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On the derivation of closed-form expressions for displacements, strains and stresses inside a poroelastic inclusion

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Abstract

This note provides the derivation of closed-form expressions for elastic displacements, strains, and stresses inside an inclusion. Jansen et al. (2019) and Wu et al. (2021) obtained correct expressions for the stresses inside an inclusion, but their derivation of these expressions contained mistakes. In this note, the correct derivation of expressions for the stresses inside an inclusion is presented and some of the results of the aforementioned studies are clarified.

1 Introduction

The linear elastic displacements, strains, and stresses due to pore pressure changes in a reservoir can be determined with inclusion theory. Eshelby [1957] first introduced inclusion theory to compute the stresses around elliptical inclusions. This approach and the closely-related nucleus of strain concept were later adopted to estimate subsidence and stress fields outside elliptical subsurface reservoirs [Geertsma, 1973, Segall, 1985, 1989, 1992, Segall et al., 1994]. Stresses inside the reservoir were considered by Segall and Fitzgerald [1998], but the stresses inside elliptical inclusion are uniform. More recently, Soltanzadeh and Hawkes [2008], Jansen et al. [2019], Lehner [2019], and Wu et al. [2021] considered the stresses inside reservoirs of various shapes. Soltanzadeh and Hawkes [2008] do not present analytical expressions for the stresses, while the other three studies do present closed-form expressions. Although all these studies presented correct final solutions for the stress field, their derivations included a conceptual step that involves mathematical subtleties and in some cases errors. The aim of this report is to clarify and correct the derivation of closed-form expressions for the stresses inside an inclusion and in particular address the derivations by Jansen et al. [2019] and Wu et al. [2021] as published in *Journal of Geophysical Research - Solid Earth*.

2 General equations of inclusion theory

A thorough explanation of inclusion theory is given by Mura [1987] and Rudnicki [2011]. Here, we will briefly go through the key equations. We consider an inclusion domain Ω within an infinite homogeneous elastic space. When the inclusion is exposed to an eigenstress σ^*_{ij} or eigenstrain ϵ^*_{ij}, there is an elastic response. The general expression for the displacements u_i due to the eigenstress or eigenstrain is given by

$$u_i(\mathbf{x}) = \int_\Omega \sigma^*_{kl}(\mathbf{x}')g_{ik,l}(\mathbf{x},\mathbf{x}')d\Omega = \int_\Omega C_{klmn}\epsilon^*_{mn}(\mathbf{x}')g_{ik,l}(\mathbf{x},\mathbf{x}')d\Omega, \tag{1}$$

where C_{ijkl} is the fourth-order stiffness tensor and g_{ij} are Green's functions, which represent the equilibrium solution for the displacement at point $\mathbf{x}$ due to a point force at another point $\mathbf{x}'$. Expressions for the Green's functions will be given later.

The total strains can be obtained from the displacements through the compatibility equations

$$\epsilon_{ij}(\mathbf{x}) = \frac{1}{2}\big(u_{j,i}(\mathbf{x}) + u_{i,j}(\mathbf{x})\big). \tag{2}$$

The elastic strains e_{ij} are defined as the difference between the total strain and the eigenstrain

$$e_{ij}(\mathbf{x}) = \epsilon_{ij}(\mathbf{x}) - \epsilon^*_{ij}(\mathbf{x}). \tag{3}$$

The stresses can be obtained from the elastic strains by Hooke's law

$$\sigma_{ij}(\mathbf{x}) = C_{ijkl}(\epsilon_{kl}(\mathbf{x}) - \epsilon^*_{kl}(\mathbf{x})) = C_{ijkl}\epsilon_{kl}(\mathbf{x}) - \sigma^*_{ij}(\mathbf{x}). \tag{4}$$

These are the general equations of inclusion theory. In this report, we will assume that the elastic properties are isotropic, such that Hooke's law can be simplified to

$$\sigma_{ij}(\mathbf{x}) = \lambda[\epsilon_{kk}(\mathbf{x}) - \epsilon^*_{kk}(\mathbf{x})]\delta_{ij} + 2\mu[\epsilon_{ij}(\mathbf{x}) - \epsilon^*_{ij}(\mathbf{x})] = \lambda\delta_{ij}\epsilon_{kk}(\mathbf{x}) + 2\mu\epsilon_{ij}(\mathbf{x}) - \sigma^*_{ij}(\mathbf{x}), \tag{5}$$

where λ and μ are Lamé's first and second parameter, respectively, and the kk subscript indicates summation. In the next section, we will apply this approach to determine the displacements, strains, and stresses resulting from fluid production or injection in subsurface reservoirs under the assumption of two-dimensional plane strain.

3 Application to plane strain poroelasticity

We represent the reservoir as an inclusion that experiences an eigenstress due to changes in pore pressure. The magnitude of the eigenstress is then

$$\sigma^*_{ij} = \alpha p \delta_{ij} \delta_\Omega, \tag{6}$$

where α is Biot's coefficient, p is the change in pore pressure, δ_{ij} is the regular Kronecker delta, and δ_Ω is a modified Kronecker delta which equals 1 inside the inclusion and 0 outside. Note that the eigenstress tensor only has non-zero values along the main diagonal (i.e, pore pressure changes do not directly induce shear stresses). In this report, we will assume that pore pressure changes are uniform across the reservoir. Hence, the eigenstress components will also be uniform across the reservoir. An expression for the eigenstrain ϵ^*_{ij} can be obtained from Equation 6 using Hooke's law. Under plane strain conditions, this yields

$$\epsilon^*_{ij} = \frac{\alpha p(1+\nu)}{3K} \delta_{ij} \delta_\Omega, \tag{7}$$

where K is the bulk modulus, which is the stress-free strain under plane strain conditions [Wang, 2000].

Substituting Equation 6 into 1 yields

$$u_i(x, y) = \alpha p \iint_\Omega \frac{\partial g_{ix}}{\partial x'} + \frac{\partial g_{iy}}{\partial y'} dx' dy', \tag{8}$$

where the Green's functions under plane strain conditions are given by Mura [1987] as

$$g_{xx}(x, y, x', y') = \frac{1}{8\pi(1-\nu)\mu} \left(\frac{(x-x')^2}{R^2} - (3-4\nu) \ln R \right), \tag{9a}$$

$$g_{yy}(x, y, x', y') = \frac{1}{8\pi(1-\nu)\mu} \left(\frac{(y-y')^2}{R^2} - (3-4\nu) \ln R \right), \tag{9b}$$

$$g_{xy}(x, y, x', y') = \frac{1}{8\pi(1-\nu)\mu} \frac{(x-x')(y-y')}{R^2}, \tag{9c}$$

with $R = \sqrt{(x-x')^2 + (y-y')^2}$. Substituting Equations 9a-9c into 8 yields

$$u_x(x, y) = \frac{D}{2} \iint_\Omega \frac{x-x'}{R^2} dx' dy' = \frac{D}{2} \iint_\Omega g_x \; dx' dy', \tag{10a}$$

$$u_y(x, y) = \frac{D}{2} \iint_\Omega \frac{y-y'}{R^2} dx' dy' = \frac{D}{2} \iint_\Omega g_y \; dx' dy', \tag{10b}$$

with $D = \frac{(1-2\nu)\alpha p}{2\pi(1-\nu)\mu}$. We define G_x and G_y as the double integrals of the Green's functions

$$G_x = \iint_\Omega g_x \; dx' dy', \tag{11a}$$

$$G_y = \iint_\Omega g_y \; dx'dy'. \tag{11b}$$

As noted by Wu et al. [2021], Fubini's theorem does not hold inside the inclusion due to the singularity in g_x and g_y at $(x, y) = (x', y')$. Therefore, changing the order of integration yields different results in this region. The proper way to solve the double integrals for points inside the inclusion is shown in Section 4. Expressions for G_x and G_y are given in Section 5 for rectangular inclusions and Section 6 for triangular inclusions.

Once the displacement field is known, the total strains ϵ_{ij} are obtained by differentiation as

$$\epsilon_{xx} = \frac{D}{2}\frac{\partial}{\partial x}\iint_\Omega g_x \; dx'dy' = \frac{D}{2}\frac{\partial G_x}{\partial x} = \frac{D}{2}G_{xx}, \tag{12a}$$

$$\epsilon_{yy} = \frac{D}{2}\frac{\partial}{\partial y}\iint_\Omega g_y \; dx'dy' = \frac{D}{2}\frac{\partial G_y}{\partial y} = \frac{D}{2}G_{yy}, \tag{12b}$$

$$\epsilon_{xy} = \frac{D}{4}\left(\frac{\partial}{\partial x}\iint_\Omega g_y \; dx'dy' + \frac{\partial}{\partial y}\iint_\Omega g_x \; dx'dy'\right) = \frac{D}{4}\left(\frac{\partial G_y}{\partial x} + \frac{\partial G_x}{\partial y}\right) = \frac{D}{2}G_{xy}. \tag{12c}$$

Note that we take the derivative of the displacements outside of the double integral. Due to the singularity in the Green's functions, Leibniz integral rule is not valid and thus switching the order of differentiation and integration is not allowed for points inside the inclusion [Mura, 1987, p. 12]. Nevertheless, earlier studies placed the derivative under the integral sign to derive Green's functions for the strains and stresses [Soltanzadeh and Hawkes, 2008, Jansen et al., 2019, Wu et al., 2021]. However, properly solving the resulting integrals (with the procedure shown in Section 4) does not yield the correct expression for the strains and stresses. This approach is only valid for points outside the inclusion. Possibly, this approach has been wrongfully adopted from studies which only considered stresses outside the inclusion. Expressions for G_{xx}, G_{yy}, and G_{xy} are presented in Section 5 and 6.

Finally, we compute the stresses from the elastic strains using Hooke's law (Equation 5). This yields

$$\sigma_{xx} = (\lambda + 2G)\epsilon_{xx} + \lambda\epsilon_{yy} - \alpha p \, \delta_\Omega, \tag{13a}$$

$$\sigma_{yy} = \lambda\epsilon_{xx} + (\lambda + 2G)\epsilon_{yy} - \alpha p \, \delta_\Omega, \tag{13b}$$

$$\sigma_{xy} = 2G\epsilon_{xy}. \tag{13c}$$

In the following sections, we present the procedure to solve the double integrals for points inside the inclusion (Section 4), present expressions for G_x, G_y, G_{xx}, G_{yy}, and G_{xy} for rectangular inclusions (Section 5) and triangular inclusion (Section 6). Since we consider linear elasticity, we can use the superposition principle to combine rectangular and triangular inclusion to recreate the geometry of a faulted reservoir. In Section 7 we present conditions which the computed strains must comply with, which is used to verify our expressions. Finally, in Section 8 we derive an alternative expression for the normal stresses that was used by Jansen et al. [2019] and Wu et al. [2021].

4 Double integral for points inside the inclusion

Outside of the inclusion, the Green's functions g_x and g_y can be integrated as usual. Inside the inclusion however, the singularity at $(x, y) = (x', y')$ complicates the matter. Therefore, we will present the procedure to obtain the correct result to the integral for points inside the inclusion here.

We consider the integral for the horizontal displacement

$$I = \iint_{\Omega} g_x \; dx'dy', \tag{14}$$

for a point (x, y) inside a rectangular inclusion

$$\Omega = \{(x', y') \in \mathbb{R}^2 \mid p \leq x' \leq q, \; r \leq y' \leq s\}. \tag{15}$$

Then, we remove a square S_δ of size $2\delta \times 2\delta$ around the singularity

$$S_\delta = \{(x', y') \in \mathbb{R}^2 \mid x - \delta \leq x' \leq x + \delta, \; y - \delta \leq y' \leq y + \delta\}, \tag{16}$$

from the original domain Ω, which yields

$$\Omega_\delta = \Omega \setminus S_\delta. \tag{17}$$

Since g_x is anti-symmetric with respect to the line $x' = x$, which indicates that for all $\delta > 0$ the principal value of the integral is equal to zero

$$P.V. \iint_{S_\delta} g_x \; dx'dy' = 0. \tag{18}$$

Next, we compute the integral in Ω_δ

$$\begin{aligned} I_\delta = \int_r^s \int_p^{x-\delta} g_x \; dx'dy' + \int_r^s \int_{x+\delta}^q g_x \; dx'dy' + \int_r^{y-\delta} \int_{x-\delta}^{x+\delta} g_x \; dx'dy' \\ + \int_{y+\delta}^s \int_{x-\delta}^{x+\delta} g_x \; dx'dy'. \end{aligned} \tag{19}$$

Since the singularity has been removed from the domain, Fubini's theorem holds and changing the order of integration does not affect this result. The result is also independent of δ. The solution to the integral in Equation 14 is then given by

$$I = I_\delta + P.V. \iint_{S_\delta} g_x \; dx'dy' = I_\delta. \tag{20}$$

This result also holds for triangular inclusions.

The Green's function for the vertical displacement g_y is anti-symmetric with respect to the line $y' = y$, and thus for all $\delta > 0$ its principal value also equals zero

$$P.V. \iint_{S_\delta} g_y \; dx'dy' = 0. \tag{21}$$

Hence, the same approach can be used for the vertical displacement by replacing g_x with g_y in the equations above.

5 Rectangular inclusion

We consider a rectangle with corners (p, r), (q, r), (q, s), and (p, s). All the expressions presented in this section are valid for points inside and outside the inclusion. Integrating the Green's functions for the displacements, while taking care around the singularity at $(x, y) = (x', y')$, yields

$$\begin{aligned} G_x = \int_r^s \int_p^q g_x \, dx' dy' = \frac{y-s}{2} \ln \left(\frac{(x-q)^2 + (y-s)^2}{(x-p)^2 + (y-s)^2} \right) - \frac{y-r}{2} \ln \left(\frac{(x-q)^2 + (y-r)^2}{(x-p)^2 + (y-r)^2} \right) \\ + (x-q) \left(\operatorname{atan} \left(\frac{y-s}{x-q} \right) - \operatorname{atan} \left(\frac{y-r}{x-q} \right) \right) \\ - (x-p) \left(\operatorname{atan} \left(\frac{y-s}{x-p} \right) - \operatorname{atan} \left(\frac{y-r}{x-p} \right) \right) \end{aligned} \tag{22}$$

$$\begin{aligned} G_y = \int_p^q \int_r^s g_y \, dx' dy' = \frac{x-q}{2} \ln \left(\frac{(x-q)^2 + (y-s)^2}{(x-q)^2 + (y-r)^2} \right) - \frac{x-p}{2} \ln \left(\frac{(x-p)^2 + (y-s)^2}{(x-p)^2 + (y-r)^2} \right) \\ + (y-s) \left(\operatorname{atan} \left(\frac{x-q}{y-s} \right) - \operatorname{atan} \left(\frac{x-p}{y-s} \right) \right) \\ - (y-r) \left(\operatorname{atan} \left(\frac{x-q}{y-r} \right) - \operatorname{atan} \left(\frac{x-p}{y-r} \right) \right) \end{aligned} \tag{23}$$

These expressions are the same as the ones given by Jansen et al. [2019].

Then, the scaled strains are obtained by taking the spatial derivatives of G_x and G_y as defined in Equation 12. This yields

$$G_{xx} = \operatorname{atan} \left(\frac{y-s}{x-q} \right) - \operatorname{atan} \left(\frac{y-r}{x-q} \right) - \operatorname{atan} \left(\frac{y-s}{x-p} \right) + \operatorname{atan} \left(\frac{y-r}{x-p} \right) \tag{24}$$

$$G_{yy} = \operatorname{atan} \left(\frac{x-q}{y-s} \right) - \operatorname{atan} \left(\frac{x-q}{y-r} \right) - \operatorname{atan} \left(\frac{x-p}{y-s} \right) + \operatorname{atan} \left(\frac{x-p}{y-r} \right) \tag{25}$$

$$G_{xy} = \frac{1}{2} \ln \left(\frac{((x-q)^2 + (y-s)^2)((x-p)^2 + (y-r)^2)}{((x-q)^2 + (y-r)^2)((x-p)^2 + (y-s)^2)} \right) \tag{26}$$

Jansen et al. [2019] found the same expressions. Wu et al. [2021] found very similar expressions, but included an extra term $-\pi \delta_\Omega$.

6 Triangular inclusion

We consider a right triangle with vertices (o, r), (p, r), and (p, s). The angle θ is given by $\theta = \operatorname{atan}\left(\frac{s-r}{p-o}\right)$. We assume that the hypotenuse lies along the line $y' = x' \tan \theta$. Using the fact that $o = r \cot \theta$, $p = s \cot \theta$, $r = o \tan \theta$, and $s = p\tan\theta$, the integrated Green's functions for the displacements in a triangular domain are

$$G_x = \int_r^s \int_{y'\cot\theta}^p g_x\, dx'dy' = \frac{y-s}{2}\ln\left((x-p)^2+(y-s)^2\right) - \frac{y-r}{2}\ln\left((x-p)^2+(y-r)^2\right)$$
$$+ (x-p)\left(\operatorname{atan}\left(\frac{y-s}{x-p}\right) - \operatorname{atan}\left(\frac{y-r}{x-p}\right)\right)$$
$$+ \sin^2\theta\left(\frac{(x-o)\cot\theta + (y-r)}{2}\ln\left((x-o^2+(y-r)^2\right)\right.$$
$$- \frac{(x-p)\cot\theta + (y-s)}{2}\ln\left((x-p)^2+(y-s)^2\right)$$
$$\left.+(x-y\cot\theta)\left(\operatorname{atan}\left(\frac{(x-o)\cot\theta+(y-r)}{x-y\cot\theta}\right) - \operatorname{atan}\left(\frac{(x-p)\cot\theta+(y-s)}{x-y\cot\theta}\right)\right)\right) \tag{27}$$

$$G_y = \int_o^p \int_r^{x'\tan\theta} g_y\, dx'dy' = \frac{x-o}{2}\ln\left((x-o)^2+(y-r)^2\right) - \frac{x-p}{2}\ln\left((x-p)^2+(y-r)^2\right)$$
$$+ (y-r)\left(\operatorname{atan}\left(\frac{x-o}{y-r}\right) - \operatorname{atan}\left(\frac{x-p}{y-r}\right)\right)$$
$$- \cos^2\theta\left(\frac{x-o+(y-r)\tan\theta}{2}\ln\left((x-o)^2+(y-r)^2\right)\right.$$
$$- \frac{x-p+(y-s)\tan\theta}{2}\ln\left((x-p)^2+(y-s)^2\right)$$
$$\left.+(y-x\tan\theta)\left(\operatorname{atan}\left(\frac{x-o+(y-r)\tan\theta}{y-x\tan\theta}\right) - \operatorname{atan}\left(\frac{x-p+(y-s)\tan\theta}{y-x\tan\theta}\right)\right)\right) \tag{28}$$

Expressions for the displacements for a triangular inclusion were not given by Jansen et al. [2019] and Wu et al. [2021].

Taking the spatial derivatives of the integrated Green's functions yields the scaled strains

$$G_{xx} = \operatorname{atan}\left(\frac{y-s}{x-p}\right) - \operatorname{atan}\left(\frac{y-r}{x-p}\right) + \frac{\sin\theta\cos\theta}{2}\ln\left(\frac{(x-o)^2+(y-r)^2}{(x-p)^2+(y-s)^2}\right)$$
$$- \sin^2\theta\left(\operatorname{atan}\left(\frac{(x-p)\cot\theta+y-s}{x-y\cot\theta}\right) - \operatorname{atan}\left(\frac{(x-o)\cot\theta+y-r}{x-y\cot\theta}\right)\right) \tag{29}$$

$$G_{yy} = \operatorname{atan}\left(\frac{x-o}{y-r}\right) - \operatorname{atan}\left(\frac{x-p}{y-r}\right) - \frac{\sin\theta\cos\theta}{2}\ln\left(\frac{(x-o)^2+(y-r)^2}{(x-p)^2+(y-s)^2}\right)$$
$$+\cos^2\theta\left(\operatorname{atan}\left(\frac{(x-p)+(y-s)\tan\theta}{y-x\tan\theta}\right) - \operatorname{atan}\left(\frac{(x-o)+(y-r)\tan\theta}{y-x\tan\theta}\right)\right) \tag{30}$$

$$G_{xy} = \frac{1}{2}\ln\left(\frac{(x-p)^2+(y-s)^2}{(x-p)^2+(y-r)^2}\right) + \frac{\sin^2\theta}{2}\ln\left(\frac{(x-o)^2+(y-r)^2}{(x-p)^2+(y-s)^2}\right)$$
$$+\sin\theta\cos\theta\left(\operatorname{atan}\left(\frac{(x-p)\cot\theta+y-s}{x-y\cot\theta}\right) - \operatorname{atan}\left(\frac{(x-o)\cot\theta+y-r}{x-y\cot\theta}\right)\right) \tag{31}$$

Again, Jansen et al. [2019] found the same expressions, but the expressions of Wu et al. [2021] included an extra term $-\pi\delta_\Omega$.

7 Verification of results

As the Green's functions are solutions to the mechanical equilibrium equations, the normal traction must be continuous everywhere. Since the eigenstress vanishes outside the inclusion, there must be a jump in the strains to maintain continuous normal traction across the interface of the inclusion. The magnitude of the required jump in strains across can be computed based only on the normal vector of the boundary [Mura, 1987, p. 39]. This approach can be used to verify our expressions for G_{xx}, G_{yy}, and G_{xy}.

The normal traction vector $\mathbf{T}$ is given by

$$\mathbf{T} = \boldsymbol{\sigma}\cdot\mathbf{n}, \tag{32}$$

with $\boldsymbol{\sigma}$ the stress tensor and $\mathbf{n}$ the outward unit vector normal to the chosen boundary of the inclusion. In our two dimensional case, this means

$$\begin{aligned} T_x &= \sigma_{xx}n_x + \sigma_{xy}n_y \\ T_y &= \sigma_{xy}n_x + \sigma_{yy}n_y \end{aligned}. \tag{33}$$

The normal traction should be continuous across the boundary of the inclusion

$$\begin{aligned} \Delta T_x &= \Delta\sigma_{xx}n_x + \Delta\sigma_{xy}n_y = 0 \\ \Delta T_y &= \Delta\sigma_{xy}n_x + \Delta\sigma_{yy}n_y = 0 \end{aligned} \tag{34}$$

where the Δ indicates the jump in the variable from just outside the inclusion to just inside the inclusion (e.g., $\Delta\sigma_{xx} = \sigma_{xx}^{out} - \sigma_{xx}^{in}$). We use Hooke's law to write the stresses in Equation 34 in terms of the strains

$$\begin{aligned} \Big[(\lambda+2\mu)\Delta\epsilon_{xx} + \lambda\Delta\epsilon_{yy} + \alpha p\Big] n_x + 2\mu\Delta\epsilon_{xy} n_y = 0 \\ 2\mu\Delta\epsilon_{xy} n_x + \Big[\lambda\Delta\epsilon_{xx} + (\lambda+2\mu)\Delta\epsilon_{yy} + \alpha p\Big] n_y = 0 \end{aligned} \tag{35}$$

The jump in strains can be expressed as

$$\Delta\epsilon_{xx} = \beta_x n_x, \tag{36a}$$

$$\Delta\epsilon_{yy} = \beta_y n_y, \tag{36b}$$

$$\Delta\epsilon_{xy} = \frac{1}{2}(\beta_x n_y + \beta_y n_x), \tag{36c}$$

where β_i represents the jump in the derivative of the displacements [Mura, 1987, p. 39]. Substituting this into Equation 35 and rearranging yields

$$\begin{aligned} \Big[(\lambda+2\mu)\beta_x n_x + \lambda\beta_y n_y\Big] n_x + \mu[\beta_x n_y + \beta_y n_x] n_y = -\alpha p n_x \\ \mu[\beta_x n_y + \beta_y n_x] n_x + \Big[\lambda\beta_x n_x + (\lambda+2\mu)\beta_y n_y\Big] n_y = -\alpha p n_y \end{aligned} \tag{37}$$

For a given boundary with known normal vector, this yields two equations for the two unknowns β_x and β_y. The general solution is

$$\beta_x = \alpha p \frac{(\lambda+\mu) n_x n_y^2 - \big((\lambda+2\mu) n_y^2 + \mu n_x^2\big) n_x}{\big((\lambda+2\mu) n_x^2 + \mu n_y^2\big)\big((\lambda+2\mu) n_y^2 + \mu n_x^2\big) - \big((\lambda+\mu) n_x n_y\big)^2}, \tag{38a}$$

$$\beta_y = \alpha p \frac{(\lambda+\mu) n_x^2 n_y - \big((\lambda+2\mu) n_x^2 + \mu n_y^2\big) n_y}{\big((\lambda+2\mu) n_x^2 + \mu n_y^2\big)\big((\lambda+2\mu) n_y^2 + \mu n_x^2\big) - \big((\lambda+\mu) n_x n_y\big)^2}. \tag{38b}$$

As an example, we consider a vertical boundary with normal vector $(n_x, n_y) = (1, 0)$. From Equation 38, we then obtain $\beta_x = -\frac{(1-2\nu)\alpha p}{2(1-\nu)\mu} = -\pi D$ and $\beta_y = 0$. Substituting these values for β into Equation 36 yields $\Delta\epsilon_{xx} = -\pi D$ and $\Delta\epsilon_{yy} = \Delta\epsilon_{xy} = 0$. This also means $\Delta G_{xx} = -2\pi$ and $\Delta G_{yy} = \Delta G_{xy} = 0$. Our expressions for G_{xx}, G_{yy}, and G_{xy} satisfy this condition.

8 Alternative expression for the normal stresses

In this section, we show that our earlier definition of the normal stresses in Equation 13 is equivalent to the definitions given by Jansen et al. [2019] and Wu et al. [2021]. For this purpose, we must first consider the stress arching ratios γ_{ij}, which are defined by Mulders [2003] and Soltanzadeh and Hawkes [2008] as the ratios of stress change over the pore pressure change multiplied by Biot's coefficient

$$\gamma_{ij} = -\frac{\sigma_{ij}}{\alpha p}. \tag{39}$$

Soltanzadeh and Hawkes [2008] state that for a cylindrical inclusion under plane strain conditions, the sum of the two stress arching ratios in the normal direction is constant

$$\gamma_{xx} + \gamma_{yy} = \frac{1-2\nu}{1-\nu}\delta_\Omega. \tag{40}$$

Wu et al. [2021] state that this holds for any inclusion under plane strain conditions, regardless of its shape. This indeed holds for our expressions for rectangular and triangular inclusions. We start with writing Equation 40 in terms of the stresses

$$-\frac{\sigma_{xx} + \sigma_{yy}}{\alpha p} = \frac{1-2\nu}{1-\nu}\delta_\Omega. \tag{41}$$

Using Hooke's law (Equation 13), the stresses can be rewritten in terms of the strains

$$-\frac{2(\lambda+\mu)(\epsilon_{xx} + \epsilon_{yy}) - 2\alpha p \delta_\Omega}{\alpha p} = \frac{1-2\nu}{1-\nu}\delta_\Omega \tag{42}$$

The strains can be written in terms of the Green's functions, which yields

$$-\frac{D(\lambda+\mu)(G_{xx} + G_{yy}) - 2\alpha p \delta_\Omega}{\alpha p} = \frac{1-2\nu}{1-\nu}\delta_\Omega. \tag{43}$$

Using $D = \frac{(1-2\nu)\alpha p}{2\pi(1-\nu)\mu}$ and $\frac{\lambda+\mu}{\mu} = \frac{1}{1-2\nu}$ and rearranging then yields

$$G_{xx} + G_{yy} = 2\pi\delta_\Omega. \tag{44}$$

Thus, the sum of G_{xx} and G_{yy} has a constant value of 2π inside the inclusion and zero outside.

Our earlier definition of the stresses (Equation 13) can be written as

$$\sigma_{xx} = \lambda(\epsilon_{xx} + \epsilon_{yy}) + 2\mu\epsilon_{xx} - \alpha p \delta_\Omega, \tag{45a}$$

$$\sigma_{yy} = \lambda(\epsilon_{xx} + \epsilon_{yy}) + 2\mu\epsilon_{yy} - \alpha p \delta_\Omega, \tag{45b}$$

or in terms of the Green's functions

$$\sigma_{xx} = \frac{\lambda D}{2}(G_{xx} + G_{yy}) + \mu D G_{xx} - \alpha p \delta_\Omega, \tag{46a}$$

$$\sigma_{yy} = \frac{\lambda D}{2}(G_{xx} + G_{yy}) + \mu D G_{yy} - \alpha p \delta_\Omega. \tag{46b}$$

Now substituting in Equation 44 yields

$$\sigma_{xx} = C G_{xx} + (\lambda\pi D - \alpha p)\delta_\Omega, \tag{47a}$$

$$\sigma_{yy} = C G_{yy} + (\lambda\pi D - \alpha p)\delta_\Omega, \tag{47b}$$

with $C = \mu D = \frac{(1-2\nu)\alpha p}{2\pi(1-\nu)}$. Finally, using $\lambda = \frac{2\mu\nu}{1-2\nu}$ and the definition of $D = \frac{(1-2\nu)\alpha p}{2\pi(1-\nu)\mu}$ gives

$$\sigma_{xx} = C(G_{xx} - 2\pi\delta_\Omega), \tag{48a}$$

$$\sigma_{yy} = C(G_{yy} - 2\pi\delta_\Omega). \tag{48b}$$

Thus, the factor $-2\pi C\delta_\Omega$ in Jansen et al. [2019] and Wu et al. [2021] contains both the contribution of the volumetric strain and the eigenstress to the normal stresses.

9 Conclusions

We presented the derivation of closed-form expressions for displacements, strains, and stresses inside a poroelastic inclusion. This derivation differs from both Jansen et al. [2019] and Wu et al. [2021], who incorrectly applied the Leibniz integral rule and integrated Green's functions for the stresses. Jansen et al. [2019] did not take note of the singularity at $(x, y) = (x', y')$ and solved the double integral incorrectly, but managed their expressions for G_{xx}, G_{yy}, and G_{xy} satisfy the necessary jump in strains and thus continuous normal traction. Wu et al. [2021] correctly solved the double integral according to Section 4, but their solution for G_{xx} and G_{yy} does not satisfy the required jump in strains. As Leibniz integral rule is not valid inside the inclusion, the Green's function for the stresses used by Jansen et al. [2019] and Wu et al. [2021] is only valid outside the inclusion. Wu et al. [2021] did obtain correct expressions for the stresses, but did so incorrectly by using two different values for the eigenstress σ^*_{ij} throughout the solution process.

Furthermore, we showed that our expressions for the stresses are equivalent to those presented by Jansen et al. [2019] and Wu et al. [2021]. Hence, while their derivations contained errors, they nevertheless obtained the correct expressions for the stresses. Both these papers included the term $-2\pi C\delta_\Omega$ in the normal stresses, but had different, incorrect explanations for its origin. Here, we showed that this term comprises of the contributions of the eigenstress and the volumetric strain to the normal stresses.

References

J. D. Eshelby. The determination of the elastic field of an ellipsoidal inclusion and related problems. *Proceedings of the Royal Society of London, Series A*, 241:376–396, 1957. URL `https://dx.doi.org/10.1098/rspa.1957.0133`.

J. Geertsma. A basic theory of subsidence due to reservoir compaction: the homogeneous case. *Verhandelingen Koninklijk Nederlandsch Geologisch Mijnbowkundig Genootschap*, 2S:43–61, 1973.

J.D. Jansen, P. Singhal, and F.C. Vossepoel. Insights from closed-form expressions for injection- and production-induced stresses in displaced faults. *Journal of Geophysical Research - Solid Earth*, 124:7193–7212, 2019. URL `https://doi.org/10.1029/2019JB017932`.

F. Lehner. An analysis of depletion-induced fault stressing - new closed-form analytical solutions. Technical report, Nederlandse Aardolie Maatschappij, 2019. URL `http://www.nam.nl/feiten-en-cijfers/onderzoeksrapporten.html`.

F.M.M. Mulders. *Modelling of Stress Development and Fault Slip in and Around a Producing Gas Reservoir.*

PhD thesis, Delft University of Technology, 2003. URL http://resolver.tudelft.nl/uuid:be742135-10d7-4d69-bdee-f808b5926065.

T. Mura. *Micromechanics of defects in solids, 2^{nd} ed.* Martinus Nijhoff Publishers, Dordrecht, 1987.

J.W. Rudnicki. Eshelby's technique for analyzing inhomogeneities in geomechanics. In Y.M. Leroy and F.K. Lehner, editors, *Mechanics of Crustal Rocks*, pages 43–72. Springer Vienna, 2011. ISBN 978-3-7091-0939-7.

P. Segall. Stress and subsidence resulting from subsurface fluid withdrawal in the epicentral region of the 1983 Coalinga earthquake. *Journal of Geophysical Research*, 90:6801–6816, 1985. URL https://doi.org/10.1029/JB090iB08p06801.

P. Segall. Earthquakes triggered by fluid extraction. *Geology*, 17:942–946, 1989. URL https://doi.org/10.1130/0091-7613(1989)017<0942:ETBFE>2.3.CO;2.

P. Segall. Induced stresses due to fluid extraction from axisymmetric reservoirs. *Pure and Applied Geophesics*, 139:535–560, 1992. URL https://doi.org/10.1007/BF00879950.

P. Segall and S.D. Fitzgerald. A note on induced stress changes in hydrocarbon reservoirs and geothermal reservoirs. *Tectonophysics*, 289:117–128, 1998. URL https://doi.org//10.1016/S0040-1951(97)00311-9.

P. Segall, J.R. Grasso, and A. Mossop. Poroelastic stressing and induced seismicity near the Lacq gas field, southwestern France. *Journal of Geophysical Research*, 99:15423–15438, 1994. URL https://doi.org/10.1029/94JB00989.

H. Soltanzadeh and C.D. Hawkes. Semi-analytical models for stress change and fault reactivation induced by reservoir production and injection. *Journal of Petroleum Science and Engineering*, 60:71–85, 2008. URL https://doi.org/10.1016/j.petrol.2007.05.006.

H. Wang. *Theory of linear poroelasticity with applications to geomechanics and hydrogeology.* Princeton University Press, 2000.

H. Wu, V. Vilarrasa, S. De Simone, M. Saaltink, and F. Parisio. Analytical solution to assess the induced seismicity potential of faults in pressurized and depleted reservoirs. *Journal of Geophysical Research - Solid Earth*, 126: e2020JB020436, 2021. URL https://doi.org/10.1029/2020JB020436.