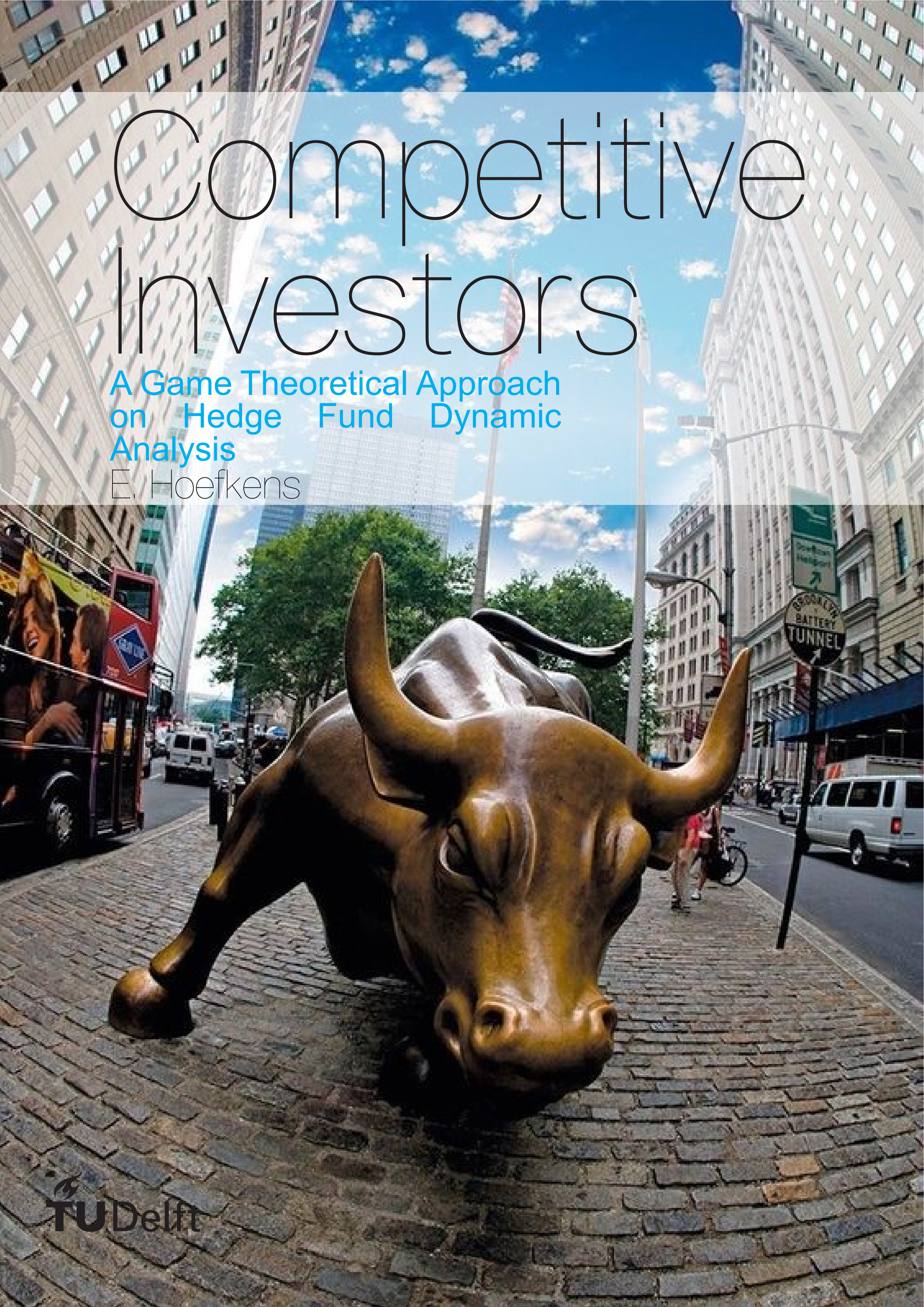
Competitive Investors

A Game Theoretical Approach on Hedge Fund Dynamic Analysis

E. Hoefkens



TU Delft

Competitive Investors

A Game Theoretical Approach on Hedge Fund Dynamic Analysis

by

E. Hoefkens

to obtain the degree of Master of Science
at the Delft University of Technology,
to be defended publicly on Friday January 29, 2021 at 11:00 AM.

Thesis committee:	Dr. R.J. Fokkink,	TU Delft, daily supervisor
	Prof.dr.ir. C.W. Oosterlee,	TU Delft, responsbile professor
	Dr.ir. G.F. Nane,	TU Delft

TU Delft
Delft University of Technology

Abstract

The Competitive Investor Game from Bell & Cover (1980) [3] and the k-Player Ranking Game from Alpern & Howard (2017) [1] are analysed in thesis. Optimal strategies have been derived and the related proofs have been given a new look. The Symmetric Multiplayer Ranking Game is considered as the general interpretation of financial competition among hedge funds. This study focused on the hedge funds that manage a Long/Short U.S. Equity strategy. Some minor evidence has been found to support the hypothesis that the studied hedge funds manage a strategy that has the objective to beat the competition in terms of annual performance in order to achieve the highest ranking.

Keywords: Hedge Fund, Long/Short U.S. Equity, Competition, Ranking, Optimal Strategy, Equilibrium

Preface

This thesis is written in order to obtain the degree of Master of Science in Applied Mathematics at the Delft University of Technology. The research has been conducted at the Delft Institute of Applied Mathematics under the supervision of Dr. R.J. Fokkink. The responsible professor of this thesis is Prof.dr.ir. C.W. Oosterlee.

I would like to thank Robbert for setting up this project and his guidance throughout the entire duration. At the first stage of the project, I enjoyed the weekly meetings as they were full with associated puzzles and solving games. At the second stage, Zoom calls became the standard. I am grateful for the freedom I received from Robbert to explore my interest in the investment world in the meantime.

Also, I would like to thank C.W. Oosterlee and G.F. Nane for taking the time to read this report and to take place in my thesis committee. Finally, I would like to thank my friends and family for their continued support over the past 5.5 years.

E. Hoefkens
Delft, January 2021

Contents

1 Introduction

1.1 Thesis Structure

The main topic of this thesis is game theory and ranking games in particular. The primary study is done on the k-Player Ranking Game from Alpern & Howard (2017) [1] and the symmetric form of that game is considered the general interpretation of financial competition among hedge funds.

The motivation behind asset management is explained and the hedge funds are introduced in this chapter. Also two historical hedge fund related events and specific strategies are highlighted. Chapter 2 will cover the hedge fund from a client's perspective and the motivation for ranking them. Further, the influence of the independent research institute Morningstar is shown and together with manager selection form the reasoning for ranking hedge funds based on performance. The necessary mathematical definitions with respect to probability and game theory are introduced in Chapter 3. The analysis of the Competitive Investor game is shown in Chapter 4 and gets expanded in Chapter 5 to the 2-player and multiplayer ranking game. Chapter 6 then aims to translate the theory of ranking games to the hedge fund landscape (and vice versa) for the data analysis.

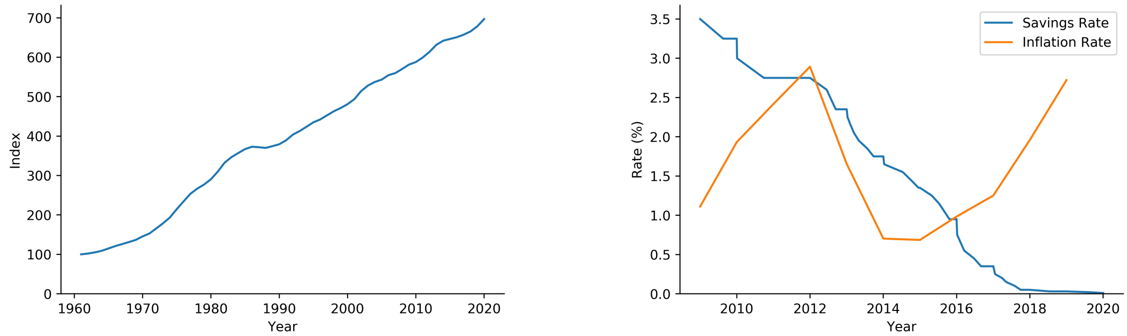
1.2 The Investment World

A euro today is worth more than a euro next year,

is a rather complex and often true statement. The decrease in the value of the euro does not mean a negative change in the currency value compared to all other currencies, but reflects a general rise in the price level of the economy. This mechanism is known as inflation. Inflation means that one can buy less goods compared to last year with the same amount of money. Inflation is country specific, not currency in particular and is determined per year (if not stated otherwise). The average inflation rate of the Netherlands from 1961 till 2020 was 3.35%[1]. This means that nowadays one needs almost €700 to buy the same goods that costed €100 in 1961. See Figure 1.1a on how the index has developed since 1961. So to maintain a so called spending power, one needs his capital to keep up with inflation. Since 1961, there only have been 2 years in which the inflation rate in the Netherlands was negative. Keeping up with inflation is particularly important regarding pensions. Suppose that one opened a bank account in the early seventies for a pension that starts in 2020. The value of that account

[1]Source Inflation Data: `https://www.inflationtool.com/`

must have grown with 3.35% on average each year in order for one to maintain the same lifestyle when retired. One way to increase the value of a bank account is with a specific savings account. Every year, the bank returns interest on one's savings account in exchange for being their customer. As this interest is added to the account on which one gets interest again the following year, the value of the account grows exponentially, just like inflation. So if the interest rate on a savings account is on average approximately equal to the inflation over a given period of time, one has equal spending power. For a long time, this was indeed the case in the Netherlands. However, looking at Figure 1.1b, tides have changed since 2016[2]. The interest rate on savings accounts dropped, while inflation only increased. So to keep up with current inflation, it requires more management than before.



(a) The inflation index of the Netherlands from 1961 - 2019.

(b) The annual inflation rate compared to the annual interest rate on a ING Lifecycle (savings) account.

Figure 1.1 – The inflation in the Netherlands compared to the interest on a savings account.

1.2.1 Asset Management

Opening a savings account is not the only way one can grow the value of a cash account. One can trade the cash for stocks, bonds, gold, bitcoin, property, art and many more other assets. Some of these assets can return interest, while other assets can increase in value or do both. If one properly manages his assets, it is possible to keep up with inflation as well and is an alternative to the savings account. The reason one needs proper management with those kinds of assets is because they are risky. Managing those risky assets is essentially equivalent to investing, investment management and asset management.

Stocks and bonds are the most well-known financial products and widely used generate excess returns. While the concepts of those two assets are commonly known, other financial products, such as options, swaps and other derivatives, can be very complex. (This complexity also caused some great problems, but this will be discussed later.) Depending on one's investment goals, there are choices to be made in terms of risk profile and type of assets. One can determine that for its own, but often a professional gets called in to assist. This can be someone from the asset management department of a bank or other specific asset management companies and advisers.

1.2.2 Investment Strategies

Investing can be done in all kinds of ways. Some examples have already been mentioned: a range of financial products, gold, bitcoin and property for example. Buying publicly traded shares of a stock is the most common way of investing a relative low amount of cash. There is

[2]Source Savings Account Interest: `https://www.ing.nl/`

a low entry threshold, they can return recurring interests in the form of dividends and increase in value, which makes them attractive. Publicly traded stocks are traded on a recognized stock exchange and can always be bought and sold during trading hours. Financial products that are traded on such a public exchange are called listed, otherwise they are non-listed assets.

Organisations that hold lots of cash, think of pension funds and insurance companies for example, need proper asset management. Pensions funds need to manage their assets in such a way that they can meet their liabilities in 30-50 years with the cash they receive right now. Everyone has different objectives when it relates to asset management and therefore, everyone will have a different strategy to try and maximize this objective.

For organisations (or individuals) that manage a significant amount of assets, it is too risky to spread one's cash between just a few assets as one can not afford the lose if one of those assets heavily declines in value (this can happen if a company defaults for example). A solution for this, is to invest in investment funds. A manager of an investment strategy seeks the best investments in a particular category and will trade all those assets in one package with the investor. The managers that are responsible for these strategies are so-called asset managers or portfolio managers and have specialistic knowledge and experience. For the service of managing the strategy, the investors pay a small percentage of their invested assets as a fee each year. When investing in a strategy, one becomes part of a collective investment and as such can benefit from the following:

1. take advantage of professionals who are be able to deliver higher returns;
2. lower transaction costs;
3. a diversified selection of products.

The differences in investment strategies can be very wide or just on a specific topic. The choice to manage a strategy in a specific direction is based on the knowledge and experience of the manager. The main distinctions are the following:

- Product type: stocks, bonds, commodities, property, etc;
- Geography: continent, country or region focus;
- Sector: only invest in assets that are active in a specific sector;
- Size: invest in assets with low or high market value;

1.2.3 Alternative Investments

Assets in the form of listed stocks and bonds and cash are considered conventional categories and the related strategies are managed in the so called mutual funds. Investing in other kinds of assets are considered alternative investments and can range from investing in your local barber shop to buying a Van Gogh painting. These are examples of investments in two single assets. However, there are also funds for these alternative investments. While a conventional investment fund in the Netherlands is regulated by the Authority of Financial Markets, an alternative fund is often not. In that way, they are able to invest in a variety of assets using non-traditional strategies. Alternative investment funds are available in a wide variety. The alternative funds that trade in all kinds of financial products are more commonly known under the name of hedge funds.

1.3 History

It was 1936 when Benjamin Graham and Jerome Newman founded the Graham-Newman Corporation. Looking back, it is considered the first hedge fund ever created. Graham has become famous by its investing strategy (and can be read about in his books Security Analysis and The Intelligent Investor). Today, his method is known as Value Investing and creates a difference between investing and speculating. Value investing is based on research and analysis and believes in the opportunities of a company. One of the most famous (and successful) investors, Warren Buffett, got inspired by Graham and his theory about value investing.

While Graham and Newman managed a successful strategy, hedge funds tend to have a bad reputation. Historical events have shown that hedge funds can impact the entire financial system of a country or take advantage of a crisis. See Section 1.3.1 and 1.3.2 for such examples, respectively.

1.3.1 LTCM

Long-Term Capital Management (LTCM)[3] is known as a hedge fund with big highs and deep lows. Initially the fund was very successful with a return of 21% in its first year, 43% in the second year and 41% in the third year. However, in its fourth year the fund lost $4.6 billion in less than four months.

LTCM was founded in 1994 and thanked it success to a new developed method that calculates the value of derivatives, namely the Black-Scholes model. After all, the researchers behind these Nobel Prize winning equations that are derived in the model, Myron S. Scholes and Robert C. Merton, were members of the board of directors of LTCM.

The fund raised more than $1 billion in capital and initially focused on bond trading. The trading strategy of the fund was to make arbitrage in convergence trades. It is a method using quantitative models to exploit deviations in the relationships between liquid securities across nations and asset classes. LTCM also traded interest rate swaps, which exchanges future interest payments of a contract for other future interest payments. See Example 1.1 to see how one can benefit from dropping interest rates.

Example 1.1. Let's say that the TU Delft has an outstanding loan of $100,000 at Bank A and pays 5% interest each year. This interest rate is fixed for the entire length of the contract. Now, the TU Delft has done some financial engineering and so happens to believe that the interest rate in the market, denoted with r, will decline in the next few years. To exploit this opportunity, the TU Delft buys an interest rate swap from Bank B to exchange the interest rate from the fixed rate of 5% with the interest from the floating rate r. See Figure 1.2 to see that the TU Delft then pays a net interest rate of r. However, Bank B does not offer to do this for free. The determination of the value of such an interest rate swap is complex and changes every moment. Because of LTCM's research, they could do this very well and buy and sell those contracts for good prices.

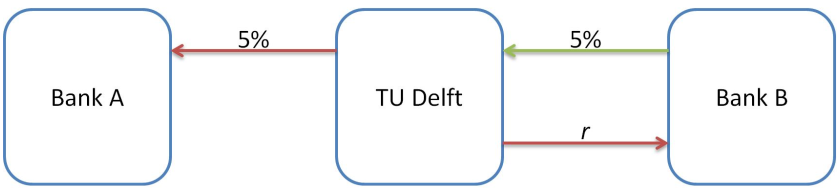


Figure 1.2 – The TU Delft buys an interest rate swap from Bank B in order to change its fixed interest rate in the contract with Bank A for a floating interest rate.

[3]Source: https://www.investopedia.com/terms/l/longtermcapital.asp

Due to the small returns generated in the arbitrage opportunities, LTCM had to highly leverage itself. Leverage is the use of borrowed money to invest. By borrowing money, bigger trades can be made with a small amount of capital. As a result, leverage magnifies the returns from favorable movements, but also magnify losses, see Example 1.2.

Example 1.2. Suppose that one has a $1000 and buys 100 shares of a stock that is worth $10 per share. If the price of the stock increases with $0.50, there is a $50 / 5% profit. However, one can also choose leverage this trade. Say that one can leverage the trade with a factor of 20, i.e. trade with a leverage of 1:20. Then another party, usually the broker of the trade, loans 19 times the money to make the trade 20 times as big. So, with $1000 and a leverage of 1:20, one can make the trade equivalent to that of a $20.000 investment. Now, if the price of the stock increases with 5%, one makes a $1000 profit on a $1000 investment, i.e. a return of 100%. The leverage thus ensures that your return multiplies with the same factor, e.g. 5% times the leverage of 20. However, this also holds for negative returns of course. A 5% drop in value of an investment with a 1:20 leverage, means that one's entire investment has become worthless as the other 95% is needed to repay the issuer of the loan.

LTCM was at his highest in 1998 with $5 billion in assets and an additional $120 billion borrowed assets. In August 1998, LTCM was holding large positions in Russian government bonds. However, when Russia defaulted on his debt, LTCM started losing hundreds of millions of dollars a day. Since LTCM had highly leveraged itself, it was in danger of defaulting on its own loans as well. They had borrowed so much money that the government of the U.S. feared that the collapse would cause a financial crisis. Therefore, in September 1998, the fund (which continued to sustain losses) was bailed out and a meltdown of the market was prevented. LTCM was too big to fail.

1.3.2 The Big Short

It is 2005 when Scion Capital's founding hedge fund manager Michael Burry, a certified medical doctor and fascinated by investments, discovers that the U.S. housing market is extremely unstable.[4] In that time, the housing market in the United States was funded with mortgages from Collateralized Debt Obligations (CDO), which are pools of money that sell loans. While these CDO's had a rating of being in the best shape possible, in fact they were poorly structured and were only to become even more riskier as interest rates would highly rise from the sold adjustable-rate mortgages. To anticipate on a market's collapse, Burry proposes to create a Credit Default Swap (CDS), allowing him to bet against those mortgage-backed securities.

CDS are contracts that enable investors to swap credit risk with another counterparty. In other words, the risk of losing your investment if a company, country, CDO (in this case) or other entity, goes bankrupt, can be bought off with a CDS. In a CDS, the buyer of the swap makes payments to the swap's seller until the maturity date of the contract. In return, the seller agrees that if the party in question defaults, the seller of the CDS will pay the buyer the security's value as well as all interest payments that would have been paid between the time of default and the security's maturity date.

Burry's bet in buying CDS, exceeding $1 billion, is accepted by major banks (as they did not see the risk) and requires paying substantial monthly premiums. The biggest investor in the fund of Scion Capital calls the bet 'wasting capital' and also many other investors demand Burry to reverse the bet and sell everything.

Eventually, the market collapses (which triggered the financial crisis of 2007-2008) and his fund's value increased by 489% with an overall profit of over $2.69 billion. The story of Burry's

[4]Source: `https://www.investopedia.com/articles/investing/020115/big-short-explained.asp`

bet and the overall situation in the market is captured by Michael Lewis in the book The Big Short.

1.4 Hedge Fund Strategies

Hedge funds are alternative investments that can use risky and creative strategies to generate returns. These funds require a larger initial investment than others, and generally are accessible only to accredited investors. That is because alternative funds require far less regulation from the government than conventional funds. Most hedge funds are illiquid, meaning that investors need to keep their money invested for longer periods of time and withdrawals tend to to happen only at certain periods of time. It is recommended that potential hedge fund investors need to understand how these funds manage their strategy and how much risk they take on when they buy into this financial product. Remark the story in Section 1.3.2 about The Big Short in which Burry damaged the trust of its investors. While no hedge funds are identical, most funds generate their returns using one or more of the (more specific) categories that are outlined below.[5]

Long/Short Equity

The concept of a long/short equity strategy is intuitive and simple. The investment research of the hedge fund turns up expected winners and expected losers in terms of stock price. Such hedge funds take long and short positions in equity and equity related derivatives to generate return. In general, hedge funds that follow a long/short strategy tend to be long-biased.

Market-Neutral

Market-neutral strategies have zero net-market exposure, i.e. the short and long positions have an equal market value. This means that the managers generate their entire return from the net-result of the chosen stocks moving in the predicted direction.

Event Driven & Merger Arbitrage

Merger arbitrage derives its returns from corporate takeover activity. That is why it is also considered an event-driven strategy. During the process of a corporate takeover a share-exchange transaction is announced. This announcement contains information about the price and magnitude of the transaction, which impacts the share price of both companies.

In the more general event-driven strategy, hedge funds buy the debt of companies that are in financial distress or have already filed for bankruptcy. Managers often focus on senior debt, which is the debt that is most likely to be repaid. An additional opportunity or hedge is to short sell the stock when the company has not yet filed bankruptcy.

Short Only

The extreme biased strategies are the short-only hedge funds. They scour through all the financials of a company and even talk to its suppliers and competitors to find any sign of trouble that is not yet been noticed by the market. Those hedge fund occasionally score a very big hit when they discover fraud or some other misbehaviour for example[6].

Quantitative

Quantitative hedge fund strategies look for patterns in historical data to make investment decisions. Quantitative analysis mostly uses mathematical and statistical modeling which rely on large data sets. Quantitative strategies can also leverage the use of the latest technology to automatically make very fast trading decisions.

[5] Source: https://www.investopedia.com/articles/investing/102113/what-are-hedge-funds.asp

[6] The fraud detection at Wirecard is a perfect and recent example.

2

Manager Selection

Investors interested in investing in (alternative) strategies (obviously) want to invest in the best fund, i.e. the fund that is ranked number 1 among its competitors, as this fund returns the best result. However, it is not always clear to say which fund performs best. When investors have different goals they want to achieve with investing, different funds can come up as the best.

2.1 Clients

The investors in a fund are the clients of the manager of the fund. The manager is the end responsible for everything concerning the fund. Just as with other businesses, clients come and go. As briefly mentioned in Section 1.2.2, clients pay a fee for the management services of a fund. This fee is (often) a fixed percentage of the invested capital. So the more clients, the higher the invested capital and thus the higher the fee. Often, the personal salary or bonus of the manager is related to the received fee and performance of the fund. In that way, the motivation of the manager is aligned with that of its clients and the company.

Of course, the fund needs to perform to grow the client base. It is best if you are ranked number 1 with respect to other funds in the specific category of your strategy. In that way, it is easier to attract new clients. The minimum investment to participate in a mutual fund can be around a few hundred to a few thousand euros. This is attractive for individual investors. The more interesting clients for fund managers are the institutional investors, like pension funds and other organisations that hold lots of cash. These institutions need to invest multiple millions and this makes them attractive potential clients. In exchange for investing more capital, institutional investors pay a smaller fee than other investors.

Where mutual funds are open to all type of investors, hedge funds are more strict and are only open to investors that satisfy specific requirements, i.e. accredited investors. These accredited investors want to invest in the best hedge fund. It is assumed that the best hedge fund is the hedge fund that will have the future performance that satisfies your conditions the best. But how do potential clients rank the hedge funds to eventually choose the number 1?

2.1.1 Manager Selection

The performance of an investment strategy is measured by the return it generates. The higher the return, the better the performance and as such, the happier the clients of the fund. It seems first hand to invest in the fund that has the best historical performance. After all, they have outperformed the other funds in the past and thus can be considered the best.

However, it has been shown for mutual funds by Goel et al. (2012) [7] that there are other indicators influencing a fund's future performance as well. Indicators like turnover, expense ratio, investment style and ownership style all affect the return of the mutual fund independently. Also, the asset management department of the Dutch investment bank Kempen states that they select the mutual funds in which they invest using more criteria than just historical performance. At Kempen, all the managers and funds are selected (and monitored) against the following criteria: Organization, Strategy, Portfolio, Performance, ESG, Governance & Operations and Terms & Conditions[7]. Kempen attaches great importance to the quality and stability of the investment team of the strategy, for example. Also proper risk management systems and a (lack of) focus on the integration of sustainability criteria can make a difference at Kempen when selecting a fund.

2.1.2 Strategy Benchmark

It is mentioned that the performance of an investment strategy is expressed in the return it generates. However, talking about performance with respect to a fund needs to be relative as every strategy has a benchmark. This benchmark represents the performance of the market in the specific category that the fund is active in. (If a manager does not communicate a benchmark for his strategy, his clients probably designate one for them themselves.)

One has probably heard of the AEX and the S&P 500 indices. The AEX is a stock market index composed of 25 Dutch companies that trade on the Euronext Amsterdam. The S&P 500 is the stock market index that measures the stock performance of 500 companies listed on stock exchanges in the United States. These are just 2 examples. There are indices for (almost) every asset category one can think of. Indices are often used as benchmarks for investment strategies. Some strategies have an absolute return as benchmark, e.g. the fund aims to have an annual return of 10%.

2.2 Morningstar

An independent research and data institute for investment funds is Morningstar. This institute is well known for their in-house developed ratings that allows to make the decision process easier for the investor:

- **Morningstar Rating**
  The Morningstar Rating assesses investment funds from 1 to 5 stars based on performance. This performance is adjusted for risk and sales charges with respect to comparable funds. Within each category, the top 10% of the funds receive 5 star ratings and the bottom 10% receive 1 star ratings. The Morningstar Rating is fully objective and based entirely on an evaluation of historical performance. Morningstar claims that the Morningstar Rating is a ”useful tool for identifying funds..., but should not be considered buy or sell signals”.

- **Analyst Rating**
  The Analyst Rating is the expression of the Morningstar forward-looking analysis of a fund. The Analyst Rating is assigned on a five-tier scale running from Gold to Negative. The top three ratings, Gold, Silver and Bronze, all indicate a positive fund analysis. The difference in these three corresponds to the level of analyst conviction in a fund’s ability to outperform its benchmark and peers through time. The Analyst Rating does not express a view on a given asset class or peer group, it seeks to evaluate each fund within the context of its objective, an appropriate benchmark and peer group. In constrast to the Morningstar Rating, the Analyst Rating is thus a qualitative measure instead of a quantitative. For funds that are not covered by analysts, Morningstar introduced the Quantitative Rating. This rating assesses funds along the same scale, but is determined

[7] Source: https://www.kempen.com/en/asset-management/

by a machine learning model based on historic ratings and results. In that way, Morningstar can cover all funds.

- **Sustainability Rating**
  The Sustainability Rating is a measure of the financial ESG (Environmental, Social and Governance) risks in a fund's portfolio with respect to the other funds in its category. The rating is calculated based on the historical holdings using the company-level ESG Risk Rating. The best 10% get a 5 Globe rating and it drops down to 1 Globe. Morningstar Rating and the Analyst Rating. Morningstar's development of this rating emphasizes the increasing importance and popularity of sustainable investing.

Morningstar claims that their Morningstar Rating should not be considered as a buy or sell signal. Blake & Morey (2000) [4] on the other hand examined "the ability of the Morningstar ratings to predict both un-adjusted and risk-adjusted returns" for U.S. domestic equity funds. The data showed that low-rated funds generally indicate relatively poor future performance, but there was little (statistical) evidence that the highest-rated funds outperform the next-to-highest and median-rated funds. Morey (2003) [10] even states that domestic equity funds with a 5-star Morningstar Ranking were "not able to load on momentum stocks as well as they did before receiving the 5-star ranking". According to Morey, investors should therefore be wary about using the 5-star rating as a negative signal for future 3-year performance.

In terms of performance, it thus has been shown that Morningstar had different prediction qualities. Low-rated funds indeed did not seem to perform (as indicated/predicted by the low rating), but high-rated funds were not always able to outperform the lower-rated funds. While Morningstar also does not recommend to consider the ratings as signals for investors, is does happen. Del Guercio & Tkac (2008) [9] applied an event-study on over 10.000 Morningstar Rating changes. They showed that not the change in the performance, but the change in the Morningstar Rating drives the flow of the fund. A change in the rating results in (economically and statistically significant) abnormal flow in the expected direction caused by the change, i.e. a positive flow for a rating upgrade and a negative flow for a rating downgrade.

The above states that investors do view Morningstar and its ratings as a quality measure for allocation decisions. Also, it confirms the reputation of Morningstar as an influential player. The one thing however is that the data used for the research mentioned above is from around the period 1995-2000. A lot has changed since then. Not only the market for investment funds, but also Morningstar has developed. For example, in July 2002, Morningstar changed its rating algorithm. It remained the case that the top 10% of funds receive the 5-star rating, but the category with respect to the ranking became the fund's more narrow investment style category (rather than all domestic equity funds). Also, it was 2016 when Morningstar introduced the Morningstar Sustainability Rating (as mentioned above). Ammann et al. (2019) [2] examined the effect of the introduction of Sustainability Rating on mutual fund flows. Strong evidence was found that investors shifted capital away from low-rated into high-rated funds as a result of the shock of the available information. Also, during the first year after the publication of the Sustainability Rating, high-rated funds receive significant higher net flows on average than an average rated fund. Low-rated funds suffer lower net flows. Switching investments in this first year mainly applied to retail investors (and funds), while institutional investors react more weakly to the publication of the rating.

So also recent research shows the impact the of Morningstar an its ratings on the behaviour of investors. Do remark that the conclusions of the studies mentioned above was all done with respect to mutual funds. This thesis shall therefore study a specific relation between hedge funds and mutual funds in Chapter 6.

2.3 Alt - Long/Short Equity - U.S.

The focus of this research will be on hedge funds that follow a long/short strategy on U.S. equity. The choice to focus on hedge funds instead of mutual funds is from the hypothesis that they form a more interesting study with respect to selecting the best fund. First, hedge funds are often not open for private investors and require a large investment for participation. This causes the choice of picking the best hedge fund the first time of being of much greater importance. The focus on the long-short equity strategy seems a natural step as this would be the to-go-to strategy when one wants to take their investment game to the next level from mutual funds. It can be frustrating to see mutual funds not taking the advantage of opportunities in the market. Hedge funds can be more creative in that kind of situations. The choice for the U.S. market is simply because of its size and it is viewed as the front runner in finance.

2.3.1 Data

It is clear that investors do not rank investment funds at performance alone (for manager selection). However, it is still considered the most important metric. After all, performance is the final result of all the work of the manager and needs to be attractive to keep existing and attract new clients.

Data is collected from Morningstar to view the returns of the hedge funds. The Premium Fund Screener from `Morningstar.com` is used to select the specific long/short equity funds in the alternative strategy bucket. See Figure A.1 and Table A.1 in Appendix A for a look on the display of the variables of the data. The data consists of the name of the hedge fund and 189 variables. These variables are specific returns, Morningstar ratings, percentage ranks, ratio's, portfolio composition and other information. See Table A.1 in Appendix A for a full overview.

2.3.2 Filtering & Ordering

A thing that needs to be sorted out when comparing hedge funds is to specify the asset class as one strategy can have be divided in multiple asset classes or shares. The difference between these various types of shares is not in the portfolio or any other strategy related variable, but mostly on the administrative side. The main distinction is the difference in the shares for private or institutional investors. As already briefly mentioned, private investors can buy shares of a fund with a relative low minimum investment and institutional investors have a large entry threshold. In return for their commitment, institutional investors pay a lower fee and get detailed reporting on the management of the strategy. So to compare the hedge funds in a fair way, it is needed to select an equal (or very comparable) asset class for every strategy. After all, the fee and structure of the asset class causes (small) changes in the return of the strategy. While some hedge funds indicate their asset class with a letter at the end of the name of the strategy (I for institutional, for example), it not clear for all hedge funds what is what. So to have a fair comparison, the asset class with the lowest fee is chosen to represent the respective strategy.

Also, the data set includes hedge funds that do not manage their own strategy, but at their turn invest in other hedge funds. These hedge funds are thus clients of other hedge funds and are known as Fund of Funds. As they do not make strategic decisions themselves, these types of funds are excluded from the data.

2.4 Ranking & Hedge Funds

It is beneficial for hedge funds to get ranked number 1 in its respective category. In that way, more clients are attracted and at the end, that is the goal of every business. The question remains if (management of the) hedge funds also see this as an important objective or not. To what extend are the hedge funds trying to beat each other?

3

Mathematical Prerequisites

Game theory is the branch of mathematics that models competition. This chapter recalls some of the basic concepts to lay the necessary groundwork for the mathematical analysis.

3.1 Game Theory

Games are played in all kinds of forms and can result in every possible payoff. An intuitive payoff of a game is that one player wins, what another player loses. This mostly applies to two-person games. The describing name of such games are two-person zero-sum games. Zero-sum indicating the that the sum of the payoff of the two players is 0. A mathematical description of a two-person zero-sum game is the so called strategic form, see Definition 3.1 from the notes of Ferguson (2014) [6].

Definition 3.1. *The strategic form of a two-person zero-sum game is given by a triplet* $(\mathcal{X}, \mathcal{Y}, \Pi)$*, where*

1. $\mathcal{X}$ *is a non-empty set, the set of strategies of Player I;*
2. $\mathcal{Y}$ *is a non-empty set, the set of strategies off Player II;*
3. Π *is a real valued function defined on* $X \times Y$*, i.e.* $\Pi(x, y) \in \mathbb{R}$ *for every* $x \in X$ *and every* $y \in Y$*.*

The interpretation of the strategic form is as follows. Player I chooses strategy x from his set of possible strategies $\mathcal{X}$, in short $x \in \mathcal{X}$, and Player II chooses strategy $y \in Y$. Of course, each player chooses his strategy unaware of the choice of the other (or both players choose simultaneously). A strategy is a complete description of what move to make in every possible situation that could occur.The chosen strategies thus fix the moves of the players at every turn in the game. So knowing the strategies, result in knowing exactly how the game is played, thus the outcome of the game is also known and so is the payoff. The function $\Pi(x, y)$ states, without lose of generality, the wins of Player I and the loses of Player II. It is thus assumed that the payoff of Player II is $-\Pi$. If Π is negative, Player I pays $|\Pi(x, y)|$ to Player II.

While it is possible to fit finite games into the strategic form, it can be very time consuming to compile the set of strategies or even a single strategy for that matter. A game like tic-tac-toe is small game, but already has quite a lot of possible situations for which one need to determine every move. So do not even start thinking about putting chess in a strategic form. See Example 3.1 for the strategic form of the Odd or Even game. A game where only one move is required.

Example 3.1. [Odd or Even] Consider a game were Player I and Player II simultaneously pick one of the numbers 1 and 2. Player I wins if the sum of the number is odd and Player II wins if the sum of the numbers is even. The payoff of the game is the sum of the numbers to the one who wins. Then the strategic form of this game is as follows:

$$\mathcal{X} = \mathcal{Y} = \{1,\ 2\},$$

$\Pi(x, y) =$

x \ y	1	2
1	-2	3
2	3	-4

$$= \begin{pmatrix} -2 & 3 \\ 3 & -4 \end{pmatrix}.$$

For simplicity, the function Π can be expressed as a matrix, where the rows represent the strategies of Player I and the columns the strategies of Player II.

At first, the Odd or Even game looks fair: both players win for 2 out of the 4 combinations and the average of the possible payoffs for Player II is equal to that of Player I. However, Player I does have an advantage in this situation. To realise this, Player I must use a so called mixed strategy for playing the game.

Elements of $\mathcal{X}$ and $\mathcal{Y}$ are considered pure strategies. There is no randomness involved with pure strategies. When one randomly chooses among the pure strategies using a probability distribution that is chosen on forehand, it is considered a mixed strategy. Note that one can also combine pure strategies at random that only differ at a specific turn in the game. Therefore, it is possible to take random decisions for every move in the game. A reason to not include randomness in pure strategies is to avoid that the set of strategies always contains an infinite number of strategies. After all, one random choice gives an infinite number of options for choosing corresponding probabilities as the subset (0,1) of the real numbers is uncountable. A two-person zero-sum game is said to be a finite game if $\mathcal{X}$ and $\mathcal{Y}$ are both finite. When playing a pure strategy, the payoff of the game is not random and only based on the interaction of your moves with those of your opponent. When playing a mixed strategy, the randomness in the moves needs us to talk about the expected payoff.

Let's have a look into a specific mixed strategy for Player I in the Odd or Even game.

Example 3.1 (Continued). As there are only 2 pure strategies in the strategy set of both players, a mixed strategy is therefore randomly choosing between playing 1 and 2 using a fixed probability distribution. Let p denote the probability that Player I will play 1. An intuitive move is to find p in such a way that the mixed strategy will always perform, no matter what Player II does. Then the following must hold for p:

$$\begin{aligned}
&\Pi_p(Player\ II\ plays\ 1) = \Pi_p(Player\ II\ plays\ 2) \\
&\Rightarrow \Pi_p(\cdot, 1) = \Pi_p(\cdot, 2) \\
&\Rightarrow p\Pi(1,1) + (1-p)\Pi(2,1) = p\Pi(1,2) + (1-p)\Pi(2,2) \\
&\Rightarrow -2p + 3(1-p) = 3p - 4(1-p) \\
&\Rightarrow 12p = 7 \\
&\Rightarrow p = \frac{7}{12}.
\end{aligned}$$

Playing 1 with probability $\frac{7}{12}$ and playing 2 with probability $\frac{5}{12}$ result in an expected payoff of $\Pi_p(\cdot, 1) = \Pi_p(\cdot, 2) = \frac{1}{12}$ for Player I. So no matter what strategy Player II chooses, Player I has an expected payoff greater than 0. The game is thus not as fair as it might seem at first.

3.1.1 Value of the Game

The maximum expected payoff that Player I is able to achieve, no matter what Player II does, is also known as the value of the game. The formal definition of the value of the game is stated by Theorem 3.1, a fundamental theorem of game theory [6].

Theorem 3.1. *Let $(\mathcal{X}, \mathcal{Y}, \Pi)$ be a finite two-person zero-sum game. Then the following statements are equivalent:*

1. *there is a number V, called the value of the game;*
2. *there is a strategy for Player I such that his expected payoff is at least V, no matter what Player II does, i.e. $\exists x \in \mathcal{X} : \Pi(x, y) \geq V \ \forall y \in \mathcal{Y}$;*
3. *there is a strategy for Player II such that his expected loss is at most V, no matter what Player I does, i.e. $\exists y \in \mathcal{Y} : \Pi(\cdot, y) \leq V \ \forall y \in \mathcal{Y}$.*

If V is negative, it means that Player I loses at most $|V|$ and Player II wins at most $|V|$.

Remark 3.1. The x and y that satisfy the requirements mentioned above in Theorem 3.1 are also known as optimal strategies. Important to note: such strategies may not be unique. Playing an optimal strategy does not mean that it is the best strategy against a specific strategy. An optimal strategy is a strategy that gives one the highest minimum payoff against any strategy.

Remark 3.2. Is V exists and is equal to 0, then the game is considered fair. If V is positive, then the game is in favor of Player I. If V is negative, then the game is in favor of Player II.

In games with 2 or more players, one may need the help of the other players to reach a solution of the game. Such a solution can be expresses as an equilibrium in the game, see Definition 3.2.

Definition 3.2. *In an equilibrium, no player has anything to gain by only changing his own strategy. (The strategies that cause an equilibrium are known as the equilibrium strategies.)*

For a 2-player game, the equilibrium strategies are the same as the optimal strategies. However, for a k-player game with $k > 2$ it holds that the equilibrium strategies are only optimal when the equilibrium is reached. Optimal strategies on the other hand perform in every situation.

3.2 Probability Theory

A move in a game can be described by a random variable. Whilst a random variable itself is not that intuitive, as it is based on the abstract construction of event spaces, probability measures and probability spaces, a cumulative distribution function does provide a clear view of the distribution (hence the name) of the possible outcomes. Let Ω denote the set of all possible outcomes, also known as the sample space. See the Definition 3.3 - 3.5 from Grimmett & Welsh (2014) [8] to describe the theoretical concept of the random variable and its cumulative distribution.

Definition 3.3. *A triple $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space if:*

(a) Ω is a non-empty set;

(b) $\mathcal{F}$ is an event space of subsets of Ω;

(c) $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F})$.

Definition 3.4. *A mapping $X : \Omega \to \mathbb{R}$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a random variable if:*

$$\{\omega \in \Omega : X(\omega) \leq x\} = \{X \leq x\} \in \mathcal{F},$$

for all $x \in \mathbb{R}$.

Definition 3.5. *Let X be a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$. The cumulative distribution function (also known as distribution or cdf) of X is the function $F_X : \mathbb{R} \to [0,1]$, defined by:*

$$F_X(x) = \mathbb{P}(X \leq x).$$

A random variable can describe a move based on randomness in the game. When there is a game that requires many random moves, it is not that great to analyse all random variables separately. Therefore, it is useful to summarize all the random moves with one random variable that represents the payoff of the game. Then the expected payoff of the strategy is represented by the expected value of the random variable. See Definition 3.6 for the determination of this metric.

Definition 3.6. *Let X be a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ and let F be the cumulative distribution function of the probability measure $\mathbb{P}$. Then the expected value of X is defined by:*

$$\mathbb{E}[X] = \int_{\Omega} X(\omega) d\mathbb{P}(\omega) = \int_{-\infty}^{\infty} x dF(x).$$

If the X is non-negative, then the following also holds:

$$\mathbb{E}[X] = \int_0^{\infty} (1 - F(x))dx = \int_0^{\infty} \mathbb{P}(X > x)dx.$$

Next, 3 additional definitions and 3 lemmas are introduced that are needed with respect to (mixed) strategies and distributions. These statements primarily are needed for the analysis of discrete, continuous or a mixture of discrete and continuous random variables. See Definition 3.7 for the requirements of a random variable. Lemma 3.1 states that a continuous random variable adopts with probability zero the same value as another (independent) random variable.

Definition 3.7. *A random variable X is continuous if its distribution function F_X can be written in the form:*

$$F_X(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^{x} f_X(u)\, du,$$

for $x \in \mathbb{R}$, for some non-negative function f_X.

Lemma 3.1. *Let X be a continuous random variable and Y an independent random variable. Then $\mathbb{P}(X = Y) = 0$.*

Proof. Sketch of the proof: Fix $y \in Y$. Then by continuity of X: $\mathbb{P}(X = y) = 0$. As this holds for every $y \in Y$, it remains that $\mathbb{P}(X = Y) = 0$. □

The counterpart of a continuous random variable are the atoms. Where continuous random variables have associated probability density, atoms have probability mass. See Definition 3.8 for the formal establishment of an atom. Then Definition 3.9 and Lemma 3.2 state that a random variable can have more than a finite amount of atoms, but not too many more.

Definition 3.8. *A measurable set A is called an atom if $\mathbb{P}(A) > 0$ and for every measurable subset $E \subseteq A$, it either has that $\mathbb{P}(E) = 0$ or $\mathbb{P}(A \setminus E) = 0$.*

Definition 3.9. *A set A is called countable if there exists a one-to-one correspondence of $\mathbb{N}$ with the elements of A.*

Lemma 3.2. *Let X be a random variable and define $A = \{a : \mathbb{P}(X = a) > 0\}$ (the set containing all atoms). Then A is a countable set.*

Proof. Define $A_n = \left\{a \in A : \mathbb{P}(X = a) \geq \frac{1}{n}\right\}$ for $n \geq 1$. Then $\bigcup_n A_n = A$ and A_n has at most n elements. So A_n is finite and therefore A is countable. □

Not only competitors, but also distributions can be ordered. When two random variables can be ordered in terms of distribution and have equal expected value, they actually have an equal distribution. See Lemma 3.3 for the specific requirements and proof of this statement.

Lemma 3.3. *Let X and Y be two random variables from the distributions F and G, respectively. If $X \leq_{st} Y$ in the sense of stochastic ordering, i.e. $F(x) \geq G(x)$ for all $x \in (-\infty, \infty)$, and $\mathbb{E}[X] = \mathbb{E}[Y]$, then X and Y have equal distribution.*

Proof. The Lemma and proof is from Shaked & Shanthikumar (2007) [12, p. 8]. Define $\hat{X} = F^{-1}(U)$ and $\hat{Y} = F^{-1}(Y)$ where U is a random variable uniformly distributed on $[0, 1]$. Denote $=_{st}$ for equality in distribution. Then $\hat{X} =_{st} X$ and $\hat{Y} =_{st} Y$ and $\mathbb{P}(\hat{X} \leq \hat{Y}) = 1$:

$$\begin{aligned} \hat{F}(x) &= \mathbb{P}(\hat{X} \leq x) = \mathbb{P}(F^{-1}(U) \leq x) = \mathbb{P}(U \leq F(x)) = F(x), \\ \hat{G}(x) &= \mathbb{P}(\hat{Y} \leq x) = \mathbb{P}(G^{-1}(U) \leq x) = \mathbb{P}(U \leq G(x)) = G(x), \\ \mathbb{P}(\hat{X} \leq \hat{Y}) &= \mathbb{P}(F^{-1}(U) \leq G^{-1}(U)) = \mathbb{P}(U \leq F(G^{-1}(U))) = 1. \end{aligned}$$

Suppose that $\mathbb{P}(\hat{X} < \hat{Y}) > 0$ holds. Then the following should hold as well:

$$\mathbb{E}[X] = \mathbb{E}[\hat{X}] < \mathbb{E}[\hat{Y}] = \mathbb{E}[Y].$$

This is a contradiction. Therefore, $\mathbb{P}(\hat{X} < \hat{Y}) = 0$ and thus $\mathbb{P}(\hat{X} = \hat{Y}) = 1$. So this implies:

$$X =_{st} \hat{X} = \hat{Y} =_{st} Y.$$

This completes the proof. □

4

Competitive Investors

Two investors compete to see which of them, starting with the same initial capital, can end up with the larger capital. The rules of the competition require that they can only do fair investments. That is, they can only invest non-negative amounts in assets whose expected return per unit invested is 1.

Suppose that the investors start with 1 unit of capital each. Thus no matter in what they invest, their expected capital at the end is equal to their initial capital. To model the choice of investments, the investors choose a distribution that represents the (random) end-value of their investment. It assumed that this can not go negative. So the players choose distributions on $[0, \infty)$ with mean 1. Let's say that Investor I chooses F with mean 1 and Investor II chooses G with mean 1[8]. Then the random variable X has distribution F and the random variable Y has distribution G, independently. Investor I wins the best if $X > Y$, Investor II wins if $X < Y$ and it is a tie if $X = Y$. The investor who wins the bet gets 1 from the other investor and there is no exchange if it is a tie. Denote Π for the payoff of Investor I. The expected payoff for Investor I is then expressed as follows:

$$\begin{aligned}\Pi(F, G) &= \mathbb{P}(X > Y) \cdot 1 + \mathbb{P}(X = Y) \cdot 0 + \mathbb{P}(X < Y) \cdot -1 \\ &= \mathbb{P}(X > Y) - \mathbb{P}(X < Y).\end{aligned}$$

Consequently, the payoff for Investor II is $-\Pi$ (as it is a zero-sum game/bet). See Game 1 for the formal definition of the game described above.

Game 1 (**The Competitive Investor Game**). Consider a two-player zero-sum game. Player I chooses a random variable X with distribution F and Player II an independent random variable Y with distribution G, such that $X, Y \geq 0$ and $\mathbb{E}[X] = \mathbb{E}[Y] = 1$. The players take a random sample from their distribution, x and y respectively. The payoff Π is 1 if $x > y$, 0 if $x = y$ and -1 if $x < y$ with respect to Player I.

The strategy space of the Competitive Investor Game is the set of all distributions that satisfy the requirements of the game. This means that each valid distribution is an element of the strategy space and thus considered as a pure strategy. Player I and II have access to the same set of distributions. So the players are equally competitive, As either player can copy the strategy of their opponent. In this case, each distribution is a pure strategy. See Example 4.1 where such a strategy is put to the test according to the Competitive Investor Game.

[8]Fun fact: any non-negative capital distribution is achievable from the initial capital 1 by a gambling scheme on fair coin tosses (Cover (1974) [5]).

Example 4.1. One can often try to solve a game by trial and error. Suppose that Player I plays the strategy that he would choose 0 with probability a half and 2 with probability a half. Denote this strategy by X and denote Y for the random variable that represents the strategy of Player II. The expected payoff for Player I would be:

$$\begin{aligned}\Pi &= \frac{1}{2}\left(\mathbb{P}(0 > Y) - \mathbb{P}(0 < Y)\right) + \frac{1}{2}\left(\mathbb{P}(2 > Y) - \mathbb{P}(2 < Y)\right) \\ &= -\frac{1}{2}\mathbb{P}(0 < Y) + \frac{1}{2}\left(\mathbb{P}(2 > Y) - \mathbb{P}(2 < Y)\right).\end{aligned}$$

What is an optimal response for Player II? If Player II chooses to play the trivial strategy, i.e. choose 1 with probability 1, he wins half of the time and the expected payoff would be 0 (for both players). In fact, he can choose any distribution that is between (and with no mass at) 0 and 2 and have an expected payoff of 0. If player II does not (randomly) choose 2 or higher, Player I will win half of the time.

To improve, let $\epsilon > 0$ arbitrarily small and now let Player II play Y with atoms at ϵ and $2+\epsilon$. The probabilities for those atoms is solved with a system of 2 equations:

$$\begin{cases}\epsilon \cdot \mathbb{P}(Y = \epsilon) + (2+\epsilon)\cdot \mathbb{P}(Y = 2+\epsilon) = 1 \\ \mathbb{P}(Y=\epsilon) + \mathbb{P}(Y = 2+\epsilon) = 1\end{cases}$$

$$\Rightarrow \mathbb{P}(Y=\epsilon) = \frac{1+2\epsilon}{2-\epsilon} \text{ and } \mathbb{P}(Y = 2+\epsilon) = \frac{1-3\epsilon}{2-\epsilon}.$$

Then the payoff of the game will be the following:

$$\begin{aligned}\Pi(F,G) &= \mathbb{P}(X > Y) - \mathbb{P}(X < Y) \\ &= \frac{1}{2}\left(\mathbb{P}(0 > Y) - \mathbb{P}(0 < Y)\right) + \frac{1}{2}\left(\mathbb{P}(2 > Y) - \mathbb{P}(2 < Y)\right) \\ &= -\frac{1}{2}\mathbb{P}(0 < Y) + \frac{1}{2}\left(\mathbb{P}(2 > Y) - \mathbb{P}(2 < Y)\right) \\ &= -\frac{1}{2} + \frac{1}{2}\left(\mathbb{P}(Y=\epsilon) - \mathbb{P}(Y = 2+\epsilon)\right) \\ &= -\frac{1}{2} + \frac{1}{2}\left(\frac{1+2\epsilon}{2-\epsilon} - \frac{1-3\epsilon}{2-\epsilon}\right) \\ &= -\frac{1}{2} + \frac{5\epsilon}{2(2-\epsilon)} \approx -\frac{1}{2}\end{aligned}$$

Of the 4 possible combinations of scores in the game, Player II only loses when he plays ϵ and Player I plays 2. This example suggests that the strategy of 0 or 2 with probability $1/2$ is too predictable. Player I can counter the strategy of Player II by choosing 2ϵ and $2+2\epsilon$ to beat player II. This indicates that an optimal strategy should be continuous.

4.1 Optimal Strategy

The example above indicates that placing probability mass at a specific point is not ideal as one can outplay this by placing his probability mass a ϵ higher. By intuition, it thus seems that it is better to choose a continuous distribution. Also, playing very high values does not seem optimal either. Every probability density (or mass) above 1 needs to be compensated with probability density under 1. For example, it is not rational to play 10 with probability $\frac{1}{10}$ and to compensate that by playing 0 with probability $\frac{9}{10}$. Playing 10 will likely result in a win, but is it worth it if you lose the other 9 times (on average)? This shows that this game is not about getting the highest score, but the players have the goal to beat the other. Relative scores matters, absolute scores not that much.

Therefore, Theorem 4.1 will be introduced as an optimal strategy for Game 1, followed by Theorem 4.2 about the uniqueness of the proposed optimal strategy. Theorems 4.2 and 4.2 are due to Bell and Cover (1988). However, the proof is adjusted to the situation of the described payoff function and complemented with the additional lemmas mentioned in Chapter 3.

Theorem 4.1 (Bell & Cover, 1988)**.** *The value of the Competitive Investor Game (Game 1) is 0 and an optimal strategy for both players is the uniform distribution on the interval* $[0, 2]$*:*

$$F^*(x) = G^*(x) = \begin{cases} \frac{x}{2}, & 0 \leq x \leq 2 \\ 1, & x \geq 2 \end{cases}.$$

Proof. Let G be an arbitrary distribution for Player II satisfying the requirements of the game and Player I plays F^*. Let X^* have distribution F^* and Y have distribution G. The density function of X^* is thus equal to $f^*(x) = \frac{1}{2}$, $0 \leq x \leq 2$. As X^* is continuous, it satisfies $\mathbb{P}(X^* = Y) = 0$ by Lemma 3.1. Then the expected payoff of strategy F^* for Player I is non-negative against any strategy of Player II:

$$\begin{aligned} \Pi(F^*, G) &= \mathbb{P}(X^* > Y) - \mathbb{P}(X^* < Y) = 1 - \mathbb{P}(X^* \leq Y) - \mathbb{P}(X^* < Y) \\ &= 1 - 2\mathbb{P}(X^* < Y) = 1 - 2\int_0^\infty \mathbb{P}(Y > x) f(x)\, dx \\ &= 1 - \int_0^2 \mathbb{P}(Y > x)\, dx \geq 1 - \int_0^\infty \mathbb{P}(Y > x)\, dx = 1 - \mathbb{E}[Y] = 0. \end{aligned} \tag{4.1}$$

Since the game is symmetrical in the players, Player II can achieve the opposite by also playing F^*. As the greatest expected payoff for both players is zero, both players play a strategy that guarantees them exactly that. F^* satisfies this optimality and is an optimal strategy for both players. The value of the game is indeed thus 0. □

Remark 4.1. Originally, the setup of the game in Bell & Cover is that the payoff of Player I is equal to $\mathbb{P}(X \geq Y)$. While this changes the value of the game, the optimal strategy remains the same. After all, both payoff functions depend on the same probabilities.

Remark 4.2. The proof of Theorem 4.1 verifies that the uniform distribution is an optimal strategy for playing the Competitive Investor Game. However, do notice that the proof reveals more than just the optimality. More specific, the end of Equation 4.1 states that the expected payoff for Player I is $1 - \mathbb{E}[Y]$. So if Player II has another restriction for the expectation of his distribution, the payoff against the uniform strategy is already known..

Remark 4.3. Game 1 is specified for distributions that have an expected value of 1. It is considered that every game of this form, but with a different equality requirement for the expected value of the distribution, is equivalent. It is just a factorization difference. For example, if both players must choose distributions with an expected value of μ, the optimal strategy would be to play uniformly on $[0, 2\mu]$. The proof of this would be fully equivalent to that of Theorem 4.1, apart from the scaling of $f(x)$ and $\mathbb{E}[Y]$ (which cancels each other) (see Equation 4.4 in the proof of Theorem 4.3 if one has that $\mathbb{E}[Y] = \mu$ as well).

Remark 4.4. Notice that changing the equality signs in the requirement of the expected value to a lesser or equal inequality ($\leq$), does not change the optimal strategies of the game. This change can be considered as a generalisation of the game and will be discussed later in this chapter.

Next, is the evaluation of the uniqueness of the optimal strategy from Theorem 4.1. See the proof of Theorem 4.2 in which the game is further analysed and also concludes that there is only one optimal strategy.

Theorem 4.2. *The uniform distribution on the interval* $[0,2]$ *is the unique optimal strategy for the Competitive Investor Game (Game 1).*

Proof. Theorem 4.1 already showed that this strategy is optimal for both players. Next to show is that there is no other strategy that can guarantee this optimality.

Let $\hat{F}$ be an arbitrary optimal strategy of Player I, i.e. $\Pi(\hat{F}, G) \geq 0$ for any arbitrary strategy G of Player II. Remark that $\Pi(F^*, \hat{F}) = 0$ must also hold. After all, the game is symmetric in the players and both strategies are optimal. Let X^* have distribution F^* and $\hat{X}$ have distribution $\hat{F}$ and recall the inequality in Equation 4.1 from Theorem 4.1:

$$\Pi(F^*, \hat{F}) = 1 - \int_0^2 \mathbb{P}(\hat{X} > x)\, dx \geq 1 - \int_0^\infty \mathbb{P}(\hat{X} > x)\, dx = 1 - \mathbb{E}[\hat{X}] = 0.$$

If $\hat{X}$ takes values larger than 2, the inequality $\geq$ changes into the strict inequality $>$. In order for the payoff to remain equal to 0, it thus must hold that $\hat{F}(2) = 1$.

Next, $\hat{F}$ is tested against three two-point distributions. Consider the following random variables:

- $Y_1 \sim \{0, c_1\}$ with distribution G_1 and $1 \leq c_1 \leq 2$,
- $Y_2 \sim \{c_2, 2\}$ with distribution G_2 and $0 \leq c_2 \leq 1$,

The corresponding probabilities for Y_1 and Y_2 are expressed in terms of c_1 and c_2, respectively, in order to keep the expectation of those random variables equal to 1. Y_1 must have that $0 \cdot \mathbb{P}(Y_1 = 0) + c_1 \cdot \mathbb{P}(Y_1 = c_1) = 1 \;\Rightarrow\; \mathbb{P}(Y_1 = c_1) = \frac{1}{c_1}$ and thus $\mathbb{P}(Y_1 = 0) = \frac{c_1 - 1}{c_1}$. For Y_2, a system of 2 equations must be solved:

$$\begin{cases} c_2 \cdot \mathbb{P}(Y_2 = c_2) + 2 \cdot \mathbb{P}(Y_2 = 2) = 1 \\ \mathbb{P}(Y_2 = c_2) + \mathbb{P}(Y_2 = 2) = 1 \end{cases}$$
$$\Rightarrow \mathbb{P}(Y_2 = c_2) = \frac{1}{2 - c_2} \text{ and } \mathbb{P}(Y_2 = 2) = \frac{1 - c_2}{2 - c_2}.$$

Following the assumption of an optimal strategy, this leads to the following inequalities:

$$\begin{aligned} \Pi(\hat{F}, G_1) \geq 0 \;&\Rightarrow \mathbb{P}(Y_1 = 0)\Pi(\hat{F}, 0) + \mathbb{P}(Y_1 = c_1)\Pi(\hat{F}, c_1) \geq 0 \\ &\Rightarrow \frac{c_1 - 1}{c_1}\left(\mathbb{P}(\hat{X} > 0) - \mathbb{P}(\hat{X} < 0)\right) + \frac{1}{c_1}\left(\mathbb{P}(\hat{X} > c_1) - \mathbb{P}(\hat{X} < c_1)\right) \geq 0 \\ &\Rightarrow (c_1 - 1)\mathbb{P}(\hat{X} > 0) + \mathbb{P}(\hat{X} > c_1) - \mathbb{P}(\hat{X} < c_1) \geq 0 \\ &\Rightarrow (c_1 - 1)\left(1 - \mathbb{P}(\hat{X} = 0)\right) + \left(1 - \mathbb{P}(\hat{X} \leq c_1)\right) - \mathbb{P}(\hat{X} < c_1) \geq 0 \\ &\Rightarrow -\left(\mathbb{P}(\hat{X} \leq c_1) + \mathbb{P}(\hat{X} < c_1)\right) - (c_1 - 1)\mathbb{P}(\hat{X} = 0) \geq -c_1 \\ &\Rightarrow \mathbb{P}(\hat{X} \leq c_1) + \mathbb{P}(\hat{X} < c_1) \leq c_1 - (c_1 - 1)\mathbb{P}(\hat{X} = 0) \\ &\Rightarrow \mathbb{P}(\hat{X} \leq c_1) + \mathbb{P}(\hat{X} < c_1) \leq c_1, \end{aligned} \tag{4.2}$$

$$\begin{aligned}
\Pi(\hat{F}, G_2) \geq 0 \ &\Rightarrow \mathbb{P}(Y_2 = c_2)\Pi(\hat{F}, c_2) + \mathbb{P}(Y_2 = 2)\Pi(\hat{F}, 2) \geq 0 \\
&\Rightarrow \frac{1}{2 - c_2}\left(\mathbb{P}(\hat{X} > c_2) - \mathbb{P}(\hat{X} < c_2)\right) + \frac{1 - c_2}{2 - c_2}\left(\mathbb{P}(\hat{X} > 2) - \mathbb{P}(\hat{X} < 2)\right) \geq 0 \\
&\Rightarrow \mathbb{P}(\hat{X} > c_2) - \mathbb{P}(\hat{X} < c_2) - (1 - c_2)\mathbb{P}(\hat{X} < 2) \geq 0 \\
&\Rightarrow \left(1 - \mathbb{P}(\hat{X} \leq c_2)\right) - \mathbb{P}(\hat{X} < c_2) - (1 - c_2)\left(1 - \mathbb{P}(\hat{X} = 2)\right) \geq 0 \\
&\Rightarrow -\left(\mathbb{P}(\hat{X} \leq c_2) + \mathbb{P}(\hat{X} < c_2)\right) + (1 - c_2)\mathbb{P}(\hat{X} = 2) \geq -c_2 \\
&\Rightarrow \mathbb{P}(\hat{X} \leq c_2) + \mathbb{P}(\hat{X} < c_2) \leq c_2 + (1 - c_2)\mathbb{P}(\hat{X} = 2),
\end{aligned} \tag{4.3}$$

Combine the end-inequality of equation 4.2 and 4.3 to state that the following must hold for $0 \leq c \leq 2$:

$$\begin{cases} \mathbb{P}(\hat{X} \leq c) + \mathbb{P}(\hat{X} < c) \leq c, & 1 \leq c \leq 2 \\ \mathbb{P}(\hat{X} \leq c) + \mathbb{P}(\hat{X} < c) \leq c + (1 - c)\mathbb{P}(\hat{X} = 2), & 0 \leq c \leq 1. \end{cases}$$

Suppose that $\mathbb{P}(\hat{X} > 2) > 0$. Then $\hat{F}$ has an atom at 0 and at 2. Example 4.1 showed that an atom in a distribution implies that that strategy can not be optimal, since an advantage can be obtained by placing probability mass at the atoms $+\epsilon$. So $\hat{F}$ must have that $\mathbb{P}(\hat{X} = 2) = 0$ and therefore satisfy the following:

$$\mathbb{P}(\hat{X} \leq c) + \mathbb{P}(\hat{X} < c) \leq c, 0 \leq c \leq 2.$$

If $\hat{F}$ is continuous, then $\mathbb{P}(\hat{X} = c) = 0$ for every $0 \leq c \leq 2$ and implies that $2\hat{F}(c) \leq c$. Now assume that $\hat{F}$ is not (entirely) continuous and that there is an atom at some c such that $2\hat{F}(c) > c$. This implies that $\mathbb{P}(\hat{X} = c) > 0$ must hold. Define $A = \{0 \leq a < 2 : \mathbb{P}(\hat{X} = a) > 0\}$ the set of atoms excluding the possibility of there being an atom at 2 (as it is already known that this is not the case). From Lemma 3.2 it is known that A then is a countable set.

Choose $a \in A$ arbitrarily. Then $\hat{F}(a) > \frac{a}{2}$ holds. Define $a' > a$ such that $a' \notin A$ and $\hat{F}(a) > \frac{a'}{2}$. This is possible by the countability of A. Then $\hat{F}(a') \geq \hat{F}(a)$ by the definition of a distribution and have that $\hat{F}(a') > \frac{a'}{2}$. This is a contradiction. Since a' is not an atom, it satisfies $\hat{F}(a') \leq \frac{a'}{2}$. So A is an empty set.

This proves that $\hat{F}$ must be a continuous distribution satisfying the following:

$$\begin{aligned} &\hat{F}(c) \leq \tfrac{c}{2}, \quad 0 \leq c \leq 2 \\ &\hat{F}(2) = 1. \end{aligned}$$

In the sense of stochastic ordering, $\hat{F}$ thus has that $\hat{F}(x) \leq F^*(x)$ for all x. Lemma 3.3 is then needed to complete the proof. Both criteria of Lemma 3.3 are satisfied for X^* and $\hat{X}$ since $\hat{F}(x) \leq F^*(x) \Rightarrow \mathbb{P}(X^* > x) \leq \mathbb{P}(\hat{X} > x)$ for all x and meet the expectation criteria for $h(x) = x$. This concludes that $\hat{F}$ and F^* have equal distribution and F^* is therefore unique. This concludes the proof of uniqueness for the optimal strategy of Player I. The proof of uniqueness for G^* of Player II follows by symmetry in the players. □

This theorem concludes the analysis of the Competitive Investor Game in which the two players are considered equally competitive.

4.2 The Asymmetric Investor Game

Next to discuss is the game where there is an inequality in the competitive level of the players. Without lose of generality, it is considered that Player I is better than Player II. This reflects in the requirements of the strategies that are allowed for each player. The expected value of the distribution of Player I is higher than that of Player II. To rule out games that are equivalent by factorization, it is assumed that Player I has a competitiveness level of $\mu > 1$ and Player II of 1. The game is thus not symmetric in the players anymore. The other rules of game remains the same and the formal definition is stated in Game 2.

Game 2 (The Asymmetric Investor Game). Consider a two-player zero-sum game. Player I chooses a random variable X with distribution F and Player II an independent random variable Y with distribution G such that $X,\ Y \geq 0$. Let $\mu > 1$ such that $\mathbb{E}[X] = \mu$ and $\mathbb{E}[Y] = 1$. Then the players take a random sample from their distribution, x and y respectively. The payoff Π is 1 if $x > y$, 0 if $x = y$ and -1 if $x < y$ with respect to Player I.

Remember that Remark 4.2 already stated some additional information about the expected payoff of the uniform strategy. It states that the expected payoff is at least $1-\mathbb{E}[Y]$ when Player I has a competitive level of 1 and Y is the random variable corresponding to Player II. Let Y have a competitive level of $\frac{1}{\mu}$, i.e. $\mathbb{E}[Y] = \frac{1}{\mu}$, where $\mu > 1$. Then the expected payoff of Player I is $1 - \mathbb{E}[Y] = \frac{\mu-1}{\mu}$. Also, this setup of the game is equivalent to that of The Asymmetric Investor Game by a factorization of $\frac{1}{\mu}$: the competitive level of Player I goes from μ to 1 and that of Player II from 1 to $\frac{1}{\mu}$. So Player I can achieve a minimum expected payoff of $\frac{\mu-1}{\mu}$ in the Asymmetric Investor Game.

Alpern & Howard (2017) [1] propose (in a more general theorem which will be discussed later) a optimal strategy for Player II that copies the strategy of Player I or gives up, i.e. plays 0. This means that Player II will have an atom in his distribution. See Theorem 4.3 for the optimal strategies of the Asymmetric Investor Game. The setup of the proof is similar to that of the one for the Competitive Investor Game.

Theorem 4.3. *The value of the Asymmetric Investor Game is* $\frac{\mu-1}{\mu}$ *and the optimal strategies for Player I and Player II, respectively, are the following:*

$$F^*(x) = \begin{cases} \frac{x}{2\mu}, & 0 \leq x \leq 2\mu \\ 1, & x \geq 2\mu \end{cases}$$

$$G^*(y) = \begin{cases} \frac{\mu-1}{\mu} + \frac{y}{2\mu^2}, & 0 \leq y \leq 2\mu \\ 1, & y \geq 2\mu \end{cases}.$$

Proof. Let G be any distribution for Player II satisfying the requirements of the game and Player I plays F^*. Let Y be the random variable with distribution G and X^* the random variable with distribution F^*. The density function of X^* is equal to $f^*(x) = \frac{1}{2\mu}$, $0 \leq x \leq 2\mu$. As X^* is continuous, it satisfies $\mathbb{P}(X^* = Y) = 0$ by Lemma 3.1. Then the expected payoff for Player I is positive against whichever strategy Player II chooses:

$$\begin{aligned}\Pi(F^*,G) &= \mathbb{P}(X^*>Y)-\mathbb{P}(X^*<Y) = 1-\mathbb{P}(X^*\leq Y)-\mathbb{P}(X^*<Y)\\ &= 1-2\mathbb{P}(X^*<Y) = 1-2\int_0^\infty \mathbb{P}(Y>x)f(x)dx\\ &= 1-\frac{1}{\mu}\int_0^{2\mu}\mathbb{P}(Y>x)dx \geq 1-\frac{1}{\mu}\int_0^\infty \mathbb{P}(Y>x)dx\\ &= 1-\frac{1}{\mu}\mathbb{E}[Y] = \frac{\mu-1}{\mu}.\end{aligned} \tag{4.4}$$

Next, the strategy G^* of Player II is tested against an arbitrary strategy F of Player I. Let X be the random variable with distribution F, Y^* the random variable with distribution G^* and Y^{**} the random variable with distribution F^*. Then the following holds:

$$\begin{aligned}\Pi(F,G^*) &= \mathbb{P}(X>Y^*)-\mathbb{P}(X<Y^*)\\ &= \mathbb{P}(Y^*=0)\left(\mathbb{P}(X>0)-\mathbb{P}(X<0)\right)+\mathbb{P}(Y^*>0)\left(\mathbb{P}(X>Y^{**})-\mathbb{P}(X<Y^{**})\right)\\ &= \frac{\mu-1}{\mu}\left(1-\mathbb{P}(X=0)\right)+\frac{1}{\mu}\left(\mathbb{P}(X>Y^{**})-(1-\mathbb{P}(X\geq Y^{**}))\right),\end{aligned} \tag{4.5}$$

where the second term of the final equation reduces to less than or equal to zero:

$$\begin{aligned}&\frac{1}{\mu}\left(\mathbb{P}(X>Y^{**})-(1-\mathbb{P}(X\geq Y^{**}))\right)\\ =&\frac{1}{\mu}\left(2\mathbb{P}(X>Y^{**})-1\right) = \frac{2}{\mu}\int_0^\infty \mathbb{P}(X>y)f^*(y)\,dy-\frac{1}{\mu}\\ =&\frac{2}{\mu}\int_0^{2\mu}\mathbb{P}(X>y)\frac{1}{2\mu}\,dy-\frac{1}{\mu} \leq \frac{1}{\mu^2}\int_0^\infty \mathbb{P}(X>y)\,dy-\frac{1}{\mu}\\ =&\frac{1}{\mu^2}\mathbb{E}[X]-\frac{1}{\mu}\\ =&0.\end{aligned} \tag{4.6}$$

So $\Pi(F,G^*) \leq \frac{\mu-1}{\mu}(1-\mathbb{P}(X=0)) \leq \frac{\mu-1}{\mu}$.

The optimal strategies thus agree in the minimum guaranteed expected payoff which at its turn agrees with the proposed value of the game. This concludes the proof. □

As already hinted, the optimal strategies indeed exist and the value of the Asymmetric Investor Game is known. When one is facing a duel and knows that the other player is better, it is thus best to choose between giving up (play 0) or equal the strategy of the opponent. On average, it is the best one can do.

A good comparison is with athletes who need to take risk. If an athlete knows that he has a tough component and only has a one time shot for the win, risk management is key. The athlete can try to copy the risk profile of his opponent adjusted to his own capabilities, i.e. play uniformly around his own average. However, he can also choose to take the risk, go for a very high score and see whether he fails (playing 0) or has a good shot at winning (following the distribution of the opponent).

Assuming that Player I knows this as well, would there be another optimal strategy as well? After all, it is shown in Example 4.1 that one can outplay the other if there is an atom in the distribution of the strategy. Theorem 4.4 shows that is not the case. Playing uniformly is the unique optimal strategy for the better player in the Asymmetric Investor game.

Theorem 4.4. *The uniform distribution on the interval* $[0, 2\mu]$ *is the unique optimal strategy in the Asymmetric Investor Game of Player I.*

Proof. An optimal strategy for Player I for the Asymmetric Investor Game has been found in Theorem 4.3. Next to prove is that this strategy is unique.

Let $\hat{F}$ be an arbitrary optimal strategy for Player I, i.e. such that $\Pi(\hat{F}, G) \geq \frac{a-1}{a}$ against any strategy G of Player II. First, remark that $\Pi(\hat{F}, G^*) = \frac{a-1}{a}$ must hold. Consider the inequality in Equation 4.6 and the final inequality of that proof. If $\hat{X}$ takes values larger than 2μ or $\mathbb{P}(X = 0) > 0$, the inequalities changes into strict inequalities. In order to remain optimal, $\hat{F}$ must therefore satisfy that $\hat{F}(2\mu) = 1$ and $\hat{F}(0) = \mathbb{P}(\hat{X} = 0) = 0$.

Next, $\hat{F}$ is tested against two-point distributions. Consider the random variables $Y_1 \sim \{0, c_1\}$, $1 \leq c_1 \leq 2a$, and $Y_2 \sim \{c_2, 2a\}$, $0 \leq c_2 \leq 1$ as the strategies for Player II. In order to keep the expectation of Y_1 and Y_2 equal to one, the corresponding probabilities are expressed in terms of c_1 and c_2, respectively. Y_1 must have that $0 \cdot \mathbb{P}(Y_1 = 0) + c_1 \cdot \mathbb{P}(Y_1 = c_1) = 1 \Rightarrow \mathbb{P}(Y_1 = c_1) = \frac{1}{c_1}$ and thus $\mathbb{P}(Y_1 = 0) = \frac{c_1 - 1}{c_1}$. For Y_2 a system of 2 equations is solved:

$$\begin{cases} c_2 \cdot \mathbb{P}(Y_2 = c_2) + 2\mu \cdot \mathbb{P}(Y_2 = 2\mu) = 1 \\ \mathbb{P}(Y_2 = c_2) + \mathbb{P}(Y_2 = 2\mu) = 1 \end{cases}$$
$$\Rightarrow \mathbb{P}(Y_2 = c_2) = \frac{2\mu - 1}{2\mu - c_2} \text{ and } \mathbb{P}(Y_2 = 2\mu) = \frac{1 - c_2}{2\mu - c_2}.$$

Following the assumption, this leads to the following inequalities:

$$\begin{aligned}
&\Pi(\hat{F}, Y_1) \geq \frac{\mu - 1}{\mu} \\
&\Rightarrow \mathbb{P}(Y_1 = 0)\Pi(\hat{F}, 0) + \mathbb{P}(Y_1 = c_1)\Pi(\hat{F}, c_1) \geq \frac{\mu - 1}{\mu} \\
&\Rightarrow \frac{c_1 - 1}{c_1}\left(\mathbb{P}(\hat{X} > 0) - \mathbb{P}(\hat{X} < 0)\right) \\
&\quad + \frac{1}{c_1}\left(\mathbb{P}(\hat{X} > c_1) - \mathbb{P}(\hat{X} < c_1)\right) \geq \frac{\mu - 1}{\mu} \\
&\Rightarrow (c_1 - 1)\,(1 - 0)) + \left(1 - \mathbb{P}(\hat{X} \leq c_1) - \mathbb{P}(\hat{X} < c_1)\right) \geq \frac{c_1(\mu - 1)}{\mu} \\
&\Rightarrow -\left(\mathbb{P}(\hat{X} \leq c_1) + \mathbb{P}(\hat{X} < c_1)\right) \geq \frac{c_1(\mu - 1)}{\mu} - c_1 \\
&\Rightarrow \mathbb{P}(\hat{X} \leq c_1) + \mathbb{P}(\hat{X} < c_1) \leq c_1 - \frac{c_1(\mu - 1)}{\mu} \\
&\Rightarrow \mathbb{P}(\hat{X} \leq c_1) + \mathbb{P}(\hat{X} < c_1) \leq \frac{c_1}{\mu}
\end{aligned} \tag{4.7}$$

$$
\begin{aligned}
&\Pi(\hat{F}, Y_2) \geq \frac{\mu-1}{\mu} \\
&\Rightarrow \mathbb{P}(Y_2 = c_2)\Pi(\hat{F}, c_2) + \mathbb{P}(Y_2 = 2\mu)\Pi(\hat{F}, 2\mu) \geq \frac{\mu-1}{\mu} \\
&\Rightarrow \frac{2\mu-1}{2\mu-c_2}\left(\mathbb{P}(\hat{X} > c_2) - \mathbb{P}(\hat{X} < c_2)\right) \\
&\quad + \frac{1-c_2}{2\mu-c_2}\left(\mathbb{P}(\hat{X} > 2\mu) - \mathbb{P}(\hat{X} < 2\mu)\right) \geq \frac{\mu-1}{\mu} \\
&\Rightarrow (2\mu-1)\left(1 - \mathbb{P}(\hat{X} \leq c_2) - \mathbb{P}(\hat{X} < c_2)\right) \\
&\quad - (1-c_2)\mathbb{P}(\hat{X} < 2\mu) \geq \frac{(\mu-1)(2\mu-c_2)}{\mu} \\
&\Rightarrow 2\mu - 1 - (2\mu-1)\left(P(\hat{X} \leq c_2) + \mathbb{P}(\hat{X} < c_2)\right) \geq \frac{(\mu-1)(2\mu-c_2)}{\mu} \\
&\quad + (1-c_2)\mathbb{P}(\hat{X} < 2\mu) \\
&\Rightarrow \mathbb{P}(\hat{X} \leq c_2) + \mathbb{P}(\hat{X} < c_2) \leq 1 - \frac{(\mu-1)(2\mu-c_2)}{\mu(2\mu-1)} - \frac{1-c_2}{2\mu-1}\mathbb{P}(\hat{X} < 2\mu).
\end{aligned} \tag{4.8}
$$

For simplicity in the notation, denote the final right part of Equation (4.8) as the following:

$$R\left(\mu, c_2, \mathbb{P}(\hat{X} < 2\mu)\right) := 1 - \frac{(\mu-1)(2\mu-c_2)}{\mu(2\mu-1)} - \frac{1-c_2}{2\mu-1}\mathbb{P}(\hat{X} < 2\mu).$$

Then notice that the following holds:

$$R\left(\mu, c_2, 1)\right) = \frac{c_2}{\mu}.$$

So if $\mathbb{P}(\hat{X} < 2\mu) = 1 \Leftrightarrow \mathbb{P}(\hat{X} = 2) = 0$ not holds, the strategy can not be optimal when following the same reasoning as in the proof of Theorem **??**. So the following is known about $\hat{F}$:

$$
\begin{aligned}
&\mathbb{P}(\hat{X} \leq c) + \mathbb{P}(\hat{X} < c) \leq \tfrac{c}{\mu} \quad , 0 \leq c < 2\mu \\
&\hat{F}(2\mu) = 1.
\end{aligned} \tag{4.9}
$$

The derived inequality above is split in two possibilities:

$$2\hat{F}(c) \leq \tfrac{c}{\mu} \quad \text{or} \quad 2\hat{F}(c) > \tfrac{c}{\mu}.$$

Consider the second option. This implies that $\mathbb{P}(\hat{X} = c) > 0$ and $\mathbb{P}(\hat{X} = c) > 0$ must hold to keep satisfying Inequality 4.7 and 4.8, respectively. Define $A = \{0 \leq a < 2\mu : \mathbb{P}(\hat{X} = a) > 0\}$ the set of atoms excluding the possibility of there being an atom in 2μ. From Lemma 3.2 it is known that this is a countable set. Choose $a \in A$ arbitrarily. Then it must satisfy the following $\hat{F}(a) > \frac{a}{2}$. Let $a' > a$ such that $a' \notin A$ and $\hat{F}(a) > \frac{a'}{2\mu}$. This is possible by the countability of A. Then $\hat{F}(a') \geq \hat{F}(a)$ by the definition of a distribution and have that $\hat{F}(a') > \frac{a'}{2\mu}$. This is a contradiction. Since a' is not an atom, it satisfies $\hat{F}(a') \leq \frac{a'}{2a}$. This statement does not hold for $a = 2\mu$ as there is no $a' > 2\mu$ for which $\frac{a'}{2\mu} < \hat{F}(2\mu) = 1$. So A is an empty set and the following is known about $\hat{F}$:

$$
\begin{aligned}
&\hat{F}(c) \leq \tfrac{c}{2\mu} \quad , 0 \leq c < 2\mu \\
&\hat{F}(2\mu) = 1.
\end{aligned} \tag{4.10}
$$

In the sense of stochastic ordering, $\hat{F}(x) \leq F^*(x)$ thus holds for all x. To complete the proof of uniqueness the result from Shaked & Shantikumar (2007) [12], i.e. Lemma 3.3, is introduced. Both criteria of Lemma 3.3 are met for X^* and $\hat{X}$ as since $\hat{F}(x) \leq F^*(x) \Rightarrow \mathbb{P}(X^* > x) \leq \mathbb{P}(\hat{X} > x)$ for all x and meet the expectation criteria for $h(x) = x$. So this concludes that $\hat{F}$ and F^* have equal distribution and F^* is therefore unique. This concludes the proof of uniqueness for the optimal strategy of Player I. □

Remark 4.5. The proof of the uniqueness of the optimal strategy of Player II is preserved for the most general form of this game.

5

Ranking Games

The players of the Competitive Investor and the Asymmetric Investor Game are limited by the expected value of their strategy. This chapter extends those games in a more general form as the k-Player Ranking Game for $k \geq 2$. This game is from Alpern & Howard (2017) [1] and introduces a more general and multiplayer form of the two-person zero-sum game that has been covered so far. Utility functions are introduced to express the risk profile of the players. One needs to battle a risk-averse player differently than a risk-seeking player.

5.1 The 2-Player Ranking Game

The 2-Player Ranking Game does not just limit the requirements of the distributions of the players to the mean, but to a limit on the moment of a mapping of the respective random variable. Player I plays the random variable X with distribution F and Player II plays the random variable Y with distribution G. Also, let ϕ and ψ be two functions. Then instead of the players being restricted by the first moment of the distribution, much broader restrictions are allowed. In particular, constraints on the generalised moment:

$$\begin{aligned} &\mathbb{E}[\phi(X)] \leq 1 \text{ for Player I,} \\ &\mathbb{E}[\psi(Y)] \leq 1 \text{ for Player II.} \end{aligned} \tag{5.1}$$

The functions ϕ and ψ have limitations though. First, to keep the game relevant, ϕ and ψ are also functions on $[0, \infty)$. Otherwise, one can leverage high values with negative values and that causes the game to lose its purpose. The same holds for decreasing functions and constant functions. Therefore, ϕ and ψ are considered strictly increasing functions. Also, it has been proved that it can be optimal for players play 0. So ϕ and ψ also satisfy $\phi(0) = \psi(0) = 0$.

Remark 5.1. The best way to think about the functions ϕ and ψ is to consider them as utility functions. A utility function summarizes the preferences of a consumer in terms of how much utility he or she gets from consuming the goods in the utility function.

There are also conditions on ϕ and ψ with respect to each other. These are needed to ensure the existence of optimal strategies in the game. The conditions are given in the formal definition of the game described above, see Equation 5.2 in Game 3. The purpose of Equation 5.2 will become clear in the proof of the theorem about optimal strategies (Theorem 5.1).

Game 3 (The 2-Player Ranking Game)**.** Two players play a zero-sum game. Let ϕ and ψ be continuous strictly increasing functions on $[0, \infty)$ with $\phi(0) = \psi(0) = 0$, satisfying:

$$\begin{aligned} \lim_{z\to\infty} \frac{1}{\psi(z)} \int_0^z \phi(x) d\psi(x) \geq 1, \\ \lim_{z\to\infty} \frac{1}{\phi(z)} \int_0^z \psi(x) d\phi(x) \geq 1. \end{aligned} \tag{5.2}$$

Player I plays X with distribution F and Player II plays Y with distribution G, such that $\mathbb{E}[\phi(X)] \leq 1$ and $\mathbb{E}[\psi(Y)] \leq 1$. Then the players take a random sample from their distribution, x and y respectively. The payoff Π is 1 if $x > y$, 0 if $x = y$ and -1 if $x < y$ with repsect to Player I.

The weighted average condition of ϕ and ψ with respect to each other (Equation 5.2) seem quite tough, but that is actually not the case. Namely, if ϕ and ψ are unbounded, meaning that $\phi(x), \psi(x) \to \infty$ if $x \to \infty$, the condition is automatically satisfied [1]. So only for bounded (strictly increasing) ϕ and ψ, the condition does not hold. Examples of that kind of functions are $f(x) = \arctan(x)$ and $f(x) = 1 - a^x$, $0 < a < 1$ for $x \geq 0$. Both f have a positive derivative on every $x \geq 0$, but are bounded by $\frac{1}{2}\pi$ and 1, respectively.

A usable form for ϕ and ψ would be as a 1-term polynomial, as every moment constraint can be put in that form. See Corollary 1 to see that this indeed always results in a qualified function for ϕ or ψ. Such functions for ϕ and ψ are used in Example 5.1.

Corollary 1. *Consider the setup of Game 3 and let* $\phi(x) = ax^n$ *and* $\psi(x) = bx^m$ *with* $a, b, n, m > 0$. *Then* ϕ *and* ψ *are well-defined (utility) functions.*

Proof. Clearly, the following holds:

$$\begin{aligned} &\phi(x) \in [0, \infty), \\ &\psi(x) \in [0, \infty), \\ &\phi(0) = \psi(0) = 0. \end{aligned}$$

Last to check is Equation 5.2:

$$\begin{aligned} \lim_{z\to\infty} \frac{1}{\psi(z)} \int_0^z \phi(x) d\psi(x) &= \lim_{z\to\infty} \frac{1}{bz^m} \int_0^z ax^n d(bx^m) = \lim_{z\to\infty} \frac{1}{bz^m} \int_0^z ax^n bmx^{m-1} dx \\ &= \lim_{z\to\infty} \frac{1}{bz^m} \int_0^z abmx^{n+m-1} dx = \lim_{z\to\infty} \frac{1}{bz^m} \left[\frac{abm}{n+m} x^{n+m}\right]_0^z \\ &= \lim_{z\to\infty} \frac{am}{n+m} z^n \to \infty \geq 1. \end{aligned}$$

The proof of the second limit is equivalent to that of the above as a and b are interchangeable, as well as are n and m:

$$\lim_{z\to\infty} \frac{1}{\phi(z)} \int_0^z \psi(x) d\phi(x) = \lim_{z\to\infty} \frac{bn}{n+m} z^m \to \infty \geq 1$$

This concludes the proof. □

Example 5.1. Consider the situation where the players are restricted by the second and third moment of their strategy, respectively. Player I plays the random variable X with distribution F

and Player II plays the random variable Y with distribution G. Player I is limited in his second moment by 3 and Player II is limited in his third moment by 5, i.e.:

$$\phi(X) = \frac{1}{3}X^2 \quad \text{such that} \quad E[\phi(X)] \leq 1 \Leftrightarrow \mathbb{E}[\tfrac{1}{3}X^2] \leq 1 \Leftrightarrow \mathbb{E}[X^2] \leq 3,$$
$$\psi(Y) = \frac{1}{5}Y^3 \quad \text{such that} \quad E[\psi(Y)] \leq 1 \Leftrightarrow \mathbb{E}[\tfrac{1}{5}Y^3] \leq 1 \Leftrightarrow \mathbb{E}[Y^3] \leq 5.$$

Using Corollary 1, it is known that ϕ and ψ satisfy the conditions of the game.

Intuitively speaking, it is probably not the best to play a distribution that includes atoms (excluding an atom at zero). This was not optimal at the previous games (the Competitive Investor Game and the Asymmetric Investor Game) and will likely still hold for this game.

In addition to playing as high as possible without compensating too mcuh, the players also need to take into account the opponent's utility function and adjust for that. Alpern & Howard (2017) [1] propose to incorporate the other player's utility function in the optimal strategy. See Theorem 5.1 for the optimal strategies of Player I and II.

Theorem 5.1. *Consider the functions ϕ and ψ from Game 3. Let b be the unique solution to the equation:*

$$\frac{1}{\psi(b)} \int_0^b \phi(x) d\psi(x) = 1 \tag{5.3}$$

and suppose without lose of generality that:

$$\frac{1}{\phi(b)} \int_0^b \psi(x) d\phi(x) \geq 1. \tag{5.4}$$

Then $F^*(x)$ and $G^*(y)$ on $[0, b]$ are optimal strategies for player Player I and Player II, respectively:

$$F^*(x) = \frac{\psi(x)}{\psi(b)},$$
$$G^*(y) = 1 - \frac{\phi(b) - \phi(y)}{\psi(b)(\phi(b) - 1)}.$$

The value of the game is $1 - \frac{2}{\psi(b)}$.

Proof. First, it is proved that b exists and is unique. Then, F^* and G^* are validated as distributions and also as strategies satisfying the requirements of the game. At last, it is showed that F^* and G^* agree in the minimum expected payoff they give to Player I and II, respectively.

Define for the (strictly increasing) functions ϕ and ψ the weighted average of $\phi(x)$ with respect to $\psi(x)$ for $0 \leq x \leq z$:

$$\kappa(z) = \frac{1}{\psi(z)} \int_0^z \phi(x) d\psi(x).$$

Then $\kappa(z) < \phi(z)$ holds and that also implies $\kappa(z)$ goes towards zero if z goes towards zero (from above) as $\phi(0) = 0$:

$$\begin{aligned} \kappa(z) &= \frac{1}{\psi(z)} \int_0^z \phi(x) d\psi(x) \\ &< \frac{1}{\psi(z)} \int_0^z \phi(z) d\psi(x) = \frac{\phi(z)}{\psi(z)} \int_0^z d\psi(x) = \frac{\phi(z)}{\psi(z)}(\psi(z) - \psi(0) = \phi(z). \end{aligned}$$

As ϕ and ψ represent utility functions, it is assumed allowed that to believe that ϕ and ψ are also continuously differentiable. Then $\kappa(z)$ is also a strictly increasing function as its derivative is positive:

$$\begin{aligned}\frac{d\kappa(z)}{dz} > 0 \Rightarrow & -\frac{\psi'(z)}{\psi(z)^2}\int_0^z \phi(x)d\psi(x) + \frac{\phi(z)\psi'(z)}{\psi(z)} > 0\\ &\Rightarrow \frac{\psi'(z)}{\psi(z)^2}\int_0^z \phi(x)d\psi(x) < \frac{\phi(z)\psi'(z)}{\psi(z)}\\ &\Rightarrow \int_0^z \phi(x)d\psi(x) < \phi(z)\psi(z),\end{aligned}$$

which holds since both the following integrals are bigger than 0 and as such derive the inequality:

$$\begin{aligned}&\int_0^z \phi(x)d\psi(x) + \int_0^z \psi(x)d\phi(x) = \phi(z)\psi(z)\\ \Rightarrow &\int_0^z \phi(x)d\psi(x) < \phi(z)\psi(z).\end{aligned}$$

So, $\kappa(z)$ is a strictly increasing function for $z \geq 0$ starting at $\kappa(0) = 0$ and Equation 5.2 ensures that $\lim_{z\to\infty} \kappa(z) \geq 1$. Hence there exits a unique solution for b (in Equation 5.3).

The conclusion above directly implies that F^* on $[0, b]$ is a well-defined distribution:

$$\begin{aligned}&F^*(x) \geq 0 \text{ for } 0 \leq x \leq b,\\ &F^*(0) = 0,\\ &F^*(b) = 1.\end{aligned}$$

Equation 5.3 and 5.4 imply that $\phi(b) > 1$ and $\psi(b) > 1$ must hold. Otherwise, the following is contradictory:

$$\begin{aligned}1 &= \frac{1}{\psi(b)}\int_0^b \phi(x)d\psi(x) < \frac{1}{\psi(b)}\int_0^b \phi(b)d\psi(x) = \frac{\phi(b)}{\psi(b)}\int_0^b d\psi(x) = \phi(b)\\ 1 &\leq \frac{1}{\phi(b)}\int_0^b \psi(x)d\phi(x) < \frac{1}{\phi(b)}\int_0^b \psi(b)d\phi(x) = \frac{\psi(b)}{\phi(b)}\int_0^b d\phi(x) = \psi(b)\end{aligned} \tag{5.5}$$

Consider the defined distribution G^*:

$$\begin{aligned}G^*(y) &= 1 - \frac{\phi(b) - \phi(y)}{\psi(b)(\phi(b) - 1)} = 1 - \frac{\phi(b) - \phi(y)}{c_1}\\ &= 1 - \frac{\phi(b)}{c_1} + \frac{\phi(y)}{c_1} = c_2 + \frac{\phi(y)}{c_1} \text{ for } c_1, c_2 \in \mathbb{R}.\end{aligned}$$

Since $c_1 > 0$ (in fact $c_1 > 1$) by Equation 5.5, G^* is a constant plus an increasing function divided by a constant. Therefore, G^* is a strictly increasing function for $y \geq 0$ and well-defined, i.e. the denominator of the fraction is not zero. It also satisfies $G(b) = 1$. Yet to determine before calling G^* a distribution, is to verify that $G^*(\psi(0)) = G^*(0) \geq 0$. From integration by parts and Equation 5.3 and 5.4, respectively, it is found that:

$$\begin{aligned}
\phi(b)\psi(b) - \phi(0)\psi(0) &= \int_0^b \phi(x)d\psi(x) + \int_0^b \psi(x)d\phi(x) \\
\Rightarrow \phi(b)\psi(b) &= \psi(b)\left(\frac{1}{\psi(b)}\int_0^b \phi(x)d\psi(x)\right) + \int_0^b \psi(x)d\phi(x) \\
\Rightarrow \phi(b)\psi(b) - \psi(b) &= \int_0^b \psi(x)d\phi(x) \geq \phi(b).
\end{aligned}$$

This leads to:

$$G^(0) = 1 - \frac{\phi(b)}{\psi(b)(\phi(b)-1)} \geq 1 - \frac{\phi(b)}{\phi(b)} = 0.$$

So, G^* is also a well-defined distribution.

Define F and G to be arbitrarily strategies for Player I and II, respectively. Let X^* have distribution F^* and X have distribution F for Player I. Also, let Y^* have distribution G^* and Y have distribution G for Player II. Then F^* and G^* agree on the value of the game when they play against G and F, respectively:

$$\begin{aligned}
\Pi(F^*,G) &= \mathbb{P}(X^* > Y) - \mathbb{P}(X^* < Y) \\
&= 1 - \mathbb{P}(X^* \leq Y) - \mathbb{P}(X^* < Y) = 1 - 2\mathbb{P}(X^* \leq Y) \\
&= 1 - 2\int_0^\infty F^*(y)dG(y) = 1 - \frac{2}{\psi(b)}\int_0^\infty \psi(y)dG(y) = 1 - \frac{2}{\psi(b)}\mathbb{E}[\psi(Y)] \geq 1 - \frac{2}{\psi(b)} \\
\Pi(F,G^*) &= \mathbb{P}(X > Y^*) - \mathbb{P}(X^* < Y^*) \\
&= \mathbb{P}(Y^* < X) - (1 - \mathbb{P}(Y^* \leq X)) \leq -1 + 2\mathbb{P}(Y^* \leq X) \\
&= -1 + 2\int_0^\infty G^*(x)dF(x) = -1 + 2\int_0^\infty 1 - \frac{\phi(b)-\phi(x)}{\psi(b)(\phi(b)-1)}dF(x) \\
&= -1 + 2\left(1 - \frac{\phi(b)}{\psi(b)(\phi(b)-1)}\right)(F(\infty) - F(0)) + \frac{2}{\psi(b)(\phi(b)-1)}\int_0^\infty \phi(x)dF(x) \\
&\leq 1 - \frac{2\phi(b)}{\psi(b)(\phi(b)-1)} + \frac{2}{\psi(b)(\phi(b)-1))}\mathbb{E}[\phi(X)] \\
&\leq 1 - \frac{2(\phi(b)+1)}{\psi(b)(\phi(b)-1)} \leq 1 - \frac{2}{\psi(b)}.
\end{aligned}$$

This concludes the proof. □

Remark 5.2. The weighted average conditions in the limit for the utility functions ϕ and ψ (Equation 5.2) are considered part of the game. One can also decide to lose this restriction and use it as an additional criteria for Theorem 5.1. In that way, the game becomes more intuitive without losing the optimal strategies for the players.

Theorem 5.1 shows the optimal strategy for the players for the 2-Player Ranking Game (with generalised moment constraints). So, the optimal strategies for the previous games (the Competitive Investor and Asymmetric Investor Game) should be able to be derived from this generalised game as well. See Corollary 2 for the derivation of the optimal strategies of The Asymmetric Investor Game.

Corollary 2. *Player I plays X with distribution F and Player II plays Y with distribution G. Without lose of generality, let $\mu > 1$ such that $\mathbb{E}[X] \leq \mu$ and $\mathbb{E}[Y] \leq 1$. To express the Asymmetric Investor game in terms of the 2-Player Ranking game, define the functions ϕ and ψ as follows:*

$$\phi(x) = \frac{x}{\mu},$$
$$\psi(x) = x$$

Proof. Clearly, from Corollary 1 it is known that ϕ and ψ satisfy all the requirements of the game with respect to those functions. Next to find is the unique solution for b:

$$\frac{1}{\psi(b)}\int_0^b \phi(x)d\phi(x) = 1 \Leftrightarrow \frac{1}{b}\int_0^b \frac{x}{\mu}dx = 1 \Leftrightarrow \int_0^b xdx = \mu b$$
$$\Leftrightarrow \frac{1}{2}b^2 = \mu b \Leftrightarrow b = 2\mu.$$

It is assumed without lose of generality that $\mu > 1$. Therefore the final property equation is satisfied as well:

$$\frac{1}{\phi(b)}\int_0^b \psi(x)d\phi(x) \geq 1 \Leftrightarrow \frac{1}{2}\int_0^{2\mu} \frac{x}{\mu}dx \geq 1 \Leftrightarrow \frac{1}{2\mu}\frac{(2\mu)^2}{2} \geq 1$$
$$\Leftrightarrow \mu \geq 1.$$

If this was not the case, the labels of players should have been switched and that would solve the issue. So, according to Theorem 5.1, Player I and II play on $[0, 2\mu]$ with the following distributions:

$$F^*(x) = \frac{\psi(x)}{\phi(b)} = \frac{x}{2\mu},$$
$$G^*(y) = 1 - \frac{\phi(b) - \phi(y)}{\psi(b)(\phi(b) - 1)} = 1 - \frac{\frac{2\mu}{\mu} - \frac{y}{\mu}}{2\mu(\frac{2\mu}{\mu} - 1)} = 1 - \frac{2 - \frac{y}{\mu}}{2\mu}$$
$$= 1 - \frac{1}{\mu} - \frac{y}{2\mu^2} = \frac{\mu - 1}{\mu} + \frac{y}{2\mu^2}.$$

The value of the game is then:

$$V = 1 - \frac{2}{\phi(b)} = 1 - \frac{2}{2\mu} = \frac{\mu - 1}{\mu}$$

This coincides 100% with what has been proved for the Asymmetric Investor Game in the previous chapter. □

Remark 5.3. For $\mu = 1$ in Corollary 2, the game directly simplifies to the Competitive Investor Game.

5.1.1 Geometry & Uniqueness

Next to determine is the uniqueness of the optimal strategies. To do this, there is a need for additional lemmas. The method of Alpern & Howard (2017) [1] to prove uniqueness is somewhat different compared to the that of Bell & Cover (1980) [3]. The use of testing against two-point distributions will still be used, but has to be extended to a geometric interpretation. See Definition 5.1 and 5.2 for two geometric principles that are needed in the proof. See Figure 5.1 for an example of Definition 5.2.

Definition 5.1. *A function f is said to be concave, if the following holds for any x and y and for any $\alpha \in [0,1]$:*

$$f(\alpha x + (1-\alpha)y) \geq \alpha f(x) + (1-\alpha) f(y).$$

Definition 5.2. *The concavification of a function f is defined as the minimum of all concave functions that are greater than or equal to f.*

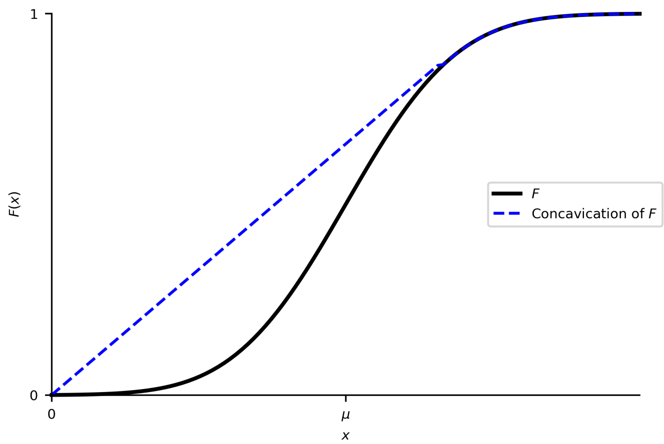


Figure 5.1 – The concavification of a distribution F.

Another lemma is needed about the mixture of two-point strategies. So far, testing against a single two-point strategy was enough to proof uniqueness for the optimal strategies in the Competitive Investor and the Asymmetric Investor Game. For this more general approach, this will not be sufficient anymore. However, Lemma 5.1 from Pinelis (2009) [11] provides a strong basis to cover all strategies.

Lemma 5.1. *Any non-negative distribution with mean μ can be expressed as a mixture of two-point distributions with mean μ and so, any mixture of pure strategies is also a mixture of two-point distributions.*

The material discussed above will be used in the first step to prove uniqueness. That is to derive a limit for the winning probability of Player II, see Lemma 5.2. It covers a very general case where Player can play any arbitrary strategy.

Lemma 5.2. *Player II plays Y with non-negative distribution G such that $\mathbb{E}[\psi(Y)] \leq \eta$ where $\psi(y)$ is a strictly increasing continuous function on $[0,\infty)$ such that $\psi(0) < \eta < \psi(\infty)$.*

Player I plays X' with distribution F' and define $F(x) = \mathbb{P}(\chi(X) \leq x)$ for the interval $[\psi(0), \psi(\infty))$.

Now let $\bar{F}$ be the concavification of F on $[\psi(0), \psi(\infty))$. Then Player II his probability of not losing can not be more than $\bar{F}(\eta)$, but Player II can either achieve this of get arbitrarily close.

Proof. Consider a two-point distribution for Player II. One atom gets placed on $u = \psi(c) < \eta$ and the other atom gets placed on d such that $\eta < \psi(d) = v$. To meet the expectation criteria, the maximum probability for Player II to assign to d is $\frac{\eta-u}{v-u}$. The probability that Player II then wins or draws against F' is the following:

$$\begin{aligned}&\mathbb{P}(Y=d)F'(d)+\mathbb{P}(Y=c)F'(c)\\=&\mathbb{P}(Y=d)F(u)+\mathbb{P}(Y=c)F(v)\\=&\frac{v-\eta}{v-u}F(u)+\frac{\eta-u}{v-u}F(v).\end{aligned} \tag{5.6}$$

Consider the graph of F against ψ, denote the points $(u, F(u))$ and $(v, F(v))$ and set a line between those two points. Because of the (linear) proportions, the height of the line at η is equal to the probability of Player II not losing, i.e. the height is equal to probability in Equation 5.6. So this height is the upper bound for the payoff of Player II when playing a two-point strategy at u and v. That means that $\bar{F}(\eta)$ is then the upper bound to what Player II can achieve by playing an optimal two-point strategy.

Lemma 5.1 states that any distribution can be expressed with two-point distributions. Therefore, $\bar{F}(\eta)$ is also the upper bound to what Player II can achieve playing any pure or mixed strategy.

Next to show is that Player II can also achieve this upper bound or get arbitrarily close. Consider a (straight) line L that does not get below $\bar{F}$ on $[\psi(0), \psi(\infty)$ and touches $\bar{F}$ at η. See Figure 5.2 for an example[9].

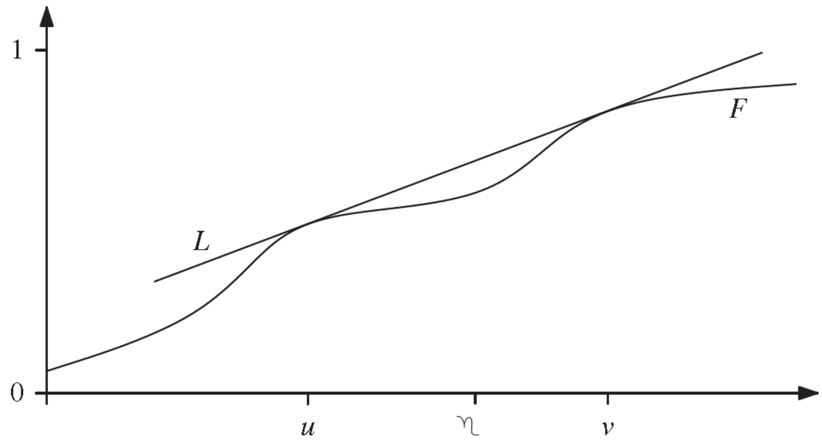


Figure 5.2 – An illustration of how the line L touches a distribution F, does not touch F at η and does not go below the concavication of F.

Then there are two possibilities:

- L also touches F at η. Then Player II can place an atom of probability 1 at η and either win or draw with probability $F(\eta) = \bar{F}(\eta)$. Player II then only draws if Player I also placed an atom at η. However, by placing a atom of probability $\frac{\eta-\psi(0)}{\eta-\psi(0)+\epsilon}$ at $\eta+\epsilon$ and the remaining probability at $\psi(0)$, Player II can get arbitrarily close to achieving $\bar{F}(\eta)$ by choosing ϵ arbitrarily small. So Player II can achieve $\bar{F}(\eta)$ if Player I did not place an atom at η or get arbitrarily close if Player I did place an atom at η.
- L does not touch F at η (as in Figure 5.2). By the properties of the distribution and concavification function, L must then touch F below and above η at at least one point. Let $u' < \eta$ and $v' > \eta$ be such two two touching points. By placing atoms at those points, Player II can achieve η provided that Player I has not placed atoms at u, v or

[9]This figure is partly adopted from Alpern & Howard (2017) [1]

both. However, by then shifting those atom to $u' + \epsilon$ and/or $v' + \epsilon$, respectively, for arbitrarily small ϵ, Player II can get arbitrarily close to achieving $\bar{F}(\eta)$. So Player II can achieve $\bar{F}(\eta)$ if Player I did not place atoms at u and v or get arbitrarily close if Player I did place an atom at one of those points.

This concludes that Player II his upper bound for the probability of not losing is $\bar{F}(\eta)$ and can also obtain this or get arbitrarily close. □

Remark 5.4. While the proof provides enough evidence to believe the theorem, it is noted that it can be tricky to believe the statements about the line L. While intuition supports the claims about L touching F or not at certain points, one can consider this as a not fully complete proof.

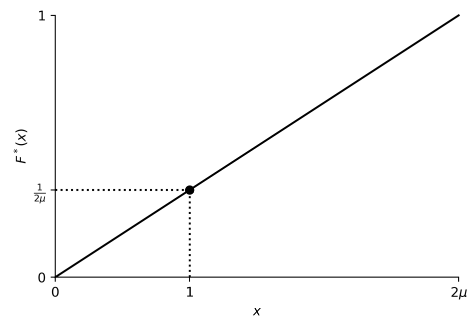
Using Lemma 5.2, the uniqueness of the optimal strategies of the 2-Player Ranking game can be proved.

Theorem 5.2. *The optimal strategies of the 2-Player Ranking Game from Theorem 5.1 are unique.*

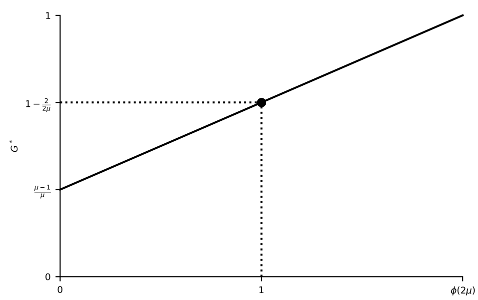
Proof. Let $\hat{F}$ be an arbitrary optimal strategy for Player I, i.e. a distribution such that $\Pi(\hat{F}, G) \geq \frac{a-1}{a}$ against any strategy G of Player II.

First, remark that $\Pi(\hat{F}, G^*) = \frac{a-1}{a}$ must hold and consider the inequality in Equation 4.6 and the final inequality of that proof. If $\hat{X}$ takes values larger than 2μ or $\mathbb{P}(X = 0) > 0$, the inequality changes into a strict inequality. In order to remain optimal, $\hat{F}$ must therefore have that $\hat{F}(2\mu) = 1$ and $\hat{F}(0) = \mathbb{P}(\hat{X} = 0) = 0$.

See Figure 5.3 for the parametric plot of the already found optimal strategies for Player I and II, respectively. Those plots will be used to deduce the uniqueness of the optimal strategies.



(a) Parametric plot of $(\psi(x), F^*(x))$



(b) Parametric plot of $(\phi(y), G^*(y))$

Figure 5.3 – The parametric plots of the optimal strategies for the 2-Player Ranking Game.

Notice that the concavification of F^* is F^* itself and the (concavification) function passes through $(1, F^*(1)) = (1, \frac{1}{2\mu})$. Assume that $\hat{F}(x) > F^*(x)$ for some $x \in [0, 2\mu]$. Then the concavification of $\hat{F}$ will pass above the point $(1, \frac{1}{2\mu})$ and Player II can thus achieve a higher probability of winning by Lemma 5.2 against this strategy. However, as $\hat{F}$ is also optimal, this should not be possible. This contradiction leads to the observation that $\hat{F}(x) \leq F^*(x)$ for all $x \in [0, 2\mu]$.

Then it only rests to introduce the result from Shaked and Shantikumar (2007) [12], i.e. Lemma 3.3, again. Both criteria of Lemma 3.3 are satisfied for X^* and $\hat{X}$ since $\hat{F}(x) \leq F^*(x) \Rightarrow$

$\mathbb{P}(X^* > x) \leq \mathbb{P}(\hat{X} > x)$ for all x and the expectation criteria is met for $h(x) = x$. So that concludes that $\hat{F}$ and F^* have equal distribution and F^* is therefore unique. This concludes the proof of uniqueness for the optimal strategy of Player I.

Next to prove is the uniqueness of the strategy of Player II. Let $\hat{G}$ be an arbitrary optimal strategy for Player II. Remark that $\Pi(F^*, \hat{G}) = \frac{a-1}{a}$ must hold and consider the inequality in Equation 4.4. If $\hat{Y}$ takes values larger than 2μ, the inequality changes into a strict inequality. In order to remain optimal, $\hat{G}$ must therefore satisfy $\hat{G}(2\mu) = 1$.

Now consider Lemma 5.2 the other way around: Player II has the fixed strategy $\hat{G}$. If $\hat{G}(y) > G^*(y)$ for some $y \in [0, 1]$, then the concavification of $\hat{G}$ must pass above the point $\left(1, 1 - \frac{2}{2\mu}\right)$, allowing Player I to play better against $\hat{G}$ compared to G^* according to Lemma 5.2. That can not be possible, so $\hat{G}(y) \leq G^*(y)$ for $y \in [0, 1]$.

The atom at 0 of G^* allows for $\hat{G}$ to possibly have that $\hat{G}(y) > G^*(y)$ for some $y \in [1, \phi(2\mu))$ while having the concavification of $\hat{G}$ going through the point $\left(1, 1 - \frac{2}{2\mu}\right)$. Lemma 5.2 states that there must be a line L above or touching $\hat{G}$ such that L touches the concavification of $\hat{G}$ at $\left(1, 1 - \frac{2}{2\mu}\right)$. Let H be the distribution function corresponding to the line L, see the dashed line in Figure 5.4 for an example of H[10].

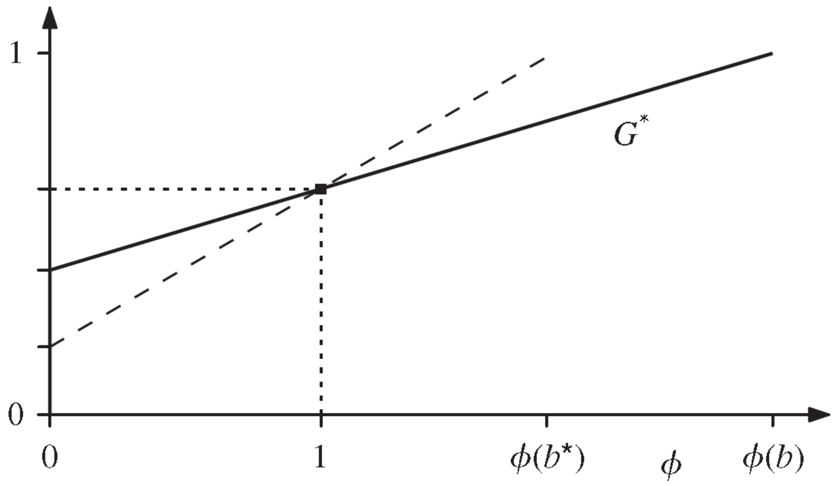


Figure 5.4

The distribution H then gives a lower bound to $\mathbb{E}[\psi(Y)]$, since $\hat{G}(x) \leq H(x)$ for all $x \geq 0$. So $\mathbb{E}_H[\psi(Y)] \leq \mathbb{E}_{\hat{G}}[\psi(Y)]$. Denote b^* for the unique value such that $b^* = min_{b>0} : H(\phi(b)) = 1$. Then the distribution H has the following characteristics:

$$
\begin{aligned}
H(x) &= 1 - \frac{\phi(b^*)}{\phi(b^*)\psi(b) - \psi(b)} + \frac{\phi(x)}{\phi(b^*)}\left(1 - \left(1 - \frac{\phi(b^*)}{\phi(b^*)\psi(b) - \psi(b)}\right)\right) \\
&= 1 - \frac{\phi(b^*)}{\phi(b^*)\psi(b) - \psi(b)} + \phi(x)\frac{1}{\phi(b^*)\psi(b) - \psi(b)} \\
dH(x) &= \frac{1}{\phi(b^*)\psi(b) - \psi(b)} d\phi(x)
\end{aligned}
$$

[10] This figure is partly adopted from Alpern & Howard (2017) [1]

The expected value of $\psi(Y)$ under the distribution H then is the following:

$$\mathbb{E}_H[\psi(Y)] = \int_0^{b^*} \psi(x) dH(x) = \frac{1}{\phi(b^*)\psi(b) - \phi(b)} \int_0^{b^*} \psi(x) d\phi(x).$$

Differentiating this with respect to $\phi(b^*)$ results in:

$$\begin{aligned}\frac{\partial \mathbb{E}_H[\psi(Y)]}{\partial \phi(b^*)} &= \frac{\psi(b)\psi(b^*)(\phi(b^*)-1) - \psi(b)\int_0^{b^*}\psi(x)d\phi(x)}{(\psi(b)(\phi(b^*)-1))^2} \\ &= \psi(b)\frac{\psi(b^*)(\phi(b^*)-1) - \int_0^{b^*}\psi(x)d\phi(x)}{(\psi(b)(\phi(b^*)-1))^2}. \end{aligned} \tag{5.7}$$

From the theorem, it is known that b is the unique solution to the following equation:

$$\frac{1}{\phi(b)} \int_0^{b} \psi(x) d\phi(x) \geq 1.$$

It has been shown that the left side of this equation is strictly increasing, so for $b^* < b$ it holds that:

$$\frac{1}{\phi(b^*)} \int_0^{b^*} \psi(x) d\phi(x) < 1.$$

Therefore it holds that the numerator of Equation 5.7 is negative for all $b^* < b$:

$$\begin{aligned}&\psi(b^*)(\phi(b^*)-1) < \int_0^{b^*} \psi(x) d\phi(x) \\ &\Rightarrow \psi(b^*)\phi(b^*) - \psi(b^*) < \phi(b^*)\psi(b^*) - \int_0^{b^*} \phi(x) d\psi(x) \\ &\Rightarrow \int_0^{b^*} \phi(x) d\psi(x) < \psi(b^*).\end{aligned}$$

So $\mathbb{E}_H[\psi(Y)]$ decreases as b^* increases until $b^* = b$. Let b^* increase till b such that $H = G^*$ and H thus has expectation 1. So for $b^* < b$, the distribution H is a lower bound for $\hat{G}$ that does not meet the expectation requirement. After all, $\mathbb{E}_H[\psi(Y)]$ decreases to 1 if b^* increases to b. The highest lower bound for the expectation is thus reached when $H = G^*$.

So to summarize. H serves as a lower bound for the expectation of $\psi(Y)$ compared to under $\hat{G}$ and preserves the optimality. However, $\psi(Y)$ under H only meets the expectation requirement of the theorem when $H = G^*$. This means that $\hat{G}(y) \leq G^*(y)$ for all $y \in [0, \phi(b))$. Following the equivalent reasoning as before with stochastic ordering, it proves that $\hat{G} = G^*$.

This concludes the proof of uniqueness for the optimal strategies of Player I and II. □

Remark 5.5. The proofs of Lemma 5.2 and Theorem 5.2 are the extension of the sketches provided by Alpern & Howard (2017) [1]. Additional lemmas, definitions and a more extensive display of the proof provides the proof in a more clear way.

5.2 Multiplayer Rankings

So far, only 2-player games have been discussed. Starting with the very specific Compeitive Investor Game and extend it to the 2-Player Ranking Game with generalised moment constraints. Following up on the 2-Player Ranking game is to allow more players. It is assumed that $k \geq 2$ players are equally competitive, have the same generalised moments constraint and aim to get the highest score. The higher a player scores, the higher he will be ranked. See Game 4 for the formal definition of the Symmetric Multiplayer Ranking Game.

Game 4 (The Symmetric Multiplayer Ranking Game). Let $\phi(x)$ be a continuous strictly increasing function on $[0,\infty)$ with $\phi(0) = 0$ and $\lim_{x\to\infty}\phi(x) \geq k$. Let $\mathcal{F}$ be the set of non-negative distributions F such that $\mathbb{E}[\phi(X)] \leq 1$ when $X \sim F \in \mathcal{F}$. In the k-player game $\Gamma = \Gamma(\mathcal{F},\ldots,\mathcal{F})$, each player i chooses a distribution $F_i \in \mathcal{F}$. The players take a random sample from their distribution and the player with highest number wins.

From the discussed 2-player games where the players are equally competitive, it is known that it is not optimal for one to place probability mass at a specific point, i.e. for one to have 1 or more atoms in his distribution. Intuitively, this will not change for a multiplayer game. When a optimal strategy exists, it will be accessible to all the players and thus all players should have equal probability of getting the highest score. In other words, there arises an equilibrium.

In a 2-player game, there is only 1 competitor to beat. In a k-player game, one needs to beat $k-1$ competitors. Therefore, one probably needs to get a significant higher score than before and compensate by playing low with a high probability. Theorem 5.3 proposes a strategy that sets the players in an equilibrium.

Theorem 5.3. *The Symmetric Multiplayer Ranking Game has an equilibrium when all players choose the distribution $F^* \in \mathcal{F}$ on $[0,\phi^{-1}(k)]$ with the following distribution function:*

$$F^*(x) = \sqrt[k-1]{\frac{\phi(x)}{k}}.$$

Proof. Clearly, the distribution function F^* is non-negative. Rewriting the distribution function finds us the function ϕ expressed in terms of our chosen F^*:

$$F^*(x) = \sqrt[k-1]{\frac{\phi(x)}{k}} \Rightarrow F^*(x)^{k-1} = \frac{\phi(x)}{k} \Rightarrow \phi(x) = kF^*(x)^{k-1}.$$

The expectation criteria is also satisfied by substituting $u = F^*(x)$:

$$\mathbb{E}_{F^*}[\phi(X)] = \int_{x=0}^{\phi^{-1}(k)} \phi(x)\; dF^*(x) = \int_{u=0}^{1} ku^{k-1}\; du = [u^k]_0^1 = 1.$$

So, F^* is an element of the strategy space. Next to show is that there exists an equilibrium. Let the first $k-1$ players choose F^*. Then the distribution function for the maximum score is the distribution F^* to the power $k-1$ by independency:

$$\mathbb{P}(\max(X_1,\ldots,X_{k-1}) \leq x) = \mathbb{P}(X_1 \leq x)\cdots\mathbb{P}(X_{k-1} \leq x) = [F^*(x)]^{k-1} = \frac{\phi(x)}{k}.$$

If the last player, player k, draws x_k, his chance of not losing is then $\min\left(\frac{\phi(x_k)}{k},1\right)$. In advance, his chance of winning will therefore be:

$$\begin{aligned}\mathbb{P}(\max(X_1,\ldots,X_{k-1}) < X_k) &\leq \mathbb{P}(\max(X_1,\ldots,X_{k-1}) \leq X_k)\\ &= \mathbb{E}_{F^*}\left[\min\left(\frac{\phi(X_k)}{k},1\right)\right] \leq \mathbb{E}_{F^*}\left[\frac{\phi(X_k)}{k}\right] \leq \frac{1}{k}.\end{aligned}$$

Player k can achieve this upper bound by choosing a continuous distribution on $[0,\phi^{-1}(k)]$ such that $\mathbb{E}[\phi(X_k)] = 1$. F^* is such a distribution. So, there exists an equilibrium where all the players have the same strategy. □

Remark 5.6. Consider the Symmetric Multiplayer Game Ranking with $\phi(x) = x$ and $k = 2$. Then this game reduces to the Competitive Investors Game. The optimal strategy is then indeed $F^*(x) = \sqrt[k-1]{\frac{\phi(x)}{k}} = \frac{x}{2}$ for $0 \leq x \leq 2$. This coincides with the previously found optimal strategy for the Competitive Investor Game.

5.2.1 Avoiding Elimination

Many competitions are designed for the players to achieve the highest score compared to the rest. Other competitions require that one does not have the lowest score in order to continue to next round, i.e. the player with the lowest score gets eliminated.

This changes the dynamics of the game. When one does not necessarily need to win, a less risky approach seems the better choice. After all, you only need to beat one player instead of $k-1$. See Game 5 for the formal definition of the game described above and Theorem 5.4 for the equilibrium strategy.

Game 5 (The Multiplayer Elimination Game)**.** Let $\phi(x)$ be a continuous strictly increasing function on $[0,\infty)$ with $\phi(0)=0$ and $\lim_{x\to\infty}\phi(x)\geq\frac{k}{k-1}$. In the k-player game $\Gamma=\Gamma(\mathcal{F},\dots,\mathcal{F})$, each player i chooses a distribution $F_i\in\mathcal{F}$. The players take a random sample from their distribution and the player with lowest score loses (or gets eliminated).

Theorem 5.4. *The Multiplayer Elimination Game reaches an equilibrium when all* k *players choose the distribution* $F^*\in\mathcal{F}$ *on* $\left[0,\phi^{-1}\left(\frac{k}{k-1}\right)\right]$ *with the following distribution function:*

$$F^*(x)=1-\sqrt[k-1]{1-\frac{k-1}{k}\phi(x)}.$$

Proof. The first thing to be checked is that F^* indeed satisfies the properties of a distribution:

$$F^*(0)=1-\sqrt[k-1]{1-\frac{(k-1)\phi(0)}{k}}=1-\sqrt[k-1]{1}=0$$

$$F^*\left(\phi^{-1}\left(\frac{k}{k-1}\right)\right)=1-\sqrt[k-1]{1-\frac{(k-1)\phi\left((\phi^{-1}\left(\frac{k}{k-1}\right)\right)}{k}}=1-\sqrt[k-1]{0}=1.$$

In order for F^* to be an increasing function, it must hold that $\sqrt[k-1]{1-\frac{k-1}{k}\phi(x)}$ is a decreasing function. As ϕ is a strictly increasing function, it holds that $1-\frac{k-1}{k}\phi(x)$ is a strictly decreasing function on $\left[0,\phi^{-1}\left(\frac{k}{k-1}\right)\right]$. Therefore, $\sqrt[k-1]{1-\frac{k-1}{k}\phi(x)}$ is indeed a decreasing function and F^* it thus a valid (continuous) distribution.

Rewrite the F^* to isolate ϕ in terms of F^* to then determine the expected value of $\phi(X)$ when X has distribution F^*:

$$\begin{aligned}
&F^*(x)=1-\sqrt[k-1]{1-\frac{(k-1)\phi(x)}{k}}\\
&\Rightarrow(1-F^*(x))^{k-1}=1-\frac{(k-1)\phi(x)}{k}\\
&\Rightarrow\phi(x)=\frac{k\left(1-(1-F_2(x))^{k-1}\right)}{k-1}=\frac{k}{k-1}-\frac{k}{k-1}\left(1-F_2(x)^{k-1}\right),
\end{aligned}$$

such that:

$$
\begin{aligned}
\mathbb{E}[\phi(X)] &= \int_{x=0}^{\phi^{-1}\left(\frac{k}{k-1}\right)} \phi(x)\ dG(x) \\
&= \int_{x=0}^{\phi^{-1}\left(\frac{k}{k-1}\right)} \frac{k}{k-1} - \frac{k}{k-1}\left(1 - F_2(x)^{k-1}\right)\ dG(x) \\
&= \int_{u=0}^{1} \frac{k}{k-1} - \frac{k}{k-1}\left(1 - u^{k-1}\right)\ du \\
&= \frac{k}{k-1}\int_{u=0}^{1} 1\ du - \frac{k}{k-1}\int_{0}^{1}(1-u)^{k-1}du \\
&= \frac{k}{k-1}\cdot 1 - \frac{k}{k-1}\left[-\frac{1}{k}(1-u)^k\right]_{u=0}^{1} \\
&= \frac{k}{k-1} - \frac{k}{k-1}\left(0 - \left(-\frac{1}{k}\right)\right) \\
&= \frac{k}{k-1} - \frac{1}{k-1} \\
&= 1.
\end{aligned}
$$

The distribution F^* thus satisfies the requirements of the game. Let $k-1$ players play F^* represented by the (independent) random variables $X_1, \dots, X_{k-1}$. Then the distribution of the minimum score of the first $k-1$ players is as follows:

$$
\begin{aligned}
\mathbb{P}(\min(X_1,\dots,X_{k-1}) \le x) &= 1 - \mathbb{P}(\min(X_1,\dots,X_{k-1}) > x) \\
&= 1 - (\mathbb{P}(X_1 > x)\cdots\mathbb{P}(X_{k-1} > x)) \\
&= 1 - \mathbb{P}(X_1 > x)^{k-1} \\
&= 1 - (1 - F_2(x))^{k-1} \\
&= 1 - \left(\sqrt[k-1]{1 - \frac{(k-1)}{k}\phi(x)}\right)^{k-1} \\
&= \frac{k-1}{k}\phi(x).
\end{aligned}
$$

If Player k plays X_k with an arbitrary distribution $F \in \mathcal{F}$, then the probability of not getting eliminated for Player k has the following upperbound:

$$
\begin{aligned}
\mathbb{P}(\min(X_1,\dots,X_{k-1}) < X_k) &\le \mathbb{P}(\min(X_1,\dots,X_{k-1}) \le X_k) \\
&= \mathbb{E}\left[\min\left(\frac{k-1}{k}\phi(X_k), 1\right)\right] \\
&\le \mathbb{E}\left[\frac{k-1}{k}\phi(X_k)\right] \\
&\le \frac{k-1}{k}.
\end{aligned}
\tag{5.8}
$$

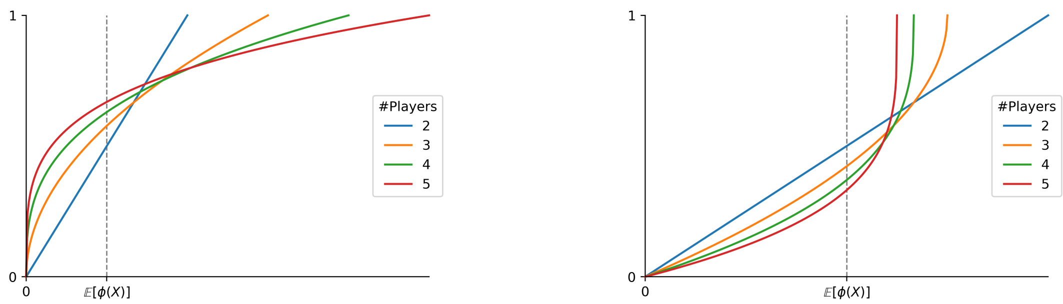
In order to achieve this bound, Player k must choose a continuous distribution on $\left[0, \phi^{-1}\left(\frac{k}{k-1}\right)\right]$ such that $\mathbb{E}[\phi(Y)] = 1$ under that distribution. The requirements of choosing a continuous distribution, the specific interval and the equality of the expected value can all be deduced from the 3 inequality signs in Equation 5.8, respectively. If not, the 3 inequalities change into strict inequalities and this has the consequence that the other competitiors are in favor.

As F^* is a distribution that satisfies all the necessary requirements, Player k can play this distribution as well and this ensures the equilibrium. □

This concludes the theory that is available on multiplayer ranking games. The 2-Player Ranking Game is completely analysed in a very general setting with known unique optimal strategies. Already, it is clear the amount of effort it takes to proof all of this. For the ranking game with $k > 2$ players, the current perception is that the equilibrium strategies for winning or not-losing the Symmetric Multiplayer Ranking Game is the only theory available.

5.3 Winning vs. Not-Losing

The difference and the change in the equilibrium strategies to win and not to lose for various players becomes much clearer when one draws them next to each other. Therefore, see Figure 5.5 for the equilibrium distribution when 2 to 5 players want to win and not lose.



(a) The equilibrium distribution when all players want to win.

(b) The equilibrium distribution when all players do not want to lose.

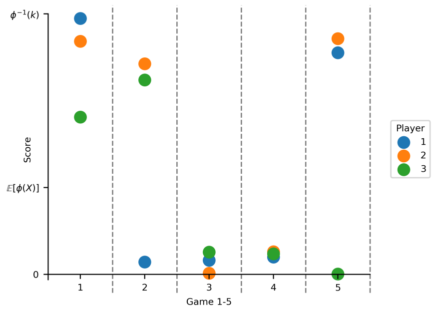
Figure 5.5 – The equilibrium distribution of the multiplayer ranking game with 2-5 players. (For this specific figure it holds that $\phi(X) = X$. Another function of ϕ changes the speed in how the distribution changes.)

The change in the equilibrium distribution is clear when the number of players in the game increase. The more players, the more leverage is used in the low scores in order for getting a higher probability of a higher score if the players aim to beat the others. When the players do not want to lose, the opposite holds. Then the players become more conservative when the number of players increase. After all, there is a high probability that another player sets a relative low score and since that is the only one you have to beat, there is no need to for higher risk than necessary. See Figure 5.6 to see the results of such an equilibrium strategies for a 3-player game.

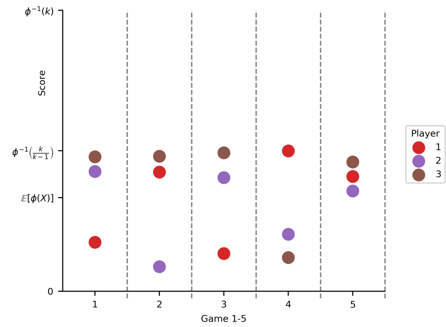
5.4 Distribution of the Maximum

Consider the equilibrium strategy of the Symmetric Multiplayer Ranking Game for k players, i.e.:

$$F^*(x) = \sqrt[k-1]{\frac{\phi(x)}{k}} \text{ for } x \in [0, \phi^{-1}(k)].$$



(a) Playing the equilibrium strategy to win.



(b) Playing the equilibrium strategy to not lose.

Figure 5.6 – Five games of the 3-player ranking game in which the players play the equilibrium strategy to win (a) and not to lose (b). (For this specific figure it holds that $\phi(X) = X$.)

Then the distribution of the maximum score of the k players is as follows:

$$\mathbb{P}(X_1 \leq x, \ \dots, \ X_k \leq x) = \mathbb{P}(X_1 \leq x)^k = F^*(x)^k = \left(\frac{\phi(x)}{k}\right)^{\frac{k}{k-1}}.$$

The distribution of the maximum thus approaches (for large k) the uniform distribution over the same score interval as the strategy. This is an important observation that will be used in the following chapter.

6

Hedge Fund Dynamics

The setup of the multiplayer ranking games needs to be translated to the hedge fund landscape in order to analyse the hedge funds in a ranking setting. Of course, the hedge fund landscape is too complicated to capture in a game. However a similar approach for an optimal strategy can be expected for hedge funds: an optimal strategy does not perform the best against any other strategy, but does provide the highest minimum ranking against all other peers

By only looking at hedge funds that are in the same specific category, the strategy set is assumed equal for the hedge funds and they are indeed in competition with respect to each other. After all, the hedge funds have access to the similar bucket of assets to invest in. So from that point of view, there is no distinction in the way the hedge funds can play versus the players of a ranking game.

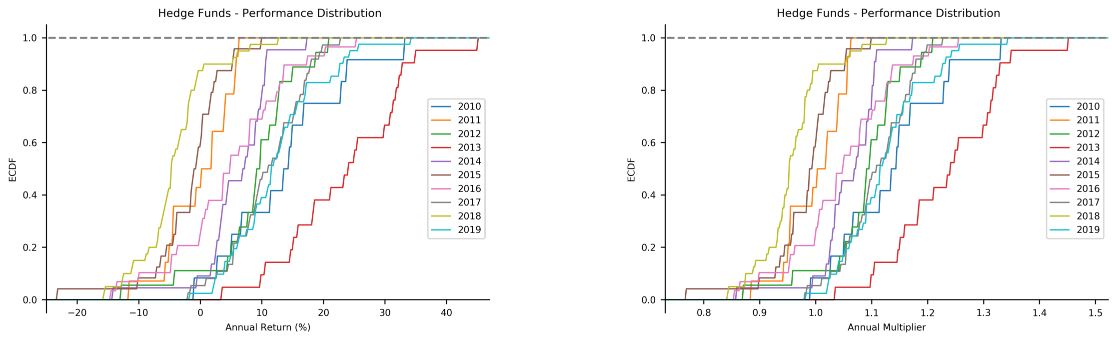
In the multiplayer ranking game, the players are limited by the utility on their strategy, i.e. $\mathbb{E}[\phi(X)] \leq 1$. This is to prevent trivial and infinite strategies and therefore to be able to construct optimal strategies as well. Also, the limit on the expectation actually represents the competitive level of the the players. In practice, hedge funds do not play a strategy that is planned around a limit on forehand. The average of the data realised from the strategy is metric that can be used as proxy for the competitive level of a hedge fund, for example. However, any measure for the competitive level of the hedge fund will probably show that they are never fully equal in practice (or otherwise can be considered as a major coincidence). For now, let's consider this assumption to be true. So all hedge funds are considered equally competitive. Also the assumption that all hedge funds share the same utility function over their performance is a heavy statement and does not relate to practice.

6.1 Annual Performance of the Hedge Funds

It can then be checked how the hedge funds played the game of trying to get the highest return. Denote $r^{i,j}$ for the annual return in % of hedge fund i in year j and $m^{i,j}$ for the annual multiplication factor over the invested capital of hedge fund i in year j, i.e.:

$$m^{i,j} = 1 + r^{i,j}$$

The annual return of a hedge fund is thus considered the score in the game where the hedge funds want to beat each other in terms of performance. The random variable that represent the score of the hedge fund is based on the risk that the hedge fund takes. See Figure 6.1 for the distribution of the returns in those two forms: as annual returns in terms of percentages and as multipliers over the capital invested in Figure 6.1a and 6.1b, respectively.



(a) The empirical distribution of $r^{\cdot,j}$ realised by the hedge funds for $j = 2010, \ldots, 2019$.

(b) The empirical distribution of $m^{\cdot,j}$ realised by the hedge funds for $j = 2010, \ldots, 2019$.

Figure 6.1 – The empirical distribution of the realised annual performance of the hedge funds over the period 2010-2019.

Since the discussed theorems in Chapter 5 have the restriction that only non-negative strategies are allowed, it is only consistent to continue with the returns expressed as multipliers to study the hedge fund landscape from a ranking game perspective.

Remark 6.1. The Competitive Investor Game is also expressed in way such that the score represents a multiplier over the invested capital. Recall that the setup of the game requires that two players invest 1 unit of capital in games with an expected payoff of 1 and the payoff can be considered the multiplication factor of that invested capital.

Figure 6.1 shows the distribution of the scores of the group of hedge funds per game, i.e. the distribution of all the realised annual returns per year. One would expect to see that although hedge funds score differently each year, the distribution of the score of the group would to some extent be the same as they play the same game each year. From this point of view, it does not seem fair to compare the score of one game to that of the other.

This is where there is difference in the hedge fund landscape versus the theoretical ranking game. In the ranking game, the situation stays the same each game. In the case of hedge funds, there is that the (lack of) activity in the market, the arise of a crisis or other events and circumstances influence the performance of the hedge funds. Therefore, to be able to say that the game does not change over the years, the performance of the hedge funds needs to be normalized in some way or another with respect to market, to each other or the amount or risk that can be taken.

6.2 Normalisation of the Performance

As the respective category of the hedge funds is the Long/Short U.S. Equity, it is only natural to have a market benchmark related to U.S. stocks. The Morningstar data already states the reported benchmark for some of the hedge funds, like the Russell 1000 Growth/Value, the S&P 1500, the S&P 500, Morningstar's own constructed indices and others. For this research, the S&P 500 index is chosen to represent the performance of the overall US equity market and thus as a possible normalization method for the performance of the hedge funds.

Next to normalizing by the S&P 500 index, two other methods are proposed to adjust the hedge fund performances over the years. These will relate to the mean and the variation of the annual performance over the years. The normalization by the S&P 500 index reflects the annual performance of the hedge funds with respect to that of the market. However, every year is different and that is not always well reflected in the annual performance. In other words, hedge funds can benefit from (short-term) movements in the market or other specific opportunities that are not reflected in the annual return of the market. Therefore it is proposed to look at the mean and the variation of the performance of the hedge funds to assess the performance relative to the group only.

6.2.1 Methods

Three methods are introduced to normalize the performance of the hedge funds over the years. The performance of the hedge funds, expressed as multipliers over the invested capital, in year j will be normalized by the normalization factors n_1^j, n_2^j and n_3^j. The normalized performance of the hedge fund i in year j by method c, denoted with $\hat{m}_c^{i,j}$, is then as follows:

$$\hat{m}_c^{i,j} = m^{i,j} \cdot n_c^j.$$

Let k_j be the number of active hedge funds (number of players) for the years $j = 2010, \ldots, 2019$. The proposed methods that take into account the performance of the S&P 500 index and the average and deviation of the performance with respect to the hedge funds, are the following:

1. **Normalisation by the S&P 500 Index**
The annual performance of the hedge funds are corrected for the performance of the U.S. stock market represented by the S&P 500 index. To quantify, the following variables are defined:

$$\begin{aligned} r_{SP}^j &= \text{the annual return (\%) of the S\&P 500 index in year } j, \\ m_{SP}^j &= 1 + r_{SP}^j \\ &= \text{the annual multiplication factor of the S\&P 500 index in year } j, \\ m_{SP} &= \left(\prod_j m_{SP}^j \right)^{\frac{1}{\#j}} \\ &= \text{the average annual multiplication factor of the S\&P 500 index over all years.} \end{aligned}$$

Then the following holds for n_1:

$$n_1^j = \frac{m_{SP}}{m_{SP}^j}.$$

This normalization adjusts the annual return of the hedge funds for the annual performance of the U.S. stock market over the years.

2. **Normalisation by the Mean of the Realised Returns**
Due to the long/short strategy, the hedge funds have the ability to perform regardless of the performance of the market. So normalizing the performance of the hedge funds with the S&P 500 index may not be the most fair representation when comparing years. Therefore, the average performance of the hedge fund group will be used as the benchmark for the realised performance of the respective year. The following variables are

defined:

$$m_{hf}^{j} = \left(\prod_{i=1}^{k_j} m^{i,j}\right)^{\frac{1}{k_j}}$$

= the annual average performance of the hedge funds in year j,

$$m_{hf} = \left(\prod_{j} m_{hf}^{j}\right)^{\frac{1}{\#j}}$$

= the average performance of all hedge funds over all years.

This implies the following for the normalization factor that normalizes by the mean:

$$n_2^j = \frac{m_{hf}}{m_{hf}^j}.$$

This normalization adjusts the annual return of the hedge funds for the average annual performance of the hedge fund group over the years.

3. **Normalisation by the Standard Deviation of the Realised Returns** The annual performances of the hedge fund group indicates about the conditions and opportunities of the respective years. However, the spread in the realised annual performance among the hedge funds says something about the opportunities and the risk taken in that year. After all, the closer the performance of the hedge funds, the less distinctive actions (could) have been made in that year. Normalizing the returns with the deviation of the returns serves as a proxy for the risk that is taken in a year. Use the following variables:

$m_{sd(hf)}^{j}$ = the standard deviation over all $m^{\cdot,j}$ in year j

$m_{sd(hf)}$ = the standard deviation over all $m_{sd(hf)}^{j}$.

The third normalization factor is then defined as follows:

$$n_3^j = \frac{m_{sd(hf)}}{m_{sd(hf)}^j}.$$

This normalization adjusts the annual return of the hedge funds for the deviation of the annual performances of the hedge fund group over the years.

Remark 6.2. It is chosen that the mean performance of S&P 500 index and that of the group performance is determined by annualizing the product of all the multipliers that represent the returns, i.e. taking the geometric mean instead of taking the arithmetic mean. In that way, a lose of 10% in one year is not yet made up for with a profit of 10% in the next year. After all, this is not how exponential growth (like inflation does) works. To make up for a lose of 10%, one would need to make multiply his invested capital with a factor of $\frac{1}{0.9} \approx 1.11$.

See Table 6.1 to see the outcome of the normalization factors for the three methods over the years. A factor smaller than 1 represents the situation that the hedge fund performance is overestimated and a factor bigger than 1 an underestimation of the performance with respect to the year and the benchmark.

j	2010	2011	2012	2013	2014	2015	2016	2017	2018	2019
k_j	12	14	18	21	22	24	29	37	40	41
n_1^j	0.988	1.112	0.979	0.857	0.997	1.120	1.014	0.931	1.187	0.866
n_2^j	0.943	1.072	0.985	0.868	1.011	1.090	1.024	0.964	1.118	0.953
n_3^j	0.775	1.397	0.960	0.737	1.219	1.166	0.779	1.219	1.265	0.980

Table 6.1 – The number of active hedge funds and the normalization factors of the three methods for the period 2010-2019 with respect to the market, the mean performance of the group and the deviation in the performance of the group.

Example 6.1. The interpretation of the normalization factors is as follows. Suppose that a hedge fund scored a return of 20% in 2013, i.e. a multiplier of 1.2. Then compared to the normalized performance of the market in 2013, that score actually is only worth $1.2 * 0.857 = 1.0284$ where 1 is the average score of the market over all the years. The hedge fund is expected to (at least) follow the benchmark. So when the market also had a good performance, the score of the hedge fund gets corrected for that.

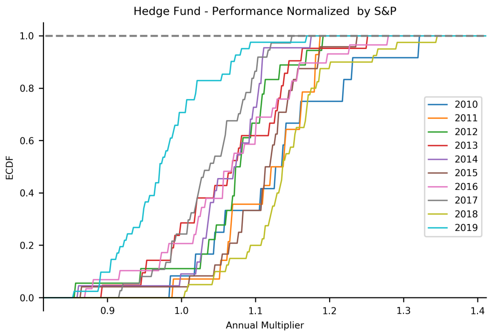
Remark 6.3. Notice the increase in the number of active hedge funds over the years. The data only consists of the hedge funds that are currently active and does not provide performances of hedge fund strategies that are currently closed. It is seen in Section 5.3 that this change in the number of players highly influences the equilibrium strategy.

For most of the years, the methods are consistent in stating if the hedge fund performance is over- of underestimated. Only in 2015, 2017 and 2018 there are different assessments. However, for the year of 2015 the difference is very small and can be neglected as the market performed just positive, while the mean and deviation of the performance of the hedge fund group were just lower than average. In 2016, the variation in the performance was significantly higher with respect to the performance of the market and the group. The most likely explanation for this was that Donald Trump won the U.S. presidency. Such an event does not highly impact the annual performance, but does causes a lot of uncertainty under the investors. Therefore a wide spread in performance is observed in that specific year. 2017 can be stated as the year of ten years after the start of the financial crisis. Also, it was a year with economic growth. The U.S. economy grew with 3,3% just in one quarter (a 3-year high) and unemployment was the lowest it has been since the year 2000 [11]. It seems that the hedge funds all captured to some extent this growth and that no special circumstances arose to make a difference. This had the result that the hedge funds on average underperformed the market (probably by their short positions) and there was not much deviation in the performance of the group.

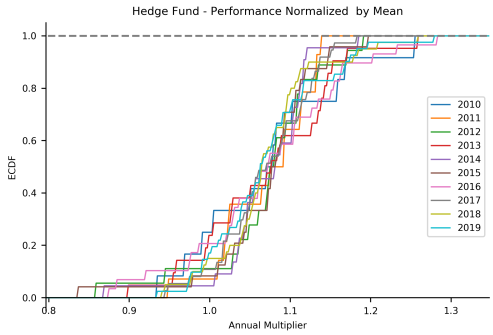
See Figure 6.2 to see how the distribution of realised performance of the hedge funds changes when normalized by the 3 methods.

The normalization methods were introduced to compare different years in a fair way. A fair comparison should hold that the at least the average performance of the group is similar to some extent and also the distributions should not vary too much. It is hard to quantify when years can be considered comparable. However, it is clear that the original performance, the performance normalized by the S&P and the performance normalized by the standard deviation do not provide a fair comparison. Whether the performance normalized by the mean is good enough to view as comparable over the years is debatable, but it is the best compared to the other methods and original performance.

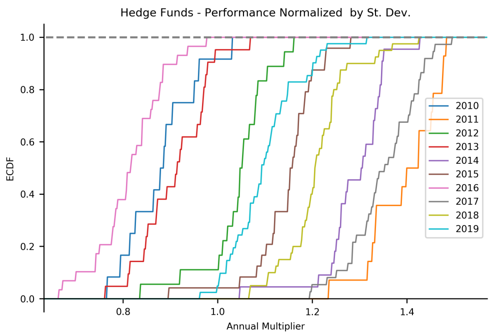
[11] Source: `https://www.cfr.org/blog/ten-most-significant-world-events-2017`



(a) The empirical distribution of $\hat{m}_1^{\cdot,j}$ realised by the hedge funds for $j = 2010, \ldots, 2019$.



(b) The empirical distribution of $\hat{m}_2^{\cdot,j}$ realised by the hedge funds for $j = 2010, \ldots, 2019$.



(c) The empirical distribution of $\hat{m}_3^{\cdot,j}$ realised by the hedge funds for $j = 2010, \ldots, 2019$.

Figure 6.2 – The empirical distributions of the normalized hedge fund performance realised by the hedge funds over the period 2010-2019.

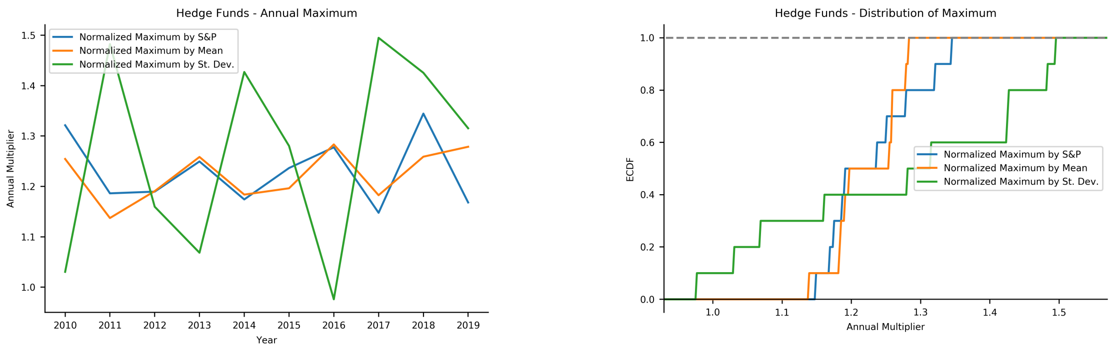
6.3 The Maximum Performance

It is hard to quantify the distribution of the (normalized) performance with respect to the ranking game perspective. It is the distribution of the maximum of the group that can be quantified. See Figure 6.3 on how the maximum of the normalized performance of the hedge funds behaves over the years and in terms of empirical distribution.

The maximum normalized by the S&P and the mean demonstrate similar movements in terms of direction and size. The maximum normalized by the standard deviation on the other hand heavily swings between relative low and high performance. The big spread is also clearly visible in the distribution of that maximum. Do keep in mind that the normalizations other than by the mean actually do not a fair comparison of the years. So there is no reasoned statement why the maximum performance normalized by the S&P and the mean show similarities.

6.3.1 Distribution of the Normalized Maximum

According to the theorem about the multiplayer ranking game, the maximum performance of the group approaches a uniformly distribution when all the players try to get the highest score in an optimal way. A uniform distribution of the maximum would indicate that the group of hedge funds indeed play like they are willing to have some low scores in exchange for higher scores in order to have all have an equal probability of getting the highest score. It is neglected that the maximum does not entirely reach the uniformity. See Figure 6.4 for the best fit of the uniform distribution to the data for all 3 methods.



(a) The maximum of the normalized hedge fund performance over the period 2010-2019.

(b) The distribution of the maximum of the normalized hedge fund performance.

Figure 6.3 – The maximum of the normalized hedge fund performance expressed over the years and in the form of an empirical distribution.

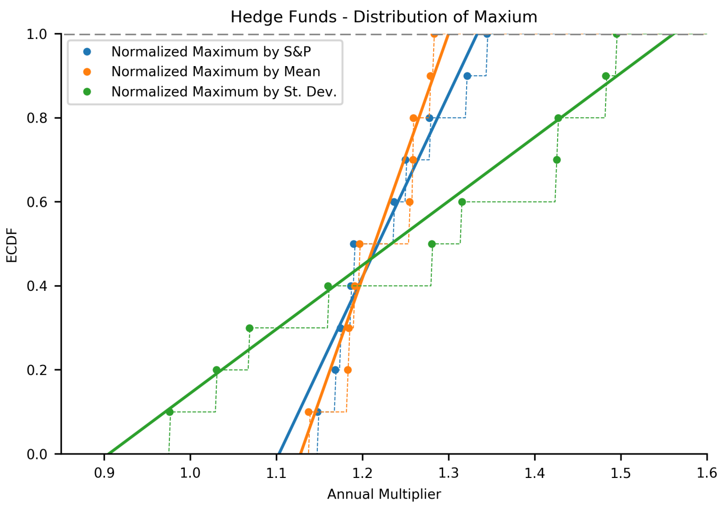


Figure 6.4 – The best linear fit for the maximum of the normalized hedge fund performance over the period 2010-2019.

At first sight, the maxima do not differ not that much from their respective best-fit uniform distribution. This is a first indication that the hedge funds may manage strategies that result in an equilibrium with respect to setting the highest rank in terms of performance, just as in the multiplayer ranking game. Such an equilibrium would imply that the every fund has an equal opportunity of winning. Looking back at the data shows that there are 9 different winners over a period of 10 years. Only 1 hedge fund managed to get the best performance compared to its peers twice. This further strengthens the suspicion that the hedge fund landscape is to some extent similar to that of the multiplayer ranking game.

6.3.2 Test Statistic

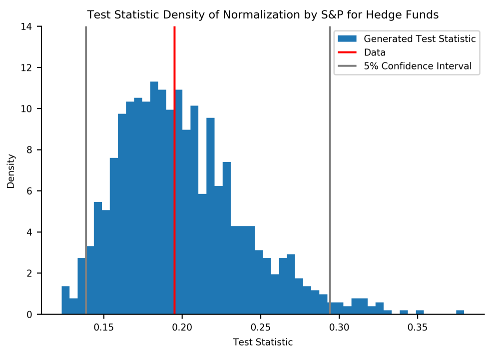
To further asses if all the maximum follow the best-fit uniform distribution, the Kolmogorov-Smirnov (K-S) distance is tested. The K-S distance measures the greatest distance from the empirical distribution to the distribution it is tested against. Again, an absolute number from the K-S does not yet provide enough information. Therefore, another 1000 tests are performed in the following way for all 3 methods:

1. Draw $V_1, \dots, V_{10}$ from the best-fit uniform distribution on the maximum of the normalized hedge fund performance;

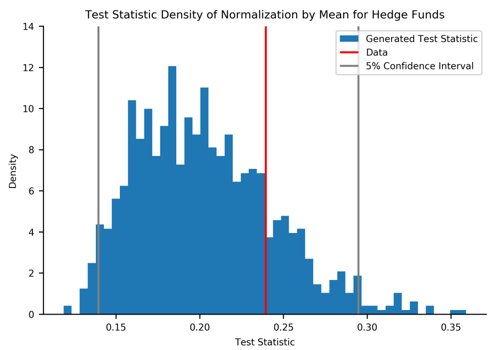
2. Determine the uniform distribution that best fits $V_1, \dots, V_{10}$ and denote this distribution with $\hat{U}$;

3. Calculate the K-S distance of $V_1, \dots, V_{10}$ and $\hat{U}$.

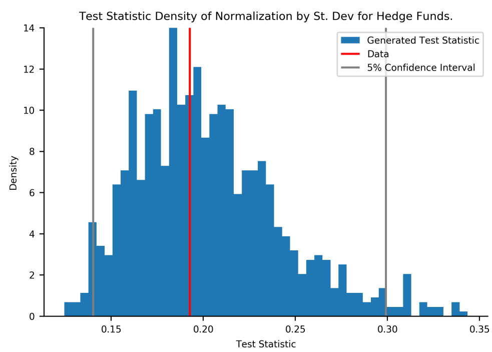
See Figure 6.5 for the result of this test compared to the K-S distance of the data.



(a) The distribution of simulated K-S distances compared to the K-S distance of the maximum performance normalized by the S&P.



(b) The distribution of simulated K-S distances compared to the K-S distance of the maximum performance normalized by the mean.



(c) The distribution of simulated K-S distances compared to the K-S distance of the maximum performance normalized by the deviation.

Figure 6.5 – Three times the distribution of 1000 simulated K-S distances with indicated 95% confidence intervals compared to the K-S distance of the maximum of the normalized performance by the three methods, respectively .

From this assessment, it can not be rejected that all the maxima are uniformly distributed. That the maximum performance normalized by the mean follows a uniform distribution indicates a similarity between the hedge funds and the multiplayer ranking game. However, the other 2 distributions of the maximum show similar results whilst the they are not in relation to the game. So the evidence that the hedge fund performance normalized by the mean can be modelled by a ranking is not that strong.

Also, the theory suggest that in order to play optimal, players must have probability density all the way down to the score of 0. For hedge funds, this is definitely not the case and the data confirms this. Each fund can have different standards, but it is most certain that the hedge funds can not afford loses of multiples of 10%, let alone multiple loses of multiples of 10%.

This is where the utility function of ϕ in the game can provide an additional parameter in translating the hedge fund performance to the interval of the game. To do that, one needs to determine the 0-point score for hedge funds. Looking at the performance normalized by the mean, the 0-point for the hedge fund performance is advised to be taken around 0.85-0.9. This has the consequence that the distribution of the maximum does not cover the same interval the as the whole performance. This also is not consistent with the game.

Another complication that in the translation is the growing number of hedge funds. This would suggest that each year should actually be considered a different game. A counterargument against this contrast is that not all hedge funds join the competition. It is much likely that not all hedge fund feel that they are in the competition (to win). Perhaps only a group of 10 hedge fund feel the pressure of beating their peers, which makes the issue of the growing number of hedge funds disappear. Further studies could try to tackle the problem of the utility function and the changing number of players.

6.4 Mutual Funds for U.S. Equity

In order to try to analyse the competition among the hedge funds in another way, an equal study is performed on mutual funds that also focus on U.S. equity. As the quantification of the competitiveness of the hedge funds is hard on an absolute basis, a study on mutual funds can give a relative comparison. The chosen set of mutual funds are the distinct funds (distinct in terms of the asset class) that invest in U.S. Large Cap Blend Equity. It is assumed that these are the funds that invest in the largest companies of the U.S. while combining value and growth stocks, i.e. the mutual funds that best represent the S&P 500 benchmark. See Figure A.2 and Table A.1 in Appendix A for a look on the display of the variables of the data of the mutual funds. The empirical distribution of the performance of the mutual funds in terms of multipliers over the invested capital can be found in Figure 6.6.

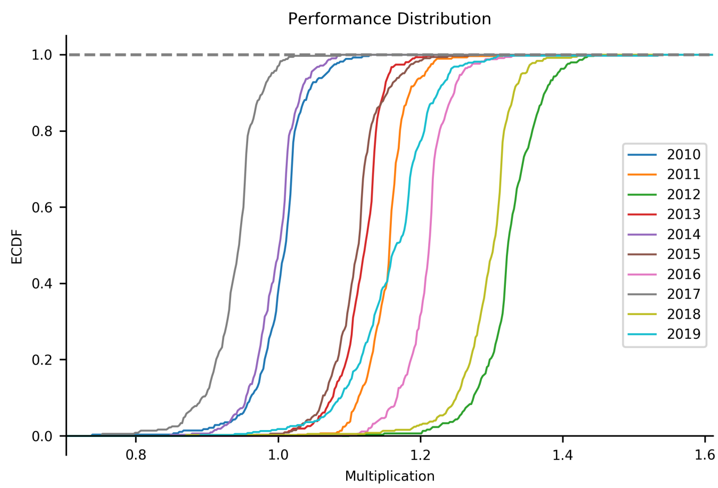


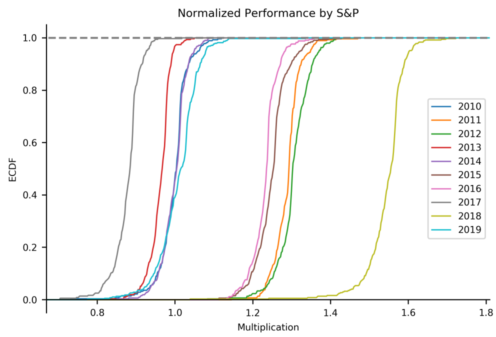
Figure 6.6 – The empirical distribution of $r^{\cdot,j}$ realised by the mutual funds for $j = 2010, \ldots, 2019$.

Just as for the hedge funds, the empirical distributions of the performance of the mutual funds must be normalized in order to compare them over the years. The construction for determining the normalization factors is the exactly the same as the one used for the hedge funds. See Table 6.2 for the normalization factors with respect to the mutual funds. (Notice that the factors related to the S&P stay the same as this is not influenced by the funds.)

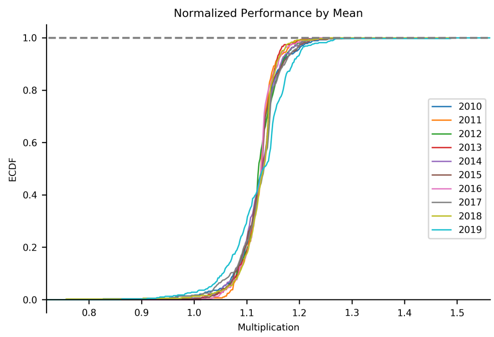
j	2010	2011	2012	2013	2014	2015	2016	2017	2018	2019
k_j	278	289	299	309	316	329	341	357	371	385
n_1^j	0.988	1.112	0.979	0.857	0.997	1.120	1.014	0.931	1.187	0.866
n_2^j	1.119	0.974	0.848	1.006	1.128	1.013	0.930	1.199	0.869	0.970
n_3^j	0.983	1.274	0.910	1.231	1.191	1.059	1.190	1.050	0.869	0.641

Table 6.2 – The normalization factors of the three methods for the period 2010-2019 for the mutual funds with respect to market, the mean performance of the group and the deviation in the performance of the group.

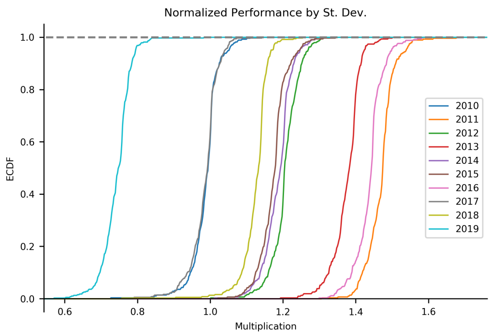
Where the normalization methods applied to the hedge funds were consistent for every year, except for two points, they are not for mutual funds. There are more years where they are not consistent than that they are. Only 2012, 2015 and 2019 agree on the under- or overestimation of the performance of the mutual funds across the 3 methods.



(a) The empirical distribution of $\hat{m}_1^{\cdot,j}$ realised by the mutual funds for $j = 2010, \ldots, 2019$.



(b) The empirical distribution of $\hat{m}_2^{\cdot,j}$ realised by the mutual funds for $j = 2010, \ldots, 2019$.



(c) The empirical distribution of $\hat{m}_3^{\cdot,j}$ realised by the mutual funds for $j = 2010, \ldots, 2019$.

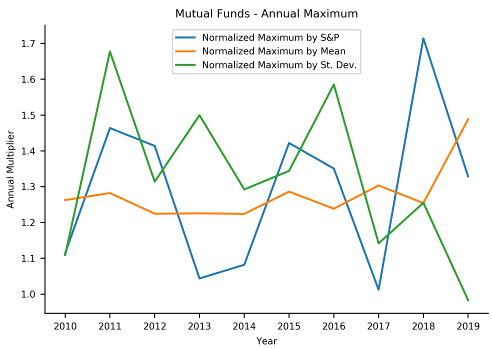
Figure 6.7 – The empirical distributions of the normalized hedge fund performance realised by the mutual funds over the period 2010-2019.

Also, the number of active mutual funds grow over time and there are significant more active mutual funds that there are hedge funds. There is a bigger growth in absolute terms compared to that of the hedge funds over the years, but this does not hold for the relative growth. From the theory on being in an equilibrium in a ranking game, it is known that the number of active players highly influence the amount of risk that needs to be taken in order to have a shot on winning.

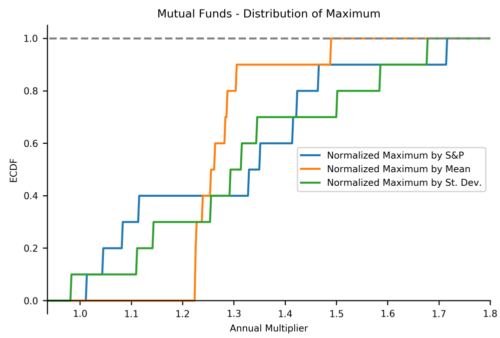
See Figure 6.7 for the empirical distributions of the performance of the mutual funds normalized by the 3 methods, respectively.

The results of the normalizations on the performance of the mutual funds are very similar to those of the hedge funds. The normalizations by the S&P and the standard deviation widely spread the years, whilst it had the intention to have similar distributions as output. The performance normalized by the mean on the other hand show that the distribution of the group is very similar ever year. Only the year of 2019 stands out from the figure as it a much wider spread than the other years (which can also be seen in Table 6.2). So just as for the hedge funds, only the performance normalized by the mean is considered a fair comparison of the years.

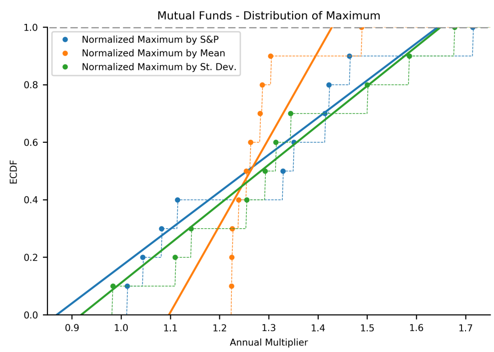
See Figure 6.8 for the summary of how the maximum of the performance of the mutual funds behave over the years in time, in distribution and complemented with the best-linear fit.



(a) The maximum of the normalized mutual fund performance over the period 2010-2019.



(b) The distribution of the maximum of the normalized mutual fund performance over the period 2010-2019.



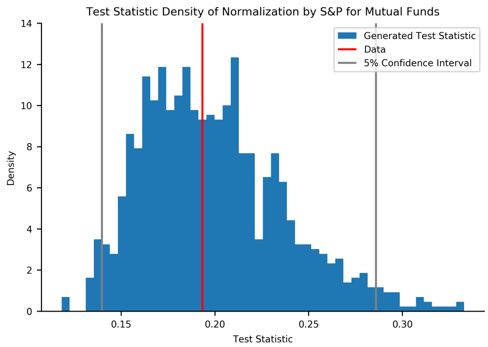
(c) The best linear fit for the maximum of the normalized hedge fund performance over the period 2010-2019.

Figure 6.8 – The maximum of the normalized mutual fund performance expressed over the years in time, in the form om an empirical distribution and added with a best-fit uniform distribution.

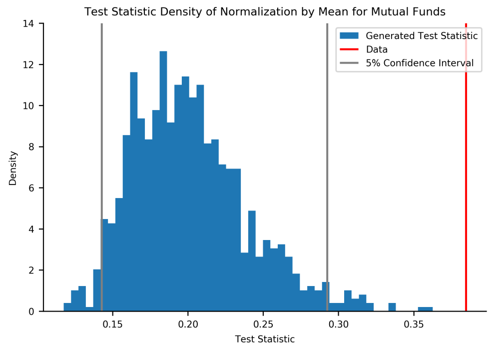
In the case of the hedge funds, the maximum normalized by the S&P and the mean behaved somewhat similar in terms of direction and size. For the mutual funds this is not the case. The maximum normalized by the mean even seems pretty consistent with an exception for the year 2019. This is also clearly visible in the best-linear fit. The line does not provide a good representation of the data because of that last year.

The maximum normalized by the standard deviation again heavily swings over time, just as was the case for the hedge funds. For that matter there is not much difference in terms of behavior. However, it is noticeable that the maximum normalized by the S&P shows similar characteristics as the maximum normalized by the standard deviation.

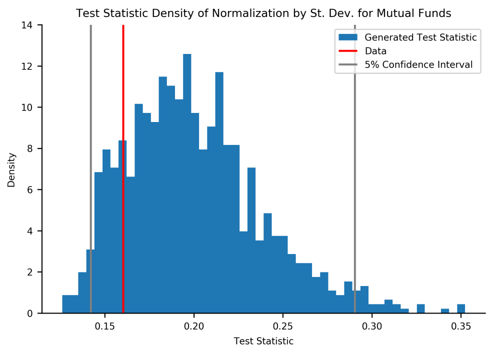
To follow-up on the best-linear fit to the data, the same K-S test statistic is performed on the maxima of the mutual funds. See Figure 6.9 for the results.



(a) The test statistic on the maximum performance of the mutual funds normalized by the S&P.



(b) The test statistic on the maximum performance of the mutual funds normalized by the mean.



(c) The test statistic on the maximum performance of the mutual funds normalized by the standard deviation.

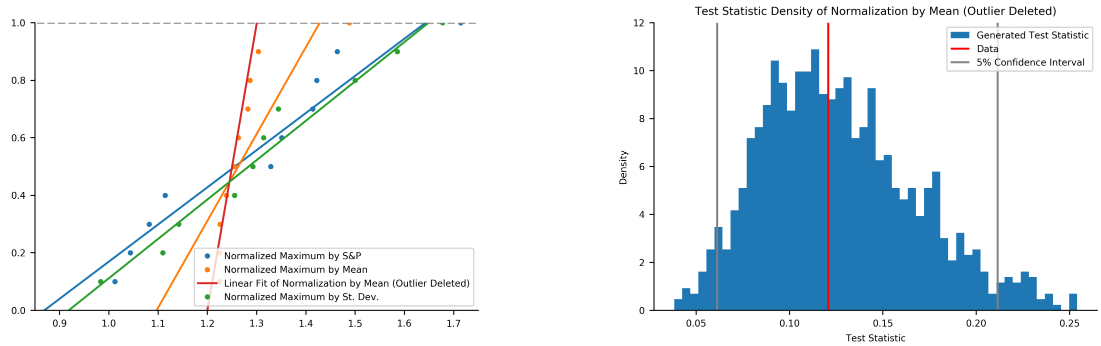
Figure 6.9 – Three times the distribution of 1000 simulated K-S distances with indicated 95% confidence intervals compared to the K-S distance of the maximum of the normalized performance of the mutual funds by the three methods, respectively

Most notable among the 3 figures is the rejection of the hypothesis that the maximum that is normalized by the mean originates from the best-fit uniform distribution as the K-S distance is outside the 95% confidence interval. However, earlier observing Figure 6.8c showed that the high K-S distance is caused by a very skewed best-fit. Treating this specific fund as an outlier in 2019, only with respect to the normalization by mean, massively changes the perspective on how this maximum behaves and the best-fit uniform distribution. See Figure 6.10 for the adjustments.

The removal of the outlier causes the best-fit uniform distribution to actually represent the data this time. Also the test statistic shows similar results compared to the the rest.

6.5 Comparison of the Maximum

So, one debatable outlier aside, there is not really a noticeable difference between the maximum performance of the hedge funds and mutual funds in terms of a uniform fit. See Figure



(a) The best linear fit for the maximum of the normalized hedge fund performance over the period 2010-2019, added with the best-fit for the maximum normalized by the mean with the outlier deleted.

(b) The distribution of 1000 simulated K-S distances compared to the K-S distance with and indicated 95% confidence interval of the maximum performance of the mutual funds normalized by the mean of the group performance with outlier deleted.

Figure 6.10 – The best linear fit and K-S distance test statistic of the maximum performance of the mutual fund normalized by the mean performance of the group with the outlier deleted

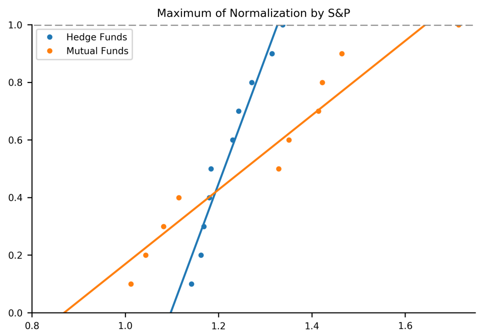
6.11 to view pair by pair how the maxima, normalized by the respective methods, differ from hedge funds compared to the mutual funds.

A quick look at Figure 6.11c shows that the distribution of the maximum that is corrected by the standard deviation of the group is very similar in terms of spread, the minimum and uniform fit. As possible explanation is by the fact that both groups operate in a very similar to equal asset class.

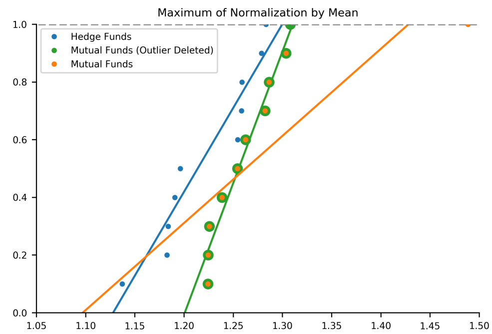
Sort of the same holds for the maxima that is normalized by the mean of their respective group (when the possible outlier is neglected). There is a similar spread, equal maximum and uniform fit for the hedge funds and the mutual funds with respect to the distribution. However, there are much more players in the mutual fund game then there are in the hedge fund game. If the mutual funds would be as competitive as the hedge funds, is should be expected that the maximum of the mutual funds would be bigger than that of the hedge funds.

The most noticeable difference among the 2 groups is the very different distribution of the maximum with respect to the market. The first observation is that the minimum and the maximum of the 2 best-fit uniform distributions are very different. Also, one can observe a split in the maximum of the mutual funds. Whilst the maximum of the hedge funds is evenly spread, the maximum of the mutual funds is centered around 1.1 or 1.4 plus a very high maximum (that originates from the year 2018, not 2019 as were the case before). So whilst the K-S distance does not remark this, it is definitely notable with respect to the uniformness of the distribution.

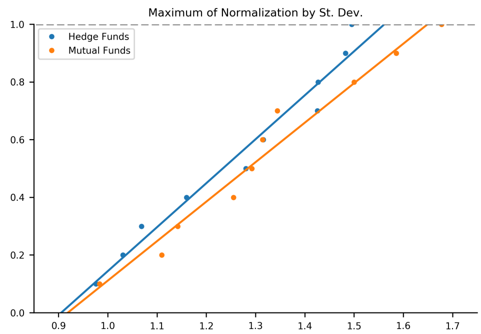
Finally, remark that the performance normalized by the mean is the only fair way to compare the performance over the years. All the figures related to the other methods are essentially only for comparison.



(a) The best linear fit of the maximum of the hedge funds and the mutual funds when normalized for the market.



(b) The best linear fit of the maximum of the hedge funds, the mutual funds and the mutual funds with the outlier deleted when normalized for the mean performance of the respective group.



(c) The best linear fit of the maximum of the hedge funds and the mutual funds when normalized for the deviation of the performance of the respective group.

Figure 6.11 – The best linear fit of the maximum of the hedge fund and the mutual fund when normalized by the 3 methods.

7

Conclusion

7.1 Summary

The world of hedge funds is a complex domain. Hedge funds are less regulated than mutual funds and their creative strategies can lead to historical events. Manager selection is key in asset management and strategies need to perform to attract clients as demonstrated in the past by the impact of the independent research institute Morningstar. It is thus of the benefit of the hedge fund to have a high ranking compared to its peers.

Ranking games were the primary research topic of this thesis. The scores that players realise in a game determines their ranking: a higher score than a competitor means a better ranking. Zero-sum games are the foundation of the simplest ranking game and introduce the value of a game. By definition, an optimal strategy in a game does not perform the best against every strategy, but does provide the highest minimum expected payoff against any strategy.

The strategy of a player is represented by the distribution of the player's score under the requirements of the game. It is not optimal from the perspective of a ranking game to have an atom in the distribution of your score if all players are equally competitive. One can outplay such strategies by placing probability mass at the same point plus ϵ. The proof of uniqueness is sketched by Bell & Cover (1980) [3] and is complemented with additional clarification in this thesis. When one is being less competitive in the 2-player game, it is uniquely optimal to copy the strategy of the opponent with maximum probability and otherwise quit.

When the payoff function in the 2-Player Ranking Game gets adjusted by the means of a utility function, the optimal strategies adjust with it. Both players will play their optimal strategies on the same interval and use the utility function of the other player to achieve the highest minimum expected payoff. The proof of uniqueness is sketched by Alpern & Howard (2017) [1] and is given a new look in this thesis.

Where the 2-player game is about beating 1 opponent, the k-player multiplayer game is about beating $k-1$ competitors. The Symmetric Multiplayer Ranking Game has an equilibrium such that all players have equal probability of winning. All players then leverage low scores with a small probability of setting a high score and getting the win. The more players, the higher the probability on a low score in order for a higher leverage for higher scores. Such a setting implies that the maximum of the score approaches the uniform distribution across the playing interval.

The hedge fund landscape in terms of performance is analysed with respect to the Symmetric Multiplayer Ranking Game and the analysis is focused on the distribution of the maximum. The normalization by the mean is needed to have comparable years in terms of distribution of the performance. The number of active hedge funds in the data heavily increases over time which results in less-comparable games. Also, the assumptions on the winner-takes-all payoff function and equal competitiveness and utility highly impact the translation. Still, the maximum normalized by the mean provides some minor evidence that the hedge funds observe competition. Also, there are 9 different winners over a period of 10 years, which coincides with the equilibrium of the ranking game.

A study on benchmark-related mutual funds is used to put the results of the hedge funds in a relative perspective. The analysis show similar results, but not as suitable to the ranking game as hedge funds. The performance of mutual funds show less winners whilst there are more players compared to the hedge funds and also a possible outlier had to be deleted in order to believe that the maximum normalized by the mean originates from a uniform distribution.

7.2 Limitations

This thesis aimed to translate a multiplayer ranking game to the world of hedge funds. While the initial idea is plausible, it is clear that the theory of ranking games is limited and that this reflects the accuracy of the translation. The following points are identified as critical points that influence the reliability of this research:

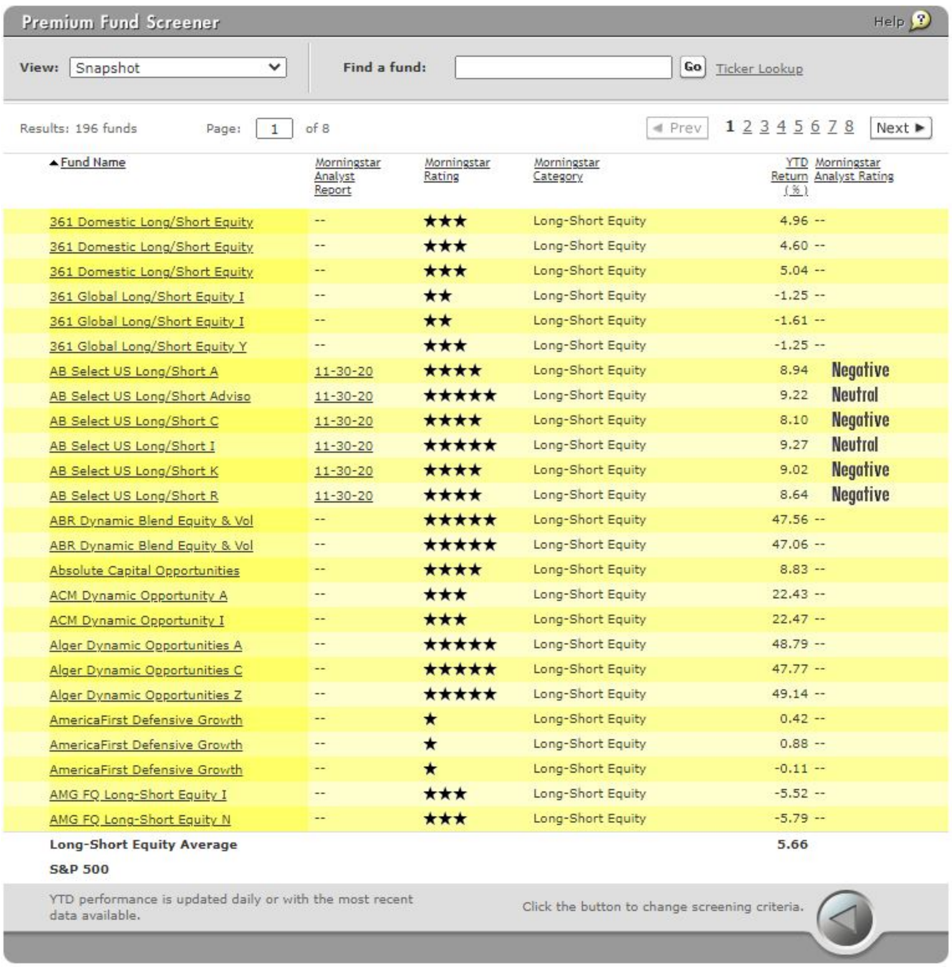
- The analysis of the hedge funds is based on the data that is retrieved from Morningstar. While the available data can be considered reliable, there is no view on the completeness of the data. Also, only the hedge fund that actively managed a strategy in 2019 are taken into account. Surely there were more hedge fund in the period of 2010 - 2019 that increased the competition, but they are not in the data.

- The translation between the Symmetric Multiplayer Ranking Game and the hedge funds landscape is to simplistic. First, the equilibrium strategies of Symmetric Multiplayer Ranking Game are based on a winner-takes-all payoff. A more suitable distribution would be that the top 10% split the winnings or some sort of winner-takes-most approach, for example. Also, perhaps an even more important assumption, is that the players do not lose anything when they have a low score. After all, the distribution of the scores in the games are based on risk, not effort. Where the game has a score of zero without consequences, this certainly does not hold for hedge funds. There is no conventional minimum for the return of a hedge funds and if there is, this would certainly have a negative impact. Also the assumptions that all hedge funds have the same utility function and competitive level does not contribute to modelling of the hedge fund performance based on ranking. It has been shown that it takes a great amount of work to only analyse the 2-Player Ranking Game. For the multiplayer ranking game, the mathematical possibilities are currently (too) limited.

It is proposed that future research focuses on the limitations mentioned above.

A

Data Overview


Premium Fund Screener

Help

View: Snapshot

Find a fund:

Go

Ticker Lookup

Results: 196 funds

Page: 1 of 8

◄ Prev 1 2 3 4 5 6 7 8 Next ►

Fund Name	Morningstar Analyst Report	Morningstar Rating	Morningstar Category	YTD Return (%)	Morningstar Analyst Rating
361 Domestic Long/Short Equity	--	★★★	Long-Short Equity	4.96	--
361 Domestic Long/Short Equity	--	★★★	Long-Short Equity	4.60	--
361 Domestic Long/Short Equity	--	★★★	Long-Short Equity	5.04	--
361 Global Long/Short Equity I	--	★★	Long-Short Equity	-1.25	--
361 Global Long/Short Equity I	--	★★	Long-Short Equity	-1.61	--
361 Global Long/Short Equity Y	--	★★★	Long-Short Equity	-1.25	--
AB Select US Long/Short A	11-30-20	★★★★	Long-Short Equity	8.94	Negative
AB Select US Long/Short Adviso	11-30-20	★★★★★	Long-Short Equity	9.22	Neutral
AB Select US Long/Short C	11-30-20	★★★★	Long-Short Equity	8.10	Negative
AB Select US Long/Short I	11-30-20	★★★★★	Long-Short Equity	9.27	Neutral
AB Select US Long/Short K	11-30-20	★★★★	Long-Short Equity	9.02	Negative
AB Select US Long/Short R	11-30-20	★★★★	Long-Short Equity	8.64	Negative
ABR Dynamic Blend Equity & Vol	--	★★★★★	Long-Short Equity	47.56	--
ABR Dynamic Blend Equity & Vol	--	★★★★★	Long-Short Equity	47.06	--
Absolute Capital Opportunities	--	★★★★	Long-Short Equity	8.83	--
ACM Dynamic Opportunity A	--	★★★	Long-Short Equity	22.43	--
ACM Dynamic Opportunity I	--	★★★	Long-Short Equity	22.47	--
Alger Dynamic Opportunities A	--	★★★★★	Long-Short Equity	48.79	--
Alger Dynamic Opportunities C	--	★★★★★	Long-Short Equity	47.77	--
Alger Dynamic Opportunities Z	--	★★★★★	Long-Short Equity	49.14	--
AmericaFirst Defensive Growth	--	★	Long-Short Equity	0.42	--
AmericaFirst Defensive Growth	--	★	Long-Short Equity	0.88	--
AmericaFirst Defensive Growth	--	★	Long-Short Equity	-0.11	--
AMG FQ Long-Short Equity I	--	★★★	Long-Short Equity	-5.52	--
AMG FQ Long-Short Equity N	--	★★★	Long-Short Equity	-5.79	--
Long-Short Equity Average				**5.66**	
S&P 500					

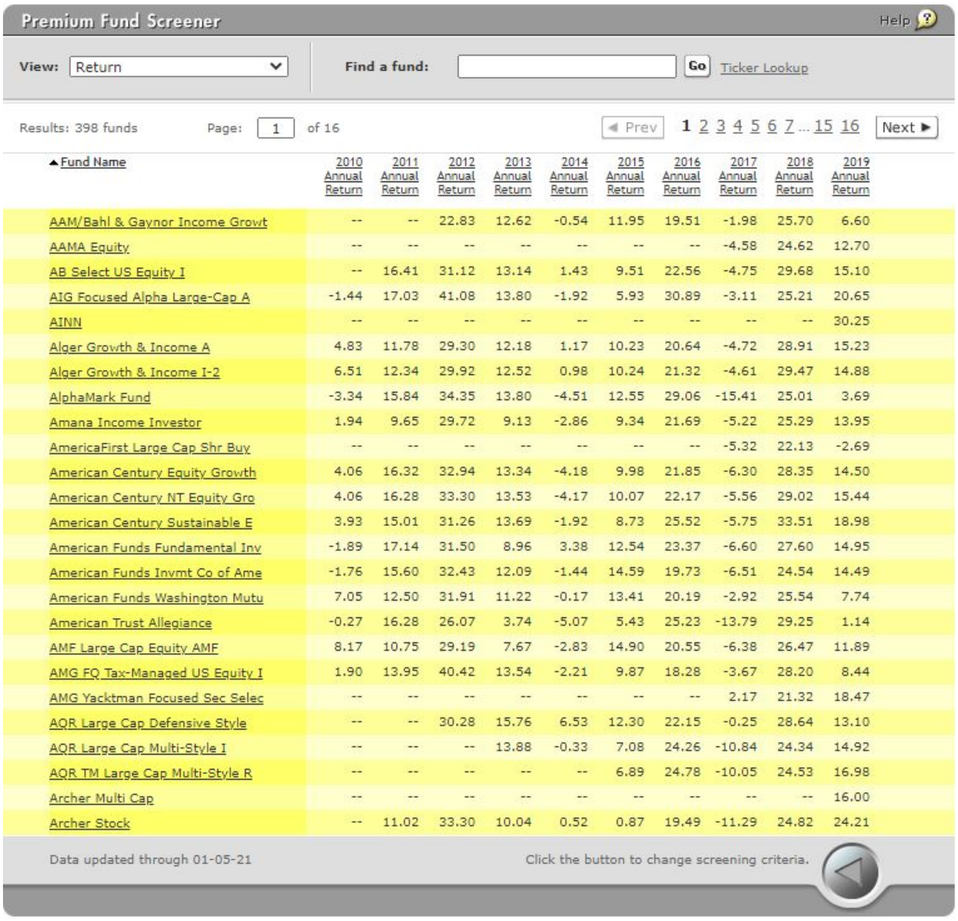
YTD performance is updated daily or with the most recent data available.

Click the button to change screening criteria.

Figure A.1 – The data of the hedge funds from the Premium Fund Screener of `Morningstar.com`

Data Variables			
% Assets in Top 10 Holdings	3-Year Return +/- Category	Classic Growth (%)	Non-US Stock (%)
10-Year % Rank	3-year Tax Cost Ratio %	Closed to All Investments	Not Classified (%)
10-Year Return (%)	4-Week % Rank	Closed to New Investors	Not Rated (%)
10-Year Return +/- Category	4-Week Return (%)	Current Price (NAV $)	Number of Holdings in Portfolio
10-year Tax Cost Ratio %	4-Week Return +/- Category	Cyclical (%)	Other (%)
12b-1 Fee (%)	5-Year % Rank	Date of Most Recent Portfolio	Other Regions (%)
15-Year % Rank	5-Year Return (%)	Developed Markets (%)	P/B Ratio
15-Year Return (%)	5-Year Return +/- Category	Distinct Asset Class	P/C Ratio
15-Year Return +/- Category	5-year Tax Cost Ratio %	Distinct Portfolio Only	P/E Ratio
15-year Tax Cost Ratio %	A (%)	Distressed (%)	P/S Ratio
1-Year % Rank	AA (%)	DTC Last Year (%)	Pacific (%)
1-Year Return (%)	AAA (%)	DTC Year Before Last (%)	Potential Cap Gains Exp (%)
1-Year Return +/- Category	AfterTax Return (no sale) 10-Year (%)	Emerging Markets (%)	Qualified Access
2010 % Rank	AfterTax Return (no sale) 1-Year (%)	Enhanced Index Funds	Redemption Fee %)
2010 Annual Return	AfterTax Return (no sale) 5-Year (%)	Europe (%)	ROA Last Year (%)
2010 Return +/- Category	AfterTax Return (no sale) Since Inception (%)	Expense Ratio (%)	ROA Year Before Last (%)
2011 % Rank	AfterTax Return (with sale) 10-Year (%)	Expense Ratio (%)2	ROE Last Year (%)
2011 Annual Return	AfterTax Return (with sale) 15-Year (%)	Front-end Load (%)	ROE Year Before Last (%)
2011 Return +/- Category	AfterTax Return (with sale) 1-Month (%)	Fund Familty Score	R-Squared
2012 % Rank	AfterTax Return (with sale) 1-Year (%)	Fund Family	SEC Yield (%)
2012 Annual Return	AfterTax Return (with sale) 3-Month (%)	Fund Inception Date	Sharpe Ratio
2012 Return +/- Category	AfterTax Return (with sale) 3-Year (%)	Fund of Funds	Slow Growth (%)
2013 % Rank	AfterTax Return (with sale) 5-Year (%)	Hard Assets (%)	Socially Conscious Funds
2013 Annual Return	AfterTax Return (with sale) YTD (%)	High Yield (%)	Speculative Growth (%)
2013 Return +/- Category	Aggressive Growth (%)	Index Funds	Standard Deviation
2014 % Rank	AIP Minimal Initial ($)	Institutional Funds	Stocks in Large-cap Blend (%)
2014 Annual Return	Alpha	Japan (%)	Stocks in Large-cap Growth (%)
2014 Return +/- Category	Average Credit Quality	Latin America (%)	Stocks in Large-cap Value (%)
2015 % Rank	Average Duration (Years)	Life Cycle Funds	Stocks in Mid-cap Blend (%)
2015 Annual Return	Average Manager Tenure (Years)	Load-Adj Return 10-Year (%)	Stocks in Mid-cap Growth (%)
2015 Return +/- Category	Average Market Cap ($Mil)	Load-Adj Return 3-Year (%)	Stocks in Mid-cap Value (%)
2016 % Rank	Average Maturity	Load-Adj Return 5-Year (%)	Stocks in Small-cap Blend (%)
2016 Annual Return	Average Moat Rating	Manager Name	Stocks in Small-cap Growth (%)
2016 Return +/- Category	Average Weighted Coupon	Manager Name2	Stocks in Small-cap Value (%)
2017 % Rank	Average Weighted Price	Mean	Style Box
2017 Annual Return	B (%)	Minimum Initial Purchase ($)	Style Box2
2017 Return +/- Category	Back-end Load (%)	Minimum Initial Purchase IRA ($)	Symbol
2018 % Rank	BB (%)	Mornigstar Rating 10 year	Total Assets ($ mil)
2018 Annual Return	BBB (%)	Mornigstar Rating 5 year	Turnover (%)
2018 Return +/- Category	Bear Market % Rank	Morningstar Analyst Rating	US & Canada (%)
2019 % Rank	Below B (%)	Morningstar Analyst Report	US Stock (%)
2019 Annual Return	Best Fit Alpha	Morningstar Rating	Yield (%)
2019 Return +/- Category	Best Fit Beta	Morningstar Rating 3 Year	YTD % Rank
3-Month % Rank	Best Fit Index	Morningstar Risk	YTD Return (%)
3-Month Return (%)	Best Fit R-Squared	Morningstar Sustainability Rating	YTD Return +/- Category
3-Month Return +/- Category	Beta	Net Margin Last Year (%)	
3-Year % Rank	Bond (%)	Net Margin Year Before Last (%)	
3-Year Return (%)	Cash (%)	No-Load Funds	

Table A.1 – All the variables that are available in the Premium Fund Screener of `Morningstar.com`



▲Fund Name	2010 Annual Return	2011 Annual Return	2012 Annual Return	2013 Annual Return	2014 Annual Return	2015 Annual Return	2016 Annual Return	2017 Annual Return	2018 Annual Return	2019 Annual Return
AAM/Bahl & Gaynor Income Growt	--	--	22.83	12.62	-0.54	11.95	19.51	-1.98	25.70	6.60
AAMA Equity	--	--	--	--	--	--	--	-4.58	24.62	12.70
AB Select US Equity I	--	16.41	31.12	13.14	1.43	9.51	22.56	-4.75	29.68	15.10
AIG Focused Alpha Large-Cap A	-1.44	17.03	41.08	13.80	-1.92	5.93	30.89	-3.11	25.21	20.65
AINN	--	--	--	--	--	--	--	--	--	30.25
Alger Growth & Income A	4.83	11.78	29.30	12.18	1.17	10.23	20.64	-4.72	28.91	15.23
Alger Growth & Income I-2	6.51	12.34	29.92	12.52	0.98	10.24	21.32	-4.61	29.47	14.88
AlphaMark Fund	-3.34	15.84	34.35	13.80	-4.51	12.55	29.06	-15.41	25.01	3.69
Amana Income Investor	1.94	9.65	29.72	9.13	-2.86	9.34	21.69	-5.22	25.29	13.95
AmericaFirst Large Cap Shr Buy	--	--	--	--	--	--	--	-5.32	22.13	-2.69
American Century Equity Growth	4.06	16.32	32.94	13.34	-4.18	9.98	21.85	-6.30	28.35	14.50
American Century NT Equity Gro	4.06	16.28	33.30	13.53	-4.17	10.07	22.17	-5.56	29.02	15.44
American Century Sustainable E	3.93	15.01	31.26	13.69	-1.92	8.73	25.52	-5.75	33.51	18.98
American Funds Fundamental Inv	-1.89	17.14	31.50	8.96	3.38	12.54	23.37	-6.60	27.60	14.95
American Funds Invmt Co of Ame	-1.76	15.60	32.43	12.09	-1.44	14.59	19.73	-6.51	24.54	14.49
American Funds Washington Mutu	7.05	12.50	31.91	11.22	-0.17	13.41	20.19	-2.92	25.54	7.74
American Trust Allegiance	-0.27	16.28	26.07	3.74	-5.07	5.43	25.23	-13.79	29.25	1.14
AMF Large Cap Equity AMF	8.17	10.75	29.19	7.67	-2.83	14.90	20.55	-6.38	26.47	11.89
AMG FQ Tax-Managed US Equity I	1.90	13.95	40.42	13.54	-2.21	9.87	18.28	-3.67	28.20	8.44
AMG Yacktman Focused Sec Selec	--	--	--	--	--	--	--	2.17	21.32	18.47
AQR Large Cap Defensive Style	--	--	30.28	15.76	6.53	12.30	22.15	-0.25	28.64	13.10
AQR Large Cap Multi-Style I	--	--	--	13.88	-0.33	7.08	24.26	-10.84	24.34	14.92
AQR TM Large Cap Multi-Style R	--	--	--	--	--	6.89	24.78	-10.05	24.53	16.98
Archer Multi Cap	--	--	--	--	--	--	--	--	--	16.00
Archer Stock	--	11.02	33.30	10.04	0.52	0.87	19.49	-11.29	24.82	24.21

Data updated through 01-05-21 | Click the button to change screening criteria.

Figure A.2 – The data of the mutual funds from the Premium Fund Screener of `Morningstar.com`

Bibliography

[1] Alpern, Steven; Howard, J.V. Winner-take-all games: The strategic optimisation of rank. *Operations Research*, 65(5):1165–1176, 2017. doi: https://doi.org/10.1287/opre.2017.1635.

[2] Ammann, Manuel; Bauer, Christopher; Fischer, Sebastiaan; Müller, Philipp. The impact of the morningstar sustainability rating on mutual fund flows. *European Financial Management*, 25(3):520–553, 2019. doi: https://doi.org/10.1111/eufm.12181.

[3] Bell, Robbert M; Cover, Thomas M. . Competitive optimality of logarithmic investment. *Mathematics Of Operations Research*, 5(2):161–166, 1980. doi: https://doi.org/10.1287/moor.5.2.161.

[4] Blake, Christopher R.; Morey, Matthew R. Morningstar ratings and mutual fund performance. *The Journal of Financial and Quantitative Analysis*, 35(3):451–483, 2000. ISSN 00221090, 17566916. URL http://www.jstor.org/stable/2676213.

[5] Cover, T.M. Universal gambling schemes and the complexity measures of Kolmogorov and Chaitin. *Technical Report, no. 12*, 1974. URL https://ci.nii.ac.jp/naid/10008214934/en/.

[6] Ferguson, Thomas S. *Game Theory, Part II, Two-Person Zero-Sum Games*. 2014. URL https://www.math.ucla.edu/~tom/math167.html.

[7] Goel, Ms. Sweta; Mani, Dr. Mukta; Sharma, Dr. Rahal. A review of performance indicators of mutual funds. *International Refereed Research Journal*, 3(4):100–107, 2012.

[8] Grimmett, Geoffrey; Welsh, Dominic. *Probability: An Introduction*. Oxford University Press, 2 edition, 2014. ISBN 978-0-19-87996-1 (hbk.).

[9] Guercio, Diane Del; Tkac, Paula A. Star power: The effect of morningstar ratings on mutual fund flow. *The Journal of Financial and Quantitative Analysis*, 43(4):907–936, 2008.

[10] Morey, Matthew R. . Kiss or death: A 5-star morningstar mutual fund rating. 2003. doi: http://dx.doi.org/10.2139/ssrn.455240.

[11] Pinelis, Iosif. Optimal two-value zero-mean disintegration of zero-mean random variables. *Electron. J. Probab.*, 14:663–727, 2009. doi: {https://doi.org/10.1214/EJP.v14-633}.

[12] Shaked, Moshe; Shanthikumar, George. *Stochastic Orders*. Springer Series in Statistics. Springer-Verlag New York, 1 edition, 1991. ISBN 978-0-387-34675-5.