**Delft University of Technology**
**Faculty of Electrical Engineering, Mathematics & Computer Science**
**Delft Institute of Applied Mathematics**

# Riemann's Explicit Formula and the Prime Number Theorem

## (Dutch title: Riemann's Expliciete Formule en de Priemgetallenstelling)

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**BSc thesis APPLIED MATHEMATICS**

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**Abstract**

This thesis presents an insight in the Riemann zeta function and the prime number theorem at an undergraduate mathematical level. The main goal is to construct an explicit formula for the prime counting function and to prove the prime number theorem using the zeta function and a Tauberian theorem. The Riemann zeta function, defined as $\sum_{n=1}^{\infty} \frac{1}{n^s}$ for $\mathrm{Re}(s) > 1$, can be continued analytically to the whole complex plane except at $s = 1$. Two proofs of this continuation were given by Bernhard Riemann in his famous article *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse* from 1859. Those proofs are studied in detail in this thesis after introducing all the required foreknowledge on the gamma function.

The prime counting function $\pi(x)$ counts the number of primes less than or equal to $x$. An explicit formula for $\pi(x)$ in terms of the nontrivial zeros of the zeta function will be constructed in a similar way as Riemann did in his article. Finally, the prime number theorem will be proved. This theorem describes the asymptotic distribution of the primes among the natural numbers: $\lim_{x\to\infty} \frac{\pi(x)\log(x)}{x} = 1$. Using the analytic continuation of the zeta function and a Tauberian theorem, the prime number theorem can be proved quite easily with only basic theory from complex analysis.

# Contents

# Chapter 1

# Introduction

## 1.1 Counting Prime Numbers

Prime numbers have been an interesting topic for mathematicians since they were first studied by the ancient Greek. Take all the natural numbers

$$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10, \ldots$$

and the prime numbers are defined as those natural numbers greater than 1 which are only divisible by 1 and itself. So, prime numbers cannot be written as the product of two smaller natural numbers. The first number greater than 1 that is not a prime number is $4 = 2 \cdot 2$, the next is $6 = 2 \cdot 3$ and so on. The first ten prime numbers are

$$2,\ 3,\ 5,\ 7,\ 11,\ 13,\ 17,\ 19,\ 23,\ 29$$

and some questions that arise are how many prime numbers are there, how do you find them and how are they distributed among the other natural numbers? The ancient Greek found out around 300 BC that there are infinitely many prime numbers. This is known as Euclid's theorem and was proved by assuming that there are only finitely many prime numbers and then deriving a contradiction. This implies that the assumption must have been wrong.

So there are infinity many primes, but which natural numbers are then prime and which are not? A simple method to find the prime numbers less than or equal to a number $n$, was invented by Eratosthenes, the chief librarian of the great library in Alexandria. The idea is to start with 2 and cross out all multiples of 2 that are less than or equal to $n$. However, do not eliminate 2 itself. Then continue with the next number, 3, and cross out the multiples. Repeat this procedure and all the numbers that remain are the prime numbers less than or equal to $n$. This method is known as the Sieve of Eratosthenes. This method can be applied for finding small prime numbers, but for large prime numbers this algorithm is highly inefficient.

An even stronger result about primes was known more than two thousand years ago and that result is so important that it is nowadays called the fundamental theorem of arithmetic.

**Theorem 1.1** (Fundamental theorem of arithmetic)**.** *Every natural number greater than 1 is either a prime number or can uniquely be represented as a product of prime numbers.*

That a natural number $n > 1$ can be written as a product of prime numbers is more or less obvious. If $n$ is not prime, then by definition $n = a \cdot b$ with $a, b < n$. If $a$ and $b$

are not prime, then you repeat this until there are only prime numbers left in the factorization. The fact that this representation of prime numbers is unique is not that obvious, however with this theorem the prime numbers can be considered as the building blocks of the natural numbers when only multiplication is used to combine the blocks. The two theorems mentioned above can be found in one of the most important works from ancient Greece, which is *The Elements* from Euclid.

The question about how the primes are distributed among the other natural numbers is hard to answer. As a start, look at the prime counting function $\pi(x)$ which is defined as the number of primes less than or equal to a real number $x$. The graph of $\pi(x)$ has jumps at every prime number and is shown in figure 1.1 for $x$ on two different domains.

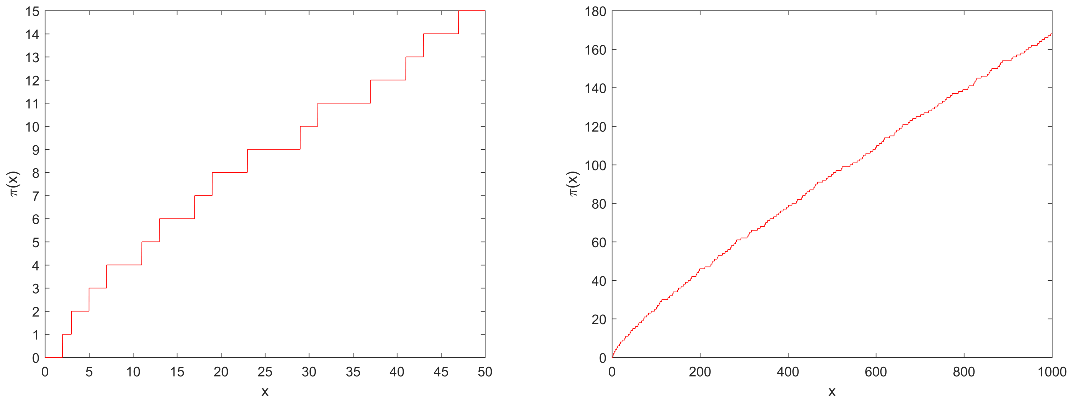


Figure 1.1: The prime counting function $\pi(x)$ for $0 \leq x \leq 50$ and $0 \leq x \leq 1000$.

The function on $0 \leq x \leq 1000$ looks smoother than that on a smaller domain. Nonetheless, the behaviour of the prime counting function is very irregular. But if we can capture an explicit formula for $\pi(x)$, then this formula might tell us something about the distribution of the primes. However, due to the irregularity and the jumps, it is not trivial that such an explicit formula exists at all!

Before getting lost in the mathematics of finding an explicit formula, an approximation of $\pi(x)$ would also be nice. This was exactly what the great mathematicians Gauss and Legendre were looking for at the end of the eighteenth century. One of the proposed approximations was

$$L(x) = \frac{x}{\log(x)},$$

where $\log(x)$ denotes the natural logarithm. In figure 1.2 both $\pi(x)$ and $L(x)$ are shown. This approximation might not be as accurate as we had hoped for, namely the difference between both graphs (the error) becomes larger as $x$ becomes larger. Nevertheless, this approximation is good if we do not look at the absolute error, but look at the *relative error* which is

$$\frac{\pi(x) - L(x)}{L(x)}.$$

The claim is that this relative error tends to zero as $x$ goes to infinity. This is known as the prime number theorem, but for now it is only a conjecture.

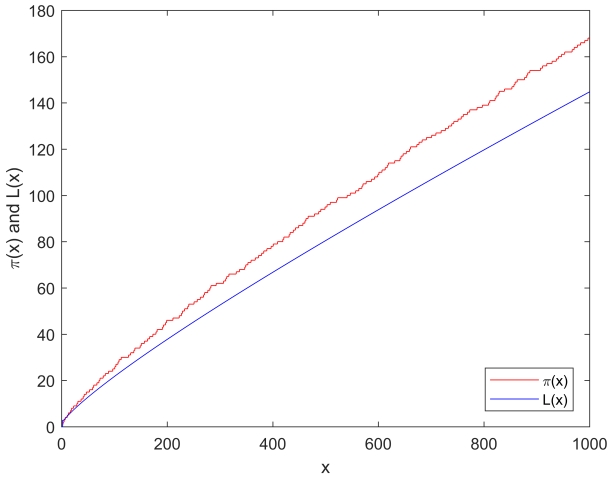


Figure 1.2: The prime counting function $\pi(x)$ and the approximation $L(x)$ for $0 \leq x \leq 1000$.

**Conjecture 1.2** (Prime number conjecture)**.** *The relative error between* $\pi(x)$ *and* $L(x)$ *goes to zero as* $x$ *goes to infinity:*

$$\lim_{x\to\infty} \frac{\pi(x) - L(x)}{L(x)} = 0$$

*or after rewriting this*

$$\lim_{x\to\infty} \frac{\pi(x)}{L(x)} = \lim_{x\to\infty} \frac{\pi(x)\log(x)}{x} = 1.$$

Before continuing on this conjecture, first a little sidetrack. The approximation $L(x)$ as in figure 1.2 is not that marvellous if you at look at the absolute error, so can we find a better one perhaps? Obviously, $L(x)$ grows too slow with respect to $\pi(x)$, meaning that the derivative of $L(x)$ is too small. Calculating this derivative using the quotient rule gives

$$L'(x) = \frac{\log x - x\frac{1}{x}}{\log^2 x} = \frac{1}{\log x} - \frac{1}{\log^2 x}.$$

The derivative can be increased by ignoring the last part with the minus sign. Then, integrating gives a function that is called $Li(x)$ of which the derivative is larger than that of $L(x)$. Define the logarithmic integral for $x > 0$ by[1]

$$Li(x) = \int_0^x \frac{1}{\log(t)} dt.$$

Figure 1.3 shows the prime counting function together with the new approximation $Li(x)$. In comparison with the other approximation $L(x)$, the logarithmic integral seems a better estimate of $\pi(x)$. In fact, $Li(x)$ is a much better estimate concerning the absolute error as can be seen in table 1.1. Regarding the relative error: it can be proven that the approximation $Li(x)$ is just a good as the approximation $L(x)$. Hence, the prime number conjecture can be reformulated as

$$\lim_{x\to\infty} \frac{\pi(x)}{Li(x)} = 1.$$

[1]Note that the integrand $\frac{1}{\log(t)}$ is not defined for $t = 1$, so the integral should be interpreted as $\lim_{\varepsilon\to 0} \left( \int_0^{1-\varepsilon} \frac{1}{\log(t)} dt + \int_{1+\varepsilon}^{x} \frac{1}{\log(t)} dt \right)$.

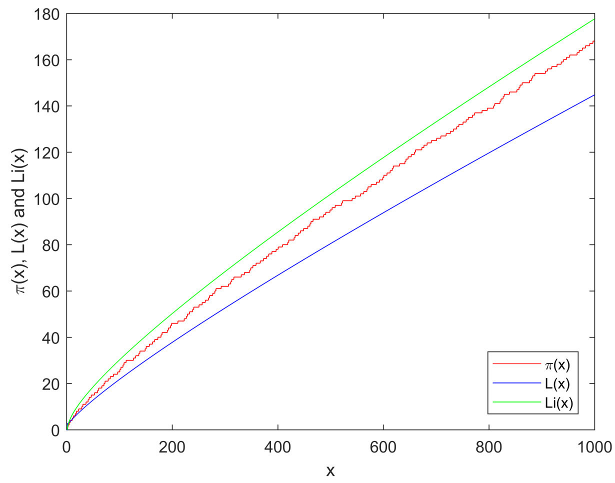


Figure 1.3: The prime counting function $\pi(x)$ and two approximations $L(x)$ and $Li(x)$ for $0 \leq x \leq 1000$.

| $x$ | $\pi(x)$ | $\pi(x) - L(x)$ | $Li(x) - \pi(x)$ |
|---|---|---|---|
| $10^6$ | $78,498$ | $6,115$ | $129$ |
| $10^8$ | $5,761,455$ | $332,774$ | $754$ |
| $10^{10}$ | $455,052,511$ | $20,758,030$ | $3,104$ |
| $10^{12}$ | $37,607,912,018$ | $1,416,706,193$ | $38,263$ |

Table 1.1: The absolute errors between $\pi(x)$ and the two approximations $L(x)$ and $Li(x)$ for some large values of $x$. The values of the errors are rounded.

Back to the proof of the prime number conjecture as formulated in conjecture 1.2. A first step towards the proof was taken by the Russian mathematician Chebyshev. He proved that

$$0.92129\ldots < \frac{\pi(x)\log(x)}{x} < 1.10555\ldots$$

holds for $x$ large enough. Even stronger, he proved that *if* the limit exists, then it would 1. But there was no proof that the limit as $x$ goes to infinity would exist. Also, the above estimate does not imply that the limit exists, namely the function $\frac{\pi(x)}{L(x)}$ might oscillate forever between $0.92129\ldots$ and $1.10555\ldots$ for $x$ very large without converging to a fixed number. So, besides the search for an explicit formula for $\pi(x)$, also a proof (or disproof) for the prime number conjecture is wanted.

Now, assuming the prime number conjecture to be true, what would that mean for the distribution of the prime numbers? Nothing could be said about the exact locations of the primes, but it can be established that the $n$th prime number $p_n$ is approximately equal to $n\log(n)$. Note that $\pi(p_n) = n$ and then the prime number conjecture gives that

$$\frac{n\log(p_n)}{p_n} \to 1 \qquad \text{as } n \to \infty. \tag{1.1}$$

Take the logarithm

$$\log\left(\frac{n\log(p_n)}{p_n}\right) \to \log(1) = 0 \qquad \text{as } n \to \infty$$

and rewrite using the properties of the logarithm

$$\log(n) + \log\big(\log(p_n)\big) - \log(p_n) \to 0 \qquad \text{as } n \to \infty.$$

Dividing by $\log(p_n)$ gives

$$\frac{\log(n)}{\log(p_n)} + \underbrace{\frac{\log\big(\log(p_n)\big)}{\log(p_n)}}_{\to 0, \text{ as } n \to \infty} - \underbrace{\frac{\log(p_n)}{\log(p_n)}}_{=1} \to 0 \qquad \text{as } n \to \infty, \tag{1.2}$$

where it should be noted that $\log\big(\log(x)\big)$ grows much slower than $\log(x)$ as $x$ tends to infinity, so that indeed their quotient goes to zero. Now, multiplying equation (1.1) and (1.2) gives after rewriting that

$$\frac{n\log(p_n)}{p_n} \cdot \frac{\log(n)}{\log(p_n)} = \frac{n\log(n)}{p_n} \to 1 \qquad \text{as } n \to \infty.$$

A fraction equals one, if and only if the numerator and denominator are equal. So from this it can be concluded that the $n$th prime number $p_n$ is approximately equal to $n\log(n)$ for large values of $n$. In fact, the larger the value of $n$, the better the approximation $n\log(n)$ is.

## 1.2 The Riemann Zeta Function

There was not much progress in the research to the distribution of the primes until a short article by Bernhard Riemann was published. Riemann (1826-1866) was a mathematician and contributed mainly to the field of differential geometry which was essential for Einstein's general theory of relativity. The branch of mathematics to which the research to prime numbers belongs is analytic number theory. Riemann's article *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse* (*On the Number of Primes Less Than a Given Magnitude*, [Riemann, 1859]) only counted eight pages and is his only contribution to the field of analytic number theory. However, this contribution was an important one: Riemann described how to obtain an explicit formula for $\pi(x)$ and introduced the mathematics which was required for proving the prime number conjecture.

The research to prime numbers was all about one function, namely

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x} = 1 + \frac{1}{2^x} + \frac{1}{3^x} + \frac{1}{4^x} + \frac{1}{5^x} + \cdots,$$

which is called the Riemann zeta function. This is an infinite sum, so the value might be infinity instead of that the sum converges to some fixed number. This function was already introduced by Euler the century before Riemann's article. It was shown that the sum converges for $x > 1$, but finding the values to which $\zeta(x)$ converges is harder. Finding the value of $\zeta(2)$ is known as the Basel problem and was solved by Euler. He proved that

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{6}.$$

The key idea of Riemann was to consider $\zeta(x)$ not as a function of a real variable $x$, but as a function of a complex variable $s$. It is common to write $s = \sigma + it$ where $\sigma$ and $t$ are real and denote the real and imaginary part of $s$ respectively. Now, all theory of complex analysis, which was still in development at Riemann's time, could be applied to obtain new results.

At first sight the zeta function has little to do with prime numbers, but there is a relation between the zeta function and the prime numbers which is quite easy to establish. Let $\mathrm{Re}(s) = \sigma > 1$, so that $\zeta(s)$ converges. We will use the idea of the Sieve of Eratosthenes, so first sieve all the terms with a factor 2. Observe that

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots \tag{1.3}$$

and

$$\frac{1}{2^s}\zeta(s) = \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \cdots . \tag{1.4}$$

Subtracting equation (1.4) from (1.3) gives that

$$\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \cdots \tag{1.5}$$

and all terms with multiples of 2 in the denominator have disappeared at the right hand side. Now sieve all the terms with a factor 3. Note that

$$\frac{1}{3^s}\left(1 - \frac{1}{2^s}\right)\zeta(s) = \frac{1}{3^s} + \frac{1}{9^s} + \frac{1}{15^s} + \frac{1}{21^s} + \cdots . \tag{1.6}$$

Subtracting equation (1.6) from (1.5) gives

$$\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{11^s} + \cdots .$$

Again, all the terms with multiples of 3 in the denominator have disappeared. Now, continue sieving all the terms with a prime in the denominator. This will finally give

$$\cdots\left(1 - \frac{1}{11^s}\right)\left(1 - \frac{1}{7^s}\right)\left(1 - \frac{1}{5^s}\right)\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1,$$

which is equivalent to

$$\zeta(s) = \frac{1}{1 - \frac{1}{2^s}} \cdot \frac{1}{1 - \frac{1}{3^s}} \cdot \frac{1}{1 - \frac{1}{5^s}} \cdot \frac{1}{1 - \frac{1}{7^s}} \cdot \frac{1}{1 - \frac{1}{11^s}} \cdots .$$

This connection between primes and the zeta function forms the foundation for the research to the distribution of the primes. This formula was found by Euler and is called the Euler product formula. The crucial step that Riemann took a century later, was making an extension of the zeta function to the whole complex plane, except for $s = 1$ where $\zeta(s)$ still diverges. This extension will be necessary for obtaining the result we want and hence will appear everywhere throughout the research to prime numbers.

Using the extension of the zeta function, together with much complex analysis, Riemann found an explicit formula for $\pi(x)$ in which the most important term was $Li(x)$, the approximation of $\pi(x)$ which was found before. This means that the other remaining terms in the expression describe the absolute error between $\pi(x)$ and $Li(x)$. So the behavior of this error tells much about the prime counting function. Unfortunately, this error depends on the zeros of the zeta function, these are the points $\rho$ in the complex plane such that $\zeta(\rho) = 0$. About the zeros $\rho$ much is unclear, because it is unknown where all these zeros lie exactly.

Despite the good ideas in Riemann's article, we should also comment on the completeness of the article. Of course, eight pages can never be enough to explain new ideas in detail and do all the complex calculations. Some gaps can be easily filled in, but there are also statements of which the truth was not trivial at all. In the course of time most statements have been established to be true, which also means that the formula for $\pi(x)$ is valid. However, as mentioned before, the expression of this formula contains the zeros of the zeta function. Riemann wrote about these zeros in his article, he expected that all the nontrivial zeros would lie on the line with real part $\frac{1}{2}$, but he was not able to prove this assertion. There are also trivial zeros, but these are relatively easy to find as the name already suggests. About the nontrivial zeros, Riemann wrote

> "Certainly one would wish for a stricter proof here; I have meanwhile temporarily put aside the search for this after some fleeting futile attempts, as it appears unnecessary for the next objective of my investigation." Translated from [Riemann, 1859].

In fact, until now nobody was able to prove or disprove this assertion. This became one of the greatest unsolved problems in mathematics which is called the Riemann hypothesis. This problem has become so important for the distribution of the primes and for problems in mathematics that the mathematician who finds a proof or disproof is awarded with one million dollars by the Clay Mathematics Institute.

Returning to the prime number conjecture, what was the influence of Riemann's article on finding a proof? The analysis of the zeta function resulted in the prime number theorem being equivalent to the zeta function having no zeros on the line with real part 1. This equivalent statement was proven in 1896, independently by the mathematicians Hadamard and de la Vallée-Poussin. This turned the prime number conjecture finally in a theorem. This was one of the highlights of mathematics in the nineteenth century.

Of course, mathematicians have not been doing nothing after achieving such a great result. The proof of the prime number theorem was long and intricate. It was based on an explicit formula for $\psi(x)$ which is a function that not only counts primes (as $\pi(x)$), but also counts powers of primes. It turned out that this function is easier to work with than $\pi(x)$. Nowadays, the explicit formula for $\pi(x)$ that Riemann found is often neglected, because the explicit formula for $\psi(x)$ is easier and in fact contains the same information.

Since the first proof of the prime number theorem, mathematicians have been looking for a less intricate one. Also, they looked for a proof that is elementary, i.e. a proof that makes no use of complex analysis. Such an elementary proof was found in 1948 by Selberg and Erdős. The most easy proofs are the ones that use so called Tauberian theorems. Donald Newman found the most easy proof up to now of the prime number theorem in the 1980's. The proof of the Tauberian theorem he used, requires in fact nothing more than Cauchy's integral formula (a standard formula for calculating integrals in complex analysis).

Having proved the explicit formula for $\pi(x)$ and the prime number theorem, it remains to find the nontrivial zeros of the zeta function, because the explicit formula could be expressed using these zeros. The investigation to a proof of the Riemann hypothesis continues until today. Many more connections to the zeta function have been found, such as the connection with random matrices and quantum mechanics.

## 1.3 Overview of this Thesis

The goal of this thesis is to give a rigorous mathematical introduction to the Riemann zeta function and the prime number theorem suitable for undergraduate mathematics students. This thesis will be mainly focused on Riemann's article in which an explicit formula for $\pi(x)$ is constructed, and the proof of the prime number theorem. Of course, many books and articles have been written about this topic, but usually not at an undergraduate level. The explicit formula for $\pi(x)$ and the prime number theorem can be proven with only basic theory about calculus, real and complex analysis. So the aim of this thesis is to write down the details of the mathematics such that it is understandable for undergraduate mathematics students familiar with real and complex analysis. Also, relevant references will be included for further reading on the subject. People without mathematical background interested in the topic can read further in [Derbyshire, 2003] or [van der Veen and van de Craats, 2011].

All the required knowledge of complex analysis, including the theory of infinite products, can be found in the appendices A and B. Chapter 2 deals with a very important function which will be encountered all the time: the gamma function. This function is in fact an extension of the factorials $n!$. Using the gamma function, several important identities will be proved, such as the reflection formula and the infinite product representation for the sine and the gamma function. This is in fact general theory which is used throughout all branches of mathematics.

In chapter 3 and 4 Riemann's article is studied. Chapter 3 gives a formal proof of the Euler product formula. Also, the two different methods Riemann gave to find the extension of the zeta function to almost the whole complex plane are studied. Chapter 4 deals with the construction of the explicit formula for $\pi(x)$. It will take several steps to find this formula and still not all details are validated in this thesis, just as in Riemann's article.

Chapter 5 and 6 have as main goal proving the prime number theorem using the zeta function. In chapter 5 all the necessary theory from analytic number theory is introduced and an equivalent form of the prime number theorem is established using the function $\psi(x)$. This equivalent statement will be proved in chapter 6. Hereby, the Tauberian theorem as proved by Newman will be used. So the final result is a proof of the prime number theorem, something which should bring joy to every mathematician.

# Chapter 2

# The Gamma Function

This chapter introduces the gamma function, which is needed for the study of the Riemann zeta function in the next chapter. First, basic properties about the gamma function are established together with Wielandt's theorem. This theorem allows us to prove an infinite product representation of the gamma function. In the last section, several important statements will be proved such as Euler's reflection formula and Legendre's duplication formula, which will be useful later.

## 2.1 Properties of the Gamma Function

**Definition 2.1.** *The gamma function* $\Gamma : \mathbb{C}_+ \to \mathbb{C}$ *with* $\mathbb{C}_+ = \{z \in \mathbb{C} : Re(z) > 0\}$ *is defined as*

$$\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} dt \tag{2.1}$$

*with* $t^{z-1} = e^{(z-1)\log t}$, $\log t \in \mathbb{R}$ *and* $Re(z) > 0$.

**Proposition 2.2.** *The gamma function is absolutely convergent and analytic on* $\mathbb{C}_+$.

*Proof.* To show the convergence, split the integral into two parts:

$$\Gamma(z) = \int_0^1 t^{z-1} e^{-t} dt + \int_1^\infty t^{z-1} e^{-t} dt. \tag{2.2}$$

Note that the following statements hold

$$|t^{z-1} e^{-t}| = t^{\mathrm{Re}(z)-1} e^{-t} < t^{\mathrm{Re}(z)-1}, \quad \text{for } t > 0 \text{ and } z \in \mathbb{C}_+$$

and

$$\int_0^1 \frac{1}{t^s} dt$$

converges absolutely for $s < 1$.
Combining these two statements gives that

$$\left| \int_0^1 t^{z-1} e^{-t} dt \right| \leq \int_0^1 |t^{z-1} e^{-t}| dt < \int_0^1 t^{\mathrm{Re}(z)-1} dt < \infty, \quad \text{for } \mathrm{Re}(z) > 0.$$

Hence the first integral of equation (2.2) converges absolutely for $z \in \mathbb{C}_+$.

For the convergence of the second integral, note that for all $x_0 > 0$ there exists a $C > 0$ such that

$$t^{\mathrm{Re}(z)-1} \leq C e^{\frac{t}{2}}$$

for all $z$ with $0 < \mathrm{Re}(z) < x_0$ and for all $t \geq 1$. From this it follows that

$$|t^{z-1}e^{-t}| = t^{\mathrm{Re}(z)-1}e^{-t} \leq Ce^{-\frac{t}{2}}$$

for $0 < \mathrm{Re}(z) < x_0$ and $t \geq 1$. The second integral of equation (2.2) becomes

$$\left|\int_1^\infty t^{z-1}e^{-t}dt\right| \leq \int_1^\infty |t^{z-1}e^{-t}|dt \leq C\int_1^\infty e^{-\frac{t}{2}}dt = 2Ce^{-\frac{1}{2}} < \infty \quad \text{for } 0 < \mathrm{Re}(z) < x_0.$$

Thus, the integral converges absolutely for all $z \in \mathbb{C}_+$.

To prove that $\Gamma(z)$ is analytic on $\mathbb{C}_+$, consider the sequence of functions $f_0, f_1, f_2, \ldots$ with $f_n : \mathbb{C}_+ \to \mathbb{C}$ for all $n \in \mathbb{N}$ defined by

$$f_n(z) = \int_n^{n+1} t^{z-1}e^{-t}dt.$$

Every $f_n$ is analytic on $\mathbb{C}_+$ since $t^{z-1}e^{-t}$ is continuous in $t \in (n, n+1)$ and analytic in $z \in \mathbb{C}_+$ (see theorem A.6). Let $D$ be a bounded subset and let $0 < \varepsilon \leq \mathrm{Re}(z) \leq K < \infty$ for all $z \in D$. Then

$$|f_0(z)| \leq \int_0^1 t^{\mathrm{Re}(z)-1}dt = \frac{1}{\mathrm{Re}(z)} \leq \frac{1}{\varepsilon} =: M_0$$

and for $n \geq 1$ and $z \in D$

$$\begin{aligned} |f_n(z)| &\leq e^{-n}\int_n^{n+1} t^{\mathrm{Re}(z)-1}dt \\ &= \frac{e^{-n}}{\mathrm{Re}(z)}\Big((n+1)^{\mathrm{Re}(z)} - \underbrace{n^{\mathrm{Re}(z)}}_{\geq 0,\text{ since } n\geq 1}\Big) \leq \frac{e^{-n}}{\varepsilon}(n+1)^K =: M_n. \end{aligned}$$

Note that $\sum_{n=0}^\infty M_n < \infty$, so by the Weierstrass $M$-test $\sum_{n=0}^\infty f_n$ converges absolutely and uniformly on $D$. If a sequence of analytic functions $f_n$ converges (locally) uniformly to $f$, then $f$ is also analytic, see theorem A.16. Applying this to the sequence of partial sums $f_0 + ... + f_n$ gives that $\Gamma(z) = \sum_{n=0}^\infty f_n(z)$ is analytic on $\mathbb{C}_+$ (see also theorem A.18).

□

The gamma function satisfies certain properties, for example with integration by parts it can be shown that

$$\Gamma(z+1) = z\Gamma(z) \quad \text{for } \mathrm{Re}(z) > 0. \tag{2.3}$$

Since $\Gamma(1) = 1$ (this can easily be seen from the definition), it follows that for $n \in \mathbb{N} \cup \{0\}$

$$\Gamma(n+1) = n!. \tag{2.4}$$

So the gamma function interpolates the factorials. The next step is to define the gamma function for the whole complex plane. An iterated application of equation (2.3) gives

$$\Gamma(z) = \frac{\Gamma(z+n+1)}{z(z+1)\cdots(z+n)}. \tag{2.5}$$

Here, $\Gamma(z)$ is only defined for $\mathrm{Re}(z) > 0$, but the right hand side is defined for $\mathrm{Re}(z) > -(n+1)$ with $z \neq 0, -1, -2, ..., -n$. All these analytic extensions for various $n$ are unique by the identity theorem for analytic functions (theorem A.20). So the gamma function can be uniquely extended as an analytic function to $\mathbb{C} \setminus S$ with $S := \{0, -1, -2, \ldots\}$ and

for $z \in \mathbb{C} \setminus S$ equation (2.3) holds.
The elements of $S$ are simple poles with residues

$$Res\,(\Gamma(z); -n) = \lim_{z \to -n} (z+n)\,\Gamma(z) \stackrel{(2.5)}{=} \frac{\Gamma(1)}{(-n)(-n+1)...(-1)} = \frac{(-1)^n}{n!}. \quad (2.6)$$

As can be seen from the definition, the gamma function also satisfies

$$|\Gamma(z)| \leq \Gamma(\text{Re}(z)), \quad \text{for } \text{Re}(z) > 0. \quad (2.7)$$

So the gamma function is bounded on any closed vertical strip in the complex plane with $0 < a \leq x \leq b$. The gamma function is completely determined by this property together with property (2.3). This is described in the following theorem as found in [Remmert, 1996].

**Theorem 2.3** (Wielandt)**.** *Let $D \subset \mathbb{C}$ be a domain containing the vertical strip*

$$V := \{z = x + yi : x, y \in \mathbb{R}, 1 \leq x < 2\} \subset \mathbb{C}.$$

*Let $f : D \to \mathbb{C}$ be an analytic function satisfying:*

1. *$f$ is bounded on $V$,*
2. *$f(z+1) = zf(z)$ for $z, z+1 \in D$.*

*Then*

$$f(z) = f(1)\Gamma(z) \quad \textit{for all } z \in D.$$

*Proof.* Similarly as for the gamma function, $f(z)$ can be extended to $\mathbb{C} \setminus S$ with $S = \{0, -1, -2, \dots\}$ using condition 2. Again $f(z+1) = zf(z)$ holds on $\mathbb{C} \setminus S$ and the residues are

$$Res(f(z); -n) = \frac{(-1)^n}{n!} f(1) \quad \text{for } -n \in S. \quad (2.8)$$

Note that $f(z)$ and $f(1)\Gamma(z)$ have the same poles and residues (compare equation (2.6) and (2.8)). This implies that the singularities are removable for the function $h : \mathbb{C} \to \mathbb{C}$ defined by $h(z) = f(z) - f(1)\Gamma(z)$. So $h$ is an entire function, which means that $h$ is analytic on the whole complex plane. Since $f(z)$ and $\Gamma(z)$ are bounded on the strip $V$ (condition 1 and equation (2.7)), $h$ is also bounded on $V$. But $h$ is also bounded on the strip $V_0 := \{z \in \mathbb{C} : 0 \leq \text{Re}(z) < 1\}$, because

- the set $V_0 \cap \{z : |\text{Im}(z)| \leq 1\}$ is compact, hence $h$ is bounded on this set,
- for $z \in V_0 \cap \{z : |\text{Im}(z)| > 1\}$ the equation $f(z) = \frac{f(z+1)}{z}$ holds. Now $f(z+1)$ is bounded since it lies in $V$. This means that $f(z)$ is bounded on $V_0 \cap \{z : |\text{Im}(z)| > 1\}$, thus $h$ is also bounded on this set.

Note that $h$ satisfies

$$h(z+1) = f(z+1) - f(1)\Gamma(z+1) = z\big(f(z) - f(1)\Gamma(z)\big) = zh(z) \quad \text{for } z \in \mathbb{C}.$$

Now, consider the function $H : \mathbb{C} \to \mathbb{C}$ defined by $H(z) = h(z)h(1-z)$. Note that $H$ is an entire function (since $h$ was also entire) and satisfies

$$H(z+1) = h(z+1)h(-z) = zh(z)\frac{h(1-z)}{-z} = -H(z).$$

So $H$ is periodic up to the sign. Also, $H$ is bounded on $V_0$, since $h(z)$ and $h(1-z)$ are bounded on $V_0$. Using the periodicity of $H$ it is obtained that $H$ is bounded on $\mathbb{C}$. Applying Liouville's theorem (theorem A.7) gives that $H(z)$ is constant, so

$$H(z) = H(1) = h(1)h(0) = \big(f(1) - f(1)\Gamma(1)\big)h(0) = 0.$$

By the definition of $H$ it follows that $h(z) \equiv 0$. So the definition $h(z) = f(z) - f(1)\Gamma(z)$ implies that $f(z) = f(1)\Gamma(z)$ for all $z \in D$. □

## 2.2 Product Representations for the Gamma Function

To continue, an infinite product representation of the gamma function is required. Therefore, the following lemma is used.

**Lemma 2.4.** *The infinite product*

$$H(z) := \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) \exp\left(-\frac{z}{n}\right)$$

*is analytic on* $\mathbb{C}$ *and* $H(z) = 0$ *if and only if* $-z \in \mathbb{N}$.

*Proof.* The zeros of $H(z)$ lie at $z = 0, -1, -2, -3, ...$, so indeed $H(z) = 0$ if and only if $-z \in \mathbb{N}$ (see remark B.1). Note that every factor of $H$ is analytic on $\mathbb{C}$. To show that $H(z)$ is analytic, use theorem B.2 and according to that theorem it remains to prove that

$$\sum_{n=1}^{\infty} \left[\left(1 + \frac{z}{n}\right) \exp\left(-\frac{z}{n}\right) - 1\right] \tag{2.9}$$

converges normally on $\mathbb{C}$.
Let $a \in \mathbb{C}$ arbitrary and let $K = \overline{B_{|a|+1}(0)}$ be a compact disk. Consider the Taylor expansion of the complex exponential:

$$\exp(-\omega) = 1 - \omega + \frac{(-\omega)^2}{2} + \mathcal{O}(\omega^3)$$

and thus

$$\omega \exp(-\omega) = \omega - \omega^2 + \frac{\omega^3}{2} + \mathcal{O}(\omega^4).$$

Combining these equations gives

$$(1 + \omega) \exp(-\omega) - 1 = -\frac{\omega^2}{2} + \mathcal{O}(\omega^3) \quad \text{for all } \omega \in \mathbb{C}.$$

The set $K$ is compact, so there exists a constant $C_K \in \mathbb{R}$ such that

$$|(1 + \omega) \exp(-\omega) - 1| \leq C_K |\omega|^2 \quad \text{for all } \omega \in K.$$

If $z \in K$, then also $\frac{z}{n} \in K$ for all $n \in \mathbb{N}$. So

$$\left|\left(1 + \frac{z}{n}\right) \exp(-\frac{z}{n}) - 1\right| \leq C_K \left|\frac{z}{n}\right|^2 \leq C_K \frac{(|a| + 1)^2}{n^2}.$$

Hence, the series in (2.9) is bounded on $K$ by $c \sum_{n=1}^{\infty} \frac{1}{n^2}$ where $c$ is a constant and this sum is finite. This implies that the series is normally convergent on $K$, but $a$ was chosen arbitrarily, so the series converges normally on $\mathbb{C}$. By theorem B.2 it follows that $H(z)$ is analytic on $\mathbb{C}$. □

Define for all $n \in \mathbb{N}$ the function $G_n : \mathbb{C} \to \mathbb{C}$ by

$$G_n(z) = z \exp(-z \log n) \prod_{j=1}^{n} \left(1 + \frac{z}{j}\right).$$

Then $G(z) := \lim_{n\to\infty} G_n(z)$ defines an analytic function on $\mathbb{C}$. Rewrite

$$\begin{aligned}
G(z) = \lim_{n\to\infty} G_n(z) &= \lim_{n\to\infty} z\exp(-z\log(n))\prod_{j=1}^{n}\left(1+\frac{z}{j}\right)\\
&= \lim_{n\to\infty} z\left(\prod_{k=1}^{n}\exp\left(\frac{z}{k}\right)\right)\exp\left(-z\log(n)\right)\prod_{j=1}^{n}\left(1+\frac{z}{j}\right)\exp\left(\frac{-z}{j}\right)\\
&= \lim_{n\to\infty} z\exp\left[z\left(\sum_{k=1}^{n}\frac{1}{k}-\log(n)\right)\right]\prod_{j=1}^{n}\left(1+\frac{z}{j}\right)\exp\left(\frac{-z}{j}\right)\\
&= z\exp(\gamma z)H(z),
\end{aligned}$$

where

$$\gamma := \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k}-\log(n)\approx 0.5772\ldots$$

is the Euler-Mascheroni constant. Lemma 2.4 gives that $G(z)$ is analytic on $\mathbb{C}$.

Just as $H$, the function $G$ has zeros at the elements of $S = \{0,-1,-2,\ldots\}$ and the gamma function has poles at the elements of $S$. Gauss showed that in fact $1/\Gamma(z) = G(z)$ as in the following proposition. The proof given here is easier than the original proof by Gauss, because the theorem of Wielandt (theorem 2.3) can be used.

**Proposition 2.5** (Gauss's product representation)**.** *For all* $z\in\mathbb{C}$

$$\frac{1}{\Gamma(z)} = G(z) = \lim_{n\to\infty}\frac{n^{-z}}{n!}z(z+1)\cdots(z+n).$$

*Proof.* Rewriting

$$\begin{aligned}
G(z) = \lim_{n\to\infty} G_n(z) &= \lim_{n\to\infty} n^{-z}z\prod_{j=1}^{n}\left(1+\frac{z}{j}\right)\underbrace{\left(\frac{j}{j}\right)}_{=1}\\
&= \lim_{n\to\infty} n^{-z}z\prod_{j=1}^{n}(j+z)\frac{1}{j}\\
&= \lim_{n\to\infty}\frac{n^{-z}}{n!}z(z+1)\cdots(z+n)
\end{aligned}$$

gives the second equality. For the first equality apply theorem 2.3 to the function $1/G(z)$. Note that $1/G(z)$ is analytic on

$$D = \{z = x+yi : x,y\in\mathbb{R}, 0<x<2\},$$

since $G$ has no zeros in $D$. This $D$ contains the strip

$$V = \{z = x+yi : x,y\in\mathbb{R}, 1\le x<2\}.$$

Proof of the two conditions of theorem 2.3:

1. For all $z\in V$ and $n\in\mathbb{N}$: $|n^{-z}| = n^{-\mathrm{Re}(z)}$ and $|z+n|\ge \mathrm{Re}(z)+n$. Hence $|G(z)|\ge G(\mathrm{Re}(z))$ for all $z\in V$. Note that the interval $[1,2]$ is compact, thus $1/G$ is bounded on this interval. This implies that $1/G$ is also bounded on $V$.

2. For all $z \in D$ with $z+1 \in D$ and for all $n \in \mathbb{N}$:

$$\begin{aligned} zG_n(z+1) &= z\frac{n^{-(z+1)}}{n!}(z+1)(z+2)\cdots(z+n+1) \\ &= \frac{z+n+1}{n}\frac{n^{-z}}{n!}z(z+1)\cdots(z+n) = \frac{z+n+1}{n}G_n(z). \end{aligned}$$

Let $n \to \infty$, then

$$zG(z+1) = G(z) \quad \text{and thus } \frac{1}{G(z+1)} = \frac{z}{G(z)}$$

for all $z \in D$ with $z+1 \in D$.

Furthermore, for all $n \in \mathbb{N}$:

$$G_n(1) = \frac{n^{-1}}{n!}(n+1)! = \frac{n+1}{n} \to 1 \text{ as } n \to \infty.$$

It follows that $G(1) = 1$ and thus $1/G(1) = 1$. By theorem 2.3 we get that $1/G(z) = \Gamma(z)$ for all $z \in D$. The gamma function has no zeros in $D$, hence $1/\Gamma(z) = G(z)$ for all $z \in D$. By theorem A.20 it follows that $1/\Gamma(z) = G(z)$ for all $z \in \mathbb{C}$. □

From the statement of proposition 2.5 a product representation for $\Gamma(z+1)$ can be recovered. This product representation will be needed later for the construction of an explicit formula for the prime counting function.

**Proposition 2.6.** *For all* $z \in \mathbb{C} \setminus \{-1,-2,-3,\dots\}$

$$\Gamma(z+1) = \lim_{n\to\infty} \frac{n!(n+1)^z}{(z+1)(z+2)\cdots(z+n)}.$$

*Proof.* Plugging $z+1$ in Gauss's product representation for $\frac{1}{\Gamma(z)}$ gives

$$\begin{aligned} \frac{1}{\Gamma(z+1)} &= \lim_{n\to\infty} \frac{n^{-(z+1)}}{n!}(z+1)(z+2)\cdots(z+n+1) \\ &= \lim_{n\to\infty} \frac{(z+1)(z+2)\cdots(z+n)}{n!} \cdot \frac{z+n+1}{nn^z} \\ &= \lim_{n\to\infty} \frac{(z+1)(z+2)\cdots(z+n)}{n!} \cdot \frac{1}{(n+1)^z} \cdot \underbrace{\frac{z+n+1}{n}}_{\to 1 \text{ as } n\to\infty} \cdot \underbrace{\frac{(n+1)^z}{n^z}}_{\to 1 \text{ as } n\to\infty} \\ &= \lim_{n\to\infty} \frac{(z+1)(z+2)\cdots(z+n)}{n!} \cdot \frac{1}{(n+1)^z} \end{aligned}$$

from which the desired product representation follows. □

## 2.3 Other Important Identities

Using the statements found in the preceding sections, Euler's reflection formula, Euler's product formula for the sine and Legendre's duplication formula can be proved.

Define a new function $f : \mathbb{C} \setminus \mathbb{Z} \to \mathbb{C}$ by $f(z) := \Gamma(z)\Gamma(1-z)$. The function $\Gamma(z)$ has simple poles for $z \in S$ and $\Gamma(1-z)$ has simple poles for $z \in \mathbb{N}$. So $f(z)$ has simple poles for all $z \in \mathbb{Z}$ and the residues are

$$Res(f;-n) = \lim_{z\to -n}(z+n)\Gamma(z)\Gamma(1-z) \overset{(2.5)}{=} (-1)^n\frac{\Gamma(2n+2)}{(2n+1)!} \overset{(2.4)}{=} (-1)^n \quad \text{for } -n \in \mathbb{Z}.$$

Also, for $z \in \mathbb{C} \setminus \mathbb{Z}$:

$$f(z+1) = \Gamma(z+1)\Gamma(-z) = z\Gamma(z)\frac{\Gamma(1-z)}{-z} = -f(z). \tag{2.10}$$

So $f$ is periodic up to the sign. Note that the above two properties also hold for the function $\frac{\pi}{\sin \pi z}$:

$$Res\left(\frac{\pi}{\sin \pi z}; -n\right) = \lim_{z \to -n} (z+n)\frac{\pi}{\sin \pi z} = \lim_{z \to -n} \frac{\pi}{\pi \cos \pi z} = (-1)^n$$

using l'Hospital's rule and

$$\frac{\pi}{\sin \pi(z+1)} = -\frac{\pi}{\sin \pi z}. \tag{2.11}$$

Instead of only sharing these two properties it turns out that $f$ and $\frac{\pi}{\sin \pi z}$ are in fact equal on $\mathbb{C} \setminus \mathbb{Z}$. This identity is known as the Euler reflection formula.

**Proposition 2.7** (Euler's reflection formula)**.** *For $z \in \mathbb{C} \setminus \mathbb{Z}$ it holds that*

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}.$$

*Proof.* The functions $\Gamma(z)\Gamma(1-z)$ and $\frac{\pi}{\sin \pi z}$ have the same poles and residues, hence the function $h : \mathbb{C} \to \mathbb{C}$ given by

$$h(z) := \Gamma(z)\Gamma(1-z) - \frac{\pi}{\sin \pi z}$$

is entire. Recall that $\Gamma(z)$ and $\Gamma(1-z)$ are bounded on $\{z \in \mathbb{C} : 0 \leq \text{Re}(z) \leq 1, |\text{Im}(z)| \geq 1\}$ (see the proof of theorem 2.3). Also, $\frac{\pi}{\sin \pi z}$ is bounded on this set, namely

$$\begin{aligned} \left|\frac{\pi}{\sin \pi z}\right| &= \left|\frac{2\pi}{\exp(\pi i z) - \exp(-\pi i z)}\right| \leq \left|\frac{2\pi}{\exp(-\pi \text{Im}(z)) - \exp(\pi \text{Im}(z))}\right| \\ &\leq \frac{2\pi}{\frac{1}{2}\exp(\pi |\text{Im}(z)|)} \leq \frac{4\pi}{\exp(\pi)}. \end{aligned}$$

Furthermore, $h$ is bounded on $\{z \in \mathbb{C} : 0 \leq \text{Re}(z) \leq 1, |\text{Im}(z)| \leq 1\}$, since this set is compact and $h$ is analytic. Using the periodicity of $h$ (which follows from equations (2.10) and (2.11)) it follows that $h$ is bounded on $\mathbb{C}$, thus by Liouville's theorem $h$ is constant. Note that $h$ is odd:

$$\begin{aligned} -h(-z) &= -\Gamma(-z)\Gamma(1+z) + \frac{\pi}{\sin(-\pi z)} = -z\Gamma(-z)\Gamma(z) - \frac{\pi}{\sin(\pi z)} \\ &= \Gamma(z)\Gamma(1-z) - \frac{\pi}{\sin(\pi z)} = h(z) \end{aligned}$$

and this implies that $h(z) = 0$ for all $z \in \mathbb{C}$ and thus

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z} \quad \text{for all } z \in \mathbb{C} \setminus \mathbb{Z}.$$

□

**Remark 2.8.** *As a consequence it is obtained that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ by plugging $z = \frac{1}{2}$ in the reflection formula.*

By combining propositions 2.5 and 2.7 the product formula for the sine is found:

$$\begin{aligned}\sin(\pi z) &= \frac{\pi}{\Gamma(z)\Gamma(1-z)} \\ &= \pi \lim_{n\to\infty} \frac{n^{-z}}{n!} z \left(\prod_{j=1}^{n}(z+j)\right) \cdot \frac{n^{-(1-z)}}{n!}(n+1-z)\prod_{j=1}^{n}(j-z) \\ &= \pi \lim_{n\to\infty} z\frac{n+1-z}{n\cdot(n!)^2}\prod_{j=1}^{n}(j^2-z^2) \\ &= \pi \lim_{n\to\infty} z\frac{n+1-z}{n}\prod_{j=1}^{n}\left(\frac{j^2-z^2}{j^2}\right) \\ &= \pi z\prod_{j=1}^{\infty}\left(1-\frac{z^2}{j^2}\right)\end{aligned}$$

for all $z \in \mathbb{C} \setminus \mathbb{Z}$. Note that for $z \in \mathbb{Z}$ both $\sin(\pi z)$ and $\prod_{j=1}^{\infty}\left(1-\frac{z^2}{j^2}\right)$ are zero.

**Proposition 2.9** (Euler's product formula for the sine)**.** *For all* $z \in \mathbb{C}$

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right).$$

For the last time we apply Wielandt's theorem to prove Legendre's duplication formula for the gamma function.

**Proposition 2.10** (Legendre's duplication formula)**.** *The following identities hold:*

$$\Gamma(z) = (2\pi)^{-\frac{1}{2}}2^{z-\frac{1}{2}}\Gamma\left(\frac{z}{2}\right)\Gamma\left(\frac{z+1}{2}\right),$$

*or equivalently*

$$\Gamma(2z) = (2\pi)^{-\frac{1}{2}}2^{2z-\frac{1}{2}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right).$$

*Proof.* Define $f(z) = 2^{z-1}\Gamma(\frac{z}{2})\Gamma(\frac{z+1}{2})$. This function is analytic on a domain containing

$$V = \{z = x + yi : x, y \in \mathbb{R}, 1 \le x < 2\}.$$

It remains to prove conditions 1 and 2 of theorem 2.3:

1. Note that $|\Gamma(z)| \le \Gamma(\mathrm{Re}(z))$ and $|2^{z-1}| = 2^{\mathrm{Re}(z)-1}$, so $f(z)$ is bounded on $V$.
2. Using the property $\Gamma(z+1) = z\Gamma(z)$ gives

$$f(z+1) = 2^z\Gamma\left(\frac{z+1}{2}\right)\Gamma\left(\frac{z}{2}+1\right) = 2\cdot\frac{z}{2}\cdot 2^{z-1}\Gamma\left(\frac{z+1}{2}\right)\frac{z}{2}\Gamma\left(\frac{z}{2}\right) = zf(z).$$

Furthermore,

$$f(1) = \Gamma\left(\frac{1}{2}\right)\Gamma(1) = \pi^{\frac{1}{2}}$$

and theorem 2.3 now gives that

$$f(z) = f(1)\Gamma(z) \implies 2^{z-1}\Gamma\left(\frac{z}{2}\right)\Gamma\left(\frac{z+1}{2}\right) = \pi^{\frac{1}{2}}\Gamma(z).$$

Rearranging this equation gives the first identity and replacing $z$ by $2z$ gives the second. □

# Chapter 3

# The Riemann Zeta Function

In Riemann's paper *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse* ([Riemann, 1859]) two (incomplete) proofs are stated on how to extend the zeta function to the whole complex plane (without $s = 1$). One proof involves contour integration to prove the functional equation of the zeta function, this proof will be worked out in section 3.2. The second proof uses the Jacobi theta function and its functional equation and will be given in section 3.3. Both proofs, which differ from how Riemann wrote them originally, that are given follow [Titchmarsh, 1986] and [Edwards, 1974]. In the last section the Riemann hypothesis is briefly discussed.

Before diving into Riemann's paper, the zeta function will be introduced in section 3.1. The Euler product formula will be formally proven, together with an analytic continuation of the zeta function to the right half-plane.

## 3.1 Riemann Zeta Function

**Definition 3.1.** *The Riemann zeta function is defined for $Re(s) > 1$ by*

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

**Proposition 3.2.** *The zeta function converges normally and is analytic in the half-plane $\{s \in \mathbb{C} : Re(s) > 1\}$.*

*Proof.* Take $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, then there exists a $\delta > 0$ such that $\mathrm{Re}(s) \geq 1 + \delta > 1$. So each term of the sum can be bounded:

$$\left| \frac{1}{n^s} \right| = \frac{1}{n^{\mathrm{Re}(s)}} \leq \frac{1}{n^{1+\delta}}$$

for $\mathrm{Re}(s) \geq 1 + \delta$ and $\sum_{n=1}^{\infty} \frac{1}{n^{1+\delta}}$ converges, hence the convergence is normal. Note that every term $\frac{1}{n^s}$ is analytic for $\mathrm{Re}(s) > 1$, hence by theorem A.18 the zeta function is analytic for $\mathrm{Re}(s) > 1$. □

As mentioned in chapter 1, it was Euler who found the value of $\zeta(2)$ by using the product formula for the sine (proposition 2.9). In general, there is a formula for the values of the zeta function for even arguments:

$$\zeta(2k) = (-1)^{k+1} \frac{(2\pi)^{2k}}{2(2k)!} B_{2k} \quad \text{with } k \in \mathbb{N},$$

where $B_n$ are the Bernoulli numbers.

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $B_n$ | 1 | $-\frac{1}{2}$ | $\frac{1}{6}$ | 0 | $-\frac{1}{30}$ | 0 | $\frac{1}{42}$ | 0 | $-\frac{1}{30}$ | $\cdots$ |

Table 3.1: The Bernoulli numbers

There are also closed forms for other values of the zeta function with real arguments greater than 1. But in general not much about them is known. For our purposes, concrete values of the zeta function are not necessary, so we will not go deeper into this.

### 3.1.1 Euler Product Formula

In chapter 1 the intuition behind the Euler product formula was shown. Now, we will proof this connection between the prime numbers and the zeta function formally.

**Proposition 3.3** (Euler product formula)**.** *For Re*$(s) > 1$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}},$$

*where p are all the prime numbers.*

*Proof.* Since $|p^{-s}| = p^{-\mathrm{Re}(s)} < 1$, the geometric series can be applied

$$(1 - p^{-s})^{-1} = \sum_{\nu=0}^{\infty} p^{-\nu s}.$$

Also, apply the Cauchy product formula for finitely many series (see corollary A.2):

$$\begin{aligned}\prod_{k=1}^{m} (1 - p_k^{-s})^{-1} &= \prod_{k=1}^{m} \sum_{\nu_k=0}^{\infty} p_k^{-\nu_k s} \\ &= \sum_{v_1=0}^{\infty} \sum_{v_2=0}^{\nu_1} \cdots \sum_{\nu_m=0}^{\nu_{m-1}} \left( p_1^{\nu_m} p_2^{\nu_{m-1}-\nu_m} \cdots p_m^{\nu_1-\nu_2} \right)^{-s}.\end{aligned}$$

By the fundamental theorem of arithmetic, each natural number can uniquely be factorized by prime numbers. Let $\mathcal{A}(m)$ be the set of all natural numbers that only have $p_1, p_2, \ldots, p_m$ in their prime factorization. Then

$$\prod_{k=1}^{m} \left(1 - p_k^{-s}\right)^{-1} = \sum_{v_1=0}^{\infty} \sum_{v_2=0}^{\nu_1} \cdots \sum_{\nu_m=0}^{\nu_{m-1}} \left( p_1^{\nu_m} p_2^{\nu_{m-1}-\nu_m} \cdots p_m^{\nu_1-\nu_2} \right)^{-s} = \sum_{n \in \mathcal{A}(m)} n^{-s}.$$

For any $N \in \mathbb{N}$ there is an $m \in \mathbb{N}$ such that the whole set $\{1, 2, ..., N\}$ is contained in $\mathcal{A}(m)$, so it is expected that as $m \to \infty$ the zeta function appears on the right hand side. Note that the sum $\sum_{n \in \mathcal{A}(m)} n^{-s}$ contains at least all the terms $n^{-s}$ for $n \leq m$ because all $n$ with $n \leq m$ can be factorized by the first $m$ prime numbers. Hence, for $\mathrm{Re}(s) > 1$

$$\begin{aligned}\left| \zeta(s) - \prod_{k=1}^{m} \left(1 - p_k^{-s}\right)^{-1} \right| &= \left| \sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n \in \mathcal{A}(m)} \frac{1}{n^s} \right| \leq \left| \sum_{n=m+1}^{\infty} n^{-s} \right| \\ &\leq \sum_{n=m+1}^{\infty} \left| n^{-s} \right| = \sum_{n=m+1}^{\infty} n^{-\mathrm{Re}(s)} \to 0 \quad \text{as } m \to \infty.\end{aligned}$$

To conclude,

$$\lim_{m\to\infty}\prod_{k=1}^{m}\left(1-p_k^{-s}\right)^{-1}=\sum_{n=1}^{\infty}n^{-s}$$

and thus for $\text{Re}(s)>1$

$$\prod_{k=1}^{\infty}\left(1-p_k^{-s}\right)^{-1}=\prod_{p\ prime}\left(1-p^{-s}\right)^{-1}=\zeta(s).$$

□

**Remark 3.4.** *From the Euler product formula it follows that the zeta function has no zeros for Re$(s)>1$, since none of factors vanish (see also remark B.1).*

### 3.1.2 Eta Function

The goal of this section is to extend the zeta function to the half-plane $\{s\in\mathbb{C}:\text{Re}(s)>0\}$. For this, the eta function (which is the alternating zeta function) is used. The eta function is defined for $\text{Re}(s)>0$ by

$$\eta(s):=1-\frac{1}{2^s}+\frac{1}{3^s}-\frac{1}{4^s}+...=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}.$$

Write the eta function in terms of the zeta function

$$\begin{aligned}\eta(s)&=\zeta(s)-2\left(\sum_{n=1}^{\infty}\frac{1}{(2n)^s}\right)=\zeta(s)-2\left(\frac{1}{2^s}\zeta(s)\right)\\&=\zeta(s)\left(1-2^{1-s}\right).\end{aligned}$$

And so

$$\zeta(s)=\frac{\eta(s)}{1-2^{1-s}}. \tag{3.1}$$

Now, using this formula the zeta function converges where the eta function converges (apart from $s=1$). We will show that the eta function converges in the right half-plane $\{s\in\mathbb{C}:\text{Re}(s)>0\}$. To prove this, the following version of the Dirichlet test is used.

**Theorem 3.5.** *Let $a_n$ and $b_n$ be complex valued sequences. Then $\sum_{n=1}^{\infty}a_nb_n$ converges, if the following are satisfied:*

*(i) there exists $M>0$ (independent of $n$) such that $\left|\sum_{k=1}^{n}a_k\right|\le M$,*

*(ii) $b_n\to 0$ as $n\to\infty$,*

*(iii) $\sum_{n=1}^{\infty}\left|b_n-b_{n+1}\right|$ is convergent.*

**Remark 3.6.** *The standard Dirichlet test, which states that $\sum_{n=1}^{\infty}a_nb_n$ converges under the conditions that $a_n\to 0$ monotonically and $|\sum_{n=1}^{N}b_n|<M$ for every $N$, cannot be applied directly in the case of the eta function. This is because the Dirichlet test requires the sequence $a_n$ to be real valued. Application of this test would be possible if $a_n=n^{-Re(s)}$ and $b_n=(-1)^{n-1}n^{-iIm(s)}$ are taken. Instead we will use the version as in theorem 3.5, which allows both $a_n$ and $b_n$ to be complex sequences.*

**Proposition 3.7.** *The eta function converges for Re$(s)>0$.*

*Proof.* Use the Dirichlet test as in theorem 3.5.
Let $a_k = (-1)^{k-1}$ and $b_k = k^{-s}$ with $\text{Re}(s) > 0$. Then the conditions of the Dirichlet test are satisfied. Namely,

$$\left|\sum_{k=1}^{n}(-1)^{k-1}\right| \leq 1$$

and

$$b_k = k^{-s} \to 0 \text{ as } k \to \infty.$$

For the third condition:

$$\begin{aligned}
\sum_{k=1}^{\infty}\left|\frac{1}{(k+1)^s} - \frac{1}{k^s}\right| &\leq \sum_{k=1}^{\infty}\int_k^{k+1}\left|\frac{d}{dt}\left(\frac{1}{t^s}\right)\right| dt = \int_1^{\infty}\left|\frac{d}{dt}\left(\frac{1}{t^s}\right)\right| dt \\
&= \int_1^{\infty}\left|\frac{-s}{t^{s+1}}\right| dt = |s|\int_1^{\infty}\frac{1}{t^{1+\text{Re}(s)}}dt \\
&= |s|\left[\frac{t^{-\text{Re}(s)}}{-\text{Re}(s)}\right]_1^{\infty} = \frac{|s|}{\text{Re}(s)}.
\end{aligned}$$

So the eta function converges for $\text{Re}(s) > 0$. □

Hence, with the identity theorem for analytic functions (theorem A.20) the following is obtained.

**Proposition 3.8.** *For Re*$(s) > 0$ *with* $s \neq 1$*, the zeta function can be continued analytically by*

$$\zeta(s) = \frac{\eta(s)}{1 - 2^{1-s}}, \quad \textit{with} \quad \eta(s) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}.$$

## 3.2 Functional Equation: Proof with Contour Integration

Riemann's first proof of the functional equation of the zeta function made use of contour integration, in which many details were omitted. In this section the full proof is given including all details. The following lemmas provide two integral identities which will be of use.

**Lemma 3.9.** *For Re*$(s) > 0$

$$\int_0^{\infty} e^{-nt}t^{s-1}dt = \frac{\Gamma(s)}{n^s}. \tag{3.2}$$

*Proof.* Apply the substitution $u = nt$. This gives for $\text{Re}(s) > 0$

$$\begin{aligned}
\int_0^{\infty} e^{-nt}t^{s-1}dt &= \int_0^{\infty} e^{-u}\left(\frac{u}{n}\right)^{s-1}\frac{du}{n} \\
&= \frac{1}{n^s}\int_0^{\infty} e^{-u}u^{s-1}du \\
&= \frac{\Gamma(s)}{n^s}.
\end{aligned}$$

□

**Lemma 3.10.** *For Re*$(s) > 1$

$$\int_0^{\infty}\frac{t^{s-1}}{e^t - 1}dt = \Gamma(s)\zeta(s). \tag{3.3}$$

*Proof.* Note that

$$\frac{1}{e^t - 1} = \frac{e^{-t}}{1 - e^{-t}} = \sum_{n=1}^{\infty} e^{-nt}$$

using the geometric series. Then interchanging sum and integral gives

$$\begin{aligned}\int_0^\infty \frac{t^{s-1}}{e^t - 1} dt &= \sum_{n=1}^{\infty} \int_0^\infty e^{-nt} t^{s-1} dt \\ &\overset{(3.2)}{=} \sum_{n=1}^{\infty} \frac{\Gamma(s)}{n^s} \\ &= \Gamma(s)\zeta(s) \quad \text{for } \mathrm{Re}(s) > 1.\end{aligned}$$

Switching the sum and integral is valid by the Fubini-Tonelli theorem:

$$\sum_{n=1}^{\infty} \int_0^\infty \left| e^{-nt} t^{s-1} \right| dt = \sum_{n=1}^{\infty} \frac{\Gamma\big(\mathrm{Re}(s)\big)}{n^s} = \zeta\big(\mathrm{Re}(s)\big)\Gamma\big(\mathrm{Re}(s)\big) < \infty.$$

□

Let $\varepsilon > 0$ and $R > 0$ such that $0 < \varepsilon < R$. Define the contour $C_{\varepsilon,R} = \Gamma_1 \cup \gamma \cup \Gamma_2$ as in figure 3.1 where

(i) $\Gamma_1$ is the curve just above the real axis from $R$ to $\varepsilon$,

(ii) $\Gamma_2$ is the curve just below the real axis from $\varepsilon$ to $R$,

(iii) $\gamma$ is a circle around the origin in positive direction with radius $\varepsilon$.

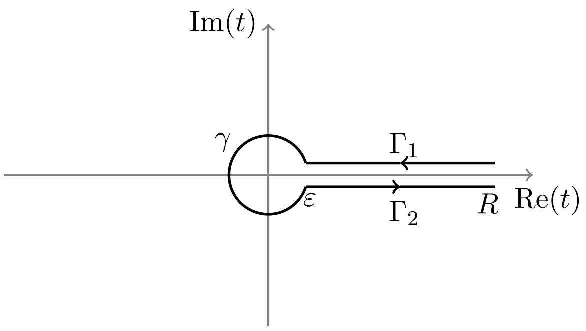


Figure 3.1: The contour $C_{\varepsilon,R}$.

**Proposition 3.11.** *For Re$(s) > 1$*

$$\lim_{\substack{\varepsilon \to 0 \\ R \to \infty}} \int_{C_{\varepsilon,R}} \frac{(-t)^{s-1}}{e^t - 1} dt = -2i \sin(\pi s)\Gamma(s)\zeta(s),$$

*where $C_{\varepsilon,R}$ is the contour defined as above.*

*Proof.* The integrand must be defined on $C_{\varepsilon,R}$, so then the branch cut of $(-t)^{s-1}$ must lie at the positive real axis. Hence, define $(-t)^{s-1} = \exp\big((s-1)\log(-t)\big)$ with $\log(t) = \log|t| + i\mathrm{Arg}(t)$, then indeed the integrand is only not defined on the positive real axis. For $t$ on $\Gamma_1$, we have that $-t$ lies just below the negative real axis, hence

$$\int_{\Gamma_1} \frac{(-t)^{s-1}}{e^t-1}dt = \int_R^{\varepsilon} \frac{(te^{-\pi i})^{s-1}}{e^t-1}dt = -e^{-\pi i(s-1)} \int_{\varepsilon}^{R} \frac{t^{s-1}}{e^t-1}dt.$$

For $t$ on $\Gamma_2$, $-t$ lies just above the negative real axis:

$$\int_{\Gamma_2} \frac{(-t)^{s-1}}{e^t-1}dt = \int_{\varepsilon}^{R} \frac{(te^{\pi i})^{s-1}}{e^t-1}dt = e^{\pi i(s-1)} \int_{\varepsilon}^{R} \frac{t^{s-1}}{e^t-1}dt.$$

In total:

$$\begin{aligned}\int_{C_{\varepsilon,R}} \frac{(-t)^{s-1}}{e^t-1}dt &= \left(\int_{\Gamma_1} + \int_{\Gamma_2} + \int_{\gamma}\right) \frac{(-t)^{s-1}}{e^t-1}dt \\ &= \left(e^{\pi i(s-1)} - e^{-\pi i(s-1)}\right) \int_{\varepsilon}^{R} \frac{t^{s-1}}{e^t-1}dt + \int_{\gamma} \frac{(-t)^{s-1}}{e^t-1}dt \\ &= -2i\sin(\pi s) \int_{\varepsilon}^{R} \frac{t^{s-1}}{e^t-1}dt + \int_{\gamma} \frac{(-t)^{s-1}}{e^t-1}dt,\end{aligned}$$

since

$$e^{\pi i(s-1)} - e^{-\pi i(s-1)} = e^{\pi i s}e^{-\pi i} - e^{-\pi i s}e^{\pi i} = -(e^{\pi i s} - e^{-\pi i s}) = -2i\sin(\pi s).$$

Now, let $\varepsilon \to 0$ and $R \to \infty$, then for $\mathrm{Re}(s) > 1$

$$\lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \left(-2i\sin(\pi s)\int_{\varepsilon}^{R} \frac{t^{s-1}}{e^t-1} + \int_{\gamma} \frac{(-t)^{s-1}}{e^t-1}dt\right) = -2i\sin(\pi s)\int_0^{\infty} \frac{t^{s-1}}{e^t-1}dt,$$

since the integral over $\gamma$ goes to zero for $\mathrm{Re}(s) > 1$. Namely, define

$$f(t) = \begin{cases} \frac{-(-t)^s}{e^t-1} & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases}$$

and using the shrinking path lemma A.13 gives that

$$\lim_{\varepsilon\to 0} \int_{\gamma} \frac{(-t)^{s-1}}{e^t-1}dt = \lim_{\varepsilon\to 0} \int_{\gamma} \frac{f(t)}{t}dt = 2\pi i f(0) = 0.$$

Finally, lemma 3.10 gives that for $\mathrm{Re}(s) > 1$

$$\lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \int_{C_{\varepsilon,R}} \frac{(-t)^{s-1}}{e^t-1}dt = -2i\sin(\pi s)\int_0^{\infty} \frac{t^{s-1}}{e^t-1}dt = -2i\sin(\pi s)\Gamma(s)\zeta(s).$$

□

Define $C$ to be the contour $C_{\varepsilon,R}$ as $\varepsilon \to 0$ and $R \to \infty$. Using Euler's reflection formula (proposition 2.7) the statement from proposition 3.11 can be rewritten as

$$\zeta(s) = \frac{i}{2\Gamma(s)\sin(\pi s)} \int_C \frac{(-t)^{s-1}}{e^t-1}dt = \frac{i\Gamma(1-s)}{2\pi} \int_C \frac{(-t)^{s-1}}{e^t-1}dt. \tag{3.4}$$

Note that $\zeta(s)$ is analytic for $\mathrm{Re}(s) > 1$ and that the integral over $C$ is analytic for all $s \in \mathbb{C}$ by theorem A.6. However, $\Gamma(1-s)$ has simple poles for $s \in \{1, 2, 3, \dots\}$ since the gamma function has simple poles in $0, -1, -2, -3, \dots$. So equation (3.4) gives an analytic continuation of the zeta function to $\mathbb{C} \setminus \{1, 2, 3, \dots\}$. Thus, the zeta function can be continued analytically to $\mathbb{C} \setminus \{1\}$ because we already established that the zeta function is analytic in $2, 3, 4, \dots$.

So the only possible pole of the zeta function is at $s = 1$. This is a simple pole with residue 1. Using the analytic continuation of the zeta function to $\mathrm{Re}(s) > 0$ as in proposition 3.8 gives that

$$\begin{aligned}\lim_{s\to 1}(s-1)\zeta(s) &= \lim_{s\to 1}\frac{s-1}{1-2^{1-s}} \cdot \lim_{s\to 1}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s} \\ &= \lim_{s\to 1}\frac{1}{2^{1-s}\log(2)} \cdot \lim_{s\to 1}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s} \qquad \text{(l'Hospital's rule)} \\ &= \frac{1}{\log(2)}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} = \frac{\log(2)}{\log(2)} = 1,\end{aligned}$$

where it is used that the power series of the logarithm

$$\log(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$$

also converges for $x = 1$.

In order to find the functional equation of the zeta function, the integral over $C$ must be evaluated. This can be done using the residue theorem (theorem A.12). Therefore, the contour $C_{\varepsilon,R}$ as in proposition 3.11 need to be extended to a closed contour $C_m = C_{\varepsilon,R} \cup \gamma_m$, see figure 3.2. Here $\gamma_m$ is a circle with radius $(2m+1)\pi$ in clockwise direction. Note that $C_m$ is a closed contour if $R = (2m+1)\pi$ for $C_{\varepsilon,R}$.

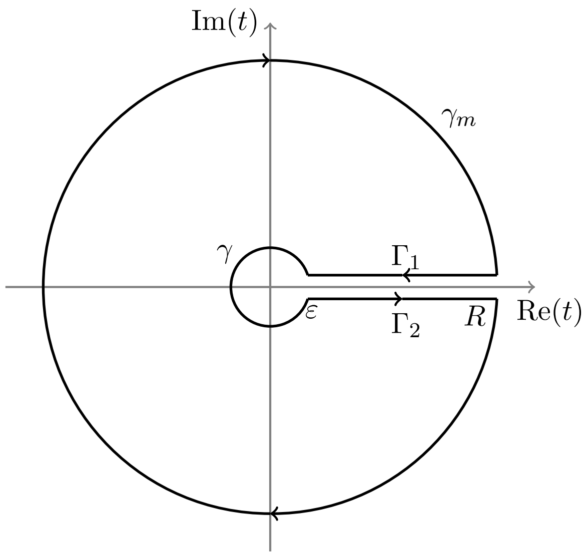


Figure 3.2: The closed contour $C_m$ with $R = (2m+1)\pi$.

For $s \in \mathbb{C}$ fixed, the poles of $\frac{(-t)^{s-1}}{e^t-1}$ as function of $t$ lying in the closed contour $C_m$ are at $\pm 2m\pi i$ for $m \in \mathbb{N}$. Note that the contour has a negative orientation, hence there is an extra minus sign. So by the residue theorem

$$\int_{C_m} \frac{(-t)^{s-1}}{e^t - 1} dt = -2\pi i \sum_{k=1}^{m} Res\left(\frac{(-t)^{s-1}}{e^t - 1}; \pm 2m\pi i\right). \tag{3.5}$$

For $m > 0$:

$$\begin{aligned} Res\left(\frac{(-t)^{s-1}}{e^t - 1}; 2m\pi i\right) &= \lim_{t \to 2m\pi i} (t - 2m\pi i) \frac{(-t)^{s-1}}{e^t - 1} \\ &= \lim_{t \to 2m\pi i} e^{(s-1)\log(-t)} \cdot \frac{t - 2m\pi i}{e^t - 1} \\ &= e^{(s-1)(\log(2m\pi) - i\frac{\pi}{2})} \cdot \frac{1}{e^{2m\pi i}} \qquad \text{(l'Hospital's rule)} \\ &= (2m\pi)^{s-1} e^{-\frac{1}{2}\pi i s} e^{\frac{1}{2}\pi i} \\ &= (2m\pi)^{s-1} i e^{-\frac{1}{2}\pi i s}. \end{aligned}$$

Similarly, for $m < 0$:

$$Res\left(\frac{(-t)^{s-1}}{e^t - 1}; -2m\pi i\right) = -(2m\pi)^{s-1} i e^{\frac{1}{2}\pi i s}.$$

Combining everything gives

$$\begin{aligned} \int_{C_m} \frac{(-t)^{s-1}}{e^t - 1} dt &= -2\pi i \sum_{k=1}^{m} Res\left(\frac{(-t)^{s-1}}{e^t - 1}; \pm 2m\pi i\right) \\ &= -2\pi i \sum_{k=1}^{m} \left[(2\pi k)^{s-1} i \left(e^{-\frac{1}{2}\pi i s} - e^{\frac{1}{2}\pi i s}\right)\right] \\ &= -2\pi i \sum_{k=1}^{m} \left[(2\pi k)^{s-1} i 2i \sin\left(-\frac{1}{2}\pi s\right)\right] \\ &= -2\pi i \sum_{k=1}^{m} \left[2(2\pi k)^{s-1} \sin\left(\frac{1}{2}\pi s\right)\right]. \end{aligned}$$

Let $\varepsilon \to 0$ and $m \to \infty$, then the integrals over $\gamma$ and $\gamma_m$ go to zero, so the integral over $C_m$ as $m \to \infty$ is equal to the integral over $C$. Plugging the result in equation (3.4) gives

$$\begin{aligned} \zeta(s) &= \frac{i\Gamma(1-s)}{2\pi} \int_C \frac{(-t)^{s-1}}{e^t - 1} dt \\ &= \frac{i\Gamma(1-s)}{2\pi} \cdot -2\pi i \sum_{k=1}^{m} \left[2(2\pi k)^{s-1} \sin\left(\frac{1}{2}\pi s\right)\right] \\ &= \Gamma(1-s) 2(2\pi)^{s-1} \sin\left(\frac{\pi s}{2}\right) \sum_{k=1}^{m} k^{-(1-s)} \\ &= \Gamma(1-s) 2(2\pi)^{s-1} \sin\left(\frac{\pi s}{2}\right) \zeta(1-s). \end{aligned}$$

It was shown in proposition 3.11 that the integral over $\gamma$ goes to zero as $\varepsilon \to 0$. It remains to show that the integral over $\gamma_m$ goes to zero as $m \to \infty$. Note that on $\gamma_m$ $|t| = (2m+1)\pi$

and that $t$ can be parametrized by $t = (2m+1)\pi e^{i\theta} = (2m+1)\pi(\cos\theta + i\sin\theta)$ with $0 < \theta < 2\pi$. Then

$$\left|e^t - 1\right| \geq \left||e^t| - 1\right| = \left|e^{\mathrm{Re}(t)} - 1\right| = \left|e^{(2m+1)\pi\cos\theta} - 1\right| \geq \left|e^{(2m+1)\pi} - 1\right|$$

and

$$\left|(-t)^{s-1}\right| = |t|^{\mathrm{Re}(s)-1} = \big((2m+1)\pi\big)^{\mathrm{Re}(s)-1},$$

which gives that

$$\begin{aligned}\left|\int_{\gamma_m} \frac{(-t)^{s-1}}{e^t - 1} dt\right| &\leq 2(2m+1)\pi^2 \max_{|t|=(2m+1)\pi} \left|\frac{(-t)^{s-1}}{e^t - 1}\right| \\ &\leq 2(2m+1)\pi^2 \frac{\big((2m+1)\pi\big)^{\mathrm{Re}(s)-1}}{e^{(2m+1)\pi} - 1} \\ &= 2\pi \frac{\big((2m+1)\pi\big)^{\mathrm{Re}(s)}}{e^{(2m+1)\pi} - 1} \to 0 \quad \text{as } m \to \infty.\end{aligned}$$

Finally, we find the functional equation of the zeta function for $s \neq 1$:

$$\zeta(s) = \Gamma(1-s)2(2\pi)^{s-1} \sin\left(\frac{\pi s}{2}\right)\zeta(1-s). \tag{3.6}$$

## 3.3 Functional Equation: Proof with Jacobi's Theta Function

The second proof of the functional equation of the zeta function given in Riemann's article uses the Jacobi theta function $\vartheta(x)$. First, we will establish some properties of $\vartheta(x)$ and the related function $\psi(x)$.

**Definition 3.12.** *The Jacobi theta function is defined for Re$(x) > 0$ by*

$$\vartheta(x) := \sum_{n=-\infty}^{\infty} e^{-n^2\pi x}.$$

*Also, for Re$(x) > 0$ define the related function*

$$\psi(x) := \sum_{n=1}^{\infty} e^{-n^2\pi x}.$$

Both functions are related by

$$\sum_{n=-\infty}^{\infty} e^{-n^2\pi x} = \sum_{n=-\infty}^{-1} e^{-n^2\pi x} + 1 + \sum_{n=1}^{\infty} e^{-n^2\pi x} = 2\sum_{n=1}^{\infty} e^{-n^2\pi x} + 1$$

and thus

$$\vartheta(x) = 2\psi(x) + 1. \tag{3.7}$$

**Proposition 3.13.** *The functions $\vartheta(x)$ and $\psi(x)$ converge absolutely for Re$(x) > 0$.*

*Proof.* With the integral test the convergence of $\psi(x)$ can be shown:

$$\begin{aligned}\left|\lim_{n\to\infty}\int_1^n e^{-y^2\pi x}dy\right| &\leq \lim_{n\to\infty}\int_1^n \left|e^{-y^2\pi x}\right|dy \\ &= \lim_{n\to\infty}\int_1^n e^{-y^2\pi\mathrm{Re}(x)}dy \\ &\leq \lim_{n\to\infty}\int_1^n e^{-y\pi\mathrm{Re}(x)}dy \\ &= \lim_{n\to\infty}\left[\frac{1}{-\pi\mathrm{Re}(x)}e^{-y\pi\mathrm{Re}(x)}\right]_1^n = \frac{1}{\pi\mathrm{Re}(x)} \quad \text{for } \mathrm{Re}(x)>0.\end{aligned}$$

Also, note that $\psi(x)$ converges normally for $\mathrm{Re}(x) > 0$, since $\left|e^{-n^2\pi x}\right| = e^{-n^2\pi\mathrm{Re}(x)}$ and $\sum_{n=1}^{\infty} e^{-n^2\pi\mathrm{Re}(x)}$ converges. Hence, by theorem A.18 $\psi(x)$ is analytic for $\mathrm{Re}(x) > 0$.

From the relation in equation (3.7) it follows that also $\vartheta(x)$ is absolutely convergent and analytic for $\mathrm{Re}(x) > 0$. □

The Jacobi theta function satisfies the following equation

$$\sqrt{x}\vartheta(x) = \vartheta(x^{-1}) \quad \text{for } \mathrm{Re}(x) > 0.$$

This property can be proven with the Poisson summation formula, but is omitted here. A proof can be found in [Edwards, 1974, Section 10.4]. Rewriting this equation for the function $\psi(x)$ using equation (3.7) gives

$$\sqrt{x}\big(2\psi(x)+1\big) = 2\psi(x^{-1})+1,$$

which is equivalent to

$$\psi(x^{-1}) = x^{\frac{1}{2}}\psi(x) + \frac{1}{2}x^{\frac{1}{2}} - \frac{1}{2}. \tag{3.8}$$

Also, the following two lemmas are needed for deriving the functional equation.

**Lemma 3.14.** *For all $x \in \mathbb{C}$ with Re$(x) \geq 1$ and $k \in \mathbb{N}$ there exists a $C \in \mathbb{R}$ such that*

$$|\psi(x)| \leq Ce^{-Re(x)}.$$

*Proof.* Let $\mathrm{Re}(x) \geq 1$ and $k \in \mathbb{N}$, then

$$\begin{aligned}|\psi(x)| = \left|\sum_{n=1}^{\infty} e^{-n^2\pi x}\right| &\leq \sum_{n=1}^{\infty} e^{-n^2\pi\mathrm{Re}(x)} \\ &= e^{-\mathrm{Re}(x)}\sum_{n=1}^{\infty} e^{-n^2\pi\mathrm{Re}(x)+\mathrm{Re}(x)} \\ &\overset{\mathrm{Re}(x)\geq 1}{\leq} e^{-\mathrm{Re}(x)}\sum_{n=1}^{\infty} e^{-n^2\pi+1} \\ &= e^{-\mathrm{Re}(x)}e\underbrace{\psi(1)}_{<\infty} = Ce^{-\mathrm{Re}(x)} \quad \text{for } C\in\mathbb{R}.\end{aligned}$$

□

**Lemma 3.15.** *For Re$(s) > 1$*

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_0^{\infty} x^{\frac{s}{2}-1}\psi(x)dx = \int_1^{\infty}\psi(x)\left(x^{-\frac{s}{2}-\frac{1}{2}} + x^{\frac{s}{2}-1}\right)dx + \frac{1}{s(s-1)},$$

*where the right hand side converges absolutely for $s \in \mathbb{C}\setminus\{0,1\}$.*

*Proof.* From lemma 3.9 with $\frac{s}{2}$ it follows that

$$\int_0^\infty e^{-nt} t^{\frac{s}{2}-1} dt = \frac{\Gamma\left(\frac{s}{2}\right)}{n^{\frac{s}{2}}}.$$

Substitute $t = n\pi x$

$$\begin{aligned}\int_0^\infty e^{-nt} t^{\frac{s}{2}-1} dt &= \int_0^\infty e^{-n^2\pi x} n^{\frac{s}{2}-1} \pi^{\frac{s}{2}-1} x^{\frac{s}{2}-1} n\pi dx \\ &= \int_0^\infty e^{-n^2\pi x} n^{\frac{s}{2}} \pi^{\frac{s}{2}} x^{\frac{s}{2}-1} dx\end{aligned}$$

and it is obtained that for $\mathrm{Re}(s) > 0$

$$\int_0^\infty e^{-n^2\pi x} x^{\frac{s}{2}-1} dx = \frac{\Gamma\left(\frac{s}{2}\right)}{n^s \pi^{\frac{s}{2}}}.$$

Summing over all $n \in \mathbb{N}$ gives that for $\mathrm{Re}(s) > 1$

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \sum_{n=1}^\infty \int_0^\infty e^{-n^2\pi x} x^{\frac{s}{2}-1} dx.$$

Interchanging limit and integral gives the first equality of the lemma. This is allowed by the dominated convergence theorem, namely $\sum_{n=1}^k e^{-n^2\pi x} x^{\frac{s}{2}-1}$ is dominated by $x^{\frac{s}{2}-1}\psi(x)$ and it remains to prove that $\int_0^\infty x^{\frac{s}{2}-1}\psi(x)dx$ converges absolutely for $\mathrm{Re}(s) > 1$. First, split the integral

$$\int_0^\infty x^{\frac{s}{2}-1}\psi(x)dx = \int_0^1 x^{\frac{s}{2}-1}\psi(x)dx + \int_1^\infty x^{\frac{s}{2}-1}\psi(x)dx. \tag{3.9}$$

For the first integral substitute $x = u^{-1}$:

$$\begin{aligned}\int_0^1 x^{\frac{s}{2}-1}\psi(x)dx &= -\int_\infty^1 \left(u^{-1}\right)^{\frac{s}{2}-1} \psi(u^{-1}) u^{-2} du \\ &\overset{(3.8)}{=} \int_1^\infty u^{-\frac{s}{2}-1} \left(u^{\frac{1}{2}}\psi(u) + \frac{1}{2}u^{\frac{1}{2}} - \frac{1}{2}\right) du.\end{aligned}$$

Plugging this in equation (3.9) gives

$$\int_0^\infty x^{\frac{s}{2}-1}\psi(x)dx = \int_1^\infty \psi(x) \left(x^{-\frac{s}{2}-\frac{1}{2}} + x^{\frac{s}{2}-1}\right) dx + \frac{1}{2}\int_1^\infty x^{-\frac{s}{2}-1}\left(x^{\frac{1}{2}} - 1\right) dx. \tag{3.10}$$

Now, evaluate the second integral

$$\frac{1}{2}\int_1^\infty x^{-\frac{s}{2}-\frac{1}{2}} - x^{-\frac{s}{2}-1} dx = \left[\frac{1}{1-s} x^{\frac{1-s}{2}} + \frac{1}{s} x^{-\frac{s}{2}}\right]_1^\infty \overset{\mathrm{Re}(s)>1}{=} \frac{1}{1-s} - \frac{1}{s} = \frac{1}{s(s-1)}$$

and plugging this back into equation (3.10) gives

$$\int_0^\infty x^{\frac{s}{2}-1}\psi(x)dx = \int_1^\infty \psi(x)\left(x^{-\frac{s}{2}-\frac{1}{2}} + x^{\frac{s}{2}-1}\right) dx + \frac{1}{s(s-1)} \quad \text{for } \mathrm{Re}(s) > 1. \tag{3.11}$$

This already gives the second equality of the lemma. Now, for the integral on the right hand side of (3.11) we can do the following estimate:

$$\begin{aligned}\left|\int_1^\infty \psi(x)\left(x^{-\frac{s}{2}-\frac{1}{2}} + x^{\frac{s}{2}-1}\right) dx\right| &\le \int_1^\infty \left|\psi(x)x^{-\frac{s}{2}-\frac{1}{2}}\right| + \left|\psi(x)x^{\frac{s}{2}-1}\right| dx \\ &\overset{\text{lemma 3.14}}{\le} \int_1^\infty Ce^{-x} x^{-\frac{\mathrm{Re}(s)}{2}-\frac{1}{2}} dx + \int_1^\infty Ce^{-x} x^{\frac{\mathrm{Re}(s)}{2}-1} dx < \infty,\end{aligned}$$

since $\int_1^\infty e^{-x}x^a dx < \infty$ for all $a \in \mathbb{R}$. So the right hand side of (3.11) is absolutely convergent for $s \in \mathbb{C} \setminus \{0,1\}$ and thus $\int_0^\infty x^{\frac{s}{2}-1}\psi(x)dx$ converges absolutely for $\text{Re}(s) > 1$, which allows the usage of the dominated convergence theorem. □

By the identity theorem for analytic functions (theorem A.20), the equation

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_1^\infty \psi(x)\left(x^{-\frac{s}{2}-\frac{1}{2}} + x^{\frac{s}{2}-1}\right)dx + \frac{1}{s(s-1)}$$

holds for all $s \in \mathbb{C} \setminus \{0,1\}$. Note that $\Gamma\left(\frac{s}{2}\right)$ has a simple pole at 0 which corresponds with the simple pole on the right hand side. Hence $\zeta(s)$ can be extended analytically to $\mathbb{C} \setminus \{1\}$. The simple pole of the right hand side for $s = 1$ should also coincide with a simple pole of $\Gamma\left(\frac{s}{2}\right)\zeta(s)$. Since $\Gamma\left(\frac{s}{2}\right)$ has no pole at $s = 1$ it follows that $\zeta(s)$ has a simple pole at $s = 1$. Also, $\Gamma\left(\frac{s}{2}\right)$ has simple poles for $s = -2k$ with $k \in \mathbb{N}$. These poles must be canceled by zeros of $\zeta(s)$ because the right hand side is defined for $s = -2k$ with $k \in \mathbb{N}$. These zeros are called the trivial zeros of the zeta function.

**Proposition 3.16.** *For $\mathbb{C} \setminus \{0,1\}$ the functional equation of $\zeta(s)$ is given by*

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

*Proof.* Substitute $1 - s$ in equation (3.11):

$$\begin{aligned}
\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) &= \int_1^\infty \psi(x)\left(x^{-\frac{1-s}{2}-\frac{1}{2}} + x^{\frac{1-s}{2}-1}\right)dx + \frac{1}{(1-s)(-s)}\\
&= \int_1^\infty \psi(x)\left(x^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}}\right)dx + \frac{1}{s(s-1)}\\
&= \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s).
\end{aligned}$$

□

**Remark 3.17.** *Note that this functional equation is equivalent to the one that was found using contour integration. Recall Legendre's Duplication formula (proposition 2.10)*

$$\Gamma(2z) = (2\pi)^{-\frac{1}{2}}2^{2z-\frac{1}{2}}\Gamma(z)\Gamma\left(z + \frac{1}{2}\right)$$

*and substituting $z = -\frac{s}{2} + \frac{1}{2}$ gives*

$$\Gamma(1-s) = (2\pi)^{-\frac{1}{2}}2^{-s+\frac{1}{2}}\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(1 - \frac{s}{2}\right).$$

*By using this identity and Euler's reflection formula (proposition 2.7) for the functional equation as in equation* (3.6) *it is obtained that*

$$\begin{aligned}
\zeta(s) &= \Gamma(1-s)2(2\pi)^{s-1}\sin\left(\frac{\pi s}{2}\right)\zeta(1-s)\\
&= 2^{-s}\pi^{-\frac{1}{2}}\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)2^s\pi^{s-1}\frac{\pi}{\Gamma\left(\frac{s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)}\zeta(1-s)\\
&= \pi^{s-\frac{1}{2}}\Gamma\left(\frac{1-s}{2}\right)\frac{\zeta(1-s)}{\Gamma\left(\frac{s}{2}\right)}.
\end{aligned}$$

*Rearranging this equation gives the identity in proposition 3.16. So both functional equations are equivalent.*

Multiply the functional equation in proposition 3.16 by $\frac{s(s-1)}{2}$ to obtain

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}+1\right)(s-1)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}+1\right)(-s)\zeta(1-s), \tag{3.12}$$

where the property $\Gamma(z+1) = z\Gamma(z)$ is used.

**Definition 3.18.** *Define for all* $s \in \mathbb{C}$

$$\xi(s) := \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}+1\right)(s-1)\zeta(s).$$

This is an entire function, because

(i) the simple pole of $\zeta(s)$ at $s = 1$ is canceled with the zero of $s - 1$,

(ii) the simple poles of $\Gamma\left(\frac{s}{2}+1\right)$ at $s = -2k$ with $k \in \mathbb{N}$ are canceled with the trivial zeros of the zeta function,

(iii) for all other $s \in \mathbb{C}$ the function $\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}+1\right)(s-1)\zeta(s)$ is analytic.

From equation (3.12) it follows that $\xi(s)$ satisfies

$$\xi(s) = \xi(1-s).$$

Zeros of $\zeta(s)$ that are not trivial can only occur if $\xi(s) = 0$. These zeros are called the nontrivial zeros and are denoted by $\rho$. By the other representation of the zeta function for $\mathrm{Re}(s) > 1$

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}$$

it follows that $\zeta(s) \neq 0$ for $\mathrm{Re}(s) > 1$. Also, from $\xi(s) = \xi(1-s)$ it follows that, apart from the trivial zeros, $\zeta(s) \neq 0$ for $\mathrm{Re}(s) < 0$. Hence, all nontrivial zeros $\rho$ must lie in the strip: $\{s \in \mathbb{C} : 0 \leq \mathrm{Re}(s) \leq 1\}$. This strip can be narrowed down to what is called the critical strip: $\{s \in \mathbb{C} : 0 < \mathrm{Re}(s) < 1\}$. That $\zeta(s) \neq 0$ for $\mathrm{Re}(s) = 1$ plays an important role in proving the prime number theorem (even stronger: it is equivalent to the prime number theorem) and this will come back in chapters 5 and 6 when we will prove the prime number theorem. The fact that $\zeta(s) \neq 0$ for $\mathrm{Re}(s) = 0$ will not be proven here.

## 3.4 The Riemann Hypothesis

All kind of estimates can be made about the nontrivial zeros in the critical strip. Riemann claimed that the number of roots $\rho$ in the critical strip with imaginary part between 0 and $T$ is approximately

$$\frac{T}{2\pi}\log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi}.$$

Riemann did not rigorously prove this assertion and neither will we in this thesis. A proof can be found in [Ingham, 1932, Ch. IV].

The strongest estimate about the location of the nontrivial zeros was conjectured by Riemann in his article. This conjecture is now know as the Riemann hypothesis and is one of the greatest unsolved problems in mathematics.

**Conjecture 3.19** (The Riemann hypothesis)**.** *All nontrivial zeros of the zeta function have real part* $\frac{1}{2}$*.*

If $\xi(\rho) = 0$ with $\rho$ in the critical strip, then the relation $\xi(s) = \xi(1-s)$ gives that $1-\rho$ is also a zero. So the nontrivial zeros of the zeta function always come in pairs. Also, from the definition of $\xi(s)$ and the fact that $\zeta(\overline{s}) = \overline{\zeta(s)}$ and $\Gamma(\overline{s}) = \overline{\Gamma(s)}$ it follows that $\xi(\overline{s}) = \overline{\xi(s)}$. So, if $\rho$ is a zero of $\xi(s)$, then $\overline{\rho}$ is also a zero of $\xi(s)$. Now, if a nontrivial zero of the zeta function $\rho_1$ does not satisfy $\text{Re}(\rho_1) = \frac{1}{2}$, then there are directly four nontrivial zeros with $\text{Re}(\rho_1) \neq \frac{1}{2}$. The other three zeros correspond to $\overline{\rho_1}$, $1-\rho_1$ and $\overline{1-\rho_1}$.

Numerically, it has been showed that at least the first $10^{13}$ nontrivial zeros indeed have real part $\frac{1}{2}$. The mathematician Hardy even proved that infinitely many zeros must lie on the line with real part $\frac{1}{2}$. Figure 3.3 shows the value $|\zeta(s)|$ on the line with $\text{Re}(s) = \frac{1}{2}$ and $0 \leq \text{Im}(s) \leq 30$. The first three zeros in the upper half-plane lie at $\frac{1}{2} + i14.134\ldots$, $\frac{1}{2} + i21.022\ldots$ and $\frac{1}{2} + i25.010\ldots$.

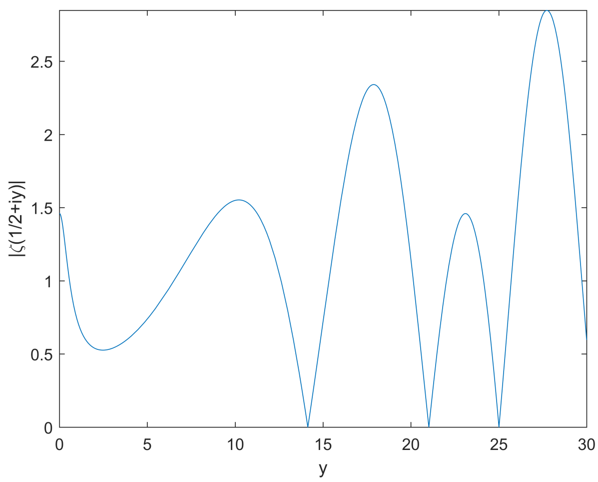


Figure 3.3: The absolute value of $\zeta(s)$ on the line with real part $\frac{1}{2}$ and imaginary part $y$ between 0 and 30.

# Chapter 4

# Explicit Formula for the Prime Counting Function

Riemann continued in his article with the construction of an explicit formula for the prime counting function, which we already briefly discussed in chapter 1.

**Definition 4.1.** *Define the prime counting function $\pi(x)$ as the number of prime numbers less than or equal to $x$ for all $x > 0$. So*

$$\pi(x) = \sum_{p \leq x} 1 = \sum_{n \leq x} u_p(n),$$

*where $p$ denote all prime numbers and*

$$u_p(n) = \begin{cases} 1 & \text{if } n \text{ is prime} \\ 0 & \text{otherwise} \end{cases}.$$

**Remark 4.2.** *Obviously, the inequality $\pi(x) < x$ holds for all $x > 0$, since $\pi(x) = x$ would mean that all numbers are prime which is not the case.*

Riemann used a slightly different function, which he called $F$. This function is the same as the prime counting function $\pi(x)$ only when there is a jump, the function takes the average value of the right and left limit. Riemann wrote that as follows

$$F(x) = \frac{F(x+0) + F(x-0)}{2}.$$

Since this function is closely related to $\pi(x)$, we will use $\pi_0(x)$ instead of $F(x)$.

**Definition 4.3.** *Define for $x > 0$*

$$\pi_0(x) = \frac{1}{2}\left(\sum_{p<x} 1 + \sum_{p \leq x} 1\right).$$

*This is exactly the same as $\pi(x)$ only at every jump the function has the value halfway. This can also be written as*

$$\pi_0(x) = \frac{1}{2}\left(\lim_{t \uparrow x} \pi(t) + \lim_{t \downarrow x} \pi(t)\right).$$

The reason for this slightly different definition is that Fourier transforms are needed in section 4.3. Recall that for an integrable function $f(x)$

$$(\mathcal{F}f)(\lambda) = \int_{-\infty}^{\infty} f(x)e^{-ix\lambda}dx$$

is the Fourier transform and

$$(\mathcal{F}^{-1}f)(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\lambda)e^{ix\lambda}d\lambda$$

is the inverse Fourier transform. Assume that $f$ is piecewise continuous and that $f$ and $\mathcal{F}f$ are integrable. Then

$$\mathcal{F}^{-1}(\mathcal{F}f)(x) = \frac{1}{2}\left(\lim_{t\uparrow x} f(t) + \lim_{t\downarrow x} f(t)\right).$$

This is why the function $\pi_0(x)$ as in definition 4.3 is required.

The construction of a formula for $\pi_0(x)$ will take several steps, but let us first describe the general idea of all the things that have to be done. The following will be treated in the next sections:

4.1 A closely related function to $\pi_0(x)$ is introduced. This function, $J(x)$, can easily be written in terms of $\pi_0(x)$. With the aid of Möbius inversion we are able to invert this formula so that $\pi_0(x)$ is written in terms of $J(x)$.

4.2 The function $\xi(s)$, as introduced in the previous chapter, can be written as an infinite product. Using this infinite product representation we can find an expression for $\log\zeta(s)$. The proof of the infinite product for $\xi(s)$ is included at the end of this chapter in section 4.6.

4.3 Also, there is a relation between $J(x)$ and $\log\zeta(s)$. First, $\log\zeta(s)$ can be written in terms of an integral involving $J(x)$. Mellin inversion will give an expression for $J(x)$ in terms of an integral with $\log\zeta(s)$.

4.4 Finally, we will substitute the expression for $\log\zeta(s)$ (found in section 4.2) in the integral which was found in section 4.3. Evaluating this integral termwise will give an explicit formula for $J(x)$.

4.5 Combining the results from section 4.1 and 4.4 gives an explicit formula for $\pi_0(x)$.

4.6 As already mentioned, this section contains a proof of the infinite product representation of $\xi(s)$.

## 4.1 $\pi_0(x)$ and $J(x)$

**Definition 4.4.** *Define for $x > 0$*

$$J(x) = \frac{1}{2}\left(\sum_{k=1}^{\infty}\sum_{p^k<x}\frac{1}{k} + \sum_{k=1}^{\infty}\sum_{p^k\leq x}\frac{1}{k}\right) = \frac{1}{2}\left(\sum_{\substack{p^k<x\\ k\in\mathbb{N}}}\frac{1}{k} + \sum_{\substack{p^k\leq x\\ k\in\mathbb{N}}}\frac{1}{k}\right),$$

*where the last identity is just a shorthand notation to avoid writing double sums every time. This function jumps with $\frac{1}{k}$ at every prime power $p^k$ and at a jump, $J(x)$ has the value halfway:*

$$J(x) = \frac{1}{2}\left(\lim_{t\uparrow x} J(t) + \lim_{t\downarrow x} J(t)\right).$$

**Remark 4.5.** *Note that for all $x > 0$ it holds that $J(x) < x$. This follows from the definition of $J(x)$ and the fact that $\pi(x) < x$ (see remark 4.2).*

**Proposition 4.6.** *A relation between $J(x)$ and $\pi_0(x)$ is given by*

$$J(x) = \sum_{1 \le k \le x} \frac{1}{k} \pi_0\big(x^{\frac{1}{k}}\big).$$

*Proof.* Rewrite $J(x)$ as

$$\begin{aligned} J(x) &= \frac{1}{2} \left( \sum_{k=1}^{\infty} \sum_{p < x^{\frac{1}{k}}} \frac{1}{k} + \sum_{k=1}^{\infty} \sum_{p \le x^{\frac{1}{k}}} \frac{1}{k} \right) \\ &= \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k} \left( \sum_{p < x^{\frac{1}{k}}} 1 + \sum_{p \le x^{\frac{1}{k}}} 1 \right) \\ &= \sum_{k=1}^{\infty} \frac{1}{k} \pi_0\big(x^{\frac{1}{k}}\big). \end{aligned}$$

Note that this sum is finite: if $k$ is large enough, then $x^{\frac{1}{k}} < 2$ so that $\pi_0(x^{\frac{1}{k}}) = 0$ (since 2 is the first prime number). Definitely, if $k > x$, then $\pi_0(x^{\frac{1}{k}}) = 0$ because $x^{\frac{1}{x}} < 2$ for all $x > 0$. Hence,

$$J(x) = \sum_{1 \le k \le x} \frac{1}{k} \pi_0\big(x^{\frac{1}{k}}\big).$$

□

### 4.1.1 Möbius Inversion

The aim is to get an expression for $\pi_0(x)$, hence the identity in proposition 4.6 needs to be inverted. This will be done using Möbius inversion. First, the heuristics behind this method is explained and after that a generalization of the Möbius inversion formula will be proved.

Define $a_k = \frac{1}{k} J\big(x^{\frac{1}{k}}\big)$ and for the sake of clarity write $\pi(x)$ for a moment instead of $\pi_0(x)$. Then obviously

$$a_1 = J(x) = \pi(x) + \frac{1}{2}\pi\big(x^{\frac{1}{2}}\big) + \frac{1}{3}\pi\big(x^{\frac{1}{3}}\big) + \frac{1}{4}\pi\big(x^{\frac{1}{4}}\big) + \cdots$$

and

$$a_2 = \frac{1}{2} J\big(x^{\frac{1}{2}}\big) = \frac{1}{2}\pi\big(x^{\frac{1}{2}}\big) + \frac{1}{4}\pi\big(x^{\frac{1}{4}}\big) + \frac{1}{6}\pi\big(x^{\frac{1}{6}}\big) + \frac{1}{8}\pi\big(x^{\frac{1}{8}}\big) + \cdots .$$

Subtracting them gives

$$a_1 - a_2 = \pi(x) + \frac{1}{3}\pi\big(x^{\frac{1}{3}}\big) + \frac{1}{5}\pi\big(x^{\frac{1}{5}}\big) + \frac{1}{7}\pi\big(x^{\frac{1}{7}}\big) + \frac{1}{9}\pi\big(x^{\frac{1}{9}}\big) + \cdots .$$

It is clear that only the odd terms remain. The idea is to add or subtract terms on the left hand side, so that on the right hand side only $\pi(x)$ remains. Note that

$$a_3 = \frac{1}{3} J\big(x^{\frac{1}{3}}\big) = \frac{1}{3}\pi\big(x^{\frac{1}{3}}\big) + \frac{1}{6}\pi\big(x^{\frac{1}{6}}\big) + \frac{1}{9}\pi\big(x^{\frac{1}{9}}\big) + \frac{1}{12}\pi\big(x^{\frac{1}{12}}\big) + \cdots .$$

Subtracting $a_3$ gives

$$a_1 - a_2 - a_3 = \pi(x) + \frac{1}{5}\pi\left(x^{\frac{1}{5}}\right) - \frac{1}{6}\pi\left(x^{\frac{1}{6}}\right) + \frac{1}{7}\pi\left(x^{\frac{1}{7}}\right) + \frac{1}{11}\pi\left(x^{\frac{1}{11}}\right) - \frac{1}{12}\pi\left(x^{\frac{1}{12}}\right) + \cdots .$$

Unfortunately, some even terms are back on the right hand side, but there are still no terms which are powers of 2 or powers of 3. Continue subtracting $a_p$ for all primes $p$ gives that only $\pi(x)$ and terms with non-pure powers of primes remain on the right hand side:

$$a_1 - \sum_p a_p = \pi(x) - \frac{1}{6}\pi\left(x^{\frac{1}{6}}\right) - \frac{1}{10}\pi\left(x^{\frac{1}{10}}\right) - \frac{1}{12}\pi\left(x^{\frac{1}{12}}\right) - \frac{1}{14}\pi\left(x^{\frac{1}{14}}\right) - \cdots .$$

For numbers that can be written as the product of two distinct prime numbers $p$ and $q$, a term with a minus sign remains. Namely, both $a_p$ and $a_q$ give a term $-\frac{1}{pq}\pi\left(x^{\frac{1}{pq}}\right)$ and $a_1 = J(x)$ contains one term $\frac{1}{pq}\pi\left(x^{\frac{1}{pq}}\right)$, so indeed $-\frac{1}{pq}\pi\left(x^{\frac{1}{pq}}\right)$ remains. To get rid of those terms, define $P \times P$ as the set containing all natural numbers which can be written as the product of two distinct primes. Then

$$a_1 - \sum_p a_p + \sum_{k \in P \times P} a_k = \pi(x) + \frac{1}{30}\pi\left(x^{\frac{1}{30}}\right) + \cdots .$$

Note that the term with $12 = 2^2 \cdot 3$ has also disappeared, because $a_6$ (which contains the term with 12) has been added. In general, all terms which do not have more than two different primes in their prime factorization have vanished. However, the term $\frac{1}{30}\pi\left(x^{\frac{1}{30}}\right)$ survives, this is because 30 arises in $a_j$ with $j = 1, 2, 3, 5, 6, 10, 15$. Three times there is a minus sign (for the primes 2, 3, 5) and all the other times there is a plus sign. Hence, one term $\frac{1}{30}\pi\left(x^{\frac{1}{30}}\right)$ remains. To continue, all numbers which are the product of three distinct primes should be subtracted again at the left hand side and so on. Finally, it is obtained that

$$a_1 - \sum_p a_p + \sum_{k \in P \times P} a_k - \sum_{k \in P \times P \times P} a_k + \cdots = \pi(x).$$

Plug in the definition of $a_k = \frac{1}{k}J\left(x^{\frac{1}{k}}\right)$ and note that the sum is finite again:

$$\sum_{1 \leq k \leq x} \frac{\mu(k)}{k} J\left(x^{\frac{1}{k}}\right) = \pi(x), \tag{4.1}$$

where $\mu(k)$ is the Möbius function defined by

$$\mu(k) = \begin{cases} 1 & \text{if } k = 1 \\ (-1)^n & \text{if } k \text{ is the product of } n \text{ distinct primes} \\ 0 & \text{otherwise} \end{cases} .$$

The above method is known as generalized Möbius inversion and the corresponding inversion formula is proved in the next theorem. The normal Möbius inversion formula is not sufficient in this case. We do not need it here, hence it is omitted. For proving this generalized formula, one result from analytic number theory about $\mu(k)$ is needed. This result is in the following lemma.

**Lemma 4.7.** *If $n \geq 1$, then*

$$\sum_{k|n} \mu(k) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases},$$

*where $k|n$ means that $k$ divides $n$.*

*Proof.* The statement is clearly true for $n = 1$. For $n > 1$, write $n$ in its factorization of prime numbers: $n = p_1^{a_1} \cdots p_r^{a_r}$. In the sum $\sum_{k|n} \mu(k)$ the only nonzero terms come from $k = 1$ and from the divisors of $n$ which are the product of distinct primes. This gives

$$\begin{aligned}\sum_{k|n} \mu(k) &= \mu(1) + \mu(p_1) + \cdots + \mu(p_r) + \mu(p_1 p_2) + \cdots + \mu(p_{r-1} p_r) + \cdots + \mu(p_1 p_2 \cdots p_r) \\ &= 1 + (-1)\binom{r}{1} + (-1)^2\binom{r}{2} + \cdots + (-1)^r\binom{r}{r} \\ &= (1-1)^r = 0.\end{aligned}$$

□

**Theorem 4.8** (Generalized Möbius inversion)**.** *Let $x > 1$ and suppose that*

$$G_x(m) = \sum_{1 \le l \le mx} F_x\left(\frac{m}{l}\right)$$

*for some $m \in \mathbb{N}$. Then*

$$\sum_{1 \le k \le x} \mu(k) G_x\left(\frac{1}{k}\right) = F_x(1).$$

*Proof.* Define for $x, y \in \mathbb{R}$

$$\mathbb{I}_{x=y} = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}.$$

Now,

$$\begin{aligned}\sum_{1 \le k \le x} \mu(k) G_x\left(\frac{1}{k}\right) &= \sum_{1 \le k \le x} \mu(k) \sum_{1 \le l \le \frac{x}{k}} F_x\left(\frac{1}{kl}\right) \\ &= \sum_{1 \le k \le x} \mu(k) \sum_{1 \le l \le \frac{x}{k}} \sum_{1 \le r \le x} \mathbb{I}_{r=kl} F_x\left(\frac{1}{r}\right) \\ &= \sum_{1 \le r \le x} F_x\left(\frac{1}{r}\right) \sum_{1 \le k \le x} \underbrace{\mu(k) \sum_{1 \le l \le \frac{x}{k}} \mathbb{I}_{l=\frac{r}{k}}}_{=\mu(k) \text{ if } k|r} \\ &= \sum_{1 \le r \le x} F_x\left(\frac{1}{r}\right) \underbrace{\sum_{k|r} \mu(k)}_{=0 \text{ if } r>1} \\ &\overset{\text{lemma 4.7}}{=} F_x(1).\end{aligned}$$

□

With this theorem equation (4.1) can be proved.

**Proposition 4.9.** *An expression for $\pi_0(x)$ in terms of $J(x)$ is*

$$\pi_0(x) = \sum_{1 \le k \le x} \frac{\mu(k)}{k} J\left(x^{\frac{1}{k}}\right).$$

*Proof.* Fix $x > 0$ and let $F_x(\frac{1}{k}) = \frac{1}{k}\pi_0(x^{\frac{1}{k}})$ and $G_x(\frac{1}{k}) = \frac{1}{k}J(x^{\frac{1}{k}})$. Note that

$$G_x(1) = J(x) = \sum_{1\le k\le x} \frac{1}{k}\pi_0(x^{\frac{1}{k}}) = \sum_{1\le k\le x} F_x\left(\frac{1}{k}\right),$$

so the condition of theorem 4.8 is satisfied. Now, it is obtained that

$$\sum_{1\le k\le x} \frac{\mu(k)}{k} J(x^{\frac{1}{k}}) = \sum_{1\le k\le x} \mu(k) G_x\left(\frac{1}{k}\right) = F_x(1) = \pi_0(x).$$

□

## 4.2 $\log \zeta(s)$ and $\xi(s)$

Recall the function $\xi(s)$ which was introduced in the previous chapter (definition 3.18):

$$\xi(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}+1\right)(s-1)\zeta(s).$$

The infinite product representation of $\xi(s)$ is

$$\xi(s) = \xi(0)\prod_{\rho}\left(1-\frac{s}{\rho}\right), \tag{4.2}$$

where $\rho$ are the zeros of $\xi(s)$ (which are the nontrivial zeros of the zeta function). The infinite product is taken in an order which pairs each root $\rho$ with the root $1-\rho$. Riemann uses this identity in his article, but was not able to prove it. Three decades later Hadamard finally proved this identity using his factorization theorem. Hadamard's factorization theorem is applicable in many more cases than only for the function $\xi(s)$. The proof of (4.2) can be simplified and is included in the last section of this chapter.

Plugging in the definition of $\xi(s)$ in the left hand side of equation (4.2) and taking the logarithm gives that for $\text{Re}(s) > 1$

$$-\frac{s}{2}\log(\pi) + \log\Gamma\left(\frac{s}{2}+1\right) + \log(s-1) + \log\zeta(s) = \log\xi(0) + \sum_{\rho}\log\left(1-\frac{s}{\rho}\right)$$

and after rewriting it is obtained that

$$\log\zeta(s) = \log\xi(0) + \sum_{\rho}\log\left(1-\frac{s}{\rho}\right) - \log\Gamma\left(\frac{s}{2}+1\right) + \frac{s}{2}\log(\pi) - \log(s-1). \tag{4.3}$$

**Remark 4.10.** *In this chapter, if there is a logarithm with a complex argument, then the principal branch of the logarithm is used.*

## 4.3 $J(x)$ and $\log \zeta(s)$

In this section, $J(x)$ and $\log\zeta(s)$ are linked, so that (together with the expression obtained from the Möbius inversion) $\pi_0(x)$ can be expressed in terms of $\log\zeta(s)$.

**Proposition 4.11.** *A relation between* $\log\zeta(s)$ *and* $J(x)$ *for* $Re(s) > 1$ *is given by*

$$\frac{\log\zeta(s)}{s} = \int_0^\infty J(x)x^{-s-1}dx.$$

*Proof.* First, note that for $\mathrm{Re}(s) > 0$

$$s \int_{p^n}^{\infty} x^{-s-1} dx = -x^{-s}\Big|_{p^n}^{\infty} = p^{-ns}. \tag{4.4}$$

From Euler's product formula (proposition 3.3) it is obtained that for $\mathrm{Re}(s) > 1$

$$\log \zeta(s) = \log \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} = -\sum_p \log\left(1 - \frac{1}{p^s}\right) = \sum_p \sum_{n=1}^{\infty} \frac{p^{-ns}}{n},$$

where it is allowed to use the Taylor expansion of the logarithm since $|p^{-s}| < 1$. Then, use equation (4.4) and interchange the integral and summations (which is allowed by the Fubini-Tonelli theorem)

$$\sum_p \sum_{n=1}^{\infty} \frac{p^{-ns}}{n} = s \sum_p \sum_{n=1}^{\infty} \frac{1}{n} \int_{p^n}^{\infty} x^{-s-1} dx = s \int_0^{\infty} \sum_{n=1}^{\infty} \sum_p \frac{1}{n} \mathbb{1}_{[p^n,\infty)}(x) x^{-s-1} dx,$$

where $\mathbb{1}_{[a,b)}(x)$ denotes the indicator function, which is 1 if $x \in [a, b)$ and zero elsewhere. The last identity is only nonzero if $p^n \leq x$ for any fixed $x > 0$. Hence, it can be rewritten as

$$s \int_0^{\infty} \sum_{n=1}^{\infty} \sum_p \frac{1}{n} \mathbb{1}_{[p^n,\infty)}(x) x^{-s-1} dx = s \int_0^{\infty} \sum_{p^n \leq x} \frac{1}{n} x^{-s-1} dx = s \int_0^{\infty} J(x) x^{-s-1} dx,$$

where the last step is allowed since the jumps of $J(x)$ occur on sets with measure zero. To conclude,

$$\frac{\log \zeta(s)}{s} = \int_0^{\infty} J(x) x^{-s-1} dx.$$

□

### 4.3.1 Mellin Inversion

The aim is to find an expression for $J(x)$ in terms of $\log \zeta(s)$ instead of the other way around. To invert the relation in proposition 4.11 the Mellin transformation is used.

**Definition 4.12.** *The Mellin transformation of a function $f(x)$ is defined as*

$$(\mathcal{M}f)(s) = \int_0^{\infty} f(x) x^{-s-1} dx$$

*and (under certain conditions) the inverse transformation is*

$$(\mathcal{M}^{-1} f)(x) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} f(s) x^s ds,$$

*where the integral is taken over a vertical line with real part $\sigma$ for some appropriate $\sigma$.*

**Remark 4.13.** *The Mellin transform is usually defined as*

$$(\mathcal{M}f)(s) = \int_0^{\infty} f(x) x^{s-1} dx.$$

*However, in order to invert the expression for $J(x)$ as in proposition 4.11, it is more convenient to define the Mellin transform as in definition 4.12.*

**Lemma 4.14.** *Write $s = \sigma + it$ and let $f(x)$ be a piecewise continuous function on $(0, \infty)$ taking the average value of the right and left limit in discontinuities. Also, assume that $f(e^\lambda)e^{-\sigma\lambda}$ is integrable for $a < \sigma < b$. If*

$$F(s) = \int_0^\infty f(x)x^{-s-1}dx$$

*is analytic for $a < \sigma < b$ and $F(s)$ goes to zero as $Im(s) \to \pm\infty$, then*

$$f(x) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} F(s)x^s ds,$$

*where $\sigma$ is chosen arbitrarily such that $a < \sigma < b$.*

*Proof.* Write $s = \sigma + it$ with $\sigma, t \in \mathbb{R}$ and $a < \sigma < b$. Substitute $x = e^\lambda$ and consider $t$ as the free variable:

$$\begin{aligned} F(\sigma + it) &= \int_0^\infty f(x)x^{-\sigma-it-1}dx = \int_{-\infty}^\infty f(e^\lambda)e^{\lambda(-\sigma-it-1)}e^\lambda d\lambda \\ &= \int_{-\infty}^\infty f(e^\lambda)e^{\lambda(-\sigma-it)}d\lambda = \frac{1}{2\pi}\int_{-\infty}^\infty g(\lambda)e^{-it\lambda}d\lambda, \end{aligned}$$

where $g(\lambda) = 2\pi f(e^\lambda)e^{-\sigma\lambda}$. Note that $g(\lambda)$ is integrable, so using the Fourier inversion formula an expression for $g(\lambda)$ is found:

$$2\pi f(e^\lambda)e^{-\sigma\lambda} = \int_{-\infty}^\infty F(\sigma + it)e^{it\lambda}dt.$$

Again substitute $x = e^\lambda$ and rewrite:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty F(\sigma + it)x^{\sigma+it}dt.$$

Doing another substitution $s = \sigma + it$ finally gives

$$f(x) = \frac{1}{2\pi}\int_{\sigma-i\infty}^{\sigma+i\infty} F(s)x^s(-i)ds = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} F(s)x^s ds,$$

where $\sigma$ is fixed such that $a < \sigma < b$. Since $F(s)$ is analytic on $a < \sigma < b$ and tends to zero as $\text{Im}(s) \to \pm\infty$, Cauchy's theorem (theorem A.4) ensures that it does not matter over which vertical line in the strip between $a$ and $b$ is integrated. □

**Remark 4.15.** *The inverse transformation of the 'ordinary' Mellin transformation (see remark 4.13) can be proven similarly, only one has to use the substitution $x = e^{-\lambda}$. Also, note that the Mellin transformation is nothing more than a Fourier transformation in different coordinates.*

**Proposition 4.16.** *An expression for $J(x)$ in terms of $\log\zeta(s)$ is*

$$J(x) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\log\zeta(s)}{s}x^s ds,$$

*where $s = \sigma + it$ with $\sigma > 1$.*

*Proof.* From proposition 4.11 it follows that for $\mathrm{Re}(s) > 1$

$$\frac{\log \zeta(s)}{s} = \int_0^\infty J(x) x^{-s-1} dx.$$

Also, $J(x)$ satisfies the conditions of lemma 4.14 and $J(e^\lambda)e^{-\sigma\lambda} \le e^{-(\sigma-1)\lambda}$ (since $J(x) \le x$ for all $x > 0$). Since $e^{-x}$ is integrable for $x > 0$, also $J(e^\lambda)e^{-\sigma\lambda}$ is integrable and thus the Fourier transform used in the proof of lemma 4.14 exists. So application of the lemma gives that for $\sigma > 1$

$$J(x) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{\log \zeta(s)}{s} x^s ds.$$

□

## 4.4 Termwise Integration

Combining equation (4.3):

$$\log \zeta(s) = \log \xi(0) + \sum_\rho \log\left(1 - \frac{s}{\rho}\right) - \log \Gamma\left(\frac{s}{2} + 1\right) + \frac{s}{2}\log(\pi) - \log(s-1) \tag{4.5}$$

and the formula for $J(x)$ from proposition 4.16:

$$J(x) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{\log \zeta(s)}{s} x^s ds$$

will hopefully give an explicit formula $J(x)$ and thus also for $\pi_0(x)$. Direct substitution and evaluating the integral termwise will not work, since this leads to a divergent integral. This can easily be seen from the term $\frac{s}{2}\log(\pi)$:

$$\frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{\frac{s}{2}\log(\pi)}{s} x^s ds = C \int_{\sigma - i\infty}^{\sigma + i\infty} x^s ds \overset{s=\sigma+it}{=} \tilde{C} \int_{-\infty}^{\infty} e^{it\log(x)} dt,$$

where $C$ and $\tilde{C}$ are constants and obviously the last integral is divergent. To avoid this problem, apply integration by parts to obtain

$$\begin{aligned}\frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{\log \zeta(s)}{s} x^s ds =& \frac{1}{2\pi i} \frac{\log \zeta(s)}{s} \frac{x^s}{\log(x)}\Big|_{s=\sigma - i\infty}^{\sigma + i\infty} \\ &- \frac{1}{2\pi i} \frac{1}{\log(x)} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{d}{ds}\left[\frac{\log \zeta(s)}{s}\right] x^s ds.\end{aligned}$$

To prove that the first term obtained after integration by parts is zero, it needs to be shown that

$$\lim_{t\to\infty} \frac{\log \zeta(\sigma \pm it)}{\sigma \pm it} x^{\sigma \pm it} = 0. \tag{4.6}$$

Note that $x^{\sigma \pm it}$ only oscillates and is thus bounded and that $\frac{1}{\sigma \pm it}$ goes to zero as $t \to \infty$. It remains to show that $\log \zeta(\sigma \pm it)$ is finite as $t \to \infty$. As seen earlier in the proof of proposition 4.11, we have for $s = \sigma + it$ with $\sigma > 1$

$$|\log \zeta(s)| = \left| \sum_p \sum_{n=1}^\infty \frac{p^{-ns}}{n} \right| \le \sum_p \sum_{n=1}^\infty \frac{p^{-n\sigma}}{n} = \log \zeta(\sigma)$$

using the Euler product formula and the Taylor expansion of the logarithm. From $\log \zeta(\sigma) < \infty$ for $\sigma > 1$ it follows that $|\log \zeta(\sigma)|$ is finite for $\sigma > 1$, which thus proves (4.6).

After integration by parts the expression for $J(x)$ is reduced to

$$J(x) = -\frac{1}{2\pi i}\frac{1}{\log(x)}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{d}{ds}\left[\frac{\log\zeta(s)}{s}\right]x^s ds.$$

Plugging expression (4.5) into this formula for $J(x)$ gives

$$J(x) = -\frac{1}{2\pi i}\frac{1}{\log(x)} I \quad \text{with} \tag{4.7}$$

$$I = \int_{\sigma-i\infty}^{\sigma+i\infty}\frac{d}{ds}\left[\frac{\log\xi(0) + \sum_\rho \log\left(1-\frac{s}{\rho}\right) - \log\Gamma\left(\frac{s}{2}+1\right) + \frac{s}{2}\log(\pi) - \log(s-1)}{s}\right]x^s ds.$$

This integral will be evaluated termwise and in each of the following subsections one term is investigated. It will turn out that all integrals are finite, so that indeed integrating termwise is allowed.

### 4.4.1 Principal Term

First, consider the term in (4.7) with $-\log(s-1)$, thus the following integral must be evaluated

$$I_1 := \frac{1}{2\pi i}\frac{1}{\log(x)}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{d}{ds}\left[\frac{\log(s-1)}{s}\right]x^s ds, \quad \sigma > 1. \tag{4.8}$$

It will turn out that this integral equals the logarithmic integral $Li(x)$ which was discussed briefly in chapter 1. This function is defined for $x > 1$ as

$$Li(x) = \lim_{\varepsilon\downarrow 0}\left(\int_0^{1-\varepsilon}\frac{dt}{\log(t)} + \int_{1+\varepsilon}^{x}\frac{dt}{\log(t)}\right),$$

which has to be interpreted as a Cauchy principal value. In order to evaluate integral (4.8), fix $x > 1$ and define

$$F(\beta) := \frac{1}{2\pi i}\frac{1}{\log(x)}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{d}{ds}\left[\frac{\log(\frac{s}{\beta}-1)}{s}\right]x^s ds.$$

Thus the goal is to evaluate $I_1 = F(1)$. Redefining $\log(\frac{s}{\beta}-1) = \log(s-\beta) - \log(\beta)$ is valid as long as $\mathrm{Re}(\beta) < \sigma$ and $\beta \in \mathbb{C}\setminus\{s : s \le 0\}$.

**Lemma 4.17.** *$F(\beta)$ is absolutely convergent for $Re(\beta) < \sigma$ and $\beta \in \mathbb{C}\setminus\{s : s \le 0\}$.*

*Proof.* Note that the part $x^s$ only oscillates and is bounded, since $|x^s| = x^\sigma$ which is constant. The derivative gives

$$\left|\frac{d}{ds}\frac{\log(\frac{s}{\beta}-1)}{s}\right| \le \left|\frac{\log(\frac{s}{\beta}-1)}{s^2}\right| + \left|\frac{1}{s(s-\beta)}\right|$$

and the integral over both parts is finite. □

The function $F(\beta)$ cannot be evaluated directly, hence we will look for another function which is equal to $F$ up to an additive constant and try to evaluate that one. Note that

$$\frac{d}{d\beta}\frac{\log(\frac{s}{\beta}-1)}{s}=\frac{1}{(\beta-s)\beta}$$

and from theorem A.6 the derivative with respect to $\beta$ and the integral can be interchanged. Also, changing the order of differentiation, which is allowed since both derivatives with respect to $s$ and $\beta$ are continuous, gives

$$\frac{d}{d\beta}F(\beta)=\frac{1}{2\pi i}\frac{1}{\beta\log(x)}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{d}{ds}\left[\frac{1}{\beta-s}\right]x^s ds.$$

Integration by parts gives

$$\begin{aligned}\frac{d}{d\beta}F(\beta)&=\frac{1}{2\pi i\beta}\frac{1}{\log(x)}x^s\frac{1}{\beta-s}\bigg|_{s=\sigma-i\infty}^{\sigma+i\infty}-\frac{1}{2\pi i\beta}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{1}{\beta-s}x^s ds\\&=-\frac{1}{2\pi i\beta}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{1}{\beta-s}x^s ds\end{aligned}$$

and this is an integral which can be evaluated.

**Lemma 4.18.** *If $Re(\beta)<\sigma$, then*

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{1}{s-\beta}x^s ds=\begin{cases}x^\beta & \textit{if } x>1\\0 & \textit{if } x<1\end{cases}.$$

*Proof.* Let $\mathrm{Re}(\beta)<\sigma$. The inverse Mellin transformation as in lemma 4.14 is given by

$$\tilde{f}(x)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\tilde{F}(s)x^s ds$$

with

$$\tilde{F}(s)=\frac{1}{s-\beta}.$$

Using the Mellin transform $\tilde{F}(s)$ is found:

$$\frac{1}{s-\beta}=\int_0^\infty\tilde{f}(x)x^{-s-1}dx.$$

Moreover, note that for $\mathrm{Re}(\beta)<\sigma$

$$\frac{1}{s-\beta}=\int_1^\infty x^{-s}x^{\beta-1}dx$$

and equating the above two integrals leads to the conlcusion that

$$\tilde{f}(x)=\begin{cases}x^\beta & \text{if } x>1\\0 & \text{if } x<1\end{cases}.$$

□

It was assumed that $x > 1$, so from the lemma it follows that

$$\frac{d}{d\beta}F(\beta) = \frac{1}{\beta}x^{\beta}.$$

We will search for a function which has the same derivative. Therefore, define the path $C^+$ as the line segment from 0 to $1-\varepsilon$, a semicircle $C_\varepsilon(1)$ around 1 with radius $\varepsilon$ in the upper half-plane and the line segment from $1+\varepsilon$ to $x$, see figure 4.1. Also, define a function which is closely related to $Li(x)$ for $\beta = 1$:

$$G(\beta) := \int_{C^+} \frac{t^{\beta-1}}{\log(t)}dt$$

for $\mathrm{Re}(\beta) > 0$. Note that $G$ is there absolutely convergent and analytic by theorem A.6. The derivative can be taken into the integral:

$$\frac{d}{d\beta}G(\beta) = \int_{C^+} \frac{d}{d\beta}\frac{t^{\beta-1}}{\log(t)}dt = \int_{C^+} t^{\beta-1}dt = \frac{t^\beta}{\beta}\bigg|_0^x = \frac{x^\beta}{\beta} = \frac{d}{d\beta}F(\beta).$$

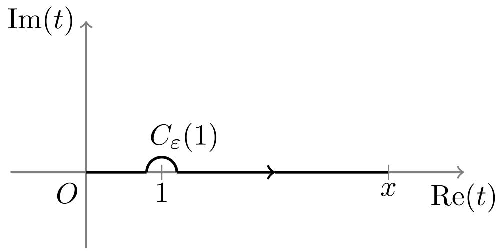


Figure 4.1: The path $C^+$.

So the derivatives of $F$ and $G$ are equal, what means that $F$ and $G$ itself only differ by a constant $c$. Since $G$ is easier to work with, this constant will be determined. Note that $F$ was defined for $\mathrm{Re}(\beta) < \sigma$ and $G$ for $\mathrm{Re}(\beta) > 0$. So $F(\beta) = G(\beta) + c$ only holds for $0 < \mathrm{Re}(\beta) < \sigma$. However, $F$ and $G$ are analytic on this strip, so by analytic continuation $F(\beta) = G(\beta) + c$ holds for $\mathrm{Re}(\beta) > 0$.
To find $c$, introduce another function

$$H(\beta) := \frac{1}{2\pi i}\frac{1}{\log(x)}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{d}{ds}\left[\frac{\log\left(1-\frac{s}{\beta}\right)}{s}\right]x^s ds,$$

where $\mathrm{Re}(\beta) < \sigma$ and $\log\left(1-\frac{s}{\beta}\right)$ is defined as $\log(s-\beta) - \log(-\beta)$ for all $\beta \in \mathbb{C} \setminus \{s : s \geq 0\}$. The functions $F$ and $H$ are both defined in the upper half-plane, so $H(\beta) - F(\beta)$ can be calculated there:

$$\begin{aligned} H(\beta) - F(\beta) &= \frac{1}{2\pi i}\frac{1}{\log(x)}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{d}{ds}\left[\frac{\log(\beta)-\log(-\beta)}{s}\right]x^s ds \\ &= \frac{1}{2\pi i}\frac{1}{\log(x)}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{d}{ds}\left[\frac{\pi i}{s}\right]x^s ds, \end{aligned}$$

where the last step is valid since the principal branch of the logarithm is used and the difference between the arguments of $\beta$ and $-\beta$ is $\pi i$ if $\beta$ lies in the upper half-plane. Now, apply integration by parts and use lemma 4.18 (with $\beta = 0$) to obtain

$$\begin{aligned}\frac{1}{2\pi i}\frac{1}{\log(x)}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{d}{ds}\left[\frac{\pi i}{s}\right]x^s ds &= \frac{1}{2\pi i}\frac{1}{\log(x)}\frac{\pi i}{s}x^s\Big|_{\sigma-i\infty}^{\sigma+i\infty} - \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\pi i}{s}x^s ds\\ &= -\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\pi i}{s}x^s ds\\ &= -\pi i.\end{aligned}$$

So it is obtained that in the upper half-plane $F(\beta) = H(\beta) + \pi i$. Before, it was found that $F$ and $G$ differ by a constant, so also $G$ and $H$ must differ by a constant. This constant will be determined by setting $\beta = u + vi$ and calculating $\lim_{v\to\infty} G(u + vi)$ and $\lim_{v\to\infty} H(u + vi)$ with $0 < u < \sigma$. The first limit is

$$\lim_{v\to\infty} G(u+vi) = \lim_{v\to\infty}\int_{C^+}\frac{t^{u+vi-1}}{\log(t)}dt.$$

Substitute $t = e^\nu$ and note that the path $C^+$ is changed to a path that starts in $-\infty$ and ends in $\log(x)$ where it avoids the singularity in 0. Using Cauchy's theorem (theorem A.4) the integral over the closed contour in figure 4.2 is zero if $\delta > 0$ is chosen such that there are no poles inside the contour. It can easily be verified that the integral from $-R + \delta i$ to $-R$ vanishes if $R \to \infty$.

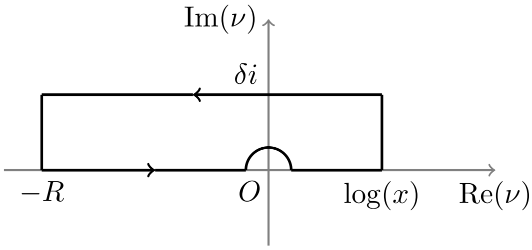


Figure 4.2: The closed contour for changing the path of integration.

So it follows that

$$G(u+vi) = \int_{-\infty+\delta i}^{\log(x)+\delta i}\frac{e^{\nu(u+vi)}}{\nu}d\nu + \int_{\log(x)+\delta i}^{\log(x)}\frac{e^{\nu(u+vi)}}{\nu}d\nu$$

for some appropriate $\delta > 0$. For the first integral substitute $\nu = z + \delta i$

$$\int_{-\infty}^{\log(x)}\frac{e^{(z+\delta i)(u+vi)}}{z+\delta i}dz = e^{-\delta v}e^{\delta u i}\int_{-\infty}^{\log(x)}\frac{e^{z(u+vi)}}{z+\delta i}dz \to 0 \quad \text{as } v\to\infty,$$

since $e^{-\delta v} \to 0$ and the integral remains finite as $v \to \infty$. For the second integral substitute $\nu = \log(x) + wi$

$$\int_0^\delta \frac{e^{(\log(x)+wi)(u+vi)}}{\log(x)+wi}(-i)dw = -ix^{u+vi}\int_0^\delta\frac{e^{-wv}e^{wui}}{\log(x)+wi}dw \to 0 \quad \text{as } v\to\infty,$$

since $e^{-wv} \to 0$ and the term in front of the integral is bounded. So

$$\lim_{v\to\infty} G(u+vi) = 0.$$

Second, evaluate the same limit for $H(u+vi)$. Work out the derivative in the definition of $H(\beta)$:

$$\frac{d}{ds}\frac{\log\left(1-\frac{s}{\beta}\right)}{s} = -\frac{\log\left(1-\frac{s}{\beta}\right)}{s^2} + \frac{1}{s(s-\beta)} = -\frac{\log\left(1-\frac{s}{\beta}\right)}{s^2} + \frac{1}{\beta(s-\beta)} - \frac{1}{\beta s}.$$

The first term becomes

$$\lim_{v\to\infty}\frac{1}{2\pi i}\frac{1}{\log(x)}\int_{\sigma-i\infty}^{\sigma+i\infty} -\frac{\log\left(1-\frac{s}{u+vi}\right)}{s^2}x^s ds.$$

Using the dominated convergence theorem ($\frac{s^\varepsilon}{s^2}x^s$ is integrable) the limit and integral can be interchanged. Note that $\log\left(1-\frac{s}{u+vi}\right)\to 0$ as $v\to\infty$, thus the above expression goes to zero as $v\to\infty$. This trick would not have worked if it was applied to $F(\beta)$, since the expression inside the log would not become zero. This is the main reason to define the function $H(\beta)$. The other two terms of the derivative give using lemma 4.18

$$\lim_{v\to\infty}\frac{1}{2\pi i}\frac{1}{\log(x)}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{x^s}{(u+vi)\big(s-(u+vi)\big)} - \frac{x^s}{(u+vi)s}ds = \lim_{v\to\infty}\left(\frac{x^{u+vi}}{u+vi} - \frac{1}{u+vi}\right).$$

Again, this goes to zero (since the numerators are bounded). To conclude:

$$\lim_{v\to\infty} G(u+vi) = 0 \quad \text{and} \quad \lim_{v\to\infty} H(u+vi) = 0$$

and thus $G(\beta) = H(\beta)$ for $0<\mathrm{Re}(\beta)<\sigma$. It was already found that $F(\beta) = H(\beta)+\pi i$, so that $\lim_{v\to\infty} F(\beta) = \pi i$. Hence, $F(\beta) = G(\beta)+\pi i$ for $0<\mathrm{Re}(\beta)<\sigma$ which means that $F(1)$ can be evaluated:

$$F(1) = G(1)+\pi i = \int_0^{1-\varepsilon}\frac{dt}{\log(t)} + \int_{C_\varepsilon(1)}\frac{dt}{\log(t)} + \int_{1+\varepsilon}^{x}\frac{dt}{\log(t)} + \pi i.$$

To evaluate the integral over the semicircle, note that $\lim_{t\to 1}\frac{t-1}{\log(t)} = 1$ by l'Hospital's rule, so at $t=1$ there is a simple pole with residue 1. Therefore, by corollary A.14 the integral over the semicircle as $\varepsilon\to 0$ is $-\pi i$. This cancels with the $\pi i$ and we are left with

$$I_1 = F(1) = \lim_{\varepsilon\downarrow 0}\left(\int_0^{1-\varepsilon}\frac{dt}{\log(t)} + \int_{1+\varepsilon}^{x}\frac{dt}{\log(t)}\right) = Li(x).$$

### 4.4.2 Term with Roots $\rho$

Next, consider the term in (4.7) with

$$\sum_\rho \log\left(1-\frac{s}{\rho}\right),$$

where $\rho$ are the zeros of $\xi(s)$. So the following integral must be evaluated

$$I_2 := -\frac{1}{2\pi i}\frac{1}{\log(x)}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{d}{ds}\left[\frac{\sum_\rho \log\left(1-\frac{s}{\rho}\right)}{s}\right]x^s ds. \tag{4.9}$$

It was unknown to Riemann whether the sum and the integral could be interchanged. He assumed it was possible and continued. For the moment we also assume it is possible and we will come back to this in section 4.5. So it is obtained that

$$I_2 = -\sum_{\rho} \frac{1}{2\pi i} \frac{1}{\log(x)} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{d}{ds} \left[ \frac{\log\left(1-\frac{s}{\rho}\right)}{s} \right] x^s ds = -\sum_{\rho} H(\rho),$$

where $H$ is as defined in the previous section. It is known that all the nontrivial zeros of the zeta function $\rho$ lie in the critical strip $\{s \in \mathbb{C} : 0 < \text{Re}(s) < 1\}$ as was found at the end of chapter 3. So all zeros $\rho$ lie in the first or fourth quadrant. For the first quadrant it was shown in the previous section that $G = H$ by calculating the limits of $G$ and $H$ as $\text{Im}(\beta) \to \infty$. Unfortunately, calculating the limit for $G(\beta)$ as $\text{Im}(\beta) \to -\infty$ is not possible because it diverges. Note, that it still holds that $H(\beta) \to 0$ as $\text{Im}(\beta) \to -\infty$. This follows from redoing the calculation in the previous section with $\text{Im}(\beta) \to -\infty$. To overcome the divergence of $G(\beta)$, redefine $G$ by changing the path of the integral a little bit:

$$G^-(\beta) = \int_{C^-} \frac{t^{\beta-1}}{\log(t)} dt,$$

where $C^-$ is the line segment from 0 to $1-\varepsilon$, a semicircle $C_\varepsilon(1)$ around 1 with radius $\varepsilon$ in the lower half-plane and the line segment from $1+\varepsilon$ to $x$, see also the path in figure 4.1 only now the semicircle is in the lower half-plane. Now calculating the limit in the same manner as in the previous section gives again that $G^-(\beta) \to 0$ as $\text{Im}(\beta) \to -\infty$. Hence, $G^- = H$ in the fourth quadrant.

Pairing the terms $\rho$ and $1-\rho$, which lie in the first and fourth quadrant respectively, gives that

$$-\sum_{\rho} H(\rho) = -\sum_{\text{Im}(\rho)>0} \left( \int_{C^+} \frac{t^{\rho-1}}{\log(t)} dt + \int_{C^-} \frac{t^{(1-\rho)-1}}{\log(t)} dt \right). \tag{4.10}$$

If $\beta$ would be real and positive, then the substitution $t = u^{\frac{1}{\beta}}$ gives

$$\int_{C^+} \frac{t^{\beta-1}}{\log(t)} dt = \int_0^{x^\beta} \frac{u^{\frac{1}{\beta}(\beta-1)}}{\frac{1}{\beta}\log(u)} \frac{1}{\beta} u^{\frac{1}{\beta}-1} du = \int_0^{x^\beta} \frac{du}{\log(u)} = Li(x^\beta) - \pi i,$$

where the path is in the upper half-plane avoiding the singularity $u = 1$. The left integral converges for $\text{Re}(\beta) > 0$, so this gives an analytic continuation of $Li(x^\beta)$ to this half-plane (where $x > 1$ is fixed). Similarly, the substitution $t = u^{\frac{1}{1-\beta}}$ gives that

$$\int_{C^-} \frac{t^{\beta-1}}{\log(t)} dt = Li(x^\beta) + \pi i,$$

where the $\pi i$ has a plus sign, because $C^-$ goes in counterclockwise direction around the pole at 1. Combining these results with equation (4.10) gives

$$I_2 = -\sum_{\rho} H(\rho) = -\sum_{\text{Im}(\rho)>0} \left( Li(x^\rho) + Li(x^{1-\rho}) \right),$$

if interchanging the sum and the integral in equation (4.9) is allowed. We comment on this matter in section 4.5. The sum is only conditionally convergent and the sum is taken in order of increasing $|\text{Im}(\rho)|$.

### 4.4.3 Constant Term

The third integral obtained from (4.7) is

$$I_3 := -\frac{1}{2\pi i}\frac{1}{\log(x)}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{d}{ds}\left[\frac{\log\xi(0)}{s}\right]x^s ds$$

and integration by parts gives

$$I_3 = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\log\xi(0)}{s}x^s ds = \log\xi(0),$$

where lemma 4.18 is used with $\beta = 0$. It remains to calculate $\xi(0)$.

**Lemma 4.19.** *The value of $\xi(s)$ in zero is $\frac{1}{2}$.*

*Proof.* By the definition of $\xi(s)$ (see definition 3.18) it follows that

$$\xi(0) = \pi^0\Gamma(1)(0-1)\zeta(0) = -\zeta(0).$$

To calculate $\zeta(0)$, use the functional equation for the zeta function as in equation (3.6)

$$\zeta(s) = \Gamma(1-s)2(2\pi)^{s-1}\sin\left(\frac{\pi s}{2}\right)\zeta(1-s)$$

and since there is a simple pole at $s=1$ with residue 1, it holds that $\lim_{s\to1}(s-1)\zeta(1) = 1$, which is equivalent to

$$\lim_{s\to1}(1-s)\zeta(1) = -1.$$

Multiplying the functional equation with $(1-s)$ gives

$$(1-s)\zeta(s) = (1-s)\Gamma(1-s)2(2\pi)^{s-1}\sin\left(\frac{\pi s}{2}\right)\zeta(1-s).$$

Take the limit and use that $\Gamma(z+1) = z\Gamma(z)$ to obtain

$$-1 = \lim_{s\to1}(1-s)\zeta(1) = \lim_{s\to1}\Gamma(2-s)2(2\pi)^{s-1}\sin\left(\frac{\pi s}{2}\right)\zeta(1-s) = 2\zeta(0).$$

Rearranging this gives $\zeta(0) = -\frac{1}{2}$ and thus $\xi(0) = \frac{1}{2}$. □

From this lemma it follows that

$$I_3 = \log\xi(0) = \log\left(\frac{1}{2}\right) = -\log(2).$$

### 4.4.4 Integral Term

The fourth integral obtained form (4.7) is

$$I_4 := \frac{1}{2\pi i}\frac{1}{\log(x)}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{d}{ds}\left[\frac{\log\Gamma\left(\frac{s}{2}+1\right)}{s}\right]x^s ds.$$

Using proposition 2.6 it follows that

$$\begin{aligned}\Gamma(s+1) &= \lim_{n\to\infty}\frac{n!(n+1)^s}{(s+1)(s+2)\cdots(s+n)}\\ &= \lim_{n\to\infty}\frac{n!}{(s+1)(s+2)\cdots(s+n)}\left(\frac{2}{1}\cdot\frac{3}{2}\cdots\frac{n+1}{n}\right)^s\\ &= \prod_{n=1}^{\infty}\frac{n}{s+n}\left(\frac{n+1}{n}\right)^s\\ &= \prod_{n=1}^{\infty}\frac{\left(1+\frac{1}{n}\right)^s}{\left(1+\frac{s}{n}\right)}.\end{aligned}$$

And thus

$$\log \Gamma\left(\frac{s}{2}+1\right) = \sum_{n=1}^{\infty}\left[-\log\left(1+\frac{s}{2n}\right)+\frac{s}{2}\log\left(1+\frac{1}{n}\right)\right],$$

which gives

$$I_4 = \frac{1}{2\pi i}\frac{1}{\log(x)}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{d}{ds}\left[\frac{\sum_{n=1}^{\infty}\left[-\log\left(1+\frac{s}{2n}\right)+\frac{s}{2}\log\left(1+\frac{1}{n}\right)\right]}{s}\right]x^s ds. \tag{4.11}$$

Again, the sum will be taken outside the integral and derivative. That this is valid will be proven at the end of this section. It is obvious that the second term in the derivative is constant, hence

$$I_4 = -\sum_{n=1}^{\infty}\frac{1}{2\pi i}\frac{1}{\log(x)}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{d}{ds}\frac{\log\left(1+\frac{s}{2n}\right)}{s}x^s ds. \tag{4.12}$$

Recall the definition of $H$,

$$H(\beta) := \frac{1}{2\pi i}\frac{1}{\log(x)}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{d}{ds}\left[\frac{\log\left(1-\frac{s}{\beta}\right)}{s}\right]x^s ds$$

for $\mathrm{Re}(\beta) < \sigma$ and $\beta \in \mathbb{C}\setminus\{s : s \geq 0\}$. Then it is clear that (4.12) equals

$$-\sum_{n=1}^{\infty} H(-2n).$$

Note that indeed $H(\beta)$ is defined on the negative real axis, but that the functions $F(\beta)$ and $G(\beta)$ are not. Hence, any relations between $H$ and $F$ or $G$ are not valid any longer because they were only true in the half-plane $\mathrm{Re}(\beta) > 0$. To get a formula for $\mathrm{Re}(\beta) < 0$ define

$$E(\beta) := -\int_x^{\infty}\frac{t^{\beta}}{\log(t)}dt,$$

which converges absolutely for $\mathrm{Re}(\beta) < 0$:

$$\int_x^{\infty}\frac{|t^{\beta-1}|}{\log(t)}dt \leq \frac{1}{\log(x)}\int_x^{\infty}t^{\mathrm{Re}(\beta)-1}dt = \frac{1}{\log(x)}\frac{1}{\mathrm{Re}(\beta)}t^{\mathrm{Re}(\beta)}\Big|_x^{\infty} < \infty.$$

Again, the derivative and integral can be interchanged by theorem A.6 to obtain

$$\frac{d}{d\beta}E(\beta) = -\int_x^{\infty}\frac{d}{d\beta}\frac{t^{\beta-1}}{\log(t)}dt = \frac{t^{\beta}}{\beta}\Big|_x^{\infty} = \frac{x^{\beta}}{\beta} = \frac{d}{d\beta}F(\beta) = \frac{d}{d\beta}H(\beta).$$

So $E(\beta)$ and $H(\beta)$ differ by a constant. Both $E$ and $H$ go to zero as $\beta \to -\infty$ (along the negative real axis). Thus $E = H$ for $\mathrm{Re}(\beta) < 0$. Hence,

$$\begin{aligned}
-\sum_{n=1}^{\infty}H(-2n) &= \sum_{n=1}^{\infty}\int_x^{\infty}\frac{t^{-2n-1}}{\log(t)}dt \\
&= \int_x^{\infty}\frac{1}{t\log(t)}\sum_{n=1}^{\infty}t^{-2n}dt \\
&= \int_x^{\infty}\frac{1}{t\log(t)}\left(\sum_{n=0}^{\infty}t^{-2n}-1\right)dt \\
&= \int_x^{\infty}\frac{t^2}{t(t^2-1)\log(t)}-\frac{t^2-1}{t(t^2-1)\log(t)}dt \\
&= \int_x^{\infty}\frac{1}{t(t^2-1)\log(t)}dt,
\end{aligned}$$

where the last integral is convergent which shows that interchanging summation and integration was allowed. So the fourth integral becomes

$$I_4 = \int_x^\infty \frac{1}{t(t^2-1)\log(t)} dt.$$

**Interchanging**

It remains to show that (4.11) and (4.12) are equal. First, show that

$$\sum_{n=1}^\infty \frac{d}{ds} \frac{\log\left(1+\frac{s}{2n}\right)}{s} \tag{4.13}$$

converges uniformly on any closed disk $|s| \leq K$, because this allows the switch of the sum and the derivative. For a fixed $s$ there is an $N \in \mathbb{N}$ such that for all $n \geq N$ it holds that $\left|\frac{s}{2n}\right| < 1$. Thus for $|s| < 2n$ the Taylor expansion

$$\log\left(1+\frac{s}{2n}\right) = \sum_{k=1}^\infty \frac{(-1)^{k+1}}{k}\left(\frac{s}{2n}\right)^k$$

converges absolutely and uniformly on every closed disk with radius less than $2n$. Now,

$$\begin{aligned} \frac{d}{ds}\frac{\log\left(1+\frac{s}{2n}\right)}{s} &= \frac{d}{ds}\frac{\sum_{k=1}^\infty \frac{(-1)^{k+1}}{k}\left(\frac{s}{2n}\right)^k}{s} \\ &= \frac{d}{ds}\sum_{k=1}^\infty \frac{(-1)^{k+1}}{k}\frac{s^{k-1}}{(2n)^k} \\ &= -\frac{1}{2}\frac{1}{4n^2} + \frac{2}{3}\frac{s}{8n^3} - \frac{3}{4}\frac{s^2}{16n^4} + \cdots . \end{aligned}$$

So the summand of the sum in (4.13) has as highest order term $n^{-2}$. Hence, by the $M$-test (4.13) converges uniformly on any closed disk with finite radius which gives that

$$\begin{aligned} \sum_{n=1}^\infty \frac{d}{ds}\frac{\log\left(1+\frac{s}{2n}\right)}{s} &= \sum_{n=1}^\infty \frac{d}{ds}\frac{\log\left(1+\frac{s}{2n}\right)+\frac{s}{2}\log\left(1+\frac{1}{n}\right)}{s} \\ &= \frac{d}{ds}\sum_{n=1}^\infty \frac{\log\left(1+\frac{s}{2n}\right)+\frac{s}{2}\log\left(1+\frac{1}{n}\right)}{s}. \end{aligned}$$

Next, show that the sum and the integral can be interchanged. The uniform convergence shows that the sum and integral can be interchanged on a finite domain

$$\int_{\sigma-iT}^{\sigma+iT} \frac{d}{ds}\left[\frac{\log\Gamma\left(\frac{s}{2}+1\right)}{s}\right] x^s ds = -\sum_{n=1}^\infty \int_{\sigma-iT}^{\sigma+iT} \frac{d}{ds}\left[\frac{\log\left(1+\frac{s}{2n}\right)}{s}\right] x^s ds \tag{4.14}$$

for finite $T$. It will be shown that the last integral is bounded by $Cn^{-2}$ for some constant $C$, so that the summation is uniformly convergent in $T$. This would imply that the limit as $T \to \infty$ and the summation can be interchanged, which then proves the equality of (4.11) and (4.12). First, do a substitution $v = \frac{s-\sigma}{2n}$ to estimate the integral in the $n$th term of the sum:

$$\begin{aligned} \int_{\sigma-iT}^{\sigma+iT} \frac{d}{ds}\left[\frac{\log\left(1+\frac{s}{2n}\right)}{s}\right] x^s ds &= \int_{-\frac{iT}{2n}}^{\frac{iT}{2n}} \frac{1}{2n}\frac{d}{dv}\left[\frac{\log\left(1+v+\frac{\sigma}{2n}\right)}{2nv+\sigma}\right] x^{\sigma+2nv} 2n dv \\ &= \frac{x^\sigma}{2n}\int_{-\frac{iT}{2n}}^{\frac{iT}{2n}} \frac{d}{dv}\left[\frac{\log\left(1+v+\frac{\sigma}{2n}\right)}{v+\frac{\sigma}{2n}}\right] x^{2nv} dv. \end{aligned}$$

Apply integration by parts to the total $n$th term (up to a minus sign):

$$\frac{1}{2\pi i}\frac{x^{\sigma}}{2n\log(x)}\int_{-\frac{iT}{2n}}^{\frac{iT}{2n}}\frac{d}{dv}\left[\frac{\log\left(1+v+\frac{\sigma}{2n}\right)}{v+\frac{\sigma}{2n}}\right]x^{2nv}dv=\frac{1}{2\pi i}\frac{x^{\sigma}}{2n\log(x)}\frac{1}{2n\log(x)}Q \quad \text{with}$$

$$Q=\frac{d}{dv}\left[\frac{\log\left(1+v+\frac{\sigma}{2n}\right)}{v+\frac{\sigma}{2n}}\right]x^{2nv}\Bigg|_{-\frac{iT}{2n}}^{\frac{iT}{2n}}-\int_{-\frac{iT}{2n}}^{\frac{iT}{2n}}\frac{d^2}{dv^2}\left[\frac{\log\left(1+v+\frac{\sigma}{2n}\right)}{v+\frac{\sigma}{2n}}\right]x^{2nv}dv.$$

Now, $Q$ has a finite maximum, so that the whole expression can be bounded by $Cn^{-2}$, where $C$ is independent of $n$ and $T$. Calculate the derivative:

$$\begin{aligned}\frac{d}{dv}\frac{\log\left(1+v+\frac{\sigma}{2n}\right)}{v+\frac{\sigma}{2n}}&=\frac{\left(v+\frac{\sigma}{2n}\right)\frac{1}{1+v+\frac{\sigma}{2n}}-\log\left(1+v+\frac{\sigma}{2n}\right)}{\left(v+\frac{\sigma}{2n}\right)^2}\\&=\frac{1}{\left(v+\frac{\sigma}{2n}\right)\left(1+v+\frac{\sigma}{2n}\right)}-\frac{\log\left(1+v+\frac{\sigma}{2n}\right)}{\left(v+\frac{\sigma}{2n}\right)^2}.\end{aligned}$$

This expression can be bounded independently of $n$ when $v$ is on the imaginary axis. Also, the second derivative

$$\frac{d}{dv}\left[\frac{1}{\left(v+\frac{\sigma}{2n}\right)\left(1+v+\frac{\sigma}{2n}\right)}-\frac{\log\left(1+v+\frac{\sigma}{2n}\right)}{\left(v+\frac{\sigma}{2n}\right)^2}\right]$$

is absolutely integrable over the imaginary axis for every $n$. So indeed the $n$th term of the sum in (4.14) can be bounded by $Cn^{-2}$ for all $T$ and thus the series converges uniformly in $T$. Hence, the limit and sum can be interchanged, which finishes the proof.

### 4.4.5 Zero Term

The last integral obtained from (4.7) is

$$I_5:=-\frac{1}{2\pi i}\frac{1}{\log(x)}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{d}{ds}\left[\frac{\frac{s}{2}\log(\pi)}{s}\right]x^s ds$$

and the derivative in the integral is zero, hence the whole term is zero.

## 4.5 The Explicit Formula

To obtain an explicit formula for the prime counting function, $\pi_0(x)$ was expressed in terms of the function $J(x)$ (proposition 4.9):

$$\pi_0(x)=\sum_{1\le k\le x}\frac{\mu(k)}{k}J\left(x^{\frac{1}{k}}\right). \tag{4.15}$$

It remained to find an expression for $J(x)$. It turned out that this function could be written as

$$J(x)=-\frac{1}{2\pi i}\frac{1}{\log(x)}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{d}{ds}\left[\frac{\log\zeta(s)}{s}\right]x^s ds.$$

Also, an expression for $\log\zeta(s)$ was established using the infinite product representation for $\xi(s)$:

$$\log\zeta(s)=\log\xi(0)+\sum_{\rho}\log\left(1-\frac{s}{\rho}\right)-\log\Gamma\left(\frac{s}{2}+1\right)+\frac{s}{2}\log(\pi)-\log(s-1),$$

where $\rho$ denotes the nontrivial zeros of the zeta function. It remained to evaluate the integral termwise, which was done separately for each term in section 4.4. Combining all these results gives that

$$\begin{aligned} J(x) &= I_1 + I_2 + I_3 + I_4 + I_5 \\ &= Li(x) - \sum_{\mathrm{Im}(\rho)>0} \big(Li(x^\rho) + Li(x^{1-\rho})\big) - \log(2) + \int_x^\infty \frac{dt}{t(t^2-1)\log(t)} \qquad \text{for } x > 1. \end{aligned}$$

Together with formula (4.15) we have finally found an explicit expression for the prime counting function. We will briefly look at the contribution of each term in $J(x)$.

(i) The function $Li(x)$ is diverging as $x \to \infty$. This term can be considered as the principal term.

(ii) The term with the nontrivial zeros of the zeta function is hard to bound because the location of the zeros is unknown. As we have seen the Riemann hypothesis conjectures that all the nontrivial zeros lie at the line with real part $\frac{1}{2}$. *Assuming* the Riemann hypothesis to be true, then it can be proven that this term grows as $\mathcal{O}\big(\sqrt{x}\log(x)\big)$ as $x \to \infty$.

(iii) The constant $\log(2) \approx 0.69$ is very small in comparison the principal term $Li(x)$ as $x \to \infty$.

(iv) The integral term also takes very small values, namely if $x \geq e$, then

$$\int_x^\infty \frac{dt}{t(t^2-1)\log(t)} \leq \int_e^\infty \frac{dt}{t(t^2-1)\log(t)} \leq \frac{1}{\log(e)} \int_e^\infty \frac{dt}{t(t^2-1)} \approx 0.07.$$

So, $J(x)$ can be approximated by the principal term $Li(x)$, which then gives the approximation for the prime counting function proposed by Riemann:

$$Ri(x) = \sum_{1\leq k\leq x} \frac{\mu(k)}{k} Li\big(x^{\frac{1}{k}}\big) = Li(x) - \frac{1}{2}Li\big(x^{\frac{1}{2}}\big) - \frac{1}{3}Li\big(x^{\frac{1}{3}}\big) - \frac{1}{4}Li\big(x^{\frac{1}{4}}\big) + \cdots .$$

Note that the first term is $Li(x)$. Only this term as approximation for the prime counting function was considered in chapter 1, but now this approximation can be improved by adding and subtracting finitely many terms of the form $\frac{1}{k}Li\big(x^{\frac{1}{k}}\big)$. Table 4.1 shows the errors between $\pi(x)$ and the three approximations

$$L(x) = \frac{x}{\log(x)}, \quad Li(x) = \int_0^x \frac{dt}{\log(t)} \quad \text{and} \quad Ri(x) = \sum_{1\leq k\leq x} \frac{\mu(k)}{k} Li\big(x^{\frac{1}{k}}\big).$$

| $x$ | $\pi(x)$ | $\pi(x) - L(x)$ | $Li(x) - \pi(x)$ | $Ri(x) - \pi(x)$ |
|---|---|---|---|---|
| $10^6$ | 78,498 | 6,115 | 129 | 29 |
| $10^8$ | 5,761,455 | 332,774 | 754 | 97 |
| $10^{10}$ | 455,052,511 | 20,758,030 | 3,104 | −1828 |
| $10^{12}$ | 37,607,912,018 | 1,416,706,193 | 38,263 | −1476 |

Table 4.1: The errors between $\pi(x)$ and the three approximations $L(x)$, $Li(x)$ and $Ri(x)$ for some large values of $x$. The values of the errors are rounded.

The error between Riemann's approximation and $\pi(x)$ changes sign, something which we have not seen yet for the other approximations. However, it has been proven by Littlewood in 1914 that also $Li(x) - \pi(x)$ changes sign infinitely many times. The first time that this will happen is around $x \approx 10^{316}$.

Riemann's article contained eight pages and did of course not cover all the details we provided in this and the previous chapter. In fact, many gaps were left open by Riemann and it took quite some time before all details were filled in. All details apart from the Riemann hypothesis which is still unproven. One of the gaps was the validity of the infinite product formula for $\xi(s)$. This was proved later by Hadamard. Another gap, which we also left open here, was whether interchanging the sum and the integral in section 4.4.2 is allowed. If this goes wrong, then all effort we put in constructing the formula for the prime counting function would be in vain. Fortunately, Von Mangoldt proved in 1905 that indeed the formula is correct. For this he did not use $J(x)$, but another function:

$$\psi(x) = \sum_{p^k < x} \log(p).$$

This function does not only count primes but also powers of primes with a weight $\log(p)$. Von Mangoldt used the same method as Riemann and found a relation between $\zeta(s)$ and $\psi(x)$:

$$-\frac{\zeta'(s)}{\zeta(s)} = s \int_1^\infty \psi(x) x^{-s-1} dx.$$

With the same method as in section 4.3 this formula can be inverted to obtain

$$\psi(x) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} -\frac{\zeta'(s)}{\zeta(s)} x^{s-1} ds$$

for $\sigma > 1$. This integral can again be evaluated using the infinite product formula for $\xi(s)$. Finally, it is obtained that

$$\psi(x) = x - \sum_\rho \frac{x^\rho}{\rho} + \sum_n \frac{x^{-2n}}{2n} - \frac{\zeta'(0)}{\zeta(0)} \tag{4.16}$$

and from this formula also the formula for $\pi_0(x)$ can be derived. The proof of this can be found in [Edwards, 1974, Ch. 3]. This only proves the correctness of the formula for $\pi_0(x)$, but not whether indeed the sum and the integral in section 4.4.2 could be interchanged. This was proved by Landau in 1908, thus the method of constructing the explicit formula as we did is correct.

In fact, the formula for $\psi(x)$ played a more important role than the explicit formula for $\pi(x)$, because it is easier to work with while it contains the same information. Also, formula (4.16) led to the first proof of the prime number theorem. It will be shown in the next chapter that

$$\lim_{x \to \infty} \frac{\pi(x) \log(x)}{x} = 1 \quad \text{if and only if} \quad \lim_{x \to \infty} \frac{\psi(x)}{x} = 1.$$

The second statement follows immediately from the formula for $\psi(x)$ if $\zeta(s)$ has no zeros on the line with real part 1. Instead of using (4.16) we will use another, more recently found method in the next chapters to prove the prime number theorem.

## 4.6 The Product Formula for $\xi(s)$

In the derivation of the expression for $\pi_0(x)$ the product formula for $\xi(s)$ was used:

$$\xi(s) = \xi(0)\prod_{\rho}\left(1-\frac{s}{\rho}\right),$$

where $\rho$ are the zeros of $\xi(s)$. The infinite product is taken in an order which pairs each root $\rho$ with the root $1-\rho$. Recall that

$$\xi(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}+1\right)(s-1)\zeta(s).$$

The infinite sum was originally a corollary of a much more general method proven by Hadamard. Now, only the product formula for $\xi(s)$ is needed, so that the proof becomes easier for only this case. This section is devoted to proving the product formula for $\xi(s)$.

First, the number of roots $\rho$ is a disk centered at $s=\frac{1}{2}$ will be estimated using Jensen's theorem. This estimate is much less precise than the estimate of Riemann about the zeros in the strip with imaginary part between 0 and $T$ as in section 3.3. However, our estimate is enough to show the convergence of the infinite product for $\xi(s)$. Also, it is proved under which conditions an even entire function is constant. Finally, using this theorem, the infinite product representation of $\xi(s)$ can be established.

### 4.6.1 Jensen's Theorem

Note that by writing $\log(s)$ for $s\in\mathbb{C}$ still the principal branch of the logarithm is meant.

**Theorem 4.20** (Jensen's Theorem)**.** *Let $f(z)$ be an analytic function on the disk $\overline{B_R(0)} = \{z : |z|\le R\}$ for some $R>0$ and suppose that $f(z)$ has no zeros on the boundary $|z|=R$. The zeros inside the disk are labeled: $z_1, z_2, \dots, z_n$ where zeros of order $k\ge 1$ (zeros that occur $k$ times) are also included $k$ times in the list. Also, assume that $f(0)\ne 0$. Then*

$$\log\left|f(0)\frac{R}{z_1}\frac{R}{z_2}\cdots\frac{R}{z_n}\right| = \frac{1}{2\pi}\int_0^{2\pi}\log\left|f(Re^{it})\right|dt.$$

*Proof.* First, assume that $f(z)$ has no zeros in $\overline{B_R(0)}$. Note that $f(z)$ might be negative. To let the logarithm be well-defined, define $\log f(z) = \log|f(0)| + \int_0^z \frac{f'(t)}{f(t)}dt$ in the disk. Using Cauchy's integral formula (theorem A.5) it is obtained that

$$\log f(0) = \frac{1}{2\pi i}\int_{|z|=R}\frac{\log f(z)}{z}dz = \frac{1}{2\pi i}\int_0^{2\pi}\frac{\log f(Re^{it})}{Re^{it}}iRe^{it}dt = \frac{1}{2\pi}\int_0^{2\pi}\log f(Re^{it})dt.$$

Note that $\log|f(z)|$ is the real part of the $\log f(z)$, hence

$$\log|f(0)| = \mathrm{Re}[\log f(0)] = \mathrm{Re}\left[\frac{1}{2\pi}\int_0^{2\pi}\log f(Re^{it})dt\right] = \frac{1}{2\pi}\int_0^{2\pi}\log|f(Re^{it})|dt.$$

This proves the case with no zeros in $\overline{B_R(0)}$. Now, assume that $f(z)$ has the zeros $z_1, z_2, \dots, z_n$ in $B_R(0)$. The function

$$F(z) := f(z)\frac{R^2-\overline{z}_1 z}{R(z-z_1)}\frac{R^2-\overline{z}_2 z}{R(z-z_2)}\cdots\frac{R^2-\overline{z}_n z}{R(z-z_n)}$$

is analytic and has no zeros in the disk $\overline{B_R(0)}$. Apply the result from above to obtain

$$\log |F(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |F(Re^{it})| dt.$$

Obviously, using the definition of $F(z)$

$$\log |F(0)| = \log \left| f(0) \frac{R}{z_1} \frac{R}{z_2} \cdots \frac{R}{z_n} \right|$$

and it remains to show that $|f(Re^{it})| = |F(Re^{it})|$, which is equivalent to proving

$$\left| \frac{R^2 - \overline{z}_i z}{R(z - z_i)} \right| = 1$$

with $z = Re^{it}$ for $i = 1, 2, ..., n$. Note that $|z| = |Re^{it}| = R$ and thus

$$\left| \frac{R^2 - \overline{z}_i z}{R(z - z_i)} \right| = \left| \frac{R^2 - \overline{z}_i z}{R(z - z_i)} \right| \underbrace{\left| \frac{\overline{z}}{R} \right|}_{=1} = \left| \frac{R^2 \overline{z} - \overline{z}_i R^2}{R^2 (z - z_i)} \right| = \left| \frac{\overline{z} - \overline{z}_i}{z - z_i} \right| = 1.$$

Combining the results gives

$$\log \left| f(0) \frac{R}{z_1} \frac{R}{z_2} \cdots \frac{R}{z_n} \right| = \frac{1}{2\pi} \int_0^{2\pi} \log \left| f(Re^{it}) \right| dt$$

as was to be proven. □

Jensen's theorem is specific for a disk around zero, but this can of course be generalized to disks centered around an arbitrary point $z_0 \in \mathbb{C}$. The theorem will change as follows:

**Corollary 4.21.** *Let $f(z)$ be an analytic function on the disk $\overline{B_R(z_0)} = \{z : |z - z_0| \leq R\}$ for some $R > 0$ and suppose that $f(z)$ has no zeros on the boundary $|z - z_0| = R$. The zeros inside the disk are labeled: $z_1, z_2, \ldots, z_n$ where zeros of order $k \geq 1$ are also included $k$ times in the list. Also, assume that $f(z_0) \neq 0$. Then*

$$\log \left| f(z_0) \frac{R}{z_1 - z_0} \frac{R}{z_2 - z_0} \cdots \frac{R}{z_n - z_0} \right| = \frac{1}{2\pi} \int_0^{2\pi} \log \left| f(Re^{it} + z_0) \right| dt.$$

*Proof.* This follows from applying theorem 4.20 to the function $g(z) = f(z - z_0)$. □

### 4.6.2 Estimate of the Number of Roots in a Disk

**Proposition 4.22.** *For sufficiently large $R > 0$, the estimate $|\xi(s)| \leq R^R$ holds in the disk $|s - \frac{1}{2}| \leq R$.*

*Proof.* By the maximum modulus principle (theorem A.8) $|\xi(s)|$ attains its maximum at the boundary of the disk. Also, the function $\xi(s)$ is entire, thus it has a power series expansion in $s = \frac{1}{2}$ which is of the form

$$\xi(s) = \sum_{n=0}^{\infty} a_{2n} \left( s - \frac{1}{2} \right)^{2n}.$$

Since $\xi(s)$ is symmetric in the line $\mathrm{Re}(s) = \frac{1}{2}$ it is obvious that all odd coefficients $a_{2n+1}$ are zero. Also, all even coefficients $a_{2n}$ are positive. We will not prove this here, but a proof can be found in [Edwards, 1974, Section 1.8]. The triangle inequality gives that

$$|\xi(s)| = \left|\sum_{n=0}^{\infty} a_{2n}\left(s - \frac{1}{2}\right)^{2n}\right| \leq \sum_{n=0}^{\infty} a_{2n}\left|s - \frac{1}{2}\right|^{2n}$$

and equality holds for $s = \frac{1}{2} + R$. So $|\xi(s)|$ has its maximum at $s = \frac{1}{2} + R$ and it suffices to show that $\xi(\frac{1}{2} + R) \leq R^R$ for $R$ large enough. Recall that $\xi(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2} + 1\right)(s-1)\zeta(s)$ and that $\zeta(s)$ decreases to 1 if $s \to \infty$ on the real line. For a given $R > 0$ choose $N \in \mathbb{N}$ such that $\frac{1}{2} + R \leq 2N < \frac{1}{2} + R + 2$. Then

$$\begin{aligned}
\xi\left(\frac{1}{2} + R\right) &\leq \xi(2N) = \pi^{-N}\Gamma(N+1)(2N-1)\zeta(2N) \\
&= \pi^{-N}N!(2N-1)\zeta(2N) \\
&\leq N^N 2N\zeta(2) = 2\zeta(s)N^{N+1} \\
&\leq 2\zeta(s)\left(\frac{1}{4} + \frac{R}{2} + 1\right)^{\frac{1}{4}+\frac{R}{2}+1} \leq R^R
\end{aligned}$$

for sufficiently large $R$. Note that $2\zeta(s)$ and $1 + \frac{1}{4}$ are relatively small in comparison to $R$, so that indeed the last inequality holds. ☐

Let $n(R)$ denote the number of roots $\rho$ of $\xi(s)$ which are in the disk $\overline{B_R\left(\frac{1}{2}\right)}$ (counted with multiplicity). Now, $n(R)$ can be estimated as in the following theorem.

**Theorem 4.23.** *For all sufficiently large $R$, the estimate $n(R) \leq 3R\log(R)$ holds.*

*Proof.* Apply Jensen's theorem to $\xi(s)$ on the disk $\overline{B_{2R}\left(\frac{1}{2}\right)}$. Note that $\xi(s)$ is entire and if there is a zero on the boundary of the disk, then there exists a $\delta > 0$ such that there are no roots on the boundary of $\overline{B_{2R+\delta}\left(\frac{1}{2}\right)}$. In this case continue working with the slightly larger disk. The centre of the disk is at $s = \frac{1}{2}$, so Jensen's theorem for arbitrary disks (corollary 4.21) gives

$$\begin{aligned}
\log\left|\xi\left(\frac{1}{2}\right)\frac{2R}{\rho_1 - \frac{1}{2}}\cdots\frac{2R}{\rho_n - \frac{1}{2}}\right| &= \log\xi\left(\frac{1}{2}\right) + \sum_{|\rho-\frac{1}{2}|\leq 2R}\log\frac{2R}{|\rho - \frac{1}{2}|} \\
&= \frac{1}{2\pi}\int_0^{2\pi}\log\left|\xi\left(2Re^{it} + \frac{1}{2}\right)\right| dt.
\end{aligned}$$

Using proposition 4.22 the last integral can be bounded by

$$\frac{1}{2\pi}\int_0^{2\pi}\log\left|(2R)^{2R}\right| dt = 2R\log(2R).$$

Consider the roots $\rho$ inside the disk with radius $R$ instead of $2R$. For these roots it holds that

$$\log\frac{2R}{|\rho - \frac{1}{2}|} \geq \log\frac{2R}{R} = \log(2).$$

Hence,

$$n(R)\log(2) = \sum_{|\rho-\frac{1}{2}|\le R} \log(2) \le \sum_{|\rho-\frac{1}{2}|\le R} \log\frac{2R}{|\rho-\frac{1}{2}|}$$
$$\le \sum_{|\rho-\frac{1}{2}|\le 2R} \log\frac{2R}{|\rho-\frac{1}{2}|} \le 2R\log(2R) - \log\xi\left(\frac{1}{2}\right).$$

Dividing by $\log(2)$ gives

$$n(R) \le \frac{2}{\log(2)} R\log(R) + 2R - \frac{\log\xi\left(\frac{1}{2}\right)}{\log(2)} \le 2R\left(\frac{\log(R)}{\log(2)}+1\right) \le 3R\log(R)$$

for sufficiently large $R$. Note that $\frac{1}{\log(2)} < \frac{3}{2}$ and thus $\frac{\log(R)}{\log(2)} + 1 < \frac{3}{2}\log(R)$ for $R$ large enough. □

### 4.6.3 Convergence of the Product

The product

$$\prod_\rho \left(1-\frac{s}{\rho}\right)$$

is taken over all roots $\rho$ where $\rho$ and $1-\rho$ are paired. So the product can be rewritten as

$$\prod_\rho \left(1-\frac{s}{\rho}\right) = \prod_{\operatorname{Im}\rho>0}\left(1-\frac{s}{\rho}\right)\left(1-\frac{s}{1-\rho}\right) = \prod_{\operatorname{Im}\rho>0}\left(1-\frac{s(1-s)}{\rho(1-\rho)}\right).$$

Note that $\zeta(s)$ is nonzero on $[0,1)$ so that there are no roots $\rho$ for $\operatorname{Im}(\rho)=0$. The infinite product converges if

$$\sum_{\operatorname{Im}\rho>0} \frac{1}{|\rho(1-\rho)|}$$

converges (see appendix B). For all roots $\rho$ the following inequality holds

$$\frac{1}{|\rho(1-\rho)|} = \frac{1}{\left|(\rho-\frac{1}{2})^2-\frac{1}{4}\right|} < C\frac{1}{\left|\rho-\frac{1}{2}\right|^2},$$

where $C$ is a constant. Hence, it suffices to show that $\sum_{\operatorname{Im}\rho>0}\left|\rho-\frac{1}{2}\right|^{-2}$ converges. Of course, this converges if the sum over all $\rho\in\mathbb{C}$ converges. This is a consequence the next theorem.

**Theorem 4.24.** *For a given $\varepsilon>0$ the series $\sum_\rho \left|\rho-\frac{1}{2}\right|^{-(1+\varepsilon)}$ converges, where $\rho$ are all roots $\rho\in\mathbb{C}$ of $\xi(\rho)=0$.*

*Proof.* Label the roots $\rho_1,\rho_2,\ldots$ in order of increasing $|\rho-\frac{1}{2}|$. Let $R_1,R_2,\ldots$ be the sequence of positive real numbers which are implicitly defined by $4R_n\log(R_n)=n$ for all $n\in\mathbb{N}$. By theorem 4.23 it follows that the number of roots in the disk $|s-\frac{1}{2}|\le R_n$ can be bounded by $3R_n\log(R_n)$ if $R_n$ is large enough for the estimate to work. Note that from the equation defining $R_n$ it follows that $\frac{n}{4}=R_n\log(R_n)$. Combining this gives

$$n(R_n) \le 3R_n\log(R_n) = \frac{3n}{4}.$$

So there are at most $\frac{3n}{4}$ roots in the $|s-\frac{1}{2}| \leq R_n$, which implies that obviously the $n$th root does not lie in the disk, thus $|\rho_n - \frac{1}{2}| > R_n$. Now,

$$\sum_n \frac{1}{\left|\rho_n - \frac{1}{2}\right|^{1+\varepsilon}} \leq \sum_n \frac{1}{R_n^{1+\varepsilon}} = \sum_n \frac{\left(4\log(R_n)\right)^{1+\varepsilon}}{n^{1+\varepsilon}}$$
$$\leq 4^{1+\varepsilon} \sum_n \frac{\log^{1+\varepsilon}(n)}{n^{1+\varepsilon}},$$

since $\log(n) = \log(4) + \log(R_n) + \log\log(R_n) \geq \log(R_n)$. For every $a > 0$ there exists $N \in \mathbb{N}$ such that $n^a > \log^a(n)$ for all $n \geq N$. Let $a = \frac{\varepsilon}{2(1+\varepsilon)}$, then

$$\sum_n \frac{1}{\left|\rho_n - \frac{1}{2}\right|^{1+\varepsilon}} \leq 4^{1+\varepsilon} \sum_n \frac{\log^{1+\varepsilon}(n)}{n^{1+\varepsilon}} \leq 4^{1+\varepsilon} \sum_n \frac{n^{\frac{\varepsilon(1+\varepsilon)}{2(1+\varepsilon)}}}{n^{1+\varepsilon}}$$
$$= 4^{1+\varepsilon} \sum_n \frac{1}{n^{1+\frac{\varepsilon}{2}}} = 4^{1+\varepsilon}\zeta(1+\frac{\varepsilon}{2}) < \infty.$$

This shows that the product converges for all $\varepsilon > 0$. □

### 4.6.4 Even Entire Functions

**Lemma 4.25.** *Let $f(s)$ be analytic on $\overline{B_R(0)}$. Let $f(0) = 0$ and define $M = \max_{|s|=R} Re[f(s)]$. Let $r < R$, then $|f(s)| \leq 2r\frac{M}{R-r}$ for all $|s| \leq r$.*

*Proof.* Consider $\phi(s) = \frac{f(s)}{s(2M-f(s))}$ and let the real and imaginary part of $f$ be denoted by $u$ and $v$ respectively. The real part of an analytic function is harmonic, hence by the maximum principle the maximum is taken at the boundary (theorem A.10 and A.11). Thus $\text{Re}[f(s)] = u(s) \leq M$ for all $s \in \overline{B_R(0)}$. This also implies that $u(s) \leq |2M - u(s)|$ on $\overline{B_R(0)}$. Now, for $|s| = R$

$$|\phi(s)| = \frac{\left(u^2+v^2\right)^{\frac{1}{2}}}{|s|\left((2M-u)^2+v^2\right)^{\frac{1}{2}}} \leq \frac{\left(u^2+v^2\right)^{\frac{1}{2}}}{R\left(u^2+v^2\right)^{\frac{1}{2}}} = \frac{1}{R}.$$

So $|\phi(s)| \leq \frac{1}{R}$ for all $s$ in the disk. Rewriting $f(s)$ in terms of $\phi(s)$ gives

$$f(s) = \frac{2Ms\phi(s)}{1+s\phi(s)}.$$

For $|s| = r < R$ the modulus of $f$ can be bounded:

$$|f(s)| = \left|\frac{2Ms\phi(s)}{1+s\phi(s)}\right| \leq \frac{2M|s\phi(s)|}{\left|1-|s\phi(s)|\right|} \leq \frac{2M\frac{r}{R}}{1-\frac{r}{R}} = \frac{2Mr}{R-r}.$$

According to the maximum modulus principle, the inequality holds for all $|s| \leq r$. □

**Theorem 4.26.** *Let $f(s)$ be an even entire function, i.e. $f(s) = f(-s)$. If for every $\varepsilon > 0$ there exists $R > 0$ such that $Re[f(s)] < \varepsilon|s|^2$ for all $s$ with $|s| \geq R$. Then, $f(s)$ is constant.*

*Proof.* Let $\varepsilon > 0$ and let $f(s)$ satisfy the conditions of the theorem. Also, $f(s) + c$ with $c \in \mathbb{C}$ satisfies the conditions. So without loss of generality it can be assumed that

$c = -f(0)$ and thus that $f(0) = 0$. Since $f$ is entire, it has a power series expansion $f(s) = \sum_{n=0}^{\infty} a_n s^n$ with $a_0 = 0$ that converges on whole $\mathbb{C}$. All the other $a_n$ satisfy

$$a_n = \frac{1}{2\pi i} \int_{|s|=\frac{R}{2}} \frac{f(s)}{s^{n+1}} ds$$

according to theorem A.5 and A.19. Thus,

$$|a_n| = \left| \frac{1}{2\pi i} \int_{|s|=\frac{R}{2}} \frac{f(s)}{s^{n+1}} ds \right| \leq \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{f\left(\frac{1}{2}Re^{it}\right)}{\left(\frac{1}{2}Re^{it}\right)^{n+1}} \frac{1}{2} iRe^{it} \right| dt = \frac{1}{2\pi} \int_0^{2\pi} \frac{\left|f\left(\frac{1}{2}Re^{it}\right)\right|}{R^n} 2^n dt.$$

Note that $\mathrm{Re}[f(s)] < \varepsilon |s|^2 = \varepsilon R^2$ on $|s| = R$ and thus $M = \max_{|s|=R} \mathrm{Re}[f(s)] = \varepsilon R^2$. Using lemma 4.25 gives

$$\begin{aligned} \frac{2^n}{R^n} \left| f\left(\frac{1}{2}Re^{it}\right) \right| &\leq \frac{2^n}{R^n} \frac{2\varepsilon R^2 \left(\frac{1}{2}R\right)}{R - \frac{1}{2}R} = 2^{n+1} \varepsilon \frac{\frac{1}{2}R^3}{R^{n+1} - \frac{1}{2}R^{n+1}} \\ &= 2^{n+1} \varepsilon \frac{\frac{1}{2}}{\frac{1}{2}R^{n-2}} = 2^{n+1} \varepsilon \frac{1}{R^{n-2}}. \end{aligned}$$

Hence,

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{\left|f\left(\frac{1}{2}Re^{it}\right)\right|}{R^n} 2^n dt \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{2^{n+1}\varepsilon}{R^{n-2}} dt = \frac{2^{n+1}\varepsilon}{R^{n-2}} \overset{n \geq 2}{\leq} 2^{n+1}\varepsilon.$$

For $n \geq 2$ the modulus of the coefficient $a_n$ can be bounded by $2^{n+1}\varepsilon$ where $\varepsilon > 0$ is arbitrary, hence $a_n = 0$ for $n \geq 2$. Also, it was assumed that $a_0 = 0$, this means that $f$ is linear: $f(s) = a_1 s$. From the fact that $f(s)$ is even, it follows that $f(s)$ must be constant. □

### 4.6.5 Product Formula

Using the results obtained in the preceding sections, the product formula can be proven. Define for $s \in \mathbb{C}$ the function

$$F(s) := \frac{\xi(s)}{\prod_\rho \left(1 - \frac{s - \frac{1}{2}}{\rho - \frac{1}{2}}\right)},$$

which is entire. The only possible singularities are at $s = \rho$, but these are canceled by the zeros in the numerator. Theorem 4.26 will be used to show that the function $\log F\left(s - \frac{1}{2}\right)$ is constant and with this the product formula of $\xi(s)$ can be proved.

#### Conditions of theorem 4.26

The function $F(s)$ is entire and has no zeros, so the logarithm is well-defined by $\log F(s) = \log F(0) + \int_0^s \frac{F'(z)}{F(z)} dz$. The condition that the real part is bounded is stated in the next theorem.

**Theorem 4.27.** *Let $\varepsilon > 0$. Then*

$$Re \log F(s) \leq \left| s - \frac{1}{2} \right|^{1+\varepsilon}$$

*for all sufficiently large $|s - \frac{1}{2}|$.*

*Proof.* Let $R$ be given and write

$$\operatorname{Re}\log F(s) = u_R(s) + v_R(s)$$

with

$$u_R(s) := \operatorname{Re}\log \frac{\xi(s)}{\prod_{|\rho-\frac{1}{2}|\leq 2R}\left(1-\frac{s-\frac{1}{2}}{\rho-\frac{1}{2}}\right)} \quad \text{and}$$
$$v_R(s) := \operatorname{Re}\log \frac{1}{\prod_{|\rho-\frac{1}{2}|> 2R}\left(1-\frac{s-\frac{1}{2}}{\rho-\frac{1}{2}}\right)}.$$

Both $u_R$ and $v_R$ are defined and analytic for all $s \in \mathbb{C}$ except for $s = \rho$ if $|\rho - \frac{1}{2}| > 2R$. For these points note that $u_R$ goes to $-\infty$ as $s \to \rho$ since $\lim_{x\downarrow 0}\log(x) = -\infty$ and that $v_R$ goes to $\infty$ as $s \to \rho$.

First, consider $u_R$ where the product of the denominator is taken over all $\rho$ such that $|\rho - \frac{1}{2}| \leq 2R$. On the circle $|s - \frac{1}{2}| = 4R$ it holds that

$$\left|1 - \frac{s-\frac{1}{2}}{\rho-\frac{1}{2}}\right| \geq \left|1 - \left|\frac{s-\frac{1}{2}}{\rho-\frac{1}{2}}\right|\right| \geq \left|1-\frac{4R}{2R}\right| = 1.$$

So the factors in the denominator are greater than 1, hence

$$u_R(s) \leq \operatorname{Re}\log\xi(s) = \log|\xi(s)|$$

and using proposition 4.22 gives

$$u_R(s) \leq \log|\xi(s)| \leq \log\left[(4R)^{4R}\right] = 4R\log(4R) \leq R^{1+\varepsilon}$$

if $R$ is large enough, that is if $4\log 4R < R^{\varepsilon}$.
For $\rho$ in the disk $|s - \frac{1}{2}| \leq 2R$, the function $u_R$ is analytic, but for $\rho$ in the annulus $2R < |s - \frac{1}{2}| \leq 4R$ there is a problem because $u_R$ is not analytic in these points. To avoid this problem, a little neighborhood around $s = \rho$ is deleted such that the function remains analytic.
Note that $u_R$ is harmonic, since it is the real part of an analytic function. By the maximum principle the maximum is attained at the boundary, which is thus the circle $|s-\frac{1}{2}| = 4R$ and the boundaries of the little neighborhoods around $s = \rho$ in the annulus $2R < |s-\frac{1}{2}| \leq 4R$. Recall that around singularities the value of $u_R$ is near $-\infty$, so that the maximum must be attained at the circle $|s-\frac{1}{2}| = 4R$. To conclude, $u_R \leq R^{1+\varepsilon}$ on the whole disk $|s-\frac{1}{2}| \leq 4R$ and thus in particular on the circle $|s - \frac{1}{2}| = R$.

Now, consider $v_R$ where the product is over all $\rho$ such that $|\rho - \frac{1}{2}| > 2R$. If $x \in \mathbb{C}$ satisfies $|x| \leq \frac{1}{2}$, then

$$\operatorname{Re}\log\frac{1}{1-x} = -\operatorname{Re}\log(1-x) = \operatorname{Re}\int_0^x \frac{1}{1-t}dt \leq \left|\int_0^x \frac{1}{1-t}dt\right| \leq |x|\max_{t\in[0,x]}\frac{1}{|1-t|} = 2|x|. \tag{4.17}$$

Also, if $\rho$ and $1-\rho$ are paired, then

$$\prod_\rho\left(1-\frac{s-\frac{1}{2}}{\rho-\frac{1}{2}}\right) = \prod_{\operatorname{Im}\rho>0}\left(1-\frac{s-\frac{1}{2}}{\rho-\frac{1}{2}}\right)\left(1+\frac{s-\frac{1}{2}}{\rho-\frac{1}{2}}\right) = \prod_{\operatorname{Im}\rho>0}\left(1-\frac{\left(s-\frac{1}{2}\right)^2}{\left(\rho-\frac{1}{2}\right)^2}\right). \tag{4.18}$$

Take in the next calculation $\operatorname{Im}\rho > 0$, then for $|s-\frac{1}{2}| = R$

$$v_R(s) = \operatorname{Re}\log \frac{1}{\prod_{|\rho-\frac{1}{2}|>2R}\left(1-\frac{s-\frac{1}{2}}{\rho-\frac{1}{2}}\right)} \le 2\sum_{|\rho-\frac{1}{2}|>2R} \frac{R^2}{|\rho-\frac{1}{2}|^2},$$

where (4.17) is applicable since $|\rho-\frac{1}{2}| > 2R$ and $|s-\frac{1}{2}| = R$ implies that $\left|\frac{s-\frac{1}{2}}{\rho-\frac{1}{2}}\right| \le \frac{1}{2}$. Now, introduce an $\varepsilon$ by rewriting the above as

$$v_R(s) \le 2\sum_{|\rho-\frac{1}{2}|>2R} \left(\frac{R}{|\rho-\frac{1}{2}|}\right)^{1-\varepsilon}\left(\frac{R}{|\rho-\frac{1}{2}|}\right)^{1+\varepsilon} \le 2\sum_{|\rho-\frac{1}{2}|>2R} \left(\frac{1}{2}\right)^{1-\varepsilon}\left(\frac{R}{|\rho-\frac{1}{2}|}\right)^{1+\varepsilon},$$

since $|\rho-\frac{1}{2}| > 2R$ implies that $\left|\frac{R}{\rho-\frac{1}{2}}\right| \le \frac{1}{2}$. Rewriting gives

$$v_R(s) \le 2^{\varepsilon}R^{1+\varepsilon}\sum_{|\rho-\frac{1}{2}|>2R}\left(\frac{1}{|\rho-\frac{1}{2}|}\right)^{1+\varepsilon} < \infty.$$

The last sum is finite by theorem 4.24 and as $R\to\infty$ the sum $\sum_{|\rho-\frac{1}{2}|>2R}\left|\rho-\frac{1}{2}\right|^{-(1+\varepsilon)} \to 0$. Also, $R$ is large so

$$2^{\varepsilon}R^{1+\varepsilon} = R(2R)^{\varepsilon} \overset{R\ge 2}{\le} R^{1+2\varepsilon}.$$

Since, $\varepsilon > 0$ can be chosen arbitrarily small, it follows that for $R$ sufficiently large and $\varepsilon$ small enough it holds that $v_R(s) \le R^{1+\varepsilon}$ for $|s-\frac{1}{2}| = R$.

For both $u_R(s)$ and $v_R(s)$ it holds that they can be bounded by $R^{1+\varepsilon}$. However, it was to be shown that $\operatorname{Re}\log F(s) = u_R(s) + v_R(s) \le R^{1+\varepsilon}$. Let $\varepsilon$ decrease to $\varepsilon'$ and choose $R$ large enough such that both $u_R$ and $v_R$ can be bounded by $R^{1+\varepsilon'}$. Now, choose $\varepsilon' > 0$ such that $2 \le R^{\varepsilon-\varepsilon'}$, then

$$\operatorname{Re}\log F(s) = u_R(s) + v_R(s) \le 2R^{1+\varepsilon'} \le R^{1+\varepsilon}$$

as was to be proven. □

The last condition of theorem 4.26 is that $\log F\big(s-\frac{1}{2}\big)$ should be even. This follows from the fact that $F\big(s-\frac{1}{2}\big)$ is even. It need to be shown that $F\big(s+\frac{1}{2}\big) = F\big(-s+\frac{1}{2}\big)$.

From the functional equation $\xi(s) = \xi(1-s)$ it is obtained that $\xi\big(s+\frac{1}{2}\big) = \xi\big(\frac{1}{2}-s\big)$. Writing the denominator of $F(s)$ as (see equation (4.18))

$$\prod_{\operatorname{Im}\rho>0}\left(1-\frac{\big(s-\frac{1}{2}\big)^2}{\big(\rho-\frac{1}{2}\big)^2}\right)$$

and plugging in $s+\frac{1}{2}$ and $s-\frac{1}{2}$ gives that

$$\prod_{\operatorname{Im}\rho>0}\left(1-\frac{s^2}{\big(\rho-\frac{1}{2}\big)^2}\right) = \prod_{\operatorname{Im}\rho>0}\left(1-\frac{(-s)^2}{\big(\rho-\frac{1}{2}\big)^2}\right).$$

To conclude, both the numerator and the denominator of $F\big(s - \frac{1}{2}\big)$ are even, thus so is $F\big(s - \frac{1}{2}\big)$.

### Applying theorem 4.26

The conditions of theorem 4.26 are satisfied, hence as a result $\log F\big(s - \frac{1}{2}\big)$ is constant and thus also $\log F(s)$ is constant. Taking the exponential gives that

$$\xi(s) = c \prod_{\rho} \left(1 - \frac{s - \frac{1}{2}}{\rho - \frac{1}{2}}\right)$$

with $c$ a constant. To eliminate this constant, divide by

$$\xi(0) = c \prod_{\rho} \left(1 - \frac{-\frac{1}{2}}{\rho - \frac{1}{2}}\right),$$

which leads to

$$\begin{aligned}
\frac{\xi(s)}{\xi(0)} &= \prod_{\rho} \left(1 - \frac{s - \frac{1}{2}}{\rho - \frac{1}{2}}\right) \left(1 - \frac{-\frac{1}{2}}{\rho - \frac{1}{2}}\right)^{-1} \\
&= \prod_{\rho} \frac{\rho - \frac{1}{2} - \big(s - \frac{1}{2}\big)}{\rho - \frac{1}{2} - \big(-\frac{1}{2}\big)} \\
&= \prod_{\rho} \frac{\rho - s}{\rho} = \prod_{\rho} \left(1 - \frac{s}{\rho}\right).
\end{aligned}$$

Finally, the desired result is found

$$\xi(s) = \xi(0) \prod_{\rho} \left(1 - \frac{s}{\rho}\right).$$

# Chapter 5

# The Prime Number Theorem

The final goal is to prove the prime number theorem, which describes the asymptotic distribution of the primes. In the coming chapters $p$ will always denote a prime number and $p_n$ will denote the $n$th prime number.

**Theorem 5.1** (Prime Number Theorem)**.** *The relative error between $\pi(x)$ and $\frac{x}{\log x}$ goes to zero as $x$ tends to infinity:*

$$\lim_{x\to\infty} \frac{\pi(x) - \frac{x}{\log x}}{\frac{x}{\log x}} = 0,$$

*or equivalently*

$$\lim_{x\to\infty} \frac{\pi(x)\log(x)}{x} = 1.$$

Recall that the prime counting function $\pi(x)$ is the number of primes less than or equal to $x$:

$$\pi(x) = \sum_{p\leq x} 1 = \sum_{n\leq x} u_p(n),$$

where

$$u_p(n) = \begin{cases} 1 & \text{if } n \text{ is prime} \\ 0 & \text{otherwise} \end{cases}.$$

In this chapter we do the preparations for proving the prime number theorem in chapter 6. The first section covers Abel summation in the discrete and continuous case. Section 5.2 introduces the Chebyshev theta and psi functions. Certain relations between those functions are proved, together with some other properties which are needed for proving the prime number theorem. In the last section an equivalent statement of the prime number theorem is established, which was already mentioned at the end of section 4.5. The theory covered in this chapter mostly comes from [Apostol, 1976] and [Jameson, 2003].

## 5.1 Abel Summation

We start with Abel summation in the discrete case, this is also called summation by parts. Let $a(k)$ and $f(k)$ be real or complex sequences. Define $A(n) = \sum_{k=1}^{n} a(k)$ with $A(0) = 0$.

**Proposition 5.2.** *Let $0 \leq m < n$. Then*

$$\sum_{k=m+1}^{n} a(k)f(k) = \sum_{k=m}^{n-1} A(k)\big[f(k) - f(k+1)\big] + A(n)f(n) - A(m)f(m)$$

*and in particular,*

$$\sum_{k=1}^{n} a(k)f(k) = \sum_{k=1}^{n-1} A(k)\big[f(k) - f(k+1)\big] + A(n)f(n).$$

*Proof.* Note that $a(k) = A(k) - A(k-1)$. Now,

$$\begin{aligned}
\sum_{k=m+1}^{n} a(k)f(k) &= \sum_{k=m+1}^{n} \big[A(k) - A(k-1)\big]f(k) \\
&= \sum_{k=m+1}^{n} A(k)f(k) - \sum_{k=m+1}^{n} A(k-1)f(k) \\
&= \sum_{k=m+1}^{n} A(k)f(k) - \sum_{k=m}^{n-1} A(k)f(k+1) \qquad (5.1) \\
&= \sum_{k=m}^{n-1} A(k)\big[f(k) - f(k+1)\big] + A(n)f(n) - A(m)f(m).
\end{aligned}$$

The second statement corresponds to the case $m = 0$. □

The Abel summation as in the previous proposition can be written in different forms. Another useful form is stated in the next proposition.

**Proposition 5.3.** *Let $0 \leq m < n$. Then*

$$\sum_{k=m}^{n} f(k)\big[a(k+1) - a(k)\big] = f(n)a(n+1) - f(m)a(m) - \sum_{k=m+1}^{n} \big[f(k) - f(k-1)\big]a(k).$$

*Proof.*

$$\begin{aligned}
\sum_{k=m}^{n} f(k)\big[a(k+1) - a(k)\big] &= \sum_{k=m}^{n} f(k)a(k+1) - \sum_{k=m}^{n} f(k)a(k) \\
&= \sum_{k=m}^{n} f(k)a(k+1) - \sum_{k=m-1}^{n-1} f(k+1)a(k+1) \\
&= f(n)a(n+1) - f(m)a(m) + \sum_{k=m}^{n-1} \big[f(k) - f(k+1)\big]a(k+1) \\
&= f(n)a(n+1) - f(m)a(m) + \sum_{k=m+1}^{n} \big[f(k-1) - f(k)\big]a(k) \\
&= f(n)a(n+1) - f(m)a(m) - \sum_{k=m+1}^{n} \big[f(k) - f(k-1)\big]a(k).
\end{aligned}$$

□

As the notation already suggested $f(k)$ could also be a function of a real or complex variable. So in the continuous version of proposition 5.2 sums will be replaced by integrals using that $f(k+1) - f(k) = \int_k^{k+1} f'(t)dt$. In the continuous case, define $A(x) = \sum_{k \leq x} a(k)$. This leads to the following theorem.

**Theorem 5.4.** *Let $y < x$ and let $f$ be continuously differentiable on $[y, x]$. Then*

$$\sum_{y<k\leq x} a(k)f(k) = A(x)f(x) - A(y)f(y) - \int_y^x A(t)f'(t)dt.$$

*Proof.* Let $m, n \in \mathbb{N}$ such that $n \leq x < n+1$ and $m \leq y < m+1$. Then $A(n) = A(x)$ and $A(m) = A(y)$. Also,

$$f(n) = f(x) - \int_n^x f'(t)dt \quad \text{and} \quad f(m+1) = f(y) + \int_y^{m+1} f'(t)dt.$$

Using this and equation (5.1) from the proof of proposition 5.2 gives

$$\begin{aligned}\sum_{y<k\leq x} a(k)f(k) &= \sum_{k=m+1}^{n} a(k)f(k) \overset{(5.1)}{=} \sum_{k=m+1}^{n} A(k)f(k) - \sum_{k=m}^{n-1} A(k)f(k+1)\\ &= \sum_{k=m+1}^{n-1} A(k)\big[f(k) - f(k+1)\big] + A(n)f(n) - A(m)f(m+1)\\ &= -\sum_{k=m+1}^{n-1} A(k)\int_k^{k+1} f'(t)dt + A(n)f(n) - A(m)f(m+1)\\ &= -\sum_{k=m+1}^{n-1} \int_k^{k+1} A(t)f'(t)dt + A(n)f(n) - A(m)f(m+1)\\ &= -\int_{m+1}^{n} A(t)f'(t)dt + A(x)f(x) - \int_n^x A(t)f'(t)dt\\ &\quad - A(y)f(y) - \int_y^{m+1} A(t)f'(t)dt\\ &= A(x)f(x) - A(y)f(y) - \int_y^x A(t)f'(t)dt.\end{aligned}$$

□

**Corollary 5.5.** *Let $f$ be continuously differentiable on $[2, x]$ and $a(1) = 0$. Then*

$$\sum_{2\leq k\leq x} a(k)f(k) = A(x)f(x) - \int_2^x A(t)f'(t)dt.$$

*Proof.* Use theorem 5.4 and note that $a(1) = 0$ implies that $A(2) = a(2)$:

$$\begin{aligned}\sum_{2\leq k\leq x} a(k)f(k) &= a(2)f(2) + \sum_{2<k\leq x} a(k)f(k)\\ &= a(2)f(2) + A(x)f(x) - A(2)f(2) - \int_2^x A(t)f'(t)dt\\ &= A(x)f(x) - \int_2^x A(t)f'(t)dt.\end{aligned}$$

□

## 5.2 Chebyshev Functions

This section introduces the Chebyshev functions $\theta(x)$ and $\psi(x)$. We prove how these functions relate to another and how they relate to $\pi(x)$. Also, it is proved that both $\theta(x)$ and $\psi(x)$ have the asymptotic behavior $\mathcal{O}(x)$ as $x \to \infty$. This result is needed for proving the prime number theorem in the next chapter.

### 5.2.1 Chebyshev Theta Function

**Definition 5.6.** *Define the Chebyshev theta function for $x > 0$ as*

$$\theta(x) := \sum_{p \leq x} \log(p) = \sum_{n \leq x} u_p(n) \log(n).$$

**Proposition 5.7.** *For all $x > 0$: $\theta(x) \leq \pi(x) \log(x)$.*

*Proof.* Let $p_1, p_2, ..., p_n$ be all the prime numbers less than or equal to $x$. Note that in this case $\pi(x) = n$. Then

$$\theta(x) = \log(p_1) + \log(p_2) + ... + \log(p_n) \leq n \log(x) = \pi(x) \log(x).$$

□

Also, an exact relation between $\theta(x)$ and $\pi(x)$ can be found.

**Proposition 5.8.** *For $x \geq 2$ two relations between $\theta(x)$ and $\pi(x)$ are given by*

$$\theta(x) = \pi(x) \log(x) - \int_2^x \frac{\pi(t)}{t} dt \tag{5.2}$$

*and*

$$\pi(x) = \frac{\theta(x)}{\log(x)} + \int_2^x \frac{\theta(t)}{t \log^2(t)} dt. \tag{5.3}$$

*Proof.* Apply corollary 5.5 with $a(n) = u_p(n)$ and $f(n) = \log(n)$:

$$\sum_{2 \leq n \leq x} u_p(n) \log(n) = \sum_{k \leq x} u_p(n) \log(x) - \int_2^x \sum_{n \leq t} u_p(n) \big[\log(t)\big]' dt.$$

and plugging in the definition for $\theta(x)$ (note that $\theta(x) = 0$ for $x < 2$) and $\pi(x)$ gives

$$\theta(x) = \pi(x) \log(x) - \int_2^x \frac{\pi(t)}{t} dt.$$

The second identity follows corollary 5.5 with $a(n) = \tilde{u}_p(n)$ and $f(n) = \frac{1}{\log(n)}$, where

$$\tilde{u}_p(n) = \begin{cases} \log(n) & \text{if } n \text{ is prime} \\ 0 & \text{otherwise} \end{cases}.$$

Note that the definitions of $\theta(x)$ and $\pi(x)$ can be rewritten as

$$\theta(x) = \sum_{n \leq x} \tilde{u}_p(n) \quad \text{and} \quad \pi(x) = \sum_{n \leq x} \frac{\tilde{u}_p(n)}{\log(n)}.$$

Corollary 5.5 gives

$$\pi(x) = \frac{\theta(x)}{\log(x)} + \int_2^x \frac{\theta(t)}{t \log^2(t)} dt.$$

□

For describing the behavior of $\theta(x)$ as $x$ tends to infinity the Landau notation is used, see appendix C.

**Theorem 5.9.** *The Chebyshev theta function has the asymptotic behavior* $\theta(x) = \mathcal{O}(x)$ *as* $x \to \infty$.

*Proof.* Recall that

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Note that for $n \in \mathbb{N}$

$$\binom{2n}{n} = \frac{(2n)!}{n!n!} = \frac{1 \cdot 2 \cdots 2n}{1^2 \cdot 2^2 \cdots n^2}.$$

Consider all the primes in the interval $[n+1, 2n]$. They only appear in the numerator, so every $p \in [n+1, 2n]$ divides $\binom{2n}{n}$. Hence, the following inequality holds

$$\prod_{n+1 \le p \le 2n} p \le \binom{2n}{n}. \tag{5.4}$$

Using the binomial formula, we get for all $n \in \mathbb{N}$ that

$$\begin{aligned} 2^{2n} = (1+1)^{2n} &= \binom{2n}{0} + \cdots + \binom{2n}{n} + \cdots + \binom{2n}{2n} \\ &\ge \binom{2n}{n} \overset{(5.4)}{\ge} \prod_{n+1 \le p \le 2n} p \\ &= \exp\left( \sum_{n+1 \le p \le 2n} \log(p) \right) = e^{\theta(2n) - \theta(n)}. \end{aligned}$$

From this it follows that

$$e^{\theta(2n) - \theta(n)} \le e^{2n \log(2)}$$

and since $t \mapsto e^t$ is increasing it is obtained that

$$\theta(2n) - \theta(n) \le 2n \log(2).$$

In particular, if $n$ is a power of 2, then

$$\theta(2^{r+1}) - \theta(2^r) \le 2^{r+1} \log(2).$$

Summing over $r = 0, 1, 2, \ldots k$ gives

$$\sum_{r=0}^{k} \theta(2^{r+1}) - \theta(2^r) \le \sum_{r=0}^{k} 2^{r+1} \log(2) = 2\log(2) \frac{1 - 2^{k+1}}{1-2} = 2^{k+2} \log(2) - 2\log(2).$$

Note that the left hand side telescopes and that $\theta(1) = 0$, so it is obtained that

$$\theta(2^{k+1}) \le 2^{k+2} \log(2) + \mathcal{O}(1).$$

Choose $k$ such that $2^k \le n \le 2^{k+1}$, then

$$\theta(n) \le \theta(2^{k+1}) \le 2^{k+2} \log(2) + \mathcal{O}(1) \le 4n \log(2) + \mathcal{O}(1).$$

This proves that $\theta(x) = \mathcal{O}(x)$. $\square$

### 5.2.2 Chebyshev Psi Function

**Definition 5.10.** *Define the Chebyshev psi function for $x > 0$ as*

$$\psi(x) := \sum_{k=1}^{\infty} \sum_{p^k \leq x} \log(p) = \sum_{n \leq x} \Lambda(n),$$

*where*

$$\Lambda(n) = \begin{cases} \log(p) & \text{if } n = p^k \text{ with } p \text{ prime and } k \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

*is the Von Mangoldt function.*

**Remark 5.11.** *This function $\psi(x)$ has nothing to do with function $\psi(x)$ that was used in section 3.3.*

The functions $\theta(x)$ and $\psi(x)$ are related by

$$\psi(x) = \sum_{k=1}^{\infty} \sum_{p \leq x^{\frac{1}{k}}} \log(p) = \sum_{k=1}^{\infty} \theta(x^{\frac{1}{k}}).$$

Note that the last infinite sum is finite since $\theta(x^{\frac{1}{k}}) = 0$ if $x^{\frac{1}{k}} < 2$. Hence,

$$\psi(x) = \sum_{k \leq \log_2(x)} \theta(x^{\frac{1}{k}}). \tag{5.5}$$

Similar to proposition 5.7, an upper bound for $\psi(x)$ can be obtained.

**Proposition 5.12.** *For all $x > 0$: $\psi(x) \leq \pi(x) \log(x)$.*

*Proof.* Let $p_1, p_2, ..., p_n$ be all the prime numbers less than or equal to $x$. For all $j \leq n$ define $k_j = \max_{p_j^k \leq x} k$ which is the number of prime powers less than or equal to $x$ corresponding to $p_j$. Note that each $p_j^k$ contributes $\log(p_j)$ to $\psi(x)$. Hence,

$$\psi(x) = k_1 \log(p_1) + ... + k_n \log(p_n).$$

Also, for every $j \leq n$: $k_j \log(p_j) = \log(p_j^{k_j}) \leq \log(x)$. Thus,

$$\psi(x) = \sum_{j=1}^{n} k_j \log(p_j) \leq n \log(x) = \pi(x) \log(x).$$

□

Before continuing with the proof that $\psi(x) = \mathcal{O}(x)$ introducing another formula for $\psi(x)$ is required. Therefore, we need the following definition.

**Definition 5.13.** *For $x > 0$ write $x = [x] + \{x\}$, where $[x] \in \mathbb{N}$ is the largest integer smaller than or equal to $x$ and $\{x\} \in [0, 1)$ is the decimal part of $x$.*

**Proposition 5.14.** *Another expression for $\psi(x)$ is*

$$\psi(x) = \sum_{p \leq x} \left[ \frac{\log(x)}{\log(p)} \right] \log(p). \tag{5.6}$$

*Proof.* Consider, as in the proof of proposition 5.12

$$\psi(x) = \sum_{j=1}^{n} k_j \log(p_j),$$

where $k_j$ is the largest possible $k$ such that $p_j^k \leq x$. Note again that

$$p_j^{k_j} \leq x \quad \Longleftrightarrow \quad k_j \log(p_j) \leq \log(x) \quad \Longleftrightarrow \quad k_j \leq \frac{\log(x)}{\log(p_j)}.$$

Since $k_j$ is the largest integer such that $p_j^{k_j} \leq x$, it is obtained that

$$k_j = \left[\frac{\log(x)}{\log(p_j)}\right],$$

which gives the desired expression. □

**Theorem 5.15.** *The Chebyshev psi function has the asymptotic behavior* $\psi(x) = \mathcal{O}(x)$ *as* $x \to \infty$.

*Proof.* Note that

$$\psi(x) = \sum_{p^k \leq x} \log(p) = \underbrace{\sum_{p \leq x} \log(p)}_{=\theta(x)} + \sum_{p^2 \leq x} \log(p) + \sum_{p^3 \leq x} \log(p) + \cdots .$$

It is proven in theorem 5.9 that $\theta(x) = \mathcal{O}(x)$, so to prove $\psi(x) = \mathcal{O}(x)$ it remains to show that the sum of the terms in $\psi(x)$ corresponding to $k \geq 2$ is not larger than $\mathcal{O}(x)$. Recall that

$$\psi(x) = \sum_{p \leq x} \left[\frac{\log(x)}{\log(p)}\right] \log(p)$$

and the sum equals $\theta(x)$ if $\left[\frac{\log(x)}{\log(p)}\right] = 1$. So for the terms corresponding to $k \geq 2$, the case $\left[\frac{\log(x)}{\log(p)}\right] > 1$ should be studied. This is equivalent to $\frac{\log(x)}{\log(p)} \geq 2$, since $[\,\cdot\,] \in \mathbb{N}$. Also,

$$\frac{\log(x)}{\log(p)} \geq 2 \quad \Longleftrightarrow \quad \log(x) \geq \log(p^2) \quad \Longleftrightarrow \quad x \geq p^2 \quad \Longleftrightarrow \quad \sqrt{x} \geq p.$$

So the terms in $\psi(x)$ with $\left[\frac{\log(x)}{\log(p)}\right] > 1$ only occur if $p \leq \sqrt{x}$, hence their contribution to $\psi(x)$ can be bounded:

$$\sum_{p \leq \sqrt{x}} \left[\frac{\log(x)}{\log(p)}\right] \log(p) \leq \sum_{p \leq \sqrt{x}} \frac{\log(x)}{\log(p)} \log(p) = \sum_{p \leq \sqrt{x}} \log(x) = \pi(\sqrt{x}) \log(x) \leq \sqrt{x} \log(x).$$

Hence,

$$\psi(x) = \mathcal{O}(x) + \mathcal{O}\big(\sqrt{x} \log(x)\big) = \mathcal{O}(x),$$

which proves the statement. □

## 5.3 Equivalent Form of the Prime Number Theorem

Instead of proving the prime number theorem using $\pi(x)$, another equivalent form will be proven. This equivalence is in the following theorem.

**Theorem 5.16.** *The following are equivalent:*

*(i)* $\displaystyle \lim_{x\to\infty} \frac{\pi(x)\log(x)}{x} = 1,$

*(ii)* $\displaystyle \lim_{x\to\infty} \frac{\theta(x)}{x} = 1,$

*(iii)* $\displaystyle \lim_{x\to\infty} \frac{\psi(x)}{x} = 1.$

*Proof.* $(i) \implies (ii)$:
Rewriting equation (5.2) from proposition 5.8 gives

$$\frac{\theta(x)}{x} = \frac{\pi(x)\log(x)}{x} - \frac{1}{x}\int_2^x \frac{\pi(t)}{t}dt.$$

So it suffices to show that

$$\frac{1}{x}\int_2^x \frac{\pi(t)}{t}dt \to 0 \quad \text{as } x \to \infty.$$

Note that $(i)$ is equivalent to

$$\lim_{x\to\infty} \frac{\frac{\pi(x)}{x} - \frac{1}{\log x}}{\frac{1}{\log x}} = 0.$$

So (see appendix C for the little-o notation)

$$\frac{\pi(x)}{x} - \frac{1}{\log(x)} = o\left(\frac{1}{\log(x)}\right)$$

and thus

$$\frac{\pi(x)}{x} = \mathcal{O}\left(\frac{1}{\log(x)}\right),$$

which means that there exists an $M > 0$ and $x_1 > 0$ such that $\frac{\pi(t)}{t} \leq M\frac{1}{\log(t)}$ for all $t \geq x_1$. Now if $t \geq x_1$, then

$$\frac{1}{x}\int_{x_1}^x \frac{\pi(t)}{t}dt \leq \frac{1}{x}\int_{x_1}^x M\frac{1}{\log(t)}dt$$

and the integral from 2 to $x_1$ is finite, which means that

$$\frac{1}{x}\int_2^x \frac{\pi(t)}{t}dt = \mathcal{O}\left(\frac{1}{x}\int_2^x \frac{1}{\log(t)}dt\right).$$

It remains to bound the last integral:

$$\begin{aligned}
\int_2^x \frac{1}{\log(t)}dt &= \int_2^{\sqrt{x}} \underbrace{\frac{1}{\log(t)}}_{\leq 1} dt + \int_{\sqrt{x}}^x \underbrace{\frac{1}{\log(t)}}_{\text{takes maximum at } \sqrt{x}} dt \\
&\leq \int_2^{\sqrt{x}} 1\cdot dt + \frac{1}{\log\sqrt{x}}\int_{\sqrt{x}}^x 1\cdot dt \\
&= \sqrt{x} - 2 + \frac{x-\sqrt{x}}{\log\sqrt{x}} \\
&= \mathcal{O}\left(\sqrt{x}\right) + \mathcal{O}\left(\frac{x}{\log\sqrt{x}}\right) = \mathcal{O}\left(\frac{x}{\log\sqrt{x}}\right).
\end{aligned}$$

Now,

$$\lim_{x\to\infty}\frac{1}{x}\int_2^x \frac{\pi(t)}{t}dt \le \lim_{x\to\infty}\mathcal{O}\left(\frac{1}{x}\frac{x}{\log\sqrt{x}}\right) = \lim_{x\to\infty}\mathcal{O}\left(\frac{1}{\log\sqrt{x}}\right) = 0,$$

which proves $(i) \implies (ii)$.
For $(ii) \implies (i)$ the proof is similar: rewriting equation (5.3) from proposition 5.8 gives

$$\frac{\pi(x)\log(x)}{x} = \frac{\theta(x)}{x} + \frac{\log(x)}{x}\int_2^x \frac{\theta(t)}{t\log^2(t)}dt$$

and it remains to show that the last term goes to zero. Note that $(ii)$ implies that $\theta(x) = \mathcal{O}(x)$, hence

$$\frac{\log(x)}{x}\int_2^x \frac{\theta(t)}{t\log^2(t)}dt = \mathcal{O}\left(\frac{\log(x)}{x}\int_2^x \frac{dt}{\log^2(t)}\right)$$

and

$$\begin{aligned}\int_2^x \frac{dt}{\log^2(t)} &= \int_2^{\sqrt{x}} \frac{dt}{\log^2(t)} + \int_{\sqrt{x}}^x \frac{dt}{\log^2(t)}\\ &\le \sqrt{x} - 2 + \frac{x-\sqrt{x}}{\log^2\sqrt{x}}\\ &= \mathcal{O}\left(\sqrt{x}\right) + \mathcal{O}\left(\frac{x}{\log^2\sqrt{x}}\right) = \mathcal{O}\left(\frac{x}{\log^2\sqrt{x}}\right).\end{aligned}$$

Now,

$$\lim_{x\to\infty}\frac{\log(x)}{x}\int_2^x \frac{\theta(t)}{t\log^2(t)}dt \le \lim_{x\to\infty}\mathcal{O}\left(\frac{\log(x)}{\log^2\sqrt{x}}\right) = \lim_{x\to\infty}\mathcal{O}\left(\frac{4}{\log(x)}\right) = 0,$$

which proves $(ii) \implies (i)$.
It remains to show $(ii) \iff (iii)$. Use that

$$0 \le \psi(x) - \theta(x) = \sum_{k\le\log_2(x)} \theta(x^{\frac{1}{k}}) - \theta(x) = \sum_{2\le k\le\log_2(x)} \theta(x^{\frac{1}{k}})$$

and that $\theta(x) \le x\log(x)$ (which follows from proposition 5.7).

$$\begin{aligned}0 \le \psi(x) - \theta(x) &\le \sum_{2\le k\le\log_2(x)} x^{\frac{1}{k}}\log\left(x^{\frac{1}{k}}\right)\\ &\le \log_2(x)\sqrt{x}\log\sqrt{x}\\ &= \frac{\log(x)}{\log(2)}\frac{\sqrt{x}}{2}\log x\\ &= \frac{\sqrt{x}\log^2(x)}{2\log(2)}.\end{aligned}$$

Hence,

$$0 \le \lim_{x\to\infty}\left(\frac{\psi(x)}{x} - \frac{\theta(x)}{x}\right) \le \lim_{x\to\infty}\frac{\log^2(x)}{2\log(2)\sqrt{x}} = 0,$$

which proves that $\frac{\psi(x)}{x}$ and $\frac{\theta(x)}{x}$ will have the same limit if it exists. □

# Chapter 6

# Proof of the Prime Number Theorem

At the end of section 4.5 we discussed how the first proof of the prime number theorem was found. This was done in 1896 independently by Hadamard and de la Vallée-Poussin. The proof was rather intricate since it used the explicit formula for the Chebyshev psi function. Finding an explicit formula for $\pi(x)$ was a lot of work as we have noticed in chapter 4 and the formula for $\psi(x)$ is derived in similar manner. Hence, we will not go through the trouble of finding another explicit formula. Luckily, the proof of the prime number theorem has been fine tuned and for the proof given here, nothing more difficult than Cauchy's integral formula is needed.

Several proofs of the prime number theorem have been found after the first one. One of the proofs was elementary (not making use of complex analysis) and was found by Selberg and Erdős in 1948. Other proofs relied on the zeta function and a Tauberian theorem. The precise meaning of a Tauberian theorem is explained in section 6.2. The Tauberian theorem that was required for proving the prime number theorem (the Wiener-Ikehara theorem) was difficult to prove and has many more applications. However, in 1980, Donald Newman found a weaker version of the general Wiener-Ikehara theorem which also leads to the proof of the equivalent statement that $\lim_{x\to\infty} \frac{\psi(x)}{x} = 1$ as in theorem 5.16 at the end of the previous chapter.

The weaker Tauberian theorem that Newman proved will be given in section 6.3. This theorem will lead to the following corollary as is described in [Korevaar, 1982] and [Ash and Novinger, 2007].

**Corollary.** *Let $f(x)$ be a nonnegative, piecewise continuous and nondecreasing function on $[1,\infty)$ such that $f(x) = \mathcal{O}(x)$ and that the integral*

$$g(z) = z\int_1^\infty f(x)x^{-z-1}dx$$

*exists for Re$(z) > 1$ and defines an analytic function. Assume that for some constant $c$*

$$g(z) - \frac{c}{z-1}$$

*has an analytic extension to a neighborhood of the line Re$(z) = 1$. Then*

$$\frac{f(x)}{x} \to c \quad \textit{as } x \to \infty.$$

This corollary is proved in section 6.4. Obviously, we want to apply it to $f(x) = \psi(x)$. All the necessary conditions for applying this corollary will be established in the first section. Combining the result of this corollary with theorem 5.16 will complete the proof of the prime number theorem in section 6.5.

## 6.1 The Zeta Function and $\psi(x)$

For applying the corollary to $f(x) = \psi(x)$ the integral

$$g(z) = z \int_1^\infty \psi(x) x^{-z-1} dx$$

is needed. In theorem 6.2 an expression for this transform is proved. First, we need the following lemma. Recall that the Von Mangoldt function was defined for $n \in \mathbb{N}$ as

$$\Lambda(n) = \begin{cases} \log(p) & \text{if } n = p^k \text{ with } p \text{ prime and } k \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}.$$

**Lemma 6.1.** *For Re$(s) > 1$ it holds that*

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{k=1}^\infty k^{-s} \Lambda(k).$$

*Proof.* Recall the Euler product formula for $\mathrm{Re}(s) > 1$:

$$\zeta(s) = \prod_{p \ prime} \frac{1}{1 - p^{-s}},$$

from which it follows that the zeta function has no zeros for $\mathrm{Re}(s) > 1$ (see remark 3.4). Define $f_n(s) = (1 - p_n^{-s})^{-1}$ which is analytic for $\mathrm{Re}(s) > 1$. Then $\sum_{n=1}^\infty f_n - 1$ converges normally and theorem B.2 gives that

$$\begin{aligned}
\frac{\zeta'(s)}{\zeta(s)} &= \sum_p \frac{\left[\left(1 - p^{-s}\right)^{-1}\right]'}{\left(1 - p^{-s}\right)^{-1}} = \sum_p \frac{-\left(1 - p^{-s}\right)^{-2} p^{-s} \log(p)}{\left(1 - p^{-s}\right)^{-1}} \\
&= -\sum_p \frac{p^{-s} \log(p)}{1 - p^{-s}} = -\sum_p \log(p) p^{-s} \sum_{k=0}^\infty p^{-sk} \\
&= -\sum_p \log(p) \sum_{k=1}^\infty p^{-sk} = -\sum_{k=1}^\infty \sum_p \log(p) \left(p^k\right)^{-s} \\
&= -\sum_{n=1}^\infty \Lambda(n) n^{-s},
\end{aligned}$$

where interchanging the order of summation is allowed since the iterated sum converges absolutely for $\mathrm{Re}(s) > 1$:

$$\sum_p \sum_{k=1}^\infty \left| p^{-sk} \log(p) \right| = \sum_p \sum_{k=1}^\infty p^{-\mathrm{Re}(s)k} \log(p) = \sum_p \frac{\log(p)}{p^{\mathrm{Re}(s)} - 1} \leq \sum_{n=1}^\infty \frac{\log(n)}{n^{\mathrm{Re}(s)} - 1}.$$

Let $\varepsilon > 0$ be small, then for $n$ large enough we have that $\log(n) < n^\varepsilon$. Hence,

$$\frac{\log(n)}{n^{\mathrm{Re}(s)} - 1} < \frac{1}{n^{\mathrm{Re}(s) - \varepsilon}}$$

and the sum

$$\sum_{n=1}^{\infty} \frac{1}{n^{\mathrm{Re}(s)-\varepsilon}}$$

converges for $0 < \varepsilon < \mathrm{Re}(s) - 1$. □

**Theorem 6.2.** *For Re*$(s) > 1$

$$-\frac{\zeta'(s)}{\zeta(s)} = s \int_1^{\infty} \psi(x) x^{-s-1} dx.$$

*Proof.* Note that by definition 5.10 of $\psi(x)$

$$\Lambda(k) = \sum_{m \leq k} \Lambda(m) - \sum_{m \leq k-1} \Lambda(m) = \psi(k) - \psi(k-1).$$

Hence,

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{k=1}^{\infty} k^{-s} \Lambda(k) = \sum_{k=1}^{\infty} k^{-s} \big(\psi(k) - \psi(k-1)\big).$$

Now, look at the partial sums and apply proposition 5.3 with $m = 1$, $f(k) = k^{-s}$ and $a(k+1) = \psi(k)$. Also, note that $a(1) = \psi(0) = 0$.

$$\begin{aligned} \sum_{k=1}^{n} k^{-s} \big(\psi(k) - \psi(k-1)\big) &= n^{-s}\psi(n) - \sum_{k=2}^{n} \psi(k-1)\big(k^{-s} - (k-1)^{-s}\big) \\ &= n^{-s}\psi(n) - \sum_{k=1}^{n-1} \psi(k)\big((k+1)^{-s} - k^{-s}\big) \\ &= n^{-s}\psi(n) + \sum_{k=1}^{n-1} \psi(k)\big(k^{-s} - (k+1)^{-s}\big). \end{aligned}$$

From proposition 5.12 follows that $\psi(x) \leq x \log(x)$, hence for $\mathrm{Re}(s) > 1$

$$\psi(n) n^{-s} \leq \frac{n \log(n)}{n^s} \to 0 \quad \text{as } n \to \infty.$$

Use the integral

$$\int_k^{k+1} x^{-s-1} dx = -\frac{1}{s} x^{-s} \Big|_k^{k+1} = -\frac{1}{s}(k+1)^{-s} + \frac{1}{s} k^{-s}$$

for the remaining sum:

$$\begin{aligned} \sum_{k=1}^{n} \psi(k)\big((k)^{-s} - (k+1)^{-s}\big) &= \sum_{k=1}^{n} \psi(k) s \int_k^{k+1} x^{-s-1} dx \\ &= \sum_{k=1}^{n} s \int_k^{k+1} \psi(x) x^{-s-1} dx \\ &= s \int_1^{n+1} \psi(x) x^{-s-1} dx, \end{aligned}$$

where in the second step $\psi(k)$ can be taken into the integral because $\psi(x)$ is constant on $[k, k+1)$. Taking limits as $n \to \infty$ finally gives that

$$-\frac{\zeta'(s)}{\zeta(s)} = s \int_1^{\infty} \psi(x) x^{-s-1} dx \quad \text{for } \mathrm{Re}(s) > 1.$$

□

Another condition of the corollary is now that

$$\frac{-\zeta'(s)}{\zeta(s)} - \frac{c}{s-1}$$

should be analytic is a neighborhood of the line $\text{Re}(s) = 1$ for some constant $c$. Therefore, an analytic continuation of the zeta function is needed. Also, the zeta function may not be zero on this line. These two properties are proved in the rest of this section.

**Theorem 6.3.** *The function $\zeta(s) - \frac{1}{s-1}$ has an analytic continuation to $Re(s) > 0$.*

*Proof.* First note that for $\text{Re}(s) > 0$

$$\int_1^\infty x^{-s} dx = \frac{1}{1-s} x^{1-s}\Big|_1^\infty = \frac{1}{s-1} \tag{6.1}$$

and

$$\int_n^x u^{-s-1} du = -\frac{1}{s} u^{-s}\Big|_n^x = -\frac{1}{s} x^{-s} + \frac{1}{s} n^{-s}. \tag{6.2}$$

Now, for $\text{Re}(s) > 1$

$$\begin{aligned} \zeta(s) - \frac{1}{s-1} &\overset{(6.1)}{=} \sum_{n=1}^{\infty} \frac{1}{n^s} - \int_1^\infty \frac{1}{x^s} dx \\ &= \sum_{n=1}^{\infty} \int_n^{n+1} \left( \frac{1}{n^s} - \frac{1}{x^s} \right) dx \end{aligned} \tag{6.3}$$

and this series converges normally for $\text{Re}(s) > 0$:

$$\begin{aligned} \left| \int_n^{n+1} \left( \frac{1}{n^s} - \frac{1}{x^s} \right) dx \right| &\overset{(6.2)}{=} \left| \int_n^{n+1} \int_n^x \frac{s}{u^{s+1}} du dx \right| \\ &\leq 1 \cdot \max_{n \leq x \leq n+1} \left| \int_n^x \frac{s}{u^{s+1}} du \right| \\ &\leq \max_{n \leq x \leq n+1} \underbrace{\left| (x-n) \max_{n \leq u \leq x} \left| \frac{s}{u^{s+1}} \right| \right|}_{\text{increasing, so maximum at } x=n+1} \\ &= \big((n+1) - n\big) \max_{n \leq u \leq n+1} \left| \frac{s}{u^{s+1}} \right| \\ &= \max_{n \leq u \leq n+1} \underbrace{\left| \frac{s}{u^{s+1}} \right|}_{\text{decreasing for Re}(s)>0} \\ &= \frac{|s|}{n^{\text{Re}(s)+1}} \end{aligned}$$

and

$$\sum_{n=1}^{\infty} \frac{|s|}{n^{\text{Re}(s)+1}}$$

converges for $\text{Re}(s) > 0$. Note that the integral in equation (6.3) is analytic for $\text{Re}(s) > 0$ by theorem A.6. Hence, the series in (6.3) is a normally convergent series of analytic functions on the right half-plane $\text{Re}(s) > 0$. By theorem A.18 it follows that

$$\zeta(s) - \frac{1}{s-1}$$

is analytic for $\text{Re}(s) > 0$. □

**Remark 6.4.** *This theorem could also be proved using the analytic continuation of the zeta function for Re*$(s) > 0$ *as derived in chapter 3 and the fact that the zeta function has a simple pole in* $s = 1$*. In chapter 3 we got an expression for the analytic continuation on the right half-plane using the eta function. However, as the above proof shows, it also possible to prove the existence of the analytic continuation only.*

**Theorem 6.5.** *The zeta function has no zeros on the line with real part 1, i.e.* $\zeta(1+it) \neq 0$ *for all* $t \in \mathbb{R}$*.*

*Proof.* For a fixed real number $t \neq 0$ introduce the function

$$\phi(\sigma) = \zeta^3(\sigma)\zeta^4(\sigma + it)\zeta(\sigma + 2it)$$

for $\sigma > 1$. From the Euler product formula it follows that for $\mathrm{Re}(s) > 1$

$$\log\big|\zeta(s)\big| = -\sum_p \log\big|1 - p^{-s}\big| = -\mathrm{Re}\left(\sum_p \mathrm{Log}\big(1 - p^{-s}\big)\right),$$

where it is used that for $w \in \mathbb{C} \setminus \{s : s \leq 0\}$

$$\mathrm{Re}\big(\mathrm{Log}(w)\big) = \mathrm{Re}\big(\log|w| + i\mathrm{Arg}(w)\big) = \log|w|.$$

Now, using the power series for the logarithm

$$\mathrm{Log}(1 - w) = -\sum_{n=1}^{\infty} \frac{w^n}{n} \quad \text{for } |w| < 1$$

gives that

$$\log\big|\zeta(s)\big| = \mathrm{Re}\left(\sum_p \sum_{n=1}^{\infty} \frac{p^{-ns}}{n}\right).$$

Hence,

$$\begin{aligned}
\log\big|\phi(\sigma)\big| &= 3\log\big|\zeta(\sigma)\big| + 4\log\big|\zeta(\sigma + it)\big| + \log\big|\zeta(\sigma + 2it)\big| \\
&= \sum_p \sum_{n=1}^{\infty} \frac{p^{-n\sigma}}{n} \mathrm{Re}\left(3 + 4p^{-int} + p^{-2int}\right) \\
&= \sum_p \sum_{n=1}^{\infty} \frac{p^{-n\sigma}}{n} \mathrm{Re}\left(3 + 4e^{-int\log(p)} + e^{-2int\log(p)}\right) \\
&= \sum_p \sum_{n=1}^{\infty} \frac{p^{-n\sigma}}{n} \Big(3 + 4\cos\big(nt\log(p)\big) + \cos\big(2nt\log(p)\big)\Big) \geq 0,
\end{aligned}$$

since that for $\theta \in \mathbb{R}$

$$\begin{aligned}
3 + 4\cos(\theta) + \cos(2\theta) &= 3 + 4\cos(\theta) + 2\cos^2(\theta) - 1 \\
&= 2\big(1 + 2\cos(\theta) + \cos^2(\theta)\big) \\
&= 2\big(1 + \cos(\theta)\big)^2 \geq 0.
\end{aligned}$$

The exponential function is increasing, hence $\log|\phi(\sigma)| \geq 0$ implies that for $\sigma > 1$

$$\big|\phi(\sigma)\big| = \big|\zeta(\sigma)\big|^3\big|\zeta(\sigma + it)\big|^4\big|\zeta(\sigma + 2it)\big| \geq 1. \tag{6.4}$$

To prove that $\zeta(1+it) \neq 0$ for every $t \in \mathbb{R}$ it suffices to consider $t \neq 0$, because at $t = 0$ the zeta function has a pole. Assume that $\zeta(1+it) = 0$ for some $t \in \mathbb{R}$ and rewrite equation (6.4) as

$$\big|(\sigma-1)\zeta(\sigma)\big|^3 \left|\frac{\zeta(\sigma+it)}{\sigma-1}\right|^4 \big|\zeta(\sigma+2it)\big| \geq \frac{1}{\sigma-1} \quad \text{for } \sigma > 1. \tag{6.5}$$

Now letting $\sigma \downarrow 1$ gives that $\big|(\sigma-1)\zeta(\sigma)\big| \to 1$ since $\zeta(s)$ has a simple pole with residue 1 at $s = 1$ (see section 3.2). Also, $\big|\zeta(\sigma+2it)\big| \to \big|\zeta(1+2it)\big|$. Note that the zeta function is analytic for $\mathrm{Re}(s) > 0$, hence it is differentiable in $1+it$, which gives that

$$\left|\frac{\zeta(\sigma+it)}{\sigma-1}\right| = \left|\frac{\zeta(\sigma+it)-\zeta(1+it)}{\sigma-1}\right| \to \big|\zeta'(1+it)\big| \quad \text{as } \sigma \downarrow 1.$$

Now, in equation (6.5) the left hand side tends to $\big|\zeta'(1+it)\big|^4\big|\zeta(1+2it)\big|$ as $\sigma \downarrow 1$, but the right hand side goes to infinity as $\sigma \downarrow 1$. This gives a contradiction, hence $\zeta(1+it) \neq 0$ for all $t \in \mathbb{R}$. □

## 6.2 Abelian and Tauberian Theorems

Consider the series of complex numbers $\sum_{n=0}^{\infty} c_n$. We call this series convergent (or summable) if the sequence of partial sums $\big(\sum_{n=0}^{k} c_n\big)_{k=0}^{\infty}$ converges to some limit. Obviously, not all series are convergent, take for example the series

$$\sum_{n=0}^{\infty}(-1)^n,$$

where the sequence of partial sums is $1, 0, 1, 0, \dots$. However, it would be natural to assign the value $\frac{1}{2}$ to this series, which is the average of the partial sums. In general, define the sequence

$$\sigma_k = \frac{s_0 + s_2 + \cdots + s_{k-1}}{k},$$

where $s_i = \sum_{n=0}^{i} c_n$ is the $i$th partial sum. Now, if the sequence $(\sigma_k)_{k=0}^{\infty}$ converges, then we call the series $\sum_{n=0}^{\infty} c_n$ Cesàro summable. Note that 'ordinary' summability implies Cesàro summability.

There is an even more general way to sum series: Abel summability (not to be confused with the Abel summation in section 5.1). A series is Abel summable if

$$\lim_{r \uparrow 1} \sum_{n=0}^{\infty} c_n r^n$$

exists. If a series is summable, then it is also Abel summable (this is called Abel's theorem). Moreover, if a series is Cesàro summable, then it is also Abel summable. It follows that

$$\text{ordinary summable} \implies \text{Cesàro summable} \implies \text{Abel summable}.$$

The theorems that show that a summation method gives the same sum as ordinary summability are called Abelian theorems. However, the implications above cannot be reversed without any further conditions. Theorems that give conditions under which the implications can be reversed are called Tauberian theorems.

We considered series to introduce Tauberian theorems, but there is also a continuous analogue with integrals instead of sums. In general, Tauberian theorems state the conditions under which 'ordinary' convergence (of sums or integrals) can be deduced from some weaker type of convergence. The Tauberian theorem that will be proved in the next section states that under certain conditions the convergence of $\int_0^\infty F(t)e^{-zt}dt$ for all $\mathrm{Re}(z) > 0$ implies that the improper integral $\int_0^\infty F(t)dt$ converges.

**Remark 6.6.** *There are more ways to assign a value to a series than the three methods discussed above. Note that $\sum_{n=1}^\infty n$ is not convergent in the sense of any of these three summability methods. However, using the analytic continuation of the zeta function we can link $\sum_{n=1}^\infty n$ with $\zeta(-1)$ and the functional equation of the zeta function* (3.6) *gives that*

$$\zeta(-1) = \Gamma(2)2(2\pi)^{-2}\sin\left(\frac{-\pi}{2}\right)\zeta(2) = -\frac{1}{2\pi^2}\cdot\frac{\pi^2}{6} = -\frac{1}{12}.$$

## 6.3 A Tauberian Theorem

**Theorem 6.7.** *Let $F(t)$ be bounded and piecewise continuous on $[0,\infty)$, so that the Laplace transform*

$$G(z) = \int_0^\infty F(t)e^{-zt}dt$$

*is well-defined and analytic throughout the open half-plane $Re(z) > 0$. Suppose that $G(z)$ can be continued analytically to a neighborhood of every point on the imaginary axis, $Re(z) = 0$. Then*

$$\int_0^\infty F(t)dt$$

*exists as an improper integral and equals $G(0)$.*

*Proof.* Let $F$ be bounded and piecewise continuous on $[0,\infty)$ and the Laplace transform $G$ is defined and analytic for $\mathrm{Re}(z) > 0$. Also, assume that $G$ has been extended analytically to a neighborhood of $\mathrm{Re}(z) = 0$, thus $G$ is analytic on a domain $U \supset \{z : \mathrm{Re}(z) \geq 0\}$. The function $F(t)$ is bounded, so without loss of generality it will be assumed that

$$|F(t)| \leq 1, \qquad \text{for all } t > 0. \tag{6.6}$$

Define for $0 < \lambda < \infty$

$$G_\lambda(z) = \int_0^\lambda F(t)e^{-zt}dt.$$

Note that by theorem A.6, $G_\lambda(z)$ is analytic on $\mathbb{C}$. The idea is to prove

$$\lim_{\lambda\to\infty}\int_0^\lambda F(t)dt = G(0), \qquad \text{or equivalently } G(0) - G_\lambda(0) \to 0 \text{ as } \lambda \to \infty \tag{6.7}$$

using the Cauchy integral formula. The path over which is integrated cannot be a circle around 0. A circle might go too far into the left half-plane and $G(z)$ might not be analytic there. Instead, for each $R > 0$, let $\delta > 0$ be large enough such that $G$ is analytic inside and on the closed contour $C_R$. This contour consists of an arc of a circle with radius $R$ and a vertical segment at $\mathrm{Re}(z) = -\delta$, see figure 6.1. The contour $C_R$ is split in two parts: $C_R^+$ is the part of $C_R$ with $\mathrm{Re}(z) > 0$ and $C_R^-$ is the part with $\mathrm{Re}(z) < 0$.

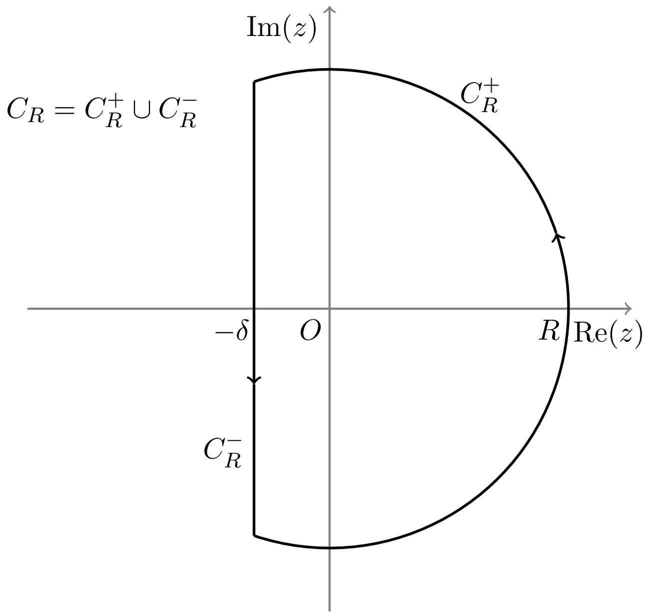


Figure 6.1: The closed contour $C_R$.

Cauchy's integral formula (theorem A.5) now gives that

$$G(0) - G_\lambda(0) = \frac{1}{2\pi i}\int_{C_R} \frac{G(z) - G_\lambda(z)}{z} dz. \tag{6.8}$$

The goal is to estimate this integral as $\lambda \to \infty$. First, try to estimate the integral using an ML-bound. Thus, for $x = \mathrm{Re}(z) > 0$

$$\begin{aligned} |G(z) - G_\lambda(z)| &= \left| \int_\lambda^\infty F(t) e^{zt} dt \right| \qquad (6.9) \\ &\leq \int_\lambda^\infty |F(t)| \left| e^{zt} \right| dt \\ &\overset{(6.6)}{\leq} \int_\lambda^\infty e^{-xt} dt \\ &= \left. \frac{e^{-xt}}{-x} \right|_{t=\lambda}^\infty \overset{\mathrm{Re}(z)>0}{=} \frac{e^{-\lambda \mathrm{Re}(z)}}{\mathrm{Re}(z)} \qquad (6.10) \end{aligned}$$

and for $x = \mathrm{Re}(z) < 0$ (the reason that only $G_\lambda$ is estimated will become clear later)

$$\begin{aligned} |G_\lambda(z)| &= \left| \int_0^\lambda F(t) e^{-zt} dt \right| \\ &\leq \int_0^\lambda e^{-xt} dt \\ &= \left. \frac{e^{-xt}}{-x} \right|_{t=0}^\lambda = \frac{e^{-\lambda \mathrm{Re}(z)} - 1}{-\mathrm{Re}(z)} \leq \frac{e^{-\lambda \mathrm{Re}(z)}}{-\mathrm{Re}(z)}. \qquad (6.11) \end{aligned}$$

The domain $U$ where $G(z)$ is analytic contains the line $\mathrm{Re}(z) = 0$. For values of $z$ near this line both estimates blow up. To avoid this problem, the factor $\frac{1}{z}$ in (6.8) will be replaced by $\frac{1}{z} + \frac{z}{R^2}$. Note that if $|z| = R$, then

$$\frac{1}{z} + \frac{z}{R^2} = \frac{R^2 + z^2}{zR^2} = \frac{|z|^2 + z^2}{zR^2} = \frac{z\bar{z} + z^2}{zR^2} = \frac{2\mathrm{Re}(z)}{R^2}. \tag{6.12}$$

In this way the $\frac{1}{\text{Re}(z)}$ in (6.10) and (6.11) will be canceled. Since

$$\frac{z\big(G(z) - G_\lambda(z)\big)}{R^2}$$

is analytic on $U$, the value of the contour integral $C_R$ will be unchanged. Also, in (6.11) $e^{-\lambda \text{Re}(z)}$ will blow up if $\text{Re}(z) < 0$. Hence, both $G(z)$ and $G_\lambda(z)$ will be multiplied by $e^{\lambda z}$. Since $e^{\lambda z}$ is entire and equals 1 at $z = 0$, Cauchy's integral formula in (6.8) can be rewritten as

$$G(0) - G_\lambda(0) = \frac{1}{2\pi i}\int_{C_R} \big(G(z) - G_\lambda(z)\big)e^{\lambda z}\left(\frac{1}{z} + \frac{z}{R^2}\right) dz. \tag{6.13}$$

Let $\varepsilon > 0$ and to prove that $G(0) - G_\lambda(0) \to 0$ as $\lambda \to \infty$ there must exist a $\lambda_0$ such that for all $\lambda \geq \lambda_0$

$$|G(0) - G_\lambda(0)| = \left|\frac{1}{2\pi i}\int_{C_R} \big(G(z) - G_\lambda(z)\big)e^{\lambda z}\left(\frac{1}{z} + \frac{z}{R^2}\right) dz\right| < \varepsilon.$$

Redoing the estimate for $x = \text{Re}(z) > 0$ gives

$$\left|\big(G(z) - G_\lambda(z)\big)e^{\lambda z}\left(\frac{1}{z} + \frac{z}{R^2}\right)\right| \overset{(6.10),(6.12)}{\leq} \frac{e^{-\lambda x}}{x}e^{\lambda x}\frac{2x}{R^2} = \frac{2}{R^2}$$

and thus by using an ML-bound

$$\left|\frac{1}{2\pi i}\int_{C_R^+} \big(G(z) - G_\lambda(z)\big)e^{\lambda z}\left(\frac{1}{z} + \frac{z}{R^2}\right) dz\right| \leq \pi R \frac{1}{2\pi}\frac{2}{R^2} = \frac{1}{R} \leq \frac{\varepsilon}{4}$$

for $R \geq \frac{4}{\varepsilon}$. For the integral over $C_R^-$, first use the triangle inequality:

$$\begin{aligned}
&\left|\frac{1}{2\pi i}\int_{C_R^-} \big(G(z) - G_\lambda(z)\big)e^{\lambda z}\left(\frac{1}{z} + \frac{z}{R^2}\right) dz\right| \\
&\leq \left|\frac{1}{2\pi i}\int_{C_R^-} G(z)e^{\lambda z}\left(\frac{1}{z} + \frac{z}{R^2}\right) dz\right| + \left|\frac{1}{2\pi i}\int_{C_R^-} G_\lambda(z)e^{\lambda z}\left(\frac{1}{z} + \frac{z}{R^2}\right) dz\right|.
\end{aligned}$$

Note that for $\text{Re}(z) > 0$ it was possible to write (see (6.9))

$$G(z) - G_\lambda(z) = \int_\lambda^\infty F(t)e^{zt}dt.$$

However, for $\text{Re}(z) < 0$ there is no explicit integral formula for $G(z)$, hence $G$ and $G_\lambda$ are treated separately. Note that the integral formula for $G_\lambda(z)$ for $\text{Re}(z) < 0$ is still valid. Recall that $G_\lambda(z)$ was analytic on $\mathbb{C}$, hence according to Cauchy's theorem A.4 the contour $C_R^-$ can be replaced by a semicircle in the left half-plane from $iR$ to $-iR$ (black contour in figure 6.2), without changing the value of the integral:

$$\left|\frac{1}{2\pi i}\int_{C_R^-} G_\lambda(z)e^{\lambda z}\left(\frac{1}{z} + \frac{z}{R^2}\right) dz\right| = \left|\frac{1}{2\pi i}\int_{\substack{|z|=R \\ \text{Re}(z)<0}} G_\lambda(z)e^{\lambda z}\left(\frac{1}{z} + \frac{z}{R^2}\right) dz\right|.$$

This change of contour is done in order to use (6.12) again. Now, redoing the estimate for

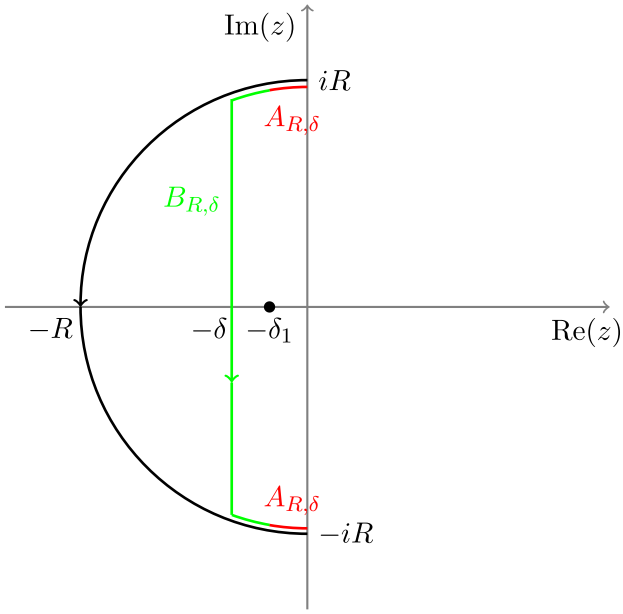


Figure 6.2: The semicircle $|z| = R$ with $\mathrm{Re}(z) < 0$ and the parts $A_{R,\delta}$ and $B_{R,\delta}$ from $C_R^-$.

$x = \mathrm{Re}(z) < 0$ gives

$$\left| G_\lambda(z)e^{\lambda z}\left(\frac{1}{z} + \frac{z}{R^2}\right)\right| \overset{(6.11),(6.12)}{\leq} \frac{e^{-\lambda x}}{|x|}e^{\lambda x}\frac{2|x|}{R^2} = \frac{2}{R^2}.$$

And by using an ML-bound

$$\left|\frac{1}{2\pi i}\int_{\substack{|z|=R\\ \mathrm{Re}(z)<0}} G_\lambda(z)e^{\lambda z}\left(\frac{1}{z} + \frac{z}{R^2}\right)dz\right| \leq \pi R\frac{1}{2\pi}\frac{2}{R^2} = \frac{1}{R} \leq \frac{\varepsilon}{4}$$

for $R \geq \frac{4}{\varepsilon}$. The last integral

$$\left|\frac{1}{2\pi i}\int_{C_R^-} G(z)e^{\lambda z}\left(\frac{1}{z} + \frac{z}{R^2}\right)dz\right|$$

is the most tricky one, since $G$ on $C_R^-$ is an analytic extension and it is not known how it actually looks like, but it is possible to find an $M_R > 0$ such that $|G(z)| \leq M_R$ on $C_R^-$. For some $\delta_1$ with $0 < \delta_1 < \delta$, the contour $C_R^-$ is split in two parts: $A_{R,\delta}$ for $\mathrm{Re}(z) \geq -\delta_1$ and $B_{R,\delta}$ for $\mathrm{Re}(z) < -\delta_1$. Both contours are shown in figure 6.2 in respectively red and green.

Note that on $B_{R,\delta}$ it holds that $|e^{\lambda z}| \leq e^{-\lambda\delta_1}$ and that $\delta \leq |z| \leq R$. Hence, by using an ML-bound

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_{B_{R,\delta}} G(z)e^{\lambda z}\left(\frac{1}{z} + \frac{z}{R^2}\right)dz\right| &\leq \frac{1}{2\pi}\pi R M_R e^{-\lambda\delta_1}\left(\frac{1}{\delta} + \frac{R}{R^2}\right)\\ &= \frac{1}{2}RM_R\left(\frac{1}{\delta} + \frac{1}{R}\right)e^{-\lambda\delta_1}.\end{aligned}$$

For fixed $R$ and $\delta_1$ this goes to zero as $\lambda \to \infty$. The integral over $A_{R,\delta}$ becomes

$$\left|\frac{1}{2\pi i}\int_{A_{R,\delta}} G(z)e^{\lambda z}\left(\frac{1}{z} + \frac{z}{R^2}\right)dz\right| \leq \frac{1}{2\pi}2R\arcsin\left(\frac{\delta_1}{R}\right)M_R\left(\frac{1}{\delta} + \frac{1}{R}\right)$$

using $|e^{\lambda z}| \leq 1$ and that the arc length of $A_{R,\delta}$ equals $2R \arcsin\left(\frac{\delta_1}{R}\right)$. So for fixed $R$, the above expression goes to zero as $\delta_1$ goes to zero.

To complete the proof, let $\varepsilon > 0$ be given and fix $\delta > 0$ such that $G$ is analytic inside and on the closed contour $C_R$. Take $R = \frac{4}{\varepsilon}$ and then recall the results from above

$$\left| \frac{1}{2\pi i} \int_{C_R^+} (G(z) - G_\lambda(z)) e^{\lambda z} \left( \frac{1}{z} + \frac{z}{R^2} \right) dz \right| \leq \frac{1}{R} = \frac{\varepsilon}{4}$$

and

$$\left| \frac{1}{2\pi i} \int_{C_R^-} G_\lambda(z) e^{\lambda z} \left( \frac{1}{z} + \frac{z}{R^2} \right) dz \right| \leq \frac{1}{R} = \frac{\varepsilon}{4}.$$

Also, choose a $\delta_1$ such that $0 < \delta_1 < \delta$ and

$$\left| \frac{1}{2\pi i} \int_{A_{R,\delta}} G(z) e^{\lambda z} \left( \frac{1}{z} + \frac{z}{R^2} \right) dz \right| < \frac{\varepsilon}{4}.$$

This is possible since the this integral goes to zero as $\delta_1 \to 0$ ($R$ is fixed). Finally,

$$\left| \frac{1}{2\pi i} \int_{B_{R,\delta}} G(z) e^{\lambda z} \left( \frac{1}{z} + \frac{z}{R^2} \right) dz \right| < \frac{\varepsilon}{4}$$

for all $\lambda \geq \lambda_0$, since this integral goes to zero as $\lambda \to \infty$ ($R$ and $\delta_1$ are fixed). So it follows that

$$|G(0) - G_\lambda(0)| = \left| \frac{1}{2\pi i} \int_{C_R} (G(z) - G_\lambda(z)) e^{\lambda z} \left( \frac{1}{z} + \frac{z}{R^2} \right) dz \right| < \varepsilon$$

for all $\lambda \geq \lambda_0$, which completes the proof. □

## 6.4 Corollary to the Tauberian Theorem

Using the Tauberian theorem the corollary can be derived which will allow us to complete the proof of the prime number theorem.

**Corollary 6.8.** *Let $f(x)$ be a nonnegative, piecewise continuous and nondecreasing function on $[1, \infty)$ such that $f(x) = \mathcal{O}(x)$ as $x \to \infty$ and that the integral*

$$g(z) = z \int_1^\infty f(x) x^{-z-1} dx$$

*exists for $Re(z) > 1$ and defines an analytic function. Assume that for some constant $c$*

$$g(z) - \frac{c}{z-1}$$

*has an analytic extension to a neighborhood of the line $Re(z) = 1$. Then*

$$\frac{f(x)}{x} \to c \quad \text{as } x \to \infty.$$

*Proof.* Let $f(x)$ and $g(z)$ be as assumed in the corollary. Define $F(t) = e^{-t} f(e^t) - c$ for $t \in [0, \infty)$. First, check if this $F$ satisfies the conditions of the Tauberian theorem. The condition $f(x) = \mathcal{O}(x)$ gives that

$$F(t) = e^{-t} \mathcal{O}(e^t) - c = \mathcal{O}(1),$$

so $F(t)$ is bounded on $[0, \infty)$, which is a hypothesis of the Tauberian theorem. Consider the Laplace transform and apply the change of variables $x = e^t$

$$\begin{aligned}
G(z) &= \int_0^\infty \big(e^{-t} f(e^t) - c\big) e^{-zt} dt \\
&= \int_1^\infty \big(x^{-1} f(x) - c\big)\, x^{-z} x^{-1} dx \\
&= \int_1^\infty f(x) x^{-z-2} dx - c \int_1^\infty x^{-z-1} dx \\
&= \int_1^\infty f(x) x^{-z-2} dx + \frac{c}{z} x^{-z}\Big|_1^\infty \\
&= \int_1^\infty f(x) x^{-z-2} dx - \frac{c}{z} \\
&= \frac{g(z+1)}{z+1} - \frac{c}{z} \\
&= \frac{1}{z+1} \left( g(z+1) - \frac{c}{z} - c \right),
\end{aligned}$$

where

$$g(z) = z \int_1^\infty f(x) x^{-z-1} dx$$

exists for $\mathrm{Re}(z) > 1$ and is analytic because of the hypothesis of the corollary. So $g(z+1)$ exists and is analytic for $\mathrm{Re}(s) > 0$. Furthermore, it follows from the assumptions that $g(z+1) - \frac{c}{z}$ has an analytic extension to a neighborhood of the line $\mathrm{Re}(z) = 0$. As a result, the same holds for $G(z)$. Thus all conditions of the Tauberian theorem are satisfied, hence it is obtained that $\int_0^\infty F(t)dt$ exists and equals $G(0)$.

Equivalently, in terms of $f$ the integrals

$$\int_0^\infty \big(e^{-t} f(e^t) - c\big) dt \overset{x=e^t}{=} \int_1^\infty \big(x^{-1} f(x) - c\big) x^{-1} dx$$

exist and are finite. To prove that $\frac{f(x)}{x} \to c$ as $x \to \infty$, it must be shown that for any $\varepsilon > 0$ there exists a constant $M$ such that

$$\frac{f(x_0)}{x_0} - c < 2\varepsilon \quad \text{and} \quad \frac{f(x_0)}{x_0} - c > -2\varepsilon$$

for all $x_0 \geq M$. Suppose that this is not the case, then for a given $\varepsilon > 0$ and $x_0 \geq M$ it is assumed that

$$\frac{f(x_0)}{x_0} \geq 2\varepsilon + c \quad \text{or} \quad \frac{f(x_0)}{x_0} \leq c - 2\varepsilon.$$

The first case, if $x \in \big[x_0, \rho x_0\big]$ with $\rho := \frac{c+2\varepsilon}{c+\varepsilon} > 1$, then

$$f(x) \geq f(x_0) \geq x_0(c + 2\varepsilon) \geq x(c + \varepsilon),$$

where in the first inequality is used that $f(x)$ is nondecreasing. Hence,

$$\begin{aligned}
\int_{x_0}^{\rho x_0} \left( \frac{f(x)}{x} - c \right) x^{-1} dx &\geq \int_{x_0}^{\rho x_0} \left( \frac{x(c+\varepsilon)}{x} - c \right) x^{-1} dx \\
&\geq \int_{x_0}^{\rho x_0} \frac{\varepsilon}{x} dx \qquad (6.14) \\
&= \varepsilon\big(\log(\rho x_0) - \log(x_0)\big) = \varepsilon \log(\rho) > 0, \quad (\rho > 1).
\end{aligned}$$

However,
$$\int_{x_1}^{x_2} \left(\frac{f(x)}{x} - c\right) x^{-1}dx \to 0 \qquad \text{as } x_1, x_2 \to \infty$$
because $\int_1^\infty \big(x^{-1}f(x) - c\big)x^{-1}dx$ is convergent. So there exists a constant $\tilde{M}$ such that for all $x_0 \geq \tilde{M}$
$$\int_{x_0}^{\rho x_0} \left(\frac{f(x)}{x} - c\right) x^{-1}dx < \varepsilon \log(\rho). \tag{6.15}$$
Now, if $x_0 \geq \max\{M, \tilde{M}\}$, then equations (6.14) and (6.15) contradict. To conclude, for $x_0$ sufficiently large $\frac{f(x_0)}{x_0} < 2\varepsilon + c$.

The second case: $\frac{f(x_0)}{x_0} \leq c - 2\varepsilon$. Let $\mu := \frac{c-2\varepsilon}{c-\varepsilon} < 1$ for $\varepsilon < c$ and let $x \in [\mu x_0, x_0]$. Then
$$f(x) \leq f(x_0) \leq x_0(c - 2\varepsilon) \leq x(c - \varepsilon).$$
Again, integrating gives the contradiction
$$\int_{\mu x_0}^{x_0} \left(\frac{f(x)}{x} - c\right) x^{-1}dx \leq \int_{\mu x_0}^{x_0} \left(\frac{x(c-\varepsilon)}{x} - c\right) x^{-1}dx = \varepsilon \log(\mu) < 0, \quad (\mu < 1).$$
□

## 6.5 The Final Result

To complete the proof of the prime number theorem, apply corollary 6.8 to $f(x) = \psi(x)$. We will check the conditions.

The Chebyshev psi function was defined for $x > 0$ as
$$\psi(x) = \sum_{p^k \leq x} \log(p).$$
Hence, $\psi(x)$ is nonnegative, piecewise continuous and nondecreasing on $[1, \infty)$ as the corollary requires. Furthermore, in theorem 5.15 it was proved that $\psi(x)$ has the asymptotic behavior $\psi(x) = \mathcal{O}(x)$ as $x \to \infty$. In theorem 6.2 we found the integral
$$-\frac{\zeta'(s)}{\zeta(s)} = s \int_1^\infty \psi(x)x^{-s-1}dx$$
for $\mathrm{Re}(s) > 1$. In theorem 6.3 it was proven that $\zeta(s)$ has an analytic continuation to the half-plane $\mathrm{Re}(s) > 0$ without $s = 1$ where it has a simple pole. Hence, $-\frac{\zeta'(s)}{\zeta(s)}$ has an analytic continuation to a neighborhood of $\{s : \mathrm{Re}(s) \geq 1, s \neq 1\}$ provided that $\frac{1}{\zeta(s)}$ has no zeros on that neighborhood. As stated several times before (see remark 3.4), $\zeta(s)$ has no zeros for $\mathrm{Re}(s) > 1$. In addition, theorem 6.5 proved that the zeta function has no zeros for $\mathrm{Re}(s) = 1$, so there exists a neighborhood of $\{s : \mathrm{Re}(s) \geq 1, s \neq 1\}$ on which $-\frac{\zeta'(s)}{\zeta(s)}$ is analytic. Since $\zeta(s)$ has a simple pole at $s = 1$, also $-\frac{\zeta'(s)}{\zeta(s)}$ has a simple pole at $s = 1$. Hence,
$$-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}$$
has an analytic extension to a neighborhood of the line $\mathrm{Re}(s) = 1$.

Corollary 6.8 now gives that

$$\lim_{x\to\infty} \frac{\psi(x)}{x} = 1,$$

which is equivalent to the prime number theorem

$$\lim_{x\to\infty} \frac{\pi(x)\log(x)}{x} = 1$$

according to theorem 5.16. This finishes the proof of the prime number theorem.

## 6.6 Final Remarks

The goals of this thesis were to construct an explicit formula for the prime counting function and to prove the prime number theorem at an undergraduate mathematical level. After a general introduction about the gamma and zeta function, we finally succeeded in constructing the formula and proving the prime number theorem. The construction of the explicit formula for $\pi(x)$ was quite technical and it took a lot of effort to prove most of the details. Unfortunately, in section 4.4 one proof of switching an integral and sum is missing. Nevertheless, the rest of the construction contains sufficient details to make it understandable for an undergraduate mathematics student.

The prime number theorem was one of the highlights of mathematics in the nineteenth century. The original proof was long and intricate since it requires the construction of an explicit formula for $\psi(x)$. This construction is similar to the construction of $\pi(x)$. Fortunately, a modern version of the proof, that was demonstrated above, only uses basic knowledge from complex analysis, which makes the proof of the prime number theorem understandable for mathematics students.

# Appendix A

# Prerequisites from Complex Analysis

This appendix contains the necessary knowledge and theorems of complex analysis. For a complete introduction to complex analysis and the proofs, see for example [Ash and Novinger, 2007], [Freitag and Busam, 2009] or [Asmar and Grafakos, 2018].

### Cauchy Product Formula

**Theorem A.1** (Cauchy product formula). *Let $a_n$ and $b_n$ be complex valued sequences such that $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are absolutely convergent series. Then*

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} a_n\right) = \sum_{n=0}^{\infty}\left(\sum_{\nu=0}^{n} a_\nu b_{n-\nu}\right).$$

**Corollary A.2.** *Let $a_{1,\nu_1}, \ldots, a_{m,\nu_m}$ be complex valued sequences such that $\sum_{\nu_1=0}^{\infty} a_{1,\nu_1}, \ldots,$ $\sum_{\nu_m=0}^{\infty} a_{m,\nu_m}$ are absolutely convergent series. Then*

$$\begin{aligned}\prod_{k=1}^{m}\left(\sum_{\nu_k=0}^{\infty} a_{k,\nu_k}\right) &= \left(\sum_{\nu_1=0}^{\infty} a_{1,\nu_1}\right)\cdots\left(\sum_{\nu_m=0}^{\infty} a_{m,\nu_m}\right)\\ &= \sum_{v_1=0}^{\infty}\sum_{v_2=0}^{\nu_1}\cdots\sum_{\nu_m=0}^{\nu_{m-1}} a_{1,\nu_m} a_{2,\nu_{m-1}-\nu_m}\cdots a_{m,\nu_1-\nu_2}.\end{aligned}$$

### Complex Integration and Theorems

**Definition A.3** (domain). *A domain is an arcwise connected non-empty open set $D \subset \mathbb{C}$.*

**Theorem A.4** (Cauchy's theorem). *Let $f$ be analytic on a simply connected domain. If $\gamma$ is a closed path in this domain, then*

$$\int_\gamma f(z)dz = 0.$$

**Theorem A.5** ((Generalized) Cauchy integral formula). *Let $f$ be analytic on a domain that contains a simple closed path $C$ with positive orientation and its interior. If $z \in \mathbb{C}$ lies in the interior of $C$, then*

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta$$

*and in particular,*

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta.$$

**Theorem A.6.** *Let $C$ be a path and $U$ an open set. Let $f(z,\zeta)$ be a function defined for $z \in U$ and $\zeta \in C$. Suppose that $f(z,\zeta)$ is continuous in $\zeta \in C$ and analytic in $z \in U$ and that the derivative $\frac{df}{dz}(z,\zeta)$ is continuous in $\zeta \in C$. Then*

$$\int_C f(z,\zeta) d\zeta$$

*is analytic in $U$ and*

$$\frac{d}{dz} \int_C f(z,\zeta) d\zeta = \int_C \frac{d}{dz} f(z,\zeta) d\zeta.$$

**Theorem A.7** (Liouville)**.** *Every entire function (i.e. an analytic function $f : \mathbb{C} \to \mathbb{C}$) which is bounded is constant.*

**Theorem A.8** (Maximum modulus principle)**.** *Let $D$ be a bounded domain and let $f$ be analytic on $D$ and continuous on the closure of $D$. Then*

*(i) $|f|$ attains its maximum on the boundary of $D$,*

*(ii) $f$ is constant if $f$ attains a maximum in $U$.*

**Definition A.9** (Harmonic function)**.** *A function $u : U \to \mathbb{R}$ is called harmonic if it has continuous partial derivatives of first and second order in $U$ and satisfies the Laplace equation*

$$\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

**Theorem A.10.** *The real and imaginary part of an analytic function defined on an open set are harmonic.*

**Theorem A.11** (Maximum modulus principle for harmonic functions)**.** *Let $U$ be a bounded domain and let $u$ be harmonic on $U$ and continuous on the closure of $U$. Then*

*(i) $u$ attains its maximum on the boundary of $U$,*

*(ii) $u$ is constant if $u$ attains a maximum in $U$.*

**Theorem A.12** (Residue theorem)**.** *Let $C$ be a simple closed positively orientated path. Suppose that $f$ is analytic on $C$ and in its interior, except for finitely many points $z_1, z_2, \ldots, z_n$ inside $C$. Then*

$$\int_C f(z) dz = 2\pi i \sum_{i=1}^{n} Res(f; z_i),$$

*where $Res(f; z_i)$ is the residue of $f$ at $z_i$.*

**Lemma A.13** (Shrinking path lemma)**.** *Assume that $f$ is a continuous, complex valued function on $\overline{B_R(z_0)}$. For $0 < \varepsilon \leq R$, let $C_\varepsilon$ be a circle with positive orientation. Then*

$$\lim_{\varepsilon \downarrow 0} \int_{C_\varepsilon} \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0).$$

**Corollary A.14.** *Assume that $f$ is analytic in a deleted neighborhood of $z_0$ with a simple pole at $z_0$. Let $\varepsilon > 0$ and let $C_\varepsilon$ be a semicircle with positive orientation. Then*

$$\lim_{\varepsilon \downarrow 0} \int_{C_\varepsilon} f(z)dz = \pi i Res(f; z_0).$$

### Convergence

**Definition A.15** (Uniform convergence)**.** *A sequence of functions $f_0, f_1, f_2, \ldots$ with $f_n : D \to \mathbb{C}$ for all $n \in \mathbb{N}$ converges uniformly to the limit $f : D \to \mathbb{C}$ if for every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|f(z) - f_n(z)| < \varepsilon$ for all $n \geq N$ and all $z \in D$.*

Uniform convergence might not hold on the whole domain, but only on a subset: the sequence $f_n$ converges locally uniformly to $f$ if for every $a \in D$ there exists a neighborhood $U$ of $a$ such that $f_n|U \cap D$ is uniformly convergent.

**Theorem A.16.** *Let $f_0, f_1, f_2, \ldots : D \to \mathbb{C}$ with $D \subset \mathbb{C}$ open be analytic functions that converge locally uniformly to the limit $f : D \to \mathbb{C}$. Then, $f$ is analytic and the sequence of derivatives $(f_n')$ converges locally uniformly to $f'$.*

This theorem can be rewritten for series of functions, namely a series of functions $f_0 + f_1 + f_2 + \cdots$ with $f_n : D \to \mathbb{C}$ for all $n \in \mathbb{N}$ with $D \subset \mathbb{C}$ converges (locally) uniformly if the sequence of partial sums $S_n := f_0 + f_1 + \ldots + f_n$ converges (locally) uniformly. If this holds, then by the above theorem $\lim_{n\to\infty} S_n = \sum_{n=1}^{\infty} f_n =: f$ is analytic.

A stronger notion than absolute and uniform convergence is normal convergence.

**Definition A.17.** *A series of functions $\sum_{n=0}^{\infty} f_n$ with*

$$f_n : D \to \mathbb{C}, \quad D \in \mathbb{C}, \quad n \in \mathbb{N}$$

*converges normally in $D$ if for each point $a \in D$ there is a neighborhood $U$ and a sequence $(M_n)_{n\geq 0}$ of nonnegative real numbers such that*

*(i)* $|f_n(z)| \leq M_n$ *for all* $z \in U \cap D$ *and all* $n \in \mathbb{N}$,

*(ii)* $\sum_{n=0}^{\infty} M_n$ *converges.*

Normal convergence is precisely the condition for the Weierstrass $M$-test. So normal convergence implies absolute and (local) uniform convergence. Hence, in the above theorem the condition of local uniform convergence can be replaced by normal convergence. For series of functions this gives:

**Theorem A.18.** *Let $f_0 + f_1 + f_2 + \cdots$ with $f_n : D \to \mathbb{C}$ for all $n \in \mathbb{N}$ a normally convergent series of analytic functions on $D \subset \mathbb{C}$ open. Then, the limit function $f$ is also analytic and $f' = f_0' + f_1' + f_2' + \cdots$.*

### Power Series and Identity Theorem

**Theorem A.19** (Power series expansion)**.** *Let $f$ be analytic on an open subset $D \subset \mathbb{C}$. Assume that for some $R > 0$ the disk $B_R(z_0)$ lies inside $D$. Then*

$$f(z) = \sum_{n=0}^{\infty} a_n \big(z - z_0\big)^n \quad \textit{with} \quad a_n = \frac{f^{(n)}(z_0)}{n!}, \quad n = 0, 1, 2, \ldots$$

*for all $z \in B_R(z_0)$.*

**Theorem A.20** (Identity theorem for analytic functions)**.** *Let $f, g : D \to \mathbb{C}$ be two analytic functions with $D \neq \emptyset$ a domain. Then, $f = g$ if and only if the set $\{z \in D : f(z) = g(z)\}$ has an accumulation point in $D$.*

**Corollary A.21** (Uniqueness of the analytic continuation)**.** *Let $D \subset \mathbb{C}$ be a domain, $M \subset D$ be a subset with at least one accumulation point in $D$ and let $f : M \to \mathbb{C}$. If there exists an analytic function $\tilde{f} : D \to \mathbb{C}$ with $\tilde{f}(z) = f(z)$ for all $z \in D$, then $\tilde{f}$ is unique with this property.*

# Appendix B

# Infinite Products

An infinite product of factors $a_n$ can be defined in terms of the complex logarithm, so that the infinite product becomes an infinite series:

$$\prod_{n=1}^{\infty} a_n := \exp \sum_{n=1}^{\infty} \mathrm{Log}(a_n)$$

with

$$\mathrm{Log}(z) = \log|z| + i\mathrm{Arg}(z) \quad \text{defined on } \mathbb{C}^- = \mathbb{C} \setminus (-\infty, 0].$$

First of all, it is assumed that the factors of the product converge to 1 (just as the sequence of terms for a convergent series go to 0). Note that the sequence $(b_n)_{n=1}^{\infty}$ with $b_i = a_i - 1$ converges to zero. Hence, there exists an $N \in \mathbb{N}$ such that $|b_n| < 1$ for all $n \geq N$. So in order to avoid the branch cut of the logarithm, define

$$\prod_{n=1}^{\infty} a_n := \prod_{n=1}^{N-1} a_n \cdot \exp \sum_{n=N}^{\infty} \mathrm{Log}(1 + b_n)$$

because $1 + b_n \in \mathbb{C}^-$ for $n \geq N$. Now, the infinite product of $a_n$ is absolutely convergent if the corresponding series $\sum_{n=1}^{\infty} \mathrm{Log}(1 + b_n)$ is absolutely convergent. The power series of $\mathrm{Log}(1 + z)$ for $|z| < 1$ is given by

$$\mathrm{Log}(1 + z) = -\sum_{n=1}^{\infty} (-1)^n \frac{z^n}{n}.$$

For $z \in \mathbb{C}$ and $|z|$ small enough (for example $|z| \leq \frac{1}{2}$) the following inequalities hold:

$$\begin{aligned}\big||\mathrm{Log}(1+z)| - |z|\big| \leq \big|\mathrm{Log}(1+z) - z\big| &= \left|\sum_{n=2}^{\infty} (-1)^n \frac{z^n}{n}\right| \leq \sum_{n=2}^{\infty} \left|\frac{z^n}{n}\right| \\ &\leq \frac{|z|^2}{2} \sum_{n=0}^{\infty} |z|^n \leq |z|^2 \leq \frac{1}{2}|z|\end{aligned}$$

using in the second to last step that according to the geometric series $\sum_{n=0}^{\infty} |z|^n \leq 2$ for $|z| \leq \frac{1}{2}$. This gives

$$\frac{|z|}{2} \leq \big|\mathrm{Log}(1 + z)\big| \leq \frac{3}{2}|z|.$$

If $\sum_{n=1}^{\infty} |b_n|$ converges, then there exists an $N \in \mathbb{N}$ such that $|b_n| < \frac{1}{2}$ for all $n \geq N$. This implies

$$\frac{|b_n|}{2} \leq \big|\mathrm{Log}(1 + b_n)\big| \leq \frac{3}{2}|b_n|. \tag{B.1}$$

Hence, $\sum_{n=1}^{\infty}\mathrm{Log}(1+b_n)$ converges. Conversely, if $\sum_{n=1}^{\infty}\mathrm{Log}(1+b_n)$ converges, then equation (B.1) also applies, thus $\sum_{n=1}^{\infty}|b_n|$ converges.
To conclude, the infinite product $\prod_{n=1}^{\infty} a_n$ converges absolutely if and only if the infinite series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}(a_n - 1)$ converges absolutely.

**Remark B.1.** *The value of the convergent infinite product is zero if and only if at least one of the factors is zero.*

For series there were theorems about when the sum is analytic provided that all terms of the series are analytic (theorem A.16 and A.18). A similar result holds for products.

**Theorem B.2.** *Let $(f_n)_{n=1}^{\infty}$ a sequence of analytic functions with $f_n : D \to \mathbb{C}$ such that $\sum_{n=1}^{\infty} f_n$ converges normally. Then, $f(z) := \prod_{n=1}^{\infty}(1+f_n(z))$ converges normally and $f(z)$ is analytic. Also, if $z \in D$ and $1 + f_n(z) \neq 0$ for all $n \in \mathbb{N}$, then*

$$\frac{f'(z)}{f(z)} = \sum_{n=1}^{\infty} \frac{f_n'(z)}{1+f_n(z)},$$

*where the series converges normally.*

# Appendix C

# Big-O and Little-o Notation

To analyse the behavior of a function when its argument tends to infinity without concerning to much about the precise details, the Landau notation will be used. Let $f$ (real or complex valued) and $g$ (real valued) be functions defined on an interval $[x_0, \infty)$. Then

$$f(x) = \mathcal{O}\big(g(x)\big) \quad \text{as } x \to \infty,$$

if there exists an $M > 0$ and $x_1 > x_0$ such that $|f(x)| \leq M|g(x)|$ for all $x \geq x_1$. Usually, just $f(x) = \mathcal{O}\big(g(x)\big)$ is written and $x \to \infty$ behind it is omitted. Some properties are

(i) $f(x) = \mathcal{O}(1)$ implies that $f$ is bounded for $x \geq x_1$,

(ii) $f(x)\mathcal{O}\big(g(x)\big) = \mathcal{O}\big(f(x)g(x)\big)$,

(iii) $\mathcal{O}\big(|k|g(x)\big) = \mathcal{O}\big(g(x)\big)$ with $k \neq 0$,

(iv) $f(x) = \mathcal{O}\big(g(x)\big)$ implies that $kf(x) = \mathcal{O}\big(g(x)\big)$ for $k \in \mathbb{R}$.

If $f_1 = \mathcal{O}(g_1)$ and $f_2 = \mathcal{O}(g_2)$, then

(v) $f_1 f_2 = \mathcal{O}(g_1 g_2)$,

(vi) $f_1 + f_2 = \mathcal{O}\big(\max\{g_1, g_2\}\big)$

The notation $f(x) = o\big(g(x)\big)$ is used if for every $\varepsilon > 0$ there exists an $x(\varepsilon) \geq x_0$ such that $|f(x)| \leq \varepsilon |g(x)|$ for all $x \geq x(\varepsilon)$. This is equivalent to

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0.$$

Some properties are

(i) $f(x) = o(1)$ implies that $\lim_{x\to\infty} f(x) = 0$,

(ii) $f = o(g)$ implies that $f = \mathcal{O}(g)$,

(iii) $f(x) = o\big(g(x)\big)$ implies that $kf(x) = o\big(g(x)\big)$ for $k \neq 0$,

(iv) $f = o(g)$ and $g = o(h)$ implies that $f = o(h)$.

If $f_1 = o(g_1)$ and $f_2 = o(g_2)$, then

(v) $f_1 f_2 = o(g_1 g_2)$.

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