Modeling the M_2 and M_4 tidal wave propagation subjected to deepening, widening and friction change in an one dimensional basin system

by

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Abstract

River estuaries are strongly modified by human interventions, such as dredging. For example the Ems river (Germany and Netherlands) has been deepened between in 1965 and 2005. As a result the sediment (sand and dirt) concentration has increased enormously [3]. The muddiness of this sediment blocks sunlight and decreases the friction of the water with the bed level. The transport of sediment into the river is mainly caused by non-linearity in tidal waves. As a result of the elevation of the bed level and the friction change of the Ems river, the amplitude of the M2 and M4 tide has changed, which was concluded in previous research. In this study the M2 and M4 tide are studied individually, subjected to deepening, widening and friction change. The main research question in this study is '*How does the propagation of the M2 and M4 tide change, subjected to deepening, change in width and change in friction?*' The result to this question can be applied to tidal rivers. To illustrate this, the results are compared to historical data of the Ems river. The sub question in this study is; '*Can historical observations of the amplitude of the M2 and M4 tides in the Ems river over the years be explained with the change in propagation of the M2 and M4 tides, subjected to deepening and friction change?*'.

The questions are answered with help of a model. The model is constructed with the one dimensional water equations, where boundary conditions are used. Next the equations can be used in two cases. In the first case, which is used to answer the main question, the variables width, depth and friction of the river basin are taken to be length independent variables. In the second case, the width, depth and friction of the river basin are taken to be length dependent variables. This second case is used to model the propagation of the M2 and M4 tide for the Ems river. For the first case an analytical and numerical solution exist. From the error, the optimal grid size for the numerical model is obtained, which is taken as $N = 100$. For the second case only a numerical solution exists.

With the model it is concluded that widening and an increasing friction cause a damping effect on the propagation of the M2 and M4 tide. Deepening, however, has a different effect on the M2 and M4 tide. It was seen that for certain value change in depth the M2 tide shows an amplification in amplitude, while the M4 tide shows an damping in amplitude for the same value change. The plots obtained in this part of the study can be used to see what is expected to happen to the amplitude of the M2 and M4 tide when a variable is varied in a tidal river basin. The latter is done for the Ems river.

An important observation made with the x-dependent model is that the amplification in amplitude from 1965 to 2005 can be explained by the decreased friction, due to increasing muddiness in the Ems river. Secondly, relatively, it was seen that the amplitude of the M4 tide shows a much greater amplification in amplitude at the beginning of the river than the amplitude of the M2 tide. This could be explained by the different change in amplitude of the M2 and M4 tide subjected to deepening, which was concluded from the x-independent model as well. However, the difference in M2 and M4 tide between 1965 and 2005 is not due to deepening on its own. Namely, the change in amplitude due to deepening highly depends on the value of the friction, which is different for the years 1965 and 2005. It is concluded that the exact amplitude of the M2 and M4 tide cannot be predicted with the x-independent model. However, it is concluded that the results from the x-independent model can be used to predict how the M2 and M4 tide will change due to deepening relatively to each other.

The results of this study are strongly influenced by the assumptions made to derive the model. Before the one dimensional, x-independent model was compared to the observations for the Ems river between 1965 and 2005 a few decisions had to be made regarding x-dependent to x-independent variables. This process needs to be further researched, before applying the x-independent model to other tidal rivers, which is left for further research.

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1

Introduction

River estuaries are important ecological environments. An estuary is a partially enclosed, coastal water body where freshwater from rivers and streams mixes with salt water from the ocean. A satellite image of an estuary is shown in figure 1.1. Estuaries, and their surrounding lands, are places of transition from land to sea. They are protected from the full force of ocean waves, winds and storms [4]. Regardless, the estuaries are strongly influenced by sediment transport in the river. Sediment is a naturally occurring material that is broken down by processes of weathering and erosion. For example, sand and silt can be carried in suspension in river water and cause the muddiness of the river to increase [2]. The transport of sediment into a river can have an enormous effect on the ecological environment of a river estuary. A good example is the Ems river estuary (Germany and Netherlands), which is known to have an increment in sediment between the 1960s and early 2000s. In particular, the sediment concentration has increased from approximately 1 kg m^{-3} to 10 kg m^{-3}, which is a significant change [3]. Another example of an estuary that experienced a strong increase in the sediment concentration is the Loire (France) [13]. The muddiness blocks sunlight, which is essential to the rooted aquatic plants that provide oxygen. In addition the suspended sediment suffocates fish eggs, increases the absorption of heat which can threaten trout and other cold or cool water fish [2]. Another disadvantage of sediment flow into the river is blocking of harbour entrances. To be short, there are enough reasons why sediment flow is important to study.

The increment of muddiness has an effect on the shear stress on the river bed. The shear stress arises from the shear force, the component of the force vector parallel to the cross section of the river. Also, the muddiness changes the turbulence in the water, which also effects the water in the river. The shear stress and turbulence are merged into one variable, namely the friction, in this study.

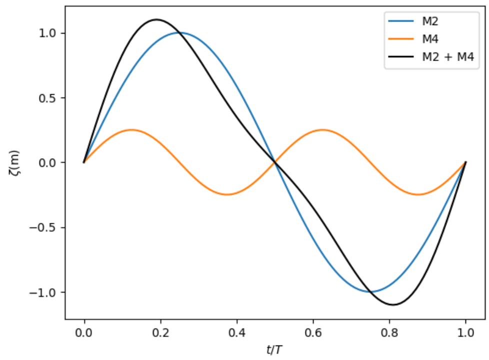
The transport of water suspension is strongly effected by tidal waves. Therefore, sediment transport is mainly caused by tides. So, tidal waves need to be investigated extensively. First it is explained what tidal waves are.

Figure 1.1: Satellite image of the Hooghly estuary, Bay of Bengal (India).

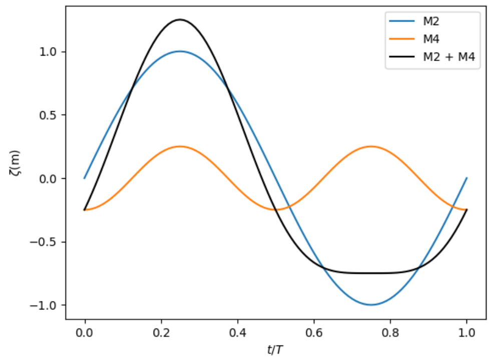
1.1. Tidal waves

Tides are the rise and fall of sea levels caused by the combined effects of the gravitational forces exerted by the Moon and the Sun, and the rotation of the Earth [12]. The tide experienced at shore, consists of an oscillation with multiple harmonics. In this study the focus will be on two of these harmonics. The main harmonic tide is the M2 tide. Here M stands for moon and two for twice a day. Therefore the period of the M2 tide is 12.42 hours. This tide is caused by the moon. Because the gravitational field created by the Moon weakens with distance from the Moon, it exerts a slightly stronger than average force on the side of the Earth facing the Moon, and a slightly weaker force on the opposite side. The Moon thus tends to "stretch" the Earth slightly along the line connecting the two bodies. The solid Earth deforms a bit, but ocean water, being fluid, is free to move much more in response to the tidal force [12].

Another harmonic is the M4 tide. This tide has a double frequency of 6.21 hours. This tide is caused by non-linear terms in the water equations that hold in an estuary. This will be explained more vividly in chapter 2. The tide has an certain velocity which oscillates at the mouth of the river. The M4 tide is especially important for sediment flow, which can be explained with figure 1.2. In these figures a negative velocity means that water is flowing out of the river and a positive velocity means that water is flowing into the river. Slow flowing water is not able to keep sediment suspended, which is why the sediment transport scales with the velocity of the water to the power of three [9]. In the figures for the velocity of the M2 and M4 tides are plotted together. The M2 and M4 tides can be in phase or out of phase. The first figure 1.2a shows the velocity of the M2 and M4 tides for when the tides are in phase. The total tide is determined by superposition of the M2 and M4 tide. It can be seen that the total tide is symmetrical around $t/T = 0.5$. Therefore the same amount of sediment is transported into the river as there is sediment transported out of the river. In figure 1.2b the tides are slightly out of phase. The result is a total tide that has a high positive velocity and a lower negative velocity, that is, there is a net sediment flow into the river. This is why the M4 tide is important for sediment flow [6].



(a) The velocity of the M2, M4 and the superposition of the two plotted for one period. Here the M2 and M4 tides are in phase, which results in an symmetrical superposition.



(b) The velocity of the M2, M4 and the superposition of the two plotted for one period. Here the M2 and M4 tides are out of phase, which results in an asymmetrical superposition.

Figure 1.2

1.2. Previous research

Many estuaries are strongly modified by human interventions in the last couple of decades. In particular channel deepening has been executed in order for big cargo ships the enter rivers. Multiple previous scientific studies have shown that deepening of the channel in the Ems river have effects on the tidal distortion. Therefore the change can be responsible for the transition from low to high sediment concentrations [3]. These studies have mostly focused on changes to the M2 tide with deepening of a river, not on the M4 tide.

This study will focus on the propagation of the M2 and M4 tide, subjected to changing parameters. In particular how the propagation differs for the M4 tide compared to the M2 tide. The parameters that are important to study are the height of the bed level of the river and the friction experienced by the water in the river. For sake of completeness of this study the width of the river shore is also taken as variable. In order to investigate the propagation of the M2 and M4 tide it is preferred to look at the variables separately. The main research question is:

How does the propagation of the M2 and M4 tide change, subjected to deepening, change in width and change in friction?

The result of this main question can be applied to tidal rivers. To illustrate this, the results are compared to historical data of the Ems river. Data for the Ems river is available for the years 1965 and 2005. Between these years dredging has let to deepening. As mentioned before, the sediment concentration has increased in this river. Therefore, the height of the bed level and the friction coefficient should have been changed between 1965 and 2005. So, it is quite interesting to apply the results of the main question to the data, as the same phenomenon could happen for similar tidal rivers, which we like to prevent. The sub research question is:

Can historical observations of the amplitude of the M2 and M4 tides in the Ems river over the years be explained with the change in propagation of the M2 and M4 tides, subjected to deepening and friction change?

1.3. Thesis structure

To be able to answer the research questions posed in the previous section, the following research approach and thesis structure are adopted. In chapter 2 the one dimensional equations, that govern fluid flow in a river and estuary, are derived. These equations hold for a one dimensional basin system. To establish these equations the boundary conditions are used, which are also explained in this chapter. After that there are two cases. In the first case the width, height and friction are taken to be constant over the length of the river. In the second case these parameters can vary over the length of the river. In chapter 3 the solutions for the equations that were derived are given for both cases. In the first case an analytical and numerical solution is implemented. For the second case only a numerical solution is given. In chapter 4 the first case is analysed and results are given. Here the results to the sub question can be found. Lastly, in chapter 5 the results for the second case are analysed, also the main question is answered in this chapter.

2
Derivation of the water equations

In this chapter the derivation for the water equations are given. Two laws are used, namely the conservation of mass and conservation of momentum. Both laws are applied to a river by applying boundary conditions. In order to be able to find a solution the conservation equations are integrated over the depth and width of the river. Then one-dimensional equations over the length of the river are obtained. After the derivation of the momentum equation a few assumptions are done and alternations are made in order to be able to solve the system of equations.

2.1. Conservation of mass equation

The first law that is used is the conservation of mass. The flow in the river is assumed to be incompressible, therefore the density change over the length of the river is ignored. The incompressible continuity equation is [10]:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{2.1}$$

Here u, v and w are the flow velocities in the Cartesian dimensions x, y and z. Here x is the variable over the length of the river, y is the variable over the width of the river and z is the variable over the height of the river. To obtain an one dimensional equation from the conservation of mass the equation is integrated over the height of the bed level and the width of the river. The height varies from $-H$, which is the bed level, to $\zeta(x,t)$, which is the water level. A schematic representation of the cross-section over the length of the river can be seen in figure 2.1. The width of the river varies from 0, which is the left side of the river, to B which is the distance to the other side of the river.

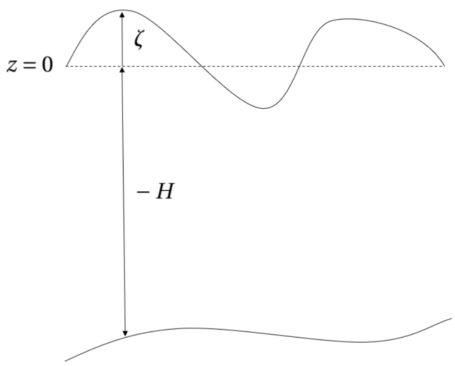


Figure 2.1: Schematic representation of the cross-sectional view over the length of the river.

The integral that needs to be evaluated can be seen in equation 2.2. Here the partial derivatives with respect to x, y and z are written as a subscript.

$$\int_{-H}^{\zeta}\int_{0}^{B}\left(u_x+v_y+w_z\right)dydz=0 \tag{2.2}$$

The derivation of this integral can be found in the appendix A.1. There the assumptions that are made in order to be able to evaluate the integrals are explained as well. The result for the equation of conservation of mass is:

$$\frac{\partial U}{\partial x}+B\frac{\partial \zeta}{\partial t}=0. \tag{2.3}$$

Here U is the width and depth averaged velocity in the x-direction.

2.2. Conservation of momentum equation

The Navier-Stokes equation is used to represent the conservation of momentum. The final result of derivation of this equation is given by [9]:

$$\frac{\partial U}{\partial t}+\frac{\partial}{\partial x}\left(\frac{U^2}{A}\right)=-gA\frac{\partial \zeta}{\partial x}-\frac{\tau_b}{\rho}P. \tag{2.4}$$

Here A is the channel cross-section area, which can also be written as $A = BH$, τ_b is the average shear stress on the boundaries and P is the wetted channel perimeter. The wetted channel perimeter is the length of the intersection of channel wetted surface with a cross sectional plane normal to the flow direction. If it is assumed that the width of the river is much greater than the height of the water column, then the width B can be taken for the parameter P, so $P \approx B$. The problem is closed mathematically by formulating the average shear stress [5].

$$\tau_b=\rho f\left|\frac{U}{A}\right|\frac{U}{A} \tag{2.5}$$

Where f is a dimensionless friction factor. There are a few non-linearities to be found in the equation that is derived from the conservation of momentum. The non-linearity's in the equation are:

- The first non-linear term is the, so called, advection of momentum term. This is the second term on the left hand side in equation 2.4, $\frac{\partial}{\partial x}\left(\frac{U^2}{A}\right)$;
- The second non-linear term in the equation is the quadratic friction term, since $\tau_b \sim U^2$.

These non-linearities cause the M4 tide. This is explained in intermezzo 2.2. These non-linearities also make it hard to obtain a solution to this problem. However, two methods are used, such that the conservation of momentum equation is approximated by a linear equation.

Intermezzo: Explanation co-oscillation tidal waves

In the introduction it was mentioned that the M4 tide is caused by an harmonic of the M2 tide. That can be explained now the conservation of momentum equation is derived. On open sea the only tide that is causing tidal waves is the M2 tide, the diurnal tide. As the wave approaches shore, the nonlinear friction term in the conservation of momentum equations is causing a second tidal wave.

The M2 tide can be written as speed $U = A\cos\omega t$, in the friction term this speed is squared. Therefore,

$$U^2 = (A\cos(\omega t))^2 = A^2\cos^2(\omega t) = A^2\left(\frac{1}{2} + \frac{1}{2}\cos(2\omega t)\right) \tag{2.6}$$

The quadratic friction causes a harmonic tide, with twice the frequency, which is the M4 tide. Therefore there are two tidal waves that enter the estuary of the river.

From the perturbation method only the leading term is taken as the equation for the conservation of momentum in the basin of the river. However, to be precise, there is a extra quadratic friction term in $O(\epsilon)$. This quadratic term again causes harmonics. Therefore, in real life, there are multiple tidal waves that change the tides inside the river. However, these other harmonics are neglected in this study.

2.2.1. Perturbation method and Scale analysis

To approximate the non-linear advection of momentum term, by a linear-term a perturbation method is used. For an introduction to perturbation techniques, see for example [7]. For the perturbation method it is needed to find which term is the least important. In order to find this least important term a scaling analysis is performed.

The equation is nondimensionalized by introducing the characteristic scale and the nondimensional quantity. The nondimensional quantities are denoted with an asterisk in the following variables;

$$\begin{aligned} t &= t^* T \\ x &= x^* L \\ U &= U^* \mu \\ A &= A^* a \\ \zeta &= \zeta^* z \\ \tau_b &= \tau_b^* \sigma \\ B &= B^* b. \end{aligned}$$

Here, T is a time scale, L is the horizontal length scale, μ is the horizontal flow velocity scale, a is the average flow area, z is the vertical length scale, σ is the shear stress scale and b is the width length scale. Next the PDE can be rewritten with the introduced non-dimensional parameters.

$$\frac{\mu}{T}U^*_{t^*} + \frac{\mu^2}{aL}\left(\frac{U^{*2}}{A^*}\right)_{x^*} = -\frac{gaz}{L}A^*\zeta^*_{x^*} - \frac{\sigma b}{\rho}\tau_b^* B^* \tag{2.7}$$

The coefficients are replaced by α, β, γ and δ.

$$\alpha U^*_{t^*} + \beta\left(\frac{U^{*2}}{A^*}\right)_{x^*} = -\gamma A^*\zeta^*_{x^*} - \delta\tau_b^* B^* \tag{2.8}$$

The characteristic values are taken for a typical tidal river and the corresponding non-dimensional parameters are presented in table 2.1.

Parameter	Dimensionless
$T \approx 40000$s	$\frac{\mu}{T} = 2.5 \cdot 10^{-5}$
$L \approx 100000$m	$\frac{\mu^2}{aL} = 1.0 \cdot 10^{-8}$
$\mu \approx 1$ ms^{-1}	$\frac{gaz}{L} = 9.8 \cdot 10^{-2}$
$a \approx 1000$ m^2	$\frac{\alpha b}{\rho} = 1.0 \cdot 10^{-2}$
$z \approx 1$ m	
$\sigma \approx 0.1$ m s^{-2}	
$b \approx 100$ m	
$g = 9.81$ ms^{-1}	
$\rho = 977$ kg m^{-3}	

Table 2.1: Characteristic values for the Ems river and the corresponding non-dimensional parameters.

The values for the parameters, which are not natural constants, are taken as for a typical tidal river. For example $T = 12h * 3600 \approx 40000$s, $a \approx BH \approx 10 * 100 = 1000$m, etc. If we fill in the values for the variables, the smallest value is $1.0 \cdot 10^{-8}$ for the dimensionless number β. From now on call $\epsilon = \beta$. This parameter is used to make an approximation for the the solutions for U and ζ. Now, the approximation for the variables U and ζ is introduced by:

$$U = U^0 + U^1 + U^3 + U^4 + \dots$$
$$\zeta = \zeta^0 + \zeta^1 + \zeta^2 + \zeta^3 + \zeta^4 + \dots$$

Here the first terms are of order 1, the second term of order ϵ, the third therm is of order ϵ^2, etc. Note that in the conservation of momentum equation we have the friction variable τ_b that is formulated as $\tau_b = C_D U^2$, where C_D is a constant. Therefore, we can write the friction as follows:

$$\tau_b = C_D \left(U^0 + U^1 + U^2 + \dots\right)^2 = C_D \left(U^{0^2} + 2U^0U^1 + \left(2U^0U^2 + U^{1^2}\right) + \dots\right) = \tau_b^0 + \tau_b^1 + \tau_b^2 + \dots \tag{2.9}$$

These approximations are substituted in the equation for momentum, see equation 2.4.

$$U_t^0 + U_t^1 + U_t^2 + \dots + \left(\frac{1}{A}\left(U^0 + U^1 + U^2 + \dots\right)^2\right)_x = -gA\left(\zeta_x^0 + \zeta_x^1 + \zeta_x^2 + \dots\right) - \frac{\tau_b^0 + \tau_b^1 + \tau_b^2 + \dots}{\rho} B \tag{2.10}$$

When the terms are sorted by order of ϵ we obtain the following equations for the first and second order.

$$\begin{aligned} \mathrm{O}(1): \quad & U_t^0 = -gA\zeta_x^0 - \tfrac{\tau_b^0}{\rho} B \\ \mathrm{O}(\epsilon): \quad & U_t^1 + \left(\tfrac{1}{A} U^{0^2}\right)_x = -gA\zeta_x^1 - \tfrac{\tau_b^1}{\rho} B \end{aligned}$$

Therefore, the simplified equation that is obtained for the conservation of momentum is

$$U_t^0 = -gA\zeta_x^0 - \frac{\tau_b^0}{\rho} B. \tag{2.11}$$

Now the equation has only one non-linearity, namely in the friction term τ_b.

2.2.2. Lorentz' linearization

The last non-linear term in the equation is the bottom shear stress. To solve this problem the, so called, Lorentz linearization is used.

$$\tau_b = \alpha U$$

Here α is the friction coefficient and U is again the width and height averaged velocity in the x-direction in the river. It is experimentally verified that the Lorentz linearization is valid for quadratic friction on the response curves for the Helmholtz mode in an almost-enclosed basin. The Helmholtz mode is characterized by uniform sea-level elevation within basins that are small compared to the tidal wavelength and co-oscillate with an adjacent sea/ocean through a narrow inlet [11].

2.3. Problem formulation

Now two equations are derived for the chosen system, with its boundary conditions. In this section the equations are shown together and the boundary conditions that are needed to find the solution are given. The two equations that are derived are:

$$\begin{aligned} U_x + B\zeta_t &= 0 \\ U_t + gA\zeta_x + \frac{\alpha U}{\rho}B &= 0. \end{aligned} \tag{2.12}$$

The subscripts represent the derivative. These equations are of leading order, obtained with the perturbation method. Here, and in the rest of this paper, the superscript ^{0} is not written for the sake of convenience. Since the tide is represented by a wave, this system of equations should represent a wave equation. The derivation for this can be found in intermezzo 2.3.

The boundary conditions over the length and height of the river are implemented in these equations. Only the lower boundary condition, at $x = 0$ and the higher boundary condition, at $x = L$ are not used yet. These boundary conditions are:

$$\begin{aligned} \zeta(0,t) &= A\cos(\omega t - \theta) \\ U(L,t) &= 0. \end{aligned} \tag{2.13}$$

Here A is the amplitude of the M2 or M4 tide at the mouth or end of the river, ω is the angular frequency of the tides and θ is the phase difference between the M2 and M4 tide.

It can be assumed that very far in the river the speed of the water is zero. This could be caused by a dam or just because the speed in the river is only caused by rain or melt water flowing to the river, which is negligible compared to the speed the tide causes in the beginning of the river.

Intermezzo: Similarity wave equation

The coupled system of equation 2.12 can be combined. The result will be a wave equation. In this intermezzo it is explained why this is the case. First the derivative with respect to x in the first equation and the derivative with respect to t in the second equation is given, the result is:

$$U_{xx} + B\zeta_{tx} = 0$$
$$U_{tt} + gA\zeta_{xt} + \left(\frac{\alpha U}{\rho} B\right)_t = 0.$$

The first equation can be rewritten into $B\zeta_{tx} = -U_{xx}$. By Clairaut's theorem we can interchange the order of partial derivatives so we get $\zeta_{tx} = \zeta_{xt}$. [8] Now this is substituted into the second equation to obtain:

$$U_{tt} = gHU_{xx} - \left(\frac{\alpha U}{\rho} B\right)_t. \tag{2.14}$$

Note that the second term on the right hand side can be written as μU_t, since α, ρ and B can be taken out of the derivative to t. Therefore

$$U_{tt} + \mu U_t = gHU_{xx}. \tag{2.15}$$

This is a second order PDE which describes the propagation of oscillations, a wave equation. The velocity of the wave is $\sqrt{gH}$. The latter will be used later in this report. The second term on the right hand side is known as the friction term in a wave equation.

3

Solutions to the one dimensional equations

In the previous chapter two coupled equations were given (see equation 2.12) that describe the river mathematically. In this chapter the solutions are given. There are two cases. In the first case the parameters H, B and α do not change over the length of the river. In this case there exists an analytical solution. In addition a numerical solution will be given. In the second case the parameters H, B and α dependent on the position in the river, so they are x-dependent. In this case only the numerical solution can be given.

3.1. Length independent parameters

In this section the analytical and numerical solutions for the first case are given. In the first case the width, height and friction are constant over the length of the river.

We start by considering the variables $U(x,t)$ and $\zeta(x,t)$ as Fourier components with frequency ω:

$$U(x,t) = Re\left(\tilde{u}(x)e^{i\omega t}\right) \tag{3.1}$$

$$\zeta(x,t) = Re\left(\tilde{\zeta}(x)e^{i\omega t}\right). \tag{3.2}$$

After use of the perturbation method, Lorentz' Linearization and substitution of $A = HB$ the equations are:

$$U_x + B\zeta_t = 0; \tag{3.3}$$

$$U_t = -gHB\zeta_x - \frac{\alpha U}{\rho}B. \tag{3.4}$$

First $U(x,t)$ and $\zeta(x,t)$ are substituted in equation 3.4. It is used that the derivative of the real part of a function is equal to the real part of the derivative of the function. Also the Fourier component is used in order to get an expression for the derivative with respect to t; time. Solving for $U(x,t)$ gives us:

$$\tilde{u}(x) = -\frac{gHB}{\left(i\omega + \frac{\alpha B}{\rho}\right)}\tilde{\zeta}_x(x). \tag{3.5}$$

The solution for $U(x,t)$ and $\zeta(x,t)$ can be substituted in equation 3.3. From equation 3.5 the derivative, $\tilde{u}_x(x)$, can be obtained. This is substituted. Then the equation for $\tilde{\zeta}(x)$ is given by:

$$-\frac{gH}{\left(i\omega + \frac{\alpha B}{\rho}\right)}\tilde{\zeta}_{xx}(x) + i\omega\tilde{\zeta}(x) = 0 \;\Rightarrow\; \tilde{\zeta}_{xx}(x) - k\tilde{\zeta}(x) = 0. \tag{3.6}$$

Where $k = \frac{-\omega^2 + i\frac{\omega\alpha B}{\rho}}{gH}$, which is the wave number. The boundary conditions given in equation 2.13 can be given in terms of $\tilde{\zeta}$ and $\tilde{u}$. To find the first boundary condition for $\tilde{u}$ equation 3.5 is used.

$$\begin{aligned} \tilde{u}(L) = 0 \qquad \Rightarrow \qquad & \tilde{\zeta}_x(L) = 0, \\ & \tilde{\zeta}(0) = A \end{aligned} \tag{3.7}$$

3.1.1. Analytical solution

In order to find a solution to the ordinary differential equation for ζ (equation 3.6), the Ansatz, $\tilde{\zeta} = e^{rx}$, is used. This is substituted into the equation, which results in $r = \sqrt{k}$ or $r = -\sqrt{k}$. Therefore, the solution is; $\tilde{\zeta}(x) = C_1 e^{\sqrt{k}x} + C_2 e^{-\sqrt{k}x}$. With C_1 and C_2 real constants.

In previous section it is explained what the boundary conditions are in terms of $\tilde{\zeta}$. These two boundary conditions are used to find the solution:

$$\tilde{\zeta}(x) = A\frac{\cosh(\sqrt{k}(x-L))}{\cosh(\sqrt{k}L)}. \tag{3.8}$$

3.1.2. Numerical solution

A numerical solution is needed for when the width, height and friction are not constant over the length of the river. In order to verify the numerical solution the numerical solution for the case where the variables are constant over the length of the river is implemented.

A numerical method is used on equation 3.6 to find the numerical solutions for the amplitude. For the second order differential term $\tilde{\zeta}_{xx}$ a second order central difference is used; $\tilde{\zeta}_{xx} = \frac{\tilde{\zeta}_{j-1} - 2\tilde{\zeta}_j + \tilde{\zeta}_{j+1}}{h^2}$, resulting in the following equation:

$$\frac{1}{h^2}\tilde{\zeta}_{j-1} - \left(\frac{2}{h^2} + k\right)\tilde{\zeta}_j + \frac{1}{h^2}\tilde{\zeta}_{j-1} = 0, \qquad j \in \{0,1,2,3,....,N-1,N\}. \tag{3.9}$$

Here h is the step size which is determined by the number of steps N, and the length of the domain L, $h = \frac{L}{N}$. The variable k is as in the previous section.

Now the two boundary conditions given in the analytical solution section, see equation 3.7 are applied to the equation as well.

$$\tilde{\zeta}_0 = A \qquad\qquad \frac{\partial\tilde{\zeta}}{\partial x} = \frac{\tilde{\zeta}_{N+1} - \tilde{\zeta}_N}{h} = 0 \;\Rightarrow\; \tilde{\zeta}_{N+1} = \tilde{\zeta}_N$$

The boundary conditions are used and the equation can be presented in matrix representation;

$$A\boldsymbol{\zeta} = \mathbf{f}.$$

Here $\boldsymbol{\zeta} = \left(\tilde{\zeta}_0, \tilde{\zeta}_1, \tilde{\zeta}_2, ..., \tilde{\zeta}_{N-1}, \tilde{\zeta}_N\right)^T$. The representations for A and $\mathbf{f}$ are presented below.

$$A = \begin{bmatrix} 1 & & & & & \\ \frac{1}{h^2} & -\left(\frac{2}{h^2}+k\right) & \frac{1}{h^2} & & & \\ & \frac{1}{h^2} & -\left(\frac{2}{h^2}+k\right) & \frac{1}{h^2} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \frac{1}{h^2} & -\left(\frac{2}{h^2}+k\right) & \frac{1}{h^2} \\ & & & & \frac{1}{h^2} & -\left(\frac{1}{h^2}+k\right) \end{bmatrix}, \qquad \mathbf{f} = \begin{bmatrix} A \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

This equation can be solved numerically using a sparse linear system solver. This is implemented in Python. The code can be found in appendix A.3.

3.1.3. Dimensionless numbers

The solution 3.8 depends on seven variables and parameters, since k depends on a lot of variables. This amount can be reduced to two parameters using dimensionless numbers. Therefore it is usefull to look at dimensionless numbers. In this section these are derived and the interpretation is given. The dimensionless numbers are obtained from the following equation.

$$\tilde{\zeta}_{xx}(x) - k\tilde{\zeta}(x) = 0 \tag{3.10}$$

Here $k = \frac{-\omega^2 + i\frac{\omega\alpha B}{\rho}}{gH}$. A scaling analysis is executed on this equation in order to obtain the dimensionless numbers. The variable $x = Lx^*$ is introduced, where L is the length the river. The derivative is then scaled to $\tilde{\zeta}_{xx} = \frac{\tilde{\zeta}_{x^*x^*}}{L^2}$. The complex indication i can be left out when a scaling analysis is done, since it has scale one.

$$\frac{\tilde{\zeta}_{x^*x^*}}{L^2} - \frac{-\omega^2 + \frac{\omega\alpha B}{\rho}}{gH}\tilde{\zeta} = 0$$

The equation is then multiplied by L^2 such that the first term is dimensionless. Then the second term is dimensionless as well. Rewriting the equation gives:

$$\tilde{\zeta}_{x^*x^*} - \frac{-\omega^2 L^2 + \frac{\omega\alpha BL^2}{\rho}}{gH}\tilde{\zeta} = 0$$

Now the two dimensionless numbers k_1 and k_2 are obtained.

$$k_1 = \frac{\omega^2 L^2}{gH} = \left(\frac{2\pi L}{\lambda}\right)^2 \qquad\qquad k_2 = \frac{\omega\alpha BL^2}{\rho gH}$$

The wavelength of a wave is known to be $\lambda = v/f$, here v is the velocity and f is the frequency. In intermezzo 2.3 it was seen that the velocity of the tidal wave is gH. The frequency is $f = \omega/2\pi$. When these two are substituted in the equation we end up with the wavelength for a tidal wave; $\lambda = 2\pi\sqrt{gH}/\omega$. When this is substituted into the equation for k_1 we get $k_1 = \left(\frac{\lambda}{L}\right)^{-2}$. This dimensionless number gives the squared ratio between the length of the river and the wavelength of the tidal wave. A k_1 that is smaller than $(2\pi)^2 \approx 39$ means that less than one wavelength of the tide can be found in the domain of the river. A k_1 greater than $(2\pi)^2 \approx 39$ means that the tidal wave has propagated one or multiple wavelengths in the river.

The second dimensionless number is a ratio of energies. To explain this a few terms in the dimensionless number need to be taken together:

- ωL can be seen as a velocity, since L has the unit m and ω has as unit s^{-1}. These parameters are taken together in parameter v;
- LB is the surface of the bed of the river. These parameters are taken together and called S;
- The potential energy is defined by $E_{pot} = mgH$, therefore the denominator scales with the potential energy, $\rho gH \sim E_{pot}$.

Then the dimensionless number k_2 can also be written as $k_2 \sim \frac{\alpha Sv}{E_{pot}}$. Now the nominator is a scale for energyloss due to friction, which scales with the friction α, S and v. The greater these individual parameters the greater the friction term. This is plausible, since the greater the bed level surface is, the more friction the tidal wave will experience. The same holds for the velocity, the greater the velocity, the more friction the water will experience. If this number is big there is a lot of friction in the system. If the number is small the potential energy is big compared to the friction term. So the bigger k_2, the more friction plays a significantly important role.

3.2. Length dependent parameters

In this section the parameters H, B and α can vary over the length of the river. From earlier research it has been seen that the equations derived in chapter 2 are still valid in this case [9]. In this section the solution method for this case is given. Note that in this section the variables B, α, H, H_x, B_x and α_x are dependent of x and therefore they are vectors. For sake of simplicity and clarity of the equations the variables are not notated in bold. We start with the same equations as in the case for independent parameters. Next, we substitute the same representation for $U(x,t)$ and $\zeta(x,t)$, see equation 3.1. The equations are divided by $e^{i\omega t}$. Now the system of equations is represented as:

$$\tilde{u}_x + i\omega B\tilde{\zeta} = 0 \tag{3.11}$$

$$i\omega\tilde{u} = -gA\tilde{\zeta}_x - \frac{\alpha\tilde{u}}{\rho}B. \tag{3.12}$$

In the previous section it was derived that equation 3.12 can be written as:

$$\tilde{u}(x) = -\frac{gHB\tilde{\zeta}_x(x)}{\left(i\omega + \frac{\alpha B}{\rho}\right)}. \tag{3.13}$$

Now the derivative of $\tilde{u}$ is calculated and sorted in $\tilde{\zeta}$, $\tilde{\zeta}_x$ and $\tilde{\zeta}_{xx}$.

$$\tilde{u}_x(x) = -\frac{g}{i\omega + \frac{\alpha B}{\rho}}\left(H_x B + HB_x - \frac{HB}{\rho\left(i\omega + \frac{\alpha B}{\rho}\right)}(\alpha_x B + \alpha B_x)\right)\tilde{\zeta}_x - \frac{gHB}{i\omega + \frac{\alpha B}{\rho}}\tilde{\zeta}_{xx} \tag{3.14}$$

Now equations 3.14 and 3.11 are used to obtain:

$$i\omega B\tilde{\zeta} + \gamma\tilde{\zeta}_x + \delta\zeta_{xx}. = 0 \tag{3.15}$$

Here $\gamma = -\frac{g}{i\omega + \frac{\alpha B}{\rho}}\left(H_x B + HB_x - \frac{HB}{\rho\left(i\omega + \frac{\alpha B}{\rho}\right)}(\alpha_x B + \alpha B_x)\right)$ and $\delta = -\frac{gHB}{i\omega + \frac{\alpha B}{\rho}}$

3.2.1. Numerical solution

There does not exist an analytical solution for equation 3.15. Therefore a numerical method is applied. The difference with the case where the variables are independent of x is that, besides the second order derivative of $\tilde{\zeta}$, there is a first order derivative. Furthermore, the variables B, γ and δ that depend on x are needed to obtain a numerical solution. For the second order derivative a second order central difference is used, for the first derivative a forward difference method is used. So $\tilde{\zeta}_{xx} = \frac{\tilde{\zeta}_{j-1} - 2\tilde{\zeta}_j + \tilde{\zeta}_{j+1}}{h^2} + \mathcal{O}(h^2)$ and $\tilde{\zeta}_x = \frac{\tilde{\zeta}_{j+1} - \tilde{\zeta}_j}{h} + \mathcal{O}(h)$. Now the variables that depend on x are chosen with the same uniform grid size as ζ. Therefore $B = (B_1, B_2, ...B_{N-1}, B_N)^T$, $\gamma = (\gamma_1, \gamma_2, ...\gamma_{N-1}, \gamma_N)^T$ and $\delta = (\delta_1, \delta_2, ...\delta_{N-1}, \delta_N)^T$. Here γ and δ are determined with H, H_x, B, B_x, α and α_x, which all depend on j. These are substituted in equation 3.15, such that:

$$\left(\frac{\delta_j}{h^2}\right)\tilde{\zeta}_{j-1} + \left(i\omega B_j - \frac{2\delta_j}{h^2} - \frac{\gamma_j}{h}\right)\tilde{\zeta}_j + \left(\frac{\delta_j}{h^2} + \frac{\gamma_j}{h}\right)\tilde{\zeta}_{j+1} = 0, \qquad j \in \{0, 1, ..., N-1, N\}. \tag{3.16}$$

The two boundary conditions that were used in the first case, see equation 3.7, are used again in this case. Therefore the f-vector given in the section above is the same for this method. The matrix A changes to:

$$\mathbf{A} = \begin{bmatrix} 1 & & & & & & \\ \left(\frac{\delta_0}{h^2}\right) & \left(\omega^2 B_1 - \frac{2\delta_1}{h^2} - \frac{\gamma_1}{h}\right) & \left(\frac{\delta_2}{h^2} + \frac{\gamma_2}{h}\right) & & & & \\ & \left(\frac{\delta_1}{h^2}\right) & \left(\omega^2 B_2 - \frac{2\delta_2}{h^2} - \frac{\gamma_2}{h}\right) & \left(\frac{\delta_3}{h^2} + \frac{\gamma_3}{h}\right) & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \left(\frac{\delta_{N-2}}{h^2}\right) & \left(\omega^2 B_{N-1} - \frac{2\delta_{N-1}}{h^2} - \frac{\gamma_{N-1}}{h}\right) & \left(\frac{\delta_N}{h^2} + \frac{\gamma_N}{h}\right) \\ & & & & \left(\frac{\delta_{N-1}}{h^2}\right) & \left(\omega^2 B_N - \frac{\delta_N}{h^2}\right) \end{bmatrix}, \qquad \mathbf{f} = \begin{bmatrix} A \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

This equation can be solved numerically using a sparse linear system solver. This is implemented in Python, the code can be found in appendix A.3.

4

Results for length independent parameters

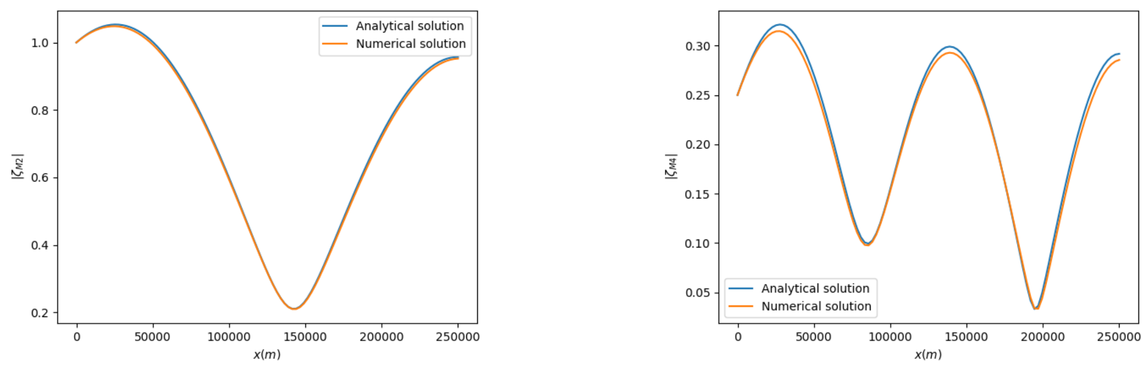
In this chapter the analytical and numerical solution for the first case are given. Here the width, height and the friction are x-independent. First, the numerical solution is compared to the analytical solution.

4.1. Analytical and numerical solution

The analytical solution and numerical solution are both implemented in Python. The code for these solutions can be found in appendix A.3. The step size is taken to be $N = 100$ for the numerical solution. With the variables given in table 4.1 the numerical and analytical solution are plotted together, both for the M2 and M4 tide.

Channel	Tide
$L = 2.5 \cdot 10^6$ m	$\omega_{M2} = 1.4 \cdot 10^{-4}$ rad s^{-1}
$B = 1 \cdot 10^2$m	$\omega_{M4} = 2.8 \cdot 10^{-4}$ rad s^{-1}
$H = 10$m	$\theta_{M2} = \theta_{M4} = 0$ rad
$\alpha = 4 \cdot 10^{-4}$ kg s^{-1} m^4	$A_{M2} = 1$ m
Constant	$A_{M4} = 0.25$ m
$g = 9.81$ ms^{-2}	
$\rho = 997$ kg m^{-3}	

Table 4.1: Characteristic values for a typical tidal river.



(a) The numerical and analytical solution for the amplitude of the M2 tide, here the friction was set to the value given in table 4.1

(b) The numerical and analytical solution for the amplitude of the M4 tide, here the friction was set to the value given in table 4.1.

Figure 4.1

It can be seen that for both the M2 as the M4 tide the numerical solution matches the analytical solution. The amplitude, $|\tilde{\zeta}|$ decreases over the length of the river, as expected, since the friction is nonzero. More checks for other settings can be found in appendix A.2.

The error, defined by the absolute difference between the absolute amplitude of the numerical solution minus the absolute amplitude of the analytical solution, is plotted over the number of grid points N, see figure 4.2. Both axis are log-scaled. It can be seen that the errors for both the M2 tide as for the M4 keep decreasing exponentially, since it shows a linear function in the log-log plot. However, for $N = 100$ the difference of the amplitude between the numerical and analytical solutions is at most 1cm. This means that the numerical solution only differs by one centimeter from the analytical solution over the whole length of the river. This is an negligible error when a tide of 1.5m is considered. For an increasing amount of grid points this maximum will only decrease. Therefore the amount of grid points that is reasonable is 100. With this value the mean squared errors for both the M2 and M4 tide are calculated. The mean squared error for the M2 tide is $4.53 \cdot 10^{-5}\text{m}^2$ and the mean squared error for the M4 tide is $3.63 \cdot 10^{-5}\text{m}^2$.

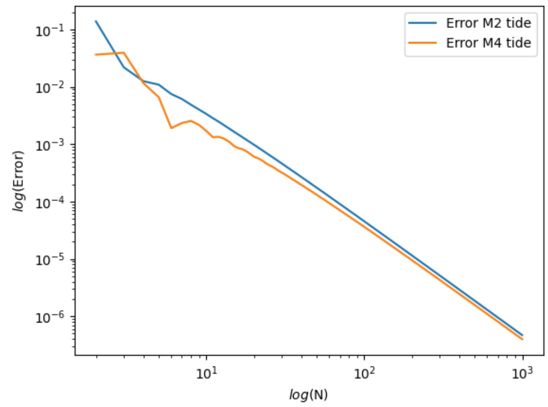


Figure 4.2: The error is plotted over the amount of grid points plot. The axis are log-scaled.

4.2. Variation in length and resonance

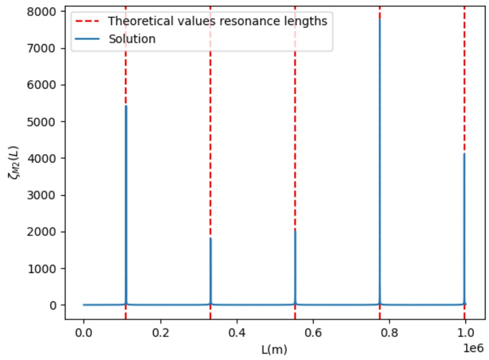
In this section the reason for amplification of an tidal wave inside a river will be illustrated using the model. The length of the river is varied and the amplitude at the beginning of the river, at $x = L$ is obtained from the numerical solution. The beginning of the river is chosen since this is an easy reference point to compare certain results. The numerical solution is used, since the numerical solution will be used later for a real life case. The default values for the variables can be found in table 4.1.

Since the tide is a wave it is expected to have resonance of the system when the anti-node of the wavefront is at the end of the river. That means that there is an amplification, or peak, in the amplitude $\tilde{\zeta}(L)$ for when the length of the river is $\frac{1}{4}\lambda$, $\frac{3}{4}\lambda$, $\frac{5}{4}\lambda$, $\frac{7}{4}\lambda$, etc, here $\lambda = \frac{\sqrt{gH}}{f}$ and $f = \frac{\omega}{2\pi}$. These values for the length of the river will be called resonance lengths from now on. When substituting the values from table 4.1 the fractions above can be calculated for the M2 and M4 tide, see table 4.2.

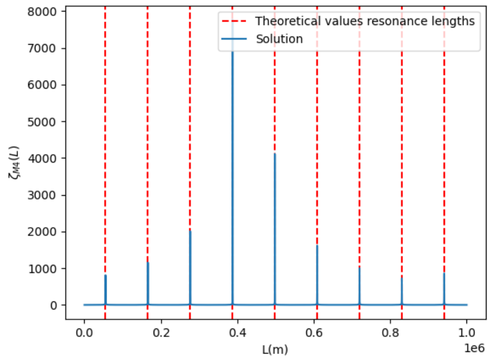
Resonance M2 tide	Resonance M4 tide
$\frac{1}{4}\lambda \approx 1.1 \cdot 10^5$	$\frac{1}{4}\lambda \approx 0.55 \cdot 10^5$
$\frac{3}{4}\lambda \approx 3.3 \cdot 10^5$	$\frac{3}{4}\lambda \approx 1.7 \cdot 10^5$
$\frac{5}{4}\lambda \approx 5.5 \cdot 10^5$	$\frac{5}{4}\lambda \approx 2.8 \cdot 10^5$
$\frac{7}{4}\lambda \approx 7.7 \cdot 10^5$	$\frac{7}{4}\lambda \approx 3.9 \cdot 10^5$

Table 4.2: Theoretical values for the resonance length of the river.

These values for the length of the river should also show an amplification in amplitude when the numerical solution is plotted. First, the friction is set to zero, such that there is no damping in the system, the figure for this can be seen in figure 4.3a for the M2 tide and in figure 4.3b for the M4 tide. On the x-axis the length of the river is given. On the y-axis the amplitude at $x = L$ is given. It can be seen that the amplitude shows an amplification for the resonance lengths of the river. It looks like the amplitude for the other values for the length of the river are zero. This is not the case, although the amplitude is very small compared to the amplitude for the resonance lengths. The theoretical values for the resonance lengths are plotted in the figures in dashed lines. Also the numerical solution at $x = L$ is plotted over the length of the river. It can be seen that the theoretical values match the amplification peaks in the numerical solution. The peaks for the numerical solution are expected to reach to infinity but since the step size is finite for the length of the river, this is impossible to accomplish numerically. Note that the wavelength for the M4 tide is not the same as the wavelength for the M2 tide, therefore the resonance lengths are different as well.



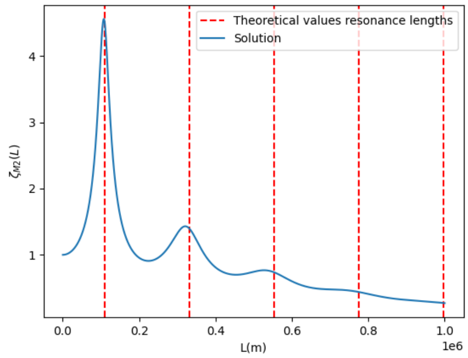
(a) The amplitude of the M2 tide at $x = L$, here the friction is set to zero. The sharp peaks in the plot represent the amplification of the amplitude.



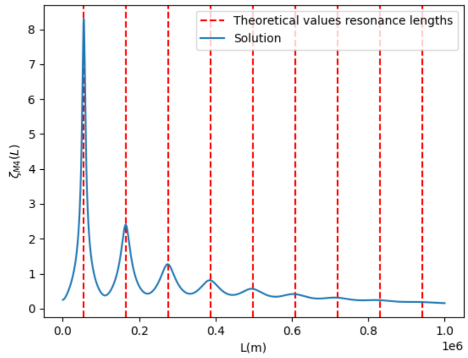
(b) The amplitude of the M4 tide at $x = L$, here the friction is set to zero. The sharp peaks in the plot represent the amplification of the amplitude.

Figure 4.3

Also the resonance plot for the case where the friction is nonzero is given, see figure 4.4a and 4.4b. It can be seen that the height of the peaks is decreasing, which can be explained by the friction term. The bigger the friction term, the faster the peaks will decrease. It can be seen that, as for the non-friction case, the theoretical resonance lengths agree with the peaks in the numerical solution. Again, the difference between the M2 and M4 tide is that the resonance lengths for the M2 tide is twice as big as the resonance lengths for the M4 tide.



(a) The amplitude of the M2 tide at $x = L$, here the friction is nonzero. The peaks in the plot represent the amplification of the amplitude. The peaks keep getting less high since the system undergoes a certain friction.



(b) The amplitude of the M4 tide at $x = L$, here the friction is nonzero. The peaks in the plot represent the amplification of the amplitude. The peaks keep getting less high since the system undergoes a certain friction.

Figure 4.4

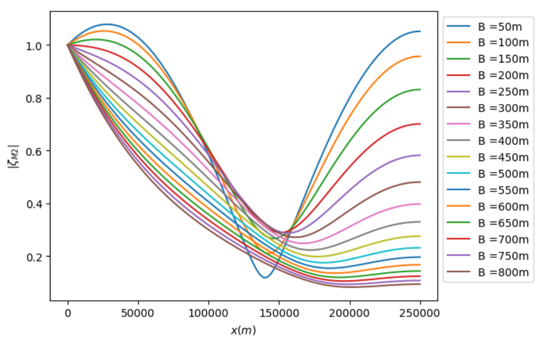
4.3. Propagation M2 and M4 tide subjected to widening, deepening and friction change

In this section, the constant parameters width, height and friction are changed and the effect on the propagation of the M2 and M4 tides are investigated. Later in this chapter the amplitude at the end of the river, at $x = L$ is analysed more vigorously.

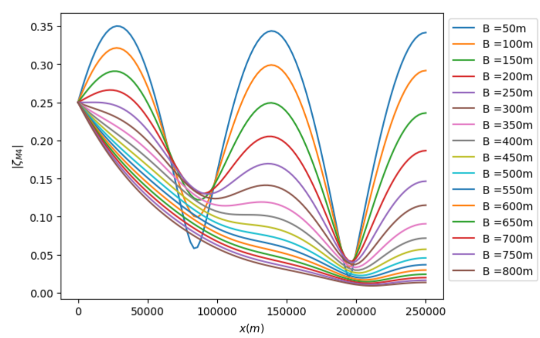
If not stated otherwise, all variables have the same values as in table 4.1. Some of the ranges taken for the parameters are not physical plausible. However, they are chosen such that the extreme cases are shown and it is more clear what it means for the amplitude when the parameters are changed.

4.3.1. Widening and friction change

First the width of the river is changed, however the width will stay constant over the length of the river. The value is varied from 50m to 800m with steps of 50m. The change of amplitude can be seen in figure 4.5a for the M2 tide and 4.5b for the M4 tide. On the x-axis the position over the length of the river can be seen and on the y-axis the amplitude is shown. It can be seen that when the width is increased both the M2 and the M4 tide are less propagated over the length of the river. When the width of the river is take to be 50m, there is no amplification or significant extra damping. When the width of the river is increased to 800m the amplitude of the tide is decreased and will be nearly zero at the end of the river. This holds for both the M2 and the M4 tide.



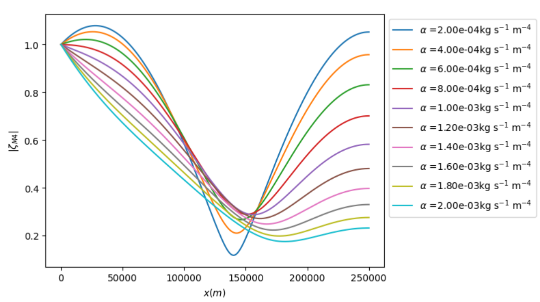
(a) The propagation of the amplitude of the M2 tide as a function of the length of the river. The propagation is plotted for a range of values for the width of the river.



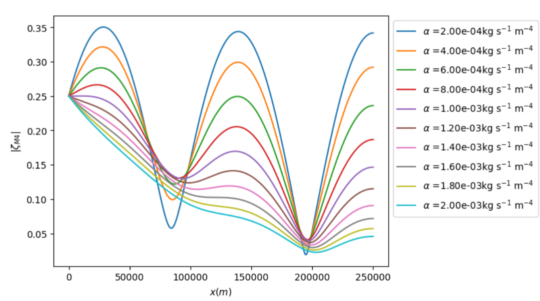
(b) The propagation of the amplitude of the M4 tide as a function of the length of the river. The propagation is plotted for a range of values for the width of the river.

Figure 4.5

Next, the friction in the river is varied. It is expected that if the friction is increased, there will be a very clear damping in the amplitude of the tides. As can be seen in figure 4.6a and 4.6b this is indeed the case. For $\alpha = 0.0002$ kg s^{-1} m^{-4} the M2 and M4 tide both nearly show no damping in amplitude. However, for $\alpha = 0.002$ kg s^{-1} m^{-4} the amplitude shows a clear damping for both the M2 tide as for the M4 tide.



(a) The propagation of the amplitude of the M2 tide as a function of the length of the river. The propagation is plotted for a range of values for the friction of the river.

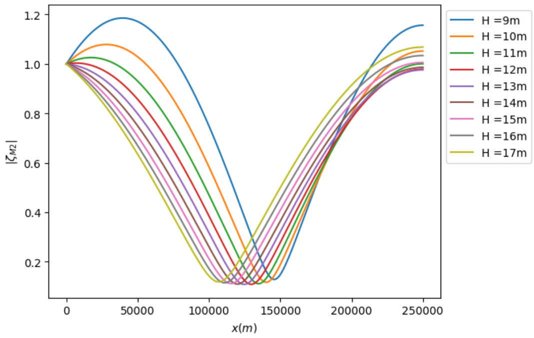


(b) The propagation of the amplitude of the M4 tide as a function of the length of the river. The propagation is plotted for a range of values for the friction of the river.

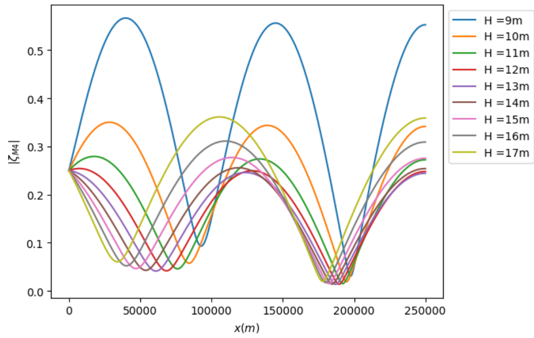
Figure 4.6

4.3.2. Deepening of the bed level

Now the depth of the river bed is considered as variable, the width and friction are constant and taken to be as in table 4.1. There is less alteration in the amplitude at the end of the river than there was when the width of the river was varied for the M2 tide. However relatively speaking, the M4 tide shows more alteration in the amplitude for the end of the river than the M2 tide. It can be observed that the deeper the river the less the amplitude of the tide is preserved over the length of the river. Also, which is more clear from the M4 amplitude plot, it can be seen that for $H = 17$ there are less oscillations than for $H = 9$. It can be concluded that for a deeper river there is less amplification in the amplitude and the wavelength of the tide is longer. This is expected since the wavelength is defined as $\lambda = \frac{c}{f}$, with $c = \sqrt{gH}$ and $f = \frac{\omega}{2\pi}$. Therefore if the depth is increased the wavelength will increase. So in the same domain the amplitude will show less oscillations for deeper rivers.



(a) The propagation of the amplitude of the M2 tide as a function of the length of the river. The propagation is plotted for a range of values for the height of the bed level of the river.



(b) The propagation of the amplitude of the M4 tide as a function of the length of the river. The propagation is plotted for a range of values for the height of the bed level of the river.

Figure 4.7

4.4. Dimensionless Numbers

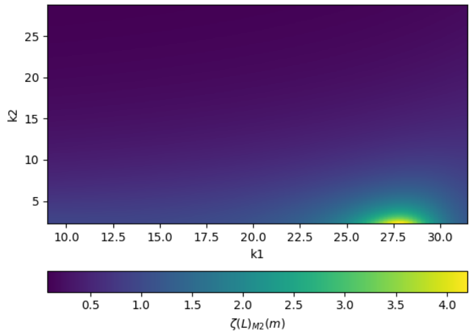
There are two dimensionless numbers found in subsection 3.1.3, repeated in equation 4.1. In this section the interpretation of these numbers are used to study the amplitude change in the M2 and M4 tide at $x = L$. The parameters that are varied are given the same values in range as in the previous section. If the parameters are not varied they are taken to be as in table 4.1, which are $H = 10$m, $B = 100$m and $\alpha = 0.0004$ kg s^{-1} m^{-4}.

$$k_1 = \frac{\omega^2 L^2}{gH}, \qquad k_2 = \frac{\omega \alpha B L^2}{\rho g H} \tag{4.1}$$

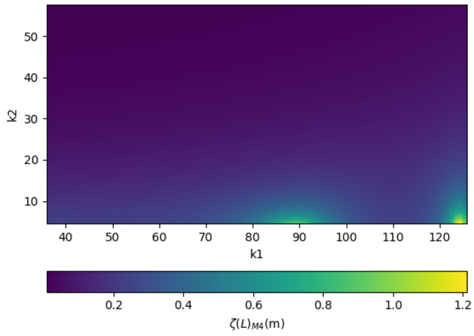
4.4.1. Variation in all variables

In previous section it was observed what happens to the propagation of the M2 and M4 tide as the width, depth and friction of the river is changed. In this section these observations will be explained using the dimensionless numbers that were derived in section 3.1.3. The dimensionless numbers change as the width, depth and friction change. When the width and the friction are changed only the second dimensionless number changes, as these parameters do not effect the first dimensionless number. The height changes both dimensionless numbers. It was seen that the amplitude at $x = L$, which is taken as the reference point, changes as well. Therefore the amplitude at $x = L$ is plotted against the dimensionless numbers, see figures 4.8a and 4.8b.

A position in the figure correspond to two dimensionless numbers k_1 and k_2, which correspond to certain values of width, depth and friction. Therefore each combination of width, depth and friction has a position in the plot. The range of the dimensionless numbers is determined by the range of width, height and friction. It can be seen that the range of the dimensionless numbers for the M2 tide differs from the range of the M4 tide. This is caused by the difference in angular frequency, which is twice as big for the M4 tide compared to the M2 tide. Lastly, there is one area of amplification in the plot for the M2 tide and two area of amplifications in the figure for the M4 tide to be seen. These areas of amplification correspond to the resonance peaks from earlier section 4.2. The plot for the amplitude of the M4 tide shows two amplifications since the frequency is twice as large. It can also be obtained that the amplitude decreases as the value of k_2 increases. This can be explained with the interpretation of this second dimensionless number. The greater the value, the more friction is dominant over the potential energy. When friction is more dominant the amplitude is damped and as a result the amplitude at the beginning of the river is lower.



(a) The amplitude of the M2 tide at the end of the river, at $x = L$, is plotted in a 2D grid. On the axis the dimensionless numbers k_1 and k_2 can be found.

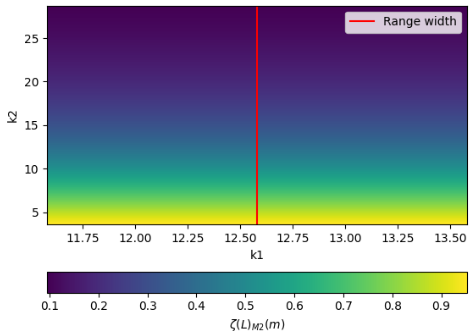


(b) The amplitude of the M4 tide at the end of the river, at $x = L$, is plotted in a 2D grid. On the axis the dimensionless numbers k_1 and k_2 can be found.

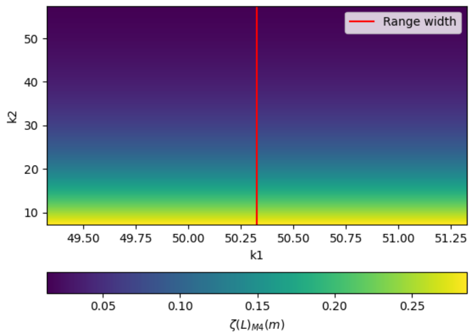
Figure 4.8

4.4.2. Changing the width and friction

As in the previous section the width is changed from 50m to 800m. It was mentioned above that only the second dimensionless number is changed. It is also mentioned that each position in 4.8a and 4.8b correspond to a certain value for the width, height and friction. Therefore a plot for the dimensionless numbers can also be made for the default values (see table 4.1) for the height and friction, while the width of the river is changed. This plot can be seen in figures 4.9a and 4.9b the range for the dimensionless numbers is presented with a red line. The line is vertically placed, since the first dimensionless number does not change. In order to be able to plot the two dimensionless number against each other, a small range in the first dimensionless number is taken. Therefore the step size in the plot for the first and second dimensionless number is different.



(a) The amplitude of the M2 tide at the end of the river, at $x = L$, is plotted in a 2D grid. On the axis the dimensionless numbers k_1 and k_2 can be found. Also a red line is plotted, this line represents the change in the second dimensionless number, whenever the width of the river is varied between $B = 50m$ and $B = 800m$.

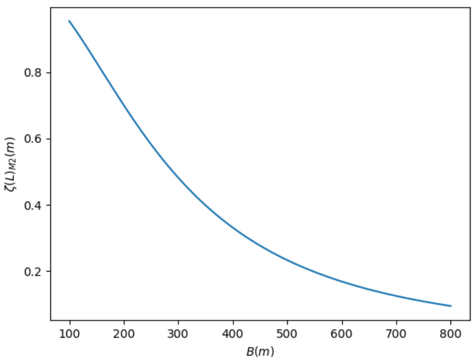


(b) The amplitude of the M4 tide at the end of the river, at $x = L$, is plotted in a 2D grid. On the axis the dimensionless numbers k_1 and k_2 can be found. Also a red line is plotted, this line represents the change in the second dimensionless number, whenever the width of the river is varied between $B = 50m$ and $B = 800m$.

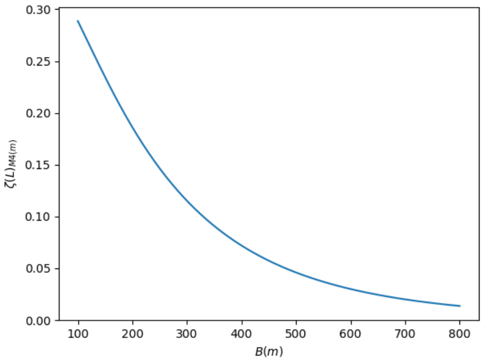
Figure 4.9

Next, the amplitude of the tides at $x = L$ is plotted, see figure 4.10a and 4.10b. The value of the first dimensionless number depends on what is taken for the height. It can be seen that for the chosen values for the height of the bed level and the friction the plots for the M2 and M4 tide do not differ much. However, it is confirmed that the wider the river, the smaller the amplitude at the beginning of the river is. This is also what was seen before in previous section 4.3.1.

In terms of dimensionless numbers; only the second dimensionless number will change due the change in width. As mentioned in section 3.1.3, when the width is increased, the bed level is increased. Therefore the energy loss due to friction is increased and is big compared to the potential energy term in the denominator of the second dimensionless number. This clarifies the decreasing amplitude of the tide at $x = L$.



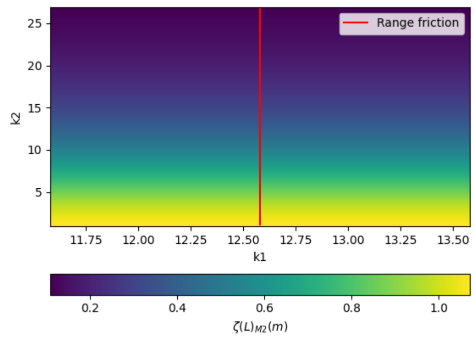
(a) The amplitude of the M2 tide is plotted over the width of the river.



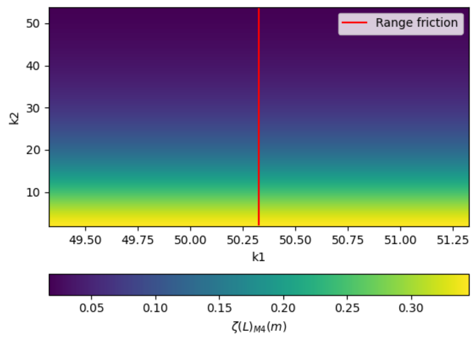
(b) The amplitude of the M4 tide is plotted over the width of the river.

Figure 4.10

The same holds for when the friction in the river is changed. The friction is changed in the range from $\alpha = 0.0001$ kg s^{-1} m^{-4} to $\alpha = 0.003$ kg s^{-1} m^{-4}. The height of the bed level and width of the river are taken as the default values given in table 4.1. As a result, only the second dimensionless number is changed. A plot for the dimensionless numbers can also be made for the default values (see table 4.1) for the height and width, while the friction of the river is changed. This plot can be seen in figures 4.11a and 4.11b. Here, the range for the dimensionless numbers is presented with a red line. As in the change of width, the line is vertically placed, since the first dimensionless number does not change. In order to be able to plot the two dimensionless number against each other, a small range in the first dimensionless number is taken. Therefore the step size in the plot for the first and second dimensionless number is different.



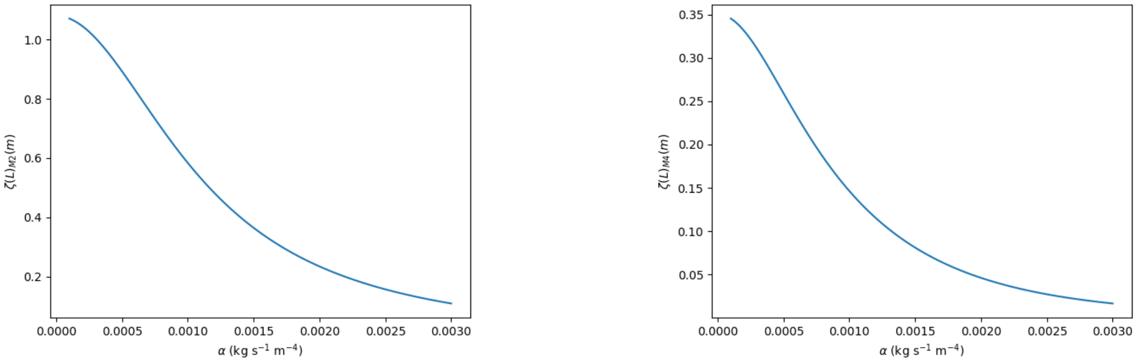
(a) The amplitude of the M2 tide at the end of the river, at $x = L$, is plotted in a 2D grid. On the axis the dimensionless numbers k_1 and k_2 can be found. Also a red line is plotted, this line represents the change in the friction, whenever the friction is varied between $\alpha = 0.00025$ ms^{-1} and $\alpha = 0.002$ ms^{-1}.



(b) The amplitude of the M4 tide at the end of the river, at $x = L$, is plotted in a 2D grid. On the axis the dimensionless numbers k_1 and k_2 can be found. Also a red line is plotted, this line represents the change in the friction , whenever the value is varied between $\alpha = 0.00025$ ms^{-1} and $\alpha = 0.002$ ms^{-1}.

Figure 4.11

Also for the change in friction the amplitudes of the tides at $x = L$ are plotted, see figure 4.12a and 4.12b. It can be seen that the M2 and M4 tide do not respond differently for a change in friction term. However, it can be seen that the higher the friction term the less the amplitude of the tide is propagated in the river. This is also clear from the second dimensionless number, it does not contribute in the value of the first dimensionless number. As the friction increases the energy loss due to friction increases. Therefore the friction dominates and the amplitude is decreased.



(a) The amplitude of the M2 tide is plotted over the friction of the river.

(b) The amplitude of the M4 tide is plotted over the friction of the river.

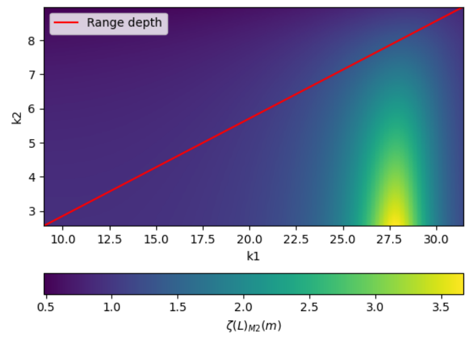
Figure 4.12

4.4.3. Changing the height of the bed level

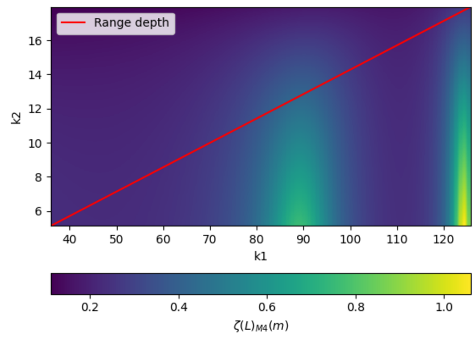
In this subsection the height of the bed level is changed in the range from 4m to 14m. The friction and width of the river are taken as a constant. As a result both dimensionless numbers are changed. Therefore the plot for the dimensionless numbers is the same as in 4.8a and 4.8b. The plot can be seen in figures 4.13a and 4.13b. The range for the dimensionless numbers is presented with a red line. Here the line is not vertical since the parameter H is present in both dimensionless numbers.

Now the amplification in the plot for M2 and M4 can be explained. In section 4.2 it was discussed that the amplitude will experience a amplification when the length of the river is at $\frac{1}{4}\lambda$, $\frac{3}{4}\lambda$, $\frac{5}{4}\lambda$, $\frac{7}{4}\lambda$, etc. The values for the dimensionless numbers in the plot where an amplification can be seen correspond to the values for $\frac{1}{4}\lambda$ in the M2 tide plot. The plot for the M2 tide shows one amplification in the amplitude at $x = L$, where the M4 tide shows two amplifications. The difference in frequency explains this difference, since the wavelength depends on the frequency. Besides that the places of amplification are not at the same distance in the M4 tide plot. This can be explained with the height dependence of the wavelength, $\lambda = \sqrt{gH}/\omega$. For example if the amplification for the amplitude at $\frac{1}{4}\lambda$. Then there will be another amplification at $\frac{1}{4}\lambda + \frac{1}{2}\lambda$. Here, λ depends on $\sqrt{H}$. Which is non-linear. Therefore the distance between the amplifications is not the same for each successive amplification area.

In terms of dimensionless numbers. When the height is changed both dimensionless numbers change, as mentioned before. The first dimensionless number decrease as the depth of the bed level is increased. Therefore the deeper the river, the less periods of the wave will be found in the river basin. This was seen by the difference in distance between the amplifications. The second dimensionless number is decreased as the depth of the river is increased. Therefore the friction-loss will be less and less important compared to the potential energy as depth is increased. This explains the observation that was made before, namely that the amplitude at $x = L$ did not change as much in amplitude for both the M2 and the M4 tide as it changed when the width and friction term were varied, see figure 4.7a.



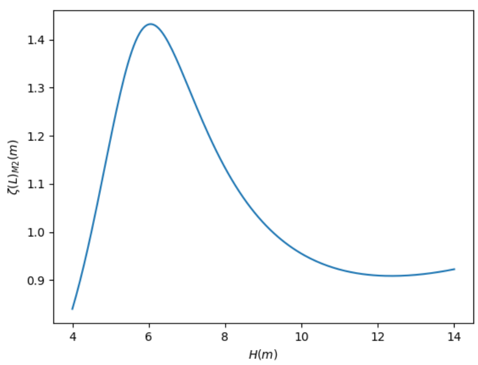
(a) The amplitude of the M2 tide at the end of the river, at $x = L$, is plotted in a 2D grid. On the axis the dimensionless numbers k_1 and k_2 can be found. Also a red line is plotted, this line represents the change in the first and second dimensionless number, whenever the heigth of the river is varied between $H = 4m$ and $H = 14m$.



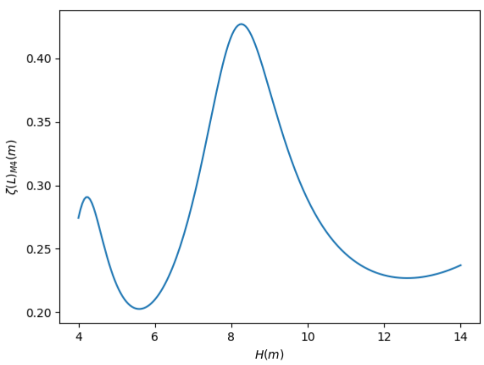
(b) The amplitude of the M4 tide at the end of the river, at $x = L$, is plotted in a 2D grid. On the axis the dimensionless numbers k_1 and k_2 can be found. Also a red line is plotted, this line represents the change in the first and second dimensionless number, whenever the heigth of the river is varied between $H = 4m$ and $H = 14m$.

Figure 4.13

Next, to make a more clear representation, the amplitude of the tides at $x = L$ is plotted, see figure 4.14a and 4.14b. It can be seen that when changing the bed level of the Ems the M2 and M4 tide change differently. For approximately $H = 6m$ the amplitude of the M2 tide at $x = L$ is at its maximum. However, the M4 tides has its peaks for $H = 4m$ and $H = 8m$ at $x = L$. Therefore, it cannot be said that the propagation of the M2 and M4 tide change the same when the height of the bed level is changed. For example if the initial bed level of a river is $H = 6m$ and it is deepened to $H = 8m$, the amplitude of the M2 is damped and, in contrary, the amplitude of the M2 tide undergoes an amplification.



(a) The amplitude of the M2 tide is plotted over the height of the river.



(b) The amplitude of the M4 tide is plotted over the height of the river.

Figure 4.14

5

Results for length dependent parameters

In this chapter the width, height and friction are considered to be not constant over the length of the river, which is physically more realistic. In particular, data from the Ems river is used to test the model that is derived. The Ems river is approximately $L = 64000m$ in length. This is the length over which the amplitude of the water is influenced by the tides at sea. This reaches from the location Knock until a dam near Herbrum. A schematic representation of the Estuary of the Ems river can be seen in figure 5.1. Knock is not located at sea but the river undergoes a significant widening and deepening, therefore it is acceptable to take this location as a boundary for $x = 0$. The variables that are known for the Ems river are presented in table 5.1.

In the last 50 years the Ems river has changed significantly [13]. The estuary underwent multiple human interventions including substantive channel deepening to support large ships [3]. These changes have caused a change in the M2 and M4 tide. In this chapter the change in M2 and M4 tide is analysed for the Ems river. Secondly, it is examined whether the model shows the same results as it did in the previous chapter.

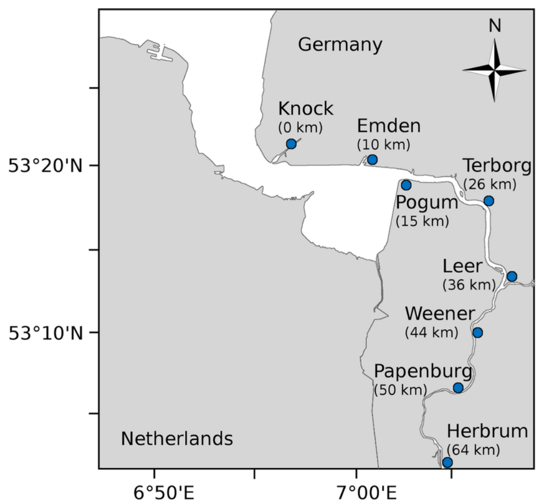


Figure 5.1: Schemetic representation of the Ems river for the first 64000m, from Knock to Herbrum.

Channel	Tide
$L = 6.4 \cdot 10^4$ m	$\omega_{M2} = 1.4 \cdot 10^{-4}$ rad. s^{-1}
	$\omega_{M4} = 2.8 \cdot 10^{-4}$ rad s^{-1}
	$\theta_{M2} = \theta_{M4} = 0$ rad
	$A_{M2} = 1.4$ m
Constant	$A_{M4} = 2.1 \cdot 10^{-1}$ m
$g = 9.81$ ms^{-2}	
$\rho = 1.0 \cdot 10^3$ kg m^{-3}	

Table 5.1: Known variables for the second case where the height, width and friction coefficient are x-dependent.

5.1. Fitting parameters to data

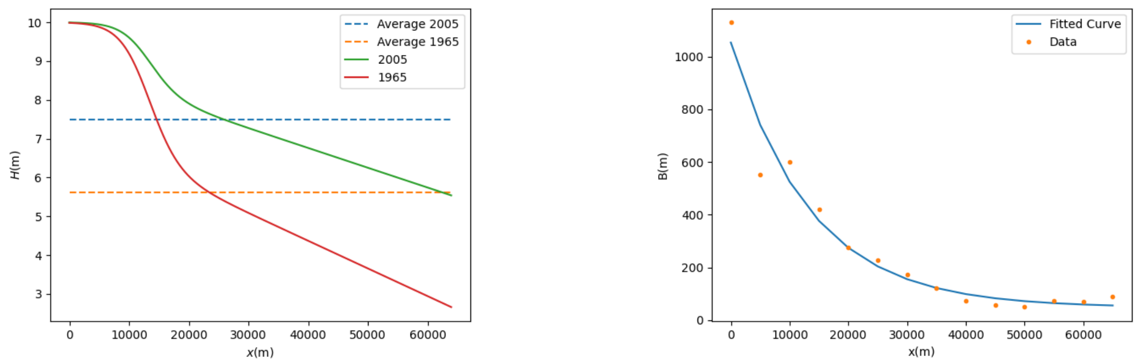
Not all parameters needed to construct the model are known, as can be seen in table 5.1. In this section the unknown variables are determined by known data for the Ems river.

5.1.1. Width

The width of the river is determined with an online Map Distance calculator [1]. Each five kilometer the width of the river is determined. A exponential curve-fit is used to obtain a function for the width, which is shown in figure 5.2b. The function that is obtained is:

$$B(x) = ae^{-bx} + c \tag{5.1}$$

Here the constants are defined as; $a = 7.7 \cdot 10^2$, $b = 5.6 \cdot 10^{-5}$ and $c = 4.0 \cdot 10^1$. It is assumed that the Ems river has not changed in width over the years. In order to compare the results from previous chapter a x-independent value for the width is needed. This value is chosen as the average of width, which is $\overline{B} = 142m$.



(a) The height of the bed level of the Ems river for the year 1965 and the year 2005. Also the average heights of the bed level of the two years is indicated with the dashed lines.

(b) The fitted curve for the development of the width B of the Ems river plotted over x. The data points that are observed is plotted as well.

Figure 5.2

5.1.2. Bed level

For the depth of the bed level of the Ems river previous research is used. [3] A function is fitted through a few measurements from 1965 and from 2005. The resulting function is;

$$H = 0.5\alpha\left(1 + \tanh\left(\frac{x - x_c}{x_l}\right)\right) + 0.5\beta x\left(1 + \tanh\left(\frac{x - x_c}{x_l}\right)\right) + \gamma \tag{5.2}$$

With the measurements from the two years different values for the constants are obtained from the fit. For 1965, $\alpha = -2.78, \beta = -7.13e-5, \gamma = 10, x_l = 5000, x_c = 13000$. For 2005, $\alpha = -1.2, \beta = -5.1e-5, \gamma = 10, x_l = 5000, x_c = 13000$. The two functions are plotted together in order to see what is changed over time, see figure 5.2a. As can be seen, the bed level of the river has increased a lot since 1965. This could be caused by gain of silt. Also the average height over the length of the river is shown for 1965 and 2005. This shows that on average the height of the bed level of the Ems river has been decreased from $\overline{H}_{1965} \approx 5.61$ m to $\overline{H}_{2005} \approx 7.49$ m in 40 years, here the average is indicated with the bar. The average value will be used to compare the results from previous chapter with the observations for the Ems river.

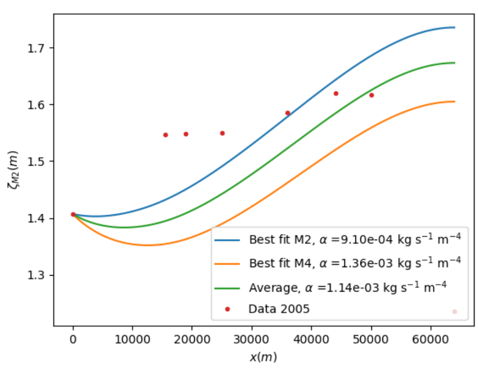
5.1.3. Friction

The description of the friction is missing in earlier research since the value highly depends on the model configuration that is chosen. However, it is known that the muddiness of the Ems increases over the length of the river. The increasing muddiness causes less turbulence in the water, which causes a reduced friction term. Now data received from observations in the Ems river in 1965 and 2005 are used to determine this friction term [3]. For the sake of simplicity a constant α will be determined, which again is used to compare the results in the previous chapter with the observations in the Ems river.

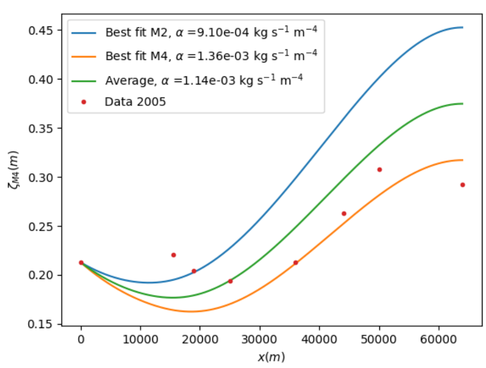
5.2. Determine friction in the Ems river

The optimal friction can be found by minimizing the difference between the numerically obtained solution for the M2 or M4 tide and the data obtained in earlier research. [3] This is done for the years 1965 and 2005 for both the M2 and M4 tide. First the optimal value for the friction for 2005 is considered. The optimal value for the friction is $\alpha_{M2} = 0.00091$ kg s^{-1} m^{-4} for the M2 tide and $\alpha_{M4} = 0.00136$ kg s^{-1} m^{-4} for the M4 tide. Since the M2 and M4 propagate in the same river domain, one value for both tides need to be defined. In figure 5.3a and 5.3b the solution to the model and the data points are given. As can be seen, the best fit for the M2 tide does not match the data for the M4 tide, and vise versa. Therefore the value will meet the two best fits half way by averaging the best fit for the M2 tide and the best fit for the M4 tide, $\alpha = 0.00114$kg s^{-1} m^{-4} . The propagation for the amplitude with this average value is also shown in the figures. The model is reasonably close to the data points. The model shows the same curve as the data points. The only data points that deviate a lot with the model is the last data point. This is the data point at the boundary of the river, at the dam in Herbrum.

The same procedure is done for the year 1965. The best value for the friction when the M2 tide is considered is $\alpha_{M2} = 0.00302$ kg s^{-1} m^{-4}. The best value for the friction when the M4 tide is considered is $\alpha_{M4} = 0.00195$ kg s^{-1} m^{-4}. The propagation of the tide for these friction values can be seen in figures 5.4a and 5.4b. First consider 5.4a, it can be seen that the best value for M2 tide has a much better range in amplitude, when the solution to the model is compared to the data. However, the shape is reserved when the value is taken as the best value for the M4 tide. The same holds for figure 5.4b. As before, in the case where the year 2005 was considered, the average value is taken as taken as the mean friction in the river. Therefore, for the year 1965, the river has a friction value of $\alpha_{M2} = 0.00249$ kg s^{-1} m^{-4}. It can be concluded that the average value of the friction has changed between 1965 and 2005. The value in 1965 was $\overline{\alpha}_{1965} = 0.00249$ kg s^{-1} m^{-4} and in 2005, $\overline{\alpha}_{2005} = 0.00114$ kg s^{-1} m^{-4}.

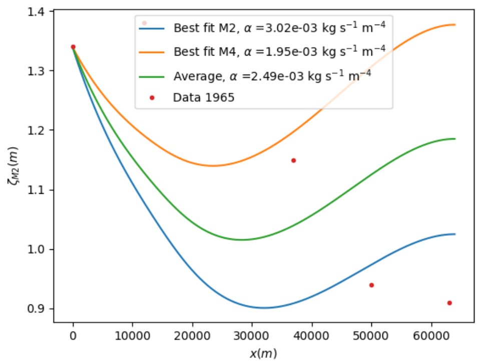


(a) The solution for the M2 amplitude from the model. The friction value is chosen as the best value for the M2 tide, the best value for the M4 tide and the average value. Also the data that is measured in 2005 is indicated with dots.

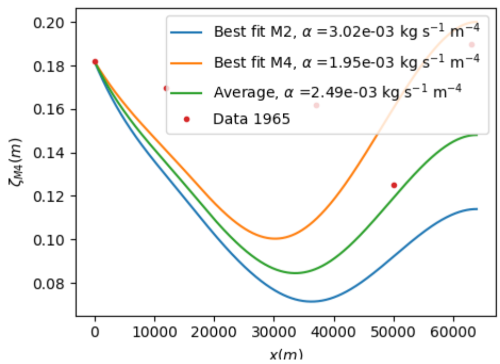


(b) The solution for the M4 amplitude from the model. The friction value is chosen as the best value for the M2 tide, the best value for the M4 tide and the average value. Also the data that is measured in 2005 is indicated with dots.

Figure 5.3



(a) The solution for the M2 amplitude from the model. The friction is chosen as the best value for the M2 tide, the best value for the M4 tide and the average value. Also the data that is measured in 1965 is indicated with dots.



(b) The solution for the M4 amplitude from the model. The friction is chosen as the best value for the M2 tide, the best value for the M4 tide and the average value. Also the data that is measured in 1965 is indicated with dots.

Figure 5.4

5.3. Propagation M2 and M4 tide in the Ems river subjected to deepening, widening and friction change

In this section the propagation of the M2 and M4 tide are shown. First some observations are made. After that these observations are explained with the results from previous chapter.

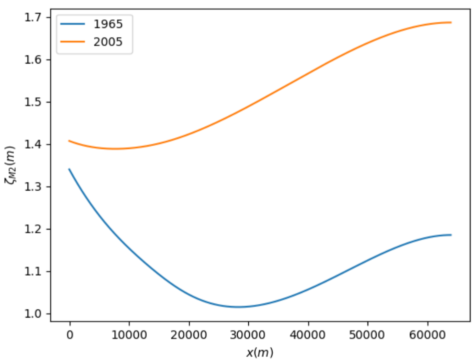
The model for x-dependent parameters is used to obtain the solution for the M2 and M4 tide for both the year 1965 and 2005. The variables that are used can be found in table 5.2. The width $B(x)$ is determined in section 5.1.1, the height $H(x)$ is determined in section 5.1.2 and the friction is determined in section 5.2. For these variables, only the friction is taken to be x-independent.

1965	2005
$L = 6.4 \cdot 10^4$	$L = 6.4 \cdot 10^4$
$B = B(x)$	$B = B(x)$
$H = H_{1965}(x)$	$H = H_{2005}(x)$
$\overline{\alpha}_{1965} = 0.00249$ kg s^{-1} m^{-4}	$\overline{\alpha}_{2005} = 0.00114$ kg s^{-1} m^{-4}

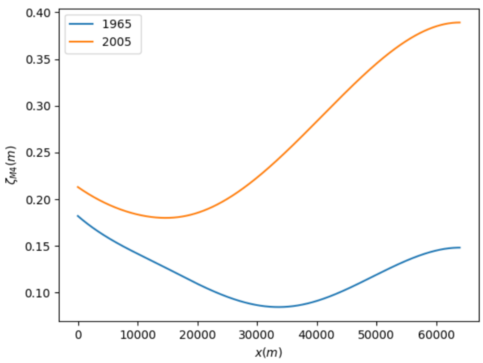
Table 5.2: Change in parameters between 1965 and 2005 for the x-dependent model.

The amplitude of the M2 and M4 tide are shown in figure 5.5a and 5.5b. The increasing amplitude for both the M2 as for the M4 tide, between 1965 and 2005, can be explained with the decreasing friction value. It was seen in section 4.4.2, that a decreasing friction value causes an increasing amplitude at $x = L$, which is the case in figure 5.5. The difference in change of the M2 and M4 tide is the amplification of the amplitude over the whole length of the river. In chapter 4 in particular the change at $x = L$ is investigated. That is why $x = L$ is considered now as well. There is a significant difference in amplification of the amplitude between the M2 amplitude compared to the M4 amplitude. To illustrate this the relative change in amplitude at $x = L$ for both tides is calculated in equation 5.5. Here RC is the relative change. It can be seen that relatively the amplitude of the M4 tide changes much more than the amplitude of the M2 tide.

$$
\begin{aligned}
RC_{M2} &= \frac{1.69 - 1.18}{1.18} \cdot 100\% \approx 42\% \\
RC_{M4} &= \frac{0.39 - 0.15}{0.15} \cdot 100\% \approx 162\%
\end{aligned} \tag{5.3}
$$



(a) The solution of the amplitude of the M2 tide, with the determined parameters for 1965 and 2005.



(b) The solution of the amplitude of the M4 tide, with the determined parameters for 1965 and 2005.

Figure 5.5

In order to compare these observations to the results from previous chapter, x-independent values for the height and width need to be chosen. For both variables the average value is used. Now the width did not change between 1965 and 2005, so $\overline{B} = 142$. The height did change over time. The average values for the height in 1965 is $\overline{H}_{1965} = 5.61$m and $\overline{H}_{2005} = 7.49$ m in 2005. The values for the friction are taken as obtained in 5.2; $\overline{\alpha}_{1965} = 0.00249$ kg s^{-1} m^{-4} and $\overline{\alpha}_{2005} = 0.00114$ kg s^{-1} m^{-4}. The overview of values for which the x-independent model is constructed can be seen in table 5.3. With this model the observations are explained.

1965	2005
$L = 6.4 \cdot 10^4$	$L = 6.4 \cdot 10^4$
$\overline{B} = 142$ m	$\overline{B} = 142$ m
$\overline{H}_{1965} = 5.61$ m	$\overline{H}_{2005} = 7.49$ m
$\overline{\alpha}_{1965} = 0.00249$ kg s^{-1} m^{-4}	$\overline{\alpha}_{2005} = 0.00114$ kg s^{-1} m^{-4}

Table 5.3: Change in parameters between 1965 and 2005.

In section 4.4.2 it was shown that the amplitude of the M2 and M4 tide does not change differently when the friction value is changed. Therefore the difference in amplification at $x = L$ can not be explained by the friction change between 1965 and 2005. In section 4.4.3, figure 4.14 shows that the change in height of the bed level can cause a different propagation of the M2 and M4 tide. The amplitude at $x = L$ is plotted over the height of the river for the values given in table 5.3, see figure 5.6. The figures show two solutions each. The solution for 1965 is computed by the model by taking the friction for 1965, and the 2005 solution is computed by taking the friction for 2005. Therefore the two lines represent the friction difference. The dashed lines in the figure represent the change of height between 1965 and 2005. It can be seen that both the amplitude of the M2 tide as the M4 tide should change at $x = L$ when comparing the height change between 1965 and 2005, see the blue line in figure 5.6. When computing the relative change, it becomes clear that the M2 and M4 tide change relatively the same subjected to deepening for the friction in 1965.

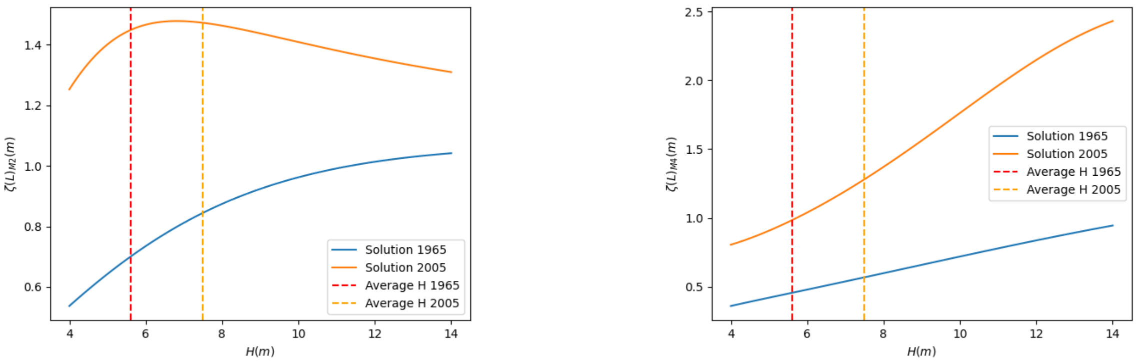
$$\begin{aligned} RC_{M2} &= \frac{0.83 - 0.70}{0.70} \cdot 100\% \approx 19\% \\ RC_{M4} &= \frac{0.56 - 0.46}{0.46} \cdot 100\% \approx 22\% \end{aligned} \tag{5.4}$$

On the contrary, the amplitude of the M4 tide changes differently compared to the M2 tide when the friction in 2005 is considered, see the orange line in the figure. The relative change for the M2 and M4 tide are calculated again.

$$\begin{aligned} RC_{M2} &= \frac{1.47 - 1.45}{1.45} \cdot 100\% \approx 1.4\% \\ RC_{M4} &= \frac{1.26 - 0.99}{0.99} \cdot 100\% \approx 27\% \end{aligned} \tag{5.5}$$

The relative change for the M2 tide is approximately 1.4 procent. It is clear relatively the M2 tide should not change subjected too deepening under friction in 2005. The relative change for the M4 tide is approximately 27 procent. Therefore the difference in amplitude change of the M2 and M4 tide in the Ems can be explained with the lower friction in 2005. However, it is not known how the friction is evolved in the years between 1965 and 2005. It can also be seen that the expected value for the amplitude at the beginning of the river for the M2 tide is approximately 0.7m in 1965 and 1.4m in 2005. The real values can be obtained from figure 5.5a, which is 1.15m in 1965 and 1.65m in 2005. There is also a difference between the expected values for the M4 tide in figure 5.6b and the real values of the amplitude of the M4 tide in figure 5.5b. This difference can be explained by the chosen x-independent values for the height and width. This decision has let to a significant difference in the amplitude. However, the relative changes still show a significant difference between the M2 and M4 tide.

It can be concluded that the difference in M2 tide and M4 tide between 1965 and 2005 is not due to deepening on its own. Namely, the change in amplitude due to deepening highly depends on the value of the friction. The exact amplitude of the M2 and M4 tide cannot be predicted with the results from the model where x-independent parameters were used. However, it can be concluded that these results can predict how the M2 and M4 tide will change due to deepening relatively to each other.



(a) The solution of the amplitude of the M2 tide, with the determined parameters for 1965 and 2005.

(b) The solution of the amplitude of the M4 tide, with the determined parameters for 1965 and 2005.

Figure 5.6

6 Discussion

In this study a lot of assumptions have been made to simplify a rivers basin system. Also a few assumptions have been made in order to obtain equations that describe a river mathematically. These assumptions are elucidated, however they can still be arguable. In this chapter these assumptions are questioned. Also, the application to real river estuary and basins is discussed.

6.1. Assumptions and decisions

To establish the equations in chapter 2 the conservation of mass and momentum formulas are integrated over the width and depth of the river. Therefore, they hold for water in an one dimensional river-basin. However, the propagation of tidal waves in estuary is not one dimensional. The one dimensional equations cause the basin to be of rectangular shape, while a river is mostly not constant in depth when the cross section is considered over the width of the river. Further research could use a two dimensional model, by integrating over the depth only. This model can then be used to investigate to what degree the results of the one-dimensional equations remain valid for the two-dimensional equations. Besides the one dimensional character of the equations a few other assumptions about the boundary conditions are made for the river basin. For example that there is no loss of mass to the sides and bed level of the river. However, it has been seen in earlier research that sediment cannot always be kept suspended and will settle on the bottom [9].

In the derivation for conservation of momentum equation it is assumed that the wetted channel parameter, P, is equal to the width of the river. However, when a rectangular basin is assumed, the parameter is $P = B+2H \approx B$. This assumption can be questioned if the height is not negligible with respect to the width. Which can be the case at the end of the river, at $x \approx L$. For example the Ems river has a width of 50m at this point, while the height of the river is approximately 4m. In further research it can be investigated if this wetted parameter causes a significant change in the solutions of the propagation of the M2 and M4 tide in a estuary.

Another assumption on the equation for conservation of momentum is that the bottom shear stress can be linearized with Lorentz' linearization for quadratic friction. In section 2.2.2 the conditions to this linearization are explained. It was said that the basin should be small compared to the tidal wavelength. The wavelengths of the M4- and M2-tide are 200km and 400km respectively. In the last chapter 5 the data of the Ems river estuary, which is approximately 60km, is compared to the model. The model uses the Lorentz' linearization. It can be questioned if this estuary is small compared to the 200km wavelength of the M4-tide that propagates in the Ems river.

After perturbation of the equation for momentum, only the leading term is taken into account. However, in section 2.2, it is explained how other harmonics also enter the river. These harmonics can change the propagation of the tide inside the river. It has been seen that the M4 tide changes differently subjected to deepening. It could be true that the extra harmonics also change differently subjected to deepening, which is not studied here. Further study could be done with these extra tidal harmonics.

Because of all these assumptions, the established results for case one, where the width, height and friction term are taken to be constant, could be questionable for real cases. However, it was seen that from the established results for the Ems river with the chosen constant values for the width, height and friction we were able to predict the relative change of the M2 and M4 tide. Therefore the assumptions that were made do not have a significant influence on the results of this study.

The observations for the amplitude of the M2 and M4 tide for the Ems river are compared to the observations from chapter 4. Before this could be done, a constant width and friction term needed to be chosen. In this study the average value is chosen. However, the width changes very rapidly in the first few kilometers of the river. The decision for the width can be researched in more detail. It is not mentioned in earlier research what the value of the friction is, for the chosen basin in this study. Therefore an optimization is executed on this parameter. The best fit for the M2 and M4 tide amplitude differ quite a lot. Yet, the average value is chosen. It is known that in the Ems river the friction coefficient decreases over the length of the river, since muddiness increases. In further research it can be decided to find a changing friction. Another decision that was made in this chapter is for the averaged height for 1965 and 2005. To compare case one to the Ems river the height of the river is averaged over the length. However, in the first 20km of the river the height of the river changes with 5m. In the other 40km the height of the bed level changes only 2.5m. This rapid change in height in the beginning stage of the river can have an strong effect on the propagation of the M2 and M4 tide. Therefore, taking the average height could be wrong. Instead, for example, a weighted average could be executed in further studies.

6.2. Application

In chapter 4 it became clear how the amplification of the M2 and M4 tide change, subjected to deepening, widening and friction change. When these parameters are known for an estuary or basin of a river, the plots can be used to see how the M2 and M4 tide react to changes in deepening, widening and friction change. However, in chapter 5 it was experienced that for the Ems river the decision making for the value of these variables can be complicated, since the changing values need to be chosen as x-independent. This is a complicated decision since the values have a strong effect on the M2 and M4 propagation in the river.

To be short, the decision making for certain values need to be further researched, before comparing the one dimensional, x-independent model to historical observations in tidal rivers.

7 Conclusion

The main question in this study was: '*How does the propagation of the M2 and M4 tide change, subjected to deepening, change in width and change in friction?*' One dimensional equations for a rectangular river basin are derived. The solution is given for two cases. In the first case the width, depth and friction are x-independent. With the model for case one the main question can be answered. The results of tjs case can be applied to tidal rivers. To illustrate this, the results are compared to observations from the model of case two. In case two the width, height and friction is x-dependent. With these results the sub question in this study is answered. The sub question was: '*Can historical observations of the amplitude of the M2 and M4 tides in the Ems river over the years be explained with the change in propagation of the M2 and M4 tides, subjected to deepening and friction change?*'

First the propagation of the amplitude for the M2 and M4 tide is plotted for multiple values of the individual parameters. It turns out that the wider the river the more the amplitude of the M2 and M4 tide is damped. The same holds for the friction. The greater the friction the more the amplitude is damped. On the contrary, when the height of the bed level of the river was varied a different result was shown. The amplitude at $x = L$ was plotted for $\alpha = 0.0004$ kg s^{-1} m^{-4}. With these values the amplitude of the M2 tide is not damped much, since the amplitude at $x = L$ does not change much when the height of the bed level was varied. Contrarily, the M4 tide does change a lot in amplitude at $x = L$. Therefore, the M2 and M4 tide react differently subjected to deepening. This is further investigated for the beginning of the river, at $x = L$. Plots are given from which it became clear that an alternation in the depth does effect the propagation of the M2 and M4 tide differently for certain values of height of the bed level. This difference was explained with changing wavelength, due to change in height, and with the difference in frequencies between the M2 and M4 tide.

With the results from the main question the historical observations for 1965 and 2005 in the Ems river are analysed. It is known that the bed level has changed from $H_{1965} = 5.61$ m to $H_{2005} = 7.49$ m on average. The first observation that was made is that the amplitude of the M2 and M4 tide both increased between the years. This can be explained with the decreasing friction over the years. Another observation that was made is that, relatively, the amplitude of the M4 tide has changed significantly more than the amplitude of the M2 tide at $x = L$ between 1965 and 2005. This is explained with the change in height of the bed level. From the plot for the amplitude at the beginning of the river, it was seen that for the change in averaged height the amplitude of the M2 tide changes relatively by the same percentage as the M4 tide when the friction in 1965 was considered. On the contrary, the amplitude of the M4 tide at the beginning of the river changes a lot relative to the M2 tide for when the friction in 2005 was considered. Therefore it is concluded that the difference in M2 and M4 tide between 1965 and 2005 is not due to deepening on its own. Namely, the change in amplitude due to deepening highly depends on the value of the friction.

And with that, the sub question can be answered. The exact amplitude of the M2 and M4 tide cannot be predicted with the results from the model where x-independent parameters were used. However, it can be concluded that these results can predict how the M2 and M4 tide in a tidal river will change due to deepening relatively to each other.

The main point of discussion is the averaging of the width, height and friction for the Ems river. In order to compare the observation in the Ems river to case one, where the height is constant over the length of the river, a constant value for the height needs to be chosen. In this study it is decided to take the average, which could be changed in future studies. Also a two dimensional model could be constructed for a river basin, with which the results from the one-dimensional model in this study can be tested.

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A

Appendix

A.1. Derivation conservation of momentum

In this section the integral 2.2 is evaluated. In the derivation the length and width integrated variables are needed, which are defined as

$$\bar{u} = \frac{1}{B}\int_0^B u dy, \qquad \hat{\bar{u}} = \frac{1}{\zeta + H}\int_{-H}^{\zeta} \bar{u} dz \tag{A.1}$$

Here $\bar{.}$ denotes the width averaged variables and $\hat{\bar{.}}$ denotes the height averaged variables. The equations A.1 also hold for v and w. Note that the integral can be broken down into three parts, u_x, v_y and w_z, since integration is a linear operator. First the integral over u_x is evaluated. The first step is to interchange the partial derivative and the integral over y, here Leibniz integration rule is used.

$$\int_{-H}^{\zeta}\int_0^B u_x dy dz = \int_{-H}^{\zeta} \frac{\partial}{\partial x}\int_0^B u dy dz$$

Now equation A.1 can be used to evaluate the integral of u over y. The variables turn into width averaged variables.

$$\int_{-H}^{\zeta} \frac{\partial}{\partial x}\int_0^B u dy dz = \int_{-H}^{\zeta} \frac{\partial}{\partial x} B\bar{u} dz$$

Next, the partial derivative is again taken out of the integral over z with Leibniz integration rule. Now since ζ is dependent of x an extra term will enter the equation.

$$\int_{-H}^{\zeta} \frac{\partial}{\partial x} B\bar{u} dz = \frac{\partial}{\partial x}\int_{-H}^{\zeta} B\bar{u} dz - B\bar{u}\frac{\partial \zeta}{\partial x}$$

Again, equation A.1 is used to evaluate the integral.

$$\frac{\partial}{\partial x}\int_{-H}^{\zeta} B\bar{u} dz - B\bar{u}\frac{\partial \zeta}{\partial x} = \frac{\partial}{\partial x}\left[B(\zeta + H)\hat{\bar{u}}\right] - B\bar{u}\frac{\partial \zeta}{\partial x}$$

Next, the term in the derivative, $\left[B(\zeta + H)\hat{\bar{u}}\right]$ is called the width and depth averaged velocity in the x-direction. This variable is given the symbol U in the rest of this report.

$$\frac{\partial}{\partial x}\left[B(\zeta + H)\hat{\bar{u}}\right] - B\bar{u}\frac{\partial \zeta}{\partial x} = \frac{\partial U}{\partial x} - B\bar{u}\frac{\partial \zeta}{\partial x}$$

So

$$\int_{-H}^{\zeta}\int_0^B u_x dy dz = \frac{\partial U}{\partial x} - B\bar{u}\frac{\partial \zeta}{\partial x}$$

The second step is to evaluate the integral over v_y. It is assumed that the velocities of the water on the boundaries at $y = B$ and $y = 0$ of the river perpendicular to the boundaries are zero, so $\int_0^B v_y dy = v(x, B, t) - v(x, 0, z) = 0$. Therefore we have that the double integral over v_y is equal to zero.

$$\int_{-H}^{\zeta}\int_{0}^{B} v_y dydz = 0$$

The third and last step is to evaluate the integral over w_z. The first step is to take out the partial derivative over z, using Leibniz integration rule.

$$\int_{-H}^{\zeta}\int_{0}^{B} w_z dydz = \int_{-H}^{\zeta}\frac{\partial}{\partial z}\int_{0}^{B} w dydz$$

Equation A.1 is used to evaluate the integral.

$$\int_{-H}^{\zeta}\frac{\partial}{\partial z}\int_{0}^{B} w dydz = \int_{-H}^{\zeta} B\bar{w}_z dz$$

Now, since this is an integral over the derivative with respect to the same variable we have;

$$\int_{-H}^{\zeta} B\bar{w}_z dz = B\left(\bar{w}(x,y,\zeta) - \bar{w}(x,y,-H)\right)$$

It is assumed that the velocity perpendicular to the bottom of the river is zero. Therefore, $\bar{w}(x,y,-H) = 0$. Also there is a condition the water-surface elevation: $\frac{\partial\zeta}{\partial t} = \bar{w} - \bar{u}\frac{\partial\zeta}{\partial x} \Rightarrow \bar{w}(x,y,\zeta) = \frac{\partial\zeta}{\partial t} + \bar{u}\frac{\partial\zeta}{\partial x}$, which is called the kinematic boundary condition, which holds for free surfaces. Therefore;

$$B\left(\bar{w}(x,y,\zeta) - \bar{w}(x,y,-H)\right) = B\left(\frac{\partial\zeta}{\partial t} + \bar{u}\frac{\partial\zeta}{\partial x}\right)$$

So, the integral over w_z is given by;

$$\int_{-H}^{\zeta}\int_{0}^{B} w_z\, dydz = B\left(\frac{\partial\zeta}{\partial t} + \bar{u}\frac{\partial\zeta}{\partial x}\right)$$

Now, all individual evaluated integrals can be put together;

$$\int_{-H}^{\zeta}\int_{0}^{B} u_x dydz + \int_{-H}^{\zeta}\int_{0}^{B} v_y dydz + \int_{-H}^{\zeta}\int_{0}^{B} w_z\, dydz = \frac{\partial U}{\partial x} - B\bar{u}\frac{\partial\zeta}{\partial x} + 0 + B\left(\frac{\partial\zeta}{\partial t} + \bar{u}\frac{\partial\zeta}{\partial x}\right)$$

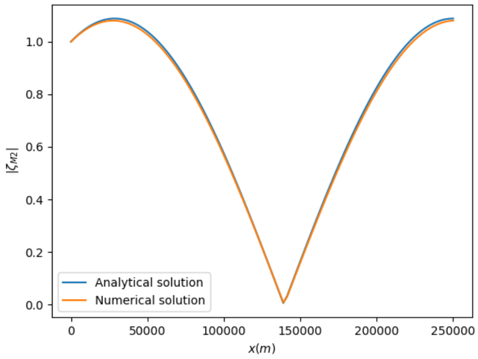
There is one term that cancels out, the integral over all the components is equal to zero. Therefore, the equation from the conservation of mass is;

$$\frac{\partial U}{\partial x} + B\frac{\partial\zeta}{\partial t} = 0 \tag{A.2}$$

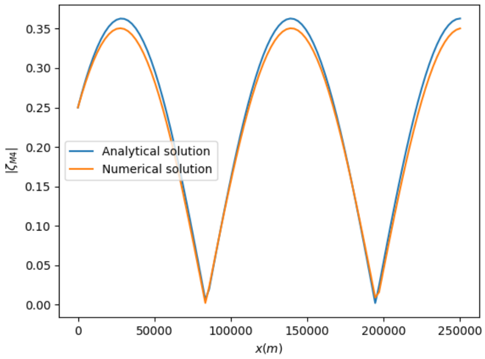
A.2. Results case 1: length independent parameters

A.2.1. Analytical and numerical solution

In the figures A.1a and A.1a below the numerical and analytical solution for the amplitude of the M2 and M4 tide are shown. Here the friction term was set to zero.

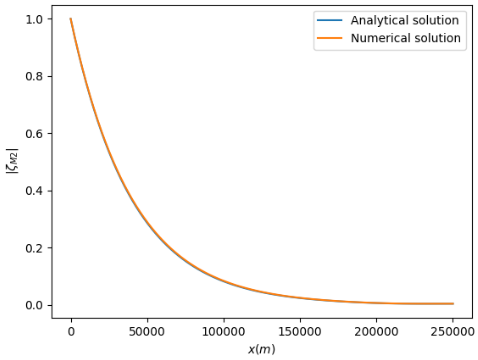


(a) The numerical and analytical solution for the amplitude of the M2 tide, here the friction was set to zero.

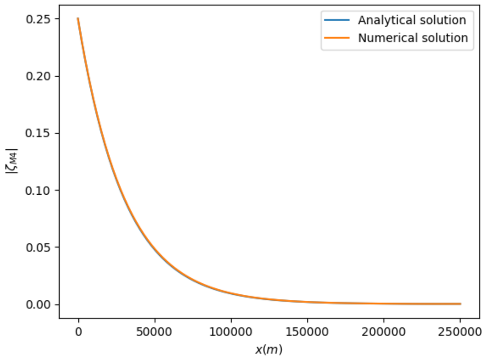


(b) The numerical and analytical solution for the amplitude of the M4 tide, here the friction was set to zero.

In the figures A.2a and A.2a below the numerical and analytical solution for the amplitude of the M2 and M4 tide are shown. Here the friction term was set to $\alpha = 0.1$, which is a very high friction coefficient.



(a) The numerical and analytical solution for the amplitude of the M2 tide, here the friction was set to $\alpha = 0.1$. This value for the friction is not plausible, but the model shows a result that is expected.



(b) The numerical and analytical solution for the amplitude of the M4 tide, here the friction was set to $\alpha = 0.1$. This value for the friction is not plausible, but the model shows a result that is expected.

A.3. Python code

In this part of the appendix the Python code is given.

The variables, numerical and analytical solutions are given here. In the subsections the code for these solutions are needed to be able to obtain the plots for the analysis, however they are not repeated in each subsection. Therefore, the code for this is:

The variables that are used and the analytical solution is:

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.linalg import solve_banded

g = 9.81
H = 10
alpha = 0.0004
B = 100
rho = 997

A_M2 = 1
A_M4 = 0.25
theta_M2 = 0
theta_M4 = 0
omega_M2 = 2*np.pi/(12.42*60*60)
omega_M4 = 2*np.pi/(6.21*60*60)

L = 250000 #length of the river
N = 100 #amount of grid points
x = np.arange(0,L,L/N)
dt = 100
t = 0

# Analytical solution


def coeffK(omega, alpha, B, rho, g, H):
    K = (-omega**2 + (1j*omega*alpha*B)/rho)/(g*H)
    return K

def c1(A,K,L):
    c1 = A_M2/(np.exp(2*np.sqrt(K)*L)+1)
    return c1

def c2(A,K,L):
    c1_M = c1(A,K,L)
    c2_M = A - c1_M
    return c2_M

def zeta(A, omega,theta, K, x, L):
    zeta_M = np.zeros(len(x))
    c1_M = c1(A,K,L)
    c2_M = c2(A,K,L)
    for i in range(0,len(x)):
        zeta_M[i] = abs(c1_M*np.exp(np.sqrt(K)*x[i]) + c2_M*np.exp(-np.sqrt(K)*x[i]))
    return zeta_M
```

The numerical solution for case one is:

```
#Numerical solution

def coeffK(omega, alpha, B, rho, g, H):
    K = (-omega**2 + (1j*omega*alpha*B)/rho)/(g*H)
```

```
    return K

def createF(A,N):
    Fvec = np.zeros(N, dtype = complex)
    Fvec[0] = A
    return Fvec



def createA_banded(L,N,omega,alpha,B,rho,g,H):
    h = L/N
    A_banded = np.ones((N,3),dtype=complex)
    for i in range(0,N):
        A_banded[i][0] = 1/h**2
        A_banded[i][1] = -(2/h**2 + coeffK(omega, alpha, B, rho , g, H))
        A_banded[i][2] = 1/h**2
    A_banded[0][0] = 0
    A_banded[0][1] = 1
    A_banded[1][0] = 0
    A_banded[-1][1] = -(1/h**2 + coeffK(omega, alpha, B, rho , g, H))
    A_banded[-1][2] = 0
    A_banded = A_banded.transpose()
    return A_banded
```

The numerical solution for case two is:

```
def createF(t):
    global A_M2, A_M4
    Fvec_M2 = np.zeros(N, dtype = complex)
    Fvec_M4 = np.zeros(N, dtype = complex)

    Fvec_M2[0] = 1.407
    Fvec_M4[0] = 0.213
    return Fvec_M2, Fvec_M4

def create_delta(omega, alpha, B, rho, g, H):
    delta = np.zeros(len(B),dtype=complex)
    for i in range(0,len(x)):
        delta[i] = -(g*H[i]*B[i]*rho)/(1j*omega*rho + alpha*B[i])
    return delta

def create_gamma(omega,alpha,B,rho,g,H):
    gamma = np.zeros(len(x),dtype=complex)
    for i in range(len(x)):
        gamma[i] = -(g*rho)/(1j*omega*rho + alpha*B[i])*(H[i]*B_x[i] + B[i]*H_x_05[i] -
            (B[i]*H[i]*(alpha_x*B[i] + alpha*B_x[i]))/(alpha*B[i] + 1j*omega*rho))
    return gamma

def createA_banded(L,N,omega,alpha,B,rho,g,H):
    h = L/N
    A_banded = np.zeros((N,3),dtype=complex)
    delta = create_delta(omega, alpha, B, rho, g, H)
    gamma = create_gamma(omega, alpha, B, rho, g, H)
    for i in range(N-1):
        A_banded[i+1][0] = delta[i]/(h**2)+ gamma[i]/h    # In deze en de volgende regels even
            opletten met i+1, i, i-1.
        A_banded[i][1] = B[i]*omega*1j -2*delta[i]/(h**2) -gamma[i]/h
        A_banded[i-1][2] = delta[i]/(h**2)
    A_banded[0][1] = 1
    A_banded[1][0] = 0
    #######Boundary Backwards Euler
```

```
    A_banded[-1][1] = 1
    A_banded[-2][2] = -1
    ########Boundary Forward Euler
    # A_banded[-1][1] = B[i]*omega*1j - delta[i]/(h**2)
    A_banded = A_banded.transpose()
    return A_banded
```

A.3.1. Variation in length and resonance

```

#Resonance M2
res = []
for i in [1/4, 3/4, 5/4, 7/4, 9/4]:
    lamb = i*(2*np.pi*np.sqrt(g*H))/(omega_M2)
    res.append(lamb)

L_min = 200
L_max = 1000000
N_L = 100
L_plot = np.arange(L_min,L_max,N_L)
zeta_L = []


for k in L_plot:
    K = coeffK(omega_M2, alpha, B, rho, g, H)
    xk = np.arange(0,k,k/N)
    zeta_M2 = zeta(A_M2,omega_M2,theta_M2_1,K,xk, k)
    zeta_L.append(zeta_M2[-1])


plt.figure()
plt.xlabel('L(m)')
plt.ylabel('$\zeta_{M2}(L)$')
for i in res:
    plt.axvline(x = i , color='b', ls = '--')
plt.plot(L_plot, zeta_L)
plt.show()

#Resonance M2

res = []
for i in [1/4, 3/4, 5/4, 7/4, 9/4, 11/4, 13/4, 15/4, 17/4]:
    lamb = i*(2*np.pi*np.sqrt(g*H))/(omega_M4)
    res.append(lamb)

L_min = 200
L_max = 1000000
N_L = 100
L_plot = np.arange(L_min,L_max,N_L)
zeta_L = []


for k in L_plot:
    K = coeffK(omega_M4, alpha, B, rho, g, H)
    xk = np.arange(0,k,k/N)
    zeta_M4 = zeta(A_M4,omega_M4,theta_M4_1,K,xk, k)
    zeta_L.append(zeta_M4[-1])


plt.figure()
```

```
plt.xlabel('L(m)')
plt.ylabel('$\zeta_{M4}(L)$')
for i in res:
    plt.axvline(x = i , color='b', ls = '--')
plt.plot(L_plot, zeta_L)
plt.show()
```

A.3.2. Analytical and numerical solution

```
A_banded_M2 = createA_banded(L,N,omega_M2,alpha,B,rho,g,H)
A_banded_M4 = createA_banded(L,N,omega_M4,alpha,B,rho,g,H)

x = np.linspace(0,L,N)
fvec_M2 = createF(A_M2,N)
fvec_M4 = createF(A_M4,N)

zeta_M2_num = abs(solve_banded((1,1),A_banded_M2,fvec_M2))
zeta_M4_num = abs(solve_banded((1,1),A_banded_M4,fvec_M4))

K_M2 = coeffK(omega_M2, alpha, B, rho, g, H)
K_M4 = coeffK(omega_M4, alpha, B, rho, g, H)

zeta_M2_an = abs(zeta(A_M2, omega_M2, 0,theta_M2,K_M2,x))
zeta_M4_an = abs(zeta(A_M4, omega_M4, 0,theta_M4,K_M4,x))

plt.figure()
plt.plot(x,zeta_M2_an, label = 'Analytical solution')
plt.plot(x,zeta_M2_num, label = 'Numerical solution')
plt.xlabel('$x(m)$')
plt.ylabel('$|\zeta_{M2}|$')
plt.legend()
plt.show()

plt.figure()
plt.plot(x,zeta_M4_an, label = 'Analytical solution')
plt.plot(x,zeta_M4_num, label = 'Numerical solution')
plt.xlabel('$x(m)$')
plt.ylabel('$|\zeta_{M4}|$')
plt.legend()
plt.show()


#Error


#Difference numerical and analytical solution
error_M2 = abs((zeta_M2_num) - (zeta_M2_an))
error_M4 = abs((zeta_M4_num) - (zeta_M4_an))
plt.figure()
plt.plot(x,error_M2, label = 'M2')
plt.plot(x,error_M4, label = 'M4')
plt.legend()
plt.title('Absolute error M2 and M4')
plt.show()



summation = 0 #variable to store the summation of differences
for i in range (0,len(x)): #looping through each element of the list
```

```
  difference = abs((zeta_M2_num[i]) - (zeta_M2_an[i])) #finding the difference between observed
      and predicted value
  summation += difference**2
MSE = summation/(len(x))
print ("The Mean Square Error for M2 is: " , MSE)


summation = 0 #variable to store the summation of differences
for i in range (0,len(x)):
    difference = abs((zeta_M4_num[i]) - (zeta_M4_an[i])) #finding the difference between
        observed and predicted value
    summation += difference**2
MSE = summation/(len(x))
print ("The Mean Square Error for M4 is: " , MSE)
```

A.3.3. Error analytical and numerical solution

```

N = np.arange(2, 1000, 1)
lst_error_M2 = np.zeros(len(N))
lst_error_M4 = np.zeros(len(N))

for i in range(len(N)):
    A_banded_M2 = createA_banded(L,N[i],omega_M2,alpha,B,rho,g,H)
    A_banded_M4 = createA_banded(L,N[i],omega_M4,alpha,B,rho,g,H)

    x = np.linspace(0,L,N[i])
    fvec_M2 = createF(A_M2,N[i])
    fvec_M4 = createF(A_M4,N[i])

    zeta_M2_num = abs(solve_banded((1,1),A_banded_M2,fvec_M2))
    zeta_M4_num = abs(solve_banded((1,1),A_banded_M4,fvec_M4))

    #Analytische oplossing
    K_M2 = coeffK(omega_M2, alpha, B, rho, g, H)
    K_M4 = coeffK(omega_M4, alpha, B, rho, g, H)

    zeta_M2_an = abs(zeta(A_M2, omega_M2, 0,theta_M2,K_M2,x))
    zeta_M4_an = abs(zeta(A_M4, omega_M4, 0,theta_M4,K_M4,x))


    #error
    summation_M2 = 0 #variable to store the summation of differences
    for j in range (0,len(x)): #looping through each element of the list
      difference_M2 = abs((zeta_M2_num[j]) - (zeta_M2_an[j])) #finding the difference between
          observed and predicted value
      summation_M2 += difference_M2**2
    MSE_M2 = summation_M2/(len(x))
    lst_error_M2[i] = MSE_M2


    summation_M4 = 0 #variable to store the summation of differences
    for k in range (0,len(x)):
        difference_M4 = abs((zeta_M4_num[k]) - (zeta_M4_an[k])) #finding the difference between
            observed and predicted value
        summation_M4 += difference_M4**2
    MSE_M4 = summation_M4/(len(x))
    lst_error_M4[i] = MSE_M4


```

```
plt.figure()
plt.plot(N,lst_error_M2, label = 'Error M2 tide')
plt.plot(N,lst_error_M4, label = 'Error M4 tide')
plt.legend()
plt.xscale('log')
plt.yscale('log')
plt.xlabel('$log$(N)')
plt.ylabel('$log$(Error)')
plt.show()
```

A.3.4. Results for length independent parameters

The code for changing friction coefficient:

```
#solutions M2 numerical
alpha = [0.0002, 0.0004, 0.0006, 0.0008, 0.0010, 0.0012, 0.0014, 0.0016, 0.0018, 0.0020 ]

# solution M4 analytical

plt.figure()
plt.xlabel('$(m)$')
plt.ylabel('$|\zeta_{M4}|$')
zeta_H_M2_analytical = []
for i in alpha:
    K = coeffK(omega_M2, i, B, rho, g, H)
    zeta_H_M2_analytical.append(zeta_t(A_M2, A_M4, omega_M2,0,theta_M2_1, theta_M4_1,K,x)[0])


for i in range(len(zeta_H_M2_analytical)):
    plt.plot(x,zeta_H_M2_analytical[i],label='alpha =' +str(alpha[i]))
    plt.legend()
plt.show()

# solution M4 analytical

plt.figure()
plt.xlabel('$(m)$')
plt.ylabel('$|\zeta_{M4}|$')
zeta_H_M4_analytical = []
for i in alpha:
    K = coeffK(omega_M4, i, B, rho, g, H)
    zeta_H_M4_analytical.append(zeta_t(A_M2, A_M4, omega_M4,0,theta_M2_1, theta_M4_1,K,x)[1])


for i in range(len(zeta_H_M4_analytical)):
    plt.plot(x,zeta_H_M4_analytical[i],label='alpha =' +str(alpha[i]))
    plt.legend()
plt.show()
```

The code for changing width:

```

B = np.arange(50,850,50)
x = np.linspace(0,L,N)

# solution M4 analytical

plt.figure()
plt.xlabel('$(m)$')
plt.ylabel('$|\zeta_{M2}|$')

```

```
zeta_H_M2_analytical = []
for i in B:
    K = coeffK(omega_M2, alpha, i, rho, g, H)
    zeta_H_M2_analytical.append(zeta_t(A_M2, A_M4, omega_M2,0,theta_M2_1, theta_M4_1,K,x)[0])


for i in range(len(zeta_H_M2_analytical)):
    plt.plot(x,zeta_H_M2_analytical[i],label='B =' +str(B[i]))
    plt.legend()
plt.show()

# solution M4 analytical

plt.figure()
plt.xlabel('$(m)$')
plt.ylabel('$|\zeta_{M4}|$')

zeta_H_M4_analytical = []
for i in B:
    K = coeffK(omega_M4, alpha, i, rho, g, H)
    zeta_H_M4_analytical.append(zeta_t(A_M2, A_M4, omega_M4,0,theta_M2_1, theta_M4_1,K,x)[1])


for i in range(len(zeta_H_M4_analytical)):
    plt.plot(x,zeta_H_M4_analytical[i],label='B =' +str(B[i]))
    plt.legend()
plt.show()
```

Code for changing height:

```

H = [9, 10 , 11, 12, 13, 14, 15, 16, 17]
x = np.linspace(0,L,N)

# solution M2 analytical

plt.figure()
plt.xlabel('$(m)$')
plt.ylabel('$|\zeta_{M2}|$')
zeta_H_M2_analytical = []
for i in H:
    K = coeffK(omega_M2, alpha, B, rho, g, i)
    zeta_H_M2_analytical.append(zeta_t(A_M2, A_M4, omega_M2, 0,theta_M2_1, theta_M4_1,K,x)[0])


for i in range(len(zeta_H_M2_analytical)):
    plt.plot(x,zeta_H_M2_analytical[i],label='H =' +str(H[i]))
    plt.legend()
plt.show()

# solution M2 analytical

plt.figure()
plt.xlabel('$(m)$')
plt.ylabel('$|\zeta_{M4}|$')
zeta_H_M4_analytical = []
for i in H:
    K = coeffK(omega_M4, alpha, B, rho, g, i)
    zeta_H_M4_analytical.append(zeta_t(A_M2, A_M4, omega_M4,0,theta_M2_1, theta_M4_1,K,x)[1])
```

```
for i in range(len(zeta_H_M4_analytical)):
    plt.plot(x,zeta_H_M4_analytical[i],label='H =' +str(H[i]))
    plt.legend()
plt.show()
```

A.3.5. Results for length independent parameters

```

M2amp = np.asarray([140.7, 154.6, 154.8, 155.0, 158.6, 161.9, 161.7, 123.5])/100
M4amp = np.asarray([21.3, 22.1, 20.4, 19.4, 21.3, 26.3, 30.8, 29.2])/100
x_waterlevel = np.asarray([0, 15.5, 19, 25, 36, 44, 50, 64])*1000


#Determine B(x) and B_x(x) in the river Eems

def func_B(x, a, b, c):
    return a*np.exp(-b*x) + c

x_data = np.arange(0,155000,5000)
B_data = [800, 552, 601, 421, 275, 229, 173, 123, 75, 59, 50, 73, 71, 90, 58, 52, 53, 56, 40,
    47, 44, 47, 41, 51, 56, 47, 51, 43, 33, 46, 38]

guess = np.array([11000,0.00007,25])
popt, pcov = curve_fit(func_B, x_data, B_data, guess)

a,b,c = popt
B = func_B(x,a,b,c)
B_x = -a*b*np.exp(-b*x)

alpha_x = np.zeros(len(x))

#Determine H(x)
# Data 1965
alpha_65 = -2.78
beta_65 = -7.13e-5
gamma_65 = 10
xl_65 = 5000
xc_65 = 13000

#Data 2005
alpha_05 = -1.2
beta = -5.1e-5
gamma = 10
xl = 5000
xc = 13000

H_05 = np.ones(len(x))
for i in range(len(x)):
    if x[i]<100000:
        H_05[i] =
            0.5*alpha_05*(1+np.tanh((x[i]-xc)/xl))+0.5*beta*x[i]*(1+np.tanh((x[i]-xc)/xl))+gamma
    else:
        H_05[i] =
            0.5*alpha_05*(1+np.tanh((100000-xc)/xl))+0.5*beta*100000*(1+np.tanh((100000-xc)/xl))+gamma
H_x_05 = np.gradient(H_05, x)


H_65 = np.ones(len(x))
for i in range(len(x)):
    if x[i]<100000:
```

```
        H_65[i] =
            0.5*alpha_65*(1+np.tanh((x[i]-xc_65)/xl_65))+0.5*beta_65*x[i]*(1+np.tanh((x[i]-xc_65)/xl_65))+gamma_
    else:
        H_65[i] =
            0.5*alpha_65*(1+np.tanh((100000-xc_65)/xl_65))+0.5*beta_65*100000*(1+np.tanh((100000-xc_65)/xl_65))+
H_x_65 = np.gradient(H_65, x)
alpha_x = 0


plt.figure()
H_mean_05 = st.mean(H_05)
H_mean_65 = st.mean(H_65)
plt.plot(x,H_mean_05*np.ones(len(x)), '--', label = 'Average 2005')
plt.plot(x,H_mean_65*np.ones(len(x)), '--', label = 'Average 1965')

plt.plot(x, H_05, label = '2005')
plt.plot(x, H_65, label = '1965')
plt.legend()
plt.xlabel('$x$(m)')
plt.ylabel('$H$(m)')
plt.show()



alpha_05_M2 = 0.00091
alpha_05_M4 = 0.00136
alpha_05_gem = (alpha_05_M2 + alpha_05_M4)/2

#solutions M2 and M4 for best M2 alpha
A_banded_M2 = createA_banded(L,N,omega_M2,alpha_05_M2,B,rho,g,H_05)
fvec = createF(t)[0]
zeta_M2_05_bestM2 = abs(solve_banded((1,1),A_banded_M2,fvec))

A_banded_M4 = createA_banded(L,N,omega_M4,alpha_05_M2,B,rho,g,H_05)
fvec = createF(t)[1]
zeta_M4_05_bestM2 = abs(solve_banded((1,1),A_banded_M4,fvec))

#solutions M2 and M4 for best M4 alpha
A_banded_M2 = createA_banded(L,N,omega_M2,alpha_05_M4,B,rho,g,H_05)
fvec = createF(t)[0]
zeta_M2_05_bestM4 = abs(solve_banded((1,1),A_banded_M2,fvec))

A_banded_M4 = createA_banded(L,N,omega_M4,alpha_05_M4,B,rho,g,H_05)
fvec = createF(t)[1]
zeta_M4_05_bestM4 = abs(solve_banded((1,1),A_banded_M4,fvec))

#solutions M2 and M4 for mean alpha
A_banded_M2 = createA_banded(L,N,omega_M2,alpha_05_gem,B,rho,g,H_05)
fvec = createF(t)[0]
zeta_M2_05_gem = abs(solve_banded((1,1),A_banded_M2,fvec))

A_banded_M4 = createA_banded(L,N,omega_M4,alpha_05_gem,B,rho,g,H_05)
fvec = createF(t)[1]
zeta_M4_05_gem = abs(solve_banded((1,1),A_banded_M4,fvec))

# plot all alphas in one figure for M2 tide
plt.figure()
plt.plot(x, zeta_M2_05_bestM2, label = 'Best fit M2, ' + r'$\alpha = 0.00091$'+ r' $ms^{-1}$')
plt.plot(x, zeta_M2_05_bestM4, label = 'Best fit M4, ' + r'$\alpha = 0.00136$'+ r' $ms^{-1}$')
plt.plot(x, zeta_M2_05_gem, label = 'Average, ' + r'$\alpha = 0.00114$'+ r' $ms^{-1}$')
plt.plot(x_waterlevel,M2amp, '.', label = 'Data 2005')
```

```
plt.legend()
plt.xlabel('$x(m)$')
plt.ylabel('$\zeta_{M2} (m)$')
plt.show()

plt.figure()
plt.plot(x, zeta_M4_05_bestM2, label = 'Best fit M2, ' + r'$\alpha = 0.00091$'+ r' $ms^{-1}$')
plt.plot(x, zeta_M4_05_bestM4, label = 'Best fit M4, ' + r'$\alpha = 0.00136$'+ r' $ms^{-1}$')
plt.plot(x, zeta_M4_05_gem, label = 'Average, ' + r'$\alpha = 0.00114$'+ r' $ms^{-1}$')
plt.plot(x_waterlevel,M4amp, '.', label = 'Data 2005')
plt.legend()
plt.xlabel('$x(m)$')
plt.ylabel('$\zeta_{M4} (m)$')
plt.show()

# 1965 ___________________________________________________________________________

M2amp = np.asarray([140.7, 154.6, 154.8, 155.0, 158.6, 161.9, 161.7, 123.5])/100
M4amp = np.asarray([21.3, 22.1, 20.4, 19.4, 21.3, 26.3, 30.8, 29.2])/100
x_waterlevel = np.asarray([0, 15.5, 19, 25, 36, 44, 50, 64])*1000


L = 64000 #length of the river
N = 1000 #amount of grid points
x = np.arange(0,L,L/N)
dt = 100
t = 0
g = 9.81
omega_M2 = 2*np.pi/(12.42*60*60)
omega_M4 = 2*np.pi/(6.21*60*60)

rho = 1020


theta_M2_1 = 0
theta_M4_1 = 0#.5*np.pi

#______________________________________________________________________________


# Data 1965
alpha_65 = -2.78
beta_65 = -7.13e-5
gamma_65 = 10
xl_65 = 5000
xc_65 = 13000

#Data 2005
alpha_05 = -1.2
beta = -5.1e-5
gamma = 10
xl = 5000
xc = 13000

H_05 = np.ones(len(x))
for i in range(len(x)):
    if x[i]<100000:
        H_05[i] =
            0.5*alpha_05*(1+np.tanh((x[i]-xc)/xl))+0.5*beta*x[i]*(1+np.tanh((x[i]-xc)/xl))+gamma
    else:
```

```
        H_05[i] = 0.5*alpha_05*(1+np.tanh((100000-xc)/xl))+0.5*beta*100000*(1+np.tanh((100000-xc)/xl))+gamma
H_x_05 = np.gradient(H_05, x)


H_65 = np.ones(len(x))
for i in range(len(x)):
    if x[i]<100000:
        H_65[i] = 0.5*alpha_65*(1+np.tanh((x[i]-xc_65)/xl_65))+0.5*beta_65*x[i]*(1+np.tanh((x[i]-xc_65)/xl_65))+gamma_
    else:
        H_65[i] = 0.5*alpha_65*(1+np.tanh((100000-xc_65)/xl_65))+0.5*beta_65*100000*(1+np.tanh((100000-xc_65)/xl_65))+
H_x_65 = np.gradient(H_65, x)
alpha_x = 0




#1965


x_waterlevel = np.asarray([0, 12000, 37000, 50000, 63000])
M2amp = np.asarray([1.34, 1.38, 1.15, 0.94, 0.91])
M4amp = np.asarray([0.182, 0.170, 0.162, 0.125, 0.190])


def createF(t):
    global A_M2, A_M4
    Fvec_M2 = np.zeros(N, dtype = complex)
    Fvec_M4 = np.zeros(N, dtype = complex)

    Fvec_M2[0] = 1.34
    Fvec_M4[0] = 0.182
    return Fvec_M2, Fvec_M4

def create_delta(omega, alpha, B, rho, g, H):
    delta = np.zeros(len(B),dtype=complex)
    for i in range(0,len(x)):
        delta[i] = -(g*H[i]*B[i]*rho)/(1j*omega*rho + alpha*B[i])
    return delta

def create_gamma(omega,alpha,B,rho,g,H):
    gamma = np.zeros(len(x),dtype=complex)
    for i in range(len(x)):
        gamma[i] = -(g*rho)/(1j*omega*rho + alpha*B[i])*(H[i]*B_x[i] + B[i]*H_x_65[i] -
            (B[i]*H[i]*(alpha_x*B[i] + alpha*B_x[i]))/(alpha*B[i] + 1j*omega*rho))
    return gamma


def createA_banded(L,N,omega,alpha,B,rho,g,H):
    h = L/N
    A_banded = np.zeros((N,3),dtype=complex)
    delta = create_delta(omega, alpha, B, rho, g, H)
    gamma = create_gamma(omega, alpha, B, rho, g, H)
    for i in range(N-1):
        A_banded[i+1][0] = delta[i]/(h**2)+ gamma[i]/h    # In deze en de volgende regels even
            opletten met i+1, i, i-1.
        A_banded[i][1] = B[i]*omega*1j -2*delta[i]/(h**2) -gamma[i]/h
        A_banded[i-1][2] = delta[i]/(h**2)
    A_banded[0][1] = 1
```

```
    A_banded[1][0] = 0
    #######Boundary Backwards Euler
    A_banded[-1][1] = 1
    A_banded[-2][2] = -1
    ########Boundary Forward Euler
    # A_banded[-1][1] = B[i]*omega*1j - delta[i]/(h**2)
    A_banded = A_banded.transpose()
    return A_banded


alpha_65_M2 = 0.00302
alpha_65_M4 = 0.00195
alpha_65_gem = (alpha_65_M2 + alpha_65_M4)/2

#solutions M2 and M4 for best M2 alpha
A_banded_M2 = createA_banded(L,N,omega_M2,alpha_65_M2,B,rho,g,H_65)
fvec = createF(t)[0]
zeta_M2_65_bestM2 = abs(solve_banded((1,1),A_banded_M2,fvec))

A_banded_M4 = createA_banded(L,N,omega_M4,alpha_65_M2,B,rho,g,H_65)
fvec = createF(t)[1]
zeta_M4_65_bestM2 = abs(solve_banded((1,1),A_banded_M4,fvec))

#solutions M2 and M4 for best M4 alpha
A_banded_M2 = createA_banded(L,N,omega_M2,alpha_65_M4,B,rho,g,H_65)
fvec = createF(t)[0]
zeta_M2_65_bestM4 = abs(solve_banded((1,1),A_banded_M2,fvec))

A_banded_M4 = createA_banded(L,N,omega_M4,alpha_65_M4,B,rho,g,H_65)
fvec = createF(t)[1]
zeta_M4_65_bestM4 = abs(solve_banded((1,1),A_banded_M4,fvec))

#solutions M2 and M4 for mean alpha
A_banded_M2 = createA_banded(L,N,omega_M2,alpha_65_gem,B,rho,g,H_65)
fvec = createF(t)[0]
zeta_M2_65_gem = abs(solve_banded((1,1),A_banded_M2,fvec))

A_banded_M4 = createA_banded(L,N,omega_M4,alpha_65_gem,B,rho,g,H_65)
fvec = createF(t)[1]
zeta_M4_65_gem = abs(solve_banded((1,1),A_banded_M4,fvec))

# plot all alphas in one figure for M2 tide
plt.figure()
plt.plot(x, zeta_M2_65_bestM2, label = 'Best fit M2, ' + r'$\alpha = 0.00302$'+ r' $ms^{-1}$')
plt.plot(x, zeta_M2_65_bestM4, label = 'Best fit M4, ' + r'$\alpha = 0.00195$'+ r' $ms^{-1}$')
plt.plot(x, zeta_M2_65_gem, label = 'Average, ' + r'$\alpha = 0.00249$'+ r' $ms^{-1}$')
plt.plot(x_waterlevel,M2amp, '.', label = 'Data 1965')
plt.legend()
plt.xlabel('$x(m)$')
plt.ylabel('$\zeta_{M2} (m)$')
plt.show()



plt.figure()
plt.plot(x, zeta_M4_65_bestM2, label = 'Best fit M2, ' + r'$\alpha = 0.00302$'+ r' $ms^{-1}$')
plt.plot(x, zeta_M4_65_bestM4, label = 'Best fit M4, ' + r'$\alpha = 0.00195$'+ r' $ms^{-1}$')
plt.plot(x, zeta_M4_65_gem, label = 'Average, ' + r'$\alpha = 0.00249$' + r' $ms^{-1}$')
plt.plot(x_waterlevel,M4amp, '.', label = 'Data 1965')
plt.legend()
plt.xlabel('$x(m)$')
```

```
plt.ylabel('$\zeta_{M4} (m)$')
plt.show()
```