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Discrete Darboux based Fast Inverse Nonlinear Fourier Transform Algorithm for Multi-solitons

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Abstract *A fast algorithm for constructing multi-solitons with linear complexity in the number of samples and eigenvalues is introduced. The algorithm is shown to be significantly faster than the conventional Darboux transform in a numerical example, with acceptable error.*

Introduction

The nonlinear Fourier transform (NFT) for the nonlinear Schrödinger equation (NSE),

$$i\partial_x q = \partial_t^2 q + 2|q|^2 q, \quad (t,x) \in \mathbb{R} \times \mathbb{R}_+, \qquad (1)$$

was first studied by Zakharov and Shabat[1]. Here, $q(x,t)$ is the slowly varying complex envelope of the electric field, x is the real spatial coordinate along the fiber and t is retarded time. The NSE is commonly used to model propagation of optical field in a loss-less single mode fiber under Kerr-type focusing nonlinearity[1]. The nonlinear Fourier spectrum corresponding to a given initial field $q(0,t)$ consists of the continuous spectrum $\hat{q}(\xi) = b(\xi)/a(\xi)$, $\xi \in \mathbb{R}$ and the discrete part $\{\zeta_k, \tilde{q}_k\} \in \mathbb{C}^2$ given by $a(\zeta_k) = 0$, $\Im(\zeta_k) \geq 0$, $\tilde{q}_k = b(\zeta_k)/\partial_\zeta a(\zeta_k)$. The discrete spectrum consists of the ordered pairs of eigenvalues ζ_k and spectral amplitudes $\tilde{q}_k$ (The precise definitions can be found in[1]). It corresponds to the solitonic component of the potential. A signal $q(x,t)$ having only solitonic components (i.e., the continuous spectrum is null) is known as a multi-soliton. The eigenvalues ζ_k do not change as the signal $q(0,t)$ travels through the fiber and the norming constants are simply multiplied by a factor $e^{-4\zeta_k^2 L}$ at $x = L$. Such a simple relationship negates the need for back-propagation, making the use of discrete spectrum interesting for information transmission. Currently there is lot of interest in communications systems based on NFT, many of which specifically use multi-solitons[2 3 4 10]. The goal of this paper is to introduce an inverse NFT algorithm for multi-solitons which is faster than the classical solution already for relatively low number of eigenvalues and also has a better asymptotic complexity compared to algorithms available in literature[11].

Generation of multi-solitons

Given a discrete spectrum $\{\zeta_k, b_k\}$, we need to compute the corresponding multi-soliton potential. This problem can be approached in several ways. The classical Darboux transform (CDT)[5] is the standard algorithm for generation of multi-solitons. The CDT has an overall complexity of $\mathcal{O}(K^2N)$ floating point operations (FLOPS), where K is the number of eigenvalues and N is the number of samples. This means that the algorithm slows down significantly for high numbers of eigenvalues. A few fast algorithms have been published[6] or are under research[11]. The fastest algorithm found in literature[6] has a complexity of $\mathcal{O}(N\log^2 N)$ but does not offer complete control over the norming constants, while the algorithm in[11] has a complexity of $\mathcal{O}(N(K + \log^2 N))$. The existing solution[11] uses a mixed framework of continuous and discrete algorithms which slows it down. In this paper we work completely in the discrete domain allowing for development of faster algorithm.

New Fast Inverse NFT for Multi-Solitons

We start by outlining the discrete version of the CDT, which will be the basis for the new algorithm. Ablowitz and Ladik[7] proposed the following general discrete scattering problem,

$$V_{n+1} = \begin{bmatrix} z + R_n S_n & Q_n + z^{-1} S_n \\ R_n + z T_n & z^{-1} + Q_n T_n \end{bmatrix} V_n, \qquad (2)$$

where $z = e^{-i\zeta h}$ is the transformed eigenvalue and h is the step-size. The discrete-time potential Q_n is related to the continuous-time potential as $q(nh) = h^{-1} Q_n + O(h^2)$. The choice $R_N = \pm Q_n^*$ and $S_n = T_n = 0$ leads to a discrete version of the Zakharov-Shabat problem. The two-dimensional eigenfunction is given by $V_n(z)$. The compatibility condition of the discrete eigenvalue problem with the corresponding time-evolution equation gives us the discrete NSE. The dependence on z and t of the terms is omitted for sake of brevity. It has been shown[7] that the discrete system in Eq. 2 can be solved with a discrete NFT, which again consists of a continuous spectrum $\tilde{q^D}(\xi) = b^D(\xi)/a^D(\xi)$, $|\xi| = 1$, and a discrete spectrum $\{z_k, \tilde{q^D}_k\}$, $a^D(z_k) = 0$, $\tilde{q^D}_k = b^D(z_k)/\partial_z a^D(z_k)$ analogous to the continuous system. Xianguo[8] derived a discrete Darboux approach for adding/removing eigenvalues using the Ablowitz-Ladik eigenvalue problem in Eq. 2 analogous to CDT[5]. Other discretization schemes of the Zakharov-Shabat problem can be found in literature[11]. In this paper, we use the Split-Magnus (SM) discretization,

$$V_{n+1} = L_n V_n, \quad L_n = \begin{pmatrix} z^{-1} & Q_{n+1/2} \\ R_{n+1/2} & z \end{pmatrix}. \tag{3}$$

Taking a hint from the structure of the discrete Darboux matrix used in[9], we make the ansatz

$$V'_n = M_n V_n, \quad M_n = \begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} \tag{4}$$

$$\begin{aligned} A_n &= z^{-k} + \sum_{j=\{-k+2,-k+4,\dots,k\}} a_n^j z^j, \\ B_n &= \sum_{j=\{-k+1,-k+3,\dots,k-1\}} b_n^j z^j, \\ C_n &= \sum_{j=\{-k+1,-k+3,\dots,k-1\}} c_n^j z^j, \\ D_n &= z^k + \sum_{j=\{-k,-k+2,\dots,k-2\}} d_n^j z^j. \end{aligned} \tag{5}$$

The discrete Darboux matrix M_n has degree k. The ' over the terms denotes the terms after the Darboux update. The initial solution of the system in Eq. 3 for $Q_{n+1/2} = R_{n+1/2} = 0$ is

$$V_n = \begin{bmatrix} z^{-n} & 0 \\ 0 & z^n \end{bmatrix}. \tag{6}$$

Following a procedure similar to the one given in[8] we can compute all the coefficients in M_n. The details are skipped due to space limitations. Then the potential at a point n can be computed from these Darboux coefficients. To arrive at the exact relation we start with the discretization in Eq. 3 after an update, which satisfies $V'_{n+1} = L'_n V'_n$. But V'_{n+1} can also be found by, $V'_{n+1} = M_{n+1} V_{n+1}$, where M_{n+1} is the discrete Darboux matrix at $n+1$. We can hence write

$$M_{n+1} V_{n+1} = L'_n M_n V_n. \tag{7}$$

Equating the coefficients of different powers of z,

$$Q'_{n+1/2} = \frac{-b_n^{(-k+1)}}{d_n^{(-k)}}, \quad R'_{n+1/2} = \frac{-c_n^{(k-1)}}{a_n^{(k)}} \tag{8}$$

$$Q'_{n+1/2} = b_{n+1}^{(k-1)}, \quad R'_{n+1/2} = c_{n+1}^{(-k+1)}. \tag{9}$$

Starting with Eq. 9 to compute the potential at $n+1/2$ from the Darboux coefficients at $n+1$, we can then use the relation,

$$V'_n = L'^{-1}_n V'_{n+1}, \tag{10}$$

to compute the Darboux coefficients at n. By iterating between the steps of computing the potential and updating the Darboux coefficients, we can compute the potential at all the staggered grid points $x_n = (n+1/2)h$ starting from $n+1$ to $-\infty$. Similarly, using Eq. 8 and

$$V'_{n+1} = L'_n V'_n, \tag{11}$$

the potential at all the staggered grid points starting from n to ∞ can be computed.
For the numerical implementation we start with a grid of 2N+1 points from -N to N. The computation of the Darboux coefficients is typically well conditioned at $n = 0$ and hence is chosen as the starting point.

Algorithm

Input: $\{\zeta_k, b_k\}$, h, N
Output: $Q_{n+1/2} = hq((n+1/2)h) + O(h^3)$

- Transform the eigenvalues $(z_k = e^{i\zeta_k h})$.
- Use Eq. 4 to find the polynomial representation of the eigenfunctions V_n at $n = 0$.
- For $n = 0, \cdots, N$ using Eq. 8 do:
  - $Q'_{n+1/2} = -b_n^{(-k+1)}/d_n^{(-k)}$
  - $R'_{n+1/2} = -Q'^*_{n+1/2}$
  - $V'_{n+1} = L'_n V'_n$
- For $n = 0, \cdots, -N$ using Eq. 9 do:
  - $Q'_{n+1/2} = b_{n+1}^{(k-1)}$
  - $R'_{n-1/2} = -Q'^*_{n-1/2}$
  - $V'_{n-1} = L'^{-1}_n V'_n$

We note that this algorithm also works for the trapezoidal discretization[11].

Complexity Analysis

The performance of the fast algorithm was compared against an efficient implementation of classical Darboux transform (Algorithm 2 in[12]) in MATLAB. Through manual counting, for K eigenvalues and N samples, the CDT algorithm requires $NK(15+11K)/2$ FLOPS while the fast algorithm requires $(81K^2)/2 + (20N + 37/2)K + 15N - 36$

FLOPS. For communication problems $K < N$ and hence the fast algorithm has a computational complexity of $\mathcal{O}(KN)$ while CDT has $\mathcal{O}(K^2N)$.

Numerical Example

As a practical example, the discrete spectrum from the experiment conducted in[10] was chosen for some arbitrary 14 bits $(K = 7)$ of data. The multi-soliton solution was constructed on the approximately 14π support [-22 22]. Based on width of the signal and speed of the DAC in[10] a step-size of 0.08 was chosen $(N = 551)$. The complexity analysis suggests that the runtime t_f of the new algorithm should be around half as large as t_{CDT}. When comparing the runtimes of actual MATLAB implementations, we found that $t_{CDT}/t_f = 1.2$. The spectrum mentioned in[10] is extended to 11 eigenvalues ($\zeta_k = [-1 + 0.45i, -0.8+0.3i, -0.6+0.45i, -0.4+0.30i, -0.2+0.45i, 0.0 + 0.30i, 0.2 + 0.45i, 0.4 + 0.30i, 0.6 + 0.45i, 0.8 + 0.30i, 1.0 + 0.45i]$) with the independently modulated spectral amplitudes ($\ln(|\tilde{q}_k|) = [11.85, 7.06, 7.69, 7.69, 5, 3.81, 5, 1.93, 1.93, -0.62, -5.43]$) for some arbitrary 22 bits of data. The new algorithm is two times faster than CDT, i.e. $t_{CDT}/t_f = 2$, for the extended discrete spectrum. The error in the generated potentials, which arises because the new algorithm works in a discretized model, was low as shown in Fig. 1.

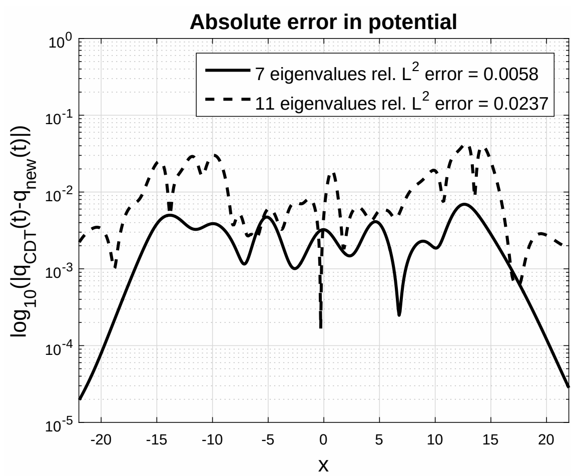

Fig. 1: The error in the potential constructed by the fast algorithm.

Conclusions

A fast inverse nonlinear Fourier transform algorithm for multi-solitons was introduced based on a fully discrete framework. The algorithm can construct the multi-soliton potentials with acceptable errors faster than CDT even if the number of eigenvalues is small, which was demonstrated with a practical example[10] and has a FLOPS complexity of $\mathcal{O}(KN)$. The stated algorithm can be extended to different discretizations of the Zakahrov-Shabat problem. In further experiments not reported here, it was observed that the numerical precision of the floating point operations needs to increased in the limit $h \to 0$ in order to avoid the break down of the new algorithm. This behaviour is under study.

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