Stochastic FitzHugh–Nagumo Equations

U. Böhm

MSc Thesis Applied Mathematics

Stochastic FitzHugh–Nagumo Equations

by

Udo Böhm

Delft University of Technology

To be defended publicly on Friday, 16 June 2023 at 15:45 PM

Student number:	5412013	
Thesis committee:	Prof. dr. M. C. Veraar	TU Delft, Supervisor
	dr. A. Agresti	ISTA
	dr. J. Bierkens	TU Delft

An electronic version of this thesis is available at
https://repository.tudelft.nlhttps://repository.tudelft.nl

Abstract

The stochastic FitzHugh–Nagumo equations are a system of stochastic partial differential equations that describes the propagation of action potentials along nerve axons. In the present work we obtain well-posedness and regularisation results for the FitzHugh–Nagumo equations with domain $\mathbb{R}^d$. We begin by considering the weak critical variational setting, where we prove global well-posedness for the case $d = 1$. We subsequently consider the strong variational setting, which allows us to extend our well-posedness results to $d \leq 4$. To prove well-posedness and regularisation for arbitrary d, we consider the FitzHugh–Nagumo equations in the $L^p(L^q)$-setting. Building on earlier results for reaction-diffusion equations, we first prove well-posedness on the d-dimensional flat torus and use bootstrapping techniques to prove instantaneous regularisation of the solution. We subsequently extend the theory for reaction-diffusion equations to the unbounded domain $\mathbb{R}^d$ to finally prove well-posedness and regularisation for the FitzHugh–Nagumo equations on $\mathbb{R}^d$.

Contents

1 Introduction

The FitzHugh–Nagumo equations are among the fundamental models in theoretical neuroscience. Their origins date back to a series of experiments conducted by Hodgkin and Huxley in the 1950s [HH52b]-[HH52c] in which they investigated the propagation of action potentials along the squid giant nerve fibre. Based on their experimental results, [HH52a] developed a mathematical model that describes the total membrane current density as a system of four ordinary differential equations. The first of these equations describes the total membrane current density I as the sum of the capacity current density $C_M \frac{\mathrm{d}V_M}{\mathrm{d}t}$ and the sum of ionic sodium, potassium, and leakage currents $I_i = g_i(V - V_K)$, $i \in \{Na, K, l\}$,

$$I = C_M \frac{\mathrm{d}V_M}{\mathrm{d}t} + \bar{g}_K n^4 (V_M - V_K) + \bar{g}_{Na} m^3 h (V_M - V_{Na}) + \bar{g}_l (V_M - V_l),$$

where $\bar{g}_i$, $i \in \{Na, K, l\}$, are constants. The remaining equations relate the ionic conductances to the potassium and sodium activations n, m, and the sodium inactivation h

$$\begin{aligned}
\frac{\mathrm{d}n}{\mathrm{d}t} &= \alpha_n(V_M)(1-n) - \beta_n(V_M)n, \\
\frac{\mathrm{d}m}{\mathrm{d}t} &= \alpha_m(V_M)(1-m) - \beta_m(V_M)m, \\
\frac{\mathrm{d}h}{\mathrm{d}t} &= \alpha_h(V_M)(1-h) - \beta_h(V_M)h.
\end{aligned}$$

FitzHugh [Fit61] suggested a simplification of the Hodgkin–Huxley model where the fast variables (V, m) and the slow variables (n, h) are each combined in a single variable u_1 and u_2, respectively. Nagumo, Arimoto and Yoshizawa [NAY62] further introduced a dependence on the space variable x that represents the distance along the nerve axon. Later elaborations introduced a stochastic input current that is typically modelled as white noise, resulting in the stochastic FitzHugh–Nagumo equations

$$\begin{cases}
\mathrm{d}u_1(t,x) = & (\nu \Delta u_1(t,x) + f(u_1(t,x)) - u_2(t,x))\,\mathrm{d}t \\
& + \sum_{k \geq 1} g_{1,k}(t,x)\mathrm{d}W_1(t) \\
\mathrm{d}u_2(t,x) = & \epsilon(u_1(t,x) - \gamma u_2(t,x))\,\mathrm{d}t,
\end{cases} \tag{1.1}$$

where ν, ϵ, γ are positive constants, f is a third-degree polynomial with positive leading coefficient, and W_1 is a U-cylindrical Brownian motion.

The FitzHugh–Nagumo equations were originally developed as a model for the propagation of action potentials along a long nerve axon, which can be approximately described as a long cable. However, in recent years there has been increasing interest in applying the FitzHugh–Nagumo model to other electrophysiological problems such as modelling the cardiac electric field (see [FS15] and [BCP09]), which require higher-dimensional models. Existing work on well-posedness of (1.1) has focussed on the variational setting (e.g., [BCP09]) and often only considers a single space dimension (e.g., [EGK21], and [HH20]). In the present work we will obtain global well-posedness and regularisation results for (1.1) for sufficiently smooth initial data. Specifically, under conditions on the parameters (p, q) that will be detailed in later sections, $\eta > 0$ small, and initial data $u_0 = (u_{1,0}, u_{2,0}) \in B^{d/q-1}_{q,p} \oplus H^{1-\eta,q}$ a.s., Equation (1.1) has a global solution $u = (u_1, u_2)$ such that

$$u \in L^p_{\mathrm{loc}}([0,\infty); H^{1,q} \oplus H^{1-\eta,q}) \cap C\left([0,\infty); B^{d/q-1}_{q,p} \oplus H^{1-\eta,q}\right),$$

and u regularises instantaneously in time and, in addition, u_1 regularises instantaneously in space,

$$u \in H^{\theta,r}_{\mathrm{loc}}\left(0,\infty; H^{1-2\theta,q} \oplus H^{2-\delta-\eta,q}\right) \quad a.s. \text{ for all } \theta \in [0,1/2), r \in (2,\infty).$$

Moreover, we obtain a priori estimates for u on any interval $[s,T], s > 0$, that is, we show that there exists a constant $N_0 > 0$ such that

$$\begin{aligned}\mathbb{E} \sup_{t\in[s,T]} \mathbb{1}_\Gamma(\|u_1(t)\|^q_{L^q(\mathbb{R}^d)} + \|u_2(t)\|^q_{L^q(\mathbb{R}^d)}) + \mathbb{E}\int_s^T \int_{\mathbb{R}^d} \mathbb{1}_\Gamma |u_1|^{q-2} |\nabla u_1|^2 \,\mathrm{d}x\mathrm{d}r \\ \leq N_0 \left(1 + \mathbb{E}\mathbb{1}_\Gamma(\|u_1(s)\|^q_{L^q(\mathbb{R}^d)} + \|u_2(t)\|^q_{L^q(\mathbb{R}^d)})\right),\end{aligned}$$

where the restriction to Γ is a technical condition required to bound the L^q-norm of u at time s.

Our approach to proving (global) well-posedness of the FitzHugh–Nagumo equations is based on the well-posedness results for reaction-diffusion equations developed in [AV22c], [AV23b], and [AV23a].

- In Section 4.1 we consider a general form of (1.1) in the weak variational setting with domain $\mathbb{R}$ as in [EGK21] and [HH20].
- In Section 4.2 we extend our results to the strong variational setting with domain $\mathbb{R}^d$, $d \leq 4$.
- In Section 5 we first consider the FitzHugh–Nagumo equations on the bounded domain $\mathbb{T}^d$ to obtain well-posedness results for higher dimensions and initial data with high regularity.
- In Section 6 we extend some of the results for reaction-diffusion equations on the bounded domain $\mathbb{T}^d$ proved in [AV23b] to the unbounded domain $\mathbb{R}^d$, which will, in particular, provide insights on the achievable regularity of solutions to the FitzHugh–Nagumo equations on $\mathbb{R}^d$.
- In Section 7 we build on the results of Sections 5 and 6 to prove global well-posedness of the FitzHugh–Nagumo equations on $\mathbb{R}^d$.

In the next section we will outline the setting and notation that will be used throughout the present work.

2 Notation, Setting and Preliminaries

We write $a \lesssim b$ for $a, b \in \mathbb{R}$ if there exists a constant $C > 0$ such that $a \leq Cb$. If there exists a constant C depending on parameters $\alpha, \beta, \ldots$ such that $a \leq Cb$, we will use the notation $a \leq C_{\alpha,\beta,\ldots} b$. For $\xi \in \mathbb{R}^d$ we write $|\xi| = \left(\sum_{j=1}^d \xi_j^2\right)^{1/2}$. The unit sphere and unit ball in $\mathbb{R}^d$ are denoted by S_1 and B_1, respectively. Moreover, we let $(a, a) = \emptyset$.

For $p \in (1, \infty)$ and $\kappa \in [0, p-1)$, $I = (a, b) \subseteq \mathbb{R}$ and X a Banach space, $w_\kappa(t)^s = |t - s|^\kappa$ is the shifted weight function with exponent κ and we let $w_\kappa(t) = w_\kappa(t)^0$, and $L^p(a, b, w_\kappa; X)$ denotes the set of all strongly measurable maps $f : I \to X$ such that the weighted L^p-norm

$$\|f\|_{L^p(a,b,w_\kappa;X)} = \left(\int_a^b w_\kappa(t)\|f(t)\|_X^p \mathrm{d}t\right)^{1/p} < \infty.$$

If, in addition, the derivative (in the distributional sense) $f' \in L^p(a, b, w_\kappa; X)$, then we write $f \in W^{1,p}(a, b, w_\kappa; X)$ and we set

$$\|f\|_{W^{1,p}(a,b,w_\kappa;X)} = \|f\|_{L^p(a,b,w_\kappa;X)} + \|f'\|_{L^p(a,b,w_\kappa;X)}.$$

We further let $L_{\mathrm{loc}}(a, b, w_\kappa, X) = \{f : (a, b) \to X :$ f is strongly measurable and for all compact $K \subseteq (a, b), \int_K w_\kappa(t)\|f(t)\|_X^p \mathrm{d}t < \infty\}$. We denote the real and complex interpolation functors by $(\cdot, \cdot)_{\kappa,p}$ and $[\cdot, \cdot]_\lambda$, respectively. We will also use the shorthand notation $X_{\kappa,p}^{\mathrm{Tr}} = (X_0, X_1)_{1-(1+\kappa)/p,p}$ and $X_p^{\mathrm{Tr}} = (X_0, X_1)_{1-1/p,p}$. The Bessel potential spaces $H^{s,q}(\mathbb{T}^d)$, $H^{s,q}(\mathbb{R}^d)$ and the Besov spaces $B_{q,p}^s(\mathbb{T}^d)$, $B_{q,p}^s(\mathbb{R}^d)$ can be defined by complex and real interpolation (see [Saw18, Section 6.6] and [Tri78]). When no confusion about the domain can arise, we will write L^q, $H^{s,q}$, $B_{q,p}^s$ instead of $L^q(\mathcal{D})$, $H^{s,q}(\mathcal{D})$, $B_{q,p}^s(\mathcal{D})$ for $\mathcal{D} \in \{\mathbb{T}^d, \mathbb{R}^d\}$. Correspondingly, we will write $L^q(\ell^2)$, $H^{s,q}(\ell^2)$, $B_{q,p}^s(\ell^2)$ instead of $L^q(\mathcal{D}; \ell^2)$, $H^{s,q}(\mathcal{D}; \ell^2)$, $B_{q,p}^s(\mathcal{D}; \ell^2)$ for $\mathcal{D} \in \{\mathbb{T}^d, \mathbb{R}^d\}$. We denote the Schwartz space by $\mathcal{S}$. Further notation will be introduced in subsequent sections as needed.

We fix a filtered probability space $(\Omega\ \mathcal{A}, \mathcal{F}, \mathbb{P})$ with σ-algebra $\mathcal{A}$, filtration $(\mathcal{F}_t)_{t\geq 0}$, and probability measure $\mathbb{P}$. We denote the progressive σ-algebra on the filtered probability space by $\mathcal{P}$.

We will regularly use the following embedding results on the domains $\mathbb{T}^d$, $\mathbb{R}^d$ (see [Tri78, Theorem 4.6.1]). Let $-\infty < s < \infty$, $1 < p_1 \leq p_2 \leq \infty$, $q \in (1, \infty)$, $\epsilon > 0$, then

$$B_{q,\infty}^{s+\epsilon} \subset B_{q,1}^s \subset B_{q,p_1}^s \subset B_{q,p_2}^s \subset B_{q,\infty}^s,$$

if $1 \leq p \leq \infty$, $1 < q_1 \leq q_2 < \infty$, $-\infty < t \leq s < \infty$, and $s - \frac{d}{q_1} \geq t - \frac{d}{q_2}$, then

$$B_{q_1,p}^s \subset B_{q_2,p}^t \quad \text{and } H^{s,q_1} \subset H^{t,q_2},$$

if $1 < p < q < \infty$, $-\infty < t \leq s < \infty$, and $s - \frac{d}{p} \geq t - \frac{d}{q}$, then

$$H^{s,p} \subset B_{q,p}^s \quad \text{and } B_{p,q}^s \subset H^{t,q},$$

and if $1 < q < \infty$, $1 \leq p \leq \infty$, $t \geq 0$, and $s - \frac{d}{q} > t$, then

$$B^s_{q,p} \subset C^t \quad \text{and } H^{s,q} \subset C^t.$$

Moreover, we will regularly use the interpolation estimates (see, e.g., [BM18])

$$\|f\|_{H^{s,q}} \lesssim \|f\|^{\lambda}_{H^{s_1,q_1}} \|f\|^{1-\lambda}_{H^{s_2,q_2}},$$

where $-\infty < s < \infty$ and $1 \leq q, q_1, q_2 \leq \infty$ and $\lambda \in (0,1)$ are such that

$$s = \lambda s_1 + (1-\lambda)s_2 \quad \text{and } \frac{1}{q} = \frac{\lambda}{q_1} + \frac{1-\lambda}{q_2}.$$

3 Well-Posedness in the Variational Setting

We summarise some results from [AV22c] that will be used in our proof of well-posedness in the variational setting. We consider quasi-linear SPDEs of the form

$$\begin{cases} \mathrm{d}u(t) + A(t,u(t))\,\mathrm{d}t & = B(t,u(t))\,\mathrm{d}W(t) \\ u(0) & = u_0, \end{cases} \tag{3.1}$$

where W is a U-cylindrical Brownian motion

Below, for Hilbert spaces U, H, we denote the set of bounded operators from U to H by $\mathcal{L}(U,H)$ and we denote the set of Hilbert-Schmidt operators from U to H by $\mathcal{L}_2(U,H)$. Moreover, for an operator $B \in \mathcal{L}_2(U,H)$ we denote $\|B\|_{\mathcal{L}_2(U,H)} = \|\!|B|\!\|_H$.

3.1 Local Well-Posedness

We make the following assumption.

Assumption 3.1 (*Assumption 3.1 in [AV22c]*)**.** *Suppose that the following conditions hold:*

1. *$A(t,v)v = A_0(t,v)v - F(t,v) - f$ and $B(t,v) = B_0(t,v)v + G(t,v) + g$, where*

   $$A_0 : \mathbb{R}_{\geq 0} \times \Omega \times H \to \mathcal{L}(V, V^*) \text{ and } B_0 : \mathbb{R}_{\geq 0} \times \Omega \times H \to \mathcal{L}(V, \mathcal{L}_2(U,H))$$

   are $\mathcal{P} \otimes \mathcal{B}(H)$-measurable, and

   $$F : \mathbb{R}_{\geq 0} \times \Omega \times V \to V^* \text{ and } G : \mathbb{R}_{\geq 0} \times \Omega \times V \to \mathcal{L}_2(U,H)$$

   are $\mathcal{P} \otimes \mathcal{B}(V)$-measurable, and $f : \mathbb{R}_{\geq 0} \times \Omega \to V^*$ and $g : \mathbb{R}_{\geq 0} \times \Omega \to \mathcal{L}_2(U,H)$ are $\mathcal{P}$-measurable maps such that a.s.

   $$f \in L^2_{loc}([0,\infty); V^*) \text{ and } g \in L^2_{loc}([0,\infty); \mathcal{L}_2(U,H)).$$

2. $\forall T > 0, n \geq 1\ \exists \theta_n > 0, M_n > 0$ *such that a.s.*

$$\langle u, A_0(t,v)u\rangle - \frac{1}{2}\|B_0(t,v)u\|_H^2 \geq \theta_n\|u\|_V^2 - M_n(1+\|u\|_H^2), \tag{3.2}$$

where $t \in [0,T], u \in V$ *and* $v \in H : \|v\|_H \leq n$.

3. *Let* $\rho_j \geq 0$ *and* $\beta_j \in (1/2, 1)$ *be such that*

$$2\beta_j \leq 1 + \frac{1}{1+\rho_j}, \quad j \in \{1, \dots, m_F + m_G\},$$

where $m_F, m_G \in \mathbb{N}$, *and suppose that* $\forall n \geq 1, T > 0$ *there exists a constant* $C_{T,n}$ *such that a.s.*

$$\|A_0(t,u)w\|_{V^*} \leq C_{T,n}(1+\|u\|_H)\|w\|_V, \tag{3.3}$$

$$\|A_0(t,u)w - A_0(t,v)w\|_{V^*} \leq C_{T,n}\|u-v\|_H\|w\|_V, \tag{3.4}$$

$$\|B_0(t,u)w\|_H \leq C_{T,n}(1+\|u\|_H)\|w\|_V, \tag{3.5}$$

$$\|B_0(t,u)w - B_0(t,v)w\|_H \leq C_{T,n}\|u-v\|_H\|w\|_V, \tag{3.6}$$

$$\|F(t,u) - F(t,v)\|_{V^*} \leq C_{T,n}\sum_{j=1}^{m_F}(1+\|u\|_{\beta_j}^{\rho_j} + \|v\|_{\beta_j}^{\rho_j})\|u-v\|_{\beta_j}, \tag{3.7}$$

$$\|F(t,u)\|_{V^*} \leq C_{T,n}\sum_{j=1}^{m_F}(1+\|u\|_{\beta_j}^{\rho_j+1}), \tag{3.8}$$

$$\|G(t,u) - G(t,v)\|_H \leq C_{T,n}\sum_{j=m_F+1}^{m_F+m_G}(1+\|u\|_{\beta_j}^{\rho_j} + \|v\|_{\beta_j}^{\rho_j})\|u-v\|_{\beta_j}, \tag{3.9}$$

$$\|G(t,u)\|_H \leq C_{T,n}\sum_{j=m_F+1}^{m_F+m_G}(1+\|u\|_{\beta_j}^{\rho_j+1}),, \tag{3.10}$$

where $t \in [0,T]$ *and* $u, v, w \in V$ *satisfy* $\|u\|_H, \|v\|_H \leq n$.

We make the following definition.

Definition 3.2 (Solution; Definition 3.2 in [AV22c])**.** Let Assumption 3.1 be satisfied and let σ be a stopping time taking values in $[0,\infty]$. Let $u : [0,\sigma)\times\Omega \to V$ be a strongly progressively measurable process.

- u is a *strong* solution to (3.1) on $[0,\sigma]\times\Omega$ if a.s. $u \in L^2_{\text{loc}}([0,\sigma); V) \cap C([0,\sigma]\cap\mathbb{R}_{\geq 0}; H)$ such that

$$F(\cdot,u) \in L^2(0,\sigma,V^*), \quad G(\cdot,u) \in L^2(0,\sigma,\mathcal{L}_2(U,H)),$$

and a.s. for all $t \in [0,\sigma)$

$$u(t) - u(0) + \int_0^t A(s,u(s))\,\mathrm{d}s = \int_0^t \mathbb{1}_{[0,\sigma)\times\Omega}B(s,u(s))\,\mathrm{d}W_s.$$

- u is a *local* solution to (3.1) if there exists an increasing sequence of stopping times $(\sigma_k)_{k\geq 1}$ such that $\lim_{k\uparrow\infty}\sigma_k = \sigma$ a.s. and $u|_{[0,\sigma_k]\times\Omega}$ is a strong solution to (3.1) on $[0,\sigma]\times\Omega$; $(\sigma_k)_{k\geq 1}$ is called a *localising sequence* for (u,σ).

- A local solution (u,σ) to (3.1) is called *unique* if for every other local solution (u',σ') for a.a. $\omega\in\Omega$ and all $t\in[0,\sigma(\omega)\wedge\sigma'(\omega))$ it holds that $u(t,\omega)=u'(t,\omega)$.

- A unique local solution (u,σ) to (3.1) is called a *maximal local solution* if for every other local solution (u',σ') it holds that a.s. $\sigma'\leq\sigma$ and for a.a. $\omega\in\Omega$ and all $t\in[0,\sigma'(\omega)$ it holds that $u(t,\omega)=u'(t,\omega)$.

- A maximal local solution (u,σ) to (3.1) is called a *global (unique) solution* if $\sigma=\infty$ a.s., in which case we write u instead of (u,σ).

Under Assumption 3.1, Theorem 3.3 in [AV22c] guarantees local well-posedness of (3.1).

Theorem 3.3 (Local Well-Posedness; Theorem 3.3 in [AV22c])**.** *Suppose that Assumption 3.1 holds. Then for every $u_0\in L^0_{\mathcal{F}_0}(\Omega;H)$, there exists a (unique) maximal solution (u,σ) to* (3.1) *such that a.s. $u\in C([0,\sigma);H)\cap L^2_{loc}([0,\sigma);V)$. Moreover, the following blow-up criterion holds*

$$\mathbb{P}\left(\sigma<\infty,\ \sup_{t\in[0,\sigma)}\|u(t)\|_H^2+\int_0^\sigma\|u(t)\|_V^2\,\mathrm{d}t<\infty\right)=0. \tag{3.11}$$

3.2 Global Well-Posedness

Under additional conditions, also global well-posedness of (3.1) can be shown. [AV22c] give the following condition on the operators A,B that guarantee global well-posedness.

Theorem 3.4 (Global Well-Posedness I; Theorem 3.4 in [AV22c])**.** *Suppose that Assumption 3.1 holds and for all $T>0$ there exist $\eta,\theta,M>0$ and a progressively measurable $\phi\in L^2((0,T)\times\Omega)$ and for any $v\in V$ and $t\in[0,T]$,*

$$\langle v,A(t,v)\rangle-(\frac{1}{2}+\eta)\|\!|B(t,v)|\!\|_H^2\geq\theta\|v\|_V^2-M\|v\|_H^2-|\phi(t)|^2. \tag{3.12}$$

Then for every $u_0\in L^0_{\mathcal{F}_0}(\Omega;H)$, there exists a (unique) global solution u to (3.1) *such that a.s. $u\in C([0,\infty);H)\cap L^2_{loc}([0,\infty);V)$. Moreover, for each $T>0$ there is a constant $C_T>0$ independent of U_0 such that*

$$\mathbb{E}\|u\|^2_{C([0,T];H)}+\mathbb{E}\|u\|^2_{L^2(0,T;V)}\leq C_T(1+\mathbb{E}\|u_0\|_H^2+\mathbb{E}\|\phi\|^2_{L^2(0,T)}). \tag{3.13}$$

4 FitzHugh-Nagumo Equations in the Variational Setting

We consider the stochastic FitzHugh–Nagumo equations

$$\begin{cases} \mathrm{d}u_1(t,x) = & (\nu\Delta u_1(t,x) + f(u_1(t,x)) - u_2(t,x))\,\mathrm{d}t \\ & + \sum_{k\geq 1}\left[b_k(t,x)\cdot\nabla u_1(t,x) + g_{1,k}(t,u(t,x))\right]\mathrm{d}W_1(t) \\ \mathrm{d}u_2(t,x) = & \epsilon(u_1(t,x) - \gamma u_2(t,x))\,\mathrm{d}t + \sum_{k\geq 1} g_{2,k}(t,u(t,x))\,\mathrm{d}W_2(t), \end{cases} \tag{4.1}$$

where W_1, W_2 are independent U-cylindrical Brownian motions, $U = \ell^2$, and the $u_1, u_2 : \mathbb{R}_{\geq 0} \times \mathbb{R}^d \to \mathbb{R}$. The classical form of the non-linear term is $f(u) = u(1-u)(u-a)$, where $a \in (0,1)$. Although it is common to consider a noise term in the first component, several authors have also considered a version of the problem with additive [ZH11; BM08] or multiplicative [Yam+19; Shi+19] noise in the second component. Therefore, here we consider the general problem with independent semi-linear noise terms in both components. Moreover, for convenience we assume that the non-linearities $(g_{1,k})_{k\geq 1}, (g_{2,k})_{k\geq 1}$ do not depend on the space variable x. However, such a dependence on x can easily be introduced at the cost of some additional additive constants when checking Assumption 3.1.

To write (4.1) in the form (3.1), we let $u = (u_1, u_2) \in V$,

$$\begin{aligned} A_0 : [0,T] \times V \times \Omega \to \mathcal{L}(V, V^*) : \\ (t,u,\omega) \mapsto \begin{pmatrix} -\nu\Delta u_1(t,x) + u_2(t,x) \\ -\epsilon u_1(t,x) + \epsilon\gamma u_2(t,x) \end{pmatrix}, \end{aligned} \tag{4.2}$$

$$\begin{aligned} F : V \to V^* : \\ u \mapsto \begin{pmatrix} f(u_1(t,x)) \\ 0 \end{pmatrix}, \end{aligned} \tag{4.3}$$

$$\begin{aligned} B_0 : [0,T] \times H \times \Omega \to \mathcal{L}(V; \mathcal{L}_2(U;H)) : \\ (t,u,\omega) \mapsto \left(u(t,x) \mapsto \begin{pmatrix} (b_k(t,x)\cdot\nabla u_1(t,x))_{k\geq 1} \\ 0 \end{pmatrix} \right), \end{aligned} \tag{4.4}$$

$$\begin{aligned} G : [0,T] \times V \times \Omega \to \mathcal{L}_2(U;H) : \\ (t,u,\omega) \mapsto \begin{pmatrix} (g_{1,k}(t,u(t,x)))_{k\geq 1} \\ (g_{2,k}(t,u(t,x)))_{k\geq 1} \end{pmatrix}, \end{aligned} \tag{4.5}$$

and we define $A(t,v) = A_0(t,v)v - F(t,v)$ and $B(t,v) = B_0(t,v)v + G(t,v)$.

4.1 Weak Setting

We first consider (4.1) in the weak setting with domain $\mathbb{R}$ as in [EGK21]. Therefore, we let

$$V \coloneqq H^1(\mathbb{R}) \oplus L^2(\mathbb{R})$$
$$H \coloneqq L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$$

$$V^* \coloneqq H^{-1}(\mathbb{R}) \oplus L^2(\mathbb{R}),$$

such that $V \hookrightarrow H = H^* \hookrightarrow V^*$, we define the Laplacian for $u_1 \in H^1(\mathbb{R})$ via

$$_{H^1(\mathbb{R})}\langle u_1, \Delta v_1 \rangle_{H^{-1}(\mathbb{R})} \coloneqq - \int_{\mathbb{R}} \nabla u_1 \nabla v_1 \mathrm{d}x,$$

and we further define the inner products

$$\begin{aligned}
(u, v)_H &\coloneqq (u_1, v_1)_{L^2(\mathbb{R})} + (u_2, v_2)_{L^2(\mathbb{R})} \\
&= \int_{\mathbb{R}} u_1 v_1 + u_2 v_2 \, \mathrm{d}x, \\
(u, v)_V &\coloneqq (u_1, v_1)_{H^1(\mathbb{R})} + \int_{\mathbb{R}} (u_2, v_2)_{L^2(\mathbb{R})} \\
&= \int_{\mathbb{R}} \nabla u_1 \nabla v_1 + u_1 v_1 \, \mathrm{d}x + \int_{\mathbb{R}} u_2 v_2 \mathrm{d}x.
\end{aligned}$$

Assumption 4.1. *We assume that the non-linearity f in* (4.1) *satisfies for any $y, y' \in \mathbb{R}$*

- $f(0) = 0$
- $\mu_f \coloneqq \sup_{y \in \mathbb{R}} f'(y) < \infty$
- $|f(y) - f(y')| \le C_1(1 + |y|^2 + |y'|^2)|y - y'|$,

There exists a constant $\nu_0 \in (0, \nu)$ such that for any $\xi \in \mathbb{R}$

- $\nu|\xi|^2 - \frac{1}{2}\sum_{k \ge 1}(b_k(t,x)^2 \xi)^2 \ge \nu_0 |\xi|^2$

and for any $y \in \mathbb{R}^2$ $g_1(y) \coloneqq (g_{1,k}(y))_{k \ge 1}$ and $g_2(y) \coloneqq (g_{2,k}(y))_{k \ge 1}$ satisfy

- $\|g_1(t,y) - g_1(t,y')\|_{\ell^2} + \|g_2(t,y) - g_2(t,y')\|_{\ell^2} \le C_1(|y_1 - y_1'| + |y_2 - y_2'|)$
- $\|g_1(t,0)\|_{L^2(\ell^2)} + \|g_2(t,0)\|_{L^2(\ell^2)} \le C_2$.

Theorem 4.2 (Global Well-Posedness)**.** *Let Assumption 4.1 hold. Then for every $u_0 \in L^0_{\mathcal{F}_0}(\Omega; H)$, there exists a (unique) global solution u to* (4.1)*. Moreover, for each $T > 0$ there is a constant $C_T > 0$ independent of u_0 such that*

$$\mathbb{E}\|u\|^2_{C([0,T];H)} + \mathbb{E}\|u\|^2_{L^2(0,T;V)} \le C_T(1 + \mathbb{E}\|u_0\|^2_H). \tag{4.6}$$

Proof. We use Theorem 3.4 to prove the claim. We begin by checking Assumption 3.1. Let $u, v, w \in V$, $\|u\|_H, \|v\|_H \le n$. For the growth bound (3.3) on A_0 we have

$$\begin{aligned}
|\langle w, A_0(t,u)v \rangle| = &\left| \int_{\mathbb{R}^d} \nu(\nabla v_1 \cdot \nabla w_1) + v_2 w_1 - \epsilon v_1 w_2 + \epsilon\gamma v_2 w_2 \, \mathrm{d}x \right| \\
\le &\nu \|\nabla v_1\|_{L^2} \|\nabla w_1\|_{L^2} + \|v_2\|_{L^2} \|w_1\|_{L^2} \\
&+ \epsilon \|v_1\|_{L^2} \|w_2\|_{L^2} + \epsilon\gamma \|v_2\|_{L^2} \|w_2\|_{L^2}
\end{aligned}$$

by Cauchy–Schwarz,

$$|\langle w, A_0(t,u)v\rangle| \le (\nu + 1 + \epsilon(1+\gamma))\|v\|_V\|w\|_V.$$

The local Lipschitz condition (3.4) holds trivially since $A_0(t,u) - A_0(t,v) = 0$.

For the growth bound (3.5) on B_0 we have

$$\begin{aligned}|\!|\!|B_0(t,u)w|\!|\!|_H^2 &= \sum_{k\ge1} \|b_k \cdot \nabla w_1\|_{L^2}^2 \\ &\le 2(\nu - \nu_0)\|\nabla w_1\|_{L^2}^2, \\ &\lesssim \|w\|_V^2.\end{aligned}$$

where we used that by Assumption 4.1 $\sum_{k\ge1}(b_k\cdot\nabla w_1)^2 \le 2(\nu-\nu_0)|\nabla w_1|^2$. The local Lipschitz condition (3.6) again holds trivially since $B_0(t,u) - B_0(t,v) = 0$.

For the growth condition (3.8) on F we use that by Assumption 4.1 $|f(u)| \le C_1(1+|u|^2)|u_1|$ so that

$$\|F(t,u)\|_{V^*} \lesssim \|u_1\|_{L^p} + \|u_1^3\|_{L^p} = \|u_1\|_{L^p} + \|u_1\|_{L^{3p}}^3.$$

We let $\rho_1 = 2$, $m_F = 1$ and use the Sobolev embeddings $L^p \hookleftarrow H^{2\beta-1}$ with $\frac{1}{p} \le 2\beta - 1 - \frac{1}{2}$ and $L^{3p} \hookleftarrow H^{2\beta-1}$ with $\frac{1}{3p} \le 2\beta - 1 - \frac{1}{2}$. The latter is most restrictive and holds for $\beta \ge \frac{2}{3}$, $p \in [1,2]$, and the requirement $2\beta \le 1 + \frac{1}{\rho+1}$ means that we additionally need $\beta \le \frac{2}{3}$. Note that the Sobolev embedding $L^{3p} \hookleftarrow H^{2\beta-1}$ limits us to $d \le 2$. We thus obtain by Sobolev embeddings with $\beta_1 = 2/3$, $\rho_1 = 2$ and Young's inequality

$$\|u_1\|_{L^p} + \|u_1\|_{L^{3p}}^3 \lesssim \|u_1\|_{H^{1/3}}^3 \lesssim 1 + \|u_1\|_{\beta_1}^{\rho_1+1}.$$

For the local Lipschitz condition (3.7) on F we use that by Assumption 4.1 $|f(u) - f(v)| \le C_1(1+|u|^2+|v|^2)|u_1 - v_1|$ so that

$$\begin{aligned}\|F(t,u) - F(t,v)\|_{V^*} &\lesssim \|F(t,u) - F(t,v)\|_{L^2} \\ &\lesssim \|u_1 - v_1\|_{L^2} + \||u_1|^2|u_1 - v_1|\|_{L^2} + \||v_1|^2|u_1 - v_1|\|_{L^2}.\end{aligned}$$

By Hölder's inequality

$$\||u_1|^2|u_1 - v_1|\|_{L^2} \le \|u_1\|_{L^6}^2\|u_1 - v_1\|_{L^6},$$

and the Sobolev embedding $H^{2\beta_1-1} \hookrightarrow L^6$ with $\beta_1 = 2/3$ gives

$$\begin{aligned}&\lesssim \|u_1\|_{H^{1/3}}^2\|u_1 - v_1\|_{H^{1/3}} \\ &= \|u_1\|_{\beta_1}^2\|u_1 - v_1\|_{\beta_1}.\end{aligned}$$

Note that the Sobolev embedding $H^{2\beta_1-1} \hookrightarrow L^6$ only holds for $d = 1$. Using $L^2 \hookleftarrow H^{1/3}$, we also have $\|u_1-v_1\|_{L^2} \lesssim \|u_1-v_1\|_{H^{1/3}} = \|u_1-v_1\|_{\beta_1}$. Combining the estimates gives

$$\|F(t,u) - F(t,v)\|_{V^*} \lesssim (1 + \|u_1\|_{H^{2\beta_1-1}}^2 + \|v_1\|_{H^{2\beta_1-1}}^2)\|u_1 - v_1\|_{H^{2\beta_1-1}}$$

$$\lesssim (1 + \|u\|_{\beta_1}^{\rho_1} + \|v\|_{\beta_1}^{\rho_1})\|u - v\|_{\beta_1}.$$

For the local Lipschitz condition (3.9) on G we use that by Assumption 4.1 $\|g_1(t,u) - g_1(t,v)\|_{\ell^2} + \|g_2(t,u) - g_2(t,v)\|_{\ell^2} \lesssim |u_1 - v_1| + |u_2 - v_2|$ so that

$$\begin{aligned}\|G(t,u) - G(t,v)\|_H \leq &\|u_1 - v_1\|_{L^2} + \|u_2 - v_2\|_{L^2} \\ \lesssim &\|u_1 - v_1\|_{H^{2\beta-1}} + \|u_2 - v_2\|_{L^2} \\ \lesssim &(1 + \|u\|_{\beta_2}^{\rho_2} + \|v\|_{\beta_2}^{\rho_2})\|u - v\|_{\beta_2},\end{aligned}$$

where we used the Sobolev embedding $L^2 \hookleftarrow H^{2\beta-1}$ with $2\beta - 1 \geq 0$, and we set $m_G = 1, \rho_2 = 1$ and $\beta_2 = 2/3$ in (3.9).

For the growth bound (3.10) on G we use that by Assumption 4.1 $\|g_1(t,0)\|_{L^2(\ell^2)} + \|g_2(t,0)\|_{L^2(\ell^2)} \leq C_2$ together with the Lipschitz condition above so that

$$\begin{aligned}\|G(t,u)\|_H &\leq \|G(t,0)\|_H + \|G(t,u) - G(t,0)\|_H \\ &\lesssim \|g_1(t,0)\|_{L^2} + \|g_2(t,0)\|_{L^2} + \|u_1\|_{L^2} + \|u_2\|_{L^2} \\ &\lesssim 1 + \|u\|_{\beta_2}^{\rho_2+1},\end{aligned}$$

by Young's inequality.

Next, we check the coercivity condition (3.2) on A_0, B_0. Let $u \in V, v \in H$, then using that $\nu|\nabla u_1|^2 - \frac{1}{2}\sum_{k\geq 1} |b_k \cdot \nabla u_1|^2 \geq \nu_0|\nabla u_1|^2$ by Assumption 4.1,

$$\begin{aligned}\langle u, A_0(t,v)u\rangle - &\frac{1}{2}\|B_0(t,v)u\|_H^2 \\ &\geq \int_{\mathbb{R}} \nu(\nabla u_1)^2 + u_2 u_1 - \epsilon u_1 u_2 + \epsilon\gamma u_2^2 \,\mathrm{d}x - \frac{1}{2}\sum_{k\geq 1}\int_{\mathbb{R}} |b_k \cdot \nabla u_1|^2 \,\mathrm{d}x \\ &\geq \nu_0\left[\|u\|_V^2 - \|u\|_H^2\right] - \frac{|1-\epsilon|}{2}\|u\|_H^2 + \epsilon\gamma\|u_2\|_{L^2(\mathbb{R}^d)}^2 \\ &\geq \nu_0\|u\|_V^2 - \left(\nu_0 + \frac{|1-\epsilon|}{2}\right)\|u\|_H^2.\end{aligned}$$

Finally, we check the coercivity condition (3.12) of Theorem 3.4. Assumption 4.1 implies that for $\eta \in (0, \nu_0/(2(\nu - \nu_0)))$ there exists a $\tilde{\nu}_0 \in (0, \nu)$ such that $\nu|\nabla u_1|^2 - \left(\frac{1}{2} + \eta\right)\sum_{k\geq 1} |b_k \cdot \nabla u_1|^2 \geq \tilde{\nu}_0|\nabla u_1|^2$, and thus

$$\langle u, A_0(t,v)u\rangle - \left(\frac{1}{2} + \eta\right)\|B_0(t,v)u\|_H^2 \geq \tilde{\nu}_0\|u\|_V^2 - \left(\tilde{\nu}_0 + \frac{|1-\epsilon|}{1}\right)\|u\|_H^2.$$

Moreover, we estimate

$$\begin{aligned}\langle u, F(t,u)\rangle &= \int_{\mathbb{R}^d} u_1(f(u_1) - f(0)) \,\mathrm{d}x \\ &\leq \int_{\mathbb{R}^d} u_1(f'(\xi)u_1) \,\mathrm{d}x,\end{aligned}$$

by the mean value theorem, and using Assumption 4.1

$$\langle u, F(t,u)\rangle \leq \mu_f\|u_1\|_{L^2}^2.$$

We already computed $\|G(t,u)\|_H \leq C_2 + C_1\|u\|_H$. Since $A(t,u) = A_0(t,u)u - F(t,u)$ and $B(t,u) = B_0(t,u)u + G(t,u)$ we combine the above estimates to get

$$\begin{aligned}\langle u, A(t,u)\rangle - &\left(\frac{1}{2}+\eta\right)\|B(t,u)\|_H^2 \\ &\geq \tilde{\nu}_0\|u\|_V^2 - \left(\tilde{\nu}_0 + \frac{|1-\epsilon|}{2} + \mu_{f'} + C_1^2(1+2\eta)\right)\|u\|_H^2 - C_2^2(2+4\eta),\end{aligned}$$

thus Theorem 3.4 applies with $\eta < \nu_0/(2(\nu-\nu_0))$. □

4.2 Strong Setting

The proof of Theorem 4.2 in the preceding section showed that the weak setting can only accommodate the FitzHugh-Nagumo equations (4.1) with dimension $d = 1$. To obtain well-posedness results for higher dimensions, we now consider (4.1) in the strong setting, where we let

$$\begin{aligned} V &\coloneqq H^2(\mathbb{R}^d) \oplus H^1(\mathbb{R}^d) \\ H &\coloneqq H^1(\mathbb{R}^d) \oplus H^1(\mathbb{R}^d) \\ V^* &\coloneqq L^2(\mathbb{R}^d) \oplus H^1(\mathbb{R}^d), \end{aligned}$$

such that $V \hookrightarrow H = H^* \hookrightarrow V^*$, and we define for $u_1 \in H^2(\mathbb{R}^d)$

$$_{H^2(\mathbb{R}^d)}\langle u_1, \Delta v_1\rangle_{L^2(\mathbb{R}^d)} \coloneqq \int_{\mathbb{R}^d} \Delta u_1 \Delta v_1 \mathrm{d}x,$$

and we further define the inner products

$$\begin{aligned} (u,v)_H &\coloneqq (u_1,v_1)_{H^1(\mathbb{R}^d)} + (u_2,v_2)_{H^1(\mathbb{R}^d)} \\ &= \int_{\mathbb{R}^d} \nabla u_1 \cdot \nabla v_1 + u_1 v_1 \,\mathrm{d}x + \int_{\mathbb{R}^d} \nabla u_2 \cdot \nabla v_2 + u_2 v_2 \,\mathrm{d}x, \\ (u,v)_V &\coloneqq (u_1,v_1)_{H^2(\mathbb{R}^d)} + (u_2,v_2)_{L^2(\mathbb{R}^d)} \\ &= \int_{\mathbb{R}^d} \sum_{|\alpha|\leq 2} \partial^\alpha u_1 \partial^\alpha v_1 \,\mathrm{d}x + \int_{\mathbb{R}^d} \nabla u_2 \cdot \nabla v_2 + u_2 v_2 \,\mathrm{d}x. \end{aligned}$$

Assumption 4.3. *We assume that the non-linearity f in* (4.1) *satisfies for any $y, y' \in \mathbb{R}$*

- $f(0) = 0$,
- $\mu_f \coloneqq \sup_{y\in\mathbb{R}} f'(y) < \infty$,
- $|f(y) - f(y')| \leq C_1(1 + |y|^2 + |y'|^2)|y - y'|$,
- $|f'(y) - f'(y')| \leq C_2(1 + |y| + |y'|)|y - y'|$,
- $|f'(y)| \leq C_3(1 + |y|^2)$,

there exists a constant $\nu_0 \in (0,\nu)$ *such that* $b = (b_k^j)_{k\geq 1, 1\leq j\leq d}$ *satisfies for any* $\xi \in \mathbb{R}^d$

- $\nu|\xi|^2 - \frac{1}{2}\sum_{k\geq 1}(b_k \cdot \xi)^2 \geq \nu_0|\xi|^2$,
- $\|b^j\|_{W^{1,\infty}(\mathbb{R}^d;\ell^2)} \leq M$,

for $y, y' \in \mathbb{R}^2$ $g_1(y) \coloneqq (g_{1,k}(t,y))_{k\geq 1}$ *and* $g_2(y) \coloneqq (g_{2,k}(t,y))_{k\geq 1}$ *satisfy*

- $\|g_1(y_1) - g_1(y_1')\|_{\ell^2} + \|g_1'(y_1) - g_1'(y_1')\|_{\ell^2} \leq C_1|y_1 - y_1'|$,

and for $u, v \in V$

- $\|g_2(u_2) - g_2(v_2)\|_{H^1(\ell^2)} \leq C_2\|u_2 - v_2\|_{H^1}$,
- $\|g_1(0)\|_{L^2(\ell^2)} + \|g_1'(0)\|_{L^2(\ell^2)} + \|g_2(0)\|_{L^2(\ell^2)} + \|g_2'(0)\|_{L^2(\ell^2)} \leq C_3$.

Remark 1. We need to impose strong restrictions on g_1, g_2 in the strong setting to ensure that the Lipschitz condition on G in Assumption 3.1 is satisfied. For instance, if we would allow g_1 to depend on u_2, our estimate for $\|\!|G|\!\|_H$ would involve terms $\|\partial_2 g_1(u_1,u_2)\nabla u_2 - \partial_2 g_1(v_1,v_2)\nabla v_2\|_{L^2}$, which cannot be factorised (via Hölder's inequality) and bounded in the form $(1 + \|u_2\|_{H^1} + \|v_2\|_{H^1})\|u_2 - v_2\|_{H^1}$ since $u_2, v_2 \in H^1$.

Our definition of a solution is analogous to Definition 3.2 in the weak setting.

Theorem 4.4 (Global Well-Posedness)**.** *Let Assumption 4.3 hold. Then for every* $u_0 \in L^0_{\mathcal{F}_0}(\Omega; H)$*, there exists a (unique) global solution* u *to* (4.1) *for* $d \leq 4$*. Moreover, for each* $T > 0$ *there is a constant* $C_T > 0$ *independent of* u_0 *such that*

$$\mathbb{E}\|u\|^2_{C([0,T];H)} + \mathbb{E}\|u\|^2_{L^2((0,T);V)} \leq C_T(1 + \mathbb{E}\|u_0\|^2_H). \tag{4.7}$$

Proof. We use Theorem 3.4 to prove the claim. We begin by checking Assumption 3.1. Let $u, v, w \in V$, $\|u\|_H, \|v\|_H \leq n$. For the growth bound (3.3) on A_0 we have

$$\begin{aligned}
|\langle w, A_0(t,u)v\rangle| \leq & \int_{\mathbb{R}^d} \nu|\Delta v_1 \Delta w_1 + \nabla v_1 \cdot \nabla w_1| + |\nabla v_2 \cdot \nabla w_1 + v_2 w_1| \\
& + \epsilon|\nabla v_1 \cdot \nabla w_2 - v_1 w_2| + \epsilon\gamma|\nabla v_2 \cdot \nabla w_2 + v_2 w_2| \,\mathrm{d}x \\
\leq & \nu(\|\Delta v_1\|_{L^2(\mathbb{R}^d)}\|\Delta w_1\|_{L^2(\mathbb{R}^d)} + \|\nabla v_1\|_{L^2(\mathbb{R}^d)}\|\nabla w_1\|_{L^2(\mathbb{R}^d)}) \\
& + \|\nabla v_2\|_{L^2(\mathbb{R}^d)}\|\nabla w_1\|_{L^2(\mathbb{R}^d)} + \|v_2\|_{L^2(\mathbb{R}^d)}\|w_1\|_{L^2(\mathbb{R}^d)} \\
& + \epsilon(\|\nabla v_1\|_{L^2(\mathbb{R}^d)}\|\nabla w_2\|_{L^2(\mathbb{R}^d)}) + \|v_1\|_{L^2(\mathbb{R}^d)}\|w_2\|_{L^2(\mathbb{R}^d)}) \\
& + \epsilon\gamma(\|\nabla v_2\|_{L^2(\mathbb{R}^d)}\|\nabla w_2\|_{L^2(\mathbb{R}^d)} + \|v_2\|_{L^2(\mathbb{R}^d)}\|w_2\|_{L^2(\mathbb{R}^d)}) \\
\lesssim & \|v\|_V\|w\|_V.
\end{aligned}$$

The local Lipschitz condition (3.4) holds trivially since $A_0(t,u) - A_0(t,v) = 0$.

For the growth bound (3.5) on B_0 we have

$$\|\!|B_0(t,u)w|\!\|^2_H \leq \sum_{k\geq 1} \|b_k \cdot \nabla w_1\|^2_{L^2} + \|\nabla(b_k \cdot \nabla w_1)\|^2_{L^2}$$

$$
\begin{aligned}
&\le \sum_{k\ge 1} \|b_k \cdot \nabla w_1\|_{L^2}^2 + \left(1+\frac{1}{\delta}\right) \|\nabla b_k \cdot \nabla w_1\|_{L^2}^2 \\
&\quad + \sum_{k\ge 1} (1+\delta) \|b_k \cdot \nabla^2 w_1\|_{L^2}^2
\end{aligned}
$$

where we introduce $\delta > 0$ to gain some extra flexibility that will be used in the proof of the coercivity condition below. Thus,

$$
\begin{aligned}
|\!|\!| B_0(t,u)w |\!|\!|_H^2 &\le \int_{\mathbb{R}^d} \sum_{k\ge 1} \sum_{i,j=1}^d b_k^i b_k^j \partial_i w_1 \partial_j w_1 \,\mathrm{d}x + \sum_{k\ge 1} \left(1+\frac{1}{\delta}\right) \|\nabla b_k \cdot \nabla w_1\|_{L^2}^2 \\
&\quad + (1+\delta) \int_{\mathbb{R}^d} \sum_{k\ge 1} \sum_{i,j,l=1}^d b_k^i b_k^j \partial_i \partial_l w_1 \partial_j \partial_l w_1 \,\mathrm{d}x \\
&\le \left(\nu - \nu_0 + M^2 \left(1+\frac{1}{\delta}\right)\right) \|\nabla w_1\|_{L^2}^2 + (\nu-\nu_0)(1+\delta) \|\Delta w_1\|_{L^2}^2,
\end{aligned}
$$

where we used in the last step that $\|b\|_{W^{1,\infty}(\ell^2)}^2 \le M^2$ and $\sum_{k\ge 1}\sum_{i,j=1}^d b_k^i b_k^j \xi_i \xi_j \le (\nu-\nu_0)|\xi|^2$ by Assumption 4.3, and $\sum_{j,l=1}^d \int_{\mathbb{R}^d} |\partial_j \partial_l w_1|^2 \,\mathrm{d}x = \|\Delta w_1\|_{L^2}^2$ by integration by parts. We thus see that

$$
\begin{aligned}
|\!|\!| B_0(t,u)w |\!|\!|_H &\lesssim \|w\|_H + \|w\|_V \\
&\eqsim C_{T,n}(1+\|w\|_V).
\end{aligned}
$$

The local Lipschitz condition (3.6) again holds trivially since $B_0(t,u)-B_0(t,v) = 0$.

For the growth condition (3.8) on F we use that by Assumption 4.3 $|f(u)| \le C_1(1+|u_1|^2)|u|$ so that

$$
\begin{aligned}
\|F(t,u)\|_{V^*} &= \|f(u_1)\|_{L^2(\mathbb{R}^d)} \\
&\le C_1(\|u_1\|_{L^2} + \|u_1^3\|_{L^2}) \\
&\lesssim C_{T,n}(1+\|u_1\|_{L^6}^3).
\end{aligned}
$$

We set $\rho_1 = 2$ and we require $\beta_1 \le 2/3$. Since $[L^2(\mathbb{R}^d), H^2(\mathbb{R}^d)]_{\beta_1} = H^{2\beta_1}$, we use the Sobolev embedding $H^{2\beta_1} \hookrightarrow L^6$ with $2\beta_1 - \frac{d}{2} \ge -\frac{d}{6}$ to get

$$
\|u_1\|_{L^6}^3 \lesssim \|u_1\|_{\beta_1}^3
$$

Hence, we have

$$
\|F(t,u)\|_{V^*} \lesssim 1 + \|u_1\|_{\beta_1}^{\rho_1+1},
$$

and thus the condition (3.8) is satisfied for $d \le 4, \beta_1 = 2/3$.

For the local Lipschitz condition (3.7) on F we use that by Assumption 4.1 $|f(u)-f(v)| \le C_1(1+|u|^2+|v|^2)|u_1-v_1|$ so that

$$
\|F(t,u)-F(t,v)\|_{V^*} \le C_1 \|(1+|u_1|^2+|v_1|^2)|u_1-v_1|\|_{L^2}
$$

$$\leq C_1(\|u_1 - v_1\|_{L^2} + \|(|u_1|^2 + |v_1|^2)|u_1 - v_1|\|_{L^2}).$$

By Hölder's inequality we have

$$\||u_1|^2|u_1 - v_1|\|_{L^2} \leq \|u_1\|_{L^6}^2 \|u_1 - v_1\|_{L^6},$$

and the Sobolev embedding $H^{2\beta_1} \hookrightarrow L^6$ with $2\beta_1 - d/2 \geq -d/6$ gives

$$\||u_1|^2|u_1 - v_1|\|_{L^2} \leq \|u_1\|_{\beta_1}^2 \|u_1 - v_1\|_{\beta_1},$$

which is satisfied for $d \leq 4, \beta_1 = 2/3$. Using $H^{2\beta_1} \hookrightarrow L^2$, we also have $\|u_1 - v_1\|_{L^2} \lesssim \|u_1 - v_1\|_{\beta_1}$ for $\beta_1 > 0$. Combining the estimates gives

$$\begin{aligned} \|F(t,u) - F(t,v)\|_{V^*} &\lesssim (1 + \|u_1\|_{H^{2\beta_1}}^2 + \|v_1\|_{H^{2\beta_1}}^2)\|u_1 - v_1\|_{H^{2\beta_1}} \\ &\lesssim (1 + \|u\|_{\beta_1}^{\rho_1} + \|v\|_{\beta_1}^{\rho_1})\|u - v\|_{\beta_1}. \end{aligned}$$

For the local Lipschitz condition (3.9) on G we use that by Assumption 4.1 $\|g_1(u_1) - g_1(v_1)\|_{\ell^2} \leq C_1|u_1 - v_1|$ and $\|g_2(u_2) - g_2(v_2)\|_{H^1} \leq C_2\|u_2 - v_2\|_{H^1}$ so that

$$\begin{aligned} \|G(t,u) - G(t,v)\|_H \lesssim &\|\nabla g_1(u_1) - \nabla g_1(v_1)\|_{L^2} + \|g_1(u_1) - g_1(v_1)\|_{L^2} \\ &+ \|g_2(u_2) - g_2(v_2)\|_{H^1} \\ \lesssim &\|u_1 - v_1\|_{L^2} + \|g_1'(u_1)\nabla u_1 - g_1'(v_1)\nabla v_1\|_{L^2} \\ &+ \|u_2 - v_2\|_{H^1}. \end{aligned}$$

Using that $\|g_1(t,u_1) - g_1(t,v_1)\|_{\ell^2} \leq C_1|u_1 - v_1|$ implies $\|g'\|_{L^\infty(\ell^2)} \leq C_1$ and by Assumption 4.1 $\|g_1'(u_1) - g_1'(v_1)\|_{\ell^2} \leq C_1|u_1 - v_1|$, we split the term

$$\begin{aligned} \|g_1'(u_1)\nabla u_1 - g_1'(v_1)\nabla v_1\|_{L^2(\ell^2)} \leq &\|g_1'(u_1)\nabla u_1 - g_1'(u_1)\nabla v_1\|_{L^2(\ell^2)} \\ &+ \|g_1'(u_1)\nabla v_1 - g_1'(v_1)\nabla v_1\|_{L^2(\ell^2)} \\ \lesssim &\|\nabla u_1 - \nabla v_1\|_{L^2} + \|u_1 - v_1\|_{L^8}\|\nabla v_1\|_{L^{8/3}} \\ \lesssim &\|u_1 - v_1\|_{H^1} + \|u_1 - v_1\|_{L^8}\|v_1\|_{H^{1,8/3}} \\ \lesssim &\|u_1 - v_1\|_{H^{3/2}}(1 + \|v_1\|_{H^{3/2}}) \\ \eqsim &\|u_1 - v_1\|_{H^{2\beta_2}}(1 + \|v_1\|_{H^{2\beta_2}}), \end{aligned}$$

where we used the Sobolev embeddings $L^8 \hookleftarrow H^{3/2}$ with $-\frac{d}{8} \leq \frac{3}{2} - \frac{d}{2}$ and $H^{1,8/3} \hookleftarrow H^{3/2}$ with $1 - \frac{d3}{8} \leq \frac{3}{2} - \frac{d}{2}$ for $d \leq 4$. We set $\beta_2 = 3/4$ and $\rho_2 = 1$, and using that $\|u_1 - v_1\|_{L^2} \lesssim \|u_1 - v_1\|_{H^{3/2}}$, we have

$$\begin{aligned} \|G(t,u) - G(t,v)\|_H \lesssim &(1 + \|u\|_{H^{2\beta_2}} + \|v\|_{H^{2\beta_2}})\|u - v\|_{H^{2\beta_2}} \\ \eqsim &(1 + \|u\|_{\beta_2}^{\rho_2} + \|v\|_{\beta_2}^{\rho_2})\|u - v\|_{\beta_2}. \end{aligned}$$

For the growth bound (3.10) on G we use that by Assumption 4.1 $\|g_1(t,0)\|_{L^2(\ell^2)} + \|g_1'(t,0)\|_{L^2(\ell^2)} + \|g_2(t,0)\|_{L^2(\ell^2)} + \|g_2'(t,0)\|_{L^2(\ell^2)} \leq C_3$ together with the Lipschitz condition above so that

$$\|G(t,u)\|_H \leq \|G(t,0)\|_H + \|G(t,u) - G(t,0)\|_H$$

$$\lesssim 1 + \|u\|_{\beta_2}^{\rho_2+1}.$$

Next, we check the coercivity condition (3.2) on A_0, B_0. Let $u \in V, v \in H$, then using the estimate for $\|B_0(t,v)u\|_H^2$ obtained above,

$$\begin{aligned}
&\langle u, A_0(t,v)u\rangle - \frac{1}{2}\|B_0(t,v)u\|_H^2 \\
&\quad \geq \nu \int_{\mathbb{R}^d} (\Delta u_1)^2 + |\nabla u_1|^2 \,\mathrm{d}x + \int_{\mathbb{R}^d} (\nabla u_2)\cdot(\nabla u_1) + u_2 u_1 \,\mathrm{d}x \\
&\quad\quad - \epsilon \int_{\mathbb{R}^d} (\nabla u_1)\cdot(\nabla u_2) + u_1 u_2 \,\mathrm{d}x + \epsilon\gamma \int_{\mathbb{R}^d} |\nabla u_2|^2 + (u_2)^2 \,\mathrm{d}x \\
&\quad\quad - \frac{1}{2}\left(\nu - \nu_0 + M^2\left(1 + \frac{1}{\delta}\right)\right)\|\nabla u_1\|_{L^2}^2 - \frac{1}{2}(\nu - \nu_0)(1+\delta)\|\Delta u_1\|_{L^2}^2 \\
&\quad = \left(\nu - \frac{1}{2}(\nu - \nu_0)(1+\delta)\right)\|\Delta u_1\|_{L^2}^2 \\
&\quad\quad + \left(\nu - \frac{1}{2}\left(\nu - \nu_0 + M^2\left(1 + \frac{1}{\delta}\right)\right)\right)\|\nabla u_1\|_{L^2}^2 \\
&\quad\quad - 2|1-\epsilon|\|u\|_H^2 + \epsilon\gamma\|u_2\|_{H^1}^2 \\
&\quad \geq \frac{1}{2}\left(\nu + \nu_0 - \delta(\nu - \nu_0)\right)\|u\|_V^2 \\
&\quad\quad - \left(\frac{1}{2}(\nu - \nu_0) + \frac{M^2}{2}\left(1 + \frac{1}{\delta}\right) + 2|1-\epsilon|\right)\|u\|_H^2 \\
&\quad =: M_n\|u\|_V^2 - \theta_n\|u\|_H^2,
\end{aligned}$$

where we used that $\int_{\mathbb{R}^d} |\Delta u_1|^2 \,\mathrm{d}x = \int_{\mathbb{R}^d} \sum_{j,k=1}^d |\partial_j\partial_k u_1|^2 \,\mathrm{d}x$ by integration by parts, and $\delta \in (0,1)$.

Finally, we check the coercivity condition (3.12) of Theorem 3.4. Since $A(t,u) = A_0(t,u)u - F(t,u)$ and our computation above already shows that $\langle u, A_0(t,u)u\rangle \geq \nu\|u\|_V^2 - (\nu + 2|1-\epsilon|)\|u\|_H^2$, we only need to obtain an estimate for $\langle u, F(t,u)\rangle$. We have

$$\begin{aligned}
\langle u, F(t,u)\rangle &= \int_{\mathbb{R}^d} \nabla u_1 \cdot \nabla u_1 f'(u_1) + u_1 f(u_1) \,\mathrm{d}x \\
&= \int_{\mathbb{R}^d} |\nabla u_1|^2 f'(u_1) + |u_1|^2 f'(\xi) \,\mathrm{d}x \\
&\leq \mu_{f'}\|u\|_H^2
\end{aligned}$$

by Assumption 4.3 and the intermediate value theorem applied to $f(u_1) - f(0)$.

We already obtained an estimate for $\|B_0(t,u)u\|_H^2$ above where we checked the growth bound (3.4) for B_0. To bound $\|G(t,u)\|_H^2$ we use

$$\begin{aligned}
\|G(t,u)\|_H^2 \leq& \|g_1(u_1)\|_{L^2(\ell^2)}^2 + \|\nabla g_1(u_1)\|_{L^2(\ell^2)}^2 + \|g_2(u_2)\|_{H^1(\ell^2)}^2 \\
\leq& \|g_1(u_1) - g_1(0)\|_{L^2(\ell^2)}^2 + \|g_1(0)\|_{L^2(\ell^2)}^2 \\
&+ \|g_1'(u_1)\|_{L^\infty(\ell^2)}^2\|\nabla u_1\|_{L^2}^2 + \|g_1'(0)\|_{L^2}^2
\end{aligned}$$

$$
\begin{aligned}
&+ \|g_2(u_2) - g_2(0)\|_{H^1}^2 + \|g_2(t,0)\|_{H^1}^2 \\
\le& C_1^2(\|u_1\|_{H^1}^2 + \|u_2\|_{H^1}^2) + 4C_3^2 \\
\le& C_1^2\|u\|_H^2 + 4C_2^2,
\end{aligned}
$$

where we used that Assumption 4.3 implies $\|g_1'(u_1)\|_{L^\infty(\ell^2)} \le C_1$.

Combining all estimates and letting $\tilde{\nu} = \nu - \nu_0$, we get

$$
\begin{aligned}
\langle u, A(t,u)\rangle - &\left(\frac{1}{2} + \eta\right) \|B(t,u)\|_H^2 \\
\ge& \langle u, A_0(t,u)\rangle - \langle u, F(t,u)\rangle - \left(\frac{1}{2} + \eta\right) \left(\|B_0(t,u)u\|_H^2 + \|G(t,u)\|_H^2\right) \\
\ge& \nu\|u\|_V^2 - (\nu + 2|1-\epsilon| + \mu_{f'})\|u\|_H^2 \\
&- \tilde{\nu}\left(\frac{1}{2} + \eta\right)(1+\delta)\|u\|_V^2 - \left(\frac{1}{2} + \eta\right)\left[\tilde{\nu} + M^2\left(1 + \frac{1}{\delta}\right)\right]\|u\|_H^2 \\
&- \left(\frac{1}{2} + \eta\right)\left(C_1^2\|u\|_H^2 + 4C_2^2\right) \\
=& \left(\nu - \tilde{\nu}\left(\frac{1}{2} + \eta\right)(1+\delta)\right)\|u\|_V^2 \\
&- \left(\nu + 2|1-\epsilon| + \mu_{f'} + \left(\frac{1}{2} + \eta\right)\left(\tilde{\nu} + M^2\left(1 + \frac{1}{\delta}\right) + C_1^2\right)\right)\|u\|_H^2 \\
&- \left((2+4\eta)C_2^2\right) \\
=:& \theta\|u\|_V^2 - M\|u\|_H^2 - |\varphi(t)|^2,
\end{aligned}
$$

and $\theta = \nu - (\nu - \nu_0)(1+2\eta)(1+\delta)/2) > 0$ for η and δ sufficiently small. □

Remark 2. Global existence of a solution to (4.1) under more restrictive assumptions than our Assumption 4.3 can often be proved using the theory for the variational setting developed in [LR15]. The approach we have taken here allows us to obtain existence results in the strong variational setting also when the weak monotonicity condition on the operators $-A, B$ (see (H2) of Chapter 4.1 in [LR15])

$$
{}_{V^*}\langle -A(u) + A(v), u - v\rangle_V + \|B(u) - B(v)\|_{L_2(U,H)} \le C\|u - v\|_H \tag{4.8}
$$

does not hold. Indeed, our Assumption 4.3 admits nonlinearities F that do not satisfy weak monotonicity, as the following example shows.

For simplicity we assume that B does not depend on (t,x), $g_1 = g_2 = 0$, and $\|B(u) - B(v)\|_H$ can be bounded by $\|u - v\|_H$. Moreover, the term ${}_{V^*}\langle -A_0(u) + A_0(v), u - v\rangle_V$ can also be bounded by $\|u - v\|_H$. Now consider $f(u) = -u^3$ and let $v_1(x) = |x|^2 \exp(-|x|^2/\lambda), v_2(x) = 0$, $w_1(x) = \exp(-|x|^2/\lambda), w_2(x) = 0$, where $\lambda > 0$, and $u_1(x) = v_1(x) + w_1(x), u_2(x) = 0$. We will show by contradiction that weak monotonicity cannot hold for this choice of u, v, that is, there is no constant $C > 0$ such that ${}_{V^*}\langle F(u) - F(v), u - v\rangle_V \le C\|u - v\|_H$

for all $\lambda > 0$. We begin with a scaling argument to simplify the necessary computations. Assume (4.8) holds, then for $\rho > 0$

$$
\begin{aligned}
{}_{V^*}\langle F(u) - F(v), u - v\rangle_V
&= \int_{\mathbb{R}^d} \nabla(f(u_1(\rho x)) - f(v_1(\rho x))) \cdot \nabla(u_1(\rho x) - v_1(\rho x))\,\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^d} (f(u_1(\rho x)) - f(v_1(\rho x)))(u_1(\rho x) - v_1(\rho x))\,\mathrm{d}x \\
&= \int_{\mathbb{R}^d} (-3u_1(\rho x)^2 \nabla u_1(\rho x) + 3v_1(\rho x)^2 \nabla v_1(\rho x)) \cdot \nabla w_1(\rho x)\,\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^d} (u_1(\rho x)^3 - v_1(\rho x)^3) w_1(\rho x)\,\mathrm{d}x \\
&= \rho \int_{\mathbb{R}^d} (-3u_1(y)^2 \nabla u_1(y) + 3v_1(y)^2 \nabla v_1(y) \cdot \nabla w_1(y)\,\mathrm{d}y \\
&\quad + \frac{1}{\rho} \int_{\mathbb{R}^d} (u_1(y)^3 - v_1(y)^3) w_1(y)\,\mathrm{d}y \\
&\lesssim \|u_1 - v_1\|_{H^1(\mathbb{R}^d)} \\
&= \int_{\mathbb{R}^d} |\nabla u_1(\rho x) - \nabla v_1(\rho x)|^2 + |u_1(\rho x) - v_1(\rho x)|^2\,\mathrm{d}x \\
&= \rho \int_{\mathbb{R}^d} |\nabla u_1(y) - \nabla v_1(y)|^2\,\mathrm{d}y + \frac{1}{\rho} \int_{\mathbb{R}^d} |u_1(y) - v_1(y)|^2\,\mathrm{d}y.
\end{aligned}
$$

Thus, dividing both sides by ρ and letting $\rho \to \infty$ shows that if weak monotonicity holds, then

$$
\int_{\mathbb{R}^d} \nabla(f(u_1) - f(v_1)) \cdot \nabla(u_1 - v_1)\,\mathrm{d}x \le \int_{\mathbb{R}^d} |\nabla(u_1 - v_1)|^2\,\mathrm{d}x.
$$

For our choice of u, v, w we have

$$
\begin{aligned}
\int_{\mathbb{R}^d} |\nabla w_1(x)|^2\,\mathrm{d}x &= \frac{16}{\lambda^2} \int_{\mathbb{R}^d} |x|^2 e^{-2|x|^2/\lambda}\,\mathrm{d}x \\
&= \frac{16}{\lambda^2} \int_{S_1} \int_0^\infty r^2 e^{-2r^2/\lambda}\,\mathrm{d}r\,\mathrm{d}S(y) \\
&= |S_1| \sqrt{\frac{\pi}{2\lambda}} \\
&\to 0
\end{aligned}
$$

as $\lambda \to \infty$, where we made a change to polar coordinates in the second step. On the other hand,

$$
\begin{aligned}
&\int_{\mathbb{R}^d} (-3u_1(x)^2 \nabla u_1(x) + 3v_1(x)^2 \nabla v_1(x)) \cdot \nabla w_1(x)\,\mathrm{d}x \\
&= \int_{\mathbb{R}^d} e^{-4|x|^2/\lambda} \left(|x|^2 \left(\frac{4}{\lambda} - \frac{4}{\lambda^2} \right) + |x|^4 \left(\frac{8}{\lambda} - \frac{12}{\lambda^2} \right) - \frac{12}{\lambda^2} |x|^6 \right) \mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{S_1} \int_0^\infty e^{-4r^2/\lambda} \left(r^2 \left(\frac{4}{\lambda} - \frac{4}{\lambda^2} \right) + r^4 \left(\frac{8}{\lambda} - \frac{12}{\lambda^2} \right) - \frac{12}{\lambda^2} r^6 \right) \mathrm{d}r \, \mathrm{d}S(y) \\
&= |S_1| \sqrt{\pi} \left(\frac{3}{256} \lambda^{3/2} - 32\sqrt{\lambda} - \frac{4}{\sqrt{\lambda}} \right) \\
&\to \infty
\end{aligned}
$$

as $\lambda \to \infty$, contradicting our assumption that (4.8) holds.

5 FitzHugh-Nagumo Equations in the $L^p(L^q(\mathbb{T}^d))$-Setting

5.1 Local Well-Posedness and Blow-Up Criteria

In Section 4 we considered the FitzHugh-Nagumo equations (4.1) in the weak and the strong variational setting. In the weak variational setting we proved (global) well-posedness for $d = 1$. By increasing the assumed differentiability of the first component in the strong variational setting we were able to prove (global) well-posedness for dimension $d \leq 4$. However, in the latter setting we needed to make restrictive assumptions on the non-linearity G, namely the components g_1 and g_2 of G could only depend on the corresponding component of u, due to the mismatch in smoothness between the two components of u. In the present section we consider the FitzHugh-Nagumo equations (4.1) in the $L^p(L^q(\mathbb{T}^d))$-setting, where the additional flexibility in the integrability of the second component of the equations allows us to obtain (global) well-posedness results under more general assumptions.

Our approach in this section follows the theory of reaction-diffusion equations on the periodic torus $\mathbb{T}^d$ developed in [AV23b]. Therefore, we consider the FitzHugh-Nagumo equations (4.1) on the periodic torus $\mathbb{T}^d$ first and attempt to generalise our results to unbounded domains in later sections. Since the second component of (4.1) does not involve a Laplace operator, the uniform ellipticity condition of Assumption 5.1(3) only holds for the first component of the system (4.1), and we need to make some adjustments to the theory of reaction-diffusion equations in [AV23b] to accommodate the second component. We let $\delta \in [1, 2)$, $\eta \in (0, 2 - \delta]$, $q \geq 2$ and consider the spaces

$$
\begin{aligned}
X_0 &= H^{-\delta,q}(\mathbb{T}^d) \oplus H^{2-\delta-\eta,q}(\mathbb{T}^d) =: X_0^1 \oplus X^2, \\
X_1 &= H^{2-\delta,q}(\mathbb{T}^d) \oplus H^{2-\delta-\eta,q}(\mathbb{T}^d) =: X_1^1 \oplus X^2, \\
X_\beta &:= [X_0, X_1]_\beta = H^{2\beta-\delta,q}(\mathbb{T}^d) \oplus H^{2-\delta-\eta,q}(\mathbb{T}^d),
\end{aligned} \tag{5.1}
$$

where $\beta \in (0, 1)$. As before, we will write $H^{-\delta,q}, H^{2-\delta,q}$ etc. instead of $H^{-\delta,q}(\mathbb{T}^d), H^{2-\delta,q}(\mathbb{T}^d)$ when no confusion can arise, and we will use the shorthand notation $X^1_\beta = [X^1_0, X^1_1]_\beta$. On these spaces we consider the operators

$$\begin{aligned}
\tilde{A}_0 &: [0,T] \times X^{\mathrm{Tr}}_{\kappa,p} \times \Omega \to \mathcal{L}(X_1, X_0) : \\
&\qquad \tilde{A}_0(t,u,\omega) = \begin{pmatrix} -\nu\Delta u_1 \\ 0 \end{pmatrix}, && (5.2)\\
A_{\mathrm{pert}} &: [0,T] \times \Omega \to \mathcal{L}(X_1, X_0) : \\
&\qquad A_{\mathrm{pert}}(t,u,\omega) = \begin{pmatrix} u_2 \\ -\epsilon u_1 + \gamma\epsilon u_2 \end{pmatrix}, && (5.3)\\
f &: X_1 \to X_0 : \\
&\qquad f(u) = \begin{pmatrix} f(u_1(t,x)) \\ 0 \end{pmatrix}, && (5.4)\\
B_0 &: [0,T] \times X^{\mathrm{Tr}}_{\kappa,p} \times \Omega \to \mathcal{L}(X_1, \gamma(U, X_{1/2})) : \\
&\qquad B_0(t,u,\omega) = \left(u(t,x) \mapsto \begin{pmatrix} (b_k(t,x)\cdot\nabla u_1(t,x))_{k\geq 1} \\ 0 \end{pmatrix} \right), && (5.5)\\
G &: [0,T] \times X_1 \times \Omega \to \gamma(U, X_{1/2}) : \\
&\qquad G(t,u,\omega) = \begin{pmatrix} (g_{1,k}(t,u(t,x)))_{k\geq 1} \\ (g_{2,k}(t,u(t,x)))_{k\geq 1} \end{pmatrix}, && (5.6)
\end{aligned}$$

and we define $A_0 = \tilde{A}_0 + A_{\mathrm{pert}}$, $A = A_0 - F$ and $B = B_0 + G$. We now extend the results of the reaction-diffusion framework of [AV23b] to the FitzHugh-Nagumo equations.

Assumption 5.1. *Let $d \geq 2$. We say that Assumption 5.1(p,q,h,δ) holds if $p \in (2,\infty)$, $q \in [2,\infty)$, $h > 1$, $\delta \in [1,2)$, $\eta \in (0, 2-\delta]$ and for $i = 1,2$ the following hold:*

1. *For each $j \in \{1,\dots,d\}$, $b^j := (b^j_k)_{k\geq 1} : \mathbb{R}_{\geq 0} \times \Omega \times \mathbb{T}^d \to \ell^2$ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{T}^d)$-measurable,*
2. *There exist $N > 0$ and $\alpha > \max\{\frac{d}{\rho}, \delta - 1\}$ with $\rho \in [2,\infty)$ such that a.s. for all $t \in \mathbb{R}_{\geq 0}$ and $j \in \{1,\dots,d\}$*
$$\|(b^j_k(t,\cdot))_{k\geq 1}\|_{H^{\alpha,\rho}(\ell^2)} \leq N, \tag{5.7}$$
3. *There exists a $\nu_0 \in (0,\nu)$ such that, a.s. for all $t \in \mathbb{R}_{\geq 0}$, $x \in \mathbb{T}^d, \xi \in \mathbb{R}^d$*
$$\sum_{j,l=1}^{d} \left(\nu\delta_{j,l} - \frac{1}{2}\sum_{k\geq 1} b^j_k(t,x) b^l_k(t,x) \right) \xi_j \xi_l \geq \nu_0 |\xi|^2, \tag{5.8}$$
4. *The map $f : \mathbb{R} \to \mathbb{R}$ is $\mathcal{B}(\mathbb{R})$-measurable and the maps $g_i := (g_{k,i})_{k\geq 1} : \mathbb{R}_{\geq 0} \times \Omega \times \mathbb{R}^2 \to \ell^2$ are $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^2)$-measurable. Moreover, for any $y, y' \in \mathbb{R}$*
$$f(\cdot, 0) = 0$$

$$\mu_{f'} \coloneqq \sup_{y\in\mathbb{R}} f'(y) < \infty,$$
$$|f(y) - f(y')| \leq C_1(1 + |y|^{h-1} + |y'|^{h-1})|y - y'|,$$
$$g_i(\cdot, 0) \in L^\infty(\mathbb{R}_{\geq 0} \times \Omega; C^1(\mathbb{T}^d; \ell^2)),$$

for any $y, y' \in \mathbb{R}^2$,

$$\|g_1(t, y) - g_1(t, y')\|_{\ell^2} \leq C_1(|y_1 - y_1'| + |y_2 - y_2'|),$$
$$\|g_2(t, y) - g_2(t, y')\|_{\ell^2} \leq C_1(|y_1 - y_1'| + |y_2 - y_2'|),$$

and for $u, v \in X_1$,

$$\|g_2(t, u) - g_2(t, v)\|_{X^2(\ell^2)} \leq C_1(\|u_1 - v_1\|_{X^2(\ell^2)} + \|u_2 - v_2\|_{X^2(\ell^2)})$$

We define a solution to the FitzHugh–Nagumo equations in the $L^p(L^q(\mathbb{T}^d))$-setting as follows.

Definition 5.2 (Solution; Definition 2.3 in [AV23b])**.** Let Assumption 5.1(p, q, h, δ) be satisfied for some $h > 1$ and let $\kappa \in [0, \frac{p}{2} - 1)$.

- Let σ be a stopping time taking values in $[0, \infty]$ and let $u : [0, \sigma) \times \Omega \to H^{2-\delta,q} \oplus H^{2-\delta-\eta,q}$ be a stochastic process. We say (u, σ) is a *local* $(p, \kappa, \delta, \eta, q)$*-solution* to (4.1) if there exists a sequence of stopping times $(\sigma_j)_{j\geq 1}$ such that
  - $\sigma_j \leq \sigma$ a.s. for all $j \geq 1$ and $\lim_{j\to\infty} \sigma_j = \sigma$ a.s.,
  - for all $j \geq 1$ the process $\mathbb{1}_{[0,\sigma_j]\times\Omega} u$ is progressively measurable,
  - a.s. for all $j \geq 1$ we have $u_i \in L^p(0, \sigma_j, w_\kappa; H^{2-\delta,q} \oplus H^{2-\delta-\eta,q})$ and
  $$f(\cdot, u) \in L^p(0, \sigma_j, w_\kappa; H^{-\delta,q} \oplus H^{2-\delta-\eta,q})),$$
  $$G(\cdot, u) \in L^p(0, \sigma_j, w_\kappa; H^{1-\delta,q}(\ell^2) \oplus H^{2-\delta-\eta,q}(\ell^2)),$$
  - a.s. for all $j \geq 1$, for all $t \in [0, \sigma_j]$ it holds
  $$\begin{aligned} u_1(t) - u_{0,1} &= \int_0^t \nu\Delta u_1 + f(u_1) - u_2 \,\mathrm{d}s \\ &\quad + \sum_{k\geq 1} \int_0^t [(b_k \cdot \nabla)u_1 + g_{1,k}(\cdot, u)] \,\mathrm{d}W_1(s) \\ u_2(t) - u_{0,1} &= \int_0^t \epsilon u_1 - \epsilon\gamma u_2 \,\mathrm{d}s \\ &\quad + \sum_{k\geq 1} \int_0^t g_{2,k}(\cdot, u) \,\mathrm{d}W_2(s) \end{aligned}$$
- (u, σ) is a $(p, \kappa, \delta, \eta, q)$*-solution* to (4.1) if for every other local $(p, \kappa, \delta, \eta, q)$-solution (u', σ') it holds that a.s. $\sigma' \leq \sigma$ and $u = u'$ on $[0, \sigma') \times \Omega$.

Assumption 5.3 (*Assumption 2.4 in [AV23b]*). *Let $d \geq 2$. Assumption 5.3(p, q, h, δ) holds if $p \in (2, \infty)$, $q \in [2, \infty)$, $h > 1$ and $\delta \in [1, \frac{h+1}{h})$ satisfy*

$$\frac{1}{p} + \frac{1}{2}\left(\delta + \frac{d}{q}\right) \leq \frac{h}{h-1}, \quad \frac{d}{d-\delta} < q < \frac{d(h-1)}{h+1-\delta(h-1)}. \tag{5.9}$$

Theorem 5.4 (Local Existence, Uniqueness, and Regularity). *Suppose that Assumption 5.1(p, q, h, δ) holds, $q > \max\left\{\frac{d}{d-\delta}, \frac{d(h-1)}{2h-\delta(h-1)}\right\}$ and that $\kappa \in [0, \frac{p}{2} - 1)$ satisfies either*

$$q < \frac{d(h-1)}{\delta} \text{ and } \frac{1+\kappa}{p} + \frac{1}{2}\left(\delta + \frac{d}{q}\right) \leq \frac{h}{h-1} \tag{5.10}$$

or

$$q \geq \frac{d(h-1)}{\delta} \text{ and } \frac{1+\kappa}{p} \leq \frac{h}{h-1}\left(1 - \frac{\delta}{2}\right). \tag{5.11}$$

Then for any $\eta \in (0, 2-\delta]$ and

$$u_0 \in L^0_{\mathcal{F}_0}\left(\Omega; B_{q,p}^{2-\delta-2\frac{1+\kappa}{p}} \oplus H^{2-\delta-\eta,q}\right) \tag{5.12}$$

there exists a unique $(p, \kappa, \delta, \eta, q)$-solution (u, σ) to (4.1) such that a.s. $\sigma > 0$ and

$$u \in L^p_{loc}([0, \sigma), w_k; H^{2-\delta,q} \oplus H^{2-\delta-\eta,q}) \cap C\left([0, \sigma); B_{q,p}^{2-\delta-2\frac{1+\kappa}{p}} \oplus H^{2-\delta-\eta,q}\right). \tag{5.13}$$

Moreover, u regularises instantaneously in time and, in addition, u_1 regularises instantaneously in space; let

$$b = \begin{cases} \infty & \text{if} \quad \delta + \eta + \frac{d}{q} \leq 2, \\ \max\{d/(\delta + \eta + \frac{d}{q} - 2), q\} & \text{if} \quad \delta + \eta + \frac{d}{q} > 2, \end{cases}$$

then

$$u \in H^{\theta,r}_{loc}\left(0, \sigma; H^{1-2\theta,\zeta} \oplus H^{2-\delta-\eta,q}\right) \quad \text{a.s. for all } \theta \in [0, 1/2), r \in (2, \infty), \zeta \in (2, b) \tag{5.14}$$

and

$$u_1 \in C^{\theta_1,\theta_2}_{loc}\left((0, \sigma) \times \mathbb{R}^d\right) \quad \text{a.s. for all } \theta_1 \in [0, 1/2), \theta_2 \in \left(0, 1 - \frac{d}{b}\right). \tag{5.15}$$

Remark 3. Note that (5.14) holds in any case for $\zeta = q$. Moreover, we prove local well-posedness for general $h > 1$, which will be used in our proof of global well-posedness. However, for the FitzHugh–Nagumo Equations (4.1) we assume $h = 3$.

The proof of Theorem 5.4 shows that the integrability in space that can be obtained is limited by the smoothness of u_2. This is reflected in our use of the Sobolev embedding $H^{2-\delta-\eta,q} \hookrightarrow H^{-1,\zeta}$ in Step 3 of the proof.

In the proof of Theorem 5.4 we will use the following lemma, the proof of which can be found in [AV23b].

Lemma 5.5 (Lemma 3.2 in [AV23b])**.** *Suppose that Assumption 5.1*(p,q,h,δ,η) *holds and let*

$$\Phi(t,v) := f(v)$$

Moreover, suppose that $q > \max\left\{\frac{d}{d-\delta}, \frac{d(h-1)}{2h-\delta(h-1)}\right\}$. *Let* $\rho_1 = h-1$ *and*

$$\beta_1 = \begin{cases} \frac{1}{2}\left(\delta + \frac{d}{q}\right)\left(1 - \frac{1}{h}\right), & \text{if } q < \frac{d(h-1)}{\delta}, \\ \frac{\delta}{2}, & \text{if } q \geq \frac{d(h-1)}{\delta}. \end{cases}$$

Then $\beta_1 \in (0,1)$ *and for* $v, v' \in X_1^1$, Φ *is a lower-order nonlinearity, that is,*

$$\|\Phi(\cdot,v) - \Phi(\cdot,v')\|_{X_0^1} \lesssim (1 + \|v\|_{X_{\beta_1}}^{\rho_1} + \|v'\|_{X_{\beta_1}}^{\rho_1})\|v - v'\|_{X_{\beta_j}}$$
$$\|\Phi(\cdot,v)\|_{X_0^1} \lesssim (1 + \|v\|_{X_{\beta_1}}^{\rho_1})\|v\|_{X_{\beta_1}}.$$

Proof of Theorem 5.4. Existence and uniqueness. We would like to apply [AV22a, Theorem 4.8]. To this end, we verify that [AV22a, Hypothesis (H)] is satisfied and that $(A_0, B_0) \in \mathcal{SMR}_{p,\kappa}^{\bullet}(T)$ for all $T \in (0,\infty)$.

By Assumption 5.1(4), the first component of f satisfies the conditions of Lemma 5.5 with β_1 as given in the lemma. Since the second component is 0, it follows that [AV22a, hypothesis (HF)] holds. For the first component of G we have by Assumption 5.1(4) for $u, v \in X_1$

$$\begin{aligned} \|g_1(u) - g_1(v)\|_{\gamma(\ell^2;X_{1/2}^1)} &\lesssim \|u_1 - v_1\|_{L^q} + \|u_2 - v_2\|_{L^q} \\ &\lesssim \|u_1 - v_1\|_{(H^{-\delta,q},H^{2-\delta,q})_{\beta_2}} + \|u_2 - v_2\|_{H^{2-\delta-\eta,q}} \\ &= \|u - v\|_{X_{\beta_2}}, \end{aligned} \tag{5.16}$$

for any $\beta_2 \in [\frac{\delta}{2}, 1)$, where we used that $\delta + \eta \in (0,2]$. Due to the linear growth of g_1 we can thus choose $\beta_2 \in (1 - \frac{1+\kappa}{p}, 1)$ sufficiently close to 1 and $\rho_2 \geq 0$ sufficiently small such that $\frac{1+\kappa}{p} \leq \frac{\rho_2+1}{\rho_2}(1-\beta_2)$ and thus (4.3) of [AV22a, (Hypothesis (HG)] holds. Similarly, for the second component of G we have by Assumption 5.1(4) for $u, v \in X_1$

$$\begin{aligned} \|g_2(u) - g_2(v)\|_{\gamma(\ell^2;X_{1/2})} &\lesssim \|u_1 - v_1\|_{H^{2-\delta-\eta,q}} + \|u_2 - v_2\|_{H^{2-\delta-\eta,q}} \\ &\lesssim \|u_1 - v_1\|_{(H^{-\delta,q},H^{2-\delta,q})_{\beta_3}} + \|u_2 - v_2\|_{H^{2-\delta-\eta,q}} \\ &= \|u - v\|_{X_{\beta_3}}, \end{aligned} \tag{5.17}$$

for any $\beta_3 \in [1 - \frac{\eta}{2}, 1)$, and since $\eta \in (0, 2-\delta]$ we can again choose $\beta_3 \in (1 - \frac{1+\kappa}{p}, 1)$ sufficiently close to 1 and $\rho_3 \geq 0$ sufficiently small such that (4.3) of

[AV22a, Hypothesis (HG)] holds, from which it follows that [AV22a, Hypothesis (HG)] also holds and we conclude that [AV22a, Hypothesis (H)] holds. To verify that $(A_0, B_0) \in \mathcal{SMR}^\bullet_{p,\kappa}$ we first note that $(\tilde{A}_0, B_0) \in \mathcal{SMR}^\bullet_{p,\kappa}(T)$; for the first component this follows from [AV21, Theorem 5.2 and Remark 5.6], and for the second component this is immediate from the definition of stochastic maximum regularity [AV21, Definition 2.3]. To establish stochastic maximum regularity of (A_0, B_0), we apply the perturbation result [AV21, Theorem 3.2]. Let $u \in X_1$ and fix $\epsilon_0 > 0$ arbitrarily small, then we have

$$\begin{aligned}\|A_{\text{pert}}u\|_{X_0} \leq& \|u_2\|_{H^{-\delta,q}} + \epsilon\|u_1\|_{H^{2-\delta-\eta,q}} + \epsilon\gamma\|u_2\|_{H^{2-\delta-\eta,q}} \\ \lesssim& (1+\epsilon\gamma)\|u_2\|_{H^{2-\delta-\eta,q}} + \epsilon\|u_1\|^{1-\eta/2}_{H^{-\delta,q}}\|u_1\|^{\eta/2}_{H^{2-\delta,q}},\end{aligned}$$

by the Gagliardo-Nirenberg inequality,

$$\begin{aligned}\leq&(1+\epsilon\gamma)\|u_2\|_{H^{2-\delta-\eta,q}} + \epsilon\left(1-\frac{\eta}{2}\right)\epsilon_0^{-\eta/(2-\eta)}\|u_1\|_{H^{-\delta,q}} \\ &+ \epsilon\frac{1}{2}\eta\epsilon_0^{2/\eta}\|u_1\|_{H^{2-\delta,q}} \\ \leq& C_{\epsilon,\gamma,\eta,\epsilon_0}\|u\|_{X_0} + \epsilon_0^{2/\eta}C_{\epsilon,\eta}\|u\|_{X_1},\end{aligned} \tag{5.18}$$

Thus, $\|A_{\text{pert}}u\|_{X_0}$ can be bounded by $\|u\|_{X_0}$ and $\|u\|_{X_1}$, where the constant for $\|u\|_{X_1}$ is arbitrarily small, and [AV21, Theorem 3.2] yields $(A_0, B_0) \in \mathcal{SMR}^\bullet_{p,\kappa}$. Finally, we check [AV22a, Assumption 3.2], which requires that there is a constant C_{A_0,B_0} such that $\|A_0\|_{\mathcal{L}(X_1,X_0)} + \|B_0\|_{\mathcal{L}(X_1,\gamma(U,X_{1/2}))} \leq C_{A_0,B_0}$. The existence of such a constant for B_0 is clear. For A_0 we have for $u \in X_1$:

$$\begin{aligned}\|A_0u\|_{X_0} \leq& \|\Delta u_1\|_{H^{-\delta,q}} + \|u_2\|_{H^{-\delta,q}} + \epsilon\|u_1\|_{H^{2-\delta-\eta,q}} + \epsilon\gamma\|u_2\|_{H^{2-\delta-\eta,q}} \\ \leq& (1+\epsilon+\epsilon\gamma)\|u\|_{X_1}\end{aligned}$$

Now [AV22a, Theorem 4.8] gives the existence of a unique (p,κ,δ,q)-solution with

$$\begin{aligned}u \in& H^{\theta,p}_{\text{loc}}([0,\sigma), w_k; H^{2-\delta-2\theta,q} \oplus H^{2-\delta-\eta,q}) \\ &\cap C\left([0,\sigma); B^{2-\delta-2\frac{1+\kappa}{p}}_{p,q} \oplus H^{2-\delta-\eta,q}\right) \quad \text{a.s.}\end{aligned}$$

for all $\theta \in [0, 1/2)$, from which the regularity properties (5.13) follow by weighted Sobolev embedding [AV22a, Proposition 2.7].

Instantaneous regularisation. Our proof follows the general lines of the proof of [AV23b, Proposition 3.1] with some adjustments for the limited regularity of u_2.

Step 1. We bootstrap regularity in time using [AV22b, Corollary 6.5 or Proposition 6.8] to show that

$$u \in \cap_{\theta\in[0,1/2)} H^{\theta,r}_{\text{loc}}(0,\sigma; H^{2-\delta-2\theta,q} \oplus H^{2-\delta-\eta,q}) \quad \text{a.s. for all } r \in (2,\infty). \tag{5.19}$$

As in [AV23b],

- If $\kappa = 0$ we choose $r \in (p, \infty)$ and $\alpha \in (0, \frac{r}{2} - 1)$ such that

$$\frac{1}{p} = \frac{1+\alpha}{r} \text{ and } \frac{1}{r} \geq \max_{j \in \{1,2,3\}} \beta_j - 1 + \frac{1}{p}$$

  and apply [AV22b, Proposition 6.8],

- If $\kappa > 0$ we let $(r, \alpha) = (p, \kappa)$ and apply [AV22b, Corollary 6.5],

and we let $X_i = Y_i = H^{2i-\delta,q} \oplus H^{2-\delta-\eta,q}$, $i \in \{0, 1\}$. The (X_0, X_1, κ, p)-setting agrees with the setting in our existence proof, so [AV22b, Hypothesis (H)] is satisfied. Moreover, [AV22b, Assumption 4.5] holds since (A_0, B_0) are independent of u, and [AV22b, Assumption 4.7] holds by [AV22b, Remark 4.8]. In the (Y_0, Y_1, α, r)-setting [AV22b, Hypothesis (H)] holds by Lemma 5.5 applied to F, the arguments given in the existence proof for G, and our choice of the parameters (α, r) above (in particular, $\frac{1}{p} = \frac{1+\alpha}{r} \leq \frac{1+\kappa}{p}$). Moreover, we check the required embeddings:

- $Y_r^{\mathrm{Tr}} = B_{q,r}^{2-\delta-2/r} \oplus H^{2-\delta-\eta,q} \hookrightarrow B_{q,p}^{2-\delta-2/p} \oplus H^{2-\delta-\eta,q} = X_r^{\mathrm{Tr}}$ by our choice of $r \geq p$,

- $Y_i = X_i$ for $i \in \{0, 1\}$.

An application of [AV22b, Corollary 6.5], or [AV22b, Proposition 6.8] if $\kappa = 0$, now gives (5.19).

Step 2. We bootstrap differentiability in space using [AV22b, Theorem 6.3] to show that

$$u \in \cap_{\theta \in [0,1/2)} H^{\theta,r}_{\mathrm{loc}}(0, \sigma; H^{1-2\theta,q} \oplus H^{2-\delta-\eta,q}) \quad \text{a.s. for all } r \in (2, \infty). \quad (5.20)$$

We may assume that $\delta \in (1, 2)$ since otherwise the result already follows from (5.19). Moreover, we choose $r > \max\left\{p, \frac{2}{2-\delta}\right\}$ such that

$$\frac{1}{r} + \frac{\delta - 1}{2} < \frac{h}{2(h-1)},$$

which is possible since $\frac{\delta-1}{2} < \frac{h}{2(h-1)}$ always holds. We consider the spaces

$$Y_i = X_i = H^{2i-\delta,q} \oplus H^{2-\delta-\eta,q} \quad \hat{Y}_i = H^{2i-1,q} \oplus H^{2-\delta-\eta,q}, i \in \{1, 2\},$$

and we set the parameters

$$\hat{r} = r, \quad \alpha = 0, \quad \hat{\alpha} = \frac{r(\delta - 1)}{2}.$$

Note that whilst [AV23b] bootstrap integrability in space first and therefore may assume that (5.11) applies, we need to consider both cases (5.10) and (5.11) in the settings (X_0, X_1, p, κ), (Y_0, Y_1, r, α), $(\hat{Y}_0, \hat{Y}_1, \hat{r}, \hat{\alpha})$. However, the choice of parameters $(r, \alpha, \hat{r}, \hat{\alpha})$ given above is sufficient in all cases. By Lemma

5.5 together with the arguments in the proof of [AV23b, Part A of Proposition 3.1] it follows that [AV22b, Hypothesis (HF)] holds in the (Y_0, Y_1, r, α) and the $(\hat{Y}_0, \hat{Y}_1, \hat{r}, \hat{\alpha})$ setting, and both are not critical. In the (Y_0, Y_1, r, α)-setting [AV22b, (HG)] holds by the same argument as in Step 1, and in the $(\hat{Y}_0, \hat{Y}_1, r, \alpha)$-setting we can repeat the computations (5.16)-(5.17) with $\hat{Y}_{1/2} = L^q \oplus H^{2-\delta-\eta,q}$ and using that $H^{1,q} \hookrightarrow H^{2-\delta-\eta,q} \hookrightarrow L^q \hookrightarrow H^{-1,q}$ since $\delta + \eta \in (0, 2]$ to show that [AV22b, (HG)] holds. Using the latter embeddings, and repeating the computations (5.18) and the arguments preceding it, we can show that $(A_0, B_0) \in \mathcal{SMR}^\bullet_{p,\kappa}$ also in the $(\hat{Y}_0, \hat{Y}_1, r, \alpha)$-setting. By [AV22b, Lemma 6.2] with $\epsilon = \frac{\delta-1}{2}$ [AV22b, (6.1)] holds. Finally, we check that the required embeddings hold:

- $Y_r^{\mathrm{Tr}} = B^{2-\delta-2/r}_{q,r} \oplus H^{2-\delta-\eta,q} \hookrightarrow B^{2-\delta-2/p}_{q,p} \oplus H^{2-\delta-\eta,q} = X_p^{\mathrm{Tr}}$ holds since $r > p$,
- There exists a $\lambda \in (0, 1)$ such that $\frac{1}{r} + \lambda \leq \frac{1}{p}$ and $Y_\lambda = H^{2\lambda-\delta,q} \oplus H^{2-\delta-\eta,q} \hookrightarrow H^{-\delta,q} \oplus H^{2-\delta-\eta,q} = X_0$ and $Y_1 = H^{2-\delta,q} \oplus H^{2-\delta-\eta,q} \hookrightarrow H^{2(1-\lambda)-\delta,q} \oplus H^{2-\delta-\eta,q} = X_{1-\lambda}$,
- $\hat{Y}_i \hookrightarrow Y_i$ holds since $\delta > 1$,
- $Y_r^{\mathrm{Tr}} = B^{2-\delta-2/r}_{q,r} \oplus H^{2-\delta-\eta,q} = B^{2-\delta-2/r}_{q,r} \oplus H^{2-\delta-\eta,q} = \hat{Y}^{\mathrm{Tr}}_{\hat{\alpha},\hat{r}}$ by the choice of parameters $(\hat{r}, \hat{\alpha})$.

An application of [AV22b, Theorem 6.3] now gives (5.20).

Step 3. We bootstrap integrability in space using [AV22b, Theorem 6.3] to show that

$$u \in \cap_{\theta\in[0,1/2)} H^{\theta,r}_{\mathrm{loc}}(0, \sigma; H^{1-2\theta,\zeta} \oplus H^{2-\delta-\eta,q}) \qquad \text{a.s. for all } r \in (2, \infty), \zeta \in (2, 4). \tag{5.21}$$

As in [AV23b], we prove the claim by an inductive argument. Specifically, we claim that there exists an $\epsilon_0 > 0$ that depends only on (r, δ, q, h, d) such that

$$\begin{aligned} &u \in \cap_{\theta\in[0,1/2)} H^{\theta,r}_{\mathrm{loc}}(0, \sigma; H^{1-2\theta,\zeta} \oplus H^{2-\delta-\eta,q}) \text{ a.s.} \\ &\quad \Longrightarrow u \in \cap_{\theta\in[0,1/2)} H^{\theta,r}_{\mathrm{loc}}(0, \sigma; H^{1-2\theta,\zeta+\epsilon_0} \oplus H^{2-\delta-\eta,q}) \text{ a.s.} \end{aligned} \tag{5.22}$$

for all $\zeta > 2$ if $\delta+\eta+\frac{d}{q} \leq 2$, and for all $\zeta+\epsilon_0 < d/(\delta+\eta+\frac{d}{q}-2)$ if $\delta+\eta+\frac{d}{q} > 2$.

Suppose the left-hand side of (5.22) holds. We now find $r_1, r_2 > p$ and $\alpha_1, \alpha_2 > 0$ such that

$$\frac{1+\alpha_1}{r_1} + \frac{1}{2}\left(\delta + \frac{d}{2}\right) < \frac{h}{h-1} \text{ and } \frac{1+\alpha_2}{r_2} < \frac{h}{h-1}\left(1 - \frac{\delta}{2}\right),$$

and we let $r = \max\{r_1, r_2\}, \alpha = \min\{\alpha_1, \alpha_2\}$. Since $2 < p \leq \zeta$, the former guarantees that (r, ζ, δ, h) and $(r, \zeta + \epsilon_0, \delta, h)$ satisfy either (5.10) or (5.11). We now consider the spaces

$$X_i = H^{2i-\delta,q} \oplus H^{2-\delta-\eta,q}, \quad Y_i = H^{2i-1,\zeta} \oplus H^{2-\delta-\eta,q},$$

$$\hat{Y}_i = H^{2i-1,\zeta+\epsilon_0} \oplus H^{2-\delta-\eta,q}, \quad i \in \{1,2\}$$

and we set the parameters

$$\hat{r} = r, \quad \hat{\alpha} = \alpha.$$

By Lemma 5.5 together with the arguments in the proof of [AV23b, Part A of Proposition 3.1] it follows that [AV22b, Hypothesis (HF)] holds in the (Y_0, Y_1, r, α) and the $(\hat{Y}_0, \hat{Y}_1, \hat{r}, \hat{\alpha})$ setting, and both are not critical. To check that [AV22a, Hypothesis (HG)] holds, we repeat the computations in (5.16)-(5.17). For the first component of G we have by Assumption 5.1(4) for $u, v \in Y_1$

$$\begin{aligned} \|g_1(u) - g_1(v)\|_{\gamma(\ell^2; Y^1_{1/2})} &= \|g_1(u) - g_1(v)\|_{L^\zeta} \\ &\lesssim \|u_1 - v_1\|_{L^\zeta} + \|u_2 - v_2\|_{L^\zeta} \\ &\lesssim \|u_1 - v_1\|_{(H^{-1,\zeta}, H^{1,\zeta})_{\beta_2}} + \|u_2 - v_2\|_{H^{2-\delta-\eta,q}} \\ &= \|u - v\|_{Y_{\beta_2}}, \end{aligned} \tag{5.23}$$

for any $\beta_2 \in [\frac{1}{2}, 1)$, where we used the Sobolev embedding $L^\zeta \hookleftarrow H^{2-\delta-\eta,q}$ with $-\frac{d}{\zeta} \leq 2 - \delta - \eta - \frac{d}{q}$ and $\delta + \eta \in (0, 2]$. Note that our use of the latter embedding result means in particular that we cannot bootstrap integrability in space if $\delta + \eta = 2$. Due to the linear growth of g_1 we can thus again choose $\beta_2 \in (1 - \frac{1+\kappa}{p}, 1)$ sufficiently close to 1 and $\rho \geq 0$ sufficiently small such that $\frac{1+\kappa}{p} \leq \frac{\rho+1}{\rho}(1 - \beta_2)$ and (4.3) of [AV22a, (Hypothesis (HG)] holds. For the second component of G we have by Assumption 5.1(4) for $u, v \in Y_1$

$$\begin{aligned} \|g_2(u) - g_2(v)\|_{\gamma(\ell^2; Y_{1/2})} &= \|g_2(u) - g_2(v)\|_{X^2_0(\ell^2)} \\ &\lesssim \|u_1 - v_1\|_{H^{2-\delta-\eta,q}} + \|u_2 - v_2\|_{H^{2-\delta-\eta,q}} \\ &\lesssim \|u_1 - v_1\|_{H^{2-\delta-\eta,\zeta}} + \|u_2 - v_2\|_{H^{2-\delta-\eta,q}} \\ &\lesssim \|u_1 - v_1\|_{(H^{-1,\zeta}, H^{1,\zeta})_{\beta_3}} + \|u_2 - v_2\|_{H^{2-\delta-\eta,q}} \\ &= \|u - v\|_{Y_{\beta_3}}, \end{aligned} \tag{5.24}$$

for any $\beta_3 \in [1 - \frac{\eta}{2}, 1)$, where we used that $H^{2-\delta-\eta,q} \hookleftarrow H^{2-\delta-\eta,\zeta}$ on the bounded domain $\mathbb{T}^d$. We can now again choose $\beta_3 \in (1 - \frac{1+\kappa}{p}, 1)$ sufficiently close to 1 and $\rho \geq 0$ sufficiently small such that (4.3) of [AV22a, (Hypothesis (HG)] holds, from which it follows that [AV22a, hypothesis (HG)] also holds and we conclude that [AV22a, hypothesis (H)] holds. We verify that $(A_0, B_0) \in \mathcal{SMR}^\bullet_{p,\kappa}$ in the same way as in our existence proof. We have $(\tilde{A}_0, B_0) \in \mathcal{SMR}^\bullet_{p,\kappa}(T)$, which for the first component follows from [AV21, Theorem 5.2 and Remark 5.6], and for the second component it is immediate from the definition of stochastic maximum regularity [AV21, Definition 2.3]. We can thus establish stochastic maximum regularity of (A_0, B_0), by applying the perturbation result [AV21, Theorem 3.2]. Let $u \in Y_1$ and fix $\epsilon_1 > 0$ arbitrarily small, then we have

$$\begin{aligned}\|A_{\mathrm{pert}}u\|_{Y_0} \leq& \|u_2\|_{H^{-1,\zeta}} + \epsilon\|u_1\|_{H^{2-\delta-\eta,q}} + \epsilon\gamma\|u_2\|_{H^{2-\delta-\eta,q}} \\ \lesssim& (1+\epsilon\gamma)\|u_2\|_{H^{2-\delta-\eta,q}} + \epsilon\|u_1\|_{H^{2-\delta-\eta,\zeta}},\end{aligned}$$

by the Sobolev embedding $H^{-1,\zeta} \hookleftarrow H^{2-\delta-\eta,q}$ with $-1-\frac{d}{\zeta} \leq 2-\delta-\eta-\frac{d}{q}$ and the embedding $H^{2-\delta-\eta,q} \hookleftarrow H^{2-\delta-\eta,\zeta}$ on the bounded domain $\mathbb{T}^d$, and by the Gagliardo-Nirenberg inequality

$$\begin{aligned}\|A_{\mathrm{pert}}u\|_{Y_0} \lesssim& (1+\epsilon\gamma)\|u_2\|_{H^{2-\delta-\eta,q}} + \epsilon\|u_1\|_{H^{-1,\zeta}}^{(\delta+\eta-1)/2}\|u_1\|_{H^{1,\zeta}}^{(3-\delta-\eta)/2} \\ \leq& (1+\epsilon\gamma)\|u_2\|_{H^{2-\delta-\eta,q}} + \frac{\epsilon}{2}(\delta+\eta-1)\,\epsilon_1^{-2/(\delta+\eta-1)}\|u_1\|_{H^{-\delta,q}} \\ &+ \frac{\epsilon}{2}(3-\delta-\eta)\epsilon_1^{2/(3-\delta-\eta)}\|u_1\|_{H^{2-\delta,q}} \\ \leq& C_{\epsilon,\gamma,\eta,\epsilon_0}\|u\|_{X_0} + \epsilon_1^{2/(3-\delta-\eta)}C_{\epsilon,\eta}\|u\|_{X_1},\end{aligned} \tag{5.25}$$

Thus, $\|A_{\mathrm{pert}}u\|_{Y_0}$ can be bounded by $\|u\|_{Y_0}$ and $\|u\|_{Y_1}$, where the constant for $\|u\|_{Y_1}$ is arbitrarily small, and [AV21, Theorem 3.2] yields $(A_0,B_0) \in \mathcal{SMR}_{p,\kappa}^{\bullet}$. Repeating the same computations with ζ replaced by $\zeta+\epsilon_0$ shows that [AV22b, Hypothesis (H)] holds if ϵ_0 is sufficiently small and that $(A_0,B_0) \in \mathcal{SMR}_{p,\kappa}^{\bullet}$ also in the $(\hat{Y}_0,\hat{Y}_1,\hat{r},\hat{\alpha})$-setting. By [AV22b, Lemma 6.2] also [AV22b, (6.1)] holds. Finally, we check that the required embeddings hold:

- $Y_r^{\mathrm{Tr}} = B_{\zeta,r}^{1-2/r} \oplus H^{2-\delta-\eta,q} \hookrightarrow B_{q,p}^{2-\delta-2/p} \oplus H^{2-\delta-\eta,q} = X_p^{\mathrm{Tr}}$ holds since $r > p, \zeta \geq q$,
- There exists a $\lambda \in (0,1)$ such that $\frac{1}{r}+\lambda \leq \frac{1}{p}$ and $Y_\lambda = H^{2\lambda,q} \oplus H^{2-\delta-\eta,q} \hookrightarrow H^{-\delta,q} \oplus H^{2-\delta-\eta,q} = X_0$ and $Y_1 = H^{1,q} \oplus H^{2-\delta-\eta,q} \hookrightarrow H^{2(1-\lambda)-\delta,q} \oplus H^{2-\delta-\eta,q} = X_{1-\lambda}$,
- $\hat{Y}_i \hookrightarrow Y_i$ since $\epsilon_0 > 0$ and $H^{2i-1,\zeta+\epsilon_0} \hookrightarrow H^{2i-1,\zeta}$ on the bounded domain $\mathbb{T}^d$,
- $Y_r^{\mathrm{Tr}} = B_{\zeta,r}^{1-2/r} \oplus H^{2-\delta-\eta,q} = B_{\zeta+\epsilon,r}^{1-2(1+\alpha)/r} \oplus H^{2-\delta-\eta,q} = \hat{Y}_{\hat{\alpha},\hat{r}}^{\mathrm{Tr}}$ holds by Sobolev embedding if

$$1-\frac{2}{r}-\frac{d}{\zeta} \geq 1-2\frac{1+\alpha}{r}-\frac{d}{\zeta+\epsilon}.$$

Since $\zeta \geq 2$, a sufficient choice for the last embedding is $\epsilon_0 \leq \frac{\alpha}{2r}$. An application of [AV22b, Theorem 6.3] now gives (5.22), and, making ϵ_0 smaller if necessary, iterating (5.22) yields (5.21).

Step 4. The result (5.15) follows by the Sobolev embeddings $H^{\theta,r} \hookrightarrow C^{\theta_1}$ if $\theta-\frac{d}{r} \geq \theta_1 > 0$ and $H^{1-2\theta,\zeta} \hookrightarrow C^{\theta_2}$ if $1-2\theta-\frac{d}{\zeta} \geq \theta_2 > 0$. By Step 1 we have $\theta_1 \in [0,1/2)$. If $\delta+\eta+\frac{d}{q} \leq 2$, we have $\theta_2 \in (0,1)$, whilst if $\delta+\eta+\frac{d}{q} > 2$, the limits on integrability in space obtained in Step 3 impose the restriction $\theta_2 \in (0, 1-(\delta+\eta+\frac{d}{q}-2))$ if the last term is positive. □

Corollary 5.6 (Local Well-Posedness in Critical Spaces; Theorem 2.7 in [AV23b])**.** *Suppose that Assumptions 5.1*(p,q,h,δ) *and 5.3*(p,q,h,δ) *hold, and set* $\kappa =: \kappa_c = p\left(\frac{h}{h-1} - \frac{1}{2}\left(\delta + \frac{d}{q}\right)\right) - 1$*. Then for any* $\eta \in (0, 2-\delta]$ *and*

$$u_0 \in L^0\left(\Omega; B_{q,p}^{\frac{d}{q}-\frac{2}{h-1}} \oplus H^{2-\delta,q}\right) \tag{5.26}$$

there exists a unique $(p,\kappa_c,\delta,\eta,q)$*-solution* (u,σ) *such that a.s.* $\sigma > 0$ *and*

$$u \in C([0,\sigma); B_{q,p}^{\frac{d}{q}-\frac{2}{h-1}} \oplus H^{2-\delta-\eta,q}) \text{ a.s.} \tag{5.27}$$

$$u \in H^{\theta,p}_{loc}\left([0,\sigma), w_{\kappa_c}; H^{2-\delta-2\theta,q} \oplus H^{2-\delta-\eta,q}\right) \text{ a.s. for all } \theta \in [0,1/2). \tag{5.28}$$

Moreover, u *regularises instantaneously in time and, in addition,* u_1 *regularises instantaneously in space; let*

$$b = \begin{cases} \infty & \text{if} \quad \delta + \eta + \frac{d}{q} \le 2, \\ \max\{d/(\delta + \eta + \frac{d}{q} - 2), q\} & \text{if} \quad \delta + \eta + \frac{d}{q} > 2, \end{cases}$$

then

$$u \in H^{\theta,r}_{loc}\left(0,\sigma; H^{1-2\theta,\zeta} \oplus H^{2-\delta-\eta,\zeta}\right) \quad \text{a.s. for all } \theta \in [0,1/2), r \in (2,\infty), \zeta \in (2,b), \tag{5.29}$$

and

$$u_1 \in C^{\theta_1,\theta_2}_{loc}\left((0,\sigma) \times \mathbb{T}^d\right) \quad \text{a.s. for all } \theta_1 \in [0,1/2), \theta_2 \in \left(0, 1 - \frac{d}{b}\right). \tag{5.30}$$

Proof. Corollary 5.6 is a direct consequence of Theorem 5.4. The proof that [AV23b, Theorem 3.1] implies their [AV23b, Theorem 2.7] carries over verbatim as it only requires checking that the conditions on (p,q,h,δ,κ_c) given in Corollary 5.6 are compatible with the conditions of Theorem 5.4. □

Theorem 5.7 (Blow-Up Criteria)**.** *Let the assumptions of Corollary 5.6 be satisfied and let* (u,σ) *be the* (p,κ_c,δ,q)*-solution to* (4.1)*. Suppose further that* $p_0 \in (2,\infty)$*,* $h_0 \ge h$*,* $\delta_0 \in [1,2)$ *are such that Assumptions 5.1*(p_0,q,h_0,δ_0) *and 5.3*(p_0,q,h_0,δ_0) *hold. Let*

$$\beta_0 = \frac{d}{q} - \frac{2}{h_0 - 1}, \quad \gamma_0 = \frac{d}{q} + \frac{2}{p_0} - \frac{2}{h_0 - 1}.$$

If $\eta_0 \in (0, 2-\delta_0]$ *is such that* $\delta + \eta = \delta_0 + \eta_0$*, then for all* $0 < s < T < \infty$

$$\begin{aligned} \mathbb{P}(\sup_{t\in[s,\sigma]} &\|u(t)\|_{B^{\beta_0}_{q,p_0} \oplus H^{2-\delta_0-\eta_0,q}} \\ &+ \|u\|_{L^{p_0}(s,\sigma; H^{\gamma_0,q} \oplus H^{2-\delta_0-\eta_0,q})} < \infty, s < \sigma < T) = 0. \end{aligned} \tag{5.31}$$

Proof. The proof follows the lines of the proof given in [AV23b]. Fix $0 < s < T < \infty$ and let (u, σ) be the (p, κ_c, δ, q)-solution to (4.1). Let $(p_0, \delta_0, h_0, \beta_0)$ be as in Theorem 5.7. Set $\kappa_{c,0} = p_0\left(\frac{h_0}{h_0-1} - \frac{1}{2}\left(\delta_0 + \frac{d}{q}\right)\right) - 1$. By the regularisation result (5.27) and the assumptions of Theorem 5.7 we have $u_2 \in C((0,\sigma); H^{2-\delta-\eta,q}) = C((0,\sigma); H^{2-\delta_0-\eta_0,q})$ for some $\eta_0 \in (0, 2-\delta_0]$. Moreover, using the regularisation result (5.29) (instead of the regularisation result (5.30) together with the embedding $C^\theta \hookrightarrow B^{\beta_0}_{q,p_0}$ for $\theta \in (\beta_0 \wedge 0, 1)$ used in [AV23b]) and the embeddings $H^{1-2\theta,q} \hookrightarrow B^{1-2\theta}_{q,\infty} \hookrightarrow B^{\beta_0}_{q,p_0}$ for a suitable $\theta \in [0, 1/2)$, which exists since $\beta_0 < 1$, we obtain

$$\mathbb{1}_{\{\sigma > s\}} u(s) \in L^0_{\mathcal{F}_s}(\Omega; B^{\beta_0}_{q,p_0} \oplus H^{2-\delta_0-\eta_0,q}).$$

We can now consider the SPDE

$$\begin{cases} \mathrm{d}v_1(t,x) = & (\nu\Delta v_1(t,x) + f(v_1(t,x)) - v_2(t,x))\,\mathrm{d}t \\ & + \sum_{k\geq 1}[b_k(t,x)\cdot\nabla v_1(t,x) + g_{1,k}(t,v(t,x))]\,\mathrm{d}W_1(t) \\ \mathrm{d}v_2(t,x) = & \epsilon(v_1(t,x) - \gamma v_2(t,x))\,\mathrm{d}t + \sum_{k\geq 1} g_{2,k}(t,v(t,x))\,\mathrm{d}W_2(t), \end{cases} \tag{5.32}$$

with $v_i(s) = \mathbb{1}_{\{\sigma > s\}} u_i(s)$, which by Theorem 5.4 has unique $(p_0, \kappa_{c,0}, \delta_0, q)$-solution (v, τ) on the interval $[s, \infty)$ such that

$$\begin{gathered} v \in H^{\theta,r}((s,\tau), w^s_\kappa; H^{1-2\theta,\zeta} \oplus H^{2-\delta-\eta,q}) \cap C([s,\tau); B^{\beta_0}_{q,p_0} \oplus H^{2-\delta_0-\eta_0,q}) \\ \forall \theta \in [0,1/2), r \in (2,\infty), \zeta \in (2,b), \end{gathered} \tag{5.33}$$

with b as given in Theorem 5.4. Applying the blow-up criterion in the abstract setting provided in [AV22b, Theorem 4.10(3)], we obtain

$$\begin{aligned} \mathbb{P}(\sup_{t\in[s,\tau]} \{\|v_1(t)\|_{B^{\beta_0}_{q,p_0}} + \|v_2(t)\|_{H^{2-\delta_0-\eta_0,q}}\} + \|v_1\|_{L^{p_0}(s,\tau;H^{\gamma_0,q})} \\ + \|v_2\|_{L^{p_0}(s,\tau;H^{2-\delta_0-\eta_0,q})} < \infty, \tau < T) = 0. \end{aligned}$$

It remains to show that the solution (v, τ) agrees with (u, σ), specifically,

$$\tau = \sigma \text{ a.s. on } \{\sigma > s\}, \quad u = v \text{ a.e. on } [s,\sigma) \times \{\sigma > s\}. \tag{5.34}$$

Note that by (5.29) and the assumption $h_0 \geq h$ we have that $(u|_{[s,\sigma)}, \mathbb{1}_{\{\sigma>s\}}\sigma + \mathbb{1}_{\Omega\setminus\{\sigma>s\}} s)$ is a $(p_0, \kappa_0, \delta_0, q)$-solution to (5.32), and by maximality of (v, τ) we have

$$\sigma \leq \tau \text{ on } \{\sigma > s\}, \quad u = v \text{ a.s. on } [s,\sigma) \times \{\sigma > s\}.$$

Applying the blow-up criteria in the abstract setting [AV22b, Theorem 4.10(3)] to u yields

$$\begin{aligned} \mathbb{P}(\sigma < T, \sup_{t\in[0,\sigma)} \{\|u_1(t)\|_{B^\beta_{q,p}} + \|u_2(t)\|_{H^{2-\delta-\eta,q}}\} \\ + \|u_1(t)\|_{L^p(0,\sigma;H^{\gamma,q})} + \|u_2(t)\|_{L^p(0,\sigma;H^{2-\delta-\eta,q})} < \infty) = 0 \end{aligned}$$

where

$$\beta = \frac{d}{q} - \frac{2}{h-1}, \quad \gamma = \frac{d}{q} + \frac{2}{p} - \frac{2}{h-1}, \quad \text{and } \kappa_c = p\left(\frac{h}{h-1} - \frac{1}{2}\left(\delta + \frac{d}{q}\right)\right) - 1.$$

The regularity result (5.33) together with the fact that Assumption 5.3 implies that $\gamma \leq 1$ give $u = v \in L^p_{\text{loc}}((s,\sigma]; H^{\gamma,q} \oplus H^{2-\delta-\eta,q})$ on $\{\sigma > s, \sigma < \tau\}$. By (5.27) with $\theta_c := \frac{\kappa_c}{p} < \frac{1}{2} - \frac{1}{p}$ and weighted Sobolev embeddings we have

$$\begin{aligned} u \in & H^{\theta_c,p}_{\text{loc}}([0,\sigma), w_{\kappa_c}; H^{2-\delta-2\theta_c,q} \oplus H^{2-\delta-\eta,q}) \\ & \subseteq L^p_{\text{loc}}([0,\sigma); H^{\gamma,q} \oplus H^{2-\delta-\eta,q}) \quad \text{a.s.,} \end{aligned}$$

and thus also $u \in L^p(0,\sigma; H^{\gamma,q} \oplus H^{2-\delta-\eta,q})$ a.s. on $\{\sigma < \tau\}$. Similarly, the regularity result (5.29) applied to v gives $u = v \in C((s,\sigma]; B^{\beta}_{p,q} \oplus H^{2-\delta-\eta,q})$ on $\{\sigma > s, \sigma < \tau\}$. By (5.27) we also have $u \in C([0,\sigma); B^{\beta}_{q,p} \oplus H^{2-\delta-\eta,q})$, from which it follows that $u \in C([s,\sigma]; B^{\beta}_{q,p} \oplus H^{2-\delta-\eta,q})$ on $\{\sigma > s, \sigma < \tau\}$. We thus get

$$\begin{aligned} \mathbb{P}(\sigma > s, \sigma < \tau) =& \mathbb{P}(\sigma > s, \sigma < \tau, \sup_{t\in[0,\sigma)} \{\|u_1(t)\|_{B^{\beta}_{q,p}} + \|u_2(t)\|_{H^{2-\delta-\eta,q}}\} \\ & + \|u_1(t)\|_{L^p(0,\sigma;H^{\gamma,q})} + \|u_2(t)\|_{L^p(0,\sigma;H^{2-\delta-\eta,q})} < \infty) \\ \leq & \mathbb{P}(\sigma < T, \sup_{t\in[0,\sigma)} \{\|u_1(t)\|_{B^{\beta}_{q,p}} + \|u_2(t)\|_{H^{2-\delta-\eta,q}}\} \\ & + \|u_1(t)\|_{L^p(0,\sigma;H^{\gamma,q})} + \|u_2(t)\|_{L^p(0,\sigma;H^{2-\delta-\eta,q})} < \infty) = 0. \end{aligned}$$

Thus, on $\{\sigma > s\}$ we have $\sigma = \tau$ as claimed in (5.34). □

5.2 Global Well-Posedness

In this section we prove global well-posedness of the FitzHugh-Nagumo Equations (4.1). We will assume that $d \geq 2$; the case $d = 1$ can be accommodated by adding a dummy variable in (4.1).

Assumption 5.8 (L^ζ-Coercivity; Version of *Assumption 4.1 in [AV23a]*)**.** *Suppose $d \geq 2$, Assumption 5.1(p, q, h, δ) holds with $h = 3$ and let $\zeta \in \{q\} \cup (q, b)$ with b as in Corollary 5.6. We say that Assumption 5.8 holds if there exist constants $\theta, M, C, > 0$ such that a.e. on $\mathbb{R}_{\geq 0} \times \Omega$ and for all $(u_1, u_2) \in C^1(\mathbb{T}^d) \oplus C(\mathbb{T}^d)$*

$$\begin{aligned} \int_{\mathbb{T}^d} |u_1|^{\zeta-2}\big(\nabla u_1 \cdot \nabla u_1 - \frac{u_1(f(u_1) - u_2)}{\zeta - 1} - \frac{1}{2}\sum_{k\geq 1}[(b_k \cdot \nabla)u_1 + g_{1,k}(\cdot, u)]^2\big)\, dx \\ \geq \theta \int_{\mathbb{T}^d} |u_1|^{\zeta-2}(|\nabla u_1|^2 - M|u_1|^2) - M|u_2|^\zeta \, dx - C. \end{aligned}$$

Remark 4. As pointed out in [AV23a], if Assumption 5.8 holds for $(u_1, u_2) \in C^1(\mathbb{T}^d) \oplus C(\mathbb{T}^d)$, it can be shown to extend to $(u_1, u_2) \in H^{1,\zeta} \oplus L^\zeta$, $\zeta \geq d$, via an approximation argument.

Lemma 5.9 (L^ζ-Coercivity for FitzHugh-Nagumo)**.** *Suppose the assumptions of Corollary 5.6 with $h = 3$ are satisfied. Then Assumption 5.8 holds for all $\zeta \in \{q\} \cup (q, b)$ with b as in Corollary 5.6.*

Proof. Let $(u_1, u_2) \in C^1(\mathbb{T}^d) \oplus C(\mathbb{T}^d)$ and fix $\zeta \in \{q\} \cup (q, b)$, then by Assumption 5.1 (4) and the mean value theorem

$$\begin{aligned}\int_{\mathbb{T}^d} -|u_1|^{\zeta-2} u_1 f(u_1)\,\mathrm{d}x &= \int_{\mathbb{T}^d} -|u_1|^{\zeta-2} u_1 u_1 f'(\xi)\,\mathrm{d}x \\ &\geq -\mu_{f'} \int_{\mathbb{T}^d} |u_1|^\zeta\,\mathrm{d}x,\end{aligned}$$

by Hölder's and Young's inequality we estimate

$$\begin{aligned}\int_{\mathbb{T}^d} |u_1|^{\zeta-2} u_1 u_2\,\mathrm{d}x &\geq -\int_{\mathbb{T}^d} |u_1|^{\zeta-1}|u_2|\,\mathrm{d}x \\ &\geq -\left(\frac{\zeta-1}{\zeta}\int_{\mathbb{T}^d} |u_1|^\zeta\,\mathrm{d}x + \frac{1}{\zeta}\int_{\mathbb{T}^d} |u_2|^\zeta\,\mathrm{d}x\right),\end{aligned}$$

and by Assumption 5.1 (4) and Hölder's and Young's inequality

$$\begin{aligned}\int_{\mathbb{T}^d} -|u_1|^{\zeta-2}\sum_{k\geq 1}(g_{1,k}(\cdot,u))^2\,\mathrm{d}x &\geq -(1+2C_1^2)\bigg(\int_{\mathbb{T}^d} |u_1|^\zeta\,\mathrm{d}x \\ &\quad + \int_{\mathbb{T}^d} |u_1|^{\zeta-2}\|g_1(0)\|_{\ell^2}^2 + |u_1|^{\zeta-2}|u_2|^2)\,\mathrm{d}x\bigg) \\ &\geq -(1+2C_1^2)\big(\left(3-\frac{4}{\zeta}\right)\int_{\mathbb{T}^d} |u_1|^\zeta\,\mathrm{d}x + \frac{2}{\zeta}\int_{\mathbb{T}^d} \|g_1(0)\|_{\ell^2}^\zeta\,\mathrm{d}x \\ &\quad + \frac{2}{\zeta}\int_{\mathbb{T}^d} |u_2|^\zeta\,\mathrm{d}x\big),\end{aligned}$$

and $\|g_1(0)\|_{L^\zeta;\ell^2}$ is bounded. Let ν be as in Assumption 5.1 (3). We may assume without loss of generality that $\nu < 1$. Fix $\epsilon_0 > 0$ such that $(1-\nu)\epsilon_0 < \nu$. We estimate

$$\begin{aligned}&\int_{\mathbb{T}^d} |u_1|^{\zeta-2}\left(|\nabla u_1|^2 - \frac{1}{2}\sum_{k\geq 1}[(b_k\cdot\nabla)u_1 + g_{1,k}(\cdot,u)]^2\right)\mathrm{d}x \\ &\quad\geq \int_{\mathbb{T}^d} |u_1|^{\zeta-2}\left(|\nabla u_1|^2 - \frac{1}{2}\sum_{k\geq 1}[|b_k|^2|\nabla u_1|^2(1+\epsilon_0) + C_{\epsilon_0}|g_{1,k}(\cdot,u)|^2\right)\mathrm{d}x\end{aligned}$$

and Assumption 5.1 (3) yields $|\nabla u_1|^2 - \frac{1+\epsilon_0}{2}\sum_{k\geq 1}|b_k|^2|\nabla u_1|^2 \geq (\nu - \epsilon_0(1-\nu))|\nabla u_1|^2$, from which it follows

$$\geq \int_{\mathbb{T}^d} |u_1|^{\zeta-2}\left((\nu-\epsilon_0(1-\nu))|\nabla u_1|^2 - \frac{1}{2}\sum_{k\geq 1} C_{\epsilon_0}|g_{1,k}(\cdot,u)|^2\right)\mathrm{d}x,$$

and we already obtained an estimate for the last term in the sum above. ☐

Theorem 5.10 (Global Existence; Theorem 4.3 in [AV23a]). *Suppose the assumptions of Corollary 5.6 are satisfied with $q > d$, $h = 3$, $\delta + \eta = 2$ and*

$$u_0 \in L^0 \left(\Omega; B^{\frac{d}{q}-1}_{q,p} \oplus L^q \right).$$

Let (u, σ) be the (p, κ_c, δ, q)-solution to (4.1)*. Then (u, σ) is a global solution, that is, $\sigma = \infty$ a.s. In particular, the regularity results* (5.27) - (5.30) *hold with $\sigma = \infty$. Moreover, there exist a constant $N_0 > 0$ such that for all $0 < s < T < \infty$ the following a prior bound holds:*

$$\begin{aligned} \mathbb{E} \sup_{t\in[s,T]} \mathbb{1}_\Gamma(\|u_1(t)\|^q_{L^q} + \|u_2(t)\|^q_{L^q}) + \mathbb{E} \int_s^T \int_{\mathbb{T}^d} \mathbb{1}_\Gamma |u_1|^{q-2} |\nabla u_1|^2 \, dx\, dr \\ \leq N_0 \left(1 + \mathbb{E}\mathbb{1}_\Gamma(\|u_1(s)\|^q_{L^q} + \|u_2(t)\|^q_{L^q})\right), \end{aligned} \tag{5.35}$$

where $\Gamma = \{\sigma > s\} \cap \{\|u_1(s)\|_{L^q} + \|u_2(s)\|_{L^q} \leq L\}$, for some $L \geq 1$. Moreover, the regularity results (5.27)-(5.30) *hold with $\sigma = \infty$ a.s.*

Remark 5. The assumptions $q > d$ and $\delta + \eta = 2$ are merely required for the application of the blow-up criteria Thoerem 5.7 in our proof of Theorem 5.10. Moreover, we note that Assumption 5.3 is compatible with the assumptions $q > d$, $h = 3$, $\delta + \eta = 2$ of Theorem 5.10. Indeed,

- By the regularity result (5.29) and the assumption $q > d$, for p sufficiently large we have $\frac{1}{p} + \frac{1}{2}\left(\delta + \frac{d}{q}\right) \leq \frac{h}{h-1} = \frac{3}{2}$,
- If $d = 2$, Assumption 5.3 also holds for $\tilde{h} > 3$ and $\delta \in [1, \frac{\tilde{h}+1}{\tilde{h}})$, so by choosing $\tilde{h} > 3$ we obtain $d \leq \frac{d}{d-\delta} < \frac{d(\tilde{h}-1)}{\tilde{h}+1-\delta(\tilde{h}-1)}$
- If $d > 2$ and $\delta > 1$, then $\frac{d}{d-\delta} < d < \frac{d}{2-\delta} = \frac{d(h-1)}{h+1-\delta(h-1)}$ holds,
- If $d > 2$ and $\delta = 1$, Assumption 5.3 also holds any $\tilde{h} > 3$, from which we obtain $\frac{d}{d-1} < d < \frac{d(\tilde{h}-1)}{2}$.

The proof of Theorem 5.10 is relies on the following lemma.

Lemma 5.11 (Energy bounds; Version of Lemma 3.8 in [AV23a]). *Suppose the assumptions of Corollary 5.6 are satisfied with $h = 3$ and let (u, σ) be the local (p, κ_c, δ, q)-solution to* (4.1)*. Then for every $\zeta \in \{q\} \cup (q, b)$, where b is as in Corollary 5.6, and for every $0 < s < T < \infty$ we have*

$$\sup_{t\in[s,\sigma\wedge T)} (\|u_1(t)\|^\zeta_{L^\zeta} + \|u_2(t)\|^\zeta_{L^\zeta}) < \infty \quad a.s.\ on\ \{\sigma > s\}, \tag{5.36}$$

$$\int_s^{\sigma\wedge T} \int_{\mathbb{T}^d} |u_1|^{\zeta-2} |\nabla u_1|^2 < \infty \quad a.s.\ on\ \{\sigma > s\}. \tag{5.37}$$

Moreover, there exists a constant $N_0 > 0$ such that for all $0 < s < T < \infty$

$$\begin{aligned}\sup_{t\in[s,\sigma\wedge T)} &\mathbb{E}\left[\mathbb{1}_{[s,\sigma)}(t)\mathbb{1}_\Gamma(\|u_1(t)\|^\zeta_{L^\zeta}+\|u_2(t)\|^\zeta_{L^\zeta})\right]\\ &+\mathbb{E}\int_s^{\sigma\wedge T}\int_{\mathbb{T}^d}\mathbb{1}_\Gamma|u_1|^{\zeta-2}|\nabla u_1|^2\,dx\,dr \\ &\le N_0\left(1+\mathbb{E}\mathbb{1}_\Gamma(\|u_1(s)\|^\zeta_{L^\zeta}+\|u_2(s)\|^\zeta_{L^\zeta})\right),\end{aligned} \tag{5.38}$$

$$\begin{aligned}\mathbb{E}\sup_{t\in[s,\sigma\wedge T)} &\mathbb{1}_\Gamma(\|u_1(t)\|^\zeta_{L^\zeta}+\|u_2(t)\|^\zeta_{L^\zeta})\\ &+\mathbb{E}\int_s^{\sigma\wedge T}\int_{\mathbb{T}^d}\mathbb{1}_\Gamma|u_1|^{\zeta-2}|\nabla u_1|^2\,dx\,dr \\ &\le N_0\left(1+\mathbb{E}\mathbb{1}_\Gamma(\|u_1(s)\|^\zeta_{L^\zeta}+\|u_2(s)\|^\zeta_{L^\zeta})\right),\end{aligned} \tag{5.39}$$

where $\Gamma = \{\|u_1(s)\|_{L^\zeta}+\|u_2(s)\|_{L^\zeta}\le L\}\cap\{\sigma > s\}$ and $L \ge 1$.

The proof of Lemma 5.11 follows the proof given in [AV23a]. However, the fact that both components of the FitzHugh-Nagumo equations (4.1) are coercive and g_1, g_2 have linear growth (compared to the general reaction-diffusion framework considered in [AV23a]) allows us to obtain bounds for $\mathbb{E}\sup_{t\in[s,\sigma\wedge T)}\mathbb{1}_\Gamma(\|u_1(t)\|^\zeta_{L^\zeta}+\|u_2(t)\|^\zeta_{L^\zeta})$ via the Burkholder-Davis-Gundy inequalities, instead of the weaker bounds for $\mathbb{E}\sup_{t\in[s,\sigma\wedge T)}\mathbb{1}_\Gamma(\|u_1(t)\|^{\lambda\zeta}_{L^\zeta}+\|u_2(t)\|^{\lambda\zeta}_{L^\zeta})$, $\lambda\in(0,1)$ obtained in [AV23a].

Proof of Lemma 5.11. We begin by establishing (5.36)-(5.37) using the generalised Itô's formula (A.1). Note that ∇u_1 and u_2 have L^ζ-integrability by (5.29) and the Sobolev embedding $H^{2-\delta-\eta,q}\hookrightarrow L^\zeta$ with $2-\delta-\eta-\frac{d}{q}\ge-\frac{d}{\zeta}$ if $b > q$ (see Remark 3). By Lemma 5.9 Assumption 5.8 holds. In order to apply Itô's formula, we will use a localisation argument. We define

$$\begin{aligned}\tau_j = &\inf\{t\in[s,\sigma) : \|u_1(t)-u_1(s)\|_{C(\mathbb{T}^d)}+\|u_1\|_{L^2(s,t;H^{1,\zeta})}\\ &+\|u_2(t)-u_2(s)\|_{L^2(s,t;H^{2-\delta-\eta,q})}\ge j\}\wedge T\end{aligned}$$

on the event $\mathcal{E} := \{\sigma > s, \|u_1(s)\|_{C(\mathbb{T}^d)}+\|u_2(s)\|_{H^{2-\delta-\eta,q}}\le j-1\}$, and we let $\tau_j = s$ on the event $\mathcal{E}^C$. Moreover, we let $\inf\emptyset = \sigma\wedge T$. Note that due to the limited regularity of u_2 provided by Theorem 5.4, we use the $H^{2-\delta-\eta,q}$-norm in our definition of τ_j instead of the $C(\mathbb{T}^d)$-norm used in [AV23a].

By the instantaneous regularisation results (5.29)-(5.30), we have $\lim_{j\to\infty}\tau_j = \sigma$. We further let

$$\Gamma_K = \{\sigma > s, \|u_1(s)\|_{L^\infty}+\|u_2(s)\|_{H^{2-\delta-\eta,q}}\le K\}\in\mathcal{F}_s.$$

Then $(\mathbb{1}_{\Gamma_K}u|_{[0,\tau_j)\times\Omega},\tau_j)$ is a local (p,κ,δ,q)-solution to (4.1). We let

$$u^{(j)}(t) = \mathbb{1}_{\Gamma_K}u(t\wedge\tau_j),$$

which is defined on the entire interval $[s,T]\times\Omega$. We thus have a.s. for all $t\in[s,T]$

$$
\begin{aligned}
u_1^{(j)}(t)-u_1^{(j)}(s)=&\int_s^t \mathbb{1}_{[s,\tau_j]\times\Gamma_K}\left[\nu\Delta u_1+f(u_1)-u_2\right]\,\mathrm{d}r\\
&+\sum_{k\geq 1}\int_s^t \mathbb{1}_{[s,\tau_j]\times\Gamma_K}\left[(b_{k,1}\cdot\nabla)u_1+g_{k,1}(\cdot,u)\right]\,\mathrm{d}W_1(r).
\end{aligned}
\tag{5.40}
$$

$$
\begin{aligned}
u_2^{(j)}(t)-u_2^{(j)}(s)=&\int_s^t \mathbb{1}_{[s,\tau_j]\times\Gamma_K}\left[\epsilon u_1-\epsilon\gamma u_2\right]\,\mathrm{d}r\\
&+\sum_{k\geq 1}\int_s^t \mathbb{1}_{[s,\tau_j]\times\Gamma_K} g_{k,2}(\cdot,u)\,\mathrm{d}W_2(r).
\end{aligned}
\tag{5.41}
$$

By Corollary 5.6 and our definition of the stopping times τ_j, $u_i^{(j)}$, $i\in\{1,2\}$, satisfy the conditions of Theorem A.1. Hence, applying the generalised Itô formula to each component of $u^{(j)}$ gives

$$
\|u_i^{(j)}(t)\|_{L^\zeta}^\zeta=\|u_i^{(j)}(s)\|_{L^\zeta}^\zeta+\zeta(\zeta-1)\mathcal{D}_i(t)+\zeta\mathcal{S}_i,
\tag{5.42}
$$

where $\mathcal{D}_i$ denotes the deterministic term

$$
\begin{aligned}
\mathcal{D}_1(t)=&\int_s^t\int_{\mathbb{T}^d}\mathbb{1}_{[s,\tau_j]\times\Gamma_K}|u_1|^{\zeta-2}\Bigg(\frac{u_1(f(u_1)-u_2)}{\zeta-1}-|\nabla u_1|^2\\
&\qquad+\frac{1}{2}\sum_{k\geq 1}[(b_k\cdot\nabla)u_1+g_{k,1}(\cdot,u)]^2\Bigg)\,\mathrm{d}x\,\mathrm{d}r,\\
\mathcal{D}_2(t)=&\int_s^t\int_{\mathbb{T}^d}\mathbb{1}_{[s,\tau_j]\times\Gamma_K}|u_2|^{\zeta-2}\left(\frac{u_2(\epsilon u_1-\epsilon\gamma u_2)}{\zeta-1}+\frac{1}{2}\sum_{k\geq 1}g_{k,2}(\cdot,u)^2\right)\mathrm{d}x\,\mathrm{d}r
\end{aligned}
$$

and $\mathcal{S}_i$ denotes the stochastic term

$$
\begin{aligned}
\mathcal{S}_1=&\sum_{k\geq 1}\int_s^t\int_{\mathbb{T}^d}\mathbb{1}_{[s,\tau_j]\times\Gamma_K}|u_1|^{\zeta-2}u_1[(b_k\cdot\nabla)u_1+g_{k,1}(\cdot,u)]\,\mathrm{d}x\,\mathrm{d}W_1(r),\\
\mathcal{S}_2=&\sum_{k\geq 1}\int_s^t\int_{\mathbb{T}^d}\mathbb{1}_{[s,\tau_j]\times\Gamma_K}|u_2|^{\zeta-2}u_2 g_{k,2}(\cdot,u)\,\mathrm{d}x\,\mathrm{d}W_2(r).
\end{aligned}
$$

Using Assumption 5.8, we obtain

$$
\begin{aligned}
\|u_1^{(j)}(t)\|_{L^\zeta}^\zeta+&\tilde{\theta}\int_s^t\int_{\mathbb{T}^d}\mathbb{1}_{[s,\tau_j]\times\Gamma_K}|u_1|^{\zeta-2}|\nabla u_1|^2\,\mathrm{d}x\,\mathrm{d}r\\
\leq&\tilde{C}_1(t-s)+\|u_1^{(j)}(s)\|_{L^\zeta}^\zeta\\
&+\tilde{M}_1\int_s^t\mathbb{1}_{[s,\tau_j]\times\Gamma_K}(\|u_1(r)\|^\zeta+\|u_2(r)\|^\zeta)\,\mathrm{d}r+\zeta\mathcal{S}_1,
\end{aligned}
\tag{5.43}
$$

where $\tilde{\theta} = \zeta(\zeta-1)\theta$, $\tilde{M}_1 = \zeta(\zeta-1)M$ and $\tilde{C}_1 = \zeta(\zeta-1)C$. Moreover, we estimate

$$\zeta(\zeta-1)\mathcal{D}_2(t) \leq \zeta\epsilon \int_s^t \mathbb{1}_{[s,\tau_j]\times\Gamma_K} \left(\frac{\zeta-1}{\zeta}\|u_2(r)\|_{L^\zeta}^\zeta + \frac{1}{\zeta}\|u_1(r)\|_{L^\zeta}^\zeta + \gamma\|u_2(r)\|_{L^\zeta}^\zeta \right) \mathrm{d}r$$
$$+ \frac{3}{2}\zeta(\zeta-1)C_1 \int_s^t \int_{\mathbb{T}^d} \mathbb{1}_{[s,\tau_j]\times\Gamma_K} |u_2|^{\zeta-2}(\|g_2(0)\|_{\ell^2}^2 + |u_1|^2 + |u_2|^2) \,\mathrm{d}x \,\mathrm{d}r,$$

and by Assumption 5.1(4), and Hölder's and Young's inequalities appplied to $\int_{\mathbb{T}^d} |u_2|^{\zeta-1}|u_1| \,\mathrm{d}x$,

$$\begin{aligned}
\zeta(\zeta-1)\mathcal{D}_2(t) \leq& \zeta\epsilon \int_s^t \mathbb{1}_{[s,\tau_j]\times\Gamma_K} \left(\frac{\zeta-1}{\zeta}\|u_2(r)\|_{L^\zeta}^\zeta + \frac{1}{\zeta}\|u_1(r)\|_{L^\zeta}^\zeta + \gamma\|u_2(r)\|_{L^\zeta}^\zeta \right) \mathrm{d}r \\
&+ \frac{3}{2}\zeta(\zeta-1)C_1 \int_s^t \mathbb{1}_{[s,\tau_j]\times\Gamma_K} \bigg(\frac{3\zeta-2}{\zeta}\|u_2(r)\|_{L^\zeta}^\zeta + \frac{2}{\zeta}\|u_1\|_{L^\zeta(\mathbb{T}^d;\ell^2)}^\zeta \\
&+ \frac{2}{\zeta}\|g_2(r,0)\|_{L^\zeta(\mathbb{T}^d;\ell^2)}^\zeta \bigg) \mathrm{d}r, \\
\leq& \tilde{M}_2 \int_s^t \mathbb{1}_{[s,\tau_j]\times\Gamma_K} (\|u_1(r)\|_{L^\zeta}^\zeta + \|u_2(r)\|_{L^\zeta}^\zeta) \,\mathrm{d}r + \tilde{C}_2(t-s),
\end{aligned}$$

by Hölder's and Young's inequalities applied to $\int_{\mathbb{T}^d} |u_2|^{\zeta-2}\|g_2(0)\|_{\ell^2}^2 \,\mathrm{d}x$, where $\tilde{M}_2 = \epsilon\zeta(1+\gamma) + 3C_1(\frac{3}{2}\zeta^2+1)$, $\tilde{C}_2 = 3(\zeta-1)C_1\|g_2(\cdot,0)\|_{L^\infty(\mathbb{R}_{\geq 0}\times\Omega;\ell^2)}$. We thus obtain

$$\begin{aligned}
\|u_2^{(j)}(t)\|_{L^\zeta}^\zeta \leq& \tilde{C}_2(t-s) + \|u_2^{(j)}(s)\|_{L^\zeta}^\zeta \\
&+ \tilde{M}_2 \int_s^t \mathbb{1}_{[s,\tau_j]\times\Gamma_K} (\|u_1(r)\|^\zeta + \|u_2(r)\|^\zeta) \,\mathrm{d}r + \zeta\mathcal{S}_2.
\end{aligned} \tag{5.44}$$

Adding (5.43) and (5.44) yields

$$\begin{aligned}
\|u_1^{(j)}(t)\|_{L^\zeta}^\zeta + \|u_2^{(j)}(t)\|_{L^\zeta}^\zeta + \tilde{\theta} \int_s^t \int_{\mathbb{T}^d} \mathbb{1}_{[s,\tau_j]\times\Gamma_K} |u_1|^{\zeta-2}|\nabla u_1|^2 \,\mathrm{d}x \,\mathrm{d}r& \\
\leq \tilde{C}(t-s) + \|u_1^{(j)}(s)\|_{L^\zeta}^\zeta + \|u_2^{(j)}(s)\|_{L^\zeta}^\zeta& \\
+ \tilde{M} \int_s^t \mathbb{1}_{[s,\tau_j]\times\Gamma_K} (\|u_1(r)\|_{L^\zeta}^\zeta + \|u_2(r)\|_{L^\zeta}^\zeta) \,\mathrm{d}r& \\
+ \zeta(\mathcal{S}_1 + \mathcal{S}_2),&
\end{aligned} \tag{5.45}$$

where $\tilde{C} = \tilde{C}_1 + \tilde{C}_2$, $\tilde{M} = \tilde{M}_1 + \tilde{M}_2$. The remaining steps of the proof carry over verbatim from [AV23a], so we only sketch these steps here.

Step 1. Firstly, taking expectations in (5.45) and applying Gronwall's lemma to the function $y(t) = \sup_{r\in[s,t]} \mathbb{E}(\|u_1^{(j)}(t)\|_{L^\zeta}^\zeta + \|u_2^{(j)}(t)\|_{L^\zeta}^\zeta)$, we obtain

$$\begin{aligned}
\sup_{r\in[s,t]} \mathbb{E}[\|u_1^{(j)}(r)\|_{L^\zeta}^\zeta + &\|u_2^{(j)}(r)\|_{L^\zeta}^\zeta] \\
&\leq e^{t\tilde{M}} \left[\mathbb{E}\|u_1^{(j)}(s)\|_{L^\zeta}^\zeta + \mathbb{E}\|u_2^{(j)}(s)\|_{L^\zeta}^\zeta + \tilde{C}t \right].
\end{aligned} \tag{5.46}$$

Step 2. Secondly, by applying Fatou's lemma to let $j \to \infty$ in (5.46) with $t = T$ and taking expectations in (5.45), we can show that there exists a constant $N_0 \geq 1$ that depends on $\tilde{\theta}, \tilde{M}, \tilde{C}, \zeta, T, \alpha_m$, and α_M such that

$$\mathbb{E}\int_s^{\sigma\wedge T} \mathbb{1}_{\Gamma_K}(\|u_1(r)\|_{L^\zeta}^\zeta + \|u_2(r)\|_{L^\zeta}^\zeta) + \mathbb{E}\int_s^{\sigma\wedge T}\int_{\mathbb{T}^d} \mathbb{1}_{\Gamma_K}|u_1|^{\zeta-2}|\nabla u_1|^2 \,\mathrm{d}x\,\mathrm{d}r \\ \leq N_0[1 + \mathbb{E}\mathbb{1}_{\Gamma_K}(\|u_1(s)\|_{L^\zeta}^\zeta + \|u_2(s)\|_{L^\zeta}^\zeta)]. \tag{5.47}$$

Step 3. Thirdly, we take the $\sup_{t\in[s,\sigma\wedge T)}$ and subsequently take expectations in (5.45) to obtain

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[s,\sigma\wedge T)}\left(\|u_1^{(j)}(t)\|_{L^\zeta}^\zeta + \|u_2^{(j)}(t)\|_{L^\zeta}^\zeta\right) + \tilde{\theta}\mathbb{E}\int_s^{\sigma\wedge T}\int_{\mathbb{T}^d} \mathbb{1}_{[s,\tau_j]\times\Gamma_K}|u_1|^{\zeta-2}|\nabla u_1|^2\,\mathrm{d}x\,\mathrm{d}r\\
\leq&\mathbb{E}(\|u_1^{(j)}(s)\|_{L^\zeta}^\zeta + \|u_2^{(j)}(s)\|_{L^\zeta}^\zeta) + \tilde{M}\mathbb{E}\int_s^{\sigma\wedge T}\mathbb{1}_{[s,\tau_j]\times\Gamma_K}(\|u_1(r)\|_{L^\zeta}^\zeta + \|u_2(r)\|_{L^\zeta}^\zeta)\,\mathrm{d}r\\
&+ \zeta\mathbb{E}\sup_{t\in[s,\sigma\wedge T)}(\mathcal{S}_1 + \mathcal{S}_2) + \tilde{C}T,
\end{aligned} \tag{5.48}$$

We first bound the innermost integral of $\mathcal{S}_1$. By Hölder's inequality and Assumption 5.3(4)

$$\begin{aligned}
&\sum_{k\geq 1}\int_{\mathbb{T}^d}\left|\mathbb{1}_{[s,\tau_j]\times\Gamma_K}|u_1|^{\zeta-2}u_1[(b_k\cdot\nabla)u_1 + g_{k,1}(\cdot,u)]\right|\,\mathrm{d}x\\
&\lesssim\left(\int_{\mathbb{T}^d}\mathbb{1}_{[s,\tau_j]\times\Gamma_K}|u_1|^\zeta\,\mathrm{d}x\right)^{1/2}\\
&\quad\times\left(\int_{\mathbb{T}^d}\mathbb{1}_{[s,\tau_j]\times\Gamma_K}|u_1|^{\zeta-2}[\|(b_k\cdot\nabla)u_1\|_{\ell^2}^2 + \|g_{k,1}(0)\|_{\ell^2}^2 + |u_1|^2 + |u_2|^2]\,\mathrm{d}x\right)^{1/2}\\
&\lesssim\left(\sup_{t\in[s,\sigma\wedge T)}\|u_1^{(j)}(t)\|_{L^\zeta}^\zeta\right)^{1/2}\\
&\quad\times\left(\int_{\mathbb{T}^d}\mathbb{1}_{[s,\tau_j]\times\Gamma_K}|u_1|^{\zeta-2}|\nabla u_1|^2\,\mathrm{d}x + \|g_1(0)\|_{L^\zeta(\ell^2)}^\zeta + \|u_1^{(j)}\|_{L^\zeta}^\zeta + \|u_2^{(j)}\|_{L^\zeta}^\zeta\right)^{1/2}
\end{aligned}$$

by Assumption 5.3(3), and Hölder's and Young's inequality. Now fix $\epsilon_0 > 0$, then by the Burkholder-Davis-Gundy inequality

$$\begin{aligned}
\mathbb{E}\sup_{t\in[s,\sigma\wedge T)}\mathcal{S}_1 \leq& C\mathbb{E}\int_s^{\sigma\wedge T}\left|\sum_{k\geq 1}\int_{\mathbb{T}^d}\mathbb{1}_{[s,\tau_j]\times\Gamma_K}|u_1|^{\zeta-2}u_1[(b_k\cdot\nabla)u_1 + g_{k,1}(\cdot,u)]\,\mathrm{d}x\right|^2\,\mathrm{d}r\\
\leq& C_T\mathbb{E}\left(\sup_{t\in[s,\sigma\wedge T)}\|u_1^{(j)}(t)\|_{L^\zeta}^\zeta\right)
\end{aligned}$$

$$\times \left(\int_s^{\sigma\wedge T} \int_{\mathbb{T}^d} \mathbb{1}_{[s,\tau_j]\times\Gamma_K} |u_1(r)|^{\zeta-2} |\nabla u_1(r)|^2 \,\mathrm{d}x\,\mathrm{d}r \right.$$
$$\left. + T\|g_1(0)\|^{\zeta}_{L^\infty(\Omega\times[s,T];\ell^2)} + \int_s^{\sigma\wedge T} \|u_1^{(j)}(r)\|^{\zeta}_{L^\zeta} + \|u_2^{(j)}(r)\|^{\zeta}_{L^\zeta}\,\mathrm{d}r \right).$$

By Assumption 5.3(4), Hölder's and Young's inequality

$$\begin{aligned}
\mathbb{E}\sup_{t\in[s,\sigma\wedge T)} \mathcal{S}_1 \le& \epsilon_0 C_{T,\zeta}\mathbb{E}\left(\sup_{t\in[s,\sigma\wedge T)} \|u_1^{(j)}(t)\|^{\zeta}_{L^\zeta}\right) \\
&+ \frac{1}{\epsilon_0} C_{T,\zeta}\left(\mathbb{E}\int_s^{\sigma\wedge T}\int_{\mathbb{T}^d} \mathbb{1}_{[s,\tau_j]\times\Gamma_K} |u_1(r)|^{\zeta-2}|\nabla u_1(r)|^2\,\mathrm{d}x\,\mathrm{d}r \right. \\
&\qquad \left. + T + \mathbb{E}\int_s^{\sigma\wedge T} \|u_1^{(j)}(r)\|^{\zeta}_{L^\zeta} + \|u_2^{(j)}(r)\|^{\zeta}_{L^\zeta})\,\mathrm{d}r \right) \\
\le& \epsilon_0 C_{T,\zeta}\mathbb{E}\sup_{t\in[s,\sigma\wedge T)} \left(\|u_1^{(j)}(t)\|^{\zeta}_{L^\zeta} + \|u_2^{(j)}(t)\|^{\zeta}_{L^\zeta} \right) \\
&+ C_{T,\zeta,\epsilon_0}\left(\mathbb{E}\int_s^{\sigma\wedge T} \|u_1^{(j)}(r)\|^{\zeta}_{L^\zeta} + \|u_2^{(j)}(r)\|^{\zeta}_{L^\zeta})\,\mathrm{d}r \right. \\
&\qquad \left. + \mathbb{E}(\|u_1^{(j)}(s)\|^{\zeta}_{L^\zeta} + \|u_2^{(j)}(s)\|^{\zeta}_{L^\zeta}) + T \right),
\end{aligned}$$

where we used the result of Step 2 to bound the integral of $|u_1|^{\zeta-2}|\nabla u_1|$.

We proceed similarly for the second stochastic term, bounding the innermost integral of $\mathcal{S}_2$. By (two applications of) Hölder's and Young's inequality and Assumption 5.3(4)

$$\begin{aligned}
&\sum_{k\ge 1}\int_{\mathbb{T}^d} \mathbb{1}_{[s,\tau_j]\times\Gamma_K} |u_2|^{\zeta-2} u_2 g_{k,2}(\cdot,u)\,\mathrm{d}x \\
&\lesssim \left(\int_{\mathbb{T}^d} \mathbb{1}_{[s,\tau_j]\times\Gamma_K} |u_2|^{\zeta}\,\mathrm{d}x \right)^{1/2} \\
&\quad \times \left(\int_{\mathbb{T}^d} \mathbb{1}_{[s,\tau_j]\times\Gamma_K} \left(|u_1|^\zeta + |u_2|^\zeta + \|g_2(0)\|^{\zeta}_{\ell^2} \right)\,\mathrm{d}x \right)^{1/2} \\
&\lesssim \left(\sup_{t\in[s,\sigma\wedge T)} \|u_2^{(j)}\|^{\zeta}_{L^\zeta} \right)^{1/2} \left(\|u_1^{(j)}\|^{\zeta}_{L^\zeta} + \|u_2^{(j)}\|^{\zeta}_{L^\zeta} + \|g_2(0)\|^{\zeta}_{L^\zeta(\ell^2)} \right)^{1/2}.
\end{aligned}$$

Fix $\epsilon_0 > 0$ as before, then by the Burkholder-Davis-Gundy inequality

$$\begin{aligned}
\mathbb{E}\sup_{t\in[s,\sigma\wedge T)} \mathcal{S}_2 \le& C\mathbb{E}\int_s^{\sigma\wedge T} \left| \sum_{k\ge 1}\int_{\mathbb{T}^d} \mathbb{1}_{[s,\tau_j]\times\Gamma_K} |u_2|^{\zeta-2} u_2 g_{k,2}(\cdot,u)\,\mathrm{d}x \right|^2 \mathrm{d}r \\
\le& C_T\mathbb{E}\left(\sup_{t\in[s,\sigma\wedge T)} \|u_2^{(j)}(t)\|^{\zeta}_{L^\zeta} \right)
\end{aligned}$$

$$
\begin{aligned}
&\times \int_s^{\sigma\wedge T} \|u_1^{(j)}(r)\|_{L^\zeta}^\zeta + \|u_2^{(j)}(r)\|_{L^\zeta}^\zeta + \|g_2(0)\|_{L^\zeta(\ell^2)}^\zeta \,\mathrm{d}r \\
\leq &\epsilon_0 C_{T,\zeta}\mathbb{E} \sup_{t\in[s,\sigma\wedge T)} \left(\|u_1^{(j)}(t)\|_{L^\zeta}^\zeta + \|u_2^{(j)}(t)\|_{L^\zeta}^\zeta\right) \\
&+ C_{T,\zeta,\epsilon_0}\left(\mathbb{E}\int_s^{\sigma\wedge T} \|u_1^{(j)}\|_{L^\zeta}^\zeta + \|u_2^{(j)}\|_{L^\zeta}^\zeta \,\mathrm{d}r + T\right),
\end{aligned}
$$

by Assumption 5.3(4), Hölder's and Young's inequality.

Fixing ϵ_0 as before and combining these results with (5.48), we have

$$
\begin{aligned}
\mathbb{E} &\sup_{t\in[s,\sigma\wedge T)} \left(\|u_1^{(j)}(t)\|_{L^\zeta}^\zeta + \|u_2^{(j)}(t)\|_{L^\zeta}^\zeta\right) \\
&\quad +\tilde{\theta}\mathbb{E}\int_s^{\sigma\wedge T}\int_{\mathbb{T}^d} \mathbb{1}_{[s,\tau_j]\times\Gamma_K}|u_1|^{\zeta-2}|\nabla u_1|^2 \,\mathrm{d}x\,\mathrm{d}r \\
&\quad \leq \epsilon_0 C_{T,\zeta}\mathbb{E}\sup_{t\in[s,\sigma\wedge T)} \left(\|u_1^{(j)}(t)\|_{L^\zeta}^\zeta + \|u_2^{(j)}(t)\|_{L^\zeta}^\zeta\right) \\
&\quad\quad + C_{T,\zeta,\epsilon_0}\Bigg(\mathbb{E}(\|u_1^{(j)}(s)\|_{L^\zeta}^\zeta + \|u_2^{(j)}(s)\|_{L^\zeta}^\zeta) \\
&\quad\quad\quad + T + \mathbb{E}\int_s^{\sigma\wedge T}(\|u_1^{(j)}(r)\|^\zeta + \|u_2^{(j)}(r)\|^\zeta)\,\mathrm{d}r\Bigg).
\end{aligned}
$$

Thus, for ϵ_0 sufficiently small, and making $\tilde{\theta}$ larger as needed, we can move the first term on the right-hand side to the left-hand side and apply Fubini's theorem the last term to obtain

$$
\begin{aligned}
\mathbb{E} &\sup_{t\in[s,\sigma\wedge T)} \left(\|u_1^{(j)}(t)\|_{L^\zeta}^\zeta + \|u_2^{(j)}(t)\|_{L^\zeta}^\zeta\right) \\
&\quad +\tilde{\theta}\mathbb{E}\int_s^{\sigma\wedge T}\int_{\mathbb{T}^d} \mathbb{1}_{[s,\tau_j]\times\Gamma_K}|u_1|^{\zeta-2}|\nabla u_1|^2 \,\mathrm{d}x\,\mathrm{d}r \\
&\quad \lesssim \mathbb{E}(\|u_1^{(j)}(s)\|_{L^\zeta}^\zeta + \|u_2^{(j)}(s)\|_{L^\zeta}^\zeta) + T \\
&\quad\quad + \int_s^T \mathbb{E}\sup_{t\in[s,\sigma\wedge r)}\left(\|u_1^{(j)}(t)\|^\zeta + \|u_2^{(j)}(t)\|^\zeta\right)\,\mathrm{d}t.
\end{aligned}
$$

Applying Gronwall's lemma to the function $y(t) = \mathbb{E}\sup_{r\in[s,\sigma\wedge t)}(\|u_1^{(j)}(r)\|^\zeta + \|u_2^{(j)}(r)\|^\zeta)$ and letting $j\to\infty$ yields the result.

Finally, (5.38) and (5.39) follow from Steps 1 to 3 with Γ_K replaced by $\Gamma = \{\sigma > s, \|u_1(s)\|_{L^\zeta} + \|u_2(s)\|_{L^\zeta} \leq L\}$, $L \geq 1$. □

Proof of Theorem 5.10. Our proof follows the lines of the proof of [AV23a, Theorem 3.2]. By assumption, (p,q,h,δ) satisfy the conditions of Corollary 5.6, and Assumption 5.3 in particular. We fix an $\epsilon_0 > 0$ sufficiently small that will be determined below and such that we can choose $h_0 \geq h = 3$ such that (since $q > d$)

$$
q = \frac{d(h_0-1)}{2}(1+\epsilon_0).
$$

We let

$$\beta_0 = \frac{d}{q} - \frac{2}{h_0 - 1},$$

and note that $\beta_0 < 0$ by our choice of h_0, and choose $p_0 > 2$ such that

$$0 > \gamma_0 = \frac{d}{q} + \frac{2}{p_0} - \frac{2}{h_0 - 1},$$

which is possible by our choice of h_0. Moreover, we choose $\delta_0 \in [1, 2)$ below and let $\eta_0 \in (0, 2-\delta_0]$ be such that $\delta_0 + \eta_0 = \delta + \eta$. Next, we verify that (p_0, q, h_0, δ_0) satisfy Assumption 5.3:

- $h_0 \geq h > 1$ by our choice of h_0,
- $\delta_0 \in [1, \frac{h_0+1}{h_0})$ for δ_0 sufficiently small,
- By our choice of p_0 $\frac{1}{p_0} < \frac{1}{h_0-1} - \frac{d}{2q}$, which implies $\frac{1}{p_0} + \frac{1}{2}\left(\delta_0 + \frac{d}{q}\right) < \frac{1}{h_0-1} + \frac{1}{2}\delta_0 < \frac{h_0}{h_0-1}$,
- Since $d \geq 2$ and $q > d$ we also have $\frac{d}{d-\delta_0} < q$ for δ_0 sufficiently small,
- Since $\frac{1+\epsilon_0}{2} \downarrow \frac{1}{2}$ as $\epsilon_0 \downarrow 0$ and $\frac{1}{2} < \frac{1}{h_0+1-\delta_0(h_0-1)}$, we can choose ϵ_0 sufficiently small so that $q < \frac{d(h_0-1)}{h_0+1-\delta_0(h_0-1)}$.

Theorem 5.7 now gives that

$$\begin{aligned}\mathbb{P}(&\sup_{t\in[s,\sigma]} \|u(t)\|_{B^{\beta_0}_{q,p_0} \oplus H^{2-\delta_0-\eta_0,q}} \\ &+ \|u\|_{L^{p_0}(s,\sigma;H^{\gamma_0,q}\oplus H^{2-\delta_0-\eta_0,q})} < \infty, s < \sigma < T) = 0.\end{aligned}$$

Next, we show that a.s.

$$\sup_{t\in[s,\sigma]} \|u(t)\|_{B^{\beta_0}_{q,p_0} \oplus H^{2-\delta_0-\eta_0,q}} + \|u\|_{L^{p_0}(s,\sigma;H^{\gamma_0,q}\oplus H^{2-\delta_0-\eta_0,q})} < \infty.$$

Our assumption that $2-\delta-\eta = 0$ means that $H^{2-\delta-\eta,q} = L^q$, and Lemma 5.11 with $\zeta = q$ gives

$$u \in L^\infty(s, \sigma \wedge T; L^q \oplus L^q).$$

Since $\gamma_0 < 0$ by our choice of p_0, we have $L^q \hookrightarrow H^{\gamma_0,q}$ so that $u \in L^{p_0}(s, \sigma \wedge T; H^{\gamma_0,q} \oplus H^{2-\delta-\eta,q})$ a.s. Moreover, by the Sobolev embedding $L^q \hookrightarrow B^{\beta_0}_{q,p_0}$ we have $u \in L^\infty(s, \sigma \wedge T; B^{\beta_0}_{q,p_0} \oplus H^{2-\delta_0-\eta_0,q})$ a.s. It thus follows that

$$\begin{aligned}\mathbb{P}(s < \sigma < T) =& \mathbb{P}(\sup_{t\in[s,\sigma]} \|u(t)\|_{B^{\beta_0}_{q,p_0} \oplus H^{2-\delta_0-\eta_0,q}} \\ &+ \|u\|_{L^{p_0}(s,\sigma;H^{\gamma_0,q}\oplus H^{2-\delta_0-\eta_0,q})} < \infty, s < \sigma < T) = 0,\end{aligned}$$

and since $\sigma > 0$ a.s., first letting $s \downarrow 0$ and subsequently letting $T \to \infty$ shows that $\mathbb{P}(\sigma < \infty) = 0$.

The a priori bound (5.35) follows from the bound (5.39) in Lemma 5.11. □

6 Local Well-Posedness of Reaction-Diffusion Equations in the $L^p(L^q(\mathbb{R}^d))$-Setting

6.1 Local well-posedness in the case $p > 2, d \geq 2$

In this section we extend the theory of reaction-diffusion equations developed in [AV23b] for the periodic torus to the unbounded domain $\mathbb{R}^d$. This will serve as a reference as we extend our theory of the FitzHugh-Nagumo equations on the periodic torus $\mathbb{T}^d$ developed in Section 5 to the unbounded domain $\mathbb{R}^d$. We consider stochastic reaction-diffusion equations of the form

$$\begin{cases} \mathrm{d}u_i(t) - \operatorname{div}(a_i \cdot \nabla u_i)\,\mathrm{d}t = & [\operatorname{div}(F_i(\cdot,u)) + f_i(\cdot,u)]\,\mathrm{d}t \\ & + \sum_{k\geq 1} [(b_{k,i}\cdot\nabla)u_i + g_{k,i}(\cdot,u)]\,\mathrm{d}W(t) \\ u_i(0) = u_{0,i}, & \end{cases} \tag{6.1}$$

for $i \in \{1,\dots,\ell\}$, where $(W_k)_{k\geq 1}$ is a sequence of standard independent Brownian motions. Our assumptions and results are largely derived from the results in [AV23b] by making adjustments to their assumptions and proofs to account for the unbounded domain $\mathbb{R}^d$ considered here. We begin by considering $p \in (2,\infty)$ and $d \geq 2$. In subsequent sections we will discuss how the conditions on the parameters change in the cases $d = 1$ and $p = 2$. Our Definition 5.2 of a (local) solution in Section 5 carries over, with some obvious adjustments, to the reaction-diffusion framework in the present section (see also [AV23b, Definition 2.3]).

Assumption 6.1 (*Assumption 2.1 in [AV23b]*)**.** *Let $d \geq 2$. Assumption 6.1(p,q,h,δ) holds if $p \in (2,\infty)$, $q \in [2,\infty)$, $h > 1$, $\delta \in [1,2)$ and for $i \in \{1,\dots,\ell\}$ the following hold:*

1. *For each $j,l \in \{1,\dots,d\}$, $a_i^{j,l} : \mathbb{R}_{\geq 0}\times\Omega\times\mathbb{R}^d \to \mathbb{R}$ and $b_i^j := (b_{k,i}^j)_{k\geq 1} : \mathbb{R}_{\geq 0}\times\Omega\times\mathbb{R}^d \to \ell^2$ are $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^d)$-measurable,*
2. *If $\delta = 1$, there exists $N > 0$ such that a.s. for all $t \geq 0$ and $j,l \in \{1,\dots,d\}$*
$$\|a_i^{j,l}\|_{L^\infty} + \|b_i^j\|_{L^\infty(\ell^2)} \leq N. \tag{6.2}$$
If $\delta > 1$ there additionally exist $\tau > \delta - 1$, $\epsilon \in (0, \tau + 1 - \delta)$ such that a.s. for all $t \geq 0$ and $j,l \in \{1,\dots,d\}$,
$$\|a_i^{j,l}\|_{C^\tau(\mathbb{R}^d)} + \|b_i^j\|_{C^\tau(\mathbb{R}^d;\ell^2)} \leq N, \tag{6.3}$$
3. *For every $s \in [0,T)$ there exist $\hat{a}_i^{j,l} : [s,T]\times\Omega \to \mathbb{R}$, $\hat{b}_i^j : [s,T]\times\Omega \to \ell^2$ such that for all $j,k \in \{1,\dots,d\}$*
$$\lim_{|x|\to\infty} \operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{t\in[s,T]} \left(|a_i^{j,l} - \hat{a}_i^{j,l}| + \|b_i^j - \hat{b}_i^j\|_{\ell^2} \right) = 0. \tag{6.4}$$

4. There exists a $\nu_i > 0$ such that, a.s. for all $t \geq 0$, $x, \xi \in \mathbb{R}^d$

$$\sum_{j,l=1}^{d} \left(a_i^{j,l}(t,x) - \frac{1}{2} \sum_{k\geq 1} b_{k,i}^{j}(t,x) b_{k,i}^{l}(t,x) \right) \xi_j \xi_l \geq \nu_i |\xi|^2, \tag{6.5}$$

5. For all $j \in \{1, \dots, d\}$ the maps

$$\begin{aligned} F_i^j, f_i &: \mathbb{R}_{\geq 0} \times \Omega \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R} \\ g_i \coloneqq (g_{k,i})_{k\geq 1} &: \mathbb{R}_{\geq 0} \times \Omega \times \mathbb{R}^d \times \mathbb{R} \to \ell^2 \end{aligned}$$

are $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R})$-measurable. Set $F_i \coloneqq (F_i^j)_{j=1}^d$ and assume that

$$\begin{aligned} F_i^j(\cdot, 0), f_i(\cdot, 0) &\in \cap_{q\geq 2, \theta \in (0,1)} L^\infty(\mathbb{R}_{\geq 0} \times \Omega; H^{\theta,q}(\mathbb{R}^d)), \\ g_i(\cdot, 0) &\in \cap_{q\geq 2, \theta \in (0,1)} L^\infty(\mathbb{R}_{\geq 0} \times \Omega; H^{\theta,q}(\mathbb{R}^d; \ell^2)), \end{aligned}$$

and a.s. for all $t \in \mathbb{R}_{\geq 0}, x \in \mathbb{R}^d, y, y' \in \mathbb{R}$

$$\begin{aligned} |f_i(t,x,y) - f_i(t,x,y')| &\lesssim (1 + |y|^{h-1} + |y'|^{h-1})|y - y'| \\ |F_i(t,x,y) - F_i(t,x,y')| &\lesssim (1 + |y|^{\frac{h-1}{2}} + |y'|^{\frac{h-1}{2}})|y - y'| \\ |g_i(t,x,y) - g_i(t,x,y')|_{\ell^2} &\lesssim (1 + |y|^{\frac{h-1}{2}} + |y'|^{\frac{h-1}{2}})|y - y'|. \end{aligned}$$

Remark 6. Part (3) of Assumption 6.1 is an extension of [AV23b, Assumption 2.1] and is needed in our proof of local existence of solutions as the domain we consider here is $\mathbb{R}^d$ instead of $\mathbb{T}^d$ as in [AV23b]. Specifically, the extension allows us to show maximal stochastic regularity of the operators involving a_i and b_i in (6.1) (see [AV21, Remark 5.7]).

Assumption 6.2 (*Assumption 2.4 in [AV23b]*). *Let $d \geq 2$. Assumption 6.2(p,q,h,δ) holds if $p \in (2, \infty)$, $q \in [2, \infty)$, $h > 1$ and $\delta \in [1, \frac{h+1}{h})$ satisfy*

$$\frac{1}{p} + \frac{1}{2}\left(\delta + \frac{d}{q}\right) \leq \frac{h}{h-1}, \quad \frac{d}{d-\delta} < q < \frac{d(h-1)}{h+1-\delta(h-1)}. \tag{6.6}$$

Definition 6.3 (Local (p,q,h,δ)-solution; *Definition 2.3 in [AV23b]*). Suppose that Assumption 6.1 holds for some $h > 1$, let $\kappa \in [0, \frac{p}{2} - 1)$.

- Let σ be a stopping time and $u = (u_i)_{i=1}^{\ell} : [0, \sigma) \times \Omega \to H^{2-\delta,q}(\mathbb{R}^d; \mathbb{R}^l)$ be a stochastic process. We say that (u, σ) is a *local (p, κ, δ, q)-solution* to (6.1) if there exists a sequence of stopping times $(\sigma_j)_{j\geq 1}$ such that
  - $\sigma_j \leq \sigma$ a.s. for all $j \geq 1$ and $\lim_{j\to\infty} \sigma_j = \sigma$ a.s.,
  - For all $j \geq 1$ the process $\mathbb{1}_{[0,\sigma_j]\times\Omega} u_i$ is progressively measurable,
  - a.s. for all $j \geq 1$ we have $u_i \in L^p(0, \sigma_j, w_\kappa; H^{2-\delta,q}(\mathbb{R}^d))$ and

$$\begin{aligned} \operatorname{div}(F_i(\cdot, u)) + f_i(\cdot, u) &\in L^p(0, \sigma_j, w_\kappa; H^{-\delta,q}(\mathbb{R}^d)) \\ (g_{k,i}(\cdot, u))_{k\geq 1} &\in L^p(0, \sigma_j, w_\kappa; H^{1-\delta,q}(\mathbb{R}^d); \ell^2), \end{aligned} \tag{6.7}$$

– a.s. for all $j \geq 1$, for all $t \in [0, \sigma_j]$ it holds that

$$
\begin{aligned}
u_i(t) - u_{0,i} = &\int_0^t \operatorname{div}(a_i \cdot \nabla u_i) + \operatorname{div}(F_i(\cdot, u)) + f_i(\cdot, u)\,\mathrm{d}s \\
&+ \sum_{k\geq 1} \int_0^t \mathbb{1}_{0,\sigma_j}[(b_{k,i} \cdot \nabla) u + g_{k,i}(\cdot, u)]\,\mathrm{d}W(s).
\end{aligned}
\tag{6.8}
$$

- (u, σ) is a (p, κ, δ, q)-*solution* to (6.1) if for any other local (p, κ, δ, q)-solution (u', σ') we have $\sigma' \leq \sigma$ and $u = u'$ on $[0, \sigma') \times \Omega$.

Remark 7. The regularity conditions (6.2)–(6.3) on the $a_i^{j,k}$ in Assumption 6.1 are necessary for the integrals in (6.8) to be well-defined. Specifically, by [AV21, Proposition 4.1(4)], if $u_0 \in L^p(0, \sigma_j, w_\kappa; H^{2-\delta,q}(\mathbb{R}^d))$, it holds a.s. that

$$
\begin{aligned}
\operatorname{div}(a_i \cdot \nabla u_i) &\in L^p(0, \sigma_j, w_\kappa; H^{-\delta,q}(\mathbb{R}^d)) \\
\operatorname{div}((b_{k,i} \cdot \nabla) u_i)_{k\geq 1} &\in L^p(0, \sigma_j, w_\kappa; H^{1-\delta,q}(\mathbb{R}^d; \ell^2)),
\end{aligned}
$$

so the stochastic integrals are well-defined as $H^{1-\delta,q}(\mathbb{R}^d)$-valued stochastic integrals, and the deterministic integrals are well-defined as $H^{-\delta,q}(\mathbb{R}^d)$-valued Bochner integrals (see [AV23b]).

Throughout the remainder of this section we set $X_0 = H^{-\delta,q}$, $X_1 = H^{2-\delta,q}$, and $X_\beta = [X_0, X_1]_\beta = H^{2\beta-\delta,q}$. Moreover, we let

$$
X^{\mathrm{Tr}}_{\kappa,p} := (X_0, X_1)_{1-(1+\kappa)/p,p} = B_{q,p}^{2-\delta-2\frac{1+\kappa}{p}}.
$$

The next theorem establishes the local existence, uniqueness and regularity of solutions to (6.1). The embeddings $H^{2i-\delta,\zeta+\epsilon_0} \hookrightarrow H^{2i-\delta,\zeta}$, $H^{2\lambda-\delta,\zeta} \hookrightarrow H^{-\delta,q}$ and $H^{2-\delta,\zeta} \hookrightarrow H^{2\lambda-\delta,q}$ for some $\lambda \in (0, 1)$ used in our proof of the regularisation results in Theorem 5.4 do not hold on the unbounded domain $\mathbb{R}^d$. Therefore, we cannot bootstrap integrability in space via [AV22b, Theorem 6.3].

Theorem 6.4 (Local Existence, Uniqueness, and Regularity; Version of Proposition 3.1 in [AV23b])**.** *Suppose that Assumption 6.1*(p, q, h, δ) *holds,* $q > \max\left\{\frac{d}{d-\delta}, \frac{d(h-1)}{2h-\delta(h-1)}\right\}$ *and that* $\kappa \in [0, \frac{p}{2} - 1)$ *satisfies either*

$$
q < \frac{d(h-1)}{\delta} \;\; \textit{and} \;\; \frac{1+\kappa}{p} + \frac{1}{2}\left(\delta + \frac{d}{q}\right) \leq \frac{h}{h-1} \tag{6.9}
$$

or

$$
q \geq \frac{d(h-1)}{\delta} \;\; \textit{and} \;\; \frac{1+\kappa}{p} \leq \frac{h}{h-1}\left(1 - \frac{\delta}{2}\right). \tag{6.10}
$$

Then for any

$$
u_0 \in L^0_{\mathcal{F}_0}\left(\Omega; B_{p,q}^{\frac{d}{q}-\frac{2}{h-1}}\right) \tag{6.11}
$$

there exists a unique (p,κ,δ,q)*-solution* (u,σ) *to* (6.1) *such that a.s.* $\sigma>0$ *and*

$$u\in L^p_{loc}([0,\sigma),w_k;H^{2-\delta,q})\cap C\left([0,\sigma);B^{2-\delta-2\frac{1+\kappa}{p}}_{p,q}\right). \tag{6.12}$$

Moreover, u regularises instantaneously in space and time,

$$u\in H^{\theta,r}_{loc}\left(0,\sigma;H^{1-2\theta,q}\right)\qquad a.s.\ for\ all\ \theta\in[0,1/2),r\in(2,\infty), \tag{6.13}$$

and, if $q>d$*, additionally*

$$u\in C^{\theta_1,\theta_2}_{loc}\left((0,\sigma)\times\mathbb{R}^d;\mathbb{R}^\ell\right)\quad a.s.\ for\ all\ \theta_1\in[0,1/2),\theta_2\in\left(0,1-\frac{d}{q}\right). \tag{6.14}$$

We say that κ is *critical* if the second term in (6.9) or (6.10) holds with equality, and the space of initial data $B^{2-\delta-2(1+\kappa)/p}_{p,q}$ is also called critical.

In our proof of Theorem 6.4 we will use the following lemma, the proof of which is a slight modification of the proof given in [AV23b] since we consider (6.1) on $\mathbb{R}^d$ instead of $\mathbb{T}^d$.

Lemma 6.5 (Lemma 3.2 in [AV23b])**.** *Suppose that Assumption 6.1*(p,q,h,δ) *holds and let*

$$\begin{aligned}\Phi(t,v)&=\Phi_0(t,v)+\Phi_1(t,v):=\operatorname{div}(F(t,v))+f(t,v)\\ \Gamma(t,v)&:=(g_k(t,v))_{k\geq 1}.\end{aligned}$$

Moreover, suppose that $q>\max\left\{\frac{d}{d-\delta},\frac{d(h-1)}{2h-\delta(h-1)}\right\}$*. Let* $\rho_1=h-1,\rho_2=\frac{h-1}{2}$ *and*

$$\beta_1=\begin{cases}\frac{1}{2}\left(\delta+\frac{d}{q}\right)\left(1-\frac{1}{h}\right), & \text{if } q<\frac{d(h-1)}{\delta},\\ \frac{\delta}{2}, & \text{if } q\geq\frac{d(h-1)}{\delta},\end{cases}$$

$$\beta_2=\begin{cases}\frac{1}{h+1}+\frac{1}{2}\left(\delta+\frac{d}{q}\right)\frac{h-1}{h+1}, & \text{if } q<\frac{d(h-1)}{2(\delta-1)},\\ \frac{\delta}{2}, & \text{if } q\geq\frac{d(h-1)}{2(\delta-1)}.\end{cases}$$

Then $\beta_1,\beta_2\in(0,1)$ *and for* $v,v'\in X_1$*,* Φ *and* Γ *are lower-order nonlinearities, that is,*

$$\begin{aligned}\|\Phi(\cdot,v)-\Phi(\cdot,v')\|_{X_0}&\lesssim\sum_{j=1}^2(1+\|v\|^{\rho_j}_{X_{\beta_j}}+\|v'\|^{\rho_j}_{X_{\beta_j}})\|v-v'\|_{X_{\beta_j}}\\ \|\Phi(\cdot,v)\|_{X_0}&\lesssim\sum_{j=1}^2(1+\|v\|^{\rho_j}_{X_{\beta_j}})\|v\|_{X_{\beta_j}}\\ \|\Gamma(\cdot,v)-\Gamma(\cdot,v')\|_{\gamma(\ell^2;X_{1/2})}&\lesssim(1+\|v\|^{\rho_2}_{X_{\beta_2}}+\|v'\|^{\rho_2}_{X_{\beta_2}})\|v-v'\|_{X_{\beta_2}}\\ \|\Gamma(\cdot,v)\|_{\gamma(\ell^2;X_{1/2})}&\lesssim(1+\|v\|^{\rho_2}_{X_{\beta_2}})\|v\|_{X_{\beta_2}}.\end{aligned}$$

Proof. By Assumption 6.1 (5)

$$F_i^j(\cdot,0), f_i(\cdot,0) \in \cap_{k\in\{1,2\}} L^\infty(\mathbb{R}_{\geq 0}\times\Omega; H^{2\beta_k-\delta,q}),$$
$$g_i(\cdot,0) \in \cap_{k\in\{1,2\}} L^\infty(\mathbb{R}_{\geq 0}\times\Omega; H^{2\beta_k,q}(\ell^2)),$$

with β_k as in Lemma 6.5, so it suffices to estimate the differences $\Phi(\cdot,v)-\Phi(\cdot,v')$ and $\Gamma(\cdot,v)-\Gamma(\cdot,v')$. For Φ_0 we have by Assumption 6.1 (5)

$$\begin{aligned}\|\Phi_0(\cdot,v)-\Phi_0(\cdot,v')\|_{H^{-\delta,q}} &\lesssim \|(1+|v|^{h-1}+|v'|^{h-1})|v-v'|\|_{H^{-\delta,q}}\\ &\lesssim \|v-v'\|_{H^{-\delta,q}} + \|(|v|^{h-1}+|v'|^{h-1})|v-v'|\|_{L^\xi}\end{aligned}$$

by the Sobolev embedding with $-\frac{d}{\xi} = -\delta - \frac{d}{q}$, and $q > \frac{d}{d-\delta}$ ensures that $\xi > 1$,

$$\lesssim \|v-v'\|_{H^{2\beta_1-\delta,q}} + (\|v\|_{L^{\xi h}}^{h-1} + \|v'\|_{L^{\xi h}}^{h-1})\|v-v'\|_{L^{\xi h}}$$

by the embedding with $H^{-\delta,q} \hookrightarrow H^{2\beta_1-\delta,q}$ since $\beta_1 > 0$, and Hölder's inequality,

$$\overset{(i)}{\lesssim} \|v-v'\|_{H^{2\beta_1-\delta,q}} + (\|v\|_{H^{2\beta_1-\delta,q}}^{h-1} + \|v'\|_{H^{2\beta_1-\delta,q}}^{h-1})\|v-v'\|_{H^{2\beta_1-\delta,q}}$$

by the Sobolev embedding with $-\frac{d}{\xi h} \leq 2\beta_1 - \delta - \frac{d}{q}$

$$\eqsim (1 + \|v\|_{X_{\beta_1}}^{\rho_1} + \|v'\|_{X_{\beta_1}}^{\rho_1}|)\|v-v'\|_{X_{\beta_1}}.$$

In (i) we consider two cases:

- If $q < \frac{d(h-1)}{\delta}$ we set $\beta_1 = \frac{1}{2}\left(\delta + \frac{d}{q}\right)\left(1-\frac{1}{h}\right)$, and the assumption $q > \frac{d(h-1)}{2h-\delta(h-1)}$ ensures that $\beta_1 \in (0,1)$ so that the non-linearity is indeed of lower order.
- If $q \geq \frac{d(h-1)}{\delta}$ we set $\beta_1 = \frac{\delta}{2}$, which at once ensures that $\beta_1 \in (0,1)$ so that the non-linearity is again of lower order.

For Φ_1 we proceed in the same manner. By Assumption 6.1 (5) we have

$$\begin{aligned}\|\Phi_1(\cdot,v)-\Phi_1(\cdot,v')\|_{H^{-\delta,q}} &= \|\mathrm{div}F(\cdot,v) - \mathrm{div}F(\cdot,v')\|_{H^{-\delta,q}}\\ &\lesssim \|F(\cdot,v)-F(\cdot,v')\|_{H^{1-\delta,q}}\\ &\lesssim \|(1+|v|^{\frac{h-1}{2}}+|v'|^{\frac{h-1}{2}})|v-v'|\|_{H^{1-\delta,q}}\\ &\lesssim \|v-v'\|_{H^{1-\delta,q}} + \|(|v|^{\frac{h-1}{2}}+|v'|^{\frac{h-1}{2}})|v-v'|\|_{L^\eta}\end{aligned}$$

by the Sobolev embedding with $-\frac{d}{\eta} = 1-\delta-\frac{d}{q}$, and $q > \frac{d}{d-\delta}$ ensures that $\eta > 1$,

$$\lesssim \|v-v'\|_{H^{2\beta_2-\delta,q}} + (\|v\|_{L^{\eta\frac{h+1}{2}}}^{\frac{h-1}{2}} + \|v'\|_{L^{\eta\frac{h+1}{2}}}^{\frac{h-1}{2}})\|v-v'\|_{L^{\eta\frac{h+1}{2}}}$$

by the embedding $H^{2\beta_2-\delta,q} \hookrightarrow H^{1-\delta,q}$ since $\beta_2 \geq \frac{1}{2}$, and Hölder's inequality,

$$\overset{(i)}{\lesssim} \|v-v'\|_{H^{2\beta_2-\delta,q}} + (\|v\|^{\frac{h-1}{2}}_{H^{2\beta_2-\delta,q}} + \|v'\|^{\frac{h-1}{2}}_{H^{2\beta_2-\delta,q}})\|v-v'\|_{H^{2\beta_2-\delta,q}}$$

by the Sobolev embedding with $-\frac{2d}{\eta(h+1)} \leq 2\beta_2 - \delta - \frac{d}{q}$

$$\eqsim (1 + \|v\|^{\rho_2}_{X_{\beta_2}} + \|v'\|^{\rho_2}_{X_{\beta_2}}|)\|v-v'\|_{X_{\beta_2}}.$$

In (i) we again consider two cases:

- If $q < \frac{d(h-1)}{2(\delta-1)}$ we set $\beta_2 = \frac{1}{h+1} + \frac{1}{2}\left(\delta + \frac{d}{q}\right)\left(\frac{h-1}{h+1}\right)$. The assumption $q \in \left(\frac{d(h-1)}{2h-\delta(h-1)}, \frac{d(h-1)}{2(\delta-1)}\right)$ ensures that $\beta_2 \in (\frac{1}{2}, 1)$ so that the non-linearity is of lower order.
- If $q \geq \frac{d(h-1)}{2(\delta-1)}$ we set $\beta_2 = \frac{\delta}{2}$, which ensures that $\beta_2 \in (\frac{1}{2}, 1)$ so that the non-linearity is of lower order. Combining the two estimates now gives the result for Φ.

  For Γ we have by Assumption 6.1 (5)

$$\begin{aligned}\|\Gamma(\cdot,v) - \Gamma(\cdot,v')\|_{\gamma(\ell^2,H^{1-\delta,q})} &= \|g(t,\cdot,v) - g(t,\cdot,v')\|_{\gamma(\ell^2,H^{1-\delta,q})} \\ &\lesssim \|(1 + |v|^{\frac{h-1}{2}} + |v'|^{\frac{h-1}{2}})|v-v'|\|_{\gamma(\ell^2,H^{1-\delta,q})} \\ &\lesssim \|v-v'\|_{H^{1-\delta,q}(\ell^2)} + \|(|v|^{\frac{h-1}{2}} + |v'|^{\frac{h-1}{2}})|v-v'|\|_{H^{1-\delta,q}(\ell^2)}\end{aligned}$$

  since $\gamma(\ell^2; H^{\zeta,q}) = H^{\zeta,q}$ for $\zeta \in \mathbb{R}$, $q \in (1,\infty)$,

$$\lesssim \|v-v'\|_{H^{1-\delta,q}(\ell^2)} + \|(|v|^{\frac{h-1}{2}} + |v'|^{\frac{h-1}{2}})|v-v'|\|_{L^{\eta}(\ell^2)}$$

  by the Sovolev embedding with $-\frac{d}{\eta} = 1 - \delta - \frac{d}{q}$, and the remaining estimates are obtained in the same manner as for Φ_1. □

Proof of Theorem 6.4. The proof of local existence and uniqueness of the (p,κ,δ,q)-solution given in [AV23b] directly carries over to the domain $\mathbb{R}^d$ considered here. By separately considering the cases $q < \frac{d(h-1)}{\delta}$, $\frac{d(h-1)}{\delta} \leq q \leq \frac{d(h-1)}{2(\delta-1)}$, and $q > \frac{d(h-1)}{2(\delta-1)}$, Lemma 6.5 shows that Assumptions *(HF)* and *(HG)* of [AV23b, Section 4.1] hold for $(F,G) = (\Phi,\Gamma)$ and that the trace space $X^{\mathrm{Tr}}_{\kappa,p}$ is critical for (6.1) if and only if

- $q < \frac{d(h-1)}{\delta}$ and $\frac{1+\kappa}{p} + \frac{1}{2}\left(\delta + \frac{d}{q}\right) = \frac{h}{h-1}$, or
- $q \geq \frac{d(h-1)}{\delta}$ and $\frac{1+\kappa}{p} = \frac{h}{h-1}\left(1 - \frac{\delta}{2}\right)$.

The existence of a unique (p,κ,δ,q)-solution now follows by applying [AV23b, Theorem 4.8] and noting that Definition 6.3 is equivalent to the definition of an L^p_κ-maximal local solution in [AV23b, Definition 4.4]. Theorem 4.8 in [AV23b] requires that the operators $(A,B) := (v \mapsto \operatorname{div}(-a(t)\cdot\nabla v), v \mapsto ((b_k(t)\cdot\nabla)v)_{k\geq 1} \in \mathcal{SMR}^\bullet_{p,\kappa}(T)$, that is, the operators have stochastic maximal regularity (see [AV21, Definition 2.3]). By Assumption 6.1, Assumption 5.1 in [AV21] is satisfied and their Theorem 5.2 together with Remark 5.7 gives that indeed $(A,B) \in \mathcal{SMR}^\bullet_{p,\kappa}$.

Our proof of the regularity results (6.13) and (6.14) follows the same lines as the proof given in [AV23b]. Assumption 6.1, and point (3) in Assumption 6.1 in combination with Remark 5.7 in [AV21] enables us to apply their Theorem 5.2 to obtain stochastic maximal regularity for (A,B).

Step 1. We bootstrap regularity in time using [AV22b, Corollary 6.5] to show that

$$u \in \cap_{\theta\in[0,1/2)} H^{\theta,r}_{\mathrm{loc}}(0,\sigma;H^{2-\delta-2\theta,q}) \quad \text{a.s. for all } r\in(2,\infty). \tag{6.15}$$

The proof for this step given in [AV23b] carries over verbatim. In particular, for the choice of spaces $X_i = Y_i = H^{2i-\delta,q}$, $i\in\{0,1\}$, the required embeddings hold since $X_i = Y_i$ and $Y^{\mathrm{Tr}}_r = B^{2-\delta-2/r}_{q,r} \hookrightarrow B^{2-\delta-2/p}_{q,p} = X^{\mathrm{Tr}}_r$ for $r\geq p$.

Step 2. We bootstrap differentiability in space using [AV22b, Theorem 6.3] to show that

$$u \in \cap_{\theta\in[0,1/2)} H^{\theta,r}_{\mathrm{loc}}(0,\sigma;H^{1-2\theta,q}) \quad \text{a.s. for all } r\in(2,\infty). \tag{6.16}$$

The proof of this step follows the lines of the proof of Step 2 in Theorem 5.4. We may assume that $\delta\in(1,2)$ since otherwise the result already follows from (6.15). We choose $r > \max\left\{p, \frac{2}{2-\delta}\right\}$ such that

$$\frac{1}{r} + \frac{\delta-1}{2} < \frac{h}{2(h-1)},$$

which is possible since $\frac{\delta-1}{2} < \frac{h}{2(h-1)}$. We consider the spaces

$$Y_i = X_i = H^{2i-\delta,q} \quad \hat{Y}_i = H^{2i-1,q}, i\in\{1,2\},$$

and let

$$\hat{r} = r, \quad \alpha = 0, \quad \hat{\alpha} = \frac{r(\delta-1)}{2}.$$

Note that this choice of parameters satisfies the assumptions of Lemma 6.5 in both cases (6.9) and (6.10), in all settings (X_0,X_1,p,κ), (Y_0,Y_1,r,α), and $(\hat{Y}_0,\hat{Y}_1,\hat{r},\hat{\alpha})$. By Lemma 6.5 together with the arguments in the proof of [AV23b, Part A of Proposition 3.1] it now follows that [AV22b, Hypothesis (H)] holds in the (Y_0,Y_1,r,α) and the $(\hat{Y}_0,\hat{Y}_1,\hat{r},\hat{\alpha})$ setting, and both are not critical. Finally, the required embeddings hold:

- $Y^{\mathrm{Tr}}_r = B^{2-\delta-2/r}_{q,r} \hookrightarrow B^{2-\delta-2/p}_{q,p} = X^{\mathrm{Tr}}_p$ holds since $r > p$,
- There exists a $\lambda\in(0,1)$ such that $\frac{1}{r}+\lambda \leq \frac{1}{p}$ and $Y_\lambda = H^{2\lambda-\delta,q} \hookrightarrow H^{-\delta,q} = X_0$ and $Y_1 = H^{2-\delta,q} \hookrightarrow H^{2(1-\lambda)-\delta,q} = X_{1-\lambda}$,

- $\hat{Y}_i \hookrightarrow Y_i$ holds since $\delta > 1$,
- $Y_r^{\mathrm{Tr}} = B_{q,r}^{2-\delta-2/r} = B_{q,r}^{2-\delta-2/r} = \hat{Y}_{\hat{\alpha},\hat{r}}^{\mathrm{Tr}}$ by the choice of parameters $(\hat{r}, \hat{\alpha})$.

An application of [AV22b, Theorem 6.3] now gives (6.16).

Step 3. The result (6.14) follows by the Sobolev embeddings $H^{\theta,r} \hookrightarrow C^{\theta_1}$ if $\theta - \frac{d}{r} \geq \theta_1 > 0$ and $H^{1-2\theta,q} \hookrightarrow C^{\theta_2}$ if $1 - 2\theta - \frac{d}{q} \geq \theta_2 > 0$. By Step 1 we have $\theta_1 \in [0, 1/2)$, and by Step 2, if $q > d$, we have $\theta_2 \in (0, 1 - \frac{d}{q})$. □

Corollary 6.6 (Local Well-Posedness in Critical Spaces; Theorem 2.7 in [AV23b])**.** *Suppose that Assumptions 6.1*(p, q, h, δ) *and 6.2*(p, q, h, δ) *hold, and set* $\kappa =: \kappa_c = p\left(\frac{h}{h-1} - \frac{1}{2}\left(\delta + \frac{d}{q}\right)\right) - 1$. *Then for any*

$$u_0 \in L^0\left(\Omega; B_{p,q}^{\frac{d}{q} - \frac{2}{h-1}}\right) \tag{6.17}$$

there exists a unique (p, κ_c, δ, q)*-solution* (u, σ) *such that a.s.* $\sigma > 0$ *and*

$$u \in C([0, \sigma); B_{p,q}^{\frac{d}{q} - \frac{2}{h-1}}) \text{ a.s.} \tag{6.18}$$

$$u \in H_{loc}^{\theta,p}\left([0, \sigma), w_{\kappa_c}; H^{2-\delta-2\theta,q}\right) \text{ a.s. for all } \theta \in [0, 1/2). \tag{6.19}$$

Moreover, u regularises instantaneously in space and time,

$$u \in H_{loc}^{\theta,r}\left(0, \sigma; H^{1-2\theta,q}\right) \qquad \text{a.s. for all } \theta \in [0, 1/2), r \in (2, \infty), \tag{6.20}$$

and, if $q > d$*, additionally*

$$u \in C_{loc}^{\theta_1,\theta_2}\left((0, \sigma) \times \mathbb{R}^d; \mathbb{R}^\ell\right) \quad \text{a.s. for all } \theta_1 \in [0, 1/2), \theta_2 \in \left(0, 1 - \frac{d}{q}\right). \tag{6.21}$$

Proof. Corollary 6.6 is a direct consequence of Theorem 6.4. The proof that Theorem 3.1 in [AV23b] implies their Theorem 2.7 carries over verbatim. □

Theorem 6.7 (Local Continuity; Proposition 3.3 in [AV23b])**.** *Let the assumptions of Theorem 6.4 be satisfied and let* (u, σ) *be the* (p, κ, δ, q)*-solution to* (6.1)*. Then there exist positive constants* (C_0, T_0, ϵ_0) *and stopping times* $\sigma_0, \sigma_1 \in (0, \sigma]$ *a.s. such that the following holds. For each* $v_0 \in L_{\mathcal{F}_0}^p(\Omega; B_{q,p}^{2-\delta-2(1+\kappa)/p}(\mathbb{R}^\ell)$ *with* $\mathbb{E}\|u_0 - v_0\|_{B_{q,p}^{2-\delta-2(1+\kappa)/p}}^p \leq \epsilon_0$ *and* (v, τ) *the* (p, κ, δ, q)*-solution to* (6.1) *with initial data* v_0 *there exists a stopping time* $\tau_0 \in (0, \tau]$ *a.s. such that for all* $t \in [0, T_0], \gamma > 0$

$$\begin{aligned}\mathbb{P}(\sup_{r\in[0,t]} &\|u(r) - v(r)\|_{B_{q,p}^{2-\delta-2(1+\kappa)/p}} \geq \gamma, \sigma_0 \wedge \tau_0 > t) \\ &\leq \frac{C_0}{\gamma^p}\mathbb{E}\|u_0 - v_0\|_{B_{q,p}^{2-\delta-2(1+\kappa)/p}}^p,\end{aligned} \tag{6.22}$$

$$\begin{aligned}\mathbb{P}(\|u-v\|_{L^p(0,t,w_\kappa;H^{2-\delta,q})} &\geq \gamma, \sigma_0\wedge\tau_0 > t)\\ &\leq \frac{C_0}{\gamma^p}\mathbb{E}\|u_0-v_0\|^p_{B^{2-\delta-2(1+\kappa)/p}_{q,p}},\end{aligned} \tag{6.23}$$

$$\mathbb{P}(\sigma_0\wedge\tau_0 \leq t) \leq C_0\left(\mathbb{E}\|u_0-v_0\|^p_{B^{2-\delta-2(1+\kappa)/p}_{q,p}} + \mathbb{P}(\sigma_1\leq t)\right). \tag{6.24}$$

Proof. The arguments from the proof of [AV22b, Theorem 4.5] used in the proof given in [AV23b] remain valid, and thus their proof carries over verbatim. □

Theorem 6.8 (Blow-Up Criteria; Theorem 2.10 in [AV23b])**.** *Let the assumptions of Theorem 6.4 be satisfied and let (u,σ) be the (p,κ_c,δ,q)-solution to* (6.1) *with $\frac{d}{q} < \frac{h+1}{h-1} - \frac{2}{p}$. Suppose further that $p_0 \in (2,\infty)$, $q_0 \in \left[q, d/(\frac{d}{q}-\delta_0)\right)$, $h_0 \geq h$, $\delta_0 \in [1,2)$ are such that Assumptions 6.1(p_0,q_0,h_0,δ_0) and 6.2(p_0,q_0,h_0,δ_0) hold. Let*

$$\beta_0 = \frac{d}{q_0} - \frac{2}{h_0-1}, \quad \gamma_0 = \frac{d}{q_0} + \frac{2}{p_0} - \frac{2}{h_0-1}, \quad b_q = \frac{d}{\frac{d}{q_0}+\frac{d}{q}-\frac{h_0+1}{h_0-1}}.$$

Then for all $0<s<T<\infty$

$$\mathbb{P}(\sup_{t\in[s,\sigma]}\|u(t)\|_{B^{\beta_0}_{q_0,p_0}} + \|u\|_{L^{p_0}(s,\sigma;H^{\gamma_0,q_0})} < \infty, s<\sigma<T) = 0, \tag{6.25}$$

and, if additionally $\frac{d}{q} < \frac{h_0+1}{h_0-1}$,

$$\mathbb{P}(\sup_{t\in[s,\sigma]}\|u(t)\|_{B^{\beta_0}_{q_1,\infty}} < \infty, s<\sigma<T) = 0 \text{ for all } q_1 \in (q_0,b_q). \tag{6.26}$$

Remark 8. [AV23b] obtain the blow-up criteria (6.25)-(6.26) under the milder conditions of Assumption 6.2(p_0,q_0,h_0,δ_0) and for $q_1>q_0$ arbitrarily large. The additional limitations on the parameters $(q_0,q_1,p_0,\delta_0,h_0)$ required here are due to the lack of spatial integrability obtained in Theorem 6.4.

The assumption $\frac{d}{q} < \frac{h+1}{h-1} - \frac{2}{p}$ guarantees that $\gamma = \frac{d}{q}+\frac{2}{p}-\frac{2}{h-1} < 1$. The assumption $\frac{d}{q} < \frac{h_0+1}{h_0-1}$ guarantees that $b_q > q_0$ so that the interval (q_0,b_q) is non-empty. Moreover, since $q_0>q$, the assumption $\frac{d}{q} < \frac{h_0+1}{h_0-1}$ is also compatible with Assumption 6.2(p_0,q_0,h_0,δ_0). The assumption $q_0 < d/(\frac{d}{q}-\delta_0)$ is required for there to exist a q_1 such that u is a $(p_0,\kappa_0,\delta_0,q_1)$-solution to (6.27).

Proof. The proof largely carries over from [AV23b] but requires some adjustments for the unbounded domain $\mathbb{R}^d$. We will only give details for the proof of (6.26) and point out where adjustments are needed when working on $\mathbb{R}^d$ instead of $\mathbb{T}^d$.

Fix $0<s<T<\infty$ and let (u,σ) be the (p,κ_c,δ,q)-solution to (6.1). Let $(q_0,p_0,\delta_0,h_0,\beta_0)$ be as in Theorem 6.8 and $b_q > q_1 > q_0$. Set $\kappa_{c,i} = p_0\left(\frac{h_0}{h_0-1} - \frac{1}{2}\left(\delta_i + \frac{d}{q_i}\right)\right) - 1$, $i=0,1$ and $\delta_1 = \delta$. We now choose $\kappa \in (\kappa_{c,0},\kappa_{c,1})$ and set $\beta = 2-\delta_0-2\frac{1+\kappa}{p_0} < \beta_0$ such that the embeddings

$$H^{1-2\theta,q} \hookrightarrow H^{1-2\theta_1,q_1} \hookrightarrow B^{\beta}_{q_1,p_0}$$

hold for some $\theta \in [0,1/2), \theta_1 \in (\frac{d}{2q} - \frac{d}{2q_1}, 1/2)$. Note that due to the unbounded domain $\mathbb{R}^d$, we cannot apply the regularisation result (6.14) together with the embedding $C^\theta \hookrightarrow B^{\beta_0}_{q_0,p_0}$ as used in [AV23b]. The existence of suitable parameters (θ, θ_1) is guaranteed by the restriction $q_1 < b_q$. By the regularisation result (6.13) we have $u \in H^{\theta,r}_{\text{loc}}\left(0,\sigma; H^{1-2\theta,q}\right)$, which with suitably chosen θ, θ_1 implies

$$\mathbb{1}_{\{\sigma > s\}} u(s) \in L^0_{\mathcal{F}_s}(\Omega; B^\beta_{q_1,p_0}).$$

The remainder of the proof proceeds as in [AV23b]. We can now consider the SPDE

$$\begin{cases} \mathrm{d}v_i(t) - \operatorname{div}(a_i \cdot \nabla v_i)\,\mathrm{d}t = & [\operatorname{div}(F_i(\cdot, v)) + f_i(\cdot, v)]\,\mathrm{d}t \\ & + \sum_{k\geq 1} [(b_{k,i} \cdot \nabla)v_i + g_{k,i}(\cdot, v)]\,\mathrm{d}W(t) \\ v_i(0) = \mathbb{1}_{\{\sigma > s\}} u(s), & \end{cases} \tag{6.27}$$

on the interval $[s, \infty)$, which by Theorem 6.4 has unique $(p_0, \kappa_0, \delta_0, q_1)$-solution (v, τ) such that

$$v \in H^{\theta,r}_{\text{loc}}\left(0,\tau; H^{1-2\theta,q_1}\right) \text{ a.s. for all } \theta \in [0,1/2), r \in (2,\infty). \tag{6.28}$$

By the choice of $\kappa < \kappa_{c,1}$ and Theorem 6.4 the space of initial data $B^\beta_{q_1,p_0}$ is not critical and we can apply the blow-up criteria in the abstract setting provided in [AV22b, Theorem 4.10(2)] to obtain

$$\mathbb{P}(\sup_{t\in[s,\tau)} \|v(s)\|_{B^\beta_{q_1,p_0}} < \infty, \tau < T) = 0,$$

which implies that

$$\mathbb{P}(\sup_{t\in[s,\tau)} \|v(s)\|_{B^{\beta_0}_{q_1,\infty}} < \infty, \tau < T) = 0$$

by the embedding $B^{\beta_0}_{q_1,\infty} \hookrightarrow B^\beta_{q_1,p_0}$ for $\beta < \beta_0$. It remains to show that the solution (v, τ) agrees with (u, σ), specifically,

$$\tau = \sigma \text{ a.s. on } \{\sigma > s\}, \quad u = v \text{ a.e. on } [s,\sigma) \times \{\sigma > s\}. \tag{6.29}$$

Note that by (6.20) and the assumptions $h_0 \geq h$ and $q_1 \leq d/(\frac{d}{q} - \delta_0)$ we have that $(u|_{[s,\sigma)}, \mathbb{1}_{\{\sigma>s\}}\sigma + \mathbb{1}_{\Omega\setminus\{\sigma>s\}}s)$ is a $(p_0, \kappa_0, \delta_0, q_1)$-solution to (6.27), and by maximality of (v, τ) we have

$$\sigma \leq \tau \text{ on } \{\sigma > s\}, \quad u = v \text{ a.s. on } [s,\sigma) \times \{\sigma > s\}.$$

Applying the blow-up criteria in the abstract setting [AV22b, Theorem 4.10(3)] to u yields

$$\mathbb{P}(\sigma < T, \sup_{t\in[0,\sigma)} \|u(t)\|_{B^\beta_{q,p}} + \|u(t)\|_{L^p(0,\sigma;H^{\gamma,q})} < \infty) = 0$$

where

$$\beta = \frac{d}{q} - \frac{2}{h-1}, \quad \gamma = \frac{d}{q} + \frac{2}{p} - \frac{2}{h-1}, \quad \text{and } \kappa_c = p\left(\frac{h}{h-1} - \frac{1}{2}\left(\delta + \frac{d}{q}\right)\right) - 1.$$

The regularity result (6.28) together with the assumption that $\gamma \le 1$ give $u = v \in L^p_{\mathrm{loc}}((s,\sigma]; H^{\gamma,q})$ on $\{\sigma > s, \sigma < \tau\}$. By (6.20) with $\theta_c := \frac{\kappa_c}{p} < \frac{1}{2} - \frac{1}{p}$ and weighted Sobolev embeddings we have

$$\begin{aligned} u \in & H^{\theta_c,p}_{\mathrm{loc}}([0,\sigma), w_{\kappa_c}; H^{2-\delta-2\theta_c,q}) \\ & \subseteq L^p_{\mathrm{loc}}([0,\sigma); H^{\gamma,q}) \quad \text{a.s.,} \end{aligned}$$

and thus also $u \in L^p(s,\sigma; H^{\gamma,q})$ a.s. on $\{\sigma < \tau\}$. Similarly, the regularity result (6.20) applied to v gives $u = v \in C((s,\sigma]; B^{\beta}_{p,q})$ on $\{\sigma > s, \sigma < \tau\}$. By (6.18) we also have $u \in C([0,\sigma); B^{\beta}_{q,p})$, from which it follows that $u \in C([s,\sigma]; B^{\beta}_{q,p})$ on $\{\sigma > s, \sigma < \tau\}$. We thus get

$$\begin{aligned} \mathbb{P}(\sigma > s, \sigma < \tau) = & \mathbb{P}(\sigma > s, \sigma < \tau, \sup_{t\in[0,\sigma)} \|u_1(t)\|_{B^{\beta}_{q,p}} + \|u_1(t)\|_{L^p(0,\sigma;H^{\gamma,q})} < \infty) \\ \le & \mathbb{P}(\sigma < T, \sup_{t\in[0,\sigma)} \|u_1(t)\|_{B^{\beta}_{q,p}} + \|u_1(t)\|_{L^p(0,\sigma;H^{\gamma,q})} < \infty) = 0. \end{aligned}$$

Thus, on $\{\sigma > s\}$ we have $\sigma = \tau$ as claimed in (6.29).

The proof of (6.25) uses the same arguments but uses the critical parameter $\kappa_{c,0}$ for the initial data $\mathbb{1}_{\{\sigma>s\}} u(s) \in B^{\beta_0}_{q_0,p_0}$ together with the blow-up criteria [AV22b, Theorem 4.10(3)] in the abstract setting, which also apply in the critical case. □

6.2 Local well-posedness in the case $p > 2, d = 1$

So far we have assumed $d \ge 2$. In this section we will comment on the changes required to prove local well-posedness and blow-up criteria when $d = 1$.

Proposition 6.9 (Local existence, uniqueness, and regularity of different settings; Proposition 6.1 in [AV23b])**.** *Let Assumption 6.1(p,q,h,δ) be satisfied for $d = 1$, and assume further that $q \ge 2$, $\frac{1}{q} - \frac{1}{h} < 2 - \delta$ and one of the following holds:*

1. $\delta + \frac{1}{q} < 2$ *and* $\frac{1+\kappa}{p} \le \frac{h}{h-1} \min\left\{1 - \frac{\delta}{2}, 1 - \frac{\delta}{2} - \frac{1}{2q} + \frac{1}{2h}, 1 - \frac{h-1}{2h}(\delta + \frac{1}{q})\right\}$,
2. $\delta + \frac{1}{q} > 2$ *and* $\frac{1+\kappa}{p} \le \frac{h}{h-1} \min\left\{1 - \frac{\delta}{2}, 1 - \frac{\delta}{2} - \frac{1}{2q} + \frac{1}{2h}\right\}$.

Then for any

$$u_0 \in L^0_{\mathcal{F}_0}(\Omega; B^{2-\delta-2(1+\kappa)/p}_{q,p}) \tag{6.30}$$

there exists a unique (p,κ,δ,q)-solution (u,σ) to (6.1) such that a.s. $\sigma > 0$ and

$$u \in L^p_{loc}([0,\sigma), w_k; H^{2-\delta,q}) \cap C\left([0,\sigma); B^{2-\delta-2\frac{1+\kappa}{p}}_{p,q}\right). \tag{6.31}$$

Moreover, u regularises instantaneously in space and time,

$$u \in H^{\theta,r}_{loc}\left(0,\sigma;H^{1-2\theta,q}\right) \qquad a.s.\ for\ all\ \theta\in[0,1/2), r\in(2,\infty), \tag{6.32}$$

$$u \in C^{\theta_1,\theta_2}_{loc}\left((0,\sigma)\times\mathbb{R}^d;\mathbb{R}^\ell\right) \quad a.s.\ for\ all\ \theta_1\in[0,1/2), \theta_2\in\left(0,1-\frac{d}{q}\right). \tag{6.33}$$

Proof. Lemma 6.5 is at the core of the proof of the local well-posedness result Theorem 6.4. Therefore, we first discuss how the conditions on β_1 and β_2 change when $d=1$. Splitting $\Phi=\Phi_0+\Phi_1$ as before, we estimate

$$\begin{aligned}\|\Phi_0(\cdot,v)-\Phi_0(\cdot,v')\|_{H^{-\delta,q}} &\lesssim \|(1+|v|^{h-1}+|v'|^{h-1})|v-v'|\|_{H^{-\delta,q}}\\ &\lesssim \|v-v'\|_{H^{2\beta_1-\delta,q}}+\|(|v|^{h-1}+|v'|^{h-1})|v-v'|\|_{L^\xi}\end{aligned}$$

by the embedding with $H^{-\delta,q}\hookrightarrow H^{2\beta_1-\delta,q}$ since $\beta_1>0$ and the Sobolev embedding with $-\frac{1}{\xi}\geq-\delta-\frac{1}{q}$, and we choose $\xi=1$,

$$\lesssim \|v-v'\|_{H^{2\beta_1-\delta,q}}+(\|v\|^{h-1}_{L^{\xi h}}+\|v'\|^{h-1}_{L^{\xi h}})\|v-v'\|_{L^{\xi h}}$$

by Hölder's inequality,

$$\lesssim \|v-v'\|_{H^{2\beta_1-\delta,q}}+(\|v\|^{h-1}_{H^{2\beta_1-\delta,q}}+\|v'\|^{h-1}_{H^{2\beta_1-\delta,q}})\|v-v'\|_{H^{2\beta_1-\delta,q}}.$$

by the Sobolev embedding with $-\frac{1}{\xi h}\leq 2\beta_1-\delta-\frac{1}{q}$, and we choose $\beta_1=\frac{1}{2}\max\{\frac{1}{q}-\frac{1}{h},0\}+\frac{\delta}{2}$ to guarantee $2\beta_1-\delta\geq 0$. We further note that we require $\frac{1}{q}-\frac{1}{h}<2-\delta$ for the non-linearity to be of lower order.

Similarly, we estimate

$$\begin{aligned}\|\Phi_1(\cdot,v)-\Phi_1(\cdot,v')\|_{H^{-\delta,q}} &\lesssim \|(1+|v|^{(h-1)/2}+|v'|^{(h-1)/2})|v-v'|\|_{H^{1-\delta,q}}\\ &\overset{(i)}{\lesssim} \|v-v'\|_{H^{1-\delta,q}}+\|(|v|^{(h-1)/2}+|v'|^{(h-1)/2})|v-v'|\|_{L^\eta}\end{aligned}$$

by the Sobolev embedding with $-\frac{1}{\eta}\geq 1-\delta-\frac{1}{q}$, and $q>\frac{d}{d-\delta}$ ensures that $\eta>1$. Thus, by the embedding $H^{2\beta_2-\delta,q}\hookrightarrow H^{1-\delta,q}$ with $\beta_2\geq\frac{1}{2}$, and Hölder's inequality,

$$\begin{aligned}&\|\Phi_1(\cdot,v)-\Phi_1(\cdot,v')\|_{H^{-\delta,q}}\\ &\quad\lesssim \|v-v'\|_{H^{2\beta_2-\delta,q}}+(\|v\|^{(h-1)/2}_{L^{\eta(h+1)/2}}+\|v'\|^{(h-1)/2}_{L^{\eta(h+1)/2}})\|v-v'\|_{L^{\eta(h+1)/2}}\\ &\quad\overset{(ii)}{\lesssim} \|v-v'\|_{H^{2\beta_2-\delta,q}}+(\|v\|^{(h-1)/2}_{H^{2\beta_2-\delta,q}}+\|v'\|^{(h-1)/2}_{H^{2\beta_2-\delta,q}})\|v-v'\|_{H^{2\beta_2-\delta,q}}\\ &\quad\eqsim (1+\|v\|^{\rho_2}_{X_{\beta_2}}+\|v'\|^{\rho_2}_{X_{\beta_2}}|)\|v-v'\|_{X_{\beta_2}}\end{aligned}$$

by the Sobolev embedding with $-\frac{2d}{\eta(h-1)}\leq 2\beta_2-\delta-\frac{d}{q}$. In (i) we use the Sobolev embedding with $-\frac{1}{\eta}\geq 1-\delta-\frac{1}{q}$ and we consider two cases to choose the value for η. If $\delta+\frac{1}{q}>2$ we let $\eta=1$, and if $\delta+\frac{1}{q}<2$ we set η via $-\frac{1}{\eta}=1-\delta-\frac{1}{q}$. In (ii) we use the Sobolev embedding with $-\frac{2}{\eta(h+1)}\leq 2\beta_2-\delta-\frac{1}{q}$, and we consider three cases:

- If $\delta + \frac{1}{q} < 2$ and $q < \frac{h-1}{2(\delta-1)}$ we set $\beta_2 = \frac{1}{h+1} + \frac{1}{2}\left(\delta + \frac{1}{q}\right)\left(\frac{h-1}{h+1}\right)$, and the assumption $q \in \left(\frac{1}{2-\delta}, \frac{h-1}{2(\delta-1)}\right)$ ensures that $\beta_2 \in (\frac{1}{2}, 1)$,
- If $\delta + \frac{1}{q} < 2$ and $q \geq \frac{h-1}{2(\delta-1)}$ we set $\beta_2 = \frac{\delta}{2}$, which ensures that $\beta_2 \in (\frac{1}{2}, 1)$,
- If $\delta + \frac{1}{q} > 2$ we set $\beta_2 = \max\{\frac{1}{2q} - \frac{1}{h+1}, 0\} + \frac{\delta}{2}$. The condition $\frac{1}{q} - \frac{1}{h} < 2 - \delta$ established before implies $\beta_2 \leq \frac{1}{2q} - \frac{1}{h+1} + \frac{\delta}{2} < \frac{1}{2q} - \frac{1}{2h} + \frac{\delta}{2} < 1$, so $\beta_2 \in [\frac{1}{2}, 1)$.

Since $\beta_2 \in [\frac{1}{2}, 1)$ in all, the non-linearity is of lower order. Combining the two estimates above now gives the result for Φ. As in the proof of Lemma 6.5 the same estimates apply to Γ. Taken together, these results establish Lemma 6.5 for the case $d = 1$ with $\rho_1 = h - 1$, $\rho_2 = \frac{h-1}{2}$, and

$$\beta_1 = \frac{1}{2}\max\left\{\frac{1}{q} - \frac{1}{h}, 0\right\} + \frac{\delta}{2},$$

$$\beta_2 = \begin{cases} \frac{1}{h+1} + \frac{1}{2}\left(\delta + \frac{1}{q}\right)\left(\frac{h-1}{h+1}\right), & \text{if } \delta + \frac{1}{q} < 2 \text{ and } q < \frac{h-1}{2(\delta-1)}, \\ \frac{\delta}{2}, & \text{if } \delta + \frac{1}{q} < 2 \text{ and } q > \frac{h-1}{2(\delta-1)}, \\ \max\left\{\frac{1}{2q} - \frac{1}{h+1}, 0\right\} + \frac{\delta}{2}, & \text{if } \delta + \frac{1}{q} > 2. \end{cases}$$

Existence of a unique (p, κ, δ, q)-solution (u, σ) to (6.1) is established as in the proof of Theorem 6.4. The version of Lemma 6.5 for the case $d = 1$ shows that the Assumptions *(HF)* and *(HG)* of [AV23b, Section 4.1] hold for $(F, G) = (\Phi, \Gamma)$ and the trace space $X^{\mathrm{Tr}}_{\kappa,p}$ is sub-critical if for $j = 1, 2$

$$\frac{1+\kappa}{p} \leq \frac{\rho_j + 1}{\rho_j} - \frac{1}{2}(1 - \beta_j). \tag{6.34}$$

The latter sub-criticality condition together with our choice of $\rho_1, \rho_2, \beta_1, \beta_2$ leads to the requirement

$$\frac{1+\kappa}{p} \leq \begin{cases} \frac{h}{h-1}\min\left\{1 - \frac{\delta}{2}, 1 - \frac{\delta}{2} - \frac{1}{2q} + \frac{1}{2h}, 1 - \frac{h+1}{h}\left(1 - \frac{\delta}{2}\right)\right\}, & \text{if } \delta + \frac{1}{2} < 2, \\ \frac{h}{h-1}\min\left\{1 - \frac{\delta}{2}, 1 - \frac{\delta}{2} - \frac{1}{2q} + \frac{1}{2h}\right\}, & \text{if } \delta + \frac{1}{2} > 2. \end{cases}$$

The existence proof is completed by applying [AV23b, Theorem 4.8] as before. The regularisation results (6.32) - (6.33) follow by repeating the arguments in the proof of Theorem 6.4. □

Also the blow-up criteria Theorem 6.8 remain valid for $d = 1$.

Theorem 6.10 (Blow-Up Criteria)**.** *Let the assumptions of Proposition 6.9 be satisfied with parameters $(p, q, h, \delta, \kappa)$ and let (u, σ) be the (p, κ, δ, q)-solution to (6.1) with $\frac{1}{q} < \frac{h+1}{h-1} - \frac{2}{p}$. Suppose further that Assumption 6.1$(p_0, q_0, h_0, \delta_0)$ holds* *with* $h_0 \geq h$ *and that* $(p_0, q_0, h_0, \delta_0, \kappa_0)$ *are such that* $q_0 \in [q, 1/(\frac{1}{q} - \delta_0))$ *and either Proposition 6.9* (1) *or* (2) *holds. Let*

$$\beta_0 = 2 - \delta_0 - 2\frac{1+\kappa_0}{p_0}, \quad \gamma_0 = 2 - \delta_0 - \frac{2\kappa_0}{p_0}, \quad b_q = \frac{1}{\frac{1}{q_0} + \frac{1}{q} - \frac{h_0+1}{h_0-1}}.$$

Then for all $0 < s < T < \infty$

$$\mathbb{P}(\sup_{t\in[s,\sigma]} \|u(t)\|_{B^{\beta_0}_{q_0,p_0}} + \|u\|_{L^{p_0}(s,\sigma;H^{\gamma_0,q_0})} < \infty, s < \sigma < T) = 0, \tag{6.35}$$

and, if additionally $\frac{1}{q} < \frac{h_0+1}{h_0-1}$,

$$\mathbb{P}(\sup_{t\in[s,\sigma]} \|u(t)\|_{B^{\beta_0}_{q_1,\infty}} < \infty, s < \sigma < T) = 0 \text{ for all } q_1 \in (q_0, b_q). \tag{6.36}$$

Proof. The proof proceeds by the same arguments as the proof of Theorem 6.8. □

6.3 Local well-posedness in the case $p = 2$

The results for the $L^p(L^q)$-setting discussed so far only consider $p > 2$. On the other hand, the variational setting considered in Section 3 corresponds to the case $p = 2$. We already obtained global well-posedness results for (3.1) in the variational setting. Therefore, it is desirable to also generalise the regularity results obtained for $p > 2$ to the boundary case $p = 2$.

We first note that the variational setting of Section 4 corresponds to the parameters $p = q = 2$, $\delta = 1$, and $\kappa = 0$ in the L^p-setting. In the case of the weak setting it is immediately clear that the (global) solution provided by Theorem 4.2 also satisfies Definition 6.3 of a (local) (p, q, h, δ)-solution if we allow $p = 2$ in Assumption 6.1. In the case of the strong setting, by the embeddings $H^{2,2} \hookrightarrow H^{1,2} \hookrightarrow L^2$ the solution provided by Theorem 4.4 also satisfies Definition 6.3 of a (local) (p, q, h, δ)-solution. Theorems 4.2 and 4.4 already show that the existence part of Theorem 6.4 holds. Hence, it remains to establish the regularity results (6.13) and (6.14).

Proposition 6.11 (Regularity for $p = 2$; Proposition 7.2 in [AV23b])**.** *Assume that* $p = q = 2$,

$$h \in \begin{cases} (1, 4], & \text{if } d = 1 \\ (1, 3], & \text{if } d = 2 \\ (1, \frac{4+d}{d}], & \text{if } d \geq 3, \end{cases} \tag{6.37}$$

and Assumption 6.1(p, q, h, δ) *holds for some* $\delta \in (1, 2)$. *If* $d = 1$, *let* $u_0 \in L^2_{\mathcal{F}_0}(\Omega; L^2)$ *and let Assumption 4.1 be satisfied, if* $d \in \{2, 3, 4\}$ *let* $u_0 \in L^2_{\mathcal{F}_0}(\Omega; H^1)$ *and let Assumption 4.3 be satisfied, and let* (u, σ) *be the* $(p, \kappa, \delta, q) = (2, 0, 1, 2)$*-solution to* (6.1) *provided by Theorem 4.2 or Theorem 4.4. Then it holds a.s. that*

$$u \in L^2_{loc}([0, \sigma); H^{1,2}) \cap C([0, \sigma); L^2).$$

Moreover, the following analogue of the instantaneous regularisation results (6.13) - (6.14) *holds:*

$$u \in H^{\theta,r}_{loc}\left(0,\sigma;H^{1-2\theta,q}\right) \qquad\qquad a.s.\ for\ all\ \theta\in[0,1/2), r\in(2,\infty),$$

and, if $d=1$*, additionally*

$$u \in C^{\theta_1,\theta_2}_{loc}\left((0,\sigma)\times\mathbb{R}^d;\mathbb{R}^\ell\right) \qquad a.s.\ for\ all\ \theta_1\in[0,1/2), \theta_2\in\left(0,\frac{1}{2}\right).$$

Note that Theorems 4.2 and 4.4 show that $\sigma=\infty$.

Proof. The first assertion of the Proposition is a direct consequence of Theorem 4.2 or Theorem 4.4. The regularisation result is proved in the same way as in [AV23b]. Therefore, we only sketch the proof here.

We first consider the case $d\geq 3$. We will apply [AV22b, Theorem 6.8] to show that the $(p,\kappa,\delta,q)=(2,0,1,2)$-solution (u,σ) coincides with the $(p,\kappa,\delta,q)=(r,\alpha,\delta_0,2)$-solution where the parameters $(r,\alpha,\delta_0,2)$ are such that the conditions of Theorem 6.4 are satisfied. To this end, we choose $\epsilon\in(0,1/2)$ such that $\delta_0:=1+\epsilon<\delta$ and we let

$$Y_i=H^{2i-1-\epsilon,2},\ X_i=H^{2i-1,2},\ p=2.$$

We choose r such that $\frac{1}{r}=\max_{j\in\{1,2\}}\beta_j-\frac{1}{2}$, where β_j are as given in Lemma 6.5, and note that this implies $r\in(2,\infty)$. Moreover, we set

$$\frac{1}{2}=\frac{1+\alpha}{r}+\frac{\epsilon}{2},$$

and note that $\alpha\in(0,\frac{r}{2}-1)$ since $\epsilon\in(0,\frac{1}{2})$. Now letting $(p,q,\kappa,\delta)=(r,2,\alpha,\delta_0)$, our assumption that $h\in(1,\frac{4+d}{d})$ implies that either condition (6.9) or condition (6.10) of Theorem 6.4 is satisfied. Repeating the arguments in the existence part of the proof of Theorem 6.4 shows that the conditions of [AV22b, Theorem 6.8] are satisfied, and we conclude from the latter result that (u,σ) coincides with the unique $(r,\alpha,\delta_0,2)$-solution to (6.1). The regularisation results of Proposition 6.11 now follow by applying Theorem 6.4 to the $(r,\alpha,\delta_0,2)$-solution to (6.1).

Next, we consider the case $d=2$. If $h\in(1,2]$, we require $\delta_0\in(1,\min\{\delta,3/2\})$ for the condition (6.9) of Theorem 6.4 to be satisfied. Hence, we choose $\epsilon\in(0,1/2)$ such that $\delta_0=1+\epsilon<\min\{\delta,3/2\}$ and repeat the arguments from the case $d\geq 3$ above. If $h\in(2,3)$, we need to modify the choice of β_1 in Lemma 6.5 to be able to apply the existence part of the proof of Theorem 6.4. Therefore, we choose $\epsilon\in(0,1/2)$ such that $\delta_0=1+\epsilon<\min\{\delta,5/3,h-1,2/(h-1)\}$, and set $\beta_1=\frac{\delta_0}{2}+\frac{1}{2}-\frac{1}{h}$. In in the Sobolev embeddings used in the proof of Lemma 6.5 we use $\xi=1$. With our choice of δ_0 the condition (6.10) of Theorem 6.4 is satisfied and we can repeat the arguments from the case $d\geq 3$ above, using our modified choice of β_1 in the existence part of the proof of Theorem 6.4.

If $h=3$, we repeat the argument for the case $d\geq 3$ with a slightly different choice of the spaces X_i and Y_i. The existence part of the proof for $d=2$ was given for the strong setting with $X_0 = L^2$ and $X_1 = H^2$. Hence, in order to repeat the arguments for the case $d \geq 3$, we require analogues of Lemma 6.5 and Theorem 6.4 for the setting $X_0 = H^{-\epsilon}$, $X_1 = H^{2-\epsilon}$ with $\epsilon \in [0, 1/2)$, $h = 3$ and $q = 2$. We let Φ and Γ as in Lemma 6.5. If $\epsilon > 0$ we estimate for Φ_0

$$
\begin{aligned}
\|\Phi_0(\cdot, v) - \Phi_0(\cdot, v')\|_{H^{-\epsilon}} &\lesssim \|(1 + |v|^{h-1} + |v'|^{h-1})|v - v'|\|_{H^{-\epsilon}} \\
&\lesssim \|v - v'\|_{H^{2\beta_1-\epsilon}} + \|(|v|^{h-1} + |v'|^{h-1})|v - v'|\|_{L^\xi}
\end{aligned}
$$

by the embedding $H^{-\epsilon} \hookrightarrow H^{2\beta_1-\epsilon}$ if $\beta_1 > 0$ and the Sobolev embedding with $-\epsilon - 1 \leq -\frac{2}{\xi}$. By Hölder's inequality,

$$
\begin{aligned}
&\|\Phi_0(\cdot, v) - \Phi_0(\cdot, v')\|_{H^{-\epsilon}} \\
&\quad\lesssim \|v - v'\|_{H^{2\beta_1-\epsilon}} + (\|v\|_{L^{\xi h}}^{h-1} + \|v'\|_{L^{\xi h}}^{h-1})\|v - v'\|_{L^{\xi h}} \\
&\quad\lesssim \|v - v'\|_{H^{2\beta_1-\epsilon}} + (\|v\|_{H^{2\beta_1-\epsilon}}^{h-1} + \|v'\|_{H^{2\beta_1-\epsilon}}^{h-1})\|v - v'\|_{H^{2\beta_1-\epsilon}}
\end{aligned}
$$

by the Sobolev embedding with $-\frac{2}{\xi h} \leq 2\beta_1 - \epsilon - 1$. Thus, it suffices to choose $\xi = 2$ and $\beta_1 = \frac{1}{2} + \frac{\epsilon}{2}$ since $h = 3$. If $\epsilon = 0$ we do not require the first Sobolev embedding since $H^0 = L^2$, the remaining steps remain unchanged with $\xi = 2$.

For Φ_1 we estimate

$$
\begin{aligned}
\|\Phi_1(\cdot, v) - \Phi_1(\cdot, v')\|_{H^{-\epsilon}} &= \|\mathrm{div} F(\cdot, v) - \mathrm{div} F(\cdot, v')\|_{H^{-\epsilon}} \\
&\lesssim \|F(\cdot, v) - F(\cdot, v')\|_{H^{1-\epsilon}} \\
&\lesssim \|(1 + |v|^{\frac{h-1}{2}} + |v'|^{\frac{h-1}{2}})|v - v'|\|_{H^{1-\epsilon}} \\
&\eqsim \|v - v'\|_{H^{1-\epsilon}} + \|(|v| + |v'|)|v - v'|\|_{H^{1-\epsilon}} \\
&\lesssim \|v - v'\|_{H^{1-\epsilon}} + (\|v\|_{L^4(\mathbb{R}^2)} + \|v'\|_{L^4(\mathbb{R}^2)})\|v - v'|\|_{H^{1-\epsilon,4}(\mathbb{R}^2)} \\
&\quad + (\|v\|_{H^{1-\epsilon,4}(\mathbb{R}^2)} + \|v'\|_{H^{1-\epsilon,4}(\mathbb{R}^2)})\|v - v'|\|_{L^4(\mathbb{R}^2)}
\end{aligned}
$$

since $h = 3$. Now using [Tay00, Chapter 2, Proposition 1.1] with $p = 2, q_1 = r_1 = q_2 = r_2 = 4$,

$$
\begin{aligned}
&\|\Phi_1(\cdot, v) - \Phi_1(\cdot, v')\|_{H^{-\epsilon}} \\
&\quad\lesssim \|v - v'\|_{H^{1-\epsilon}} + (\|v\|_{L^4(\mathbb{R}^2)} + \|v'\|_{L^4(\mathbb{R}^2)})\|v - v'|\|_{H^{1-\epsilon,4}(\mathbb{R}^2)} \\
&\qquad + (\|v\|_{H^{1-\epsilon,4}(\mathbb{R}^2)} + \|v'\|_{H^{1-\epsilon,4}(\mathbb{R}^2)})\|v - v'|\|_{L^4(\mathbb{R}^2)} \\
&\quad\lesssim \|v - v'\|_{H^{1-\epsilon}} + (\|v\|_{H^{2\beta_2-\epsilon,2}(\mathbb{R}^2)} + \|v'\|_{H^{2\beta_2-\epsilon,2}(\mathbb{R}^2)})\|v - v'|\|_{H^{2\beta_2-\epsilon,2}(\mathbb{R}^2)}
\end{aligned}
$$

by the Sobolev embeddings $H^{2\beta_2-\epsilon,2} \hookrightarrow L^4$ with $-\frac{1}{2} \leq 2\beta_2 - \epsilon - 1$ and $H^{2\beta_2-\epsilon,2} \hookrightarrow H^{1-\epsilon,2}$ with $-\epsilon \leq 2\beta_2 - \epsilon - 1$. Thus, it suffices to choose $\beta_2 = \frac{3}{4}$. For Γ we use that

$$
\begin{aligned}
\|\Gamma_1(\cdot, v) - \Gamma_1(\cdot, v')\|_{\ell^2;\gamma(X_{1/2})} &\eqsim \|g(\cdot, v) - g(\cdot, v')\|_{\gamma(\ell^2;H^{1-\epsilon})} \\
&\lesssim \|(1 + |v|^{\frac{h-1}{2}} + |v'|^{\frac{h-1}{2}})|v - v'|\|_{H^{1-\epsilon}},
\end{aligned}
$$

from which we obtain an estimate in the same way as for Φ_1 with $\beta_2 = \frac{3}{4}$. Note that the condition

$$\frac{1+\kappa}{p} \leq (1-\beta_j)\frac{1+\rho_j}{\rho_j}$$

for Assumptions *(HF)* and *(HG)* of [AV23b, Section 4.1] to hold for $(F, G) = (\Phi, \Gamma)$ then becomes $\frac{1+\kappa}{p} \leq \min\left\{\frac{1}{2}, \frac{3}{4}(1-\epsilon)\right\}$. The proof of Theorem 6.4 for the setting $X_0 = H^{-\epsilon}$, $X_1 = H^{2-\epsilon}$, $h = 3$ and $q = 2$ now follows the same steps as before.

We are now ready to repeat the reasoning for the case $d \geq 3$. We first choose $\epsilon \in (0, 1/2)$, then set $\frac{r}{2} = \max_{j\in\{1,2\}\beta_j - \frac{1}{2} = \frac{1}{4}}$ and choose $\alpha \in (0,1) = (0, 1-\frac{r}{2})$ such that

$$\frac{1}{2} = \frac{1+\alpha}{r} + \frac{\epsilon}{2}.$$

If we let $(p, q, \kappa, \delta) = (r, 2, \alpha, \epsilon)$, we can repeat the arguments of the existence part of the proof of Theorem 6.4 with spaces $X_0 = H^{-\epsilon}$ and $X_1 = H^{2-\epsilon}$ to show that the conditions of [AV22b, Theorem 6.8] are satisfied. We conclude from the latter result that (u, σ) coincides with the unique $(r, \alpha, \epsilon, 2)$-solution to (6.1). The regularisation results of Proposition 6.11 now follow by applying Theorem 6.4 to the $(r, \alpha, \epsilon, 2)$-solution to (6.1).

Finally, we consider the case $d = 1$. We use the proof of Proposition 6.9 with suitable parameters to establish the regularity results. To this end, we choose $\epsilon \in (0, 1/2)$ such that

$$\delta_0 = 1 + \epsilon < \min\left\{\delta, \frac{2h}{h-1} - \frac{3}{2}, 1 + \frac{1}{h}, \frac{1}{2} + \frac{2}{h}\right\}.$$

The latter guarantees that

$$\frac{1}{2} < \tfrac{h}{h-1}\min\left\{1 - \tfrac{\delta}{2}, 1 - \tfrac{\delta}{2} - \tfrac{1}{2q} + \tfrac{1}{2h}, 1 - \tfrac{h-1}{2h}(\delta + \tfrac{1}{q})\right\}.$$

Letting β_1 and β_2 as in the proof of Proposition 6.9, we can choose $r > 2$, such that $\frac{1}{r} = \max_{j\in\{1,2\}} \beta_j - \frac{1}{2}$. Moreover, we set

$$\frac{1}{2} = \frac{1+\alpha}{r} + \frac{\epsilon}{2},$$

where $\alpha \in (0, \frac{r}{2} - 1)$. Thus, the case $\delta_0 + \frac{1}{2} < 2$ and $\frac{1+\alpha}{r} \leq \frac{h}{h-1}\min\left\{1 - \frac{\delta_0}{2}, 1 - \frac{\delta_0}{2} - \frac{1}{4} + \frac{1}{2h}, 1 - \frac{h-1}{2h}(\delta_0 + \frac{1}{2})\right\}$ of Proposition 6.9 is satisfied with parameters $(p, q, \kappa, \delta) = (r, 2, \alpha, \delta_0)$. We can now repeat the arguments from the case $d \geq 3$ above. □

7 FitzHugh-Nagumo Equations in the $L^p(L^q(\mathbb{R}^d))$-Setting

With the results on local and global well-posedness of reaction-diffusion equations on $\mathbb{R}^d$ in hand, we now return to the problem of well-posedness of the

FitzHugh-Nagumo equations on the unbounded domain $\mathbb{R}^d$ with $d > 1$. We begin by establishing analogues of the existence and regularisation results Theorem 5.4, Corollary 5.6, and the blow-up criteria 5.7.

7.1 Local Well-Posedness and Blow-Up Criteria

We again consider the FitzHugh-Nagumo equations (4.1). As in Section 5, since the second component of the equation does not involve a Laplace operator, the uniform ellipticity condition of Assumption 7.1(4) only holds for the first component of the system (4.1). Therefore, we need to make some adjustments to the theory developed for reaction-diffusion equations in Section 6. We let $\delta \in [1, 2)$, $\eta \in (0, 2-\delta]$, $q \geq 2$ and we consider the spaces

$$\begin{aligned} X_0 &= H^{-\delta,q}(\mathbb{R}^d) \oplus H^{2-\delta-\eta,q}(\mathbb{R}^d) =: X_0^1 \oplus X^2, \\ X_1 &= H^{2-\delta,q}(\mathbb{R}^d) \oplus H^{2-\delta-\eta,q}(\mathbb{R}^d) =: X_1^1 \oplus X^2, \\ X_\beta &:= [X_0, X_1]_\beta = H^{2\beta-\delta,q}(\mathbb{R}^d) \oplus H^{2-\delta-\eta,q}(\mathbb{R}^d), \end{aligned} \tag{7.1}$$

where $\beta \in (0, 1)$. We will again use the shorthand notation $H^{-\delta,q}, H^{2-\delta,q}$ etc. instead of $H^{-\delta,q}(\mathbb{R}^d), H^{2-\delta,q}(\mathbb{R}^d)$ when no confusion can arise, and we will write $X_\beta^1 = [X_0^1, X_1^1]_\beta$. On these spaces we consider the operators $\tilde{A}_0, A_{\text{pert}}, f, B_0, G$, which we define as in (5.2). We again let $A_0 = \tilde{A}_0 + A_{\text{pert}}$, $A = A_0 - F$ and $B = B_0 + G$. Moreover, we make the following assumptions

Assumption 7.1. *Let $p \in (2, \infty)$, $q \in [2, \infty)$, $\delta \in [1, 2)$ and for $i = 1, 2$ the following hold:*

1. *For each $j, \in \{1, \dots, d\}$, $b^j := (b_k^j)_{k\geq 1} : \mathbb{R}_{\geq 0} \times \Omega \times \mathbb{R}^d \to \ell^2$ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable,*
2. *If $\delta = 1$, there exists $N > 0$ such that a.s. for all $t \geq 0$ and $j \in \{1, \dots, d\}$*
$$\|b^j\|_{L^\infty(\ell^2)} \leq N, \tag{7.2}$$
If $\delta > 1$ there additionally exist $\tau > \delta - 1$, $\epsilon \in (0, \tau + 1 - \delta)$ such that a.s. for all $t \geq 0$ and $j \in \{1, \dots, d\}$,
$$\|b^j\|_{C^\tau(\mathbb{R}^d;\ell^2)} \leq N, \tag{7.3}$$
3. *For every $s \in [0, T)$ there exist $\hat{b}^j : [s, T] \times \Omega \to \ell^2$ such that for all $j \in \{1, \dots, d\}$*
$$\lim_{|x|\to\infty} \operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{t\in[s,T]} \|b^j - \hat{b}^j\|_{\ell^2} = 0, \tag{7.4}$$
4. *There exists a $\nu_0 \in (0, \nu)$ such that, a.s. for all $t \geq 0$, $x, \xi \in \mathbb{R}^d$*
$$\sum_{j,l=1}^{d} \left(\nu\delta_{j,l} - \frac{1}{2} \sum_{k\geq 1} b_k^j(t, x) b_k^l(t, x) \right) \xi_j \xi_l \geq \nu_0 |\xi|^2, \tag{7.5}$$

5. *The map $f:\mathbb{R}\to\mathbb{R}$ is $\mathcal{B}(\mathbb{R})$-measurable and the maps $g_i \coloneqq (g_{k,i})_{k\geq 1}: \mathbb{R}_{\geq 0}\times\Omega\times\mathbb{R}\to\ell^2$ are $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{B}(\mathbb{R})$-measurable. Moreover,*

$$\begin{aligned}
f(\cdot,0) &= 0,\\
\mu_{f'} &\coloneqq \sup_{u\in\mathbb{R}} f'(u) < \infty,\\
|f(u)-f(v)| &\leq C_1(1+|u|^{h-1}+|v|^{h-1})|u-v|,\\
g_i(\cdot,0) &\in \cap_{q\geq 2,\theta\in(0,1)} L^\infty(\mathbb{R}_{\geq 0}\times\Omega; H^{\theta,q}(\mathbb{R}^d;\ell^2)),\\
\|g_1(t,u)-&g_1(t,v)\|_{\ell^2}\\
&\leq C_1(|u_1-v_1|+|u_2-v_2|)\\
\|g_2(t,u)-&g_2(t,v)\|_{\ell^2} \leq C_1(|u_1-v_1|+|u_2-v_2|)\\
\|g_2(t,u)-&g_2(t,v)\|_{X^2(\ell^2)}\\
&\leq C_1(\|u_1-v_1\|_{X^2(\ell^2)}+\|u_2-v_2\|_{X^2(\ell^2)})
\end{aligned}$$

Our definition of a solution is analogous to Definition 5.2 in Section 5 for the domain $\mathbb{T}^d$.

Assumption 7.2 (*Assumption 2.4 in [AV23b]*)**.** *Let $d\geq 2$. Assumption 7.2(p,q,h,δ) holds if $p\in(2,\infty)$, $q\in[2,\infty)$, $h>1$ and $\delta\in[1,\frac{h+1}{h})$ satisfy*

$$\frac{1}{p}+\frac{1}{2}\left(\delta+\frac{d}{q}\right)\leq\frac{h}{h-1},\qquad \frac{d}{d-\delta}<q<\frac{d(h-1)}{h+1-\delta(h-1)}. \tag{7.6}$$

Theorem 7.3 (Local Existence, Uniqueness, and Regularity)**.** *Suppose that Assumption 5.1(p,q,h,δ) holds, $q>\max\left\{\frac{d}{d-\delta},\frac{d(h-1)}{2h-\delta(h-1)}\right\}$ and that $\kappa\in[0,\frac{p}{2}-1)$ satisfies either*

$$q<\frac{d(h-1)}{\delta}\ \textit{and}\ \frac{1+\kappa}{p}+\frac{1}{2}\left(\delta+\frac{d}{q}\right)\leq\frac{h}{h-1} \tag{7.7}$$

or

$$q\geq\frac{d(h-1)}{\delta}\ \textit{and}\ \frac{1+\kappa}{p}\leq\frac{h}{h-1}\left(1-\frac{\delta}{2}\right). \tag{7.8}$$

Then for any $\eta\in(0,2-\delta]$ and

$$u_0\in L^0_{\mathcal{F}_0}\left(\Omega; B_{q,p}^{2-\delta-2\frac{1+\kappa}{p}}\oplus H^{2-\delta-\eta,q}\right) \tag{7.9}$$

there exists a unique (p,κ,δ,η,q)-solution (u,σ) to (4.1) such that a.s. $\sigma>0$ and

$$u\in L^p_{loc}([0,\sigma),w_k;H^{2-\delta,q}\oplus H^{2-\delta-\eta,q})\cap C\left([0,\sigma);B_{q,p}^{2-\delta-2\frac{1+\kappa}{p}}\oplus H^{2-\delta-\eta,q}\right). \tag{7.10}$$

Moreover, u regularises instantaneously in time and, in addition, u_1 regularises instantaneously in space,

$$u \in H^{\theta,r}_{loc}\left(0,\sigma; H^{1-2\theta,q} \oplus H^{2-\delta-\eta,q}\right) \qquad a.s.\ for\ all\ \theta \in [0,1/2), r \in (2,\infty), \tag{7.11}$$

and, if $q > d$, additionally

$$u_1 \in C^{\theta_1,\theta_2}_{loc}\left((0,\sigma)\times\mathbb{R}^d;\mathbb{R}\right) \qquad a.s.\ for\ all\ \theta_1 \in [0,1/2), \theta_2 \in \left(0, 1-\frac{d}{q}\right). \tag{7.12}$$

The proof of Theorem 5.4 largely carries over to the proof of Theorem 7.3. Therefore, we only point out which adjustments are required to accommodate the unbounded domain $\mathbb{R}^d$. As stated for our proof of Theorem 6.4, the embeddings $H^{2i-\delta,\zeta+\epsilon_0} \hookrightarrow H^{2i-\delta,\zeta}$, $H^{2\lambda-\delta,\zeta} \hookrightarrow H^{-\delta,q}$ and $H^{2-\delta,\zeta} \hookrightarrow H^{2\lambda-\delta,q}$ for some $\lambda \in (0,1)$ used in our proof of the regularisation results in Theorem 5.4 do not hold on the unbounded domain $\mathbb{R}^d$. In addition, the computations (5.24) used in Step 3 of our proof of Theorem 5.4 to establish [AV22b, Hypothesis (HG)] in the (Y_0, Y_1, r, α)-setting and the $(\hat{Y}_0, \hat{Y}_1, \hat{r}, \hat{\alpha})$-setting relied on the fact that $\mathbb{T}^d$ is a bounded domain. Therefore, we cannot bootstrap integrability in space via [AV22b, Theorem 6.3].

Proof. Existence and uniqueness. We apply [AV22a, Theorem 4.8]. To this end, we verify that [AV22a, Hypothesis (H)] is satisfied and that $(A_0, B_0) \in \mathcal{SMR}^\bullet_{p,\kappa}(T)$ for all $T \in (0,\infty)$.

By Assumption 7.1(5), the first component of f satisfies the conditions of Lemma 6.5 with $\Phi = \Phi_1 = f$, $\Gamma = 0$, and β_1 as given in the lemma. Since the second component is 0, it follows that [AV22a, hypothesis (HF)] holds. For G the computations in (5.16) - (5.17) and subsequent comments carry over verbatim, and thus [AV22a, hypothesis (H)] holds. To verify that $(A_0, B_0) \in \mathcal{SMR}^\bullet_{p,\kappa}$ we again note that $(\tilde{A}_0, B_0) \in \mathcal{SMR}^\bullet_{p,\kappa}(T)$; for $\tilde{A}_0$ this follows from [AV21, Theorem 5.2 and Remark 5.6], and for B_0 this follows from Assumption 7.1(3) together with [AV21, Theorem 5.2 and Remark 5.7]. The computations in (5.18) carry over verbatim to the unbounded domain $\mathbb{R}^d$, and [AV21, Theorem 3.2] yields $(A_0, B_0) \in \mathcal{SMR}^\bullet_{p,\kappa}$. [AV22a, Assumption 3.2] is verified as in the proof of Theorem 5.4, and existence and uniqueness of the local solution (u,σ) follows from [AV22a, Theorem 4.8]. The regularity properties (7.10) follow by weighted Sobolev embedding [AV22a, Proposition 2.7].

Instantaneous regularisation. Steps 1 - 2 of the proof of Theorem 5.4 carry over verbatim since none of the embedding results used there make use of the fact that $\mathbb{T}^d$ is a bounded domain. The regularity result (7.12) is obtained in the same way as in Step 3 of the proof of Theorem 6.4. □

Corollary 7.4 (Local Well-Posedness in Critical Spaces; Theorem 2.7 in [AV23b])**.** *Suppose that Assumptions 7.1(p,q,h,δ) and 7.2(p,q,h,δ) hold, and set $\kappa =$:*

$\kappa_c = p\left(\frac{h}{h-1} - \frac{1}{2}\left(\delta + \frac{d}{q}\right)\right) - 1$. *Then for any* $\eta \in (0, 2-\delta]$ *and*

$$u_0 \in L^0\left(\Omega; B_{q,p}^{\frac{d}{q}-\frac{2}{h-1}} \oplus H^{2-\delta,q}\right) \tag{7.13}$$

there exists a unique $(p, \kappa_c, \delta, \eta, q)$*-solution* (u, σ) *such that a.s.* $\sigma > 0$ *and*

$$u \in C([0,\sigma); B_{q,p}^{\frac{d}{q}-\frac{2}{h-1}} \oplus H^{2-\delta-\eta,q}) \text{ a.s.} \tag{7.14}$$

$$u \in H_{loc}^{\theta,p}\left([0,\sigma), w_{\kappa_c}; H^{2-\delta-2\theta,q} \oplus H^{2-\delta-\eta,q}\right) \text{ a.s. for all } \theta \in [0,1/2). \tag{7.15}$$

Moreover, u regularises instantaneously in time and, in addition, u_1 *regularises instantaneously in space,*

$$u \in H_{loc}^{\theta,r}\left(0,\sigma; H^{1-2\theta,\zeta} \oplus H^{2-\delta-\eta,q}\right) \quad \text{a.s. for all } \theta \in [0,1/2), r \in (2,\infty), \tag{7.16}$$

and, if $q > d$*, additionally*

$$u_1 \in C_{loc}^{\theta_1,\theta_2}\left((0,\sigma) \times \mathbb{T}^d\right) \quad \text{a.s. for all } \theta_1 \in [0,1/2), \theta_2 \in \left(0, 1 - \frac{d}{q}\right). \tag{7.17}$$

Proof. Corollary 7.4 is immediate from Theorem 7.3. □

Our proof of Theorem 6.8 in Section 6 showed that the lack of spatial integrability for solutions to reaction-diffusion equations (6.1) imposes considerable limitations on the parameters (q_0, q_1) for which blow-up criteria can be obtained. As we have seen in the proof of Theorem 5.7 in Section 5, the lack of regularisation for the second component of the FitzHugh-Nagumo equations (4.1) further limits us to applying the blow-up criteria [AV22b, Theorem 4.10(3)] in the abstract setting. Therefore, we only state the equivalent of Theorem 5.7.

Theorem 7.5 (Blow-Up Criteria)**.** *Let the assumptions of Corollary 7.4 be satisfied and let* (u, σ) *be the* (p, κ_c, δ, q)*-solution to* (4.1)*. Suppose further that* $p_0 \in (2, \infty)$*,* $h_0 \geq h$*,* $\delta_0 \in [1, 2)$ *are such that Assumptions 7.1*(p_0, q, h_0, δ_0) *and 7.2*(p_0, q, h_0, δ_0) *hold. If* $\eta_0 \in (0, 2-\delta_0]$ *is such that* $\delta + \eta = \delta_0 + \eta_0$*, then for all* $0 < s < T < \infty$

$$\begin{aligned} \mathbb{P}(&\sup_{t\in[s,\sigma]} \|u(t)\|_{B_{q,p_0}^{\beta_0} \oplus H^{2-\delta_0-\eta_0,q}} \\ &+ \|u\|_{L^{p_0}(s,\sigma; H^{\gamma_0,q} \oplus H^{2-\delta_0-\eta_0,q})} < \infty, s < \sigma < T) = 0. \end{aligned} \tag{7.18}$$

Proof. The proof of Theorem 5.7 carries over verbatim with $\zeta = q$ in (5.33) since none of the embedding results used there make use of the fact that $\mathbb{T}^d$ is a bounded domain. □

7.2 Global Well-Posedness

In this section we prove global well-posedness of the FitzHugh-Nagumo Equations (4.1) on $\mathbb{R}^d$. As in Section 5, we will assume that $d \geq 2$; the case $d = 1$ can be accommodated by considering (4.1) on $\mathbb{R} \times \mathbb{T}$ and adding a dummy variable. Due to the limited spatial integrability of the solution to (4.1) provided by Corollary 7.4, we will formulate all assumptions and proofs with $\zeta = q$.

Assumption 7.6 (L^q-Coercivity; Version of *Assumption 4.1 in [AV23a]*)**.** *Suppose $d \geq 2$, Assumption 7.1(p, q, h, δ) holds with $h = 3$. We say that Assumption 7.6 holds if there exist constants $\theta, M, C, > 0$ such that a.e. on $\mathbb{R}_{\geq 0} \times \Omega$ and for all $(u_1, u_2) \in \mathcal{S}(\mathbb{R}^d) \oplus \mathcal{S}(\mathbb{R}^d)$*

$$\int_{\mathbb{R}^d} |u_1|^{q-2}\big(\nabla u_1 \cdot \nabla u_1 - \frac{u_1(f(u_1) - u_2)}{q-1} - \frac{1}{2}\sum_{k\geq 1}[(b_k \cdot \nabla)u_1 + g_{1,k}(\cdot, u)]^2\big)\, dx$$
$$\geq \theta \int_{\mathbb{R}^d} |u_1|^{q-2}(|\nabla u_1|^2 - M|u_1|^2) - M|u_2|^q \, dx - C.$$

Remark 9. As pointed out in [AV23a], if Assumption 5.8 holds for $(u_1, u_2) \in \mathcal{S}(\mathbb{R}^d) \oplus \mathcal{S}(\mathbb{R}^d)$, it can be shown to extend to $(u_1, u_2) \in H^{1,q} \oplus L^q$ via an approximation argument.

Lemma 7.7 (L^q-Coercivity for FitzHugh-Nagumo)**.** *Suppose the assumptions of Corollary 5.6 with $h = 3$ are satisfied. Then Assumption 5.8 holds.*

Proof. The proof of Lemma 5.9 carries over verbatim if we replace the assumption $(u_1, u_2) \in C^1(\mathbb{T}^d) \oplus C(\mathbb{T}^d)$ with $(u_1, u_2) \in \mathcal{S}(\mathbb{R}^d) \oplus \mathcal{S}(\mathbb{R}^d)$. □

Theorem 7.8 (Global Existence; Theorem 4.3 in [AV23a])**.** *Suppose the assumptions of Corollary 7.4 are satisfied with $q > d$, $h = 3$, $\delta + \eta = 2$ and*

$$u_0 \in L^0\left(\Omega; B_{q,p}^{\frac{d}{q}-1} \oplus L^q\right).$$

Let (u, σ) be the (p, κ_c, δ, q)-solution to (4.1)*. Then (u, σ) is a global solution, that is, $\sigma = \infty$ a.s. In particular, the regularity results* (7.14) - (7.17) *hold with $\sigma = \infty$. Moreover, there exists a constant $N_0 > 0$ such that for all $0 < s < T < \infty$ the following a priori bound holds:*

$$\begin{aligned}\mathbb{E} \sup_{t\in[s,T]} \mathbb{1}_\Gamma(\|u_1(t)\|_{L^q}^q + \|u_2(t)\|_{L^q}^q) + \mathbb{E}\int_s^T \int_{\mathbb{R}^d} \mathbb{1}_\Gamma |u_1|^{q-2}|\nabla u_1|^2 \, dx\, dr \\ \leq N_0\left(1 + \mathbb{E}\mathbb{1}_\Gamma(\|u_1(s)\|_{L^q}^q + \|u_2(t)\|_{L^q}^q)\right),\end{aligned} \tag{7.19}$$

where $\Gamma = \{\sigma > s\} \cap \{\|u_1(s)\|_{L^q} + \|u_2(s)\|_{L^q} \leq L\}$, for some $L \geq 1$. Moreover, the regularity results (7.14)-(7.17) *hold with $\sigma = \infty$ a.s.*

The proof of Theorem 7.8 relies on the following lemma.

Lemma 7.9 (Energy bounds; Version of Lemma 3.8 in [AV23a])**.** *Suppose the assumptions of Corollary 5.6 are satisfied with $h = 3$ and $q > d$, and let (u, σ) be the local (p, κ_c, δ, q)-solution to (4.1). Then for every $0 < s < T < \infty$ we have*

$$\sup_{t \in [s, \sigma \wedge T)} \|u_1(t)\|_{L^q}^q + \|u_2(t)\|_{L^q}^q < \infty \quad a.s.\ on\ \{\sigma > s\}, \tag{7.20}$$

$$\int_s^{\sigma \wedge T} \int_{\mathbb{R}^d} |u_1|^{q-2} |\nabla u_1|^2 < \infty \quad a.s.\ on\ \{\sigma > s\}. \tag{7.21}$$

Moreover, there exists a constant $N_0 > 0$ such that for all $0 < s < T < \infty$ and $k \geq 1$

$$\begin{aligned} &\sup_{t \in [s, \sigma \wedge T)} \mathbb{E}\left[\mathbb{1}_{[s,\sigma)}(t) \mathbb{1}_\Gamma (\|u_1(t)\|_{L^q}^q + \|u_2(t)\|_{L^q}^q)\right] \\ &\qquad + \mathbb{E} \int_s^{\sigma \wedge T} \int_{\mathbb{R}^d} \mathbb{1}_\Gamma |u_1|^{q-2} |\nabla u_1|^2 \, dx \, dr \\ &\qquad \leq N_0 \left(1 + \mathbb{E} \mathbb{1}_\Gamma (\|u_1(s)\|_{L^q}^q + \|u_2(s)\|_{L^q}^q)\right), \end{aligned} \tag{7.22}$$

$$\begin{aligned} &\mathbb{E} \sup_{t \in [s, \sigma \wedge T)} \mathbb{1}_\Gamma (\|u_1(t)\|_{L^q}^q + \|u_2(t)\|_{L^q}^q) \\ &\qquad + \mathbb{E} \int_s^{\sigma \wedge T} \int_{\mathbb{R}^d} \mathbb{1}_\Gamma |u_1|^{q-2} |\nabla u_1|^2 \, dx \, dr \\ &\qquad \leq N_0 \left(1 + \mathbb{E} \mathbb{1}_\Gamma (\|u_1(s)\|_{L^q}^q + \|u_2(s)\|_{L^q}^q)\right), \end{aligned} \tag{7.23}$$

where $\Gamma = \{\|u_1(s)\|_{L^q} + \|u_2(s)\|_{L^q} \leq L\} \cap \{\sigma > s\}$ and $L \geq 1$.

Proof of Lemma 7.9. By our assumption that $q > d$, the regularity result (7.17) applies for some $\theta_2 > 0$. Therefore, we can define the stopping times τ_j as in the proof of Lemma 5.11, and (7.16) and (7.17) give that $\lim_{j \to \infty} \tau_j = \sigma$. By (7.16) and the embedding $H^{2-\delta-\eta, q} \hookrightarrow L^q$, ∇u_1 and u_2 have L^q-integrability, which enables us to apply the generalised Itô formula A.1. The remainder of the proof of Lemma 5.11 carries over verbatim. □

Proof of Theorem 7.8. The proof of Theorem 5.10 carries over verbatim. □

8 Discussion

In the present work we developed (global) well-posedness results for the stochastic FitzHugh–Nagumo equations. Applying well-posedness results for stochastic evolution equations in the critical variational setting allowed us to prove the existence of a global solution as well as regularisation properties for the case $d = 1$ in the weak variational setting. Moving to the strong variational setting further allowed us to prove existence and regularity results for $d \leq 4$, but required us to impose restrictive assumptions on the noise terms. In particular, we had to assume that the semi-linear noise terms of the two component equations only depend on the corresponding component of the solution process. By considering the FitzHugh–Nagumo equations in the $L^p(L^q)$-setting, first for the bounded domain $\mathbb{T}^d$ and subsequently for the unbounded domain $\mathbb{R}^d$, we were able to prove (global) existence of a solution under linear growth assumptions on the semi-linear noise terms. Moreover, we were able to prove instantaneous regularisation of the solution in the first component, and to obtain a priori estimates for the solution on arbitrary intervals $[s, T]$ with $0 < s < T < \infty$.

One limitation of our results is that our proofs of global well-posedness require us to assume high spatial integrability of the initial data that grows with the dimension of the domain. The latter is due to two facts. Firstly, as our investigation of local well-posedness of reaction-diffusion equations on $\mathbb{R}^d$ showed, existing methods for bootstrapping spatial integrability fail on unbounded domains. Secondly, the absence of spatial regularisation in the second component of the FitzHugh–Nagumo equations limits the spatial regularisation that can be obtained for the first component, and thus for the entire solution to the system. While the former problem might be addressed by weakening the assumptions of the well-posedness results in the abstract setting (i.e., [AV22b, Theorem 6.3, Corollary 6.5 and Proposition 6.8]) on which our proofs rely, the latter problem is inherent in the FitzHugh–Nagumo equations and can likely not be resolved.

An open problem that we were not able to address here is the question whether and under which conditions the a priori bounds obtained in Theorems 5.10 and 7.8 hold with $s = 0$. [AV23a] in their Lemma 3.8 show that for reaction-diffusion equations, for sufficiently regular initial data one can indeed take $s = 0$. Another open problem that we could not address here is establishing compatibility of the solution obtained in the $L^p(L^q(\mathbb{R}^d))$-setting with the solution obtained in the variational setting. [AV23b] present such a compatibility result (for the bounded domain $\mathbb{T}^d$) in their Proposition 3.5; they show that if a solution to a given reaction-diffusion equation exists under two sets of parameters $(p_1, q_1, \kappa_1, \delta_1, h_1)$ and $(p_2, q_2, \kappa_2, \delta_2, h_2)$, then the two solutions coincide (i.e., $\sigma_1 = \sigma_2$ and $u_1 = u_2$ a.e. on $[0, \sigma_1) \times \Omega$). Our results Theorems 6.9 and 6.11 show that existence of local solutions in the boundary cases $d = 1$ and $p = q = 2$ can also be obtained in the $L^p(L^q(\mathbb{R}^d))$-setting. However, since the proof of [AV23b, Proposition 3.5] relies on the instantaneous regularisation of solutions to reaction-diffusion equations, we were not able to prove an analogue of their result for the FitzHugh–Nagumo equations, neither on $\mathbb{T}^d$ nor on $\mathbb{R}^d$.

A Appendix

A.1 Generalised Itô formula

Theorem A.1 (Generalised Itô formula; Extension of Proposition A.1 in [DHV16]). *Let $\mathcal{D} \in \{\mathbb{T}^d, \mathbb{R}^d\}$, $\zeta \geq 2$. Suppose that the assumptions of either Corollary 5.6 or Corollary 7.4 hold and that $v = (v_1, v_2)$ is a local $(p, \kappa, \delta, \eta, q)$-solution to* (4.1) *such that*

$$v \in C([0,T]; L^\zeta(\mathcal{D})) \cap L^2(0,T; H^{1,\zeta}(\mathcal{D})) \oplus C([0,T]; L^\zeta(\mathcal{D})).$$

Moreover, let

$$\phi = \begin{pmatrix} f(v_1) - v_2 \\ \epsilon(v_1 - \gamma v_2) \end{pmatrix} \quad \Phi = \begin{pmatrix} \nu \nabla v_1 \\ 0 \end{pmatrix} \quad \psi_k = \begin{pmatrix} b_k \cdot \nabla v_1 + g_{1,k}(\cdot, v) \\ g_{2,k}(\cdot, v) \end{pmatrix}$$

so that v_i, $i \in \{1, 2\}$, satisfies a.s.

$$\mathrm{d}v_i = \phi_i(t)\,\mathrm{d}t + \mathrm{div}(\Phi_i(t))\,\mathrm{d}t + \sum_{k\geq 1} \psi_{i,k}(t)\,\mathrm{d}W_i(t) \text{ on } [0,T] \times \Omega, \quad v_i(0) = v_{0,i}$$

in $H^{-1,\zeta}(\mathcal{D})$. Then a.s. and for all $t \in [0,T]$

$$\begin{aligned} \int_{\mathcal{D}} |v_i(t)|^\zeta \,dx = \int_{\mathcal{D}} |v_{0,i}|^\zeta \,\mathrm{d}x &+ \int_0^t \int_{\mathcal{D}} \zeta |v_i(s)|^{\zeta-2} v_i(s) \phi_i(s)\,\mathrm{d}x\,\mathrm{d}s \\ &- \int_0^t \int_{\mathcal{D}} \zeta(\zeta-1) |v_i(s)|^{\zeta-2} \nabla v_i(s) \cdot \Phi_i(s)\,\mathrm{d}x\,\mathrm{d}s \\ &+ \sum_{k\geq 1} \int_0^t \int_{\mathcal{D}} \zeta |v_i(s)|^{\zeta-2} v_i(s) \psi_{i,k}(s)\,\mathrm{d}x\,\mathrm{d}W_i(s) \\ &+ \frac{1}{2} \sum_{k\geq 1} \int_0^t \int_{\mathcal{D}} \zeta(\zeta-1) |v_i(s)|^{\zeta-2} |\psi_{i,k}(s)|^2\,\mathrm{d}x\,\mathrm{d}s. \end{aligned} \tag{A.1}$$

Proof. Assume first that $\mathcal{D} = \mathbb{T}^d$ or supp v_i is bounded. Let $(\delta_m)_{m\geq 1} \subset C_c^\infty(\mathcal{D})$ be a Dirac sequence and let $v_{i,m} = v_i * \delta_m$, and similarly for $v_{0,i,m}, \phi_{i,m}, \Phi_{i,m}$ and $\psi_{i,k,m}$. Then a.s. for all $t \in [0,T], x \in \mathcal{D}$

$$v_{i,m}(t,x) - v_{0,i,m} = \int_0^t \phi_{i,m}(s)\,\mathrm{d}s + \int_0^t \mathrm{div}(\Phi_{i,m}(s))\,\mathrm{d}s + \sum_{k\geq 1} \int_0^t \psi_{i,k,m}(s)\,\mathrm{d}W_i(s).$$

We now let $\xi \in C_b^2(\mathbb{R})$ be such that $\xi(x) = |x|^\zeta$ for $|x| \leq N$ for some constant N. We apply the 1-dimensional Itô formula to $\xi(x)$ (using the boundedness of

$v_{i,m}$) and integrate over x to obtain

$$\begin{aligned}\int_{\mathcal{D}} |v_{i,m}(t)|^\zeta \,\mathrm{d}x = \int_{\mathcal{D}} |v_{i,m}(0)|^\zeta \,\mathrm{d}x + \int_0^t \int_{\mathcal{D}} \zeta |v_{i,m}(s)|^{\zeta-2} v_{i,m}(s)\phi_{i,m}(s)\,\mathrm{d}x\,\mathrm{d}s \\ - \int_0^t \int_{\mathcal{D}} \zeta(\zeta-1)|v_{i,m}(s)|^{\zeta-2} \nabla v_{i,m}(s)\cdot \Phi_{i,m}(s)\,\mathrm{d}x\,\mathrm{d}s \\ + \sum_{k\geq 1}\int_0^t \int_{\mathcal{D}} \zeta |v_{i,m}(s)|^{\zeta-2} v_{i,m}(s)\psi_{i,k,m}(s)\,\mathrm{d}x\,\mathrm{d}W_i(s) \\ + \frac{1}{2}\sum_{k\geq 1}\int_0^t \int_{\mathcal{D}} \zeta(\zeta-1)|v_{i,m}(s)|^{\zeta-2}|\psi_{i,k,m}(s)|^2\,\mathrm{d}x\,\mathrm{d}s.\end{aligned} \tag{A.2}$$

As in the proof given in [DHV16], using the fact that ξ has bounded derivatives up to second order, the embedding $L^p \hookrightarrow L^q$ for $q \leq p$ on finite measure spaces, and the assumptions on v, it follows that the deterministic terms converge

$$\begin{aligned}|v_{i,m}|^\zeta &\to |v_i|^\zeta \\ |v_{i,m}|^{\zeta-2} v_{i,m}\phi_{i,m} &\to |v_i|^{\zeta-2} v_i \phi_i \\ |v_{i,m}|^{\zeta-2}\nabla v_{i,m}\cdot \Phi_{i,m} &\to |v_i|^{\zeta-2}\nabla v_i \cdot \Phi_i \\ |v_{i,m}|^{\zeta-2}\sum_{k\geq 1}|\psi_{i,k,m}|^2 &\to |v_i|^{\zeta-2}\sum_{k\geq 1}|\psi_{i,k}|^2\end{aligned}$$

in $L^1(\Omega\times[0,T]\times\mathcal{D})$ as $m\to\infty$, and that the integrand of the stochastic term converges

$$|v_{i,m}|^{\zeta-2} v_{i,m}\psi_{i,k,m} \to |v_i|^{\zeta-2} v_i \psi_{i,k}$$

in $L^2(\Omega\times[0,T]\times\mathcal{D})$ as $m\to\infty$, from which convergence of the stochastic integral follows by the Burkholder-Davis-Gundy inequality. Thus, up to extracting a subsequence, each term in (A.2) converges a.e. in $\Omega\times[0,T]$ to the corresponding term in (A.1).

If $\mathcal{D}=\mathbb{R}^d$ and supp v_i is unbounded, we let $\chi_K \in C_c(\mathbb{R}^d)$ be a smooth cutoff function with $\chi_K = 1$ on $B(0,K)$ and $\chi_K = 0$ on $\mathbb{R}^d \setminus B(0,K+1)$. Denote by $v_{i,m}^{(K)} = \chi_K v_{i,m}$ and similarly for $v_{0,i,m}^{(K)}, \phi_{i,m}^{(K)}, \Phi_{i,m}^{(K)}$ and $\psi_{i,k,m}^{(K)}$. Hence, multiplying by χ_K^ζ, applying the 1-dimensional Itô formula to $\xi(x)$ with a suitable constant N and integrating over x gives

$$\begin{aligned}\int_{\mathbb{R}^d} |v_{i,m}^{(K)}(t)|^\zeta \,\mathrm{d}x = \int_{\mathbb{R}^d} |v_{i,m}(0)^{(K)}|^\zeta \,\mathrm{d}x + \int_0^t \int_{\mathbb{R}^d} \zeta |v_{i,m}^{(K)}(s)|^{\zeta-2} v_{i,m}^{(K)}(s)\phi_{i,m}^{(K)}(s)\,\mathrm{d}x\,\mathrm{d}s \\ - \int_0^t \int_{\mathbb{R}^d} \zeta(\zeta-1)|v_{i,m}^{(K)}(s)|^{\zeta-2}(\chi_K\nabla v_{i,m}(s))\cdot \Phi_{i,m}^{(K)}(s)\,\mathrm{d}x\,\mathrm{d}s \\ + \sum_{k\geq 1}\int_0^t \int_{\mathbb{R}^d} \zeta |v_{i,m}^{(K)}(s)|^{\zeta-2} v_{i,m}^{(K)}(s)\psi_{i,k,m}^{(K)}(s)\,\mathrm{d}x\,\mathrm{d}W_i(s)\end{aligned}$$

$$+\frac{1}{2}\sum_{k\geq 1}\int_0^t\int_{\mathbb{R}^d}\zeta(\zeta-1)|v_{i,m}^{(K)}(s)|^{\zeta-2}|\psi_{i,k,m}^{(K)}(s)|^2\,\mathrm{d}x\,\mathrm{d}s.$$

Repeating the arguments above, we obtain that a.s. in $(\omega,t)\in\Omega\times[0,T]$

$$\begin{aligned}\int_{\mathbb{R}^d}|v_i^{(K)}(t)|^\zeta\,\mathrm{d}x=&\int_{\mathbb{R}^d}|v_{0,i}^{(K)}|^\zeta\,\mathrm{d}x+\int_0^t\int_{\mathbb{R}^d}\zeta|v_i^{(K)}(s)|^{\zeta-2}v_i^{(K)}(s)\phi_i^{(K)}(s)\,\mathrm{d}x\,\mathrm{d}s\\&-\int_0^t\int_{\mathbb{R}^d}\zeta(\zeta-1)|v_i^{(K)}(s)|^{\zeta-2}(\chi_K\nabla v_i(s))\cdot\Phi_i^{(K)}(s)\,\mathrm{d}x\,\mathrm{d}s\\&+\sum_{k\geq 1}\int_0^t\int_{\mathbb{R}^d}\zeta|v_i^{(K)}(s)|^{\zeta-2}v_i^{(K)}(s)\psi_{i,k}^{(K)}(s)\,\mathrm{d}x\,\mathrm{d}W_i(s)\\&+\frac{1}{2}\sum_{k\geq 1}\int_0^t\int_{\mathbb{R}^d}\zeta(\zeta-1)|v_i^{(K)}(s)|^{\zeta-2}|\psi_{i,k}^{(K)}(s)|^2\,\mathrm{d}x\,\mathrm{d}s,\end{aligned}$$

and letting $K\to\infty$ gives the result. □

References

[AV21] Antonio Agresti and Mark Veraar. *Stochastic maximal $L^p(L^q)$-regularity for second order systems with periodic boundary conditions.* arXiv:2106.01274v3. 2021.

[AV22a] Antonio Agresti and Mark Veraar. "Nonlinear parabolic stochastic evolution equations in critical spaces part I. Stochastic maximal regularity and local existence". In: *Nonlinearity* 35.8 (2022), pp. 4100–4210. ISSN: 0951-7715.

[AV22b] Antonio Agresti and Mark Veraar. "Nonlinear parabolic stochastic evolution equations in critical spaces part II: Blow-up criteria and instataneous regularization". In: *J. Evol. Equ.* 22.2 (2022), Paper No. 56, 96. ISSN: 1424-3199. DOI: 10.1007/s00028-022-00786-7. URL: https://doi-org.proxy.uba.uva.nl/10.1007/s00028-022-00786-7.

[AV22c] Antonio Agresti and Mark Veraar. *The critical variational setting for stochastic evolution equations.* arXiv:2206.00230v2. 2022.

[AV23a] Antonio Agresti and Mark Veraar. "Reaction-diffusion equations with transport noise and critical superlinear diffusion: Global well-posedness of weakly dissipative systems". In: (2023). arXiv:2301.06897v1.

[AV23b] Antonio Agresti and Mark Veraar. "Reaction-diffusion equations with transport noise and critical superlinear diffusion: Local well-posedness and positivity". In: *J. Differential Equations* 368 (2023), pp. 247–300. ISSN: 0022-0396. DOI: 10.1016/j.jde.2023.05.038. URL: https://doi-org.proxy.uba.uva.nl/10.1016/j.jde.2023.05.038.

[BCP09] Y. Bourgault, Y. Coudiére and C. Pierre. "Existence and uniqueness of the solution for the bidomain model used in cardiac electrophysiology". In: *Nonlinear Anal. Real World Appl.* 10 (2009), pp. 458–482. DOI: 10.1016/j.nonrwa.2007.10.007.

[BM08] Stefano Bonaccorsi and Elisa Mastrogiacomo. "Analysis of the stochastic FitzHugh-Nagumo system". In: *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* 11.3 (2008), pp. 427–446. ISSN: 0219-0257. DOI: 10.1142/S0219025708003191. URL: https://doi-org.proxy.uba.uva.nl/10.1142/S0219025708003191.

[BM18] H. Brezis and P. Mironescu. "Gagliardo–Nirenberg inequalities and non-inequalities: The full story". In: *Ann. Inst. Henri Poincare (C) Anal. Non Lineaire* 35.5 (2018), pp. 1355–1376. DOI: 10.1016/j.anihpc.2017.11.007. URL: https://doi.org/10.1016/j.anihpc.2017.11.007.

[DHV16] Arnaud Debussche, Martina Hofmanová and Julien Vovelle. “Degenerate parabolic stochastic partial differential equations: quasilinear case”. In: *Ann. Probab.* 44.3 (2016), pp. 1916–1955. ISSN: 0091-1798. DOI: 10.1214/15-AOP1013. URL: https://doi.org/10.1214/15-AOP1013.

[EGK21] Katharina Eichinger, Manuel V Gnann and Christian Kuehn. *Multiscale analysis for traveling-pulse solutions to the stochastic FitzHugh-Nagumo equations.* arXiv:2002.07234v2. 2021.

[Fit61] R. FitzHugh. “Impulses and physiological states in theoretical models of nerve membrane”. In: *Biophys. J.* 1.6 (1961), pp. 445–466. DOI: 10.1016/s0006-3495(61)86902-6.

[FS15] P. C. Franzone and G. Savare. *Degenerate evolution systems modeling the cardiac electric field at micro-and macroscopic level.* Evolution equations, semigroups and functional analysis. Springer, Cham, 2015, pp. vi+266. ISBN: 978-3-319-22353-7; 978-3-319-22354-4. DOI: 10.1007/978-3-319-22354-4. URL: https://doi-org.proxy.uba.uva.nl/10.1007/978-3-319-22354-4.

[HH20] C H S Hamster and H J Hupkes. “Stability of traveling waves for systems of reaction-diffusion equations with multiplicative noise”. In: *SIAM Journal of Mathematical Analysis* 52.2 (2020), pp. 1386–1426. DOI: https://doi.org/10.1137/18M1226348.

[HH52a] A. L. HODGKN and A. F. HUXLEY. “A quantitative description of membrane current and its application to conduction and excitation in nerve”. In: *J. Physiol.* 117.4 (1952), pp. 500–544. DOI: 10.1113/jphysiol.1952.sp004764.

[HH52b] A. L. HODGKN and A. F. HUXLEY. “Currents carried by sodium and potassium ions through the membrane of the giant axon of Loligo”. In: *J. Physiol.* 116.4 (1952), pp. 449–472. DOI: 10.1113/jphysiol.1952.sp004717.

[HH52c] A. L. HODGKN and A. F. HUXLEY. “Measurement of current-voltage relations in the membrane of the giant axon of Loligo”. In: *J. Physiol.* 116.4 (1952), pp. 424–448. DOI: 10.1113/jphysiol.1952.sp004716.

[LR15] Wei Liu and Michael Röckner. *Stochastic partial differential equations: an introduction.* Universitext. Springer, Cham, 2015, pp. vi+266. ISBN: 978-3-319-22353-7; 978-3-319-22354-4. DOI: 10.1007/978-3-319-22354-4. URL: https://doi-org.proxy.uba.uva.nl/10.1007/978-3-319-22354-4.

[NAY62] J. Nagumo, S. Arimoto and S. Yoshizawa. “An active pulse transmission line simulating nerve axon”. In: *Proc. IRE* 50.10 (1962), pp. 2061–2070. DOI: 10.1109/JRPROC.1962.288235.

[Saw18] Y. Sawano. *Theory of Besov spaces.* Vol. 56. Developments in Mathematics. Springer, Singapore, 2018.

[Shi+19] Lin Shi et al. "Asymptotic behavior of stochastic FitzHugh-Nagumo systems on unbounded thin domains". In: *J. Differential Equations* 267.7 (2019), pp. 4373–4409. ISSN: 0022-0396. DOI: 10.1016/j.jde.2019.05.002. URL: https://doi-org.proxy.uba.uva.nl/10.1016/j.jde.2019.05.002.

[Tay00] Michael E. Taylor. *Tools for PDE*. Vol. 81. Mathematical Surveys and Monographs. Pseudodifferential operators, paradifferential operators, and layer potentials. American Mathematical Society, Providence, RI, 2000, pp. x+257. ISBN: 0-8218-2633-6. DOI: 10.1090/surv/081. URL: https://doi-org.proxy.uba.uva.nl/10.1090/surv/081.

[Tri78] Hans Triebel. *Interpolation theory, function spaces, differential operators*. Vol. 18. North-Holland Mathematical Library. North-Holland Publishing Co., Amsterdam-New York, 1978, p. 528. ISBN: 0-7204-0710-9.

[Yam+19] Marius E. Yamakou et al. "The stochastic FitzHugh-Nagumo neuron model in the excitable regime embeds a leaky integrate-and-fire model". In: *J. Math. Biol.* 79.2 (2019), pp. 509–532. ISSN: 0303-6812. DOI: 10.1007/s00285-019-01366-z. URL: https://doi-org.proxy.uba.uva.nl/10.1007/s00285-019-01366-z.

[ZH11] Yan Zheng and Jian-hua Huang. "Stochastic stability of FitzHugh-Nagumo systems perturbed by Gaussian white noise". In: *Appl. Math. Mech. (English Ed.)* 32.1 (2011), pp. 11–22. ISSN: 0253-4827. DOI: 10.1007/s10483-011-1389-7. URL: https://doi-org.proxy.uba.uva.nl/10.1007/s10483-011-1389-7.