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Improving the global analysis of mechanical systems via parallel computation of basins of attraction

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Abstract

Numerical integrations represent a time-consuming element in the long-term dynamics analysis of mechanical systems. This limits the resolution of the computations and the dimension of the system to be investigated numerically. In fact, even pushing memory resources to their thresholds, only few tools can deal with higher-dimensional systems. This work illustrates, in a preliminary manner, the results that can be obtained reducing the aforementioned constraints thanks to the implementation of algorithms based on a parallel computing approach. In particular, by focusing on basins of attraction, four applications are discussed. i) The full domain of attraction for a four-dimensional (4D) system describing a linear oscillator coupled with a nonlinear absorber is calculated. ii) The variation of a safe basin with respect to the system dimension is then analyzed. It is highlighted how 4D and 3D analyses provide more confident results with respect to 2D analyses. iii) The parametric variation of a 2D system with a reduced step is performed by building a 3D representation which allows to highlight a smooth transition between the states. iv) A convergence study of a basin of attraction resolution is carried out. The integrity factor is used as a comparison measure.
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Keywords: Basins of attraction; dynamical integrity; parallel computing; multidimensional systems

1. Introduction

In modern investigations of dynamical systems it is realized that it is no longer possible to determine only the attractors, other unstable orbits (saddles, homoclinic, heteroclinic, etc.), and to detect their local bifurcations[1]. In fact, even if a solution is stable, it may be not visible in practice because it has a small neighborhood of safe initial conditions[2]. Furthermore, the attractor may disappear suddenly, by means of a crisis[3]. These (and other, indeed) phenomena call for a global analysis of the system at hand, to be added to the "classical" local analyses - which have to be done in any case.

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The key tool for developing global analysis are the basins of attractions, which are the subsets of the phase space that share the same attractor, i.e. the ensemble of initial states whose orbits converge to the same attractor when the time goes to infinity.

The determination of basins of attraction by means of analytical techniques is possible only in few and paradigmatic cases. In general, they can be determined only by numerical techniques, and their sizes and shapes are thus *a priori* unpredictable: they can surround the attractor with a large and compact area, or be tight and stretched; can fill a portion of the state space by presenting an intermediate size with tongues towards certain directions, or be characterized by a riddled behaviour caused by the bubbling of attractors[4]; their boundaries can be smooth or fractal[5], etc.. Attempts to classify basins into general classes have been presented in literature[6] with the goal of permitting easier comparison.

The concept of basin of attraction has been later extended to the "safe basin", i.e. the subset of initial conditions sharing a common property, not necessarily the convergence toward an attractor[1,7].

Studying the properties (robustness, compactness, etc.) of basins of attraction (or, more generally, of safe basin) is the goal of the dynamical integrity theory, which has been recently developed starting from the observation that "classical" (Lyapunov) stability is not enough for practical purposes[7]. In fact, if the surrounding basin of attraction is not "large and compact" enough, even small perturbations - which are always present in everyday applications - can lead the system to a different attractor, resulting in a different, unexpected and often unwanted dynamical behaviour.

A key tool of dynamical integrity is the definition of a measure that provides magnitude of the safe basin, which faces with the problem of the intrinsically unsafe fractal basins (since they imply sensitivity to initial conditions, which is unwanted in common applications). Thus, in addition of the magnitude of the safe basin (so called GIM), other measures of the dynamical integrity have been introduced in the past to rule out the fractal parts[1,2,7,8].

The reduction of the safety and the erosion of the basins of attraction of a guyed tower model are evaluated in[9]. Dynamical integrity analysis of parallel-plate Micro-Electro-Mechanical Systems (MEMS) have also been performed in the recent past[10]. The erosion of the basin of attraction for electrostatic microactuators due to both the amplitude and the frequency of the actuation voltage has been evaluated by[11]. The device presents an high sensitivity to the initial conditions, and modification in the excitation leads to a reduction of the smoothness of the boundary of the basins.

The safety and dynamic integrity of a parametrically excited cylindrical shell is undertaken in[12] by analyzing the evolution of the various basins of attraction in the four-dimensional (4D) phase space. Projections of a 4D phase space, describing the oscillations and stability of the same mechanical model have been also proposed[13]: with the use of basins of attraction the authors highlight the instability phenomena that may arise under loading conditions such as a parametric excitation of flexural modes, and the escape phenomenon from the pre-buckling potential well.

By using 2D cross sections of the 5D basins of attraction, erosion profiles and integrity measures for the parametrically excited noncontacting atomic force microscopy problem are obtained in[14]. In[15] the sizes variation of basins of attraction is analysed in a periodically forced pendulum with oscillating support in the case of time-varying dissipation.

The importance of basins of attraction as a global analysis tool is proved by the countless applications in the engineering fields[16], although other examples such as in economics, are not uncommon[17]. Both continuous[18,19] and discrete[20] systems are tackled. Two coupled logistic maps presenting partially riddled basins in systems with chaotic saddle located between two attractors are shown in[21]. The dynamics of dices rolling is undertaken in[22] showing basins of attraction of different cube die faces.

Beside the massive use of basin of attraction, the state of the art is still essentially based on two dimensional cross sections also for high-dimension systems[23]. Globally, only sections of the real phase space are represented, which, incidentally, may lead to overlook rare attractors or chimera states, which can have an important role in the global system dynamics[24,25]. This is basically due to the limited computational resources available.

To advance in the direction of determining full basins of attraction of higher dimensional systems, various attempts have been done in the past[26,27].

The authors have contributed by developing an algorithm that exploits parallel computation for faster determination of higher order, fully dimensional, basins of attraction[28,29]. These works are continued here by applying the algorithm to some specific problems of interest in mechanical systems. The goal is *not* a thorough investigation of the basins of attraction, of their behaviour, and of the consequences for mechanical engineering. This is left for future work.

Here we are only aimed at showing in a *preliminary*, and necessarily non exhaustive, way, the powerfulness of the developed algorithm for global analysis of systems of practical interest.

2. Summary of the algorithm for parallel computing of basins of attraction

There are two main approaches for the calculation of basins of attraction. Starting from a discretization of the state space, the grid-of-starts method time-integrates all points in the discretized domain up to the steady-state behaviour[30]. Differently, an approximation of the real trajectory is built defining a mapping[31] between cells in the so called cell-to-cell mapping method (for a detailed review of cell-mapping methods we refer to[32]).

Based on these algorithms, the data collected in this paper has been evaluated in a relatively small cluster (12 nodes cluster equipped with 8-core Intel Xeon CPUs ($E5-2630v3$, 2.4 GHz) and 128 GB RAM each) splitting the domain grid across multiple cores. Several parallel processes can evaluate simultaneously different cells of the domain overcoming the inner seriality of the cell-mapping method. The details of the algorithm can be found in[29] where are also given indications on how set up the processes to obtain an optimal computational time. The distributed memory approach makes compulsory a coherence between the local chunks of the total domain: a post processing share of informations, performed locally by each process, aligns all the parts of the grid towards the same attractors leading to a coherent basin.

3. Applications

By exploiting the advantages given by the parallel computing in terms of elaboration time and resources administration, we show some higher dimensional basins of attractions, and related applications, which are necessary for the global analysis of the considered systems.

3.1. Multi-Degree-of-Freedom basins of attraction

The first case we consider consists of determining the basins of attraction of higher dimensional systems, which was the original motivation for using parallel computing for basins of attraction, and which is also one of the challenges for the future applications of dynamical integrity.

The analysed system is a linear primary oscillator coupled with a nonlinear adsorber[33]. A 4D basin is completely determined in the whole phase space where its dynamics develop.

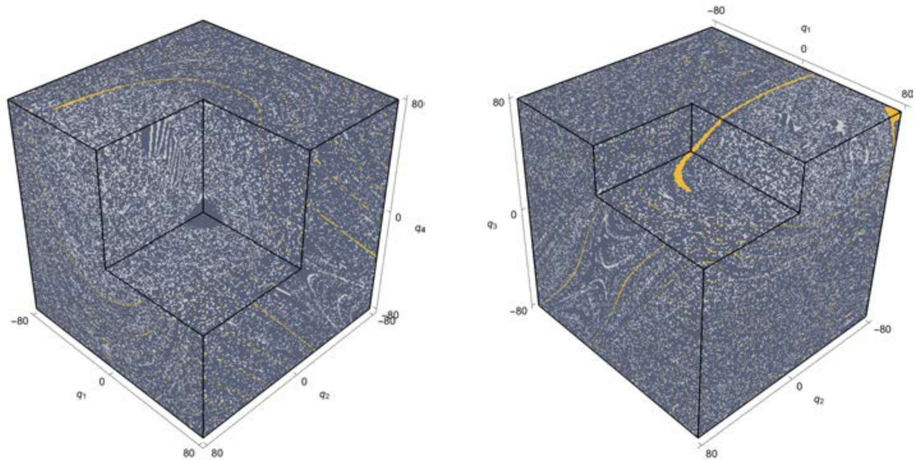
The mechanical system is governed by the 4 first-order differential equations:

$$\begin{cases} \dot{q}_1 = q_2\,, \\ \dot{q}_2 = -2\gamma_1 q_2 - 2\epsilon\hat{\gamma}_N\,(q_2 - q_4) - q_1 - \epsilon\Omega^2\,(q_1 - q_3) - F^2\epsilon\Omega_N^2(q_1 - q_3)^3 + \sin(\omega\tau)\,, \\ \dot{q}_3 = q_4\,, \\ \dot{q}_4 = -2\hat{\gamma}_N\,(q_4 - q_2) - \Omega^2\,(q_3 - q_1) - F^2\Omega_N^2(q_3 - q_1)^3. \end{cases} \quad (1)$$

The variables $q_{1,2}$ and $q_{3,4}$ represents the position and velocity of the primary oscillator and the nonlinear tuned mass damper, respectively. The parameters used for the presented example are $\gamma_1 = 0.02$, $\epsilon = 0.1$, $\hat{\gamma}_N = 0.002$, $\Omega = 0$, $F = 0.35$, $\Omega_N = 0.09$ and $\omega = 1.8$. The chosen parameters, as explained in detail in[34], lead to a configuration with three stable attractors, two of which with period-1 and one of period-3.

We build 4D basins of attraction of the hypercube $q_i \in [-80, 80]$, $i = 1, .., 4$, which is shown to contain the 3 attractors and the most interesting part of their basins, each dimension is discretized in 200 intervals, leading to a total number of $1'600'000'000$ cells and a step size of 0.804020.

Figure 1 shows the basins wrapping the two attractors with period-1, in the two subfigures illustrating 3D sections of the 4D basin, the missing dimension is fixed in q_3 in Fig. 1a, and in q_4 in Fig. 1b, respectively. The attractor in Fig. 1a presents a compact central region surrounded by a large region where the other basins intermix. Conversely the period-1 attractor in Fig. 1b is within a thin shell-shaped basin with a reduced distance from other basins.



(a) (b)

Fig. 1: 3D sections of the calculated 4D basins. The figures are sliced at the position of the attractor. (a) Period-1 attractor located at $\mathbf{q}^T = \{-0.014478, -0.802730, 0.000992, 0.000029\}$; Section shown $[\cdot, \cdot, 0.000992, \cdot]$. (b) Period-1 attractor located at $\mathbf{q}^T = \{-6.740523, 7.223642, 47.422095, -56.577836\}$; Section shown $[\cdot, \cdot, \cdot, -56.577836]$.

3.2. Accuracy of Integrity Measures computed on sub-sections

Here we address the problem of how Integrity Measures calculated on 2D sections of the basins of attraction - which what is usually done in the past - are accurate with respect to the "true" measure computed on the full dimensional phase space. In other words, we study how approximate is the safety evaluation based on Integrity Measures computed on sections of the whole phase space.

The Local Integrity Measure (LIM), which is the radius of the largest hyper-sphere centred in the attractor and entirely contained in the basin (thus ruling out the fractal parts from the integrity evaluation), is used. The measure is computed on the basin of the period-1 attractor shown in Fig. 1a, and also for higher values of the parameter F.

In Fig. 2 is reported the LIM calculated in 2D cross sections of the 4D basin. By looking at the $[q_1, q_2]$ plane (Fig. 2f) the central area appears compact, while the real shape of the basin is much more jagged when seen from sections in the other planes (Fig. 2a-e). This is yet a demonstration that lower dimension sections of the basins lose important information, namely that the "other" dimensions can be important in basins morphology, at least from a qualitative point of view.

Table 1: LIM variations by increasing the external force F, considering 2D, 3D and 4D basins.

F	0.35	0.40	0.45
4D LIM	8.3556	7.1463	6.4822
3D LIM $[q_1, q_2, q_3]$	14.2473 (+70.51%)	12.2991 (+72.10%)	10.9359 (+68.71%)
3D LIM $[q_1, q_2, q_4]$	8.6968 (+4.08%)	7.8776 (+10.23%)	6.4822 (+0.0%)
3D LIM $[q_1, q_3, q_4]$	9.8472 (+17.85%)	8.5846 (+20.13%)	7.4562 (+15.03%)
3D LIM $[q_2, q_3, q_4]$	8.6968 (+4.08%)	7.4562 (+4.34%)	6.4822 (+0.0%)
2D LIM $[q_4, q_3]$	11.2563 (+34.72%)	9.6482 (+35.01%)	8.1994 (+26.49%)
2D LIM $[q_4, q_2]$	8.6968 (+4.08%)	7.9186 (+10.81%)	6.4822 (+0.0%)
2D LIM $[q_4, q_1]$	10.0743 (+20.57%)	9.0964 (+27.29%)	7.9594 (+22.79%)
2D LIM $[q_3, q_2]$	15.2552 (+82.57%)	12.9641 (+81.41%)	11.8712 (+83.14%)
2D LIM $[q_3, q_1]$	14.8254 (+77.43%)	13.0885 (+83.15%)	10.9359 (+68.71%)
2D LIM $[q_2, q_1]$	14.8254 (+77.43%)	12.5592 (+75.74%)	11.8712 (+83.14%)

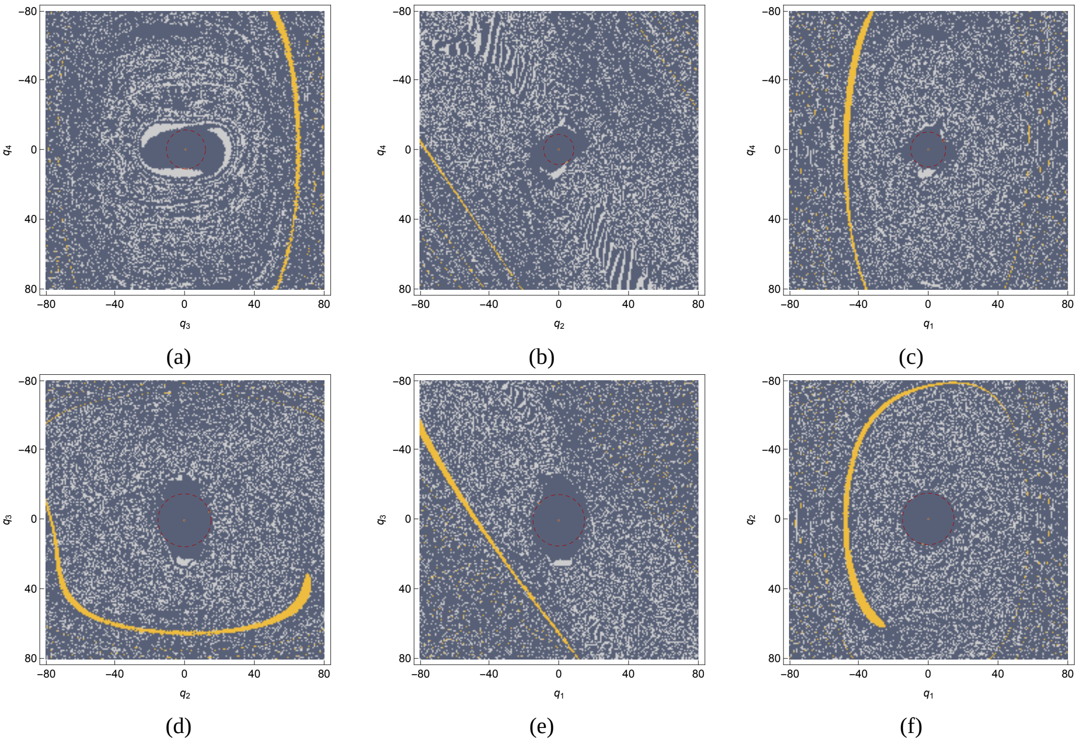


Fig. 2: 2D sections of the basins of attraction with the LIM highlighted by the circles. $F = 0.35$. (a) Plane $q_4 - q_3$; (b) Plane $q_4 - q_2$; (c) Plane $q_4 - q_1$; (d) Plane $q_3 - q_2$; (e) Plane $q_3 - q_1$; (f) Plane $q_2 - q_1$.

This impression is confirmed numerically by the values of the integrity measure reported in Tab. 1, it reports the LIM calculated by accounting for 2,3 and 4 dimensions of the basin. The main observation is that lower dimension sections always overestimate the "true" LIM, so that they are not on the safe side. In some cases this overestimation can be important also from a quantitative point of view (e.g. +83.15%). The results suggest that the $[q_2, q_4]$ plane is the most reliable and it can be noted that for $F = 0.45$ a cross section provides the same LIM of the improved measure, but this is expected to be casual and in any case can be checked only if the full measure is calculated. The data in Tab. 1 highlights a decreasing trend of the LIM measure under increased force amplitudes ($F = 0.40$ and $F = 0.45$) that represents the basin erosion.

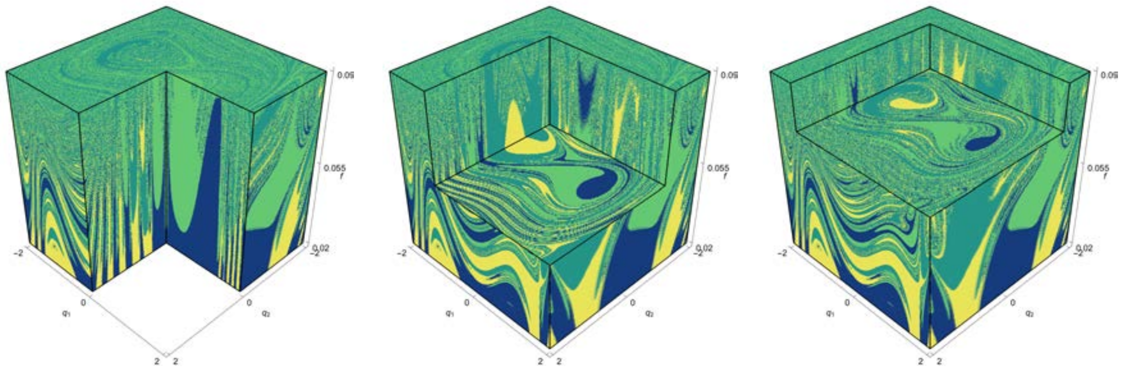
3.3. *Parametric variation analysis*

In this section we investigate the Parametric Variation (PV) of the basins of attraction of the forced vibrations of a two-well Duffing oscillator, which are governed by the equations

$$\begin{cases} \dot{q}_1 = q_2\,, \\ \dot{q}_2 = -2\zeta q_2 + q_1 - \gamma q_1^3 + f\cos(\Omega t)\,, \end{cases} \tag{2}$$

where the linear viscous damping is $\zeta = 0.025$ and the other constant parameters are $\gamma = 1$, $\Omega = 1.2$. q_1 is the position of the oscillator while q_2 describes the velocity at each instant of time.

A parametric variation is performed with respect to the excitation amplitude ($f \in [0.02, 0.09]$) and within a domain $q_{1,2} \in [-2, 2]$. The coordinates and the parameter are discretized into 500 steps, so that the total grid is composed of 500^3 cells. In Fig. 3 are reported three different sections for the computed basin. In the Fig. 3a it is possible to observe the transition between the region with two attractors and a basin where four attractors coexist. Indeed, for



(a) (b) (c)

Fig. 3: Sliced 3D views of the domain of parametric variation of basins of attraction of Eq. (2). It is remarked that only 2D sections at $f = const.$ are basins of attraction.

low values of f only two, one per potential well, nonresonant, small amplitude attractors govern the dynamics. By increasing the excitation amplitude, two resonant (large amplitude) attractors appear due to the nonlinear resonance, so that now there are four attractors, two per potential well. It is also shown that increasing the excitation amplitude (from the bottom to the top in Fig. 3), the basins become more and more fractal.

Figures 3b,c illustrate the modification of the areas and the shapes of resonant and non resonant attractors with respect to f; the fractal tongues around the compact basin that wrap the attractors are clearly visible in Fig. 3c.

It must be remarked that, strictly speaking, those reported in Fig. 3 are not basins of attraction, since f is not a variable but rather a parameter. Only "horizontal" sections at $f = const.$ are basins of attraction. However, the pictures reported in Fig. 3 are very useful in providing an accurate analysis of the basins erosion for increasing amplitude, which will eventually provide the f critical threshold for safe use of the present oscillator. They investigate the variation of the properties of the working domain with respect to a specific parameter trying to represent the corresponding changes and metamorphoses.

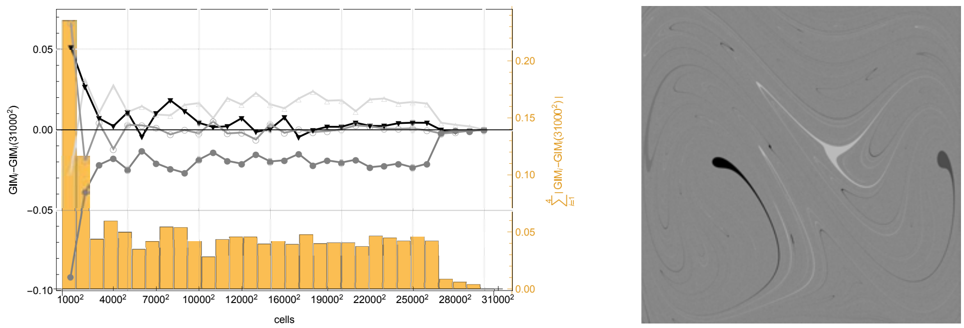
The use of basins of attraction algorithm to determine simultaneously the evolution of the basins is an appealing feature, which overcomes the previous analyses where the basins evolution was determined at discrete (and commonly spread) values of the parameter, or was determined by focusing only on the variation of the basin boundaries with respect to a starting basin.

3.4. Basins' accuracy

The next point that we wish to address is how the basin accuracy increases by augmenting the discretization of the phase space. More precisely, we want to investigate when it reaches a value that can be considered accurate enough. With this objective, the basins of attraction for fixed values of the parameters have been built by increasing the number of the cells, for a fixed window in the phase space.

While in the previous sections we used the LIM, in this section we employ the Global Integrity Measure (GIM), which is the hyper volume of the basin (so that in 2D domains it reduces to an area and in 3D domains to a volume). This choice is motivated by the fact that here we are not interested in the integrity (which indeed is poorly measured by the GIM), i.e. we do not want to rule out the fractal parts. We are instead interested in the whole basins, and GIM is thus a proper measure for this specific objective.

We present the variation of the GIM with respect to the resolution on which the basin is calculated. The analysis is applied to the system of Eq. (2) for a domain $q_1 \in [-1.4, 1.4]$ and $q_2 \in [-1, 1]$. The amplitude and the frequency of the periodic excitation are respectively set to $f = 0.13$ and $\Omega = 1.15$; others parameters are unvaried with respect to the application of Sect. 3.3. This configuration presents four different stable attractors with multiple fractal tongues around each compact basin (Fig. 4b). The family of curves shown in Fig. 4a illustrates the variation of the area of the four basins by increasing the number of cells in the square grid used to discretize the set of the initial conditions. All



(a) (b)

Fig. 4: a) Convergence behaviour for the basin of attraction of Eq.(2) ($\Omega = 1.15$, $f = 0.13$). The GIM value for each basin i, for $i = 1, ..4$, are normalized with respect to the total area and expressed as percentage. b) The four basins of attraction and the fractal area.

the points refer to a reference basin calculated with 31000×31000 cells and considered as “exact” for our purposes. In Fig. 4a it is also reported the total variation of the basins by means of a bars chart.

The variation of the four basins depicts three characteristic regions, in which the basins shapes change accordingly. A first increment of resolution drives the basin toward a row convergence since the actual resolution is not enough to estimate correctly the areas (first two bars). The convergence behaviour is then governed by a mixing in the number of cells for each attractor, as the formation of multiple fractal tongues is observed between the grids 3000^2 - 27000^2. A further increment of resolution refines the boundaries but the basin can be considered globally determined as the difference no longer increases.

4. Conclusion

It has been shown by means of some practical applications the importance of extending the actual limits in the computation of basins of attraction. In fact, in order to understand properly the shape of the basin and to perform more reliable integrity analyses additional dimensions must be accounted for.

The results have been obtained using a parallel algorithm in the framework of a distributed memory approach, which is considered a viable tool for the global analysis of dynamical systems.

We investigated the effect of increasing the dimensions of the safe basins, and fully dimensional basins have been computed and compared with those obtained by considering low dimensional (typically 2D) sections of the phase space. A more close view on the real shape of multidimensional basins can be obtained. It has been shown that in general the integrity measure computed on low dimensional sections of the full basins overestimates the compact part surrounding the attractor, and thus underestimates the safety of the system.

A parametric variation of the basins of attraction of a classical Duffing oscillator has been performed to illustrate the powerfulness of the proposed approach to accurately investigate the erosion of the basins. The basins accuracy has been then investigated, and it has been shown that the convergence of the basin toward the “exact” one passes through three stages with respect to the increasing resolution. A reliable basin with the formation of the minor fractal tongues has required about one million of cells.

Large dimensional basins do not affect only the computation, that can be addressed by using the parallel implementations, but also the management and the visualization of big data. This makes the post processing involved and more specific tools able to handle matrices in the order of GigaBytes must be adopted.

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