Bachelor Thesis

Magic Card Trick Analysis

By

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Laymen's summary

The magical "Cheney card trick" will be presented in this thesis using a mathematical approach. This trick is performed by a magician and an assistant in the following way: the magician leaves the room, and the assistant lets the audience draw 5 cards from a standard deck of 52 cards. Then, the assistant returns one of the 5 drawn cards to the audience and places the remaining 4 cards on the table. The magician can enter the room once the 4 cards have been placed on the table. The magician looks at the 4 cards on the table and, by these 4 cards, identifies the card that was given to the audience.

The underlying mathematics of this trick is this thesis's primary research goal. We will analyse this trick's exact working and discover the mathematical concepts. Introducing mathematical definitions and theorems will help us prove specific properties of the trick. We will discuss ways to extend the trick by using a larger deck of cards than a standard deck of 52 cards. In the trick's working, specific orders of the 4 cards on the table will be studied. Additionally, we will introduce a step-by-step process that can be used to find a protocol that the magician and the assistant can use to perform the trick. Ultimately, this thesis will provide insight into how this specific card trick and mathematics are tied together to create a fantastic illusion.

Summary

The magical "Cheney card trick" will be presented in this thesis using a mathematical approach. This trick is performed by a magician and an assistant in the following way: the magician leaves the room, and the assistant lets the audience draw 5 cards from a standard deck of 52 cards. Then, the assistant returns one of the 5 drawn cards to the audience and places the remaining 4 cards on the table. The magician can enter the room once the 4 cards have been placed on the table. The magician looks at the 4 cards on the table and, by these 4 cards, identifies the card that was given to the audience.

This thesis explores the underlying mathematical principles. The trick's workings will first be explained, after which an algorithm for performing the trick will be given. In the analysis of the trick, we noticed that the size of the deck that we performed the trick with could be expanded. Given that we draw n cards from a deck of size d, we introduce an upper bound. This upper bound on the deck size is: $d \leq n! + n - 1$. Furthermore, we introduce Birkhoff-von-Neumann's theorem and Hall's marriage theorem. Using these two theorems, we will prove that a convention for the magician and the assistant to perform the trick always exists when we attain the introduced upper bound. We will prove the existence of a convention and provide algorithms to perform the trick with a deck of cards attaining the introduced upper bound. While performing the trick, a specific order of the cards appears more often than others. When analysed, we find that a less preferable order of the cards is only necessary for approximately 10 per cent of the drawn hands. Additionally, a theorem of Gale-Shapley about stable and optimal matches is introduced, adding an extra dimension to Hall's marriage theorem. Gale-Shapley's algorithm is then introduced, and the possibility of finding a stable match between hands and messages is explored. Altogether, this thesis aims to find an answer to the question: *What mathematical principles and ideas underlie the "Cheney card trick"?*

Contents

1 Introduction

“Magic’s just science that we do not understand yet.”— Arthur C. Clarke. This quote applies to this thesis, as we will thoroughly investigate the underlying mathematics of the "Cheney card trick". Magic and mathematics can go hand in hand, especially regarding card tricks. Some card tricks depend on trickery or sleight of hand. There are card tricks, however, where elegant mathematical concepts underlie them. The card trick we will investigate in this thesis is one of those card tricks. It was invented by Fitch Cheney in 1950, in this thesis, we will refer to it as the "Cheney card trick". A magician and an assistant can perform it in the following way.
The magician leaves the room, and the assistant asks the audience to draw 5 cards from a standard deck of 52 cards. The 5 cards are returned to the assistant, who looks at the 5 cards. The assistant gives the audience one card back, the 3 ♡. The assistant places the remaining 4 cards on the table. The cards read from left to right: Q ♡, 7♢, 7♠, A♣. The magician now enters the room. The magician reviews the 4 cards and tells the audience which card they have: the 3 ♡.

1.1 Objective and scope of this thesis

As we described, this trick does not involve trickery or fast-handedness. How can the magician know the fifth card from the 4 cards lying on the table? In this thesis, this trick will be analysed extensively. The trick itself is used for its ability to amaze. However, it often goes unnoticed what large amount of elegant mathematical concepts underlie it. This thesis will answer the question: *What mathematical principles and ideas underlie the "Cheney card trick"?* We will introduce mathematical concepts that can be applied to the card trick. These concepts will be used to prove theorems concerning the card trick and will also be considered further, finding related results.

1.2 Outline of the content of this thesis

In Section 2, we review the necessary literature for this thesis. We will consider two articles by Kleber and Vakil [2002] and Simonson and Holm [2003]. These articles have previously researched the "Cheney card trick", and we will state their findings. In Subsections 2.1 and 2.2, Hall’s marriage theorem by Hall [1935] and Birkhoff-von-Neumann’s theorem by Birkhoff [1946] will be introduced. These theorems will be important in our analysis of the trick and reveal some of the mathematical background underlying the card trick.
Section 3 will begin the mathematical analysis of the "Cheney card trick". After properly introducing the card trick, we examine the trick’s exact workings using a standard deck of 52 cards. Even though it is said that "A magician never reveals their secrets", we will dive into the exact details of the trick. However, I encourage readers of this thesis to figure out how the trick works before reading the explanation.
Subsection 3.1 includes a complete explanation of how the trick can be performed. We also explore how the trick could be performed differently and include an algorithm for performing the trick.
In Subsection 3.2, we start our analysis by introducing two relevant definitions concerning the card trick. Using these definitions and the theorems we have introduced, we will derive and prove relevant lemmas and theorems concerning the card trick.
In Subsection 3.3, we introduce a general upper bound on the deck size based on the number of cards drawn. When drawing 5 cards, we find that the number of cards in the deck is bounded by 124.
Subsection 3.4 introduces an important theorem concerning the "Cheney card trick". After introducing the upper bound on the deck size, we will prove that when we attain this upper bound, there exists a convention the magician and the assistant can use to perform the trick. This can be done by pairing all possible hands to a specific message using the theorems we introduced in our literature review.
Having established that a convention exists, the next question is whether the magician and the assistant can devise a specific convention to perform the card trick using a larger deck of cards.
In Subsection 3.5, we introduce a way to write an integer p within a specific range as a sum of factorials. We include three algorithms for the assistant and the magician to perform the trick. The first algorithm entails the process of ordering the cards to represent a number p using the previously introduced method. Finally, we prove that these three algorithms together form a way to perform the trick where n cards are drawn from a deck of size $d \leq n! + n - 1$. This provides a constructive proof of the existence of a convention. With a thorough understanding, this convention can even be performed without using scrap paper or trying to remember much information. In Section 4, we will take a different approach to the card trick. After proving the theorems and providing a specific method to perform the trick using n cards drawn from a deck where $d \leq n! + n - 1$, we will return to the original ’Cheney card trick’ with 5 cards drawn from a standard deck of 52 cards.
When performing the card trick, we have a less favourable way of ordering the cards to give a path to the hidden card.

We will investigate the probability of this less favourable arrangement by examining certain hands and assessing how the suits can be dealt with in each hand. Afterwards, we will analyse the corresponding probabilities of specific hands and draw a conclusion. Combining the results, we find that we are only sometimes required to use this unfavourable ordering method.

In Section 5, Hall's marriage theorem will be further considered. Specifically, we will discuss the Gale-Shapley algorithm, which introduces an additional condition for Hall's marriage theorem. We shall consider their algorithm and explore how it can be used. We will explore the possibility of applying this algorithm to find a convention which we can use to perform the "Cheney card trick". This proves to be quite a comprehensive computational process. However, the algorithm does provide us with insight into how we find a specific convention to perform the card trick with.

We conclude our research in Section 6. In Subsection 6.1, we state our most important findings and summarise the mathematical concepts we applied in our analysis of the "Cheney card trick".

In Subsection 6.2, we introduce parts of this research which could be investigated further.

Altogether, we investigate the mathematical principles within the "Cheney card trick" and include results relating to the underlying mathematics. We begin by reviewing the existing literature concerning the "Cheney card trick" in the next section.

2 Literature review

This thesis is based on the following articles written by: Kleber and Vakil [2002] and Simonson and Holm [2003]. The article of Kleber and Vakil [2002] describes the card trick performed with 5 cards drawn from a standard deck of 52 cards. The article tells us about the trick's first appearance in the book "Math Miracles". This book credits the trick to William Fitch Cheney Jr., also known as "Fitch", a mathematician at MIT. The trick is often referred to as the "Cheney card trick". The article explains how the 52-card trick can be performed and suggests the possibility of a trick using a larger deck of cards. It includes a brief derivation of the upper bound on the deck size and gives a sketch of a proof of the existence of a strategy when this upper bound is attained. After this is done, an actual strategy for performing the trick is provided, where 5 cards are drawn from a deck with a maximum size of 124 cards. The article ends with a combinatorial question about how many strategies exist for maximum deck size.
The article by Simonson and Holm [2003] delves deeper into the topic, providing a more mathematical approach to the card trick. The authors start by tackling the problem of finding a strategy for performing the trick. In order to find a strategy, all the possibilities in which a hand can be drawn and the possibilities of choosing the hidden card are considered. It is quickly realised that to find a strategy, something more ingenious is to be thought of. When trying different combinations for every hand, it is necessary to backtrack. Backtracking occurs when choosing a hidden card and ordering the remaining cards to give a path to this hidden card. When we consider a particular hand and notice that all the different orders of that hand to give a path to a hidden card are depleted, we need to backtrack. We have to consider our previous orders of cards and change them. This results in changing many combinations, making it a comprehensive and exhausting process.
The article of Simonson and Holm [2003] also introduces lower and upper bounds for the number of cards in the deck used in the trick. The upper bound is first defined as the number of cards in the deck for which we can find a trick to perform. Every time a trick is found for a deck of a certain size, the upper bound is increased to this number of cards. The question is posed of whether there is a limit to the upper bound and if it can be reached. A computer model finds a strategy for the trick when performed with 3 cards drawn from a deck of 8 cards. Another strategy is found, and a question about the uniqueness of the strategies is posed. It is noticed that the two different strategies for the trick contain some symmetries where every ordered pair is switched in order. A general upper bound is introduced for the size of the deck of cards. Since there exists a strategy when drawing 3 cards from a deck of 8 cards equalling the general upper bound, the article poses the following question: "Does a strategy exist for the card trick when we draw n cards from a deck of cards which size attains a specific upper bound?" The authors prove the existence of a strategy using bipartite graphs and Hall's marriage theorem. The article concludes with the same convention as in the article by Kleber and Vakil [2002], to perform the trick with 5 cards drawn from a deck of 124 cards.
After considering these two articles, we are ready to introduce the two theorems needed in our analysis of the "Cheney card trick". The first theorem we introduce is Hall's.

2.1 Hall's marriage theorem and its application

Philip Hall was an English mathematician who lived during the 20th century. Hall introduced his marriage theorem in his paper "On representatives of subsets". He concerned himself with the existence of a system of distinct representatives. A system of distinct representatives was first introduced by König [1916] and is defined as follows:

Definition 2.1. Given any set S and any finite system of subsets of S: $T_1, T_2, \ldots, T_m$.
A Complete set of Distinct Representatives (C.D.R) is a set of m distinct elements of S: $a_1, a_2, \ldots, a_m$ such that:

$$a_i \in T_i \quad \text{for each } i = 1, 2, \ldots, m.$$

We may say a_i represents T_i, where every T_i does not need to be finite or distinct from each other. Given this definition, we can now recite Hall's marriage theorem:

Theorem 2.1. *Hall's marriage theorem, by Hall [1935]. Given any set S and any finite system of subsets of S:*

$$T_1, T_2, \ldots, T_m. \tag{2.1}$$

In order for a C.D.R. of the sets shown in Equation 2.1 to exist, it is sufficient that for each $k = 1, \ldots, m$ any selection of k of the sets 2.1 shall contain between them at least k elements of S.

Theorem 2.1 has a straightforward application; it can be used to investigate whether a group of men and women can be paired with one another. Suppose a group of men all have a list of women they prefer to marry. The women do not have a list of preferences and would prefer to marry with every man. Theorem 2.1 can then be used to verify whether it is possible to marry all these men to a woman. Hall tells us that if each subset of k men has at least k women in their combined list of preferences, then every man can be married to a woman. This is also where Theorem 2.1 gets most of its recognition and why it is often referred to as "Hall's marriage theorem". To understand this theorem better, we will consider the following example.

Example 2.1. Suppose we have a set S of 5 men, $S = \{M_1, M_2, M_3, M_4, M_5\}$. There are 5 women $\{W_1, W_2, W_3, W_4, W_5\}$, of which every man has 2 women he prefers. The preferences of the men are given as subsets of these women.

$$M_1 :\{W_1, W_2\}, \quad M_2 :\{W_1, W_3\}, \quad M_3 :\{W_3, W_4\}$$
$$M_4 :\{W_3, W_4\}, \quad M_5 :\{W_4, W_5\}$$

We will check Hall's condition to verify whether we can find a complete set of distinct representatives for the men.
For $k = 1$, all subsets of 1 man have at least 1 woman in their preferences.
For $k = 2$, every union of 2 subsets contains at least 2 elements, as each man prefers 2 women.
For $k = 3$, we will look at the unions of preferences of 3 men. For all but one of the unions of the preferences of 2 men, there are already 3 different women in their combined preferences. M_3 and M_4, however, prefer the same 2 women. Adding a different man to this union increases their combined set of preferred women to at least 3 women since all 3 other men prefer at least 1 additional woman.
For $k = 4$, we will consider the 5 different unions of preferences.

$$M_1, M_2, M_3, M_4 \text{ prefer } W_1, W_2, W_3, W_4$$
$$M_1, M_2, M_3, M_5 \text{ prefer } W_1, W_2, W_3, W_4, W_5$$
$$M_1, M_2, M_4, M_5 \text{ prefer } W_1, W_2, W_3, W_4, W_5$$
$$M_1, M_3, M_4, M_5 \text{ prefer } W_1, W_2, W_3, W_4, W_5$$
$$M_2, M_3, M_4, M_5 \text{ prefer } W_1, W_3, W_4, W_5$$

All subsets of 4 men have at least 4 women in their combined preferences.
For $k = 5$, the combined preference of all 5 men is the set: $\{W_1, W_2, W_3, W_4, W_5\}$, which has 5 elements. We can conclude that the condition of Theorem 2.1 holds for all possible subsets of the men's preferences. Therefore, a complete set of distinct representatives exists. To find a complete set of distinct representatives, we must select a woman from each man's preference list such that no two men are paired with the same woman. We can pair M_1 to W_2, M_2 to W_1, M_3 to W_3, M_4 to W_4 and M_5 to W_5, bringing us to the following complete set of distinct representatives:

$$a_1 = W_2$$
$$a_2 = W_1$$
$$a_3 = W_3$$
$$a_4 = W_4$$
$$a_5 = W_5.$$

This way the set $\{a_1, a_2, a_3, a_4, a_5\}$, is a set that represents the men M_1, M_2, M_3, M_4, M_5.
By applying Theorem 2.1 to the problem, we have determined the possibility of pairing each man to a different woman according to their preferences, ensuring a unique match for each man.

In our analysis of the card trick in Section 3, we will use Hall's marriage theorem to pair all possible drawn hands to specific messages given by the cards on the table.

2.2 Birkhoff-von-Neumann's matrix decomposition and its application

Birkhoff-von-Neumann introduced their theorem about doubly stochastic matrices in 1946. To better understand their theorem, we will first explain the terms included in the theorem. A *doubly stochastic matrix* is a square matrix of non-negative real numbers, where each column and row sums to 1. A *convex combination* is a linear combination (in this case of matrices) where all coefficients are non-negative and sum to 1. A *permutation matrix* is a square $\{0,1\}$-matrix where every row and column contains exactly one 1, and its remaining entries are 0. We can now introduce the theorem.

Theorem 2.2. *Birkhoff-von-Neumann's theorem, by Birkhoff [1946]*
Every doubly stochastic matrix is a convex combination of permutation matrices.

Theorem 2.2, tells us that any doubly stochastic matrix $A \in \mathbb{R}^{n\times n}$ can be written in the form $A = \sum_{i=1}^{n} c_i * P_i$ with $c_i \in \mathbb{R} \geq 0$ constants and $P_i \in \mathbb{R}^{n\times n}$ permutation matrices, where $\sum_{i=1}^{n} c_i = 1$. We will consider an example to understand better what Theorem 2.2 entails.

Example 2.2. Suppose we have a doubly stochastic matrix $A = \frac{1}{3}\begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

We can write this matrix as the convex combination of permutation matrices:

$$A = \frac{1}{3}\begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.2}$$

The convex combination in which matrix A can be written is often referred to as the Birkhoff-von-Neumann decomposition. Theorem 2.2 is very helpful in numerous fields of mathematics, such as combinatorial optimisation, game theory and matrix theory.

Having introduced these two theorems and the previous research on this topic, we are ready to start our analysis on the "Cheney card trick".

3 Mathematical approach of the "Cheney card trick"

We want to analyse the "Cheney card trick" using a mathematically based approach. First start by explaining the exact workings of the trick. Afterwards relevant definitions concerning the trick will be defined. Using these definitions, we can introduce bounds on the deck size and prove whether a trick can be performed when we attain these bounds. We will prove this using both a non-constructive and constructive proof.

3.1 The "Cheney card trick" using a standard deck of 52 cards

The trick is performed by a magician and an assistant, using a standard deck of 52 cards. The magician leaves the room, and the assistant lets the audience draw 5 cards from a standard deck of 52 cards. The assistant returns one card to the audience for them to see, this card is the hidden card for the magician. The assistant places the remaining 4 cards on the table in a specific order for the magician to see. The magician enters the room, reviews the cards on the table, and names the hidden card given to the audience, which is then revealed.

To start our analysis of the trick, we will number the cards which do not have a number yet from 1 to 13, where the ace represents the number 1, the jack the number 11, the queen the number 12 and the king the number 13. The trick begins by drawing 5 random cards from a deck of 52 cards, which are then given to the assistant. The assistant now has 5 cards; among these, there will always be at least two cards of the same suit. This is a direct application of the pigeonhole principle. The pigeonhole principle was introduced in 1834 by Dirichlet [1834] and called "Dirichlet's box principle". The principle states that if i items are put into j containers, with $i > j$, then at least one of the containers contains more than one item. Since the assistant has 5 cards and there are only 4 suits, which correspond to the items and containers respectively, we will always have at least two cards of the same suit.

We thus have at least 2 cards of the same suit. Of these cards having the same suit, we can use 2 for the first step of our trick. We will let one of these cards be the hidden card, and use the other to declare the suit of the hidden card. The card that declares the suit is placed on the table first and will be the card shown most left on the table.

Now there are 3 cards left, which need to be placed on the table. These 3 cards can be ordered in 3! = 6 different ways. We can couple these 6 different orders to a number between 1 and 6 as follows: we categorise the cards as low (L), middle (M), and high (H) based on their numerical values. When two cards have the same number, we order them according to the suit precedence used in the card game bridge: Clubs ♣$<$ Diamond ♢$<$ Hearts ♡$<$Spades ♠. For example, when we have a 5♠ and a 5♢, the 5♠ is higher in order than the 5♢. We couple the different orders to numbers as follows:

LMH to 1	LHM to 2	MLH to 3	
MHL to 4	HLM to 5	HML to 6	(3.1)

This way, we can declare a number between 1 and 6 using the remaining 3 cards.

How can we give a path to the hidden card using a number between 1 and 6? There are 13 cards in a suit. We visualise these cards within a suit as if arranged in a circular manner, starting back at the ace once we pass the king. The assistant will give a path from the card declaring the suit to the hidden card by adding a number between 1 and 6. Since there are at least 2 cards of the same suit, we can always reach one of these cards by adding a number between 1 and 6 to the other, going past the king and starting back at the ace if necessary.

Suppose we have drawn the following 2 cards of the same suit: 4♠ and K♠. The king represents the number 13, and we cannot add a number between 1 and 6 to the 4 to reach the number 13. We can, however, give a path from the K♠ to the 4♠ by adding 4 to the K♠ and reach the 4♠ this way. The assistant will choose the 4♠ as the hidden card and place the king as the left card of the 4 cards placed on the table. The three cards on the right must represent the number 4, which can be achieved by placing these cards in the order "MHL" (middle, high, low).

We have seen how to declare the suit by placing a card of the same suit on the left. Additionally, we can declare the number of the hidden card by adding a number between 1 and 6 to the card declaring the suit. We are now ready to consider the example given in the introduction.

Example 3.1. The 5 cards the audience gives the assistant are the 3♡, Q♡, A♣, 7♢ and 7♠. There are two cards of hearts; the assistant must choose one to be the hidden card. The assistant can give a path from the Q♡ to the 3♡ by adding 4. Therefore, the Q♡ is put on the table, and the 3♡ is given to the audience to be the hidden card. There are 3 remaining cards that need to be ordered to represent the number 4. The assistant, therefore, orders them as middle, high, low, which can be done by ordering them as 7♢, 7♠, A♣. After the 4 cards have been arranged on the table, the magician can enter the room. After reviewing the 4 cards on the table, the magician observes the Q♡ and the 3 cards to the right representing the number 4. The magician can then determine and reveal the hidden card to be the 3♡.

The trick works quite well when performing it with 5 cards drawn from a deck of 52 cards. With a little practice the process can be done fluently and does not need to take a long time. However, how we perform the trick is not unique. How the cards are ordered to represent a number between 1 and 6, as shown in Equation 3.1, is simply a choice. One could, for example, order the cards based first on the card's precedence in bridge, and if two of the same suit appear, then order them based on their number. Also the card representing the suit of the hidden card can be put in another position.
While performing the trick, passing beyond the king seems less preferable. It is easier to add a number between 1 and 6 and retrieve a number below 13 to give a path to the hidden card. How we prefer to perform the trick can be formalised as an algorithm. In this algorithm the preference of not passing beyond the king is added. The algorithm for performing the trick is as follows:

Input : A hand of 5 cards drawn from a standard deck of cards
Output: Hidden card and an arrangement of the 4 cards on the table
1: Let every card, which does not have a numerical value, represent a number as follows: ace: 1, jack: 11, queen: 12 and king: 13;
if *There are 2 cards of the same suit* x_1 *and* x_2 *from the set of drawn cards, where* $|x_1 - x_2| = y \leq 6$ **then**
  i. Pick cards x_1 and x_2 and place the card with the lower numerical value on the table, let the other card be the hidden card;
  ii. Label the remaining 3 cards as low (L), middle (M), and high (H) based on their numerical values, where the cards having the same number are ordered using the suits precedence in bridge: $\clubsuit < \diamondsuit < \heartsuit < \spadesuit$;
  iii. Order the 3 remaining cards to represent the number y as in Equation 3.1 and place these 3 ordered cards to the right of the card already on the table.
end
else
  i. Pick two cards of the same suit w_1 and w_2 from the set of drawn cards;
  $y = |w_1 - w_2|$;
  ii. Place the higher card of w_1 and w_2 on the table, let the other card be the hidden card;
  iii. Label the remaining 3 cards as low (L), middle (M), and high (H) based on their numerical values, where the cards having the same number are ordered using the suits precedence in bridge: $\clubsuit < \diamondsuit < \heartsuit < \spadesuit$;
  iv. Order the 3 remaining cards to represent the number $(13 - y)$ as in Equation 3.1 and place these 3 ordered cards to the right of the card already on the table.
end

Algorithm 1: Picking the hidden card and ordering the remaining 4 cards

The magician can use the following algorithm to determine the hidden card.
Input : A sequence of 4 cards in a specific order
Output: Suit and number of the hidden card

1: Let every card which does not have a numerical value represent a number as follows: ace: 1, jack: 11, queen: 12 and king: 13;
2: Let k be the suit of the card on the left and l the number of the card on the left;
3: Label the remaining 3 cards as low (L), middle (M), and high (H) based on their numerical values, where the cards having the same number are ordered using the suits precedence in bridge: $\clubsuit < \diamondsuit < \heartsuit < \spadesuit$;
4: Determine a number y by the order in which these 3 cards are placed, based on Equation 3.1;
5: $z = l + y$;
6: **if** $z > 13$ **then**
  $z = z - 13$
end
else
  $z = z$
end
7: The hidden card has suit k and number z, where for z the numbers 1,11,12,13 represent ace, jack, queen and king respectively;

Algorithm 2: Determination of the hidden card in the "Cheney card trick"

What is often not observed initially is that the assistant chooses the card given back to the audience. If the assistant would not carefully pick this card, the trick would not be possible as the 4 cards on the table only have $4! = 24$ different orders to be put in. The assistant could then not give a path to hidden card, as the hidden card can be one of 48 cards.

3.2 Introduction of relevant definitions

To take a more mathematical approach, we introduce definitions concerning the trick. These definitions apply to the card trick where we draw n cards from a deck of d cards. The introduction of these definitions allows us to look at the trick in a more discrete and mathematical-based way. The definitions we will need are the following:

Definition 3.1. A *message* is a map given by the $n-1$ cards on the table to the hidden card.

Definition 3.2. A *convention* is a set of messages such that all possible hands have a message, where no hands have the same message.

3.3 Upper bound on the deck size

We can now start analysing the bounds on the deck size and introduce a lemma on the upper bound on the deck size. The minimum deck size is relatively straightforward. For the trick to be performed, we need to be able to draw n cards, so the lower bound for performing the trick is a deck, which contains n cards. We will now introduce a lemma regarding the upper bound on the deck size.

Lemma 3.1. Consider $n \in \mathbb{N}$ as the number of cards drawn and $d \in \mathbb{N}$ as the deck size. The upper bound on the deck size is given by:

$$d \leq n! + n - 1.$$

Proof. We consider $n \in \mathbb{N}$ to be the number of cards drawn and $d \in \mathbb{N}$ to be the size of the deck of cards. The trick starts by drawing n cards, which can be ordered in $n!$ different ways. We can thus give $n!$ different messages using the drawn cards. When performing the trick, we put 1 card of the n drawn cards back in the deck as the hidden card. We thus remove $n-1$ cards from the deck of cards. Therefore, we have $d-(n-1)$ cards, which can be the hidden card.
The total number of cards the hidden card can be, is bounded by the number of different messages. We can not have one message give a path to two different hidden cards. This brings us to the following expression $d-(n-1) \leq n!$. From this expression, we can conclude the following:

$$d \leq n! + n - 1, \tag{3.2}$$

which is what we wanted to prove. □

We have proven the upper bound on the deck size to be $d \leq n! + n - 1$. In the following subsections, we will prove the existence of a convention when attaining the upper bound on the deck size. We will do this using both a non-constructive and a constructive proof. We will start by recalling two essential theorems introduced in Subsections 2.1 and 2.2.

3.4 A non-constructive proof of the existence of a convention

We introduced Hall's marriage theorem in Subsection 2.1 as Theorem 2.1. This theorem can be used to marry a group of men to a group of women, given that for every subset of i men, their combined preferences contain at least i women. We will use this theorem to prove that we can attain a marriage between every hand and message. This way, we can ensure there exists a unique message for every hand and prove the existence of a convention.
In Subsection 2.2, we also introduced Birkhoff-von-Neumann's theorem as Theorem 2.2. The following is a corollary of this theorem:

Corollary 3.1. If a matrix contains only non-negative integers and has constant row and column sums, it can be written as a linear combination of permutation matrices.

Proof. Let A be an arbitrary square matrix containing only positive integers, with rows and columns summing to k. Then $\frac{1}{k}A$ is a doubly stochastic matrix, and by Theorem 2.2, we can write a doubly stochastic matrix as the sum of permutation matrices P_i.

$$\frac{1}{k}A = \sum_{i=1}^{r} c_i P_i \Rightarrow A = k\sum_{i=1}^{r} c_i P_i = \sum_{i=1}^{r} k c_i P_i.$$

□

We can now introduce and prove the theorem about the existence of a convention.

Theorem 3.1. *Assume $n \in \mathbb{N}$ the number of cards drawn and $d \in \mathbb{N}$ the deck size. If it holds that $d \leq n! + n - 1$, then a convention exists.*

Proof. To prove the existence of a convention, when $d \leq n! + n - 1$ holds, we split the proof into two parts. We begin by proving the existence of a convention for $d = n! + n - 1$. After proving this, we can derive that a convention also exists when $d < n! + n - 1$.
We start by considering the number of different hands that can be drawn and the number of different messages that can be sent. We draw n cards from a deck of d cards where we do not take the order of these n cards into account. The number of possible hands is then given by:

$$\binom{d}{n} = \frac{d!}{n!(d-n)!}. \tag{3.3}$$

For the messages, the order of the cards matters. Therefore, to determine the number of different messages, we need to consider the number of different ways $(n-1)$ cards can be drawn from a deck of d cards, but also multiply this by the number of ways to arrange $(n-1)$ cards; $(n-1)!$. The number of possible messages is then given by:

$$(n-1)! \cdot \binom{d}{n-1} = \frac{d!}{(d-n+1)!}. \tag{3.4}$$

We observe the number of possible hands and possible messages to be equal. A short derivation is given below, Equations 3.3 and 3.4 have equal numerators, we can prove the equality by showing the denominators to be equal. We consider a deck of size $n! + n - 1$, therefore, we fill in $d = n! + n - 1$

$$\begin{aligned}(d-n+1)! &= (d-n)!(d-n+1) \\ &= (d-n)!((n!+n-1)-n+1) \\ &= (d-n)!n! \end{aligned} \tag{3.5}$$

We create a square null-matrix $A \in \mathbb{R}^{\binom{d}{n} \times \frac{d!}{(d-n+1)!}}$. The rows of A are indexed by the number of possible hands. The columns are indexed by the number of possible messages. By Observation 3.5, this matrix is square. Every entry in this matrix refers to a specific hand of n cards and a specific message of $(n-1)$ cards. To prove the existence of a convention, we use Hall's marriage theorem to pair every row with a specific column. Thereby pairing all possible hands to a message.
We set an entry of matrix A to 1 if the message's $(n-1)$ cards are contained in the hand of n cards. We now have a square 0,1-matrix of size $\binom{d}{n} \times \frac{d!}{(d-n+1)!}$. We consider an example shown in Figure 3.1 to better understand how such a matrix looks. In this figure, the matrix corresponds to the trick where we draw 2 cards from a deck of 3 cards. This matrix is of size 3×3 as we can 3 draw 3 possible messages and hands from a deck of 3 cards.

$$\begin{array}{c} \\ (1,2) \\ (1,3) \\ (2,3) \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

Figure 3.1: Example matrix A with $n = 2$, $d = 3$ where the columns are indexed by the possible messages and the rows by the possible hands

Since every message is contained in $d - (n-1) = n!$ hands and every hand contains $d - (n-1) = n!$ messages, there are $n!$ 1's in every row and column. We now have a square matrix A, where all individual rows and columns sum up to a constant $n!$. By Corollary 3.1, we can write such a matrix as the sum of permutation matrices.
As stated before, a permutation matrix contains a 1 in every row and column where every other entry is 0. When considering such a permutation matrix, we notice that for every subset of k rows (hands), we can pair at least k eligible columns (messages) to these rows. Therefore, the condition of Theorem 2.1 holds, and we can attain a marriage between every hand and message.
We conclude there exists a convention for a deck size $d = n! + n - 1$ since, for every hand there is a message, where no hands have the same message.

We still need to prove a convention exists for $d < n! + n - 1$. We consider the number of messages and hands. The number of different hands and messages is given in equations 3.3 and 3.4, respectively. When $d < n! + n - 1$, we observe the number of messages to be larger than the number of hands. A derivation as before is given below, we only consider the numerators since the denominators ($d!$) are equal.

$$\begin{aligned}(d-n+1)! &= (d-n)!(d-n+1)\\ &< (d-n)!((n!+n-1)-n+1)\\ &= (d-n)!n! \end{aligned} \tag{3.6}$$

The numerator of the number of messages is smaller than the numerator of the number of hands. We thus have more messages than hands when $d < n! + n - 1$. By our proof for $d = n! + n - 1$ and there now being more messages than hands when $d < n! + n - 1$, we conclude there also exists a convention when $d < n! + n - 1$. We have proven the existence of a convention for a deck size $d \leq n! + n - 1$. □

To prove this is the maximum number of cards we can perform a trick with, we prove that we cannot give a convention when the deck size exceeds $n! + n - 1$. We will do so by introducing and proving the following lemma:

Lemma 3.2. Assume $n \in \mathbb{N}$ the number of cards drawn and $d \in \mathbb{N}$ the deck size. If it holds that $d > n! + n - 1$, then there exists no convention.

Proof. Assume $n \in \mathbb{N}$ the number of cards drawn and $d \in \mathbb{N}$ the deck size. As in our proof of Theorem 3.1, we have the number of different hands equal to

$$\binom{d}{n} = \frac{d!}{n!(d-n)!} \tag{3.7}$$

The number of different messages is equal to

$$(n-1)! \cdot \binom{d}{n-1} = \frac{d!}{(d-n+1)!} \tag{3.8}$$

If it holds that $d > n! + n - 1$, then the number of messages is smaller than the number of hands. A derivation is given below, we only consider the numerators as the denominators ($d!$) are equal.

$$\begin{aligned}(d-n+1)! &= (d-n)!(d-n+1)\\ &> (d-n)!((n!+n-1)-n+1)\\ &= (d-n)!n! \end{aligned} \tag{3.9}$$

Since the denominator of the number of different messages is greater than the denominator of the number of different hands, we can conclude that the number of messages is smaller than the number of different hands. This way hands must have the same message, as there are more hands than messages. Therefore, a convention does not exist. □

We have proven the existence of a convention using a non-constructive proof. This leads to the question of whether we can give a specific convention that works for n cards and a deck size $d \leq n! + n - 1$. As it turns out, there is a convention that serves as a constructive proof. In the following subsection, we will explain this specific convention.

3.5 A constructive proof of the existence of a convention

We have considered the convention for performing the "Cheney card trick" using 52 cards, as explained in Subsection 3.1. The convention for the "Cheney card trick" accounted for 4 different suits containing 13 different cards. The deck of 52 cards could therefore not easily be extended to a larger deck size. However, using a different convention, we can increase the number of cards in this deck to 124 cards, as proven by our non-constructive proof of the existence of a convention in Subsection 3.4.

In this subsection, we introduce an actual convention to perform the trick with n cards drawn from a deck of $d \leq n! + n - 1$ cards. We will describe this convention, as given in the article by Kleber and Vakil [2002], in detail. The cards used in this trick are no longer suited but numbered from 0 to $(d - 1)$.

We introduce three algorithms for this convention: Algorithm 3, 4 and 5, respectively. The assistant and magician can use these algorithms to pick and recover the hidden card.

Algorithm 4 picks the hidden card and uses Algorithm 3 to order the remaining cards after the hidden card is picked.

Algorithm 5 finds the hidden card based on the ordered sequence of cards. Before introducing these algorithms, we provide a method used by Algorithm 3 to represent a number p. This number p will be represented using the *factorial number system*, which expresses integers as the sum of factorials. We will introduce and prove Lemma 3.3 concerning this representation. In the proof of Lemma 3.3, we have included key insights from an expository paper by Del Vigna.

Lemma 3.3. Let $n, p \in \mathbb{N}$ and $0 \leq p \leq (n-1)! - 1$. Then p can be uniquely written as $\sum_{j=1}^{n-2} z_j \cdot j!$, where $0 \leq z_j \leq j \leq n-2$.

To prove this lemma we will first introduce the process of ordering the cards to represent a number p which can be determined using the sum shown in Lemma 3.3. To represent p using Lemma 3.3, we need to assign values to z_j. The value of z_j is the number of times we can remove the corresponding factorial $j!$ before p becomes a negative number. Algorithm 3 outlines the procedure for determining the values of z_j. After all values of z_j are determined the cards are arranged in a specific order to represent these values.
The algorithm begins by ordering the cards in ascending order and labelling them $k_0 < k_1 < \ldots < k_{n-2}$. In an iterative process, where we start at index $(n-2)$ and move in descending order to index 1, we determine the values of z_j. These are obtained using the *floor* function. The $floor$ of a number z is the greatest integer less than or equal to z. Once these values are obtained, the cards are arranged iteratively, starting with the card having the lowest value. Each card k_j is then placed in a position such that the number of cards lying to the right of k_j and having a lower value than k_j corresponds to the value of z_j.

Input : Sequence of $n-1$ numbered cards and a number p
Output: Ordered set of $n-1$ numbered cards communicating a number p

1: Sort the $(n-1)$ cards in ascending order: $k_0 < k_1 < \ldots < k_{n-2}$;
2: **for** $j \in \{(n-2), ..., 1\}$ **do**
  $z_j = floor(\frac{p}{j!})$
  $p = p - z_j \cdot j!$
end
3: Place card k_0 on the table;
4: **for** $j \in \{1, ..., (n-2)\}$ **do**
  Place card k_j on the table in the z_j'th position from the right, where the most right position is 0
end

Algorithm 3: Reordering $(n-1)$ cards to communicate a number p

In step 2 the values of z_j are explicitly determined corresponding to a given number p. Using this step we can now give a constructive proof of Lemma 3.3.

Proof. To prove that an integer p in the range $0 \leq p \leq (n-1)! - 1$, can be written as $\sum_{j=1}^{n-2} z_j \cdot j!$, where $0 \leq z_j \leq j \leq n-2$ and the z_j's are determined using step 2 of Algorithm 3, we use mathematical induction.
Base case $n = 2$:
The range of p is $0 \leq p \leq (n-1)! - 1 = (2-1)! - 1 = 0 \Rightarrow p = 0$.
We have $z_0 = 0$ and $z_1 = 0$ by step 2 of Algorithm 3. We can write $p = 0$ as $\sum_{j=1}^{0} z_j \cdot j! = 0$, where $0 \leq z_j \leq j \leq n-1 = 2-1 = 1$.
Inductive step:
Assume that Lemma 3.3 holds for some integer $n = k \geq 2$, where any integer p in the range $0 \leq p \leq (k-1)! - 1$ can thus be written as:

$$p = \sum_{j=1}^{k-2} z_j \cdot j!, \text{ where } 0 \leq z_j \leq j \leq k-2 \text{ and the values of } z_j \text{ are determined by step 2 of Algorithm 3.} \tag{3.10}$$

We want to prove that Equation 3.10 also holds for $n = k + 1$.
For $n = k + 1$, and an arbitrary integer p in the range $0 \leq p \leq (k)! - 1$, we want to prove that p can be written as:

$$p = \sum_{j=1}^{k-1} z_j \cdot j!, \text{ where } 0 \leq z_j \leq j \leq k-1. \tag{3.11}$$

In the process of step 2 in Algorithm 3 the integer p is given. It is as follows:

1. It starts with $j = k - 1$
2. z_{k-1} is determined as $z_{k-1} = floor(\frac{p}{(k-1)!})$

3. The number p is updated to be $p = p - z_{k-1} \cdot (k-1)!$

After the z_{k-1} is determined it will continue with $j = k-2$ till $j = 1$. Let p' be the updated p after the first iteration of the process. For p' we now have

$$0 \leq p' = p - z_{k-1} \cdot (k-1)! < (k-1)!. \tag{3.12}$$

p' is thus an integer in the range $0 \leq p \leq (k-1)! - 1$, which fits the induction hypothesis. Applying the induction hypothesis we can write p' as:

$$p' = \sum_{j=1}^{k-2} z_j \cdot j!, \text{ where } 0 \leq z_j \leq j \leq k-2. \tag{3.13}$$

Thus, the original p can be expressed as:

$$p = z_{k-1} \cdot (k-1)! + p' = z_{k-1} \cdot (k-1)! + \sum_{j=1}^{k-2} z_j \cdot j!. \tag{3.14}$$

This expression for p is of the desired form:

$$p = \sum_{j=1}^{k-1} z_j \cdot j!, \tag{3.15}$$

where $0 \leq z_j \leq j \leq k-1$ and all the z_j can be determined using step 2 of Algorithm 3. This completes the inductive step.

Finally, to prove uniqueness, assume p can be expressed as:

$$p = \sum_{j=1}^{n-2} z_j \cdot j! = \sum_{j=1}^{n-2} z_j' \cdot j!, \text{ where } z_j \neq z_j' \text{ for some } j. \tag{3.16}$$

Subtracting these 2 sums gives:

$$\sum_{j=1}^{n-2} (z_j - z_j') \cdot j = 0 \tag{3.17}$$

In this sum $-j \leq z_j - z_j' \leq j$ for all j. Let k be the largest index, for which $z_j \neq z_j'$. We derive the following equation:

$$0 = \sum_{j=1}^{k} (z_j - z_j') \cdot j! = (z_k - z_k') \cdot k! + \sum_{j=1}^{k-1} (z_j - z_j') \cdot j!. \tag{3.18}$$

Without loss of generality we can assume $z_k - z_k' > 0$ and derive that $(z_k - z_k') \cdot k! \geq k!$ for $k \geq 0$.

For the last part of Equation 3.18 we have:

$$\sum_{j=1}^{k-1} (z_j - z_j') \cdot j! \leq \sum_{j=1}^{k-1} j \cdot j! \text{ as } -j \leq z_j - z_j' \leq j \tag{3.19}$$

From this we derive the following:

$$\begin{aligned} \sum_{j=1}^{k-1} j \cdot j! &= \sum_{j=1}^{k-1} ((j+1)! - j!) \\ &= (2! - 1!) + (3! - 2!) + ... + ((k-1)! - (k-2)!) + (k! - (k-1)!) \\ &= -1! + k!, \text{ as all terms apart from } -1! \text{ and } k! \text{ cancel out.} \end{aligned} \tag{3.20}$$

It follows that $\sum_{j=1}^{k-1} (z_j - z_j') \cdot j! \leq k! - 1 < k!$. Together with Equation 3.18 we arrive at a contradiction as:

$$(z_k - z_k') \cdot k! \geq k! \text{ and } \sum_{j=1}^{k-1} (z_j - z_j') \cdot j! < k!, \tag{3.21}$$

and conclude that the representation of p is unique. □

Using Algorithm 3, we can thus order the remaining $(n-1)$ cards to form a number p in the range $0 \leq (n-1)!-1$. Which number p needs to be, is determined using Algorithm 4.
Algorithm 4 describes how the assistant selects the hidden card and sorts the remaining cards. Performing the trick is done using a deck of $d \leq n! + n - 1$ cards, where n is the number of cards drawn by the audience. Algorithm 4 has the drawn hand of n cards as input. The process starts by ordering the cards in ascending order based on their numerical values, labelling them by $c_0 < c_1 < ... < c_{n-1}$ and taking their sum. The hidden card is picked as card c_i where i is the lowest non-negative number congruent to the sum modulo n. Then a number p is determined using the following equation:

$$p = floor(\frac{c_i - i}{n}). \tag{3.22}$$

We will give a short derivation showing that p is a number between 0 and $(n-1)! - 1$, such that we can write it as the sum of factorials using Lemma 3.3. In Equation 3.22, the expression $c_i - i$ adjusts for cards drawn with a lower number than the hidden card. Therefore, $c_i - i$ is a number between 0 and $(d - n)$.

$$\begin{aligned} 0 \leq floor(\frac{c_i - i}{n}) \leq floor(\frac{(d-n)}{n}) \leq floor(&\frac{((n! + n - 1) - n)}{n}) \\ &= floor(\frac{n! - 1}{n}) \\ &= floor(\frac{n!}{n} - \frac{1}{n}) \\ &= floor((n-1)! - \frac{1}{n}) \\ &= (n-1)! - 1 \end{aligned} \tag{3.23}$$

Since p is a number between 0 and $(n-1)! - 1$, p can be uniquely written as:

$$\sum_{j=1}^{n-2} z_j \cdot j!, \text{ where } 0 \leq z_j \leq j \leq n-2. \tag{3.24}$$

After the assistant has picked the hidden card and ordered the remaining cards using Algorithms 3 and 4, the magician can use Algorithm 5 to determine the hidden card. Algorithm 5 describes the process of determining the hidden card and has the set of $(n-1)$ ordered cards as input. First, a number q congruent to the negative sum of the $(n-1)$ remaining cards is determined. Then, the same number p as in Algorithm 4 is computed by the order of the cards. Using p and q, we find the number $r = n \cdot p + q$. The final step in determining the hidden card involves an iterative process, where the number of cards on the table that are lower in value than the hidden card is added to r. After having introduced the three algorithms to pick and recover the hidden card, we will prove that Algorithm 5 always finds the hidden card picked by Algorithm 4.
The algorithms to perform the trick are as follows:

Input : Sequence of n numbered cards
Output: Hidden card c_i and an arrangement of the $(n-1)$ remaining cards to signal a number p

1: Sort the cards in ascending order: $c_0 < c_1 < \ldots < c_{n-1}$ based on their numerical values;
2: $s = \sum_{j=0}^{n-1} c_j$;
3: Let i be the lowest non-negative number such that $i \equiv s \pmod{n}$;
4: Pick the card c_i to be the hidden card;
5: $p = floor(\frac{c_i - i}{n})$;
6: Use Algorithm 3 to arrange the remaining $(n-1)$ cards to represent the number p

Algorithm 4: Picking the hidden card and ordering the remaining cards

Input : Sequence of $(n-1)$ numbered cards in a specific order
Output: The number r on the hidden card

1: Label the cards on the table in ascending order by $k_0 < k_1 < \ldots < k_{n-2}$;
2: $x = \sum_{j=0}^{n-2} k_j$;
3: Determine the lowest non-negative number q such that $q \equiv -x \pmod{n}$;
4: Compute $p = \sum_{j=1}^{n-2} z_j \cdot j!$, where z_j is the number of cards positioned to the right of k_j and also having a lower number than k_j;
5: Compute $r = n \cdot p + q$;
6: Let $t = 0$;
7:for $j \in \{0, ..., (n-2)\}$ **do**
  if $k_j \leq r + j$ **then**
    $t = t + 1$
  end
end
8: $r = r + t$

Algorithm 5: Determination of the hidden card

Lemma 3.4. Algorithms 3, 4 and 5 form a convention to perform the card trick. Where n cards are drawn from a deck of size $d \leq n! + n - 1$ and numbered from 0 to $(d-1)$.

Proof. Algorithm 4 picks the hidden card c_i and orders the $(n-1)$ remaining cards to represent a number p using Algorithm 3. Algorithm 4, thus has a card c_i and the ordered set of $(n-1)$ cards as output. The ordered set of $(n-1)$ cards is the input of Algorithm 5. Algorithm 5 has a number r as output. We will prove that with the set of ordered cards as input, the output of Algorithm 5 is the hidden card picked by Algorithm 4.
Algorithm 4 picks the hidden card c_i based on the lowest non-negative number i congruent to the sum s of the n cards drawn. This way, we have the following:

$$i \equiv s \pmod{n}. \tag{3.25}$$

In step 2 of Algorithm 5, the sum of the $(n-1)$ remaining cards is x; it thus holds that: $x + c_i = s$. Together with Equation 3.25, we derive the following equation:

$$x + c_i = s \equiv i \pmod{n} \Rightarrow -x \equiv c_i - i \pmod{n}. \tag{3.26}$$

In step 3 of Algorithm 5, a number q is determined to be the lowest non-negative number such that:

$$q \equiv -x \pmod{n}. \tag{3.27}$$

Combining Equations 3.26 and 3.27 we derive the following expression:

$$q \equiv -x \equiv c_i - i \pmod{n}. \tag{3.28}$$

q is thus also the lowest non-negative number congruent to $c_i - i$. If we recall Equation 3.22 where we determined p, and combine it with Equation 3.28 we can derive the following:

$$p = floor(\frac{c_i - i}{n}) = \frac{c_i - i - q}{n}. \tag{3.29}$$

The number c_i is thus decomposed by Algorithm 4 as:

$$c_i = n \cdot p + q + i. \tag{3.30}$$

The number q is determined in step 3 of Algorithm 5 using the expression provided in Equation 3.27. In step 4 of Algorithm 5, we determine the number p using the same expression used to form p in Algorithm 4. With p and q determined, the number r is introduced as $r = n \cdot p + q$.
Then the number $t = 0$ and we iterate over all the cards on the table in ascending order. For every card lower than or equal to $r + j$, where j is the index of the card k_j, t is increased by 1. Since the hidden card is not on the table, we never add 1 to t for cards with a higher value than the hidden card. We will, therefore, only add the number of cards with a lower value than the hidden card. As t is the number of cards on the table with a lower value than the hidden card, we conclude that $t = i$. The last step is adding t to r, resulting in $r = n \cdot p + q + t$.
Since $t = i$, we have:

$$r = n \cdot p + q + t = n \cdot p + q + i = c_i, \tag{3.31}$$

and we can conclude that the hidden card picked by Algorithm 4 is found by Algorithm 5. This concludes our proof. □

Before concluding this section, we provide an example using this pseudocode for the case where 6 cards are drawn from a deck of 725 cards numbered from 0 to 724.

Example 3.2. The audience draws the following six cards for the assistant to pick the hidden card from and order the remaining cards: {12, 456, 455, 102, 185, 642}. We apply Algorithm 4.

1. Ordering them gives: $12 < 102 < 185 < 455 < 456 < 642$.
   We have; $c_0 = 12$, $c_1 = 102$, $c_2 = 185$, $c_3 = 455$, $c_4 = 456$ and $c_5 = 642$.

2. $s = \sum_{j=0}^{n-1} c_j = 12 + 102 + 185 + 455 + 456 + 642 = 1852$.

3. The lowest non-negative number congruent to $s \pmod 6$ is 4, we have $i = 4$.

4. We pick $c_4 = 456$ to be the hidden card.

5. $p = floor(\frac{456-4}{6}) = 75$.

6. Using Algorithm 3, we order the remaining cards to represent the number 75.

7. Sorting the remaining 5 cards in ascending order as $k_0 < k_1 < ... < k_4$ gives $k_0 = 12$, $k_1 = 102$, $k_2 = 185$, $k_3 = 455$ and $k_4 = 642$.

8. We must determine the values for $z_1, ...z_4$, where $p = 75$.
   For $j = 4$, we have $p = 75$, thus $z_4 = floor(\frac{75}{4!}) = 3$, now we have $p = 75 - 3 \cdot 4! = 3$.
   For $j = 3$, we have $p = 3$, thus $z_3 = floor(\frac{3}{3!}) = 0$, now we have $p = 3 - 0 \cdot 3! = 3$.
   For $j = 2$, we have $p = 3$, thus $z_2 = floor(\frac{3}{2!}) = 1$, now we have $p = 3 - 1 \cdot 2! = 1$.
   For $j = 1$, we have $p = 1$, thus $z_1 = floor(\frac{1}{1!}) = 1$.

9. We have $z_1 = 1$, $z_2 = 1$, $z_3 = 0$ and $z_4 = 3$.
   We put k_0 on the table;
   The most right position we refer to as position 0.
   k_1 is put on position 1 from the right, we now have k_1, k_0.
   We now put k_2 on position 1 from the right and have k_1, k_2, k_0.
   We put k_3 on position 0 and have k_1, k_2, k_0, k_3.
   We put k_4 on position 3 from the right and arrive at: k_1, k_4, k_2, k_0, k_3 giving 102, 642, 185, 12, 455.
   This way, we have 1 card lower and lying to the right of k_1, giving $z_1 = 1$; 1 card lower and lying to the right of k_2, giving $z_2 = 1$; 0 cards lower and lying to the right of k_3, giving $z_3 = 0$; and 3 cards lower and lying to the right of k_4, giving $z_4 = 3$.

Using Algorithms 3 and 4, we picked the hidden card $c_4 = 456$ and arranged the remaining cards in a specific order representing the number 75. Now, the magician can use Algorithm 5 to determine the hidden card. The input is the following sequence of ordered cards: {102, 642, 185, 12, 455}.

1. We label the cards in ascending order by $k_0 < k_1 < ... < k_{n-2}$, with thus $k_0 = 12$, $k_1 = 102$, $k_2 = 185$, $k_3 = 455$, $k_4 = 642$.

2. $x = \sum_{j=0}^{n-2} k_j = 12 + 102 + 185 + 455 + 642 = 1396$.

3. $q = 2$, this is the lowest non-negative number such that $q \equiv -1396 \pmod 6$.

4. We observe that $z_1 = 1$, $z_2 = 1$, $z_3 = 0$ and $z_4 = 3$ when we look at the order of the 5 cards on the table. Computing the sum $p = \sum_{j=1}^{n-2} z_j \cdot j! = 1 \cdot 1 + 1 \cdot 2 + 0 \cdot 6 + 3 \cdot 24 = 75$. Thus $p = 75$.

5. $r = n \cdot p + q = 6 \cdot 75 + 2 = 452$.

6. We let $t = 0$.

7. For $j = 0$, we have $k_0 = 12 \leq r + 0 = 452$, thus $t = 0 + 1 = 1$.
   For $j = 1$, we have $k_1 = 102 \leq r + 1 = 453$, thus $t = 1 + 1 = 2$.
   For $j = 2$, we have $k_2 = 185 \leq r + 2 = 454$, thus $t = 2 + 1 + 3$.
   For $j = 3$, we have $k_3 = 455 \leq r + 3 = 455$, thus $t = 3 + 1 = 4$.
   For $j = 4$, we have $k_4 = 642 > r + 4 = 456$.

8. $r = r + t = 452 + 4 = 456$

Algorithm 5 finds the card 456 as the hidden card, which is exactly the hidden card picked by Algorithms 3 and 4.

This example concludes this section of our mathematical analysis of the trick. In the next section, we will revisit the "Cheney card trick", where 5 cards are drawn from a standard deck of 52 cards. We will investigate the event where we must give a path surpassing the king and revert to the lower numbers.

4 Distributions within the drawn hands of the "Cheney card trick"

We reexamine the "Cheney card trick" using a standard deck of 52 cards. When performing the trick as described in Subsection 3.1, we notice an uneven distribution in situations where we must use either the highest or the lower cards of a suit to declare the suit of the hidden card. Specifically, we are rarely forced to place the highest card of a suit on the table to declare the suit of the hidden card and then revert to lower numbers, surpassing the king. In this section, we will analyse the probability of being obligated to use the highest card and thus surpass the king.
First, we need to establish some specifications. If we have the option to place a lower card of a suit on the table to indicate the suit of the hidden card, we will always choose to do so. For example, if we are dealt the 3♠, 5♠ and K♠. One should note that both the king♠ and the 3♠ can be used to declare the suit of the hidden card. In this case, we will always choose the 3♠ to declare the suit of the hidden card, which is the 5♠ in this case. To determine the probability of being obligated to use the higher card to declare the suit, we will begin by categorising distinct hands and determining their probabilities.

4.1 Possible hands and their probabilities

When dealt 5 cards from 4 different suits, there are 6 possible combinations in which the suits can appear in a hand. We are concerned only with the number of cards of each suit, not the specific suits themselves. We aim to determine the probabilities of obtaining these six distinct hands. To do this, we must first consider the number of different ways in which the distinct hands can be drawn. All possible hands and the number of ways the suits can appear in a drawn hand are listed in Table 4.1. The numbers in the left column refer to how the suits can be ordered within a drawn hand, not looking at specific suits. The right column gives the corresponding number of ways of drawing each hand. To get a better understanding of how this table can be retrieved, we will explain how we found the number of ways to draw the following hands: {5,0,0,0}, {4,1,0,0} and {3,1,1,0}.
When drawing the hand {5,0,0,0}, we have drawn 5 cards of the same suit. There are 4 different suits that we can draw these 5 cards from and, thus, 4 ways of drawing this hand.
When drawing the hand {4,1,0,0}, we have to draw 4 cards of the same suit and 1 card of a different suit. Again, we can draw the 4 cards of the same suit, where there are 4 different suits to draw these cards from. The other card we can draw from the 3 remaining suits. We thus have $4 \cdot 3 = 12$ ways of drawing the hand {4,1,0,0}.
When drawing the hand {3,1,1,0}, we have 3 cards of the same suit and twice 1 card of different suits. As before, we can draw the 3 cards of the same suit in 4 different ways. For the first card of a different suit, we can draw from 3 different suits. For the other card of a different suit we can draw from the 2 remaining suits. However, the number of ways we can draw the hand {3,1,1,0} is equal to $\frac{4 \cdot 3 \cdot 2}{2} = 12$. We divide by 2 as we can substitute the suits of the 2 single cards. We can determine the number of ways to draw the remaining three hands similarly.

Hand	Number of ways of drawing this hand
5,0,0,0	4
4,1,0,0	12
3,2,0,0	12
3,1,1,0	12
2,2,1,0	12
2,1,1,1	4

Table 4.1: Number of ways of drawing distinct hands

We can now determine the probabilities of drawing these distinct hands. These probabilities are calculated using combinations, where we draw a specific number of cards from a suit consisting of 13 cards. By multiplying these combinations, we determine the number of ways to draw each type of hand. Subsequently, to determine the probability of each hand, we divide the number of ways to draw that hand, by the number of ways to draw any 5-card hand from the deck.

$$\mathbb{P}(\text{Hand} = 5,0,0,0) = \frac{4\binom{13}{5}\binom{13}{0}\binom{13}{0}\binom{13}{0}}{\binom{52}{5}} = \frac{4\cdot 1287}{2598960} = \frac{5148}{2598960} \approx 0.0019808$$

$$\mathbb{P}(\text{Hand} = 4,1,0,0) = \frac{12\binom{13}{4}\binom{13}{1}\binom{13}{0}\binom{13}{0}}{\binom{52}{5}} = \frac{12\cdot 715\cdot 13}{2598960} = \frac{111540}{2598960} \approx 0.042917$$

$$\mathbb{P}(\text{Hand} = 3,2,0,0) = \frac{12\binom{13}{3}\binom{13}{2}\binom{13}{0}\binom{13}{0}}{\binom{52}{5}} = \frac{12\cdot 286\cdot 78}{2598960} = \frac{267696}{2598960} \approx 0.10312$$

$$\mathbb{P}(\text{Hand} = 3,1,1,0) = \frac{12\binom{13}{3}\binom{13}{1}\binom{13}{1}\binom{13}{0}}{\binom{52}{5}} = \frac{12\cdot 286\cdot 13\cdot 13}{2598960} = \frac{580008}{2598960} \approx 0.22317$$

$$\mathbb{P}(\text{Hand} = 2,2,1,0) = \frac{12\binom{13}{2}\binom{13}{2}\binom{13}{1}\binom{13}{0}}{\binom{52}{5}} = \frac{12\cdot 78\cdot 78\cdot 13}{2598960} = \frac{949104}{2598960} \approx 0.36519$$

$$\mathbb{P}(\text{Hand} = 2,1,1,1) = \frac{4\binom{13}{2}\binom{13}{1}\binom{13}{1}\binom{13}{1}}{\binom{52}{5}} = \frac{4\cdot 78\cdot 13\cdot 13\cdot 13}{2598960} = \frac{685464}{2598960} \approx 0.26375$$

Table 4.2 outlines the 6 different combinations and their probabilities. The numbers in the left column indicate the ways in which the suits are drawn within the hand. The right column shows the probability of drawing each type of hand.

Hand	Probability
5,0,0,0	0.0019808
4,1,0,0	0.042917
3,2,0,0	0.10300
3,1,1,0	0.22317
2,2,1,0	0.36519
2,1,1,1	0.26375

Table 4.2: Table of different hands and their probabilities

4.2 Probability of putting down the highest card of a suit

When examining the probability of being obligated to put down the highest card of a suit and surpassing the king, we only need to look into the last two cases of Table 4.2. When we have three or more cards of the same suit, we never have to put down the highest of these three. Suppose we have three cards of the same suit $c_1 < c_2 < c_3$. We are never obligated to put down c_3 to give a path to c_1 since the distance between c_1 and c_2 or between c_2 and c_3 is at most 6. For the cases where we have more than 3 cards of the same suit, it is evident that we are not obligated to put down the highest card.

We thus only consider the last two cases in Table 4.2. These cases refer to the hands in which we have exactly two cards of the same suit and the hands in which we have two pairs of cards from different suits. We consider the probability of being obligated to lie down the highest card and surpass the king. We will refer to the event of lying down the highest card of the suit and surpassing the king as "Highest card". We will refer to drawing a specific hand as before, "Hand $=$ 2,1,1,1", if we draw the suits in that specific distribution.

We can now start determining the probability of having to put down the highest card: $\mathbb{P}$(Highest card). The probability can be split into two parts as follows:

$$\begin{aligned}\mathbb{P}(\text{Highest card}) = {} & \mathbb{P}(\text{Highest card} \mid \text{Hand} = 2,1,1,1)\cdot\mathbb{P}(\text{Hand} = 2,1,1,1) \\ & + \mathbb{P}(\text{Highest card} \mid \text{Hand} = 2,2,1,0)\cdot\mathbb{P}(\text{Hand} = 2,2,1,0). \end{aligned} \tag{4.1}$$

We will examine the first expression: $\mathbb{P}(\text{Highest card} \mid \text{Hand} = 2,1,1,1)$. In this hand, we draw exactly two cards of the same suit. Therefore, this is the probability that two cards within the same suit differ in value by more than 6; $\mathbb{P}(\text{Two cards within the same suit differ in value by more than 6})$. We will address this latter probability as p, thus

$$p = \mathbb{P}(\text{Two cards within the same suit differ in value by more than 6}). \tag{4.2}$$

We have $\mathbb{P}(\text{Hand} = 2,1,1,1) \approx 0.26375$ as in Table 4.2. We can determine the probability p simply by counting the possibilities. We draw 2 cards from a deck of 13 cards of the same suit, where we do not take the order into account. The number of ways this can be done is equal to the following:

$$\binom{13}{2} = 78. \tag{4.3}$$

We thus find 78 different hands of two cards. The hands containing two of the same suit, where we are forced to put down the higher card, are the following pairs:

13 and the $1, 2, 3, 4, 5, 6$;	12 and the $1, 2, 3, 4, 5$;
11 and the $1, 2, 3, 4$;	10 and the $1, 2, 3$;
9 and the $1, 2$;	8 and the 1;

There are 21 in total, thus $p = \frac{21}{78} = \frac{7}{26} \approx 0.26923$.
When drawing the hand in which the suits are distributed as "2,1,1,1", we have exactly two cards of the same suit, and we arrive at the following derivation:

$$\begin{aligned}\mathbb{P}(\text{Highest card} \mid \text{Hand} = 2,1,1,1) \cdot \mathbb{P}(\text{Hand} = 2,1,1,1) &= p \cdot \mathbb{P}(\text{Hand} = 2,1,1,1)\\ &= 0.26923 \cdot 0.26375\\ &= 0.071009. \end{aligned} \tag{4.4}$$

For the hands in which the suits are distributed as 2,2,1,0, we can use a similar derivation. We have $\mathbb{P}(\text{Hand} = 2,2,1,0) \approx 0.36519$ as in Table 4.2. The small difference this time is that we square the probability p. This probability corresponds to two cards within the same suit, differing in value by more than 6. We square this probability because we have drawn two of the same suit twice, giving the following:

$$\begin{aligned}\mathbb{P}(\text{Highest card} \mid \text{Hand} = 2,2,1,0) \cdot \mathbb{P}(\text{Hand} = 2,2,1,0) &= p^2 \cdot \mathbb{P}(\text{Hand} = 2,2,1,0)\\ &= 0.26923^2 \cdot 0.36519\\ &= 0.026467. \end{aligned} \tag{4.5}$$

In Equation 4.1, we saw the probability $\mathbb{P}(\text{Highest card})$ was equal to the sum of Equations 4.4 and 4.5. Summing these probabilities gives: $0.07101 + 0.02645 = 0.09753$. We can conclude that when drawing a hand of 5 cards, we are obligated to put down the highest card in 9.753 per cent of the cases.

5 A deeper look into Hall's marriage theorem

We have introduced Birkhoff and Hall's theorems, revealing the mathematical background necessary to prove the existence of a convention. In this section, we take a deeper look at an extension of Hall's marriage theorem and put it in a broader context. We introduce Gale-Shapley's definitions about stability and optimality and introduce their algorithm. These link directly to Hall's marriage theorem. It adds an extra dimension by allowing the groups to give ranked preferences of the other group, where they thus rank the individuals they prefer to marry.

5.1 Gale-Shapley's theorem for stable and optimal marriages

As described before, Hall's marriage theorem can be used in the context of arranging marriages, ensuring that every man or woman is paired with someone they prefer. Hall's marriage theorem provides insight into the existence of such marriages. A *marriage* refers to the complete set of the men coupled to the women. This theorem establishes that a marriage exists if, for any subset of men, the set of women they prefer is at least as large as the subset of men itself.
The article of Gale-Shapley enhances the matchmaking process. Both men's and women's preferences are considered, asking them to rank the individuals they prefer. By taking this ranking in preferences into account, we arrive at a different problem. The problem is now about stable marriages. Can we pair every man with a woman, such that no man and woman both prefer to be paired with each other over their current partners? The paper by Gale [1962] posed the same problem. D. Gale and L. Shapley introduced the problem as the matching of students and colleges, where both the students and the colleges have a ranking in their preference of one another. Definitions of stability and optimality are introduced. We will give these same definitions, relating them to the marriage of men and women.

Definition 5.1. Stability by Gale [1962]. A marriage between a complete set of men and women is *stable* if no man and woman would both prefer to be paired with each other over their current partners.

Definition 5.2. Optimality by Gale [1962]. A stable marriage between the complete set of men and women is *optimal* if every man or woman is at least as well off under it, as under any other stable marriage.

All men and women thus have a list of their preferences. There can be no ties in their preferences. Also, men and women can be excluded from these preferences if they prefer not to marry certain individuals. This relates to how we posed the problem in Subsection 2.1, where a man would not be paired with a woman he did not prefer. Gale and Shapley pose the question, if it is possible, for any pattern of preferences, to find a stable set of marriages? To better understand the problem, we will consider an example as given by Gale [1962].

Example 5.1. We have the following matrix shown in Figure 5.1 containing the preferences of 3 men, respectively α, β and γ, and 3 women, respectively A, B and C. The matrix indices contain first the preference of the men and second the preference of the women. Thus, man α ranks woman A first, and woman A ranks man α third. Note that there are 3! = 6 possible marriages when there are 3 men and women.

	A	B	C
α	1, 3	2, 2	3, 1
β	3, 1	1, 3	2, 2
γ	2, 2	3, 1	1, 3

Figure 5.1: Matrix containing the preferences, given by Gale [1962]

We consider the possible marriages from Figure 5.1 and list them in Table 5.1

Married pairs	Corresponding preferences	Stability condition	Optimality
(α, A), (β, B), (γ, C)	(1,3), (1,3), (1,3)	Stable	Optimal for the men
(α, B), (β, C), (γ, A)	(2,2), (2,2), (2,2)	Stable	Not optimal
(α, C), (β, A), (γ, B)	(3,1), (3,1), (3,1)	Stable	Optimal for the women
(α, A), (β, C), (γ, B)	(1,3), (2,2), (3,1)	Unstable	Not optimal
(α, B), (β, A), (γ, C)	(2,2), (3,1), (1,3)	Unstable	Not optimal
(α, C), (β, B), (γ, A)	(3,1), (1,3), (2,2)	Unstable	Not optimal

Table 5.1: Table of possible marriages and their stability and optimality

From Table 5.1, we can easily observe all possible marriages. There are 3 stable marriages and of these stable marriages 2 are optimal: in the first marriage, all men are paired to their first choice, and in the third marriage, all women are paired to their first choice. Therefore, these are optimal by Definition 5.2, since every man or woman is better off under this marriage than the other possible marriages. In the second marriage, all men and women are paired with their second choice. Therefore, this is a stable but not optimal marriage for both groups, as two marriages exist in which either group is paired to their first choice.
The last three marriages are unstable. Looking at the fourth marriage, we observe that γ and A would rather be paired to each other than to their current partner. Therefore, the marriage is unstable by Definition 5.1. This also applies to the fifth and sixth marriage, where a man and woman would rather be paired with another than their current partner.

This example poses a thought-provoking question: Does a stable marriage always exist? This is investigated explicitly by Gale [1962], where they derived the following theorem:

Theorem 5.1. *Existence of stable marriages, given by Gale [1962]*
A stable set of marriages always exists.

A constructive proof is given in the way of an iterative process, which is now referred to as the Gale-Shapley Algorithm. This algorithm guarantees stability. Also, all men and women are paired with each other when the number of men and women is equal. A question about optimality can be posed again. When men initiate proposals in this algorithm, the article by Gale [1962] tells us that the algorithm secures an optimal set of marriages for the men. Conversely, if women take the initiative, it ensures optimality for their obtained marriage. In the next subsection, we will consider this algorithm, linking it to finding a convention for the card trick using a deck of 124 cards as well.

5.2 Gale-Shapley's algorithm for finding an optimal marriage

Suppose we have a set of men and women, each with a list of preferences of the opposite gender, without any ties in their preferences. Both men and women can leave out individuals of the opposite gender. The Gale-Shapley Algorithm presented in Algorithm 6 is a method for finding a stable set of marriages. The algorithm works as follows: we let the men propose to the women in this case. Where each man proposes to the woman he ranks the highest. If this woman is already taken, there are two possibilities: either the woman ranks her current man lower than the man proposing to her and the man proposing switches with the current partner, or the man proposes to the woman he ranks second highest. This process continues until all men have found a partner, or till the men that are still free have no women to propose to anymore.

```
Input  : A list of ranked preferences of the opposite gender for all men and women
Output: A stable matching optimal for the men
Initialise all men and women as free;
while there exists a free man who still has a woman to propose to, pick a free man m do
    Let w be m's highest-ranked woman to whom he has not yet proposed;
    if w is free then
        (m, w) become engaged;
    else
        some pair (m', w) already exists;
        if w prefers m to m' then
            (m, w) become engaged, and m' becomes free;
        else
            (m', w) remain engaged;
        end
    end
end
return the output is a list of married pairs which is optimal for the men;
```

Algorithm 6: Gale-Shapley Algorithm (Men-Proposing)

By initially swapping the roles of men and women, we can attain an optimal set of marriages for the women.
We shall put the algorithm into perspective by applying it to a matrix of preferences between men and women. Although we can consider the preferences from Example 5.1, we have already found the optimal marriages in that case. Therefore, we will consider the following new example, applying the algorithm to a matrix containing the preferences of 5 men and 5 women. In this example, the existence of a marriage is a given.

Example 5.2. We consider the preferences of 5 men, respectively $\alpha, \beta, \gamma, \delta, \epsilon$ and 5 women, respectively A, B, C, D, E. Their preferences are stated in Figure 5.2. We have also added the possibility of rather not marrying a specific individual. This is marked by an "×". The first number indicates a specific man's preference ranking for the women. The second number indicates a specific woman's preference ranking for the men.

$$\begin{array}{c} \\ \alpha \\ \beta \\ \gamma \\ \delta \\ \epsilon \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 1,1 & 2,3 & 4,2 & 5,3 & 3,1 \\ 2,2 & 4,5 & 1,3 & \times,\times & 3,\times \\ 3,5 & 4,4 & \times,\times & 2,2 & 1,2 \\ 2,4 & 1,2 & 4,4 & 3,1 & 5,3 \\ 4,3 & 1,1 & 3,1 & 2,4 & 5,4 \end{bmatrix} \end{array}$$

Figure 5.2: Preference matrix where the entries refer to the preference of the men and women

In applying Algorithm 6 to this group of men and women, we will let the men propose to the women. We begin by initialising all men and women as free. We will continue as long as there are free men who still have to propose to certain women.

1. At the start, there are 5 free men. We pick man α to start and let him propose to his first choice woman A. She is free, so we marry man α to woman A.
2. There are now 4 free men. We pick man β and let him propose to his first choice, woman C. She is free, so we marry man β to woman C.
3. There are now 3 free men. We pick man γ and let him propose to his first choice, woman E. She is free, so we marry man γ to woman E.
4. There are now 2 free men. We pick man δ and let him propose to his first choice, woman B. She is free, so we marry man δ to woman B.
5. There is now 1 free man, ϵ. We let him propose to his first choice, woman B. However, woman B is already married to man δ. Since she prefers ϵ, we marry man ϵ to a woman B, and man δ becomes free.
6. We now have 1 free man, δ. He has already proposed to woman B, so we move on to his highest-ranked woman to whom he has not yet proposed. This is woman A, she is already married to man α, whom she prefers. We then move to his next highest-ranked woman, woman D. She is free, so we marry man δ to woman D.

There are no free men left, we have thus found the following marriage of the men and women:

$$\alpha \text{ to } A, \qquad \beta \text{ to } C, \qquad \gamma \text{ to } E, \qquad \delta \text{ to } D, \qquad \epsilon \text{ to } B. \tag{5.1}$$

This is a stable marriage optimal for the men. An optimal marriage for the women can be achieved by letting the women propose to the men.

Applying this algorithm to the card trick by using it to find a convention proves to be quite challenging. The problem is that this algorithm is computationally extensive when applying it to our card trick. We can consider the matrix used in our proof of Theorem 3.1. This matrix corresponds to the hands and messages that must be paired. The matrix corresponding to 2 cards drawn from a deck of 3 cards is given in Figure 3.1. This matrix, however, becomes very large when we draw 5 cards from a deck of 124 cards. The corresponding matrix has approximately $\binom{124}{5} \approx 2.25 \times 10^8$ rows and columns corresponding to potential hands and messages that must be paired. Gale [1962] tells us that their algorithm takes at most $x^2 - 2x + 2$ stages, where x is the number of men we want to pair to a woman. Thus, unfortunately, if we have $x = 2.25 * 10^8$ hands to pair to a message, this will computationally be quite extensive.

What if we put aside the computational intensity of the process? Can we then arrive at a convention using this algorithm? If we put it aside, we are left with the adding of preferences to the messages given a specific hand and to the hands given a specific message.

Adding the preferences is a long process, where there does not seem to be a logical way of doing this. We could specify certain preferences, such as preferring messages, in ascending order for every hand. Unfortunately, having a convention where all messages are in ascending order is not always possible. We can pair most of the messages in

ascending order to a hand. Eventually, we will run out of available messages in ascending order to pair to a hand. When we consider a deck of maximum size, we will have an equal number of possible messages and hands. All possible messages, therefore, need to be paired to a hand if we use deck of maximum size.
Also ties are not allowed between preferences, as the algorithm picks in order of the preferences. If we allow ties in the preferences, the algorithm must be extended to pick one of the options where the preferences are equal. In the process of the algorithm in Example 5.1, we observe that not all options of pairing the men to a woman are considered. We could have considered man α to have all women ranked as his number 1. If the algorithm had picked woman A to be paired with man α, there would not have been a problem. However, being able to pick a woman when there are ties in preference, would need to be implemented in the algorithm.
As it is hard to find a logical way of adding the preferences, we could choose to add the preferences randomly. We do not allow any ties of preferences for messages given a specific hand and vice versa. Algorithm 6 will then provide us with a marriage of all the hands and messages, giving a specific convention. If we add the preferences at random, we will have a convention that is not easy to perform since, for every hand, a specific but random message is to be used.
Finding a convention that can be performed or is easy to remember, is a difficult task using Algorithm 6. It only guarantees that we have a stable marriage between hands and messages according to the added preferences. To illustrate that the algorithm could work and find a convention, we will apply it to the trick where we draw 2 cards from a deck of 3 cards. We will consider the following matrix, which corresponds to the trick where we draw $n = 2$ cards from a deck of $d = 3$ cards. The columns are indexed by the possible messages, and the rows by the possible hands.

$$\begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} (1,2) \\ (1,3) \\ (2,3) \end{array} & \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

Figure 5.3: Example matrix with $n = 2$ cards drawn from a deck of 3 cards, where the columns are indexed by the possible messages and the rows by the possible hands

Adding the preferences can be done accordingly giving the following matrix:

$$\begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} (1,2) \\ (1,3) \\ (2,3) \end{array} & \begin{bmatrix} 1,2 & 2,1 & \times,\times \\ 2,1 & \times,\times & 1,2 \\ \times,\times & 1,2 & 2,1 \end{bmatrix} \end{array}$$

Figure 5.4: Example matrix with $n = 2$ card drawn from a deck of 3 cards, where the columns are indexed by the possible messages and the rows by the possible hands. All hands and messages are given a preference of each other

Applying the algorithm to this matrix and letting the hands propose to the messages would result in the following convention, where the messages are coupled to the hands as follows:

$$1 \text{ to } (1,2) \qquad 2 \text{ to } (2,3) \qquad 3 \text{ to } (1,3). \tag{5.2}$$

The algorithm basically gives us a permutation matrix based on the introduced preferences, which we can then use as a convention for our card trick. This example is fairly straightforward, but it helps illustrate the process when we set aside the computational intensity of adding preferences and using the algorithm.
Although the algorithm does not directly relate to our card trick, it suggests a potential method for finding a convention and provides deeper insight into Hall's marriage theorem and its potential extensions.

6 Conclusion and suggestions for further research

6.1 Conclusion

In our introduction, we posed our research question: *What mathematical principles and ideas underlie the "Cheney card trick"?* To answer this question, let us summarise the sections of this thesis and conclude what mathematical principles we have come across and in which branches of mathematics they are used.

In Section 3, we started by explaining the "Cheney card trick", where we draw 5 random cards from a standard deck of 52 cards. We noted that we always have at least 2 cards of the same suit. Of these, we let one be the hidden card, and the other specify the suit of this card. Using the 3 cards left, we specify a number between 1 and 6 to give a path to the number of the hidden card. Also, we designed and analysed Algorithms 1 and 2 to perform the "Cheney card trick". Algorithm 1 can be used to choose the hidden card and order the 4 remaining card. Algorithm 2 can be used to retrieve the hidden card based using the 4 ordered cards on the table.

To start our mathematical analysis, we introduced the definitions of a *message* and a *convention*, which helped us approach the card trick mathematically.

We found that we could expand the deck size of the "Cheney card trick" to drawing n cards from a deck of size $d \leq n! + n - 1$.

With the introduction of this upper bound, we proved Theorem 3.1, demonstrating the existence of a convention at this upper bound. This proof relied on two key theorems: Birkhoff-von-Neumann and Hall's marriage theorem. We constructed a 0,1-matrix with rows corresponding to drawn hands and columns to messages, setting entries to 1 if the cards of a message were included in a hand. This matrix was expressed as a linear combination of permutation matrices. Each permutation matrix indicated a specific message to be used for every hand drawn, confirming the existence of a convention.

In Section 3, we also proved Lemma 3.3 telling us that any integer $0 \leq p \leq (n-1)! - 1$ could be uniquely written as the sum of factorials. We introduced Algorithm 3, which provided a method to order $n-1$ cards to form a number p using Lemma 3.3. Using Algorithm 3, we proved Lemma 3.3. We introduced Algorithms 4 and 5. The first could be used by the assistant to pick the hidden card and order the cards using Algorithm 3. The second could be used by the magician to determine the hidden card by the ordered sequence of cards. Together these three algorithms proved to form a convention which could be used to perform the card trick using a deck of cards of size $d \leq n! + n - 1$.

In Section 3, we used mathematical principles from various fields of mathematics. The main theme was discrete mathematics, as we used definitions and lemmas to prove theorems about the "Cheney card trick." However, these proofs also involved Hall's marriage theorem and Birkhoff-von Neumann's theorem, which have applications in set theory and matrix theory. The introduction and analysis of the algorithms fits into algorithm design and analysis, making this section a section in which various mathematical principles are addressed.

In Section 4, we looked into specific messages that the 4 cards, put on the table by the assistant, could give. In the convention we used for the trick, we were sometimes forced to give a message where we passed beyond the king.

We analysed the event where, in giving a message, we had to put down the highest card of a suit and surpass the king. We determined the probability of this event by considering distinct hands and their probabilities. We identified in which of these drawn hands the event occurred and calculated their probabilities. The total probability of the event was 0.09753. In this section, the use of combinations and permutations to determine the probabilities played a significant role. The primary field of mathematics in this section was probability theory.

In Section 5, we further considered Hall's marriage theorem. We introduced the research conducted by Gale and Shapley, who added an extra dimension by allowing preferences of the men and women of one another. Allowing these ranked preferences led to the problem of stable marriages. A marriage would be stable if no man and woman would prefer to be paired with each other over their current partners. Given these preferences, a stable marriage always exists between these men and women. This was proven by introducing Gale-Shapley's algorithm.

This algorithm can be used to find a stable marriage between men and women who have given preferences for one another. This marriage would be optimal for the group that initially proposed to the other group. We used Hall's marriage theorem to prove the existence of a marriage between possible hands and messages, therefore proving the existence of a convention. Gale-Shapley's algorithm can find a specific convention when preferences are added to the hands and messages. This proved to be computationally quite extensive. However, the algorithm will find a convention if we ignore this complexity.

In section 5, we did not encounter any new mathematical fields. However, Gale-Shapley's theorem can be applied in set theory, and the analysis of their algorithm is interesting in the context of the "Cheney card trick" and finding a convention.

Overall, the mathematical principles we used are found in the following fields of mathematics: discrete mathematics, set theory, matrix theory, algorithm design and analysis, and probability theory.

In conclusion, the "Cheney card trick" is a fascinating combination of magic and mathematics. By exploring Hall's marriage theorem, the Birkhoff-von-Neumann theorem, and the Gale-Shapley algorithm, we have uncovered the elegant mathematical principles that can be used to analyse the "Cheney card trick".

6.2 Suggestions for further research

While this study has provided significant insight into the "Cheney card trick", revealing the underlying mathematics, there remain topics that could further be investigated. Finding a specific convention, for example, proves to be quite challenging. Is finding a different convention that can be performed by a magician and an assistant when attaining the upper bound possible? The way in which the "Cheney card trick" is performed is not unique. For example, different combinations can be used to define a number between 1 and 6. Is there another convention that does not resemble the "Cheney card trick"? I would recommend further research on this.
The assistant chooses the card returned to the audience when performing the "Cheney card trick". Suppose the audience chooses this card themselves, and it is thus a random card. Using the convention we introduced in Subsection 3.1, what is the probability of being able to perform the trick when the audience chooses the hidden card? In how many cases of the drawn hands can we give a path, using the 4 cards left in the hand, to this random hidden card? In Subsection 4.1, we used the table shown in Figure 4.2, which could also help determine this probability.
It might also be interesting to think of a specific convention to perform the trick where the hidden card is selected randomly. An upper bound for this trick should be lower than the upper bound we introduced. Is it possible to use an altered version of the conventions we introduced to perform such a trick?
As a final recommendation, I suggest exploring other mathematical results related to this card trick. Additionally, it can be interesting to investigate other magic tricks that incorporate the mathematical concepts discussed in this thesis.

While mathematics and magic may seem like distinct domains, they can blend together very well when approached from a different perspective. When we don't understand the workings of a mathematical card trick, it can be experienced as real magic.

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