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Cantor sets for regular continued fractions and Lüroth series

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Abstract

In this thesis, we consider the summation of Cantor sets. After a brief introduction to these sets, specifically focusing on the Cantor Middle Third set, we explore the relevance of Cantor sets in various number expansions, including r-ary expansions, regular continued fraction expansions, and Lüroth series expansions. Marshall Hall, an American mathematician, extensively studied regular continued fractions, leading to significant discoveries, such as his theorem that every real number can be expressed as the sum of two regular continued fractions with partial quotients less than or equal to 4. This thesis extends Hall's investigations to Lüroth series expansions, aiming to establish analogous results to Hall's theorem.

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Chapter 1

Introduction

In 1883, Georg Cantor, a German mathematician, introduced the Cantor sets. Among these sets, the Cantor Middle Third set stands out as the most elementary. This set is constructed by iteratively removing the open middle thirds from a series of line segments. Initially, the open interval $(\frac{1}{3}, \frac{2}{3})$ is removed from the interval $[0, 1]$, resulting in two remaining line segments: $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$. This process repeats indefinitely, with each remaining segment having its open middle third removed in subsequent iterations. The Cantor Middle Third set comprises all points within the interval $[0, 1]$ that remain undeleted throughout this infinite procedure.

When we take the sum of Cantor sets, significant results emerge. For instance, a notable result is that if C represents the Cantor Middle Third set, then $C+C$ equals the interval $[0, 2]$, as will be proven in Section 2.2. This result motivated the mathematician Marshall Hall to further investigate the summation of Cantor sets. Hall utilized regular continued fractions, a specific method for expanding numbers. A regular continued fraction associated with $x \in \mathbb{R}$ is an expression of the form:

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}}$$

where $a_0 \in \mathbb{Z}$, $x - a_0 \in [0, 1)$ and $a_i \in \mathbb{N}$ for $i > 0$. Here, a_i are referred to as partial quotients. Further explanation and properties of regular continued fractions are discussed in Chapter 3.

In 1947, Hall proved a significant theorem stating that every real number can be represented as the sum of an integer and two regular continued fractions, each having partial quotients less than or equal to 4. To be precise, he defines $F(m)$ as the set of those real numbers x having a regular continued fraction expansion with arbitrary $a_0 \in \mathbb{Z}$ and partial quotients $a_i \leq m$, where m is a positive integer. Hall's theorem is then stated as

$$F(4) + F(4) = \mathbb{R}.$$

Chapter 5 provides a proof of this theorem, and several generalizations by researchers such as Cusick [2], Divis [4], Astels [1], and Hlavka [9] are discussed.

In Chapter 6, we investigated whether Hall's result can be obtained for other number expansions besides regular continued fractions. In this thesis, we considered standard Lüroth series expansions, which are introduced in Chapter 4. We conjecture that a theorem analogous to Hall's theorem holds for Lüroth series. In chapter 6, we will delve into the details and formulate a theorem regarding this conjecture.

Chapter 2

R-ary expansions

Numbers can be represented in various ways. Among these ways, the most familiar approach is through decimal expansion. The decimal numeral system, constantly utilized in our daily lives to represent numbers, serves as a fundamental example of a positional system. A positional number system indicates that the value of a digit within a number depends on its position. Derived from the Latin word 'decima,' meaning ten, it employs digits from 0 to 9, with powers of 10 serving as the basis. For example, consider the number 243. In the decimal numeral system, this number is written as follows:

$$243 = 2 \cdot 10^2 + 4 \cdot 10^1 + 3 \cdot 10^0 \tag{2.1}$$

In addition to the decimal system, another widely utilized positional notation is the binary numeral system. In this system, numbers are represented using only two digits, 0 and 1, with the base being 2. For example, the number 26 in the binary numeral system is represented as:

$$26 = 1 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0 \tag{2.2}$$

The decimal system and the binary system are examples of r-ary number expansions where $r = 10$ for the decimal system and $r = 2$ for the binary system. In general, a r-ary expansion of a number x where $x \in [0, 1)$, with r in $\mathbb{Z}$ and $r \geq 2$, is given by:

$$x = \sum_{n=1}^{\infty} \frac{a_n}{r^n} \tag{2.3}$$

where a_n belongs to the set $\{0, 1, \ldots, r-1\}$.

If x is a real number with $0 \leq x < 1$, for every r in $\mathbb{Z}$ and $r \geq 2$, there is a sequence of integers (a_n) such that equation (2.3) holds. Conversely, if (a_n) is any sequence of integers from the set $\{0, 1, \ldots, r-1\}$, the series $\sum_{n=0}^{\infty} \frac{a_n}{r^n}$ with r in Z and $r \geq 2$ converges to a real number x where $0 \leq x < 1$. We will also write this as $x = [0, a_1 a_2 \ldots]_r$.

To prove that if x is a real number with $0 \leq x \leq 1$, for every r in $\mathbb{Z}$ and $r \geq 2$, there is a sequence of integers (a_n) such that equation (2.3) holds, we define the function $T_r : [0, 1) \to [0, 1)$ by:

$$T_r(x) = rx \mod 1.$$

In other words,

$$T_r(x) = rx - a_1(x), \tag{2.4}$$

where $a_1 = a_1(x) = \lfloor rx \rfloor \in \{0, 1, \ldots, r-1\}$ is the first digit of x. Now, define $a_n = a_n(x) = \lfloor T_r^{n-1}(x) \rfloor$ for $n \in \mathbb{N}$. Throughout this thesis, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x, and $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. Then, from (2.4), we see that:

$$x = \frac{a_1}{r} + \frac{1}{r} T_r(x) = \frac{a_1}{r} + \frac{1}{r}\left(\frac{a_2}{r} + \frac{1}{r} T_r^2(x)\right) = \frac{a_1}{r} + \frac{a_2}{r^2} + \frac{1}{r^2} T_r^2(x),$$

and after k steps, we find

$$x = \sum_{n=1}^{k} \frac{a_n}{r^n} + \frac{1}{r^n} T_r^k(x). \tag{2.5}$$

Since $0 \leq \frac{T_r}{r^n} \leq \frac{1}{r^n} \to 0$ as $n \to \infty$, it follows from equation (2.5) that

$$x = \sum_{n=1}^{\infty} \frac{a_n}{r^n}.$$

We can then rewrite equation (2.3) for an arbitrary x in $\mathbb{R}$ as

$$x = \lfloor x \rfloor + \sum_{n=1}^{\infty} \frac{a_n}{r^n} = \lfloor x \rfloor + 0, a_1 a_2 .... \tag{2.6}$$

2.1 3-ary expansion

One way to view the Cantor Middle Third set is in terms of ternary expansions, r-ary expansions where $r = 3$. Given x where $0 \leq x \leq 1$, there is a sequence of integers $(a_n)_{n=1}^{\infty}$, $a_n \in \{0, 1, 2\}$ such that the series

$$x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$$

converges to x. In other words, we can associate x to $[0, a_1 a_2 \ldots]_3$, where $a_n \in \{0, 1, 2\}$.

Consider the ternary expansion for a fixed $x \in [0, 1]$. We know that x then lies in exactly one of the intervals $I_0 := \left[0, \frac{1}{3}\right]$, $I_1^0 := \left(\frac{1}{3}, \frac{2}{3}\right)$, or $I_2 := \left[\frac{2}{3}, 1\right]$. Now, if $a_1 = 1$, one way to think about that, is to say that x then belongs to I_1^0. Similarly, if $a_2 = 1$, x then belongs to either $(\frac{1}{9}, \frac{2}{9})$, $(\frac{4}{9}, \frac{5}{9})$, or $(\frac{7}{9}, \frac{8}{9})$. Then it is clear that if $a_1 \neq 1$ and $a_2 \neq 1$, we must have x belonging to

$$[0, \tfrac{1}{9}] \cup [\tfrac{2}{9}, \tfrac{1}{3}] \cup [\tfrac{2}{3}, \tfrac{7}{9}] \cup [\tfrac{8}{9}, 1].$$

Going further along in the ternary expansion and not using the digit 1 has the effect of breaking each of the previous intervals into equal thirds and throwing the middle third away. Thus we conclude that the Cantor Middle Third set is defined as the subset of the interval $[0, 1]$ comprising all numbers x that possess a ternary expansion where each digit a_n belongs to the set $\{0, 2\}$ for all n. In other words, it consists of numbers whose ternary expansion does not contain the digit 1.

2.2 Proof of $C + C = [0, 2]$

In this subsection, we will prove in two different ways that the sum of two Cantor Middle Third sets covers the entire interval $[0, 2]$. We define a sum of two sets A and B to be $A + B = \{a + b : a \in A \text{ and } b \in B\}$. This is a somewhat surprising result since the Cantor Middle Third set has the property of having a Lebesque measure of zero, which is demonstrated in Section 2.2.3.

2.2.1 A proof of $C + C = [0, 2]$ using 3-ary expansions

The proof of $C + C = [0, 2]$ provided in this subsection utilizes the expression of numbers x in C as a ternary expansion, using only digits a_n from the set $\{0, 2\}$, as discussed in Section 2.1.

Consider two elements of the Cantor set, $a \in C$, $b \in C$. According to Section 2.1, a and b can be written as:

$$a = \sum_{n=1}^{\infty} \frac{a_n}{3^n}, \quad b = \sum_{n=1}^{\infty} \frac{b_n}{3^n}$$

where a_n, b_n are elements of $\{0, 2\}$ for each $n \in \mathbb{N}$.

Define $x = \sum_{n=1}^{\infty} \frac{x_n}{3^n}$ where $x_n := \frac{a_n + b_n}{2}$. Then every x_n is an element of $\{0, 1, 2\}$ since $x_n = 0$ if $a_n = b_n = 0$, $x_n = 2$ if $a_n = b_n = 2$, and $x_n = 1$ otherwise. Clearly we have that:

$$x = \sum_{n=1}^{\infty} \frac{x_n}{3^n} \in [0, 1].$$

And thus it follows that:

$$a + b = 2 \cdot x = 2 \cdot \sum_{n=1}^{\infty} \frac{x_n}{3^n} \in [0, 2]$$

Hence, we have that $C + C \subseteq [0, 2]$.

Conversely, to show $[0, 2] \subseteq C + C$, it is enough to show $[0, 1] \subseteq \frac{1}{2}C + \frac{1}{2}C$, where we define $\frac{1}{2}C = \{\frac{1}{2}c \,|\, c \in C\}$. Let x be an arbitrary element of the interval [0,1]. Then x can be written as:

$$x = \sum_{n=1}^{\infty} \frac{x_n}{3^n}, \qquad \text{where } x_n \in \{0, 1, 2\}.$$

Thus, here $[0, x_1, x_2, \ldots]$ represents the ternary expansion of x. Observe that $a \in \frac{1}{2}C$ if and only if there exists $t \in C$ such that $a = \frac{1}{2}t$. Hence, we have that $a \in \frac{1}{2}C$ can be written as:

$$a = \sum_{n=1}^{\infty} \frac{a_n}{3^n}, \qquad \text{where } a_n \in \{\frac{1}{2} \cdot 0 = 0, \frac{1}{2} \cdot 2 = 1\}.$$

Now define $a = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$ and $b = \sum_{n=1}^{\infty} \frac{b_n}{3^n}$ in the following way:

For each $n \in \mathbb{N}$, $a_n = 0$ if $x_n = 0$ and $a_n = 1$ if $x_n = 1, 2$ and $b_n = 0$ if $x_n = 0, 1$ and $b_n = 1$ if $x_n = 2$.

Thus $a, b \in \frac{1}{2}C$ and for each $n \in \mathbb{N}$, $a_n + b_n = 0$ if $x_n = 0$, $a_n + b_n = 1$ if $x_n = 1$, and $a_n + b_n = 2$ if $x_n = 2$. Therefore $x = a + b \in \frac{1}{2}C + \frac{1}{2}C$ and hence $[0, 1] \subseteq \frac{1}{2}C + \frac{1}{2}C$. We conclude that $[0, 2] \subseteq C + C$, which completes the proof. [11]

2.2.2 A graphical proof of $C + C = [0, 2]$

For the second proof of $C + C = [0, 2]$, we will demonstrate that each line of the form $x + y = a$, where a belongs to $[0, 2]$, intersects the Cartesian product $C \times C$. If (c, b) denotes the point of intersection, then it follows directly $a = b + c$, where both b and c belong to the Cantor Middle Third set. The Cartesian product $C \times C$ can be constructed similarly to the Cantor Middle Third set itself, noting that for any $a \in [0, 2]$ the line $x + y = a$ intersects the square $[0, 1] \times [0, 1]$ in at least one point (and infinitely many points if there is more than one intersection point). Now divide the square $[0, 1] \times [0, 1]$ into 9 squares of side length $\frac{1}{3}$; see Figure 2.1. Remove the 5 ‘central’ squares, corresponding to the blank squares in Figure 2.1. Note that for any $a \in [0, 2]$ the line $x + y = a$ intersects at least one of the four remaining squares. Choose one of these squares with which the line $x + y = a$ intersects; then the process repeats itself, but now on a smaller scale. In Figure 2.1, a line $x + y = a$ is drawn where a can take any value between 0 and 2 and the described process is illustrated.

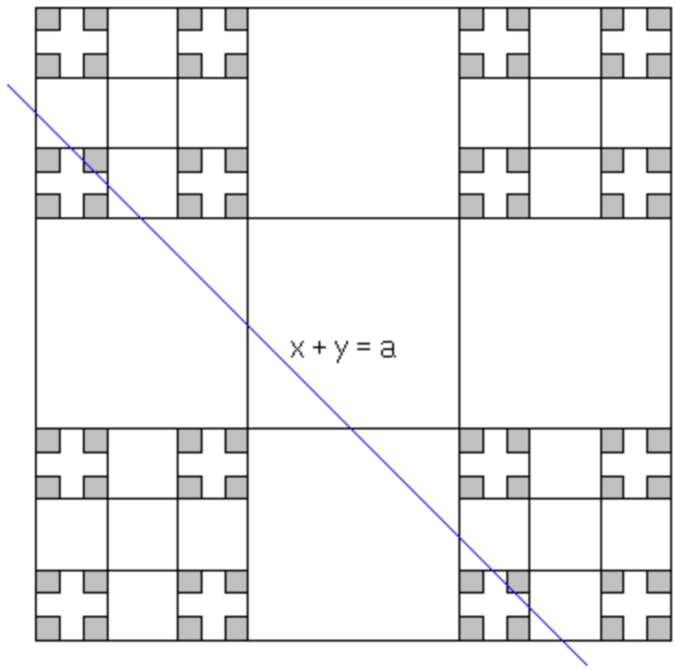

Figure 2.1: A line $x + y = a$ where $a \in [0, 2]$ and 189 squares

We denote any of the squares out of the four 'corner' squares with a side length of $\frac{1}{3}$ that intersects the line $x + y = a$ as S_1. Similarly, any line with a slope of 1 that intersects the square S_1 will intersect at least one of the 'corner' squares with a side length of $\frac{1}{9}$ that are contained within S_1. We denote one of these smaller squares as S_2 and by construction we have that $S_2 \subset S_1$.

Using the described method, we can continue constructing sets S_i where the sequence (S_i) is a nested sequence of closed sets. Moreover, the interval $[0, 2]$ is compact since it is closed and bounded. We have that a topological space is compact if and only if the intersection of every sequence of closed sets with the finite intersection property is non-empty. Here, a sequence of closed sets is said to have the finite intersection property if every finite subsequence has a non-empty intersection. Thus, because of the compactness of the interval $[0, 2]$, for the decreasing sequence (S_i), it holds that the intersection of any finite subsequence is not empty; it corresponds to the smallest square in the subsequence. Consequently, the squares $\{S_i\}$ have a common point. This point belongs to both $C \times C$ and the line $x + y = a$. This holds for every point a in the interval $[0, 2]$ and thus we have proven that $C + C = [0, 2]$. [5]

2.2.3 Lebesque measure of the Cantor set

The property of the Cantor Middle Third set having a Lebesgue measure of zero can be demonstrated in two different ways. For the first method, notice that from the construction of the Cantor Middle Third set, we remove 2^{n-1} disjoint intervals at each step $n > 0$, each having a length of $\left(\frac{1}{3}\right)^n$. Thus, we remove a total length of

$$\sum_{n=1}^{\infty} 2^{n-1} \cdot \left(\frac{1}{3}\right)^n = \frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n = \frac{1}{3} \cdot \frac{1}{1-\frac{2}{3}} = \frac{1}{3} \cdot 3 = 1$$

from the interval $[0,1]$. Since the Lebesgue measure of the interval $[0,1]$ is equal to 1, the Cantor Middle Third set must have a Lebesgue measure of zero.

Another way to see that the Cantor Middle Third set has Lebesgue measure zero is to use Ergodic Theory. Underlying the ternary expansion is the map $T_3(x) = 3x(\text{mod}1)$, and one easily can show that the system $([0,1), \mathcal{B}, T_3, \lambda)$, where $\mathcal{B}$ is the collection of Borel sets of $[0,1)$, and λ is Lebesgue measure on $[0,1)$, which is T_3-invariant, is ergodic. Due to this it follows from Birkhoff's Individual Ergodic Theorem that for Lebesgue almost every $x \in [0,1)$ the frequency of the digit 1 is $\frac{1}{3}$. Since in C there are **no** numbers x with a digit 1, we conclude that C is part of nullset, and therefore a nullset. For more details, see [3].

Chapter 3

Continued fraction expansions

3.1 Introducing the regular continued fraction expansion

In Chapter 2, we examined r-ary expansions with $r = 2$, 3, and 10 in particular. Another way to represent a number is using a continued fraction. This thesis almost exclusively examines a specific type of continued fractions; the *regular continued fractions*. From now on, we refer to a regular continued fraction as RCF. A RCF associated with $x \in \mathbb{R}$ is an expression of the form:

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}} \tag{3.1}$$

where $a_0 \in \mathbb{Z}$, $x - a_0 \in [0, 1)$ and $a_i \in \mathbb{N}$ for $i > 0$. The numbers a_i are called the partial quotients, and the notation used for the RCF-expansion of x from (3.1) is $x = [a_0; a_1, a_2, \dots]$.

3.1.1 The Euclidean algorithm

Continued fractions have remarkable properties, one of them is that x is a rational number if and only if x has a finite RCF expansion. Since rational numbers are numbers that can be written in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$, we have that $\frac{p}{q}$ determines a finite RCF and that the continued fraction $[a_0; a_1, a_2, \dots, a_n]$ represents a rational number $\frac{p}{q}$. To prove this, we will demonstrate how the Euclidean algorithm for finding the greatest common divisor can be used to find the RCF of a rational number x. The Euclidean algorithm proceeds as follows.

1. Start with two non-negative integers, r_0 and r_1, where $r_0 \geq r_1$.
2. Compute the remainder r_2 of dividing r_0 by r_1 and express it as $r_0 = a_1 \cdot r_1 + r_2$ with $0 \leq r_2 < r_1$.
3. If r_2 equals zero, the algorithm stops. Otherwise, repeat the following steps.
4. Take r_1 as the new r_0 and r_2 as the new r_1.
5. Compute again the remainder r_3 by dividing r_0 by r_1 and express it as $r_1 = a_2 \cdot r_2 + r_3$, with $0 \leq r_3 < r_2$.
6. Continue this iterative process until the remainder equals zero. At some point, this occurs because the process terminates after a finite number of steps.

Now, suppose we want to find the RCF expansion of the rational number $x = \frac{117}{31}$. Notice that:

$$117 = 3 \cdot 31 + 24.$$

Rewriting this gives us:

$$\frac{117}{31} = 3 + \frac{24}{31} = 3 + \frac{1}{\frac{31}{24}}.$$

.
We have that:

$$31 = 1 \cdot 24 + 7,$$

which again gives us that:

$$\frac{31}{24} = 1 + \frac{7}{24}.$$

Plugging this in in the equation for $\frac{117}{31}$, we get that:

$$\frac{117}{13} = 3 + \frac{1}{1 + \frac{24}{7}}.$$

Continuing using the Euclidean algorithm, we get the following equations:

$$24 = 3 \cdot 7 + 3,$$

$$7 = 2 \cdot 3 + 1,$$

$$3 = 3 \cdot 1 + 0.$$

Since we obtain a zero remainder in the last equation, the Euclidean algorithm stops, and by substitution, we obtain the RCF of $x = \frac{117}{31}$:

$$\frac{117}{13} = 3 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{3}}}}$$

This process works for *any* rational number and thus we have demonstrated that any rational number can be written as a finite RCF.

3.2 The Gauss map T

The RCF operator also known as the Gauss map $T : [0, 1) \to [0, 1)$ is defined by:

$$T(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor \tag{3.2}$$

for $x \neq 0$ and $T(0) = 0$. The map T is illustrated in Figure 3.1.

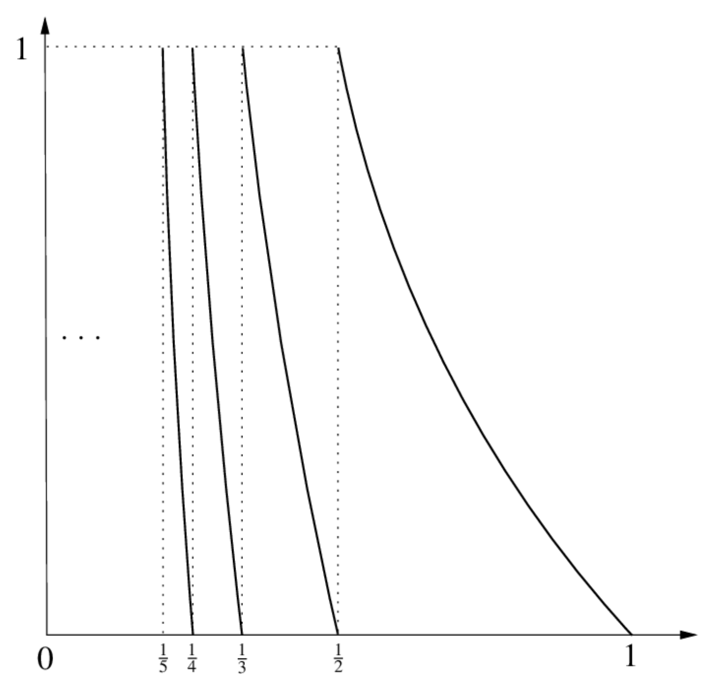

Figure 3.1: The Gauss map T

Now, let $y \in \mathbb{R} \setminus \mathbb{Q}$ and choose $a_0 \in \mathbb{Z}$ such that $x := y - a_0 \in [0, 1)$. If we iterate the function T, we have:

$$\begin{aligned}
T_0(x) &= x, \\
T_1(x) &= T(T_0(x)) = \frac{1}{T_0(x)} - \left\lfloor \frac{1}{T_0(x)} \right\rfloor. \\
T_n(x) &= \frac{1}{T^{n-1}(x)} - \left\lfloor \frac{1}{T^{n-1}(x)} \right\rfloor \\
a_n(x) &:= \left\lfloor \frac{1}{T^{n-1}(x)} \right\rfloor \\
T^{n-1}(x) &= \frac{1}{a_n + T^n(x)} \\
x &= \frac{1}{a_1 + T_1(x)} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + T_2(x)}} = \dots = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\dots + \cfrac{1}{a_n + T_n(x)}}}}
\end{aligned}$$

for $n \geq 1$. Also, it holds that for all $n \geq 0$, $T_n(x) \in (0, 1)$.

Now, let

$$a_n = a_n(x) := \left\lfloor \frac{1}{T^{n-1}(x)} \right\rfloor$$

for $n \geq 1$. This implies that for all $n \geq 1$, $T^{n-1}(x) = \frac{1}{a_n + T^n(x)}$.

Then, we have:

$$x = \frac{1}{a_1 + T_1(x)} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + T_2(x)}} = \dots = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\dots + \cfrac{1}{a_n + T_n(x)}}}}.$$

The Gauss map T thus generates, for each x, a sequence of digits $(a_n)_{n \geq 0}$. If x is a rational number, this sequence is finite.

If x is an irrational number whose infinite continued fraction expansion is $[0; a_1, a_2, a_3, \dots]$, one can truncate the continued fraction expansion at level n and obtain a rational number $\frac{p_n}{q_n}$ given by:

$$\frac{p_n}{q_n} = [a_0; a_1, a_2, \dots, a_n].$$

The sequence of rational numbers $\frac{p_0}{q_0} = [a_0;]$, $\frac{p_1}{q_1} = [a_0; a_1]$, $\frac{p_2}{q_2} = [a_0; a_1, a_2]$, ..., $\frac{p_n}{q_n} = [a_0; a_1, \ldots, a_n]$, are referred to as the *convergents* of a continued fraction, where n denotes its length. So we define for x the n-th convergent with $n \in \mathbb{N}$ by:

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}.$$

We will show that these convergents provide a good approximation of an irrational number x in Section 3.4.

3.3 Quality of approximation

Let $(a_n)_{n\geq 0}$ be a sequence of positive real numbers, and let p_n and q_n be defined by the following recurrence relations for $n \geq 1$:

$$p_n = a_n p_{n-1} + p_{n-2},$$
$$q_n = a_n q_{n-1} + q_{n-2}$$

where $p_0 = q_{-1} = 1$ and $p_{-1} = q_0 = 0$.

Then the n^{th} convergent is given by:

$$[a_0, a_1, a_2, \ldots, a_n] = \frac{p_n}{q_n} \tag{3.3}$$

where p_n and q_n are defined as above. To prove this, we use induction. For the case where $n = 1$, we have that

$$\frac{p_1}{q_1} = \frac{a_1 p_0 + p_{-1}}{a_1 q_0 + q_{-1}} = \frac{a_1 \cdot 1 + 0}{a_0 \cdot 0 + 1} = [a_0].$$

Moreover, assume equation (3.3) holds for $n = k$, then for $n = k + 1$ we have that

$$\begin{aligned}
[a_0, a_1, a_2, \ldots, a_k, a_{k+1}] &= [a_0, a_1, a_2, \ldots, a_k + \frac{1}{a_{k+1}}] \\
&= \frac{(a_k + \frac{1}{a_{k+1}})p_{k-1} + p_{k-2}}{(a_k + \frac{1}{a_{k+1}})q_{k-1} + q_{k-2}} \\
&= \frac{a_{k+1}(a_k p_{k-1} + p_{k-2}) + p_{k-1}}{a_{k+1}(a_k q_{k-1} + q_{k-2}) + q_{k-1}} \\
&= \frac{a_{k+1} p_k + p_{k-1}}{a_{k+1} q_k + q_{k+1}} \\
&= \frac{p_{k+1}}{q_{k+1}}.
\end{aligned}$$

Therefore, we proved by induction that equation (3.3) holds. We also have that the following equation holds

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}. \tag{3.4}$$

Once again, we prove equation (3.4) by induction. For the base case, when $n = 1$, it holds that:

$$\begin{aligned} p_1 q_0 - p_0 q_1 &= p_1 \cdot 0 - 1 \cdot q_1 \\ &= -q_1 = -(a_1 q_0 + q_{-1}) = -(a_1 \cdot 0 + 1) \\ &= -1 = (-1)^{-1}. \end{aligned}$$

Furthermore, if (3.4) holds for $n = k$, then for $n = k + 1$, we obtain:

$$\begin{aligned} p_{k+1} q_k - p_k q_{k+1} &= (a_{k+1} p_k + p_{k-1}) q_k - p_k (a_{k+1} q_k + q_{k-1}) \\ &= p_{k-1} q_k - p_k q_{k-1} = -(-1)^{k-1} = (-1)^k = (-1)^{(k+1)-1}. \end{aligned}$$

And thus, through induction, we have demonstrated that equation (3.4) holds. Now, let $t_n = [a_n; a_{n+1}, \ldots]$ for $n \geq 0$. Then, for $x = [a_0; a_1, a_2, \ldots, a_n]$, we have:

$$x = \frac{p_{n-2} + t_n p_{n-1}}{q_{n-2} + t_n q_{n-1}}. \tag{3.5}$$

This can be proven again using induction. First, observe that

$$t_{n+1} = \frac{1}{t_n - a_n},$$

since we have that

$$t_n = [a_n; a_{n+1}, \ldots] = a_n + \frac{1}{t_{n+1}}.$$

For the case where $n = 0$, we obtain

$$\frac{p_{-2} + t_0 p_{-1}}{q_{-2} + t_0 q_{-1}} = \frac{0 + t_0 \cdot 1}{1 + t_0 \cdot} = t_0 = [a_0] = x.$$

And thus, the base case holds. Moreover, if equation (3.5) holds for $n = k$, we have that for $n = k + 1$ that

$$\begin{aligned} \frac{p_{k-1} + t_{k+1} p_k}{q_{k-1} + t_{k+1} q_k} &= \frac{\frac{1}{t_k - a_k} p_k + p_{k-1}}{\frac{1}{t_k - a_k} q_k + q_{k-1}} \\ &= \frac{\frac{1}{t_k - a_k} a_k p_{k-1} + p_{k-2} + p_{k-1}}{\frac{1}{t_k - a_k} a_k q_{k-1} + q_{k-2} + q_{k-1}} \\ &= \frac{p_{n-2} + t_n p_{n-1}}{q_{n-2} + t_n q_{n-1}} = x. \end{aligned}$$

This proves equation (3.5). Finally, we establish the following bound for the accuracy of the convergents in approximating the value of the continued fraction:

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n^2} \tag{3.6}$$

The proof of equation (3.6) proceeds as follows:

$$\begin{aligned}\left|x-\frac{p_n}{q_n}\right| &= \left|\frac{p_{n-1}+t_{n+1}p_n}{q_{n-1}+t_{n+1}q_n}-\frac{p_n}{q_n}\right| \\ &= \left|\frac{q_n(p_{n-1}+t_{n+1}p_n)-p_n(q_{n-1}+t_{n+1}q_n)}{q_n(q_{n-1}+t_{n+1}q_n)}\right| \\ &= \left|\frac{q_np_{n-1}-q_{n-1}p_n}{q_n(q_{n-1}+t_{n+1}q_n)}\right| \\ &= \left|\frac{(-1)^n}{q_n(q_{n-1}+t_{n+1}q_n)}\right| \\ &= \frac{1}{q_n(q_{n-1}+t_{n+1}q_n)}\end{aligned}$$

Moreover, since $t_{n+1} > a_{n+1}$ by definition and since the q_i's are strictly increasing by definition, we obtain

$$\left|x-\frac{p_n}{q_n}\right| \le \frac{1}{q_n(q_{n-1}+t_{n+1}q_n)} < \frac{1}{q_nq_{n+1}} < \frac{1}{q_n^2}.$$

We also observe that for any $\alpha, \beta \in \mathbb{R}$ and continued fraction expansions:

$$\begin{aligned}x &= [0; a_1, a_2, \ldots, a_n, \alpha], \\ y &= [0; a_1, a_2, \ldots, a_n, \beta],\end{aligned}$$

the following relationship holds:

$$|x-y| = \frac{|\alpha-\beta|}{q_k\left(\alpha+\frac{q_{n-1}}{q_n}\right)\left(\beta+\frac{q_{n-1}}{q_n}\right)}. \tag{3.7}$$

This is the case since:

$$\begin{aligned}|x-y| &= |[0; a_1, a_2, \ldots, a_n, \alpha] - [0; a_1, a_2, \ldots, a_n, \beta]| \\ &= \frac{\alpha p_n + p_{n-1} - \beta p_n + p_{n-1}}{\alpha q_n + \frac{q_{n-1}}{q_n} \beta q_n + \frac{q_{n-1}}{q_n}} \\ &= \frac{(\alpha p_n + p_{n-1})(\beta q_n + \frac{q_{n-1}}{q_n}) - (\alpha q_n + \frac{q_{n-1}}{q_n})(\beta p_n + p_{n-1})}{(\alpha q_n + \frac{q_{n-1}}{q_n})(\beta q_n + \frac{q_{n-1}}{q_n})} \\ &= \frac{\alpha(p_nq_{n-1} - p_{n-1}q_n) + \beta(p_{n-1}q_n - p_nq_{n-1})}{q_n(\alpha + \frac{q_{n-1}}{q_n})(\beta + \frac{q_{n-1}}{q_n})} \\ &= \frac{|(-1)^n\alpha + (-1)^{n+1}\beta|}{q_n(\alpha + \frac{q_{n-1}}{q_n})(\beta + \frac{q_{n-1}}{q_n})} \\ &= \frac{|\alpha-\beta|}{q_n(\alpha + \frac{q_{n-1}}{q_n})(\beta + \frac{q_{n-1}}{q_n})}\end{aligned}$$

where we used Equations (3.3) and (3.4).

3.4 Periodic continued fractions

A *periodic continued fraction* is a RCF whose partial quotients eventually repeat from some point onwards, i.e., there exists some positive integers N, h such that

$$a_n = a_{n+h}$$

for all $n > N$. In general, a periodic continued fraction has the form

$$[b_0; b_1, b_2, \dots] = [b_0; \dots, b_N, a_0, a_1, \dots, a_{h-1}, a_0, a_1, \dots, a_{h-1}, a_0, \dots].$$

We write this as

$$[b_0; \dots, b_N, \overline{a_0, \dots, a_{h-1}}]$$

where $b_0; b_1, \dots, b_N$ is the pre-period and $a_0, a_1, \dots, a_{h-1}$ represents the period. Furthermore, a *quadratic irrational* α is defined as an irrational number that is a root of a quadratic equation, which has the following form

$$ax^2 + bx + c = 0$$

where $a \neq 0$ and $a, b, c \in \mathbb{Z}$.

Now, consider the periodic continued fraction $\alpha = [\overline{1, 2}]$. Suppose we want to determine its convergence. We have that

$$\alpha = 1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \dots}}}}$$

We can rewrite this as

$$\alpha = 1 + \frac{1}{2 + \frac{1}{\alpha}} = 1 + \frac{1}{\frac{2\alpha+1}{\alpha}} = 1 + \frac{\alpha}{2\alpha + 1} = \frac{3\alpha + 1}{2\alpha + 1}.$$

Thus we have that $2\alpha^2 - 2\alpha - 1 = 0$, which is a quadratic equation with two roots where α corresponds to the positive root so we have that

$$\alpha = \frac{1 + \sqrt{3}}{2}$$

and thus we have seen that this periodic continued fraction α can be written as a quadratic irrational. Similarly, every periodic continued fraction represents a quadratic irrational, and conversely, every quadratic irrational corresponds to a periodic continued fraction. Thus we have that an infinite RCF is periodic if and only if it represents a quadratic irrational.

L. Euler proved one side of this theorem: if x is irrational with a periodic continued fraction expansion, then x is a quadratic irrational. Euler's theorem is straightforward to prove. Let $x = [b_0, \ldots, b_N, \overline{a_0, \ldots, a_{h-1}}]$ and define $\alpha = [\overline{a_0, \ldots, a_{h-1}}]$. Then by the periodicity of the expansion, we have that $\alpha = [a_0, \ldots, a_{h-1}, \alpha]$. Using equation (3.5), we obtain

$$\alpha = \frac{p_{n-2} + \alpha p_{n-1}}{q_{n-2} + \alpha q_{n-1}}$$

which is a quadratic equation for α. Since α is an infinite continued fraction, it is irrational. Thus, α is a quadratic irrational. Furthermore, $x = [b_0, \ldots, b_N, \alpha]$ is also a rational function in α. Combining this function with α being a root of a quadratic equation, we conclude that x is also a root of another quadratic equation. Since x is an irrational number, we deduce that x is a quadratic irrational, completing Euler's proof.

Joseph Lagrange proved the converse of this theorem, which has a less trivial proof that I won't fully elaborate on here. Nonetheless, the underlying idea is as follows. Suppose $x \in \mathbb{Q}$ satisfies the quadratic equation

$$ax^2 + bx + c = 0,$$

where $a, b, c \in \mathbb{Z}$. Let $[a_0, a_1, a_2, \ldots]$ be the RCF of x, and let t_n be its n^{th} convergent. Then, we have $x = [a_0, a_1, a_2, \ldots, a_{n-1}, t_n]$. By equation (3.5), we can express x as

$$x = \frac{p_{n-1} + t_n p_n}{q_{n-1} + t_n q_n}.$$

Substituting this expression for x into the quadratic equation, we obtain

$$A_n t_n^2 + B_n t_n + C_n = 0,$$

where

$$\begin{aligned} A_n &= a p_{n-1}^2 + b p_{n-1} q_{n-1} + c q_{n-1}^2, \\ B_n &= 2a p_{n-1} p_{n-2} + b(p_{n-1} q_{n-2} + p_{n-2} q_{n-1}) + 2c q_{n-1} q_{n-2}, \\ C_n &= a p_{n-2}^2 + b p_{n-2} q_{n-2} + c p_{n-2}^2. \end{aligned}$$

Note that A_n, B_n, and C_n are integers, $C_n = A_{n-1}$, and

$$B^2 - 4A_n C_n = (b^2 - 4ac)(p_{n-1} q_{n-2} - q_{n-1} p_{n-2})^2 = b^2 - 4ac.$$

Note that $A_n \neq 0$ since otherwise $\frac{p_{n-1}}{q_{n-1}}$ would be a rational root of the quadratic equation $ax^2 + bx + c = 0$, contradicting the fact that x is irrational. The remainder of the proof involves an elaborate computation that bounds each of A_n, B_n, and C_n independently of n. Assuming this, there are only a finite number of possibilities for the triples $(A_{n_1}, B_{n_1}, C_{n_1}), (A_{n_2}, B_{n_2}, C_{n_2}), (A_{n_3}, B_{n_3}, C_{n_3})$. Hence, we can choose n_1, n_2, n_3 such that

$$(A_{n_1}, B_{n_1}, C_{n_1}) = (A_{n_2}, B_{n_2}, C_{n_2}) = (A_{n_3}, B_{n_3}, C_{n_3}).$$

This implies that each of $t_{n_1}, t_{n_2}, t_{n_3}$ is a root of (say)

$$A_{n_1}t^2 + B_{n_1}t + C_{n_1}.$$

As a result, two of them must be equal. Thus, $t_{n_1} = t_{n_2}$ (say), and

$$t_{n_1} = [a_{n_1}, a_{n_1} + 1, \ldots],$$

$$t_{n_2} = [a_{n_2}, a_{n_2} + 1, \ldots],$$

and the continued fraction becomes periodic. The complete proof can be found in [7].

3.5 Nearest integer continued fractions

The *nearest integer continued fraction* (NICF) is a type of real continued fraction that permits negative integers as partial quotients. Unlike the RCF, where rounding down is used, the algorithm for computing the NICF of a real number x involves rounding to the nearest integer. In cases such as when $x = 2,5$, we round to the smallest integer, thus setting $x = 2$ as the nearest integer.

Given a real number $x \in [-\frac{1}{2}, \frac{1}{2})$, its continued fraction to the nearest integer is of the form

$$x = \cfrac{e_1}{a_1 + \cfrac{e_2}{a_2 + \cfrac{e_3}{a_3 + \cfrac{e_4}{a_4 + \ldots}}}}$$

so $x = [0; \frac{e_1}{a_1}, \frac{e_2}{a_2}, \frac{e_3}{a_3}, \ldots]$. The partial quotients a_n and the signs $e_n = \pm 1$ are determined by the NICF operator $T_{\frac{1}{2}} : [-\frac{1}{2}, \frac{1}{2}) \to [-\frac{1}{2}, \frac{1}{2})$, which is defined by

$$T_{\frac{1}{2}} = \left|\frac{1}{x}\right| - \left\lfloor \left|\frac{1}{x}\right| + \frac{1}{2} \right\rfloor \tag{3.8}$$

for $x \neq 0$ and $T_{\frac{1}{2}}(0) = 0$. The signs $e_n = \pm 1$ are set according to whether $T_{\frac{1}{2}}^{n-1}(x)$ is positive or not and the partial quotients a_n are given by

$$a_n := \left\lfloor \frac{e_n}{T_{\frac{1}{2}}^{n-1}(x)} + \frac{1}{2} \right\rfloor.$$

The NICF provides either a better or equal approximation for a real number x compared to the RCF. We can illustrate this with an example by comparing the RCF with the NICF of the number π. The RCF of π is equal to

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \dots}}}$$

where in the third step we have that $\left\lfloor \frac{1}{0{,}06...} \right\rfloor = \lfloor 15{,}996... \rfloor$. Since the RCF rounds down, we assign a_2 to be 15. However, the fraction 15.996... is much closer to 16 than to 15. Therefore, selecting a_2 as 16 would have been a more logical choice, which is precisely what the NICF would have done in this situation. If we follow this approach for each step, it ultimately provides us with a better approximation for the number π, and generally for a real number x.

Chapter 4

Lüroth Series expansions

4.1 The standard Lüroth Series expansion

In Chapter 2 and 3 we have seen two distinct methods for representing numbers. In 1883, Jacob Lüroth introduced an expansion for real numbers $x \in \mathbb{R}$ given by

$$x = a_0 + \frac{1}{a_1} + \frac{1}{a_1(a_1-1)a_2} + \ldots + \frac{1}{a_1(a_1-1)\ldots a_{n-1}(a_{n-1}-1)a_n} + \ldots \quad (4.1)$$

where $a_o \in \mathbb{Z}$ and $a_n \in \mathbb{N}$, $a_n \geq 2$. We write this as: $x = \langle a_0; a_1, a_2, \ldots, a_n, \ldots \rangle$. While this expansion is recognized as the *standard Lüroth expansion*, in this paper it is referred to simply as the *Lüroth expansion*. The Lüroth expansion is generated by the Lüroth map $L : [0,1] \to [0,1]$ defined by

$$L(x) := \left\lfloor \frac{1}{x} \right\rfloor \left(\left\lfloor \frac{1}{x} \right\rfloor + 1 \right) x - \left\lfloor \frac{1}{x} \right\rfloor, \quad \text{for } x \neq 0$$

and $L(0) := 0$. In Figure 4.2, the operator L is illustrated.

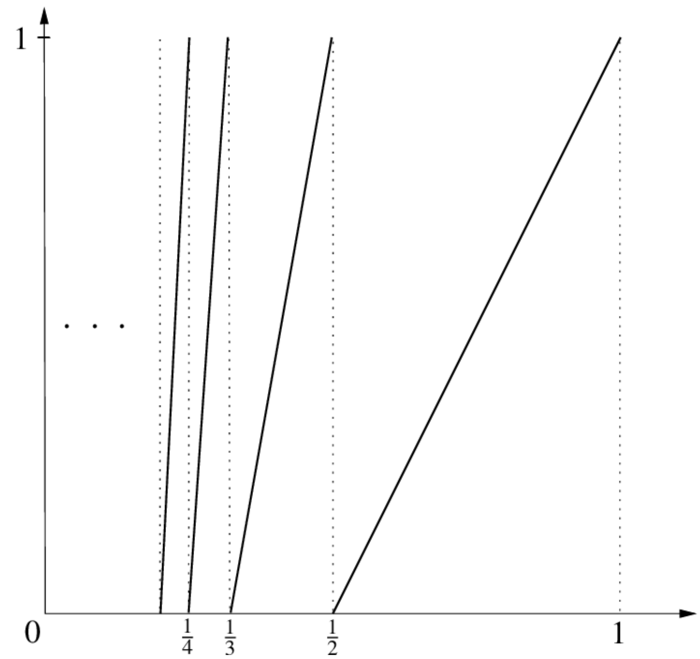

Figure 4.1: Lüroth map L

The Lüroth expansion, as provided in equation (4.1), then arises when defining

$$a_1 = a_1(x) := \left\lfloor \frac{1}{x} \right\rfloor + 1$$

so $x \in I_k := (\frac{1}{k}, \frac{1}{k+1}]$, if and only if $a_1(x) = k$. Moreover, when $L^n(x) \neq 0$, we have that

$$a_n = a_n(x) := a_1(L^n(x))$$

Rewriting equation (4.1) in terms of the partial quotients a_i gives us $L(x) = a_1(a_1-1)x - (a_1-1)$, and in general

$$L^n(x) = a_n(a_n-1)L^{n-1}(x) - (a_n - 1)$$

Thus we find that

$$\begin{aligned} x &= \frac{1}{a_1} + \frac{1}{a_1(a_1-1)} L(x) \\ &= \frac{1}{a_1} + \frac{1}{a_1(a_1-1)} \left(\frac{1}{a_2} + \frac{1}{a_2(a_2-1)} L^2(x) \right) \\ &= \frac{1}{a_1} + \frac{1}{a_1(a_1-1)} \left(\frac{1}{a_2} + \frac{1}{a_2(a_2-1)} (\frac{1}{a_3} + \frac{1}{a_3(a_3-1)} L^3(x)) \right) \\ &= \dots \end{aligned}$$

Continuing in this manner yields the Lüroth expansion provided in (4.1). Since $a_n \geq 2$ and $0 \leq L^n(x) \leq 1$, one easily sees that the infinite series from (4.1) converges to x.

4.2 The alternating Lüroth Series expansion

An alternative to the standard Lüroth expansion is the *alternating* Lüroth expansion, defined for $x \in (0,1]$ as follows

$$x = \frac{1}{a_1 - 1} - \frac{1}{a_1(a_1-1)(a_2-1)} + \ldots + \frac{(-1)^{k+1}}{a_1(a_1-1)\ldots(a_{k-1}-1)(a_k-1)} + \ldots \tag{4.2}$$

where $a_k \geq 2$ and $k \geq 1$. The alternating series expansion is generated by the operator $L_A : [0,1] \to [0,1]$ defined by

$$L_A(x) := 1 + \left\lfloor \frac{1}{x} \right\rfloor - \left\lfloor \frac{1}{x} \right\rfloor \left(\left\lfloor \frac{1}{x} \right\rfloor + 1 \right) x, \quad \text{for } x \neq 0$$

and $L(0) := 0$. Note that $L_A(x) = 1 - L(x)$, see also Figure 4.2.

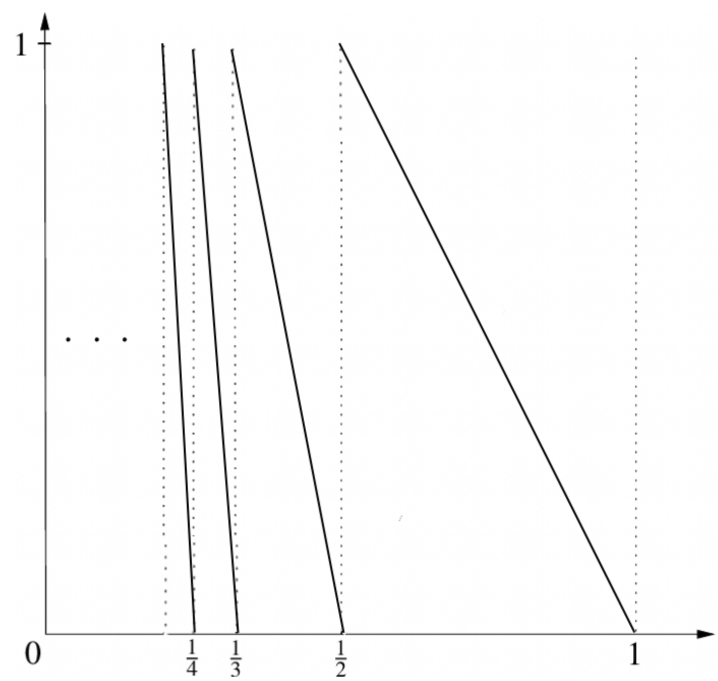

Figure 4.2: Alternating Lüroth map L_A

Chapter 5

Hall's theorem

In 1947, M. Hall Jr. proved in [6] that

$$F(4) + F(4) = \mathbb{R} \tag{5.1}$$

where $F(m)$ is defined as the set of irrational numbers with partial quotients $a_i \leq m$ for $i \geq 1$ and $a_0 \in \mathbb{Z}$. Moreover, define for $N \in \mathbb{Z}$: $F^n(m) = \{[a_0; a_1, a_2, a_3, \ldots] \in \mathbb{R} : a_0 = n, \forall i \geq 1 : 1 \leq a_i \leq m\}$ and thus $F(m) = \bigcup_{n \in \mathbb{Z}} F^n(m)$.

In this chapter, we prove Hall's Theorem, based on the proof provided by Hall (see [6]). We begin by establishing some basic theorems concerning the summation of Cantor Sets, which are necessary for proving Hall's theorem.

5.1 Cantor sets

As mentioned in Chapter 1, the most simple Cantor set, the Cantor Middle Third set, is created by recursively removing the open middle thirds of a set of line segments.

More generally, let $A \subseteq \mathbb{R}$ be a closed and bounded interval and consider the open and disjoint subintervals $\{C_k^n \subset A : n \in \mathbb{Z}_{\geq 0}, k \in \mathbb{Z}_{\geq 0}, k < 2^n\}$. Furthermorwe, the A_k^n and the C_k^n are ordered, so we have that $A_1^n < A_2^n < \cdots < A_{2^n-1}^n$ and $C_1^n < C_2^n < \cdots < C_{2n-1}^n$. We then define the subsets A_n^k recursively as follows:

1. $A_0^0 = A$ and $C_0^1 \subset A$
2. $C_k^n \subset A_k^{n-1}$ and $A_{2k}^n = \{x \in A_k^{n-1} \setminus C_k^n : \forall c \in C_k^n, x < c\}$
3. $C_k^n \subset A_k^{n-1}$ and $A_{2k+1}^n = \{x \in A_k^{n-1} \setminus C_k^n : \forall c \in C_k^n, x > c\}$

Note that for $n \in \mathbb{N}$ and $k < 2^n$, the sets C_k^n satisfy:

$$C_k^n \subset A_k^{n-1}.$$

See also Figure 5.1 below. This might suggest that the sets (C_k^n) are given from the start. This is not necessarily the case; for the *Cantor Middle Third* set C these sets (C_k^n) are given recursively by the construction of C. We refer to the family of sets $\{C_k^n : n \in \mathbb{Z}_{\geq 0}, k \in \mathbb{Z}_{\geq 0}, k < 2^n\}$ as *Cantor gaps* for A.

If we define A_k^n in this way, and for $A \subseteq \mathbb{R}$, we define $l(A) = |\sup(A) - \inf(A)|$ if $\sup(A)$ and $\inf(A)$ exist, and $l(A) = \infty$ otherwise, then we set $a_k^n = l(A_k^n)$ and $c_k^n = l(C_k^n)$. Moreover, a General Cantor Point Set is defined as

$$L(A, \{C_k^n : n \in \mathbb{Z}_{\geq 0}, k \in \mathbb{Z}_{\geq 0}, k < 2^n\}) = A \setminus \left(\bigcup_{n \in \mathbb{Z}_{\geq 0}, k \in \mathbb{Z}_{\geq 0}, k < 2^n} C_k^n \right).$$

For simplicity, we write $L(A, \{C_k^n : n \in \mathbb{Z}_{\geq 0}, k \in \mathbb{Z}_{\geq 0}, k < 2^n\})$ as $L(A)$. This General Cantor Point Set together with the corresponding Cantor Gaps for A can be visualized as a tree in the following way.

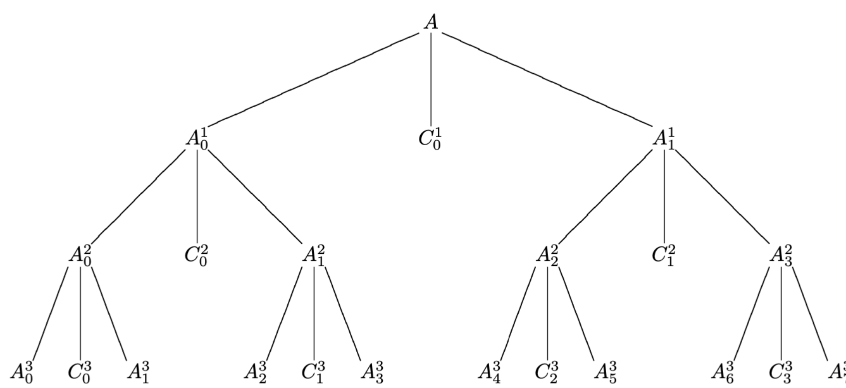

Figure 5.1: The first three layers of this process

In the context of a tree structure, we designate set A_0^0 as the root. In the remainder of this chapter, the following theorem will be proved.

Theorem 1. *Let $A = [a_1, a_2]$ and $B = [b_1, b_2]$ be closed bounded intervals of $\mathbb{R}$, where $a = a_2 - a_1$ and $b = b_2 - b_1$. Suppose $\{C_k^n\}$ and $\{D_k^n\}$ are Cantor Gaps for A and B, respectively. In addition, define $e = \min(a, b)$ and suppose that the following conditions hold:*

(C1) *for every $n > 0$ and $k < 2^n$, one has $l(C_k^n) := c_k^n \leq \min\{a_n^{2k}, a_n^{2k+1}\}$ and $l(D_k^n) := d_k^n \leq \min\{b_n^{2k}, b_n^{2k+1}\}$.*

(C2) $\frac{1}{3} \leq \frac{a}{b} \leq 3$.

Then we have that $L(A) + L(B) = A + B$.

To prove Theorem 1, we first introduce several definitions and lemmas. We start by showing that Condition **(C1)** holds.

Let $A = [a_1, a_2]$ and $B = [b_1, b_2]$ be two closed intervals, and let $e = \min(a_2 - a_1, b_2 - b_1)$. Then we define,

$$\overline{[A, B]} = [a_2 + b_2 - 2e, a_2 + b_2]$$
$$\underline{[A, B]} = [a_1 + b_1, a_1 + b_1 + 2e]$$

.

Note that we obviously have that

$$L(A) + L(B) \subseteq \underline{[A, B]} \cup \overline{[A, B]}.$$

The next lemma gives an important tool in the proof of *Hall's Theorem.*

Lemma 2. *Suppose* A *and* B *are closed bounded subsets of* $\mathbb{R}$*, where* $l(A) = a$ *and* $l(B) = b$*. Let* C *and* D *be open subsets of* A *and* B *respectively. Define*

$$A_1 = \{a \in A : a < c \text{ for all } c \in C\},$$

$$A_2 = \{a \in A : a > c \text{ for all } c \in C\},$$
$$B_1 = \{b \in B : b < c \text{ for all } d \in D\},$$
$$B_2 = \{b \in B : b > c \text{ for all } d \in D\}.$$

If $l(C) \leq \min\{l(A_1), l(A_2)\}$ *and* $l(D) \leq \min\{l(B_1), l(B_2)\}$*, then for any* $\gamma \in \underline{[A, B]} \cup \overline{[A, B]}$*, one of the following holds:*

1. $\gamma \in \underline{[A, B_1]} \cup \overline{[A, B_1]}$,

2. $\gamma \in \underline{[A, B_2]} \cup \overline{[A, B_2]}$,

3. $\gamma \in \underline{[A_1, B]} \cup \overline{[A_1, B]}$,

4. $\gamma \in \underline{[A_2, B]} \cup \overline{[A_2, B]}$.

Proof. By symmetry of cases, assuming $a \leq b$ is sufficient. Let $A = [a_1, a_2]$ and $B = [b_1, b_2]$, then $e = a_2 - a_1 = a$. Therefore,

$$\begin{aligned} \underline{[A, B]} &= [a_1 + b_1, a_1 + b_1 + 2a] = [a_1 + b_1, 2a_2 - a_1 + b_1] \\ \overline{[A, B]} &= [a_2 + b_2 - 2a, a_2 + b_2] = [2a_1 - a_2 + b_2, a_2 + b_2] \end{aligned} \tag{5.2}$$

Consider the intervals $B_1 = [b_1, x]$ and $B_2 = [y, b_2]$. Depending on the lengths of the intervals A, B_1, and B_2, we encounter four distinct cases.

i. $l(A) \leq \min\{l(B_1), l(B_2)\}$.
In this case $e = a_2 - a_1 = a$, (5.2) and

$$\underline{[A, B_1]} = [a_1 + b_1, 2a_2 - a_1 + b_1] = \underline{[A, B]},$$

$$\overline{[A, B_2]} = [2a_1 - a_2 + b_2, a_2 + b_2] = \overline{[A, B]}.$$

Therefore, we have $\gamma \in \underline{[A, B_1]} \cup \overline{[A, B_1]}$ or $\gamma \in \underline{[A, B_2]} \cup \overline{[A, B_2]}$, which correspond to cases *1* and *2*.

ii. $l(B_2) < l(A) \leq l(B_1)$.
As in case (i) we have that $e = a_2 - a_1$, and therefore

$$\underline{[A, B_1]} = \underline{[A, B]} \tag{5.3}$$

Moreover, for $\overline{[A, B_1]}$, $e = a_2 - a_1$ and for $\overline{[A, B_2]}$, $e' = b_2 - y$, where $e' < e$, which gives

$$\begin{aligned} \overline{[A, B_1]} &= [a_2 + x - 2a, a_2 + x] = [2a_1 - a_2 + x, a_2 + x], \\ \overline{[A, B_2]} &= [a_2 + b_2 - 2e', a_2 + b_2] = [a_2 - b_2 + 2y, a_2 + b_2]. \end{aligned} \tag{5.4}$$

Due to the assumptions that $\max\{l(A_1), l(A_2)\} \geq l(C)$ and $\max\{l(B_1), l(B_2)\} \geq l(D)$,

$$y - x = l(D) \leq l(B_2) = b_2 - y,$$

so $2y \leq x + b_2$ and therefore $a_2 - b_2 + 2y \leq a_2 + x$, hence we conclude from (5.4) that $\overline{[A, B_1]}$ and $\overline{[A, B_2]}$ overlap. Furthermore, we have that $x \leq b_2$, hence $2a_1 - a_2 + x \leq 2a_1 - a_2 + b_2$. But then

$$\overline{[A, B]} = [2a_1 - a_2 + b_2, a_2 + b_2] \subseteq [2a_1 - a_2 + x, a_2 + b_2] = [A, B_1] \cup [A, B_2].$$

This and (5.3) again correspond to cases *1* or *2*.

iii. $l(B_1) < l(A) \leq l(B_2)$.
This problem is analogous to case (ii). We then have that

$$\overline{[A, B]} = \overline{[A, B_2]},$$

and

$$\underline{[A, B]} \subseteq \underline{[A, B_1]} \cup \underline{[A, B_2]}.$$

Thus, this situation corresponds to case *1* or *2*.

iv. $\max\{l(B_1), l(B_2)\} < l(A)$.
Since by assumption, $x - b_1 \geq y - x$, in this case, $l(A) > y - x$ and $3l(A) > (x - b_1) + (y - x) + (b_2 - y) = b_2 - b_1 = l(B)$. Therefore, $3a_2 - 3a_1 \geq b_2 - b_1$, yielding

$$2a_1 - a_2 + b_2 \leq 2a_2 - a_1 + b_1.$$

From (5.2) this implies that $\underline{[A,B]}$ and $\overline{[A,B]}$ overlap, and that therefore $A+B=\underline{[A,B]}\cup\overline{[A,B]}$. If we now look at the following four intervals:

$$\underline{[A,B_1]}=[a_1+b_1,a_1-b_1+2x],$$
$$\underline{[A,B_2]}=[a_1+y,a_1+2b_2-y],$$
$$\overline{[A,B_1]}=[a_2+2b_1-x,a_2+x],$$
$$\overline{[A,B_2]}=[a_2-b_2+2y,a_2+b_2],$$

Because $x-b_1\geq y-x$, and therefore $2x-b_1>y$, from which we see that

$$a_1-b_1+2x\geq a_1+y,$$

and it follows that $\underline{[A,B_1]}$ and $\underline{[A,B_2]}$ overlap. Moreover, since $b_2-y\geq y-x$, and therefore $x\geq 2y-b_2$, from which we see that

$$a_2+x\geq a_2-b_2+2y,$$

which implies that $\overline{[A,B_1]}$ and $\overline{[A,B_2]}$ overlap. Because $x-b_1\geq y-x$ and $b_2-y\geq y-x$, and given that $l(B)=(x-b_1)+(y-x)+(b_2-y)$, we have $3l(B_1)\geq l(B)$ or $3l(B_2)\geq l(B)$.

If $3l(B_1)\geq l(B)$, and because $l(A)\leq l(B)$ by assumption, we have that:

$$a_2-a_1\leq b_2-b_1 \text{ and therefore } a_2-a_1\leq 3(b_1-x),$$
$$a_2+2b_1-x\leq a_1-b_1+2x.$$

Thus we have that $\underline{[A,B_1]}$ and $\overline{[A,B_1]}$ overlap.

Analogously, if $3l(B_2)\geq l(B)$, then $\underline{[A,B_2]}$ and $\overline{[A,B_2]}$ overlap.

Hence we conclude that γ lies in one of those four intervals, which completes the proof of this lemma.

□

Now we will prove that under the same setup as in Theorem 1 and given **(C1)**, we have that

$$L(A)+L(B)=\underline{[A,B]}\cup\overline{[A,B]}=[a_1+b_1,a_1+b_1+2e]\cup[a_2+b_2-2e,a_2+b_2].$$

Proof. By construction, it is clear that $L(A)+L(B)\subseteq\underline{[A,B]}\cup\overline{[A,B]}$.

For the other direction, suppose that $\gamma\in\underline{[A,B]}\cup\overline{[A,B]}$. By Lemma 2, there exists a decreasing sequence $(A_1,B_1),(A_2,B_2),(A_3,B_3),\ldots$ where for all $i,j>0$, $A_{i+1}\subseteq A_i$, $B_{j+1}\subseteq B_j$, and there are $m,n,k,l>0$ such that $A_i=A_k^m$, $B_j=B_l^n$.

In addition, for every $i>0$, $\gamma\in\overline{[A_i,B_i]}$, or $\gamma\in\underline{[A_i,B_i]}$. There are two cases:

1. $\lim_{i\to\infty} l(A_i) = 0$ and $\lim_{j\to\infty} l(B_j) = 0$. Take in this case $\alpha = \lim \max(A_i)$, $i \to \infty$ and $\beta = \lim \max(B_j), j \to \infty$. Then $\alpha \in L(A)$, $\beta \in L(B)$, and $\gamma = \alpha + \beta$.

2. $\lim_{i\to\infty} l(A_i) = s \neq 0$ or $\lim_{j\to\infty} l(B_i) = t \neq 0$. Suppose initially that $t > s$. Then, there exist $i, j > 0$ such that $A_i = [a_1, a_2]$, $l(A_i) = s$, and $B_j = [b_1, b_2]$, $l(B_j) = t$. According to Lemma 2, the sequence $(A_1, B_1), (A_2, B_2), (A_3, B_3), \ldots$ could have been selected in such a way that B_j cannot be further partitioned, i.e., $B_j \subseteq L(B)$. Thus, $\gamma \in A_i + B_j = [a_1 + b_1, a_2 + b_2]$.

   (a) If $a_1 + b_1 \leq \gamma \leq a_1 + b_1 + t$, set $\alpha = a_1$ and $\beta = \gamma - a_1$.

   (b) If $a_1 + b_1 + t \leq \gamma \leq a_2 + b_2$, set $\alpha = a_2$ and $\beta = \gamma - a_2$.

   In both scenarios, $\alpha \in L(A)$ since it represents an endpoint of the interval A_i, and $\beta \in L(B)$. Thus, $\gamma = \alpha + \beta \in L(A) + L(B)$.

   Now, suppose $t = s > 0$. Then, there exist $i, j > 0$ such that $l(A_i) = t = l(B_j)$. According to Lemma 2.2.5, A_i and B_j could have been selected such that they are no longer divisible, i.e., $A_i \subseteq L(A)$ and $B_j \subseteq L(B)$. Therefore, $\gamma \in A_i + B_j \subseteq L(A) + L(B)$.

□

Now, also suppose Condition **(C2)** holds. If $\frac{1}{3} \leq \frac{a}{b} \leq 3$, then $b \leq 3a$ and $a \leq 3b$. Therefore, $(a_2 - a_1) + (b_2 - b_1) \leq a + b \leq 4a$ and $(a_2 - a_1) + (b_2 - b_1) \leq a + b \leq 4b$. Also, $(a_2 - a_1) + (b_2 - b_1) \leq 4e$. Thus, $a_2 + b_2 - 2e \leq a_1 + b_1 + 2e$.

We thus conclude, $L(A) + L(B) = [a_1 + b_1, a_1 + b_1 + 2e] \cup [a_2 + b_2 - 2e, a_2 + b_2] = [a_1 + b_1, a_2 + b_2] = A + B$, which proves Theorem 1. □

5.2 Proof of Hall's theorem

In this section, Hall's theorem, $F(4) + F(4) = \mathbb{R}$, will be demonstrated by introducing a set A for which the Cantor set $L(A)$ of A equals $F(4)$ and by applying Theorem 1.

5.2.1 $L(A) = F^0(4)$ with $A = [\frac{1}{2}(\sqrt{2} - 1), 2(\sqrt{2} - 1)]$

We will use the closed and bounded interval $A = [\frac{1}{2}(\sqrt{2} - 1), 2(\sqrt{2} - 1)]$ as the root of a tree for which it holds that $L(A) = F^0(4)$. To demonstrate that A is a suitable interval for this problem, we will first show that $F^0(4)$ is contained in A and then obtain the set $F^0(4)$ as a Cantor set of A.

Two infinite continued fractions which differ in any partial quotient will be unequal, and if $\alpha = [a_0; a_1, a_2, \ldots, a_{n-1}, a, \ldots]$ and $\beta = [a_0; a_1, a_2, \ldots, a_{n-1}, b, \ldots]$ where if n is even and $a < b$, then $\alpha < \beta$, while $\alpha > \beta$ if n is odd, see [8] for a proof. Consequently, it holds that $\max(F^0(m)) = [0; \overline{1, m}]$ and $\min(F^0(m)) = [0; \overline{m, 1}]$. From Section 3.4, we have that these numbers are quadratic irrationals, namely:

$$\max(F^0(m)) = \frac{\sqrt{m^2+4m} - m}{2}$$

$$\min(F^0(m)) = \frac{\sqrt{m^2+4m} - m}{2m}.$$

Specifically, we have

$$\max(F^0(4)) = 2(\sqrt{2} - 1)$$

and

$$\min(F^0(4)) = \frac{1}{2}(\sqrt{2} - 1).$$

Thus, it follows that $F^0(4) \subseteq \left[\frac{1}{2}(\sqrt{2}-1), 2(\sqrt{2}-1)\right] = [0.2071068\ldots, 0.8284271\ldots]$.

Now, we will construct a tree similar to the one depicted in Figure 5.1, with root $A = \left[\frac{1}{2}(\sqrt{2}-1), 2(\sqrt{2}-1)\right]$, where $L(A) = F^0(4)$. The tree appears as follows:

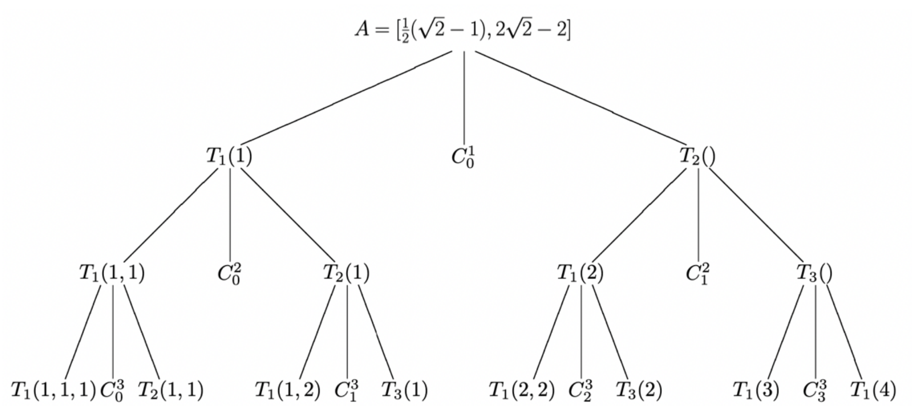

Figure 5.2: The first four layers of the three

Here,

1. $T_1(b_1, b_2, \ldots, b_k) = \{[0; b_1, \ldots, b_k, a_{k+1}, a_{k+2}, \ldots] : 1 \leq a_j \leq 4 \text{ for all } j > k\}$,

2. $T_2(b_1, b_2, \ldots, b_k) = \{[0; b_1, \ldots, b_k, a_{k+1}, a_{k+2}, \ldots] : 2 \leq a_{k+1} \leq 4, 1 \leq a_j \leq 4 \text{ for all } j > k+1\}$,

3. $T_3(b_1, b_2, \ldots, b_k) = \{[0; b_1, \ldots, b_k, a_{k+1}, a_{k+2}, \ldots] : 3 \leq a_{k+1} \leq 4, 1 \leq a_j \leq 4 \text{ for all } j > k+1\}$,

where $k \geq 0$, $b_1, b_2, \ldots, b_k$ are fixed integers $1 \leq b_i \leq 4$ and $[S]$ is defined to be the smallest closed interval containing S.

Defining the sets T_i for $i = 1, 2, 3$ in this way gives us, for example, the sets

$$T_1(1) = \{[0; 1, a_2, a_3, a_4, \ldots] : 1 \leq a_j \leq 4 \text{ for all } j > 1\} = \left[[0; 1, \overline{1, 4}], [0; \overline{1, 4}]\right],$$

and

$$T_3() = \{[0; a_1, a_2, a_3, a_4, \ldots] : 3 \leq a_1 \leq 4, 1 \leq a_j \leq 4 \text{ for all } j > 1\} = \left[[0; \overline{4, 1}], [0; 3, \overline{4, 1}]\right].$$

Observe that the sets T_i for $i = 1, 2, 3$ are subsets of $F^0(4)$, as all partial quotients in these sets belong to the set $\{1, 2, 3, 4\}$. The idea behind constructing the tree is essentially based on the fact that a partial quotient a_j with $j \geq 0$ of a regular continued fraction can take any integer value, and we partition these values based on whether the a_j is either equal to 1, equal to a number in the set $\{2, 3, 4\}$, or equal to a number greater than or equal to 5. This process repeats recursively, allowing us to partition the entire interval A. To be more precise, each of the sets T_i for $i = 1, 2, 3$ are divided into two closed subintervals separated by a Cantor Gap in the following way:

1. An interval of type T_1 can be divided into two intervals: one of type T_1 with $a_{k+1} = 1$, $1 \leq a_j \leq 4$ for $j > k + 2$, and another of type T_2 with $2 \leq a_{k+1} \leq 4$, $1 \leq a_j \leq 4$ for $j > k + 2$. The corresponding Cantor Gap then contains all other points of the interval of type T_1 that are not of these two types, namely the points with partial quotients $a_j \notin \{1, 2, 3, 4\}$ for $j > k$.

2. An interval of type T_2 is divided into an interval of type T_1 with $a_{k+1} = 2$, where $1 \leq a_j \leq 4$ for $j > k + 2$, and one of type T_3 with $3 \leq a_{k+1} \leq 4$, where $1 \leq a_j \leq 4$ for $j > k + 2$.

3. An interval of type T_3 is divided into one interval of type T_1 with $a_{k+1} = 3$, where $1 \leq a_j \leq 4$ for $j > k + 2$, and another interval of type T_1 with $a_{k+1} = 4$, where $1 \leq a_1 \leq 4$ for $j > k + 2$.

When dividing the sets A, T_1, T_2, and T_3 in this way, this recursively defines the complete binary tree where the first four layers are shown in Figure 5.2.1. For example, the first layer of the tree given in Figure 5.2.1 will consist of

$$\begin{aligned} T_1(1) &= [[0; 1, \overline{1, 4}], [0; \overline{1, 4}]] = [0.546718\ldots, 0.8284271\ldots], \\ T_2() &= [[0; \overline{4, 1}], [0; 2, \overline{4, 1}]] = [0.2071068\ldots, 0.453082\ldots], \\ C_0^1 &= A \setminus (T_1(1) \cup T_2()) = [0.453082\ldots, 0.546718\ldots], \end{aligned}$$

where root A is divided into the sets $T_1(1)$, $T_2()$ and the Cantor Gap C_0^1:

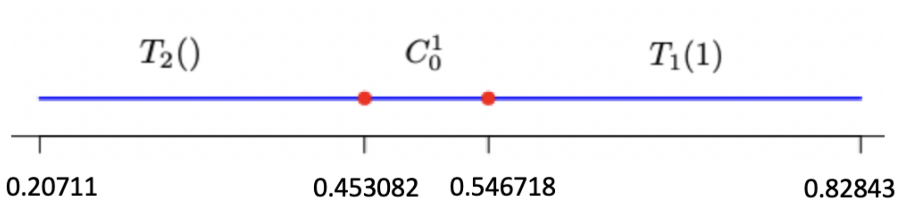

Figure 5.3: The interval A divided into $T_1(1), C_0^1$ and $T_2()$

The intervals T_i for $i = 1, 2, 3$ and C_k^n form a General Cantor Point Set containing all points where the regular continued fraction expansion only contain partial quotients less than or equal to 4. Consequently, we can conclude that $L(A) = F^0(4)$.

5.2.2 Applying Theorem 1 on $F(4) + F(4)$

In Section 5.2.1 we have demonstrated that $L(A) = F^0(4)$ where $A = [\frac{1}{2}(\sqrt{2} - 1), 2(\sqrt{2}-1)]$. If conditions **(C1)** and **(C2)** from Theorem 1 are met for $L(A) = F^0(4)$, then $F^0(4)+F^0(4) = A+A = [\sqrt{2}-1, 4(\sqrt{2}-1)] = [0.4142136\ldots, 1.656854\ldots]$, of length $3(\sqrt{2} - 1) = 1.242641\ldots$.

We will start by examining Condition **(C1)**, which states:

(C1) for every $n > 0$ and $k < 2^n$, one has
$c_k^n \le \min\{a_n^{2k}, a_n^{2k+1}\}$ and $d_k^n \le \min\{b_n^{2k}, b_n^{2k+1}\}$.

Recall that, for example, a_n^{2k} is defined as the length of the interval A_n^{2k}.
Let $\frac{p_{k-1}}{q_{k-1}}$ and $\frac{p_k}{q_k}$ be the last two convergents of $[0; b_1, \ldots, b_k]$, implying that $\frac{p_k}{q_k} = [0; b_1, \ldots, b_k]$, as discussed in Section 3.2. Additionally, define $\xi = \frac{1}{2}(\sqrt{2}+1) = [1; \overline{4, 1}]$. Then ξ satisfies the following relations:

$$4\xi = \left[4; \overline{1, 4}\right] = 2(\sqrt{2} + 1) \tag{5.5}$$

$$\frac{1}{\xi} = \left[0; \overline{1, 4}\right] = 4\xi - 4. \tag{5.6}$$

Let $b_1, b_2, \ldots, b_k \in \{1, 2, 3, 4\}$, then there are three cases based on how the sets T_i for $i = 1, 2, 3$ were divided into two closed subintervals separated by a Cantor Gap in Section 5.2.1.

1. For case 1, denote by t_1, c, t_2 respectively the lengths of the intervals $T_1(b_1, \ldots, b_k, 1)$, $T_2(b_1, \ldots, b_k)$ and the corresponding Cantor Gap C. It

follows that the extremes of $T_1(b_1, b_2, \ldots, b_k)$ are,

$$\mu = \left[0; b_1, b_2, \ldots, b_k, \overline{1, 4}\right] = \frac{\xi p_k + p_{k-1}}{\xi q_k + q_{k-1}}$$
$$\nu = \left[0; b_1, b_2, \ldots, b_k, \overline{4, 1}\right] = \frac{4\xi p_k + p_{k-1}}{4\xi q_k + q_{k-1}}$$

where if k is even, $\mu = \max(T_1)$ and $\nu = \min(T_1)$ and if k is odd, $\mu = \min(T_1)$ and $\nu = \max(T_1)$. Then by Equation (3.7), it follows that

$$t_1 = \frac{1 + \frac{1}{\xi} - \xi}{q_k(1 + \frac{1}{\xi} + \frac{q_{k-1}}{q_k})(\xi + \frac{q_{k-1}}{q_k})}$$

Similarly, upon computation, we obtained values for c and t_2, yielding:

$$c = \frac{2 + \frac{1}{4\xi} - 1 - \frac{1}{\xi}}{q_k(2 + \frac{1}{4\xi} + \frac{q_{k-1}}{q_k})(1 + \xi + \frac{q_{k-1}}{q_k})}$$
$$t_2 = \frac{4\xi - 2 - \frac{1}{4\xi}}{q_k(4\xi + \frac{q_{k-1}}{q_k})(2 + \frac{1}{\xi} + \frac{q_{k-1}}{q_k})}$$

Since the relation obtained in Equation (5.6) holds and since $0 \leq q_{k-1} < q_k$ and thus $0 \leq \frac{q_{k-1}}{q_k} < 1$, we have that:

$$\frac{c}{t_1} = \frac{(4 - 3\xi)(\xi + \frac{q_{k-1}}{q_k})}{(3\xi - 3)(1 + \xi + \frac{q_{k-1}}{q_k})}$$
$$\frac{c}{t_2} = \frac{(4 - 3\xi)(4\xi + \frac{q_{k-1}}{q_k})}{(3\xi - 1)(4\xi - 3 + \frac{q_{k-1}}{q_k})}.$$

Observe that $\frac{c}{t_1}$ as a function of $\frac{q_{k-1}}{q_k}$ takes its maximum at $\frac{q_{k-1}}{q_k} = 1$, while $\frac{c}{t_2}$ as a function of $\frac{q_{k-1}}{q_k}$ takes its maximum at $\frac{q_{k-1}}{q_k} = 0$. Consequently,

$$\frac{c}{t_1} \leq \frac{(4 - 3\xi)(\xi + 1)}{(3\xi - 3)(2 + \xi)} < 1$$
$$\frac{c}{t_2} \leq \frac{(4 - 3\xi)(4\xi - 1)}{(3\xi - 1)(4\xi - 3)} < 1.$$

Thus, we find that $c < t_1$ and $c < t_2$ and thus Condition **(C1)** is satisfied for first type subdivisions.

Cases 2 and 3 are proven in a similar manner to case 1; therefore, only the results are presented.

2. Denote by t_1, c, t_2 the lengths of the intervals $T_1(b_1, \ldots, b_k, 2)$, $T_3(b_1, \ldots, b_k)$

and the corresponding Cantor Gap C, respectively. Then we have that

$$t_1 = \frac{2+\frac{1}{\xi}-2-\frac{1}{4\xi}}{q_k(2+\frac{1}{\xi}+\frac{q_{k-1}}{q_k})(2+\frac{1}{4\xi}+\frac{q_{k-1}}{q_k})},$$
$$c = \frac{3+\frac{1}{4\xi}-2-\frac{1}{\xi}}{(2+\frac{1}{\xi}+\frac{q_{k-1}}{q_k})(3+\frac{1}{4\xi}+\frac{q_{k-1}}{q_k})},$$
$$t_2 = \frac{4\xi-3-\frac{1}{4\xi}}{q_k(3+\frac{1}{4\xi}+\frac{q_{k-1}}{q_k})(4\xi+\frac{q_{k-1}}{q_k})},$$

satisfying $c < t_1$ and $c < t_2$ and thus Condition **(C1)** is satisfied for this case.

3. Denote again by t_1, c, t_2 the lengths of the intervals $T_1(b_1, \ldots, b_k, 3)$, C, and $T_1(b_1, \ldots, b_k, 3)$, respectively. Then

$$t_1 = \frac{3+\frac{1}{\xi}-3-\frac{1}{4\xi}}{q_k(3+\frac{1}{\xi}+\frac{q_{k-1}}{q_k})(3+\frac{1}{4\xi}+\frac{q_{k-1}}{q_k})},$$
$$c = \frac{4+\frac{1}{4\xi}-3-\frac{1}{\xi}}{(3+\frac{1}{\xi}+\frac{q_{k-1}}{q_k})(4+\frac{1}{4\xi}+\frac{q_{k-1}}{q_k})},$$
$$t_2 = \frac{4\xi-4-\frac{1}{4\xi}}{q_k(4+\frac{1}{4\xi}+\frac{q_{k-1}}{q_k})(4\xi+\frac{q_{k-1}}{q_k})},$$

Again, Condition **(C1)** is satisfied for this subdivision.

We thus conclude that Condition **(C1)** is satisfied for all the subdivisions of A.

Furthermore, Condition **(C2)**, which states:

$$\textbf{(C2)} \quad \frac{1}{3} \leq \frac{a}{b} \leq 3,$$

is also satisfied since $\frac{a}{b} = \frac{a}{a}$ in our case. This means that Theorem 1 can be applied. Hence $L(A) + L(A) = A + A$ and thus $F^0(4) + F^0(4) = [\sqrt{2}-1, 4\sqrt{2}-4]$. This implies that every element within $[\frac{1}{2}, \frac{3}{2}]$ can be written as the sum of two elements of $F^0(4)$. Consequently, it follows that every element of $\mathbb{R}$ can be written as the sum of two elements of $F^n(4)$, and thus we conclude that $F(4) + F(4) = \mathbb{R}$, which completes the proof of Hall's theorem. □

Using similar techniques, analogous results were obtained by Cusick [2], Divis [4], Astels [1], and Hlavka [9]. In 1971, T. Cusick proved in [2] that every real number is representable as a sum of two real numbers, each of which has a fractional part whose continued fraction expansion contains no partial quotient less than 2. In 1973, B. Divis showed in [4] that

$$F(3) + F(3) + F(3) = \mathbb{R}$$

and that

$$F(3) + F(3) \neq \mathbb{R}.$$

Moreover, in 1975, J. Hlavka generalized Hall's result to the case of different sets $F(m)$ and $F(n)$ in [9]. He proved that

$$F(m) + F(n) = \mathbb{R}$$

holds for (m, n) equal to $(2, 7)$ or $(3, 4)$, but does not hold for (m, n) equal to $(2, 4)$. In 1999, S. Astels [1] examined the difference of two sets $F(m)$ and $F(n)$. Note that if A is a set of real numbers, we define $-A$ by

$$-A = \{-a \mid a \in A\}$$

and denote $A + (-B)$ by $A - B$. He showed that

$$F(m) - F(n) = \mathbb{R}$$

holds if (m, n) equals $(2, 5)$, $(3, 3)$, or $(3, 4)$.

Moreover, in 2013, N. Oswald and J. Steuding proved in [10] that every $x \in \mathbb{R}$ can be written as the sum of an integer and at most $\lfloor \frac{b+1}{2} \rfloor$ NICF (see Chapter 3.5) each having partial quotients of at least b.

Chapter 6

Hall-type results for Lüroth Series

In this chapter, we will demonstrate how Hall's result can be applied to Lüroth series in a similar way as we did in Chapter 5 with regular continued fractions. Define for $N \in \mathbb{Z}$:

$$F_L^n(m) = \{\langle a_0; a_1, a_2, a_3, \ldots \rangle \in \mathbb{R} : a_0 = n, \forall i \geq 1 : 2 \leq a_i \leq m\}$$

and $F_L(m) = \bigcup_{n \in \mathbb{Z}} F_L^n(m)$. We will show that the following equation holds:

$$F_L(4) + F_L(4) = \mathbb{R}. \tag{6.1}$$

Note that for the Lüroth series, the digits a_i for $i \geq 1$ are greater or equal to 2, whereas for the RCF, the partial quotients are greater or equal to 1. This means we only need 3 digits for $F_L(4)$ in (6.1) instead of 4 partial quotients for $F(4)$ in (5.1).

As can be seen in Figure 4.1, for the Lüroth map L, it holds that if x and y have the same first digit and

$$\text{if } x < y \text{ then } L(x) < L(y). \tag{6.2}$$

We have chosen to examine the standard Lüroth series instead of the alternating Lüroth series due to the fact that this equation holds, causing the digits of the Lüroth series to behave differently from the partial quotients of the RCF. Moreover, the alternating Lüroth series is essentially the linear version of the RCF, as shown in Figure 4.2, and therefore we expect Hall-type results for the alternating Lüroth series would be more similar to Hall-type results for the RCF. Nevertheless, it would be interesting to look into this in further research.

As a consequence of Equation (6.2), it holds that $\max(F_L^0(m)) = \langle 0; \overline{2} \rangle$ and $\min(F_L^0(m)) = \langle 0; \overline{m} \rangle$. Therefore, we have that $\max(F_L^0(4)) = \langle 0; \overline{2} \rangle = 1$ and

$\min(F_L^0(4)) = \langle 0; \overline{4} \rangle = 0.\overline{27}$. It follows that $F_L^0(4) \subseteq [\langle 0; \overline{4} \rangle, \langle 0; \overline{2} \rangle] = [0.\overline{27}, 1]$. Thus, in this case, we will look at the set $A = [0.\overline{27}, 1]$ in order to apply Theorem 1 again.

We will now construct a tree similar to the one depicted in Figure 5.1, with root $A = [0.\overline{27}, 1]$, where $L(A) = F_L^0(4)$. Let us first define the following sets:

1. $T_1(b_1, \ldots, b_k) = \{\langle 0; b_1, \ldots, b_k, a_{k+1}, \ldots \rangle : a_{k+1} = 2, 2 \leq a_j \leq 4 \text{ for all } j > k+1\}$,

2. $T_2(b_1, \ldots, b_k) = \{\langle 0; b_1, \ldots, b_k, a_{k+1}, \ldots \rangle : 3 \leq a_{k+1} \leq 4, 2 \leq a_j \leq 4 \text{ for all } j > k+1\}$,

where $k \geq 0$, $b_1, b_2, \ldots, b_k$ are fixed integers $2 \leq b_i \leq 4$.

Note that because Equation (6.2) holds, the union of the disjoint intervals $T_1() = [\frac{1}{2}, 1]$ and $T_2() = [0.\overline{27}, \frac{1}{2}]$ completely covers the set $A = [0.\overline{27}, 1]$. This implies that the first subdivision of the tree, where we divide the root A into $T_1()$ and $T_2()$, will not result in a corresponding Cantor Gap C. The first two layers of the tree then appear as follows:

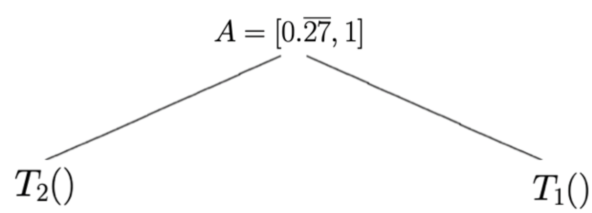

Figure 6.1: The first two layers of the tree

However, in the third layer of the tree, the first two Cantor Gaps appear when dividing the intervals $T_1()$ and $T_2()$ as follows:

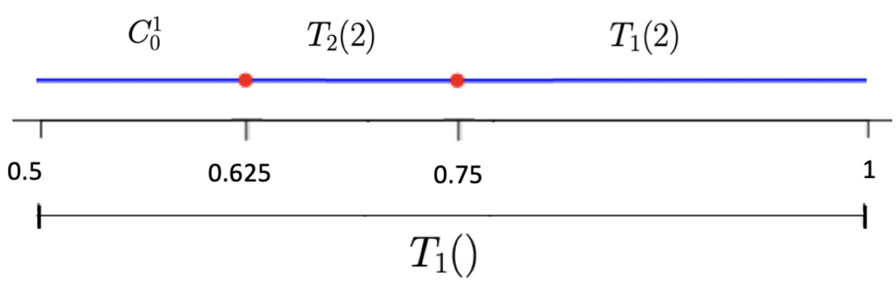

Figure 6.2: The interval $T_1()$ divided into $T_1(2)$, $T_2(2)$ and C_0^1

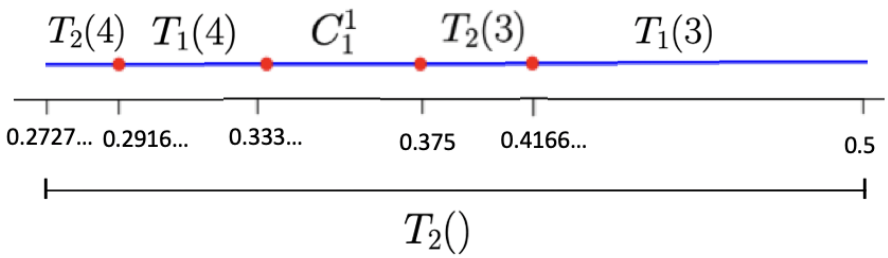

Figure 6.3: The interval $T_2()$ divided into $T_1(3)$, $T_2(3)$, $T_1(4)$, $T_2(4)$ and C_1^1

These subdivisions, together with Figure 6.1 lead to the first three layers of the tree:

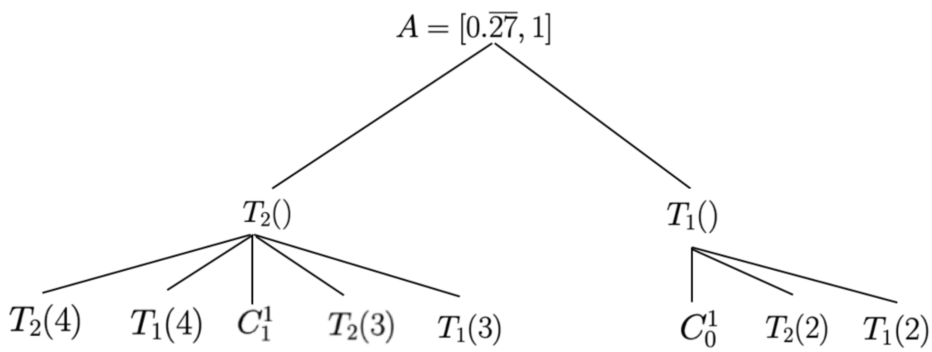

Figure 6.4: The first three layers of the tree

To generalize the process of building the tree, each of the sets T_i for $i = 1, 2$ are divided in the following way:

1. An interval of type T_1 is divided into three intervals: one of type T_1 with $b_k = 2$, one of type T_2 with $b_k = 2$ and the corresponding Cantor Gap C.

2. An interval of type T_2 is divided into 5 intervals, namely one interval of type T_1 with $b_k = 3$, one interval of type T_1 with $b_k = 4$, one of type T_2 with $b_k = 3$, one of type T_2 with $b_k = 4$ and the corresponding Cantor Gap C.

When dividing the sets T_i for $i = 1, 2$ in this way, this recursively defines the complete binary tree with root $A = [0.\overline{27}, 1]$, where the first three layers are shown in Figure 6.4. The intervals T_i for $i = 1, 2$ and C_k^m form a General Cantor Point Set containing all points where the Lüroth series only contain digits less than or equal to 4. Consequently, we can conclude that $L(A) = F_L^0(4)$. Moreover, if conditions **(C1)** and **(C2)** from Theorem 1 are met for $L(A) = F_L^0(4)$, then Equation (6.1) follows.

Condition **(C2)**, which states:

$$\textbf{(C2)} \quad \frac{1}{3} \leq \frac{a}{b} \leq 3,$$

is satisfied since $\frac{a}{b} = \frac{a}{a}$ in this case. The only thing left is to show condition **(C1)** is satisfied. We will demonstrate that **(C1)** holds for the third layer of the tree and for the subdivision of $T_1(2)$ and our *conjecture* is that condition **(C1)** holds in general. However, we intend to work on this conjecture shortly.

Let us now check condition **(C1)** for the two Cantor Gaps in the third layer of the tree given in Figure 6:

1. $l(C_0^1) = \frac{5}{8} - \frac{1}{2} = \frac{1}{8}$, $l(T_2(2)) = \frac{3}{4} - \frac{5}{8} = \frac{3}{8}$ and $l(T_1(2)) = 1 - \frac{3}{4} = \frac{1}{4}$. Thus it holds that
$$l(C_0^1) = \tfrac{1}{8} < \min\{\tfrac{3}{8}, \tfrac{1}{4}\} = \tfrac{1}{4}.$$

2. $l(C_1^1) = \frac{3}{8} - \frac{1}{3} = \frac{1}{24}$, $l(T_1(3) \cup T_2(3)) = \frac{1}{2} - \frac{3}{8} = \frac{1}{8}$ and $l(T_1(4) \cup T_2(4)) = \frac{1}{3} - 0.\overline{27} = 0.\overline{06}$. Thus it holds that
$$l(C_1^1) = \tfrac{1}{24} < \min\{\tfrac{1}{8}, 0.\overline{06}\} = 0.\overline{06}.$$

We conclude that condition **(C1)** is satisfied for this layer of the tree. To convince the reader that our conjecture holds, we also examine one of the intervals at the next level of the tree, specifically $T_1(2)$. According to the earlier description of the tree construction, the segment of the fourth layer containing $T_1(2)$ will be:

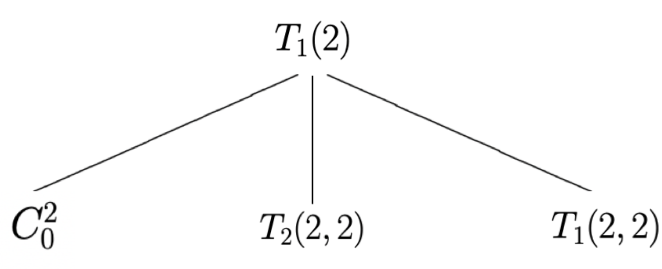

Figure 6.5: The segment of the fourth layer containing $T_1(2)$

Here, the interval $T_1(2)$ is divided as follows:

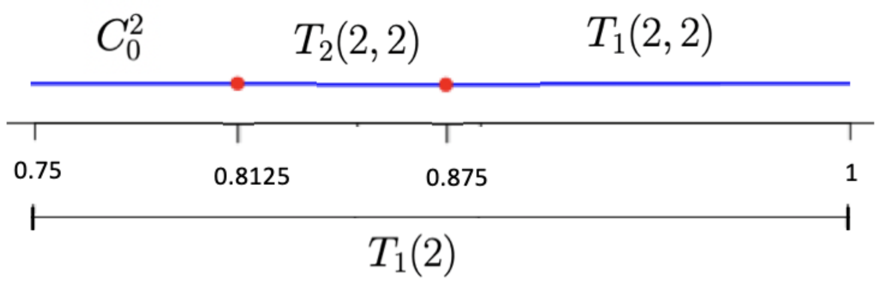

Figure 6.6: The interval $T_1(2)$ divided into $T_1(2,2)$, $T_2(2,2)$ and C_0^2

From Figure 6.6, we see that $l(C_0^2) = \frac{13}{16} - \frac{3}{4} = \frac{1}{16}$, $l(T_2(2,2)) = \frac{7}{8} - \frac{13}{16} = \frac{1}{16}$ and $l(T_1(2,2)) = 1 - \frac{7}{8} = \frac{1}{8}$. Thus it holds that

$$l(C_0^2) = \tfrac{1}{16} \leq \min\{\tfrac{1}{16}, \tfrac{1}{8}\} = \tfrac{1}{16}$$

and again condition **(C1)** is satisfied for this specific part of the tree. Given that the rest of the tree is constructed in the same way as depicted in Figures 6.4 and 6.5, this example broadly suggests that condition **(C1)** is valid for any interval of type T_1. We expect that the same applies to the case where we have an interval of type T_2, and therefore, we expect our conjecture to hold. This still needs to be formally written down in a general form, which we intend to do shortly. However, if this proves to be the case, we can conclude that Theorem 1 applies, which means Equation (6.1) is proven.

Chapter 7

Conclusion

This thesis provided an introduction of Cantor sets, examined various number expansions, such as r-ary expansions, regular continued fractions, and Lüroth series and analyzed how the summation of Cantor sets can be related to these number expansions. For this, a foundational understanding of constructing Cantor sets and properties related to these number expansions were needed.

Initially, we demonstrated through ternary expansions and a graphical proof how the sum $C + C$, where C is the Cantor Middle Third set, equals the interval $[0, 2]$. This fundamental result served as a key motivation for Marshall Hall to delve deeper into the summation of Cantor sets. In 1947, M. Hall proved in [6] that

$$F(4) + F(4) = \mathbb{R},$$

where $F(m)$ is defined as the set of irrational numbers with partial quotients $a_i \leq m$ for $i \geq 1$ and $a_0 \in \mathbb{Z}$. For the proof of this result, several lemmas and theorems were introduced and proved, and a tree was constructed that demonstrates that $F(4)$ is the Cantor set of $A = [\frac{1}{2}(\sqrt{2} - 1), 2(\sqrt{2} - 1)]$.

Moreover, by defining

$$F_L(m) = \{\langle a_0; a_1, a_2, a_3, \ldots \rangle \in \mathbb{R} : a_0 \in \mathbb{Z}, \forall i \geq 1 : 2 \leq a_i \leq m\},$$

and constructing a tree to verify that $F_L(4)$ forms the Cantor set of $A = [0.\overline{27}, 1]$, we can apply the same lemmas and theorems. This establishes the following Hall-type result for Lüroth series:

$$F_L(4) + F_L(4) = \mathbb{R}.$$

Not all aspects of the proof for this equation have been finalized. However, we are confident in our conjecture that the equation holds, and we intend to confirm this shortly. Additionally, further studies could examine whether Hall's result and similar results by Cusick [2], Divis [4], Astels [1], and Hlavka [9] can be applied to other number expansions, such as r-ary expansions or the alternating Lüroth expansion.

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