb{P}\left(e^{tX} \geq e^{t\alpha}\right) \tag{262}$$

Lastly, the Chernoff Bound follows by applying Theorem A.1 to (262). □

Remark A.1. *Typically, the Chernoff Bound is optimized over* t*. In fact, the optimization can be traced back to Cramér [6] in* (4)*; we see that we optimize over* t *in the rate function This results in some* t *working better than others. As a consequence, it follows that we might not have chosen the optimal* $\beta(n)$ *in* (91)*.*

Theorem A.3. *(Chebyshev Inequality)*
Let X *be a random variable with finite mean* μ *and existing variance* σ^2*. For all* $k > 0$*, we then have*

$$\mathbb{P}\left(|X - \mu| \geq k\sigma\right) \leq \frac{1}{k^2}. \tag{263}$$

Proof. We are going to begin by squaring the inequality over which we take the probability in (A.3). Unlike the proof of the Chernoff Bound, Theorem A.2, this transformation does not preserve order relations. However, since we are looking at $|X - \mu|$, the probability remains the same; i.e.

$$\mathbb{P}\left((X - \mu)^2 \geq \sigma^2 k^2\right) \tag{264}$$

$$= \mathbb{P}\left(X - \mu \geq \sigma k\right) + \mathbb{P}\left(X - \mu \leq -\sigma k\right), \tag{265}$$

$$= \mathbb{P}\left(|X - \mu| \geq \sigma k\right). \tag{266}$$

Now, we apply Markov's Inequality on (264), yielding

$$\mathbb{P}\left((X - \mu)^2 \geq \sigma^2 k^2\right) \leq \frac{\mathbb{E}[(X - \mu)^2]}{\sigma^2 k^2}. \tag{267}$$

Note that $\mathbb{E}[(X - \mu)^2] = var(X) = \sigma^2$, therefore, (267) becomes:

$$\mathbb{P}\left((X - \mu)^2 \geq \sigma^2 k^2\right) \leq \frac{1}{k^2}. \tag{268}$$

□

B Partial Integration

In this appendix we provide the Partial Integration Theorem used in the proof of Lemma 6.4

Theorem B.1. *(Partial Integration)*
Let X be a random variable, let $\alpha > 0$ and let $q_1, q_2 \in \mathbb{R}$, such that $q_1 < q_2$. Then

$$\begin{aligned}\mathbb{E}\left[\exp(\alpha X)\,\mathbb{1}_{\{q_1 \le X \le q_2\}}\right] = \alpha \int_{q_1}^{q_2} \exp(\alpha t)\,\mathbb{P}(X \ge t)dt + \exp(\alpha q_1)\mathbb{P}(X \ge q_1) \\ - \exp(\alpha q_2)\mathbb{P}(X \ge q_2). \end{aligned} \tag{269}$$

Proof. We begin by writing out the expectation (269).

$$\mathbb{E}\left[\exp(\alpha X)\,\mathbb{1}_{\{q_1 \le X \le q_2\}}\right] = \int_{q_1}^{q_2} \exp(\alpha t)\, f_X(t)dt, \tag{270}$$

where $f_X(t)$ is the density function of X. Next, we apply standard partial integration on (270), yielding

$$\int_{q_1}^{q_2} \exp(\alpha t)\, f_X(t)dt \tag{271}$$

$$= |\exp(\alpha t)\,\mathbb{P}(X \le t)|_{q_1}^{q_2} - \alpha \int_{q_1}^{q_2} \exp(\alpha t)\,\mathbb{P}(X \le t)dt, \tag{272}$$

$$= |\exp(\alpha t)\,(1 - \mathbb{P}(X \ge t))|_{q_1}^{q_2} - \alpha \int_{q_1}^{q_2} \exp(\alpha t)\,(1 - \mathbb{P}(X \ge t))dt, \tag{273}$$

$$\begin{aligned} &= \exp(\alpha q_2)(1 - \mathbb{P}(X \ge q_2)) - \exp(\alpha q_1)(1 - \mathbb{P}(X \ge q_1)) \\ &\quad - \alpha \int_{q_1}^{q_2} \exp(\alpha t)dt + \alpha \int_{q_1}^{q_2} \exp(\alpha t)\mathbb{P}(X \ge t)dt, \end{aligned} \tag{274}$$

$$\begin{aligned} &= \alpha \int_{q_1}^{q_2} \exp(\alpha t)\,\mathbb{P}(X \ge t)dt + \exp(\alpha q_1)\mathbb{P}(X \ge q_1) \\ &\quad - \exp(\alpha q_2)\mathbb{P}(X \ge q_2); \end{aligned} \tag{275}$$

since

$$\alpha \int_{q_1}^{q_2} \exp(\alpha t)dt = \exp(\alpha q_2) - \exp(\alpha q_1). \tag{276}$$

□

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