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Benders decomposition-based optimization of train departure frequencies in metro networks

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***Abstract*— Timetables determine the service quality for passengers and the energy consumption of trains in metro systems. In metro networks, a timetable can be made by designing train departure frequencies for different periods of the day, which is typically formulated as a mixed-integer linear programming (MILP) problem. In this paper, we first apply Benders decomposition to optimize the departure frequencies considering time-varying passenger origin-destination demands in metro networks. An ϵ-optimal Benders decomposition approach is subsequently used to reduce the solution time further. The performance of both methods is illustrated in a simulation-based case study using a grid metro network. The results show that both the classical Benders decomposition approach and the ϵ-optimal Benders decomposition approach can significantly reduce the computation time for the optimization of train departure frequencies in metro networks. In addition, the ϵ-optimal Benders decomposition approach can further reduce the solution time compared to the classical Benders decomposition approach when the problem scale increases while maintaining an acceptable level of performance.**

I. INTRODUCTION

Metro systems have become essential to urban transportation, providing millions of people with fast, efficient, and sustainable travel options, especially in large cities. The metro system is particularly critical in densely populated urban areas, where an efficient and reliable timetable is paramount for passenger satisfaction and the energy efficiency of the metro system.

Efficient train scheduling approaches enable metro systems to optimize energy consumption, reduce waiting times, and adjust transport capacity to meet passenger demands of different periods. A nonlinear programming problem (NLP) was formulated in [1] to minimize the time passengers spend and the energy consumption of trains in a metro line, for which an iterative convex programming approach was proposed. A bi-directional train line was considered in [2], and a Lagrangian-based method was proposed to solve the resulting NLP problem. An adaptive large neighborhood search algorithm was developed in [3] for the timetable scheduling problem of a rail rapid transit line so as to create convenient timetables for passengers considering a dynamic demand pattern. To improve the efficiency of passenger-centric timetable scheduling in metro networks, a simplified model was developed in [4], where the resulting optimization problem is solved in a moving horizon manner for real-time timetable scheduling.

In metro networks, trains typically operate with a relatively short headway, and thus the train departure frequency, which refers to the number of trains departing from a line per time unit, is crucial for the transport capacity of metro networks. To handle time-varying passenger origin-destination demands, it is necessary to implement effective strategies for optimizing departure frequencies in real time. Previous studies, such as [5], have utilized heuristic and exact methods to optimize train capacities and line frequencies within metro networks. Similarly, [6] applied mixed-integer nonlinear programming (MINLP) to optimize train capacities and line frequencies in urban metro networks. A passenger absorption model was proposed in [7] to optimize the departure frequency of trains of each line in metro networks, and the resulting problem was formulated as a mixed-integer linear programming (MILP) problem.

Timetable scheduling models often involve non-continuous variables, resulting in non-convex optimization problems that can be time-consuming to solve. Benders decomposition is regarded as an efficient methodology to solve MILP problems where the large-scale MILP problem is divided into two small-scale problems to reduce the computational burden [8]–[10]. Benders decomposition has also been used in railway timetable scheduling problems. Taking into account the uncertain passengers transfer time in metro networks, a generalized Benders decomposition approach was developed in [11] to efficiently solve the resulting MILP problem. A logic-based Benders decomposition approach that can reuse the precomputed logic Benders cuts to reduce the computation burden of the timetable rescheduling problem was proposed in [12]. In [13], the solution time of the Benders decomposition algorithm was reduced by splitting the algorithm solution process into three steps to address the fact that the relation between routing and scheduling variables is absent in the master problem. The proposed Benders decomposition approaches in [11], [12], and [13] were all shown to reduce the solution time significantly; however, passenger origin-destination (OD) demands were not considered explicitly.

This paper deals with the train departure frequency optimization problem in metro networks based on the model developed in [7], which can explicitly include time-varying OD passenger demands. The main contribution of this paper is twofold: (1) Benders decomposition-based algorithms are used in the train departure frequency optimization prob-

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lem to reduce the computational burden; (2) several Benders decomposition-based algorithms are compared on a simulation-based case study, which can facilitate the method selection when solving train departure frequency optimization problems.

The remainder of this paper is structured as follows. In Section II, a problem formulation is given. In Section III, the classical and ϵ-optimal Benders decomposition algorithms used for the passenger absorption model are discussed. Simulation results are provided in Section IV. Finally, conclusions are given in Section V.

II. Problem Formulation

In this work, the model proposed in [7] is used to optimize train departure frequencies. Time-varying passenger demands are approximated using piecewise constant functions in the model, allowing a balanced trade-off between solution time and accuracy. We briefly introduce the model and the corresponding optimization problem below, and for more details on the model, we refer to paper [7], [14].

In the passenger absorption model, the planning time window is divided into several periods, and passenger OD demands are assumed to be constant in each period. The total travel time of passengers within a given planning time window is estimated by:

$$
\begin{aligned}
J_{\text{time}} = & \sum_{k=k_0}^{k_0+N-1} \sum_{p \in P} \Big(n_p(k)T + n_p^{\text{depart}}(k)\bar{r}_p + n_p^{\text{arr,tra}}(k)\theta_p^{\text{trans}} \Big) \\
& + \sum_{p \in P} n_p(k_0+N)T,
\end{aligned} \tag{1}
$$

where N denotes the number of periods in the planning time window; P is the set of all platforms in the metro network; T is the length of a period; $n_p(k)$ denotes the number of passenger waiting at platform p at the start of period k; $n_p^{\text{depart}}(k)$ represents the number of passenger departing from platform p during period k; $n_p^{\text{arr,tra}}(k)$ denotes the number of passengers arriving at platform p with the intention of transferring to another platform during period k; and θ_p^{trans} is the average travel time for passengers transferring from platform p. In the metro network, trains travel a predetermined route, stopping at every platform. The average travel time for a train departing from platform p to the next platform on its route is denoted as $\bar{r}_p$. The energy consumption of trains in the planning time window is estimated by:

$$
J_{\text{cost}} = \sum_{k=k_0}^{k_0+N-1} \sum_{p \in P} f_p(k)\bar{E}_p, \tag{2}
$$

where $f_p(k)$ is the departure frequency at platform p during period k, and $\bar{E}_p$ denotes the average operational costs associated with dispatching a train from platform p towards the next platform on its route. The optimization problem is given as:

$$
\min \quad J = J_{\text{time}} + \zeta J_{\text{cost}}, \tag{3a}
$$

subject to

$$
f_p(k) = \frac{T-\gamma_p}{T} l_p(k-\delta_p) + \frac{\gamma_p}{T} l_p(k-\delta_p-1), \tag{3b}
$$

$$
f_p(k) \le f_p^{\max}, \tag{3c}
$$

$$
C_p(k) = f_p(k)C_{\max} - \sum_{m \in S} n_{p,m}^{\text{train}}(k), \tag{3d}
$$

$$
n_{p,m}(k+1) = n_{p,m}(k) + \lambda_{p,m}(k)T + n_{p,m}^{\text{arr,tra}}(k) - n_{p,m}^{\text{absorb}}(k), \tag{3e}
$$

$$
n_p^{\text{wait}}(k) = n_p(k) + \lambda_p(k)T + n_p^{\text{arr,tra}}(k), \tag{3f}
$$

$$
n_p^{\text{absorb}}(k) = \min\Big(C_p(k), n_p^{\text{wait}}(k)\Big), \tag{3g}
$$

$$
n_{p,m}^{\text{absorb}}(k) = \alpha_{p,m}(k) n_p^{\text{absorb}}(k), \tag{3h}
$$

$$
n_{p,m}^{\text{train}}(k) = \frac{T-\bar{r}_p^{\text{pla}}}{T} n_{\text{p}^{\text{pla}}(p,m)}^{\text{depart}}(k) + \frac{\bar{r}_p^{\text{pla}}}{T} n_{\text{p}^{\text{pla}}(p,m)}^{\text{depart}}(k-1), \tag{3i}
$$

$$
n_{p,\text{sta}(p)}^{\text{alight}}(k) = n_{p,m}^{\text{train}}(k), \tag{3j}
$$

$$
n_{p,m \in S/\{\text{sta}(p)\}}^{\text{alight}}(k) = n_{p,q,m}^{\text{trans}}(k), \tag{3k}
$$

$$
n_{p,m}^{\text{depart}}(k) = n_{p,m}^{\text{train}}(k) - n_{p,m}^{\text{alight}}(k) + n_{p,m}^{\text{absorb}}(k), \tag{3l}
$$

$$
n_{q,p,m}^{\text{trans}}(k) = \chi_{q,p,m}(k) n_{q,m}^{\text{train}}(k), \tag{3m}
$$

$$
n_{p,m}^{\text{arr,tra}}(k) = \sum_{q \in \text{pla}(p)} \Big(\frac{T-\theta_{q,p}^{\text{trans}}}{T} n_{q,p,m}^{\text{trans}}(k) + \frac{\theta_{q,p}^{\text{trans}}}{T} n_{q,p,m}^{\text{trans}}(k-1) \Big), \tag{3n}
$$

$$
k = k_0, k_0+1, ..., k_0+N-1,
$$

where ζ is a weight used to balance both objectives; $l_p(k)$ denotes the train departure frequency of the starting platform of the line on which platform p lies; $\delta_p = \lfloor \psi_p / T \rfloor$ and $\gamma_p = \psi_p - \delta_p T$, with ψ_p denoting the average travel time for train between departing from a starting platform of a line and departing from another platform p of that same line; $f_p^{\max}$ denotes the maximum train departure frequency of platform p; $C_p(k)$ represents the remaining capacity on a train at platform p during period k with $C_{\max}$ being the maximum capacity of a train; $n_{p,m}^{\text{train}}(k)$ is the number of passengers on board of trains at platform p with destination m during period k; $n_{p,m}(k)$ denotes the number of passenger waiting at platform p with destination m during period k; $\lambda_{p,m}(k)$ is the passenger arrival rate at platform p with destination m during period k; $n_{q,p,m}^{\text{arr,tra}}(k)$ denotes the number of transferring passengers arriving at platform q to transfer to platform p with destination m during period k; $n_{p,m}^{\text{absorb}}(k)$ represents the number of passengers who board a train at platform p with destination m during period k; $n_p^{\text{wait}}(k)$ denotes the number of passengers waiting for a train at platform p with destination m during period k; and $n_{p,m}^{\text{absorb}}(k)$ denotes the number of passengers alighting a train at platform p with destination m during period k. Parameter $\alpha_{p,m}$ is the relative fraction of passengers that board a train at platform p whose destination is station m; and $\chi_{q,p,m}$ is the relative fraction of passengers arriving at platform q with destination m, who will transfer from platform q to platform p.

Note that (3g) is a nonlinear function, and we can use the method in [15] to transform (3g) into linear inequalities. Then, we obtain an MILP problem for train departure frequency optimization; for a more elaborate explanation of the resulting MILP problem, we refer to [7].

The solution time of directly solving this MILP problem is significant. Therefore, this paper aims to present approaches to solve the resulting MILP in a time-efficient manner.

III. Benders Decomposition-based Train Departure Frequency Optimization

Benders decomposition [8] is an efficient method for solving large-scale optimization problems involving both continuous and discrete variables. In Benders decomposition, an optimization problem is divided into a master problem and a dual sub-problem that can be solved independently. The master problem is formulated as an MILP problem to determine the integer variables, while the dual sub-problem is formulated as a linear programming problem. The dual sub-problem is either feasible and bounded, after which a so-called optimality cut is added to the master problem, or is unbounded, after which a feasibility cut is added to the master problem.

A. Classical Benders Decomposition for Train Departure Frequency Optimization

The classical Benders decomposition [8] is applied in this section for the MILP problem (3) described in Section II. In this paper, according to the definition used in [8], $l_p(k)$ and $\delta_p^{\text{absorb}}(k)$ are the so-called "complicating variables", as they are integer and binary variables, respectively. the MILP problem is non-convex due to these variables. Since T, $\gamma_p(k)$, and $\delta_p(k)$ are all parameters, it follows from (3b) that once $l_p(k)$ is given, $f_p(k)$ is also known. We define a vector $\mathbf{y}(k_0)$ to collect the integer variables $l_p(k)$, binary variables $\delta_p^{\text{absorb}}(k)$, and $f_p(k)$ in the planning time window starting from period k_0. Then, all other variables related to the number of passengers in the planning time window starting from period k_0 are collected in a vector $\mathbf{x}(k_0)$. For compactness, we can write problem (3) as:

$$\min_{\mathbf{x}(k_0),\mathbf{y}(k_0)} \quad J = \mathbf{c}^{\mathrm{T}}(k_0)\mathbf{x}(k_0) + \mathbf{g}^{\mathrm{T}}(k_0)\mathbf{y}(k_0) \tag{4a}$$

$$\text{s.t.} \quad A(k_0)\mathbf{x}(k_0) + B(k_0)\mathbf{y}(k_0) = \mathbf{b}(k_0), \tag{4b}$$

$$D(k_0)\mathbf{x}(k_0) + E(k_0)\mathbf{y}(k_0) \leq \mathbf{d}(k_0), \tag{4c}$$

$$\mathbf{x}(k_0) \in \mathbb{R}^{n_1}, \tag{4d}$$

$$\mathbf{y}(k_0) \in \mathbb{Y}^{n_2}, \tag{4e}$$

$$\mathbf{x}(k_0) \geq 0, \tag{4f}$$

where (4a) represents the objective function (3a), (4b) collects the equality constraints, (4c) collects the inequality constraints, and $\mathbb{Y}^{n_2}$ defines the feasible set for $\mathbf{y}(k_0)$.

By fixing $\mathbf{y}(k_0)$ as $\bar{\mathbf{y}}(k_0)$ in Benders decomposition, the sub-problem turns into a linear programming problem, and by using duality theory and introducing dual variables $\mathbf{u}_1(k_0)$ and $\mathbf{u}_2(k_0)$, the dual sub-problem becomes:

$$\max_{\mathbf{u}_1(k_0),\mathbf{u}_2(k_0)} J_{\text{dsp}} = \mathbf{u}_1^{\mathrm{T}}(k_0)\big(B(k_0)\bar{\mathbf{y}}(k_0) - \mathbf{b}(k_0)\big) + \mathbf{u}_2^{\mathrm{T}}(k_0)\big(E(k_0)\bar{\mathbf{y}}(k_0) - \mathbf{d}(k_0)\big) + \mathbf{g}^{\mathrm{T}}(k_0)\bar{\mathbf{y}}(k_0) \tag{5a}$$

$$\text{s.t.} \quad \mathbf{u}_1^{\mathrm{T}}(k_0)A(k_0) + \mathbf{u}_2^{\mathrm{T}}(k_0)D(k_0) = \mathbf{c}^{\mathrm{T}}(k_0), \tag{5b}$$

$$\mathbf{u}_1(k_0) \in \mathbb{R}^{m_1}, \tag{5c}$$

$$\mathbf{u}_2(k_0) \in \mathbb{R}^{m_2}_{\geq 0}. \tag{5d}$$

If the feasible set of (5) is not empty, the dual sub-problem can be either unbounded or feasible for any arbitrary choice of $\bar{\mathbf{y}}(k_0)$. If the dual sub-problem is unbounded, there exists a pair of extreme rays $\bar{\mathbf{r}}_{q1}(k_0) \in \mathbb{Q}_1$ and $\bar{\mathbf{r}}_{q2}(k_0) \in \mathbb{Q}_2$, with $\mathbb{Q}_1$ and $\mathbb{Q}_2$ being the sets of extreme rays, for which $\bar{\mathbf{r}}_{q1}^{\mathrm{T}}(k_0)(B(k_0)\mathbf{y}(k_0) - \mathbf{b}(k_0)) + \bar{\mathbf{r}}_{q2}^{\mathrm{T}}(k_0)(E(k_0)\mathbf{y}(k_0) - \mathbf{d}(k_0)) > 0$. To avoid this, the following feasibility cut is added to the master problem:

$$\bar{\mathbf{r}}_{q1}^{\mathrm{T}}(k_0)\big(B(k_0)\mathbf{y}(k_0) - \mathbf{b}(k_0)\big) + \bar{\mathbf{r}}_{q2}^{\mathrm{T}}(k_0)\big(E(k_0)\mathbf{y}(k_0) - \mathbf{d}(k_0)\big) \leq 0. \tag{6}$$

While there may be multiple possible extreme rays which lead to unboundedness in the dual sub-problem, only one pair of extreme rays is used for the feasibility cut.

If a feasible and bounded solution can be found for dual sub-problem (5), the solution for the dual variables can be denoted as the extreme points, i.e., $\bar{\mathbf{u}}_{e1}(k_0) \in \mathbb{E}_1$ and $\bar{\mathbf{u}}_{e2}(k_0) \in \mathbb{E}_2$, with $\mathbb{E}_1$ and $\mathbb{E}_2$ being the sets of extreme points. We use J_{dsp} to denote the value of the objective function of the dual sub-problem. The optimal value of the objective function provides an upper bound of the original optimization problem, which is denoted as U_{ub}. For the ith iteration of the Benders decomposition algorithm, the upper bound is updated as follows: $U_{\text{ub}}^i = \min\left(U_{\text{ub}}^{i-1}, J_{\text{dsp}}^i\right)$. In addition, an optimality cut is added to the master problem:

$$\bar{\mathbf{u}}_{e1}^{\mathrm{T}}(k_0)(B(k_0)\mathbf{y}(k_0) + \mathbf{b}(k_0)) - \bar{\mathbf{u}}_{e2}^{\mathrm{T}}(k_0)(E(k_0)\mathbf{y}(k_0) - \mathbf{d}(k_0)) \geq -\eta. \tag{7}$$

Finally, the master problem (MP) is formulated as:

$$\min_{\mathbf{y}(k_0),\eta} \quad J_{\text{mp}} = \mathbf{g}^{\mathrm{T}}(k_0)\mathbf{y}(k_0) + \eta \tag{8}$$

$$\text{s.t.} \quad (4\text{e}), (6), (7)$$

The solution $\bar{\mathbf{y}}(k_0)$ to the master problem is used for dual sub-problem (5) in the next iteration and is also used to update the lower bound: $U_{\text{lb}}^i = \min\left(U_{\text{lb}}^{i-1}, J_{\text{mp}}^i\right)$, where J_{mp} denotes the objective function value of the master problem.

The procedure of the classical Benders decomposition-based train departure frequency optimization algorithm used is presented in Algorithm 1.

B. ϵ-Optimal Benders Decomposition for Train Departure Frequency Optimization

To reduce the computation time of the master problem, [16] proposed a variant of Benders decomposition where the master problem stops as soon as a feasible solution is found, as opposed to an optimal solution. The algorithm is then guaranteed to terminate in a finite number of steps, as there is a finite number of optimal dual solutions for the sub-problem. Like the classical Benders decomposition, the

Algorithm 1 Classical Benders decomposition-based train departure frequency optimization algorithm

Input: α, ζ, N, P, S, $\theta_{q,p}^{\text{trans}}$, and $\bar{E}_p$; $\bar{r}_p$, ψ_p, $\chi_{q,p,m}(k)$; estimated values of $\beta_{j,p,m}(k)$, $\lambda_{j,m}^{\text{station}}(k)$, and $\alpha_{p,m}(k)$
Set initial values:
$U_{\text{ub}}^0 \leftarrow \infty$, $U_{\text{lb}}^0 \leftarrow -\infty$, $\bar{f}_p(k) \leftarrow 0$, $\bar{\delta}_p^{\text{absorb}}(k) \leftarrow 0$, $i \leftarrow 0$
Output: $f_p(k)$, $\delta_p^{\text{absorb}}(k)$
while $U_{\text{ub}}^i - U_{\text{lb}}^i \geq \alpha$ **do**
  $i \leftarrow i+1$
  Solve (5) using $\bar{f}_p(k)$ and $\bar{\delta}_p^{\text{absorb}}(k)$
  if (5) is feasible and bounded **then**
    Obtain J_{dsp}, $\bar{\mathbf{u}}_{e1}(k_0)$, and $\bar{\mathbf{u}}_{e2}(k_0)$
    Update upper bound:
    $U_{\text{ub}}^i \leftarrow \min\left(U_{\text{ub}}^{i-1}, J_{\text{dsp}}^i\right)$
    Add optimality cut (7) using extreme points
  else if (5) is feasible but unbounded **then**
    Compute extreme rays $\bar{\mathbf{r}}_{q1}(k_0)$ and $\bar{\mathbf{r}}_{q2}(k_0)$
    Add feasibility cut (6) using extreme rays
  end if
  Solve (8) to obtain new $\bar{f}_p(k)$, and $\bar{\delta}_p^{\text{absorb}}(k)$
  Update lower bound:
  $U_{\text{lb}}^i \leftarrow \min\left(U_{\text{lb}}^{i-1}, J_{\text{mp}}^i\right)$
end while

feasible solution to the master problem is used for the dual sub-problem of the next iteration. Since the solution to the master problem is no longer optimal, the master problem no longer provides a valid lower bound. Instead, the ϵ-optimal Benders algorithm terminates when the master problem cannot produce a feasible solution. The master problem is then turned into a feasibility problem in the form of (9) instead of an optimization problem and hence is generally easier to handle, especially for large-scale problems.

$$\mathbf{g}^{\mathrm{T}}(k_0)\mathbf{y}(k_0) + \eta \leq U_{\text{ub}}(1-\epsilon) \quad \text{s.t.} \quad (4e), (6), (7) \tag{9}$$

where $\epsilon \in (0,1)$ is the slackness variable. A higher value for ϵ might result in faster convergence to the solution at the cost of a potentially worse solution.

A potential drawback of the ϵ-optimal Benders decomposition algorithm is that it may require more iterations than the classical Benders decomposition algorithm, as the non-optimal solutions to the master problem may also lead to non-optimal Benders cuts.

The detailed procedure of ϵ-optimal Benders decomposition-based train departure frequency optimization algorithm is shown in Algorithm 2.

IV. Case Study

In this section, we conduct a case study to compare the Benders decomposition-based algorithms for train departure frequency optimization.

A. Set-up

The metro network that is used for the case study is shown in Fig. 1, which consists of 21 stations, 60 platforms, and 6

Algorithm 2 ϵ-optimal Benders decomposition-based train departure frequency optimization algorithm

Input: α, ζ, N, P, S, $\theta_{q,p}^{\text{trans}}$, $\bar{E}_p$, and ϵ; $\bar{r}_p$, ψ_p, $\chi_{q,p,m}(k)$; estimated values of $\beta_{j,p,m}(k)$, $\lambda_{j,m}^{\text{station}}(k)$, and $\alpha_{p,m}(k)$
Set initial values:
$U_{\text{ub}}^0 \leftarrow \infty$, $\bar{f}_p(k) \leftarrow 0$, $\bar{\delta}_p^{\text{absorb}}(k) \leftarrow 0$, $i \leftarrow 0$
Output: $f_p(k)$, $\delta_p^{\text{absorb}}(k)$
while $U_{\text{ub}}^i \geq \alpha$ **do**
  $i \leftarrow i+1$
  Solve (5) using $\bar{f}_p(k)$ and $\bar{\delta}_p^{\text{absorb}}(k)$
  if (5) is feasible and bounded **then**
    Obtain J_{dsp}, $\bar{\mathbf{u}}_{e1}(k_0)$, and $\bar{\mathbf{u}}_{e2}(k_0)$
    Update upper bound:
    $U_{\text{ub}}^i \leftarrow \min\left(U_{\text{ub}}^{i-1}, J_{\text{dsp}}^i\right)$
    Add optimality cut (7) using extreme points
  else if (5) is feasible but unbounded **then**
    Compute extreme rays $\bar{\mathbf{r}}_{q1}(k_0)$ and $\bar{\mathbf{r}}_{q2}(k_0)$
    Add feasibility cut (6) using extreme rays
  end if
  Solve (9)
  if (9) is feasible **then**
    Obtain new $\bar{f}_p(k)$, and $\bar{\delta}_p^{\text{absorb}}(k)$
  else if (9) is infeasible **then**
    Break while loop
  end if
end while

bidirectional lines. The number on top of each link in Fig. 1 represents the average travel time between two stations and is used to determine the parameters $\bar{r}_p$ and ψ_p.

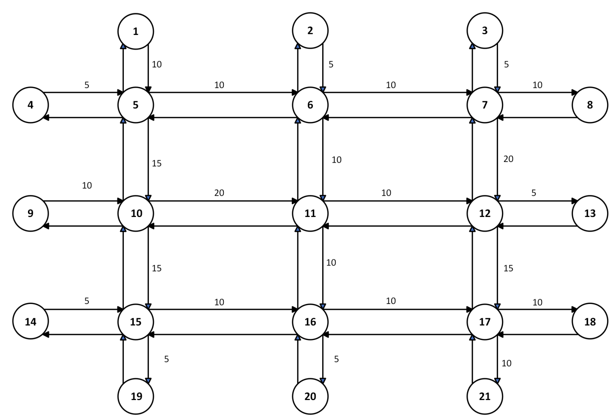

Fig. 1. Railway operations planning

We consider time-varying passenger OD demands, and passenger demands are considered to be constant for one period. The length of each period is 60 minutes. The cost per train run E_p is a function associated with the travel time $\bar{r}_p$. The values of the parameters are given in Table I.

In general, parameters $\alpha_{p,m}$ and $\chi_{q,p,m}$ can be estimated using historical data. However, since we use a fictional metro network, the values for $\beta_{j,p,m}$, $\alpha_{p,m}$, and $\chi_{q,p,m}$ are computed considering the average travel time between

TABLE I

PARAMETER VALUES

Parameter	Value
Stop criterion α	1
Transfer time θ_p^{trans}	1 [min]
Capacity $C_{\max}$	2000 passengers
Operational cost E_p	$2 \cdot \bar{r}_p$
Max departure frequency $f_p^{\max}$	20
Weight ζ	1000
Epsilon benders ϵ	0.01, 0.05, and 0.1

stations and assuming that passengers will always choose the shortest path to their destination in terms of time spent in the metro network.

We first apply the classical Benders decomposition-based algorithm for optimizing train departure frequencies. The total computation time consists of solving the dual sub-problem and master problem, updating the upper and lower bound, generating optimality and feasibility cuts, and obtaining extreme rays when necessary. The computation time for the ϵ-optimal Benders decomposition-based algorithm is computed in the same manner as for the classical Benders decomposition-based algorithm. Three different ϵ values are compared, i.e. 0.01, 0.05, and 0.1. For comparison, the resulting MILP problem is also directly solved by using `gurobi`, i.e., a state-of-the-art commercial solver for mixed integer programming problems.

The algorithms will be compared based on the objective function value and the required computation time. All the simulations are conducted using Matlab R2021a on a MacBook Pro 2017 with 2.3 GHz Dual-core Intel Core i5 processor and 8GB RAM. For the direct MILP algorithm, we use the commercial solver `gurobi` v9.5.2rc0 (mac64[x86]). Simulations are done for different planning time windows, i.e., from 2 ($N = 2$) to 6 hours ($N = 6$). The time limit for solving the resulting train departure frequency optimization problem for all the algorithms is set to 2 hours.

B. Results

The simulation results for all methods can be seen in Table II, where N.A. is used to indicate that no solution was found within 2 hours. For the sake of simplicity, we use "Gurobi" for the results obtained by solving the MILP problem using `gurobi`, "Benders" to denote the classical Benders decomposition algorithm, and "ϵ-Benders" to denote the ϵ-optimal Benders decomposition algorithm.

From Table II, we can find that when $N = 2$, `gurobi` has a better performance than both Benders decomposition-based methods. However, this changes when $N = 4$; the solution time of `gurobi` is significantly higher than that of both the Benders algorithm and the ϵ-Benders algorithm. For $N = 4$ and $N = 6$, both `gurobi` and the classical Benders decomposition algorithm cannot find the solution within 2 hours. The ϵ-optimal Benders decomposition algorithm outperforms the classical Benders decomposition algorithm when $N = 6$ in terms of solution time; this is because the master problem increases significantly in computation complexity with each

TABLE II

COMPARISON OF DIFFERENT METHODS

N	Method	Objective function value	CPU time [s]
2	`gurobi`	1.70×10^5	16.1
	Classical Benders	1.70×10^5	66.9
	ϵ-Benders (0.01)	1.70×10^5	79.4
	ϵ-Benders (0.05)	1.71×10^5	70.6
	ϵ-Benders (0.1)	1.87×10^5	69.2
4	`gurobi`	4.00×10^5	6230.4
	Classical Benders	4.00×10^5	388.9
	ϵ-Benders (0.01)	4.00×10^5	358.4
	ϵ-Benders (0.05)	4.14×10^5	305.1
	ϵ-Benders (0.1)	4.14×10^5	291.3
6	`gurobi`	N.A.	N.A.
	Classical Benders	N.A.	N.A.
	ϵ-Benders (0.01)	6.80×10^5	1771.2
	ϵ-Benders (0.05)	7.07×10^5	714.4
	ϵ-Benders (0.1)	7.14×10^5	687.9

added feasibility or optimality cut. When $N = 6$, the classical Benders decomposition algorithm cannot find the solution within 2 hours due to the master problem taking too long. The ϵ-optimal Benders decomposition can significantly reduce the computational complexity of the master problem.

TABLE III

SIMULATION RESULTS FOR BENDERS DECOMPOSITION APPROACHES

N	Method	Iterations	t_{sub} [s]	t_{ray} [s]	t_{mas} [s]
2	Classical Benders	43	24.8	33.9	8.2
	ϵ-Benders (0.01)	51	32.4	36.5	10.4
	ϵ-Benders (0.05)	47	26.7	35.4	8.4
	ϵ-Benders (0.1)	44	25.3	35.9	7.9
4	Classical Benders	85	96.1	145.6	147.3
	ϵ-Benders (0.01)	113	141.3	158.8	58.3
	ϵ-Benders (0.05)	99	108.6	157.9	38.6
	ϵ-Benders (0.1)	97	103.3	149.9	38.0
6	Classical Benders	N.A	N.A.	N.A.	N.A.
	ϵ-Benders (0.01)	169	279.2	349.2	1142.8
	ϵ-Benders (0.05)	151	235.5	349.4	129.4
	ϵ-Benders (0.1)	147	237.4	355.7	94.8

To further illustrate the results, the number of iterations and the total time spent in each part of the algorithm are given in Table III. The evolution process of the different algorithms for $N = 4$ is also given. The convergence of the upper and lower bound is shown in Fig.2 for the classical Benders decomposition algorithm. The upper bound of the classical Benders decomposition algorithm changes only once. The dual sub-problem is unbounded for all other iterations. Since the ϵ-optimal Benders decomposition algorithm does not produce a valid lower bound, only the evolution of the upper bound is provided. The evolution of the upper bound for the different ϵ values is displayed in Fig.3 The upper bound of the ϵ-optimal Benders decomposition algorithm changes several times; the lower the value of ϵ, the more times the upper bound changes. As the master problem of the ϵ-optimal Benders decomposition algorithms becomes a feasibility problem, the computation time of solving the master problem is reduced.

The simulation shows that the ϵ-optimal Benders de-

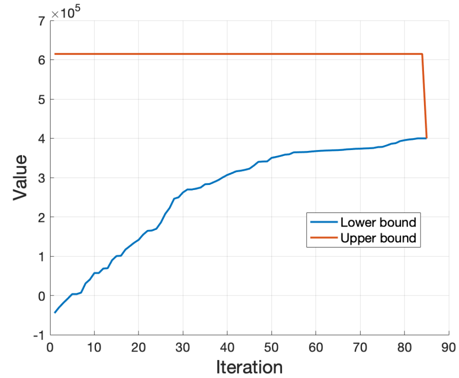

Fig. 2. Convergence upper bound and lower bound of classical Benders decomposition algorithm ($N = 4$)

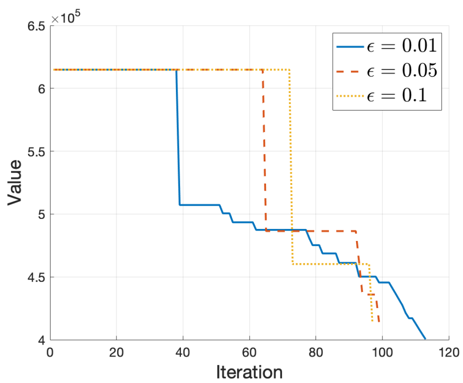

Fig. 3. Evolution of upper bound of ϵ-optimal Benders decomposition algorithm for different ϵ values ($N = 4$)

composition algorithm is suitable for real-time optimization of train departure frequencies in metro networks. When simulating for multiple cycles, increasing the number of (integer) variables and constraints leads to long solution times for `gurobi`. While the classical Benders decomposition approach outperforms `gurobi` in terms of solution time when the number of cycles is four or higher, the high number of feasibility cuts required leads to a computationally complex master problem, which is a bit time consuming. The ϵ-optimal Benders decomposition algorithm has been shown to be able to provide a solution in a relatively short time at the cost of some accuracy. By changing the value for ϵ, train operations can make a balanced trade-off between solution time and performance.

V. Conclusions

The optimization of the departure frequencies in metro networks can be formulated as a mixed-integer linear programming problem. This paper has applied the Benders decomposition approach to reduce the computational burden of the train departure frequency optimization problem. To further improve the efficiency of the Benders decomposition-based approach, an ϵ-optimal strategy is used, which reduces the solution time by turning the master problem of the Benders decomposition into a feasibility problem. Simulation results indicate that the Benders-decomposition-based methods can reduce the computational time of train departure frequency optimization problems when the problem size increases. The ϵ-optimal Benders decomposition algorithm can further reduce the solution time of the classical Benders decomposition algorithm when the problem size increases.

In the future, we will focus on further reducing the computation time of Benders decomposition-based approaches. The potential approaches are to improve the efficacy of feasibility cuts and generate them based on the problem's structure.

References

[1] Y. Wang, B. Ning, T. Tang, T. J. van den Boom, and B. De Schutter, "Efficient real-time train scheduling for urban rail transit systems using iterative convex programming," *IEEE Transactions on Intelligent Transportation Systems*, vol. 16, no. 6, pp. 3337–3352, 2015.

[2] E. Hassannayebi, S. H. Zegordi, and M. Yaghini, "Train timetabling for an urban rail transit line using a lagrangian relaxation approach," *Applied Mathematical Modelling*, vol. 40, no. 23-24, pp. 9892–9913, 2016.

[3] E. Barrena, D. Canca, L. C. Coelho, and G. Laporte, "Single-line rail rapid transit timetabling under dynamic passenger demand," *Transportation Research Part B: Methodological*, vol. 70, pp. 134–150, 2014.

[4] X. Liu, A. Dabiri, Y. Wang, and B. De Schutter, "Modeling and efficient passenger-oriented control for urban rail transit networks," *IEEE Transactions on Intelligent Transportation Systems*, vol. 24, no. 3, pp. 3325–3338, 2023.

[5] A. De-Los-Santos, G. Laporte, J. A. Mesa, and F. Perea, "Simultaneous frequency and capacity setting in uncapacitated metro lines in presence of a competing mode," *Transportation Research Procedia*, vol. 3, pp. 289–298, 2014.

[6] D. Canca, E. Barrena, A. De-Los-Santos, and J. L. Andrade-Pineda, "Setting lines frequency and capacity in dense railway rapid transit networks with simultaneous passenger assignment," *Transportation Research Part B: Methodological*, vol. 93, pp. 251–267, 2016.

[7] X. Liu, A. Dabiri, and B. De Schutter, "Timetable scheduling for passenger-centric urban rail networks: Model predictive control based on a novel absorption model," in *2022 IEEE Conference on Control Technology and Applications (CCTA)*, IEEE, 2022, pp. 1147–1152.

[8] J. F. Benders, "Partitioning procedures for solving mixed-variables programming problems," *Numerische Mathematik*, vol. 4, no. 1, pp. 238–252, 1962.

[9] R. Rahmaniani, T. G. Crainic, M. Gendreau, and W. Rei, "The Benders decomposition algorithm: A literature review," *European Journal of Operational Research*, vol. 259, no. 3, pp. 801–817, 2017.

[10] J. Fu, A. Núñez, and B. De Schutter, "Accelerated optimal maintenance scheduling for generation units on a truthful platform," *Energy Reports*, vol. 8, pp. 9777–9786, 2022.

[11] Y. Hu, S. Li, M. M. Dessouky, L. Yang, and Z. Gao, "Computationally efficient train timetable generation of metro networks with uncertain transfer walking time to reduce passenger waiting time: A generalized Benders decomposition-based method," *Transportation Research Part B: Methodological*, vol. 163, pp. 210–231, 2022.

[12] F. Leutwiler, G. B. Filella, and F. Corman, "Accelerating logic-based Benders decomposition for railway rescheduling by exploiting similarities in delays," *Computers & Operations Research*, vol. 150, p. 106 075, 2023.

[13] K. Keita, P. Pellegrini, and J. Rodriguez, "A three-step Benders decomposition for the real-time railway traffic management problem," *Journal of Rail Transport Planning & Management*, vol. 13, p. 100 170, 2020.

[14] X. Liu, A. Dabiri, J. Xun, and B. De Schutter, "Bi-level model predictive control for metro networks: Integration of timetables, passenger flows, and train speed profiles," 2023, submitted to journal.

[15] H. P. Williams, *Model Building in Mathematical Programming*. John Wiley & Sons, 2013.

[16] A. M. Geoffrion and G. W. Graves, "Multicommodity distribution system design by Benders decomposition," *Management Science*, vol. 20, no. 5, pp. 822–844, 1974.