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Chapter 7
Distributed Stochastic Thermal Energy Management in Smart Thermal Grids

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Abstract This work presents a distributed stochastic energy management framework for a thermal grid with uncertainties in the consumer demand profiles. Using the model predictive control (MPC) paradigm, we formulate a finite-horizon chance-constrained mixed-integer linear optimization problem at each sampling time, which is in general non-convex and hard to solve. We then provide a unified framework to deal with production planning problems for uncertain systems, while providing a-priori probabilistic certificates for the robustness properties of the resulting solutions. Our methodology is based on solving a random convex program to compute the uncertainty bounds using the so-called scenario approach and then, solving a robust mixed-integer optimization problem with the computed randomized uncertainty bounds at each sampling time. Using a tractable approximation of uncertainty bounds, the proposed formulation retains the complexity of the problem without chance constraints. We also present two distributed approaches that are based on the alternating direction method of multipliers (ADMM) to solve the robust mixed-integer problem. The performance of the proposed methodology is illustrated using Monte Carlo simulations and employing two different problem formulations: optimization over input sequences (open-loop MPC) and optimization over affine feedback policies (closed-loop MPC).

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7.1 Introduction

Smart Thermal Grids (STGs) represent a new concept in the energy sector that involves the use of the smart grid concept in thermal energy networks connecting several households and greenhouses (agents) to each other via a transport line of thermal energy. One of the major challenges in sustainable energy systems is to improve the efficiency, reliability, and sustainability of the production and the distribution of energy. STGs can contribute to obtaining sustainable energy systems by introducing a reliable production plan using renewable energy sources such as solar or geothermal energy and provide efficient large-capacity storage options. This results in a reduction of carbon dioxide (CO_2) emissions, improved energy efficiency, and the implementation of renewable energy systems [1].

In an STG setting, the agents have a potential to contribute to the overall energy balance. Every agent fulfills the role of a consumer when it demands more energy than it produces with its production units (e.g. micro-combined heat and power), and fulfills the role of a producer when the demand is less than the production of its production units [2]. Since the major energy consumption is typically used for thermal purposes, the motivation for STGs can be both economical and environmental. A better price is achieved with less energy transport when the resources are used more efficiently, while the thermal energy losses are reduced.

We therefore foresee a shift towards a situation where a large number of small scale agents (e.g. utility companies and independent users) have more impact on the energy balance of the grid, while their optimal decisions are made by considering the thermal demand profiles, which are uncertain. The planning of thermal energy production to match supply and demand is challenging since predictions on the thermal energy demand are not perfect. This highlights the necessity of formulating stochastic variants of standard day-ahead planning problems in the grid, while providing probabilistic guarantees regarding the satisfaction of smart grid system constraints.

Model predictive control (MPC) is one of the most widely used advanced control design methods that can handle constraints on both inputs and states, and can obtain an optimal control sequence that minimizes a given objective function subject to the model and operational constraints in a receding horizon fashion. One way to treat uncertainty is to use a robust MPC formulation [3–5], which provides a control law that satisfies the problem constraints for all admissible uncertain variables by assuming that the uncertainty is bounded. However, the resulting solution tends to be conservative since all uncertainty realizations are treated equally. Stochastic MPC offers an alternative approach to achieve a less conservative solution, thereby the system constraints are treated in a probabilistic sense (chance constraints), meaning that the constraints need to be satisfied only probabilistically up to a pre-assigned level to reduce the conservatism of robust MPC. An effective solution to address such problems is to employ randomized algorithms that require substituting the chance constraint with a finite number of hard constraints corresponding to samples of the uncertainty set. Randomized MPC approximates stochastic MPC via the so-called scenario approach (see [6] and the references therein), and if the underlying optimization problem is convex with respect to the decision variables, finite sample guarantees can be provided for a desired confidence level of constraint fulfillment.

In this work we cannot employ the well-known scenario approach due to the fact that the underlying problem is not convex (mixed-integer program). The main challenge here is in the presence of the uncertain thermal energy demands to compute a discrete (binary) variable vector that corresponds to the on-off status of the generating units, and a continuous variable vector that is related to the amount of thermal energy that each unit should produce to satisfy a given demand level at each sampling time. Instead, we propose a two-step procedure that is based on a mixture of randomized and robust optimization [7]. We first determine a probabilistic bounded set of uncertainties that is guaranteed to include a given percentage of uncertainty realizations. Then, we use the obtained set in a deterministic robust MPC approach. Note that the first step leads to a convex sub-problem even if the original problem contains binary variables. In this way we can have similar results in terms of confidence level of constraint fulfillment as in the standard scenario approach. Using a tractable approximation of uncertainty bounds, the deterministic robust formulation leads to a tractable problem for each sampling time. A framework for stochastic linear systems using a combination of randomization and robust optimization was introduced in [8]. In this work instead we introduce a new framework for stochastic hybrid linear systems that leads to stochastic mixed-integer optimization problems. Due to the large number of agents in a large-scale multi-agent setting as in this work, computational burden may also become an issue. In this line, distributed approaches are seen to be more suitable, and also, more flexible and scalable than the centralized counterpart [9]. We therefore propose two distributed schemes that can be applied to solve the resulting tractable problem using the alternating direction method of multipliers (ADMM). The main contributions of this work are as follows:

- A technical description of smart thermal grids with uncertainties in the consumer demand profiles as an optimization problem formulation.
- The problem formulation leads to a finite-horizon chance-constrained mixed-integer linear program at each sampling time. To solve this problem, we first formulate an auxiliary problem to obtain a bounded set for the uncertainty. Using the scenario approach, the result of this sub-problem is a subset of the uncertainty space that contains a portion of the probability mass of the uncertainty with high confidence level. We then solve a robust version of the initial problem subject to the uncertainty confined in the obtained set. Note that our method does not restrict the underlying probability distribution of uncertainty as in robust optimization methods and it is only assumed that the uncertainties are independent and identically distributed.
- To guarantee that the resulting problem is solvable, we develop a tractable scheme based on the dependency of the constraint functions on the uncertainty sets.
- Both the open-loop stochastic MPC formulation and the closed-loop affine feedback policies of stochastic MPC formulation are described and used to illustrate a performance of the proposed methodology using Monte Carlo simulations.

- We provide two distributed ADMM schemes to solve the resulting tractable optimization problem: (1) a fully distributed and (2) a distributed with coordination schemes.

It is important to highlight that this work is based on the conference paper published in [10] and thesis report in [11]. The layout of this work is as follows: Sect. 7.2 provides a general stochastic MPC framework for the problem of uncertain smart thermal grids. In Sect. 7.3 a tractable methodology is developed and probabilistic performance guarantees are provided. The distributed ADMM schemes are formulated in 7.4. In Sect. 7.5, we demonstrate the efficiency of the proposed methodology through a numerical example. Finally, Sect. 7.6 provides some concluding remarks and directions for future work.

7.2 Problem Formulation

This section provides a brief description of smart thermal grids with multiple agents that can be producers and consumers of power and heat in a smart grid setting. The goal of the agents is to match the local consumption and production to avoid transport losses in the network and improve energy efficiency.

System Description

Consider a regional thermal grid consisting of N agents (households, greenhouses). We describe the model of a single agent that is facilitated with a micro-combined heat and power plant (micro-CHP), a boiler, and a heat storage. Each agent can be both producer and consumer which is known as the prosumer concept. This model introduces the technical constraints of each agent and the coupling between such agents in the network. Moreover, for every transaction of thermal energy in the smart grid, there are several heat exchangers located near the corresponding agents and we assume that the heat exchangers do not add additional costs to the heat production for the sake of simplicity.

For a day-ahead planning production problem of each agent, we consider a finite horizon $N_t = 24$ problem with hourly steps, and introduce the subscript t in our notation to characterize the value of the quantities for a given time instance $t \in \{0, 1, \dots, N_t - 1\}$. For each sampling time t of the problem for all agents $i \in \{1, 2, \dots, N\}$, we define the main vector of control decision variables to be

$$u^m_{i,t} := [p_{\mathbf{g},t}, p_{\mathbf{ug},t}, p_{\mathbf{dg},t}, h_{\mathbf{g},t}, h_{\mathbf{b},t}, h_{\mathbf{im},t}, c^{\mathbf{su}}_{\mathbf{g},t}, c^{\mathbf{su}}_{\mathbf{b},t}, z_{\mathbf{g},t}, z_{\mathbf{b},t}]^\top \in \mathbb{R}^{10},$$

where $p_{\mathbf{g},t}$, $h_{\mathbf{g},t}$ denote the electrical power and heat production by the micro-CHP, $p_{\mathbf{ug},t}$, $p_{\mathbf{dg},t}$ relate to the up and down spinning of electrical power by the micro-CHP, $h_{\mathbf{b},t}$, $h_{\mathbf{im},t}$ correspond to the heat provided by the boiler and imported heat from external parties during period of high heat demand, $c^{\mathbf{su}}_{i,t} := [c^{\mathbf{su}}_{\mathbf{g},t}, c^{\mathbf{su}}_{\mathbf{b},t}]^\top$ is a vector that contains the startup cost of micro-CHP and boiler, and $z_{i,t} := [z_{\mathbf{g},t}, z_{\mathbf{b},t}]^\top$ are auxiliary variables needed to model the minimum up and down times of each micro-CHP and boiler, respectively. Moreover, $v_{i,t} : [v_{\mathbf{g},t}, v_{\mathbf{b},t}]^\top \in \{0, 1\}^{N_v=2}$ is a binary vector of dimension 2 and denotes the on-off status of each micro-CHP and boiler for each agent i at step t. We call the difference between the level of heat storage and the forecast of heat demand $h^{\mathbf{f}}_{\mathbf{d},t}$ the imbalance error $x_{i,t} \in \mathbb{R}^{N_x=1}$ at agent i, defined as

$$x_{i,t} = h_{\mathbf{s},t} - h^{\mathbf{f}}_{\mathbf{d},t} \ , \tag{7.1}$$

where $h_{\mathbf{s},t}$ represents the heat storage level (assuming there are no thermal losses in the conversion and storage system). The heat storage level has the following dynamics:

$$h_{\mathbf{s},t+1} = \eta_{\mathbf{s}} x_{i,t} + \eta_{\mathbf{s}} \left(h_{\mathbf{g},t} + h_{\mathbf{b},t} + h_{\mathbf{im},t} + \sum_{j \in N_{-i}} (1 - \alpha_{ij}) h^{ij}_{\mathbf{ex},\mathbf{t}} \right) ,$$

where $\eta_{\mathbf{s}} \in (0, 1)$ and $\alpha_{ij} \in (0, 1)$ denote the efficiency of storage and the heat loss coefficient due to transportation between agent i and j, respectively. N_{-i} is the set of neighbors of agent i and is given by

$$N_{-i} \subseteq \{1, 2, \ldots, N\} \backslash \{i\} \ .$$

In order for an agent to contribute to the local balancing of heat by exchanging heat with neighbors, we define an auxiliary control variable vector $u^a_{i,t} \in \mathbb{R}^{|N_{-i}|}$ with elements $h^{ij}_{\mathbf{ex},\mathbf{t}}$ denoting the exchanged heat between agent i and other adjacent agents $j \in N_{-i}$. Notice that $h^{ij}_{\mathbf{ex},\mathbf{t}}$ can have either positive or negative values depending on if agent i imports or exports heat from or to agent j, respectively. By substituting $h_{\mathbf{s},t}$ in (7.1), one can derive the dynamical behavior of imbalance $x_{i,t}$ that is given by

$$x_{i,t+1} = A_i x_{i,t} + B_i u_{i,t} + w_{i,t} \ , \tag{7.2}$$

where $B_i = \eta_{\mathbf{s}}[b_1^\top, b_2^\top]^\top$, with $b_1 = [0, 0, 0, 1, 1, 1, 0, 0, 0, 0] \in \mathbb{R}^{10}$, $b_2 \in \mathbb{R}^{|N_{-i}|}$ containing elements $(1 - \alpha_{ij})$, and $A_i = \eta_{\mathbf{s}}$. The complete vector of control decision variables is $u_{i,t} = [u^{m\top}_{i,t}, u^{a\top}_{i,t}]^\top \in \mathbb{R}^{N_{u_i}}$ with $N_{u_i} = 10 + |N_{-i}|$ for every agent i at each step t. By definition (7.1), $w_{i,t} := -h^{\mathbf{f}}_{\mathbf{d},t+1} \in \mathbb{R}^{N_w=1}$ corresponds to the forecast heat demand in the next time step. We now consider that the only uncertainty is due to the deviation of the actual heat demand from its forecast value and therefore, $w_{i,t}$ represents an uncertain parameter for every hour and for each agent.

Our goal is to find the control input $u_{i,t}$ for each agent i such that the imbalance error stays a small positive value for all steps $t \in \{0, 1, \ldots, N_t - 1\}$ at minimal production cost and satisfying physical constraints. We associate an economic linear cost function with each agent i at step t as

$$J_i(u_{i,t}) = c_i^\top u_{i,t}^m \ , \tag{7.3}$$

where c_i is a cost vector and is defined as

$$c_i := [c_{\mathbf{gas}}\eta_{\mathbf{CHP}}^{-1}, c_{\mathbf{up}}, c_{\mathbf{dp}}, 0, c_{\mathbf{b}}\eta_{\mathbf{b}}^{-1}, c_{\mathbf{im}}, 1, 1, 0, 0]^\top \in \mathbb{R}^{10}.$$

$c_{\mathbf{gas}}$ relates to the cost of natural gas that is used by micro-CHP and $\eta_{\mathbf{CHP}}, \eta_{\mathbf{b}} \in (0, 1)$ are the efficiency of a micro-CHP and a boiler at each step, respectively. $c_{\mathbf{up}}, c_{\mathbf{dp}}$ denote the cost of up and down spinning electrical power production by micro-CHP, respectively. We define $p_{\mathbf{ug},t}, p_{\mathbf{dg},t}$ to be up and down spinning variables that are related to the amount of surplus and needed electrical power, respectively, in each agent at each step with respect to the local power demand. The cost of heat generated by a boiler is $c_{\mathbf{b}}$, and the cost of imported heat from an external party is $c_{\mathbf{im}}$. The seventh and eighth entry of c_i represent the start-up costs. The cost associated with the thermal energy produced by the micro-CHP is considered to be zero due to the fact that the electrical power and thermal energy generated by a micro-CHP are coupled by $h_{\mathbf{g},t} = \frac{\eta_{\mathbf{h}}}{\eta_{\mathbf{p}}} p_{\mathbf{g},t}$ where $\eta_{\mathbf{h}}, \eta_{\mathbf{p}} \in (0, 1)$ are the efficiency of a micro-CHP for production of thermal energy and electrical power, respectively.

The resulting optimization problem for an agent i is given by

$$\min_{\{u_{i,t}, v_{i,t}\}_{t=0}^{N_t-1}} \sum_{t=0}^{N_t-1} J_i(u_{i,t}) \tag{7.4a}$$

subject to:

1. Startup cost constraints for $t = 0, 1, \ldots, N_t - 1$:

$$c_{i,t}^{su} \geq \Lambda^{su}(v_{i,t} - v_{i,t-1}) \ , \ c_{i,t}^{su} \geq 0 \ , \tag{7.4b}$$

where Λ^{su} is a diagonal matrix including the startup costs of each micro-CHP and boiler.

2. Production and transportation capacity constraints for $t = 0, 1, \ldots, N_t - 1$:

$$v_{\mathbf{g},t} p_{\mathbf{g}}^{\min} \leq p_{\mathbf{g},t} \leq p_{\mathbf{g}}^{\max} v_{\mathbf{g},t} \ , \tag{7.4c}$$

$$v_{\mathbf{g},t} h_{\mathbf{g}}^{\min} \leq h_{\mathbf{g},t} \leq h_{\mathbf{g}}^{\max} v_{\mathbf{g},t} \ , \tag{7.4d}$$

$$v_{\mathbf{b},t} h_{\mathbf{b}}^{\min} \leq h_{\mathbf{b},t} \leq h_{\mathbf{b}}^{\max} v_{\mathbf{b},t} \ , \tag{7.4e}$$

$$h_{\mathbf{im},t}^{\min} \leq h_{\mathbf{im},t} \leq h_{\mathbf{im},t}^{\max} \ , \tag{7.4f}$$

$$h_{\mathbf{ex},t}^{\min} \leq h_{\mathbf{ex},t}^{ij} \leq h_{\mathbf{ex},t}^{\max} \ , \ \forall i, j \in N_{-i} \tag{7.4g}$$

where $p_{\mathbf{g}}^{\min}, p_{\mathbf{g}}^{\max}$ denote the minimum and maximum electrical power production capacities of each micro-CHP, $h_{\mathbf{g}}^{\min}, h_{\mathbf{g}}^{\max}$ relate to the minimum and maximum heat production capacities of each micro-CHP, $h_{\mathbf{b}}^{\min}, h_{\mathbf{b}}^{\max}$ are the minimum and maximum heat production capacities of each boiler, $h_{\mathbf{im}}^{\min}, h_{\mathbf{im}}^{\max}$ are the minimum and maximum available heat capacities of each external party, $h_{\mathbf{ex}}^{\min}$, $h_{\mathbf{ex}}^{\max}$ represent the minimum and maximum transportation capacities of neighbors.

3. Balance constraints for heat exchanged with neighbors for $t = 0, 1, \ldots, N_t - 1$:

$$h_{\mathbf{ex},t}^{ij} + h_{\mathbf{ex},t}^{ji} = 0 \; . \; \forall i, j \in N_{-i} \tag{7.4h}$$

4. Up and down spinning electrical power constraints for $t = 0, 1, \ldots, N_t - 1$:

$$-p_{\mathbf{dg},t} \leq p_{\mathbf{g},t} - p_{\mathbf{d},t} \leq p_{\mathbf{ug},t}, \;\; p_{\mathbf{ug},t} \geq 0, \, p_{\mathbf{dg},t} \geq 0, \tag{7.4i}$$

where $p_{\mathbf{d},t}$ is a local electrical power demand for each agent $i \in \{1, \ldots, N\}$.

5. Ramping capacity constraint for all $t = 0, 1, \ldots, N_t - 1$:

$$-p_{\mathbf{g}}^{\mathbf{down}} \leq p_{\mathbf{g},t} - p_{\mathbf{g},t-1} \leq p_{\mathbf{g}}^{\mathbf{up}} \; , \tag{7.4j}$$

where $p_{\mathbf{g}}^{\mathbf{down}}$, $p_{\mathbf{g}}^{\mathbf{up}}$ denote the down and up capacity of decreasing and increasing electrical power of a micro-CHP within two consecutive periods, respectively. Note that this constraint is considered just for the electrical power of micro-CHP due to the fact that heat can be produced within each step.

6. Status change constraints for $t = 0, 1, \ldots, N_t - 1$:

$$\begin{aligned} & z_{i,t} \geq v_{i,t} - v_{i,t-1} \; , \; z_{i,t} \geq 0 \; , \\ & \sum_{\tau=t+1-\Delta t_{\mathbf{up}}}^{t} z_{i,\tau} \leq v_{i,t} \; , \forall t \in \{\Delta t_{\mathbf{up}}, \ldots, N_t - 1\} \; , \\ & \sum_{\tau=t+1}^{t+\Delta t_{\mathbf{down}}} z_{i,\tau} \leq 1 - v_{i,t} \; , \forall t \in \{1, \ldots, N_t - 1 - \Delta t_{\mathbf{down}}\} \; , \end{aligned} \tag{7.4k}$$

where $\Delta t_{\mathbf{up}}, \Delta t_{\mathbf{down}} \in \mathbb{R}_+$ denote the minimum time an agent needs to change status of the micro-CHP and the boiler.

7. Probabilistic constraint:

$$\mathbb{P}(x_{i,t+1} \geq 0 \; , \; \forall t \in \{0, 1, \ldots, N_t - 1\}) \geq 1 - \epsilon \; , \tag{7.4l}$$

where $\epsilon \in (0, 1)$ is the admissible constraint violation parameter. Given $x_{i,0}$ is equal to $x_i(k)$ which is the current state measurement at real time k. This constraint implies that the imbalance error should be a positive value at minimum production cost for all heat demand realizations with high probability $1 - \epsilon$.

The proposed optimization problem (7.4) is a finite-horizon multi-stage, chance-constrained mixed-integer linear program, whose stages are coupled by the startup binary (7.4b), ramping (7.4j), status change (7.4k) and imbalance error (7.4l) constraints.

Open-Loop Stochastic MPC

In order to formulate a stochastic MPC for the overall smart thermal grid imbalance problem, we first extend the optimization problem (7.4) for all agents in the grid. Let us define $X_t := [x_{1,t}^\top, \ldots, x_{N,t}^\top]^\top \in \mathbb{R}^{N_x N}$, $U_t := [u_{1,t}^\top, \ldots, u_{N,t}^\top]^\top \in \mathbb{R}^{N_u}$, where $N_u = \sum_{i=1}^{N} N_{u_i}$, and $V_t := [v_{1,t}^\top, \ldots, v_{N,t}^\top]^\top \in \mathbb{R}^{N_v N}$ to be the state, control input and binary variables of the grid, respectively. We also define $W_t := [w_{1,t}^\top, \ldots, w_{N,t}^\top]^\top \in \mathbb{R}^{N_w N}$ to be an uncertainty vector of all the agents. The grid cost $\boldsymbol{J}(U_t)$ at step t is assumed to be the sum of the individual costs for all agents

$$\boldsymbol{J}(U_t) = C^\top U_t = \sum_{i=1}^{N} J_i(u_{i,t}) ,$$

where $C := [c_1^\top, \ldots, c_N^\top]^\top$. The dynamics of the imbalance error of all agents in the grid can be expressed as

$$X_{t+1} = AX_t + BU_t + W_t , \tag{7.5}$$

where $A = \text{diag}(A_1, \ldots, A_N)$ and $B = \text{diag}(B_1, \ldots, B_N)$. The uncertain variable vector $W_t \in \mathbb{R}^N$ is defined on a probability space Δ. It is assumed that Δ is endowed with the Borel $\sigma-$algebra and $\mathbb{P}$ is a probability measure defined over Δ. It is important to note that for our study we only need a finite number of instances of W_t, and we do not require the probability space Δ and the probability measure $\mathbb{P}$ to be known explicitly.

To illustrate the advantages obtained by adopting the policies that were discussed at the end of the preceding section, we first need to introduce some compact notations for the overall system dynamics evolution along the finite time horizon. Consider the following vectors of state, control input, binary variables, and uncertainty parameter matrices.

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{N_t} \end{bmatrix}, \; \mathbf{U} = \begin{bmatrix} U_0 \\ U_1 \\ \vdots \\ U_{N_t-1} \end{bmatrix}, \; \mathbf{V} = \begin{bmatrix} V_0 \\ V_1 \\ \vdots \\ V_{N_t-1} \end{bmatrix}, \; \mathbf{W} = \begin{bmatrix} W_0 \\ W_1 \\ \vdots \\ W_{N_t-1} \end{bmatrix}.$$

The imbalance error dynamics for all agents over the prediction horizon can be now written as

$$\mathbf{X} = \mathbf{A}X_0 + \mathbf{B}\mathbf{U} + \mathbf{H}\mathbf{W} ,$$

where

$$\mathbf{A} = \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^{N_t} \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} B & 0 & \cdots & 0 \\ AB & B & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 \\ A^{N_t-1}B & \cdots & AB & B \end{bmatrix}, \ \mathbf{H} = \begin{bmatrix} I & 0 & \cdots & 0 \\ A & I & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 \\ A^{N_t-1} & \cdots & A & I \end{bmatrix}.$$

The initial state values are defined by $X_0 := X(k) = [x_1^\top(k), \ldots, x_N^\top(k)]^\top \in \mathbb{R}^{N_x N}$ and it is assumed that the current state measurement at real time k is given to each agent together with the the forecasted thermal energy demand for the next day step-wise. The objective function can be expressed by $\mathbf{J}(\mathbf{U}) = \mathbf{C}^\top \mathbf{U}$, where $\mathbf{C} = \mathbf{1}^{N_t} \bigotimes C$ using the Kronecker product.

We are now in a position to define the optimization problem for the overall smart thermal grid as follows:

$$\min_{\mathbf{U}} \quad \mathbf{J}(\mathbf{U}) \tag{7.6a}$$

$$\text{s.t.} \quad \mathbb{P}_{\mathbf{W}}(\mathbf{A}X_0 + \mathbf{BU} + \mathbf{HW} \geq 0) \geq 1 - \epsilon \ , \ X_0 = X(k) \, , \tag{7.6b}$$

$$\mathbf{EU} + \mathbf{FV} + \mathbf{P} \leq 0 \ , \ \mathbf{W} \in \Delta^N \tag{7.6c}$$

where $\mathbf{E}$, $\mathbf{F}$ and $\mathbf{P}$ are matrices of appropriate dimensions. Notice that $\mathbb{P}_{\mathbf{W}}$ depends on the string of uncertain scenario realizations. The solution of (7.6) is the optimal planned input sequence $\{U_0^*, V_0^*, \ldots, U_{N_t-1}^*, V_{N_t-1}^*\}$. Based on the MPC paradigm the current input is set to $\{U(k), V(k)\} := \{U_0^*, V_0^*\}$ and we proceed in a receding horizon fashion. This means (7.6) is solved at each step t by using the current measurement of the state $X(k)$. Due to the presence of chance constraints, the feasible set is in general non-convex and hard to determine explicitly. We describe a tractable formulation to solve (7.6) by using robust randomization techniques in Sect. 7.3.

Closed-Loop Stochastic MPC

In the presence of uncertainty, the problem of finding the optimal state feedback policy becomes quite challenging. One way to tackle this problem is to look for a sub-optimal solution by parameterizing the control input variables. As a first approach, one can directly parameterize the control input variables as an affine function of the uncertainty

$$U_t = \overline{U}_t + \sum_{j=0}^{t-1} \theta_{t,j} W_j \ ,$$

where $\overline{U}_t$ and $\theta_{t,j}$ are optimization variables. In this way the designed closed-loop control system is equivalent to an open-loop control system with a feedforward uncertainty compensator [12, 13]. Consider the following finite-horizon chance-constrained mixed-integer linear program by adopting a feedback control policy (7.7d) that is affine in the uncertainty samples.

$$\min_{\overline{\mathbf{U}},\mathbf{G},\mathbf{V}} \quad \mathbf{J}(\mathbf{U}) \tag{7.7a}$$

$$\text{s.t.} \quad \mathbb{P}_{\mathbf{W}}(\mathbf{A}X_0 + \mathbf{BU} + \mathbf{HW} \geq 0) \geq 1 - \epsilon\ , \ \ X_0 = X(k)\,, \tag{7.7b}$$

$$\mathbf{EU} + \mathbf{FV} + \mathbf{P} \leq 0\ , \tag{7.7c}$$

$$\mathbf{U} = \overline{\mathbf{U}} + \mathbf{GW}\ , \ \ \mathbf{W} \in \Delta^N \tag{7.7d}$$

where matrices $\overline{\mathbf{U}}$ and $\mathbf{G}$ are given by

$$\overline{\mathbf{U}} = \begin{bmatrix} \overline{U}_0 \\ \overline{U}_1 \\ \vdots \\ \overline{U}_{N_t-1} \end{bmatrix}, \ \mathbf{G} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \theta_{1,0} & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 \\ \theta_{N_t-1,0} & \cdots & \theta_{N_t-1,N_t-2} & 0 \end{bmatrix}.$$

Notice that each element of $\mathbf{G}$ has dimension $\mathbb{R}^{N_u \times N_w N}$.

To solve (7.6) and (7.7), we have to transform the chance-constrained problem to a tractable one without introducing any assumptions on $\mathbb{P}$ and its moments. Hence, we follow a randomization-based approach. The proposed procedure in [6] is called scenario approach that allows to substitute the chance constraints with a finite number of hard constraints corresponding to scenarios of the uncertainty and provides a probabilistic guarantee, if the underlying problem is convex with respect to the decision variables. The number of scenarios of the uncertainty realizations N_s that needs to be extracted must satisfy

$$N_s \geq \frac{2}{\epsilon}\left(d + \ln\frac{1}{\beta}\right), \tag{7.8}$$

where $\epsilon \in (0, 1)$ is a desired level of constraint violations, d is the number of decision variables and $\beta \in (0, 1)$ is a desired confidence level with which the drawn scenarios lead to a feasible solution [14, 15]. Unfortunately, we cannot follow this approach, due to the binary vector $\mathbf{V}$. Even if the convexity condition were satisfied, the number of scenarios that one needs to generate grows linearly with the number of decision variables, thus hampering the applicability of the method to large-scale problems [8]. For example, the number of decision variables in the proposed open-loop stochastic MPC formulation (7.6) is $d = (N_u + N_v N)N_t$, and the number of decision variables in the proposed closed-loop affine uncertainty feedback policy stochastic MPC formulation (7.7) is $d + d_G$, where $d_G = N_u N_w N \frac{(N_t-1)N_t}{2}$. Due to the high dimension of decision space, we cannot even employ the extensions to non-convex problems in [17]. To overcome this difficulty, we propose a tractable methodology based on the results of [7] in Sect. 7.3.

7.3 Centralized Stochastic MPC

In this section, we use the results in [3] to approximate the chance constraints that appear in the proposed formulations (7.6) and (7.7). We then develop a tractable methodology to reformulate the proposed robust formulations. The approximation is done in a way to provide a feasible solution for all scenarios of the uncertainty realizations with probabilistic guarantees. In the first step, a bounded set that contains the uncertainty realizations with a specific probability of violations is constructed. We then formulate a robust optimization problem with respect to that set and show that the solution is guaranteed to be feasible for the initial chance constrained problems (7.6) and (7.7) with the desired level of confidence.

Randomization-Based Reformulation

Define $\mathcal{B}_i(\gamma)$ to be a bounded set of uncertainty realizations and we assume that it is an axis-aligned hyper-rectangle for each agent i. Note that the choice of a hyper-rectangle is not restrictive and any convex set with convex volume could have been chosen instead [7]. We parametrize $\mathcal{B}_i(\gamma) := \times_{t=0}^{N_t-1}[\underline{\gamma}_t, \overline{\gamma}_t]$ by $\gamma = (\underline{\gamma}, \overline{\gamma}) \in \mathbb{R}^{2N_t}$, where $\underline{\gamma} = (\underline{\gamma}_0, \ldots, \underline{\gamma}_{N_t-1}) \in \mathbb{R}^{N_t}$ and $\overline{\gamma} = (\overline{\gamma}_0, \ldots, \overline{\gamma}_{N_t-1}) \in \mathbb{R}^{N_t}$. Consider now the following chance-constrained optimization problem

$$\begin{aligned} \min_{\gamma} \quad & \sum_{t=0}^{N_t-1} \overline{\gamma}_t - \underline{\gamma}_t \\ \text{s.t.} \quad & \mathbb{P}(w_i \in \Delta \mid w_{i,t} \in [\underline{\gamma}_t, \overline{\gamma}_t],\ \forall t \in \{0, \ldots, N_t - 1\}) \geq 1 - \epsilon. \end{aligned} \tag{7.9}$$

By construction the problem (7.9) is a convex program and we can apply the standard scenario approach to obtain a solution as follows.

$$\begin{aligned} \min_{\gamma} \quad & \sum_{t=0}^{N_t-1} \overline{\gamma}_t - \underline{\gamma}_t \\ \text{s.t.} \quad & w_{i,t}^j \in [\underline{\gamma}_t, \overline{\gamma}_t]\,, \begin{cases} \forall t \in \{0, \ldots, N_t - 1\} \\ \forall j \in \{1, \ldots, N_s\} \end{cases}, \end{aligned} \tag{7.10}$$

where N_s is the required number of scenarios (7.8) for each agent $i \in \{1, \ldots, N\}$ with $d = N_t N$. The optimal solution of (7.10) γ^* is a feasible solution for the problem (7.9) with confidence $1 - \beta$.

Determine $\mathcal{B}_i(\gamma^*)$ for all i and define $\mathfrak{B}^* := \{\mathcal{B}_1(\gamma^*), \ldots, \mathcal{B}_N(\gamma^*)\}$ and pose the robust counterpart of the problems (7.6)–(7.7) where $\mathbf{W} \in \mathfrak{B}^* \cap \Delta^N$. Note that the robust counterparts of (7.6)–(7.7) are not randomized programs and instead, they are finite-horizon robust mixed-integer linear problems where the constraints have to be satisfied for all values of the uncertainty inside $\mathfrak{B}^* \cap \Delta^N$. It is worth to mention that any feasible solution of the robust counterparts of (7.6)–(7.7) is a feasible solution for the problems (7.6) and (7.7) with at least confidence of $1 - \beta$.

The robust counterpart problems are tractable and equivalent to mixed-integer linear programs, since the uncertainty is bounded in a convex set [18]. It is shown in [18] that the robust problems are tractable and remain in the same class as the original problems, e.g. robust mixed-integer programs remain mixed-integer programs, for a certain class of uncertainty sets. This is achieved under the assumptions that the constraint functions are linear and homogeneous with respect to the uncertainty vector. In the sequel, we describe a tractable scheme for the robust counterparts of (7.6)–(7.7).

Tractable Robust Reformulation

Following the methodology outlined in the previous section, we first define $\boldsymbol{\gamma^o} := [\gamma_0^o, \gamma_1^o, \dots, \gamma_{N_t-1}^o] \in \mathbb{R}^{NN_t}$ to be a vector whose elements are the middle points of the hyper-rectangle $\mathfrak{B}^*$ and is defined as $\boldsymbol{\gamma^o} = 0.5(\overline{\boldsymbol{\gamma}}^* + \underline{\boldsymbol{\gamma}}^*)$ and each element of $\boldsymbol{\gamma^o}$ represents a vector for all agents $i \in \{1, 2, \dots, N\}$. Consider now the following tractable reformulations of the proposed robust counterpart of problems (7.6) and (7.7).

$$\min_{\mathbf{U},\mathbf{V}} \quad \mathbf{J}(\mathbf{U}) \tag{7.11a}$$

$$\text{s.t.} \quad \mathbf{A}X_0 + \mathbf{B}\mathbf{U} + \mathbf{H}\boldsymbol{\gamma^o} + \boldsymbol{\eta} \geq 0\,, \quad X_0 = X(k)\,, \tag{7.11b}$$

$$\mathbf{E}\mathbf{U} + \mathbf{F}\mathbf{V} + \mathbf{P} \leq 0\,, \tag{7.11c}$$

where $\boldsymbol{\eta} := [\eta_0, \eta_1, \dots, \eta_{N_t-1}] \in \mathbb{R}^{NN_t}$ is a vector with each element $\eta_t \in \mathbb{R}^N$ denoting a bound for the worst-case uncertainty realizations at step t for all agents $i \in \{1, 2, \dots, N\}$. We refer to Proposition 7.1 below that shows how to achieve this bound. We next present a tractable reformulation of the proposed robust counterpart of problem (7.7).

$$\min_{\overline{\mathbf{U}},\mathbf{G},\mathbf{V}} \quad \mathbf{J}(\mathbf{U}) \tag{7.12a}$$

$$\text{s.t.} \quad \mathbf{A}X_0 + \mathbf{B}\mathbf{U} + \mathbf{H}\boldsymbol{\gamma^o} + \boldsymbol{\eta} \geq 0\,, \quad X_0 = X(k)\,, \tag{7.12b}$$

$$\mathbf{E}\mathbf{U} + \mathbf{F}\mathbf{V} + \mathbf{P} \leq 0\,, \tag{7.12c}$$

$$\mathbf{U} = \overline{\mathbf{U}} + \mathbf{G}\boldsymbol{\gamma^o} + \boldsymbol{\eta_g}\,, \tag{7.12d}$$

$$\eta_{g,t} \leq [\mathbf{G}(\overline{\boldsymbol{\gamma}}^* - \boldsymbol{\gamma^o})]_t\,, \forall t \in \{0, 1, \dots, N_t - 1\}, \tag{7.12e}$$

$$\eta_{g,t} \leq [\mathbf{G}(\underline{\boldsymbol{\gamma}}^* - \boldsymbol{\gamma^o})]_t\,, \forall t \in \{0, 1, \dots, N_t - 1\}, \tag{7.12f}$$

where $\boldsymbol{\eta_g} := [\eta_{g,0}, \eta_{g,1}, \dots, \eta_{g,N_t-1}] \in \mathbb{R}^{NN_t}$ is a vector with each element $\eta_{g,t} \in \mathbb{R}^N$. The following proposition shows the link between the tractable problems (7.11) and (7.12), and the proposed formulations (7.6) and (7.7), respectively.

Proposition 7.1 *If the tractable problems (7.11) and (7.12) have an optimal solution, where $\boldsymbol{\eta}$ is obtained by solving the following problem*

$$\max_{\boldsymbol{\eta}\in\mathbb{R}^{NN_t}} \ \boldsymbol{\eta}$$
$$s.t.\ \eta_t \leq [\mathbf{H}(\overline{\boldsymbol{\gamma}}^* - \boldsymbol{\gamma}^{\boldsymbol{o}})]_t \ , \forall t \in \{0, 1, \ldots, N_t - 1\},$$
$$\eta_t \leq [\mathbf{H}(\underline{\boldsymbol{\gamma}}^* - \boldsymbol{\gamma}^{\boldsymbol{o}})]_t \ , \forall t \in \{0, 1, \ldots, N_t - 1\}, \tag{7.13}$$

then it is a feasible solution for the chance-constrained problems (7.6) and (7.7) with at least $1 - \beta$ *confidence level, respectively.*

Proof It is shown in [19, Proposition 1] that any feasible solution of the robust counterparts of (7.6)–(7.7) is a feasible solution of the initial chance-constrained problems (7.6) and (7.7), respectively. Therefore, we have to show that the proposed tractable problems (7.11) and (7.12) are equivalent with the robust counterparts of (7.6)–(7.7). Consider the following robust constraint,

$$0 \leq \mathbf{A}X_0 + \mathbf{BU} + \mathbf{HW} \ , \ \forall \mathbf{W} \in \mathfrak{B}^* \cap \Delta^N \ ,$$

that can be written in an equivalent format using the linearity and homogeneity assumption of the constraint with respect to the uncertainty, leading to

$$\begin{aligned} 0 \leq & \mathbf{A}X_0 + \mathbf{BU} + \mathbf{H}(\boldsymbol{\gamma}^{\boldsymbol{o}} + \Delta\boldsymbol{\gamma}) = \\ & \mathbf{A}X_0 + \mathbf{BU} + \mathbf{H}\boldsymbol{\gamma}^{\boldsymbol{o}} + \mathbf{H}\Delta\boldsymbol{\gamma} \ , \ \forall \Delta\boldsymbol{\gamma} \in [\underline{\boldsymbol{\gamma}}^* - \boldsymbol{\gamma}^{\boldsymbol{o}}, \boldsymbol{\gamma}^{\boldsymbol{o}} - \overline{\boldsymbol{\gamma}}^*] \ . \end{aligned}$$

We need to introduce the vector $\boldsymbol{\eta} := [\eta_0, \eta_1, \ldots, \eta_{N_t-1}] \in \mathbb{R}^{NN_t}$ with each element $\eta_t \in \mathbb{R}^N$ representing a bound for $\mathbf{H}\Delta\boldsymbol{\gamma}$. Consider now the worst-case uncertainty realizations to be $(\overline{\boldsymbol{\gamma}}^* - \boldsymbol{\gamma}^{\boldsymbol{o}})$ and $(\underline{\boldsymbol{\gamma}}^* - \boldsymbol{\gamma}^{\boldsymbol{o}})$. We pose the problem (7.13) to find bound $\boldsymbol{\eta}$. Using this bound leads to

$$\begin{aligned} 0 &\leq \mathbf{A}X_0 + \mathbf{BU} + \mathbf{H}\boldsymbol{\gamma}^{\boldsymbol{o}} + \boldsymbol{\eta} \\ &\leq \mathbf{A}X_0 + \mathbf{BU} + \mathbf{H}\boldsymbol{\gamma}^{\boldsymbol{o}} + \mathbf{H}\Delta\boldsymbol{\gamma} \ = \ \mathbf{A}X_0 + \mathbf{BU} + \mathbf{H}(\boldsymbol{\gamma}^{\boldsymbol{o}} + \Delta\boldsymbol{\gamma}). \end{aligned}$$

The proof is completed. ■

Remark 7.1 Note that we can use the same approach by introducing $\eta_{g,t}$ to be the worst-case superposition of the uncertainty realizations with the following constraints:

$$\eta_{g,t} \leq \sum\nolimits_{j=0}^{t-1} \theta_{t,j}[(\overline{\boldsymbol{\gamma}}^* - \boldsymbol{\gamma}^{\boldsymbol{o}})]_j \ , \ \eta_{g,t} \leq \sum\nolimits_{j=0}^{t-1} \theta_{t,j}[(\underline{\boldsymbol{\gamma}}^* - \boldsymbol{\gamma}^{\boldsymbol{o}})]_j \ .$$

Robust Randomized Model Predictive Control

The proposed procedure of a robust randomized MPC is summarized in Algorithm 4. We compare our proposed methodology to illustrate its performance against a hybrid approach as a benchmark, where the generating unit status problem is solved deterministically, meaning that we initialize $\boldsymbol{\gamma}^{\boldsymbol{o}} \equiv \mathbf{W}^{\text{forecast}}$, $\boldsymbol{\eta} \equiv 0$ in (7.11), (7.12) with the forecast value of the energy demand and solve the deterministic variant of the problems. At the next step, we fix the on-off status of the generating units (and also the startup cost and auxiliary variables) to the binary vector computed by the previous deterministic program, and formulate a stochastic production planning problem. We refer to this as the Benchmark approach and the steps are summarized in Algorithm 5.

Algorithm 4 Robust Randomized MPC

1: Fix $X_0 = X(k)$, $\epsilon \in (0, 1)$, $\beta \in (0, 1)$ ▷ the initial (current) state measurement, level of constraint violations and confidence level of the agents, respectively.
2: Generate N_s scenarios (7.8) with $d = 2N_t N$ and establish $\mathfrak{B}^*$ by solving the optimization problem (7.10).
Open-loop
3: Solve (7.11) and determine an optimal solution $\mathbf{U}^*$, $\mathbf{V}^*$. Apply the first optimal solution $U(k) := U_0^\star$, $V(k) := V_0^\star$ to the STG agents.
Affine Uncertainty Feedback
4: Solve (7.12) and determine an optimal solution $\overline{\mathbf{U}}^*$, $\mathbf{G}^*$, $\mathbf{V}^*$. Apply the first optimal solution $U(k) := U_0^\star + [\mathbf{G}^*\boldsymbol{\gamma}^{\boldsymbol{o}} + \boldsymbol{\eta}_{\boldsymbol{g}}]_0$, $V(k) := V_0^\star$ to the STG agents.
5: $k \leftarrow k + 1$
6: Go to step 1.

Algorithm 5 Benchmark Approach

Deterministic Generating Unit Status
1: Fix $X_0 = X(k)$ and $\boldsymbol{\gamma}^{\boldsymbol{o}} \equiv \mathbf{W}^{\text{forecast}}$, $\boldsymbol{\eta} \equiv 0$ ▷ the initial (current) state measurement and no heat demand prediction error.
Open-loop
2: Solve (7.11) and determine an optimal solution $\mathbf{V}^*_{\text{OLP}}$.
Affine Uncertainty Feedback
3: Solve (7.12) and determine an optimal solution $\mathbf{V}^*_{\text{AUF}}$.
Stochastic Production Planning
4: Fix $\epsilon \in (0, 1)$, $\beta \in (0, 1)$ and consider $\boldsymbol{\gamma}^{\boldsymbol{o}} \equiv \mathbf{W}^{(j)}$, $\boldsymbol{\eta} \equiv 0$, $j = 1, \ldots, N_s$ ▷ level of constraint violations and confidence, respectively.
Open-loop
5: Generate N_s scenarios (7.8) with $d = N_u N_t$.
6: Fix $V(k) = V^*_{\text{OLP},0} \rightarrow$ Solve (7.11) and determine an optimal solution $\mathbf{U}^*$. Apply the first optimal solution $U(k) := U_0^\star$ to the STG agents. Go to step 2.
Affine Uncertainty Feedback
7: Generate N_s scenarios (7.8) with $d = N_t(Nu + N_w N(N_t - 1)/2)$.
8: Fix $V(k) = V^*_{\text{AUF},0}$ and solve (7.12) and determine an optimal solution $\overline{\mathbf{U}}^*$, $\mathbf{G}^*$. Apply the first optimal solution $U(k) := \overline{U}_0^\star + [\mathbf{G}^*\boldsymbol{\gamma}^{\boldsymbol{o}} + \boldsymbol{\eta}_{\boldsymbol{g}}]_0$ to the STG agents.
9: $k \leftarrow k + 1$
10: Go to step 1.

Remark 7.2 The proposed framework solves a stochastic mixed-integer program and it does not necessarily lead to a less conservative approach than the direct scenario approach [14], due to the fact that the number of required scenarios (7.8) is a function of the dimension of the decision variables. The decision variable size in our framework is proportional to the uncertainty dimension and in case of high uncertainty dimension, the advantage of our solution comes at the expense of a more conservative performance.

7.4 Distributed Stochastic MPC

In this section, we formulate two distributed approaches that are based on the alternating direction method of multipliers (ADMM) [20] to solve the robust randomized MPC (Algorithm 4). Due to the existence of the balance constraints (7.4h), problem (7.11) is not trivially separable. Therefore, we decompose the problem by considering the dual problem that is associated with problem (7.11). Furthermore, the ADMM approach is considered since the cost function is not strictly convex. In this section we present two ADMM formulations, which are a fully distributed scheme and a distributed scheme with a coordinator. Note that due to space constraints, we only address the distributed formulation of (7.11). However, problem (7.12) can also be solved in a distributed fashion with the same approach. Based on the proposed distributed formulations, Steps 3 and 4 in Algorithm 4 can be done in distributed manner.

Fully Distributed Scheme

Problem (7.11) can be expressed in a compact form as

$$\min_{\{\tilde{\boldsymbol{u}}_i,\tilde{\boldsymbol{v}}_i\}_{i=1}^N} \quad \sum_{i=1}^{N} \boldsymbol{J}_i(\tilde{\boldsymbol{u}}_i) \tag{7.14a}$$

$$\text{subject to} \quad \tilde{\boldsymbol{u}}_i \in \mathcal{L}_{u,i}, \quad \tilde{\boldsymbol{v}}_i \in \mathcal{L}_{v,i}, \tag{7.14b}$$

$$\tilde{\boldsymbol{u}}_i^a + \sum_{j \in N_{-i}} \mathbf{G}_{ij}\tilde{\boldsymbol{u}}_j^a = 0, \quad \forall i \in \{1, \ldots, N\}, \tag{7.14c}$$

where for each agent i, $\boldsymbol{J}_i(\tilde{\boldsymbol{u}}_i) = \sum_{t=0}^{N_t-1} J_i(u_{i,t})$, $\tilde{\boldsymbol{u}}_i = \left[u_{i,0}^\top, \ldots, u_{i,N_t-1}^\top\right]^\top$, $\tilde{\boldsymbol{v}}_i = \left[v_{i,0}^\top, \ldots, v_{i,N_t-1}^\top\right]^\top$, and $\tilde{\boldsymbol{u}}_i^a = \left[u_{i,0}^{a\top}, \ldots, u_{i,N_t-1}^{a\top}\right]^\top$. Furthermore, $\mathcal{L}_{u,i}$ and $\mathcal{L}_{v,i}$ are the sets defined by the local constraints, i.e., (7.4b)–(7.4g), (7.4i)–(7.4l). Additionally, (7.14c) represents the coupling constraints, where $\mathbf{G}_{ij}$, for each $j \in N_{-i}$, is defined appropriately according to (7.4h). We therefore can reformulate the problem as:

$$\min_{\{\tilde{\boldsymbol{u}}_i,\tilde{\boldsymbol{y}}_i,\tilde{\boldsymbol{v}}_i\}_{i=1}^N} \quad \sum_{i=1}^{N} \boldsymbol{J}_i(\tilde{\boldsymbol{u}}_i) + \boldsymbol{\psi}_i(\tilde{\boldsymbol{y}}_i) \tag{7.15a}$$

$$\text{subject to} \quad \tilde{\boldsymbol{u}}_i \in \mathcal{L}_{u,i}, \quad \tilde{\boldsymbol{v}}_i \in \mathcal{L}_{v,i}, \tag{7.15b}$$

$$\tilde{\boldsymbol{u}}_i^a - \tilde{\boldsymbol{y}}_i = 0, \quad \forall i \in \{1, \ldots, N\}\,, \tag{7.15c}$$

where $\boldsymbol{\psi}_i(\tilde{\boldsymbol{y}}_i)$ corresponds to a convex indicator function such that $\psi_i(\tilde{\boldsymbol{y}}_i) = 0$ if $\tilde{\boldsymbol{y}}_i = -\sum_{j\in N_{-i}} \mathbf{G}_{ij}\tilde{\boldsymbol{u}}_j^a$ and $\psi_i(\tilde{\boldsymbol{y}}_i) = +\infty$ otherwise. Moreover, note that $\tilde{\boldsymbol{y}}_i = [\boldsymbol{y}_{i,0}^\top, \ldots, \boldsymbol{y}_{i,N_t-1}^\top]^\top \in \mathbb{R}^{N_t|N_{-i}|}$ for all $i \in \{1, \ldots, N\}$, are auxiliary variables. Consider the augmented Lagrangian of this problem as follows:

$$\begin{aligned}\boldsymbol{L}_\rho &= \sum_{i=1}^{N} \Big(\boldsymbol{J}_i(\tilde{\boldsymbol{u}}_i) + \boldsymbol{\psi}_i(\tilde{\boldsymbol{y}}_i) + \tilde{\boldsymbol{\lambda}}_i^\top(\tilde{\boldsymbol{u}}_i^a - \tilde{\boldsymbol{y}}_i) + \frac{\rho}{2}\|\tilde{\boldsymbol{u}}_i^a - \tilde{\boldsymbol{y}}_i\|_2^2 \Big) \\ &= \sum_{i=1}^{N} \boldsymbol{L}_{\rho,i}(\tilde{\boldsymbol{u}}_i, \tilde{\boldsymbol{y}}_i, \tilde{\boldsymbol{\lambda}}_i).\end{aligned}$$

where $\tilde{\boldsymbol{\lambda}}_i = [\boldsymbol{\lambda}_{i,0}^\top, \ldots, \boldsymbol{\lambda}_{i,N_t-1}^\top]^\top \in \mathbb{R}^{N_t|N_{-i}|}$ for all $i \in \{1, \ldots, N\}$ are the Lagrange multipliers and $\rho > 0$ is the penalty parameter. Hence, the augmented dual problem associated with Problem (7.15) is

$$\max_{\{\tilde{\boldsymbol{\lambda}}_i\}_{i=1}^N} \min_{\{\tilde{\boldsymbol{u}}_i, \tilde{\boldsymbol{y}}_i, \tilde{\boldsymbol{v}}_i\}_{i=1}^N} \sum_{i=1}^{N} \boldsymbol{L}_{\rho,i}(\tilde{\boldsymbol{u}}_i, \tilde{\boldsymbol{y}}_i, \tilde{\boldsymbol{\lambda}}_i) \tag{7.16a}$$

$$\text{subject to} \quad \tilde{\boldsymbol{u}}_i \in \mathcal{L}_{u,i}, \quad \tilde{\boldsymbol{v}}_i \in \mathcal{L}_{v,i}, \quad \forall i \in \{1, \ldots, N\}, \tag{7.16b}$$

$$\tilde{\boldsymbol{y}}_i = -\sum_{j\in N_{-i}} \mathbf{G}_{ij}\tilde{\boldsymbol{u}}_j^a, \quad \forall i \in \{1, \ldots, N\}. \tag{7.16c}$$

We are now in a position to provide the ADMM steps that solves this problem in an iterative fashion:

1. Updating $\tilde{\boldsymbol{u}}_i$ and $\tilde{\boldsymbol{v}}_i$ for all $i \in \{1, \ldots, N\}$:

$$\begin{aligned}\{\tilde{\boldsymbol{u}}_i^{(q+1)}, \tilde{\boldsymbol{v}}_i^{(q+1)}\} \in \operatorname*{argmin}_{\tilde{\boldsymbol{u}}_i, \tilde{\boldsymbol{v}}_i} \quad & \boldsymbol{L}_{\rho,i}(\tilde{\boldsymbol{u}}_i, \tilde{\boldsymbol{y}}_i^{(q)}, \tilde{\boldsymbol{\lambda}}_i^{(q)}) \\ \text{subject to} \quad & \tilde{\boldsymbol{u}}_i \in \mathcal{L}_{u,i}, \quad \tilde{\boldsymbol{v}}_i \in \mathcal{L}_{v,i}.\end{aligned}$$

2. Sending $\tilde{\boldsymbol{u}}_i^{a(q+1)}$ to the neighbors, $j \in N_{-i}$, for all $i \in \{1, \ldots, N\}$.
3. Receiving $\tilde{\boldsymbol{u}}_j^{a(q+1)}$ from the neighbors, $j \in N_{-i}$, for all $i \in \{1, \ldots, N\}$.
4. Updating $\tilde{\boldsymbol{y}}_i^{(q+1)}$ for all $i \in \{1, \ldots, N\}$:

$$\begin{aligned}\tilde{\boldsymbol{y}}_i^{(q+1)} = \operatorname*{argmin}_{\tilde{\boldsymbol{y}}_i} \quad & \boldsymbol{L}_{\rho,i}(\tilde{\boldsymbol{u}}_i^{(q+1)}, \tilde{\boldsymbol{y}}_i, \tilde{\boldsymbol{\lambda}}_i^{(q)}) \\ \text{subject to} \quad & \tilde{\boldsymbol{y}}_i = -\sum_{j\in N_{-i}} \mathbf{G}_{ij}\tilde{\boldsymbol{u}}_j^{a(q+1)},\end{aligned}$$

   which implies

$$\tilde{\boldsymbol{y}}_i^{(q+1)} = -\sum_{j\in N_{-i}} \mathbf{G}_{ij}\tilde{\boldsymbol{u}}_j^{a(q+1)}. \tag{7.17}$$

5. Updating $\tilde{\boldsymbol{\lambda}}_i$ for all $i \in \{1, \ldots, N\}$ via a gradient method:

$$\tilde{\boldsymbol{\lambda}}_i^{(q+1)} = \tilde{\boldsymbol{\lambda}}_i^{(q)} + \rho \left(\tilde{\boldsymbol{u}}_i^{a(q+1)} - \tilde{\boldsymbol{y}}_i^{(q+1)} \right),$$

where $\rho > 0$.

The algorithm stops when

$$\left\| \begin{bmatrix} \tilde{\boldsymbol{u}}_1^{a(q)} - \tilde{\boldsymbol{y}}_1^{(q)} \\ \vdots \\ \tilde{\boldsymbol{u}}_N^{a(q)} - \tilde{\boldsymbol{y}}_N^{(q)} \end{bmatrix} \right\|_2 < \nu,$$

for a small $\nu > 0$. Note that the steps of updating $\tilde{\boldsymbol{y}}_i$ and $\tilde{\boldsymbol{\lambda}}_i$ are fully distributed among the agents. Moreover, not all decision variables but only $\tilde{\boldsymbol{u}}_i^a$, which contains $h_{\mathbf{ex},t}^{ij}$, needs to be communicated between the agents.

One issue of an iterative algorithm, such as this method, is that it possibly requires a large number of iterations before the stopping criterion is met. In this regard, we apply warm start to reduce the number of iterations. It is done by using the solutions obtained in the previous sampling time since it often gives a good enough approximation [20]. For instance, consider that $\tilde{\boldsymbol{y}}_i^{(q)} = \tilde{\mathbf{y}}_i$ and $\tilde{\boldsymbol{\lambda}}_i^{(q)} = \tilde{\mathbf{z}}_i$ are the solutions obtained at the last iteration (q) at sampling time t. We initialize $\tilde{\boldsymbol{y}}_i^{(0)}$ and $\tilde{\boldsymbol{\lambda}}_i^{(0)}$ for the next sampling time as $\tilde{\boldsymbol{y}}_i^{(0)} = \left[\tilde{\mathbf{y}}_i^\top(2 : N_t - 1) \;\; \mathbf{0}_{1\times|N_{-i}|}\right]^\top$ and $\tilde{\boldsymbol{\lambda}}_i^{(0)} = \left[\tilde{\mathbf{z}}_i^\top(2 : N_t - 1) \;\; \mathbf{0}_{1\times|N_{-i}|}\right]^\top$.

Distributed Scheme with Coordination

The second ADMM method is formulated by perceiving the problem as an optimal exchange problem [20]. In this regard, we can consider restating problem (7.11) as follows:

$$\min_{\{\tilde{\boldsymbol{u}}_i, \tilde{\boldsymbol{v}}_i\}_{i=1}^N} \quad \sum_{i=1}^{N} \boldsymbol{J}_i(\tilde{\boldsymbol{u}}_i) \tag{7.18a}$$

$$\text{subject to} \quad \tilde{\boldsymbol{u}}_i \in \mathcal{L}_{u,i}, \quad \tilde{\boldsymbol{v}}_i \in \mathcal{L}_{v,i}, \tag{7.18b}$$

$$\sum_{i=1}^{N} \mathbf{K}_i \tilde{\boldsymbol{u}}_i^a = 0, \tag{7.18c}$$

in which (7.18c) represents the balance constraints (7.4h). This formulation is different from 7.14 since there is only one global coupling constraint (7.18c) instead of N coupling constraints (7.14c). We can then follow the unscaled form of ADMM for such problems as provided in [20] that consists of the following iterations:

1. Updating of $\tilde{\boldsymbol{u}}_i$ and $\tilde{\boldsymbol{v}}_i$:

$$\{\tilde{\boldsymbol{u}}_i^{(q+1)}, \tilde{\boldsymbol{v}}_i^{(q+1)}\} \in \underset{\tilde{\boldsymbol{u}}_i, \tilde{\boldsymbol{v}}_i}{\operatorname{argmin}} \ \Big\{ \boldsymbol{J}_i(\tilde{\boldsymbol{u}}_i) + \boldsymbol{\lambda}^{(q)\top} \left(\mathbf{K}_i \tilde{\boldsymbol{u}}_i^{a(q)}\right) + \frac{\rho}{2} \left\| \mathbf{K}_i \tilde{\boldsymbol{u}}_i^{a(q)} + \left(\tilde{\boldsymbol{y}}^{(q)} - \mathbf{K}_i \tilde{\boldsymbol{u}}_i^{a(q)}\right) \right\|_2^2 \Big\}$$
$$\text{subject to} \quad \tilde{\boldsymbol{u}}_i \in \mathcal{L}_{u,i}, \quad \tilde{\boldsymbol{v}}_i \in \mathcal{L}_{v,i},$$

2. Updating of $\tilde{\boldsymbol{y}}$:

$$\tilde{\boldsymbol{y}}^{(q+1)} = \sum_{i=1}^{N} \mathbf{K}_i \tilde{\boldsymbol{u}}_i^{a(q+1)}.$$

3. Updating of $\tilde{\boldsymbol{\lambda}}$:

$$\tilde{\boldsymbol{\lambda}}^{(q+1)} = \tilde{\boldsymbol{\lambda}}^{(q)} + \rho \tilde{\boldsymbol{y}}^{(q+1)}.$$

In this approach, although the optimization problem is solved in a distributed fashion, the process of updating the auxiliary decision variable $\tilde{\boldsymbol{y}}$ and the Lagrange multiplier $\tilde{\boldsymbol{\lambda}}$ requires a coordinator that receives the decision of $\tilde{\boldsymbol{u}}_i^a$ at each iteration from all agents.

7.5 Numerical Study

We carried out Monte Carlo simulations and a comparison with the Benchmark Algorithm 5 to illustrate the performance of our proposed Algorithm 4 (robust randomized MPC) for both open-loop and closed-loop formulations. All optimization problems were solved using the solver BNB via the MATLAB interface YALMIP [21].

Simulation Setup

In this simulation study we consider a small thermal grid with three agents as an example. Figure 7.1 depicts the connections between each agent and their local components. Each agent has a micro-CHP, a boiler and a thermal storage. In the proposed model the difference between the level of thermal storage and local thermal energy demand (imbalance errors) is defined as the state of the local agent. The thermal storage level of agent one, two and three are presented in Fig. 7.1 using h_s^1, h_s^2, h_s^3, respectively. There are also three lines between agents indicating that thermal energy exchange is possible. We assume that an external party is available for all agents to provide thermal energy.

The proposed Algorithm 4 and the Benchmark approach are applied to the example provided in Fig. 7.1 with $N = 3$. We solve a day-ahead production planning problem for an uncertain thermal grid with $N_t = 24$ and hourly steps. It is assumed that the up and down capacity of decreasing and increasing electrical power are $p_{\mathbf{g}}^{\mathbf{up}} = p_{\mathbf{g}}^{\mathbf{down}} = p_{\mathbf{g}}^{\max}/3$ and the minimum time for a change of production unit status $(\Delta t_{\mathbf{up}}, \Delta t_{\mathbf{down}})$ is 2 h. Table 7.1 contains all parameters that are considered for the

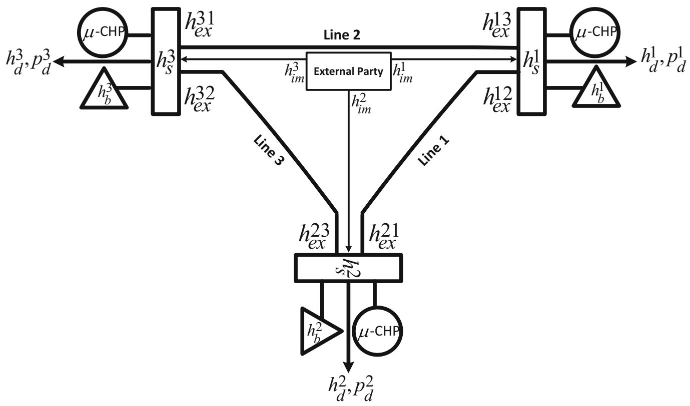

Fig. 7.1 Three-agent (households, greenhouses) thermal grid example. Each agent has a $\mu-$CHP, a boiler and a thermal storage. h_s^1, h_s^2, h_s^3 are related to the local thermal storage in agent one, two and three, respectively. There are also three lines between agents indicating that thermal energy exchange is possible. It is considered to have an external party available for all agents to provide thermal energy

Table 7.1 Parameters with their symbols and values

Parameter	Value	Unit
$p_{\mathbf{g}}^{\max}$, $h_{\mathbf{g}}^{\max}$, $p_{\mathbf{g}}^{\min}$, $h_{\mathbf{g}}^{\min}$	120.0, 120.0, 0.0, 0.0	[KW]
$h_{\mathbf{b}}^{\max}$, $h_{\mathbf{im}}^{\max}$, $h_{\mathbf{b}}^{\min}$, $h_{\mathbf{im}}^{\min}$	120.0, 120.0, 0.0, 0.0	[KW]
$h_{\mathbf{ex}}^{\max}$, $h_{\mathbf{ex}}^{\min}$, $h_{\mathbf{s},0}^{i}$	20.0, −20.0, 10.0	[KW]
$\eta_{\mathbf{CHP}}$, $\eta_{\mathbf{h}}$, $\eta_{\mathbf{p}}$	0.25, 0.7, 0.3	–
$\eta_{\mathbf{s}}$, $\eta_{\mathbf{b}}$, α_{ij}	0.85, 1.0, 0.25	–
$c_{\mathbf{gas}}$, $c_{\mathbf{up}}$, $c_{\mathbf{dp}}$, $c_{\mathbf{b}}$, $c_{\mathbf{im}}$	45.0, 0.0, 100.0, 45.0, 300.0	e
$\Lambda^{\mathbf{su}}$ (micro-CHP, boiler)	diag(60.0, 120.0)	e
ϵ, β	0.1, 0.0001	–

example in Fig. 7.1. In order to generate scenarios for the thermal energy demand error, we used a Markov chain based model (we refer the reader to [22] for more details). Moreover, we consider to have different forecast of energy demand profiles for each agent. We construct scenarios of uncertain demand profiles assuming that the realization changes randomly to represent historical uncertain demand data.

Simulation Results: Centralized Stochastic MPC

Figure 7.2 shows imbalance error trajectories for different agents. Due to the definition of the imbalance error in Eq. (7.1), our goal is to minimize these errors. This means that the requested thermal energy demand is provided for each agent at each step with the desired level of violation ϵ as in Eq. (7.4l). The initial value for the storage level in each agent is considered to be 10 [KW]. In Fig. 7.2 the 'blue', 'red' and 'green' lines are related to the imbalance error profiles (x_1, x_2, x_3) in the first,

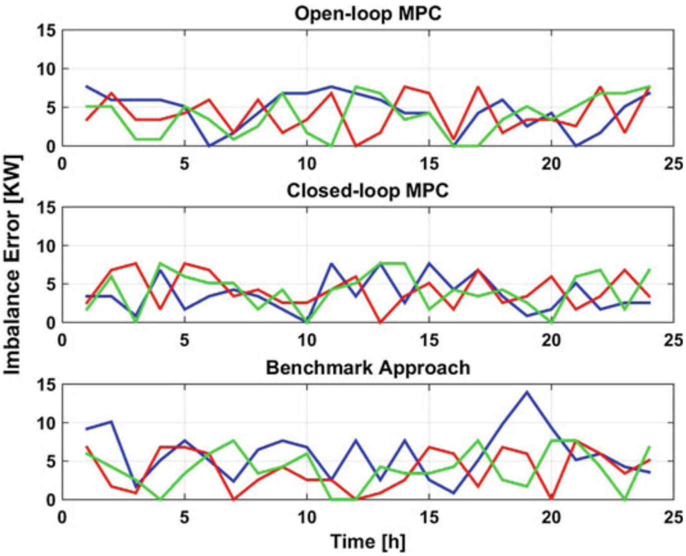

Fig. 7.2 Imbalance error trajectories. 'Blue' lines show the imbalance error x_1 in the first agent and x_2 the imbalance error in the second agent is shown by 'Red' lines. 'Green' lines represent the imbalance error x_3 in the third agent. The first, second and third sub-figures are related to the results of open-loop MPC, closed-loop MPC considering affine uncertainty feedback, and the Benchmark approach, respectively

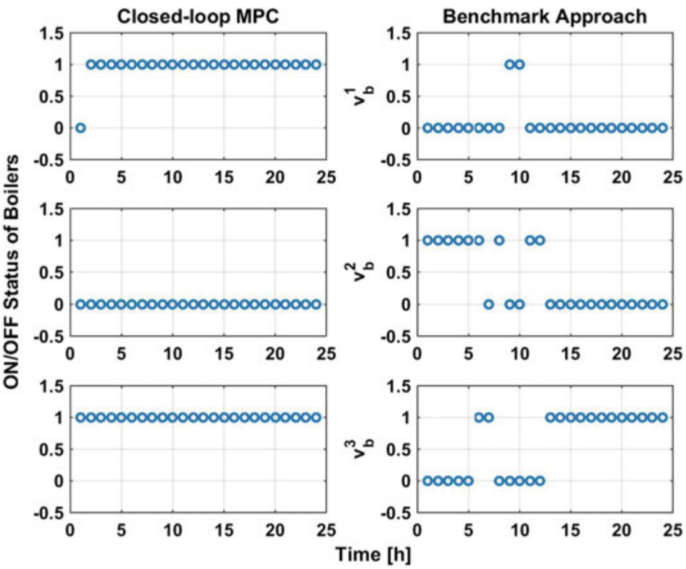

Fig. 7.3 ON/OFF status of boilers. The first, second and third sub-figures are related to agent 1, 2, and 3, respectively. The sub-figures on the left are the results of closed-loop MPC considering affine uncertainty feedback, and the sub-figures on the right are the results of the Benchmark approach

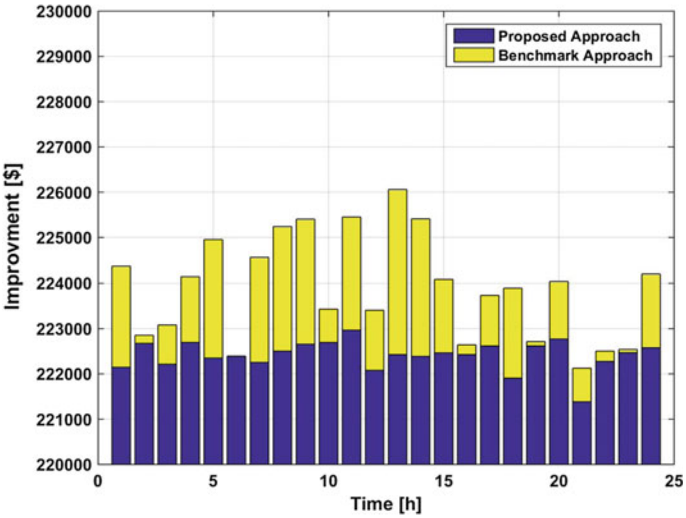

Fig. 7.4 Relative cost improvement (expressed in $) of the closed-loop MPC approach (affine uncertainty feedback) over the Benchmark approach

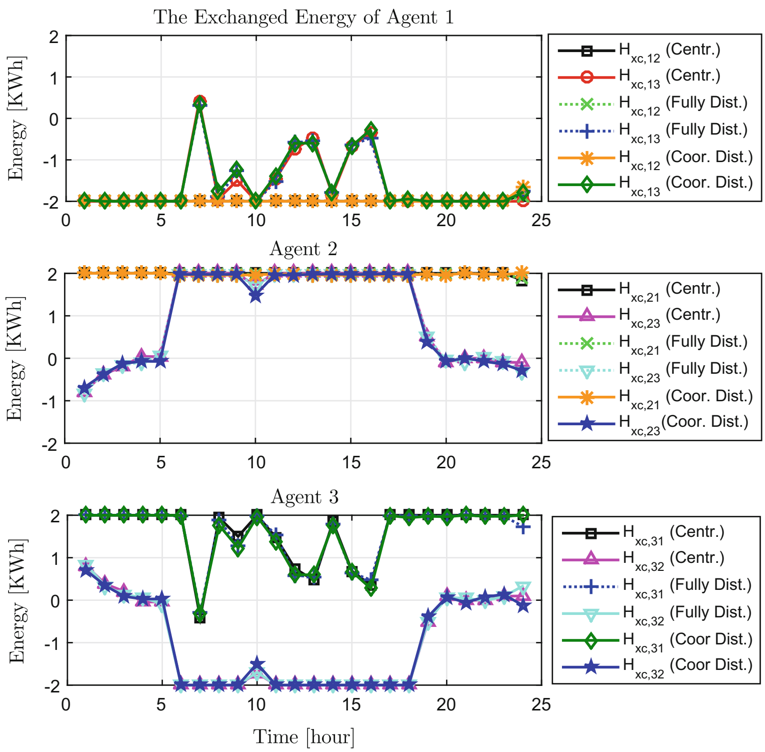

Fig. 7.5 The thermal energy exchanged between agents

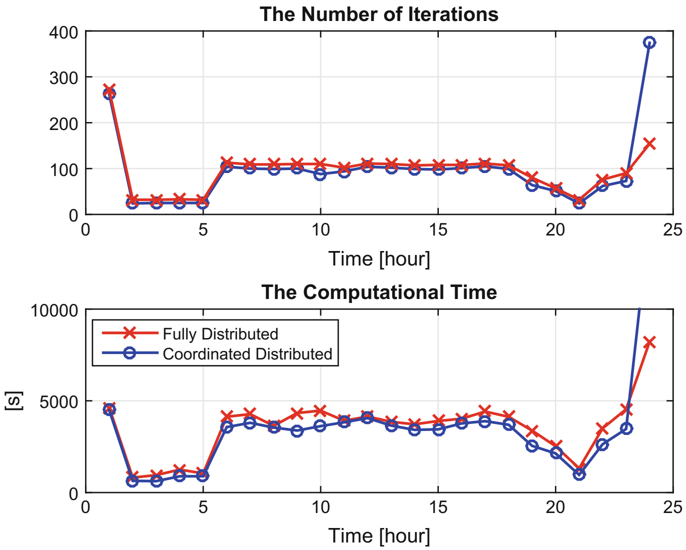

Fig. 7.6 The number of iterations and the computational time required by the distributed approaches at each sampling time

second and third agent. The top sub-figure shows the result of optimizing directly over input sequences in each sampling time (open-loop MPC), the middle depicts the result of closed-loop MPC considering affine uncertainty feedback, and the last sub-figure shows the result of the Benchmark approach. In Fig. 7.3 the ON/OFF status of boilers at each sampling time using blue for the first, second and third agent are shown, respectively. The left-hand-side sub-figures are related to the closed-loop MPC considering affine uncertainty feedback, and the-right-hand side ones show the results of the Benchmark approach. In Fig. 7.4 the relative cost improvement between the cost generated by the closed-loop MPC considering affine uncertainty feedback and the cost generated by the Benchmark approach in each sampling time is shown.

The proposed Algorithm 4 offers a better working plan for production units as well as an hourly-based production cost improvement compared to the Benchmark approach. The improvement in terms of cost is due to the scheduling flexibility offered by the proposed algorithm, where the binary variables are solved together with the production planning problem, allowing us to identify more optimal working status for production units.

Simulation Results: Distributed Stochastic MPC

We also compare the performance of the distributed approaches with the centralized one. In summary, the distributed approaches are able to find quite similar solutions to the centralized one. Moreover, Fig. 7.5 shows the satisfaction of the balance constraints in some agents when the distributed approaches are employed. Furthermore, Fig. 7.6 shows the number of iterations and the computational time required by the distributed approaches to solve the problem at each sampling time. It can be seen that both distributed methods have similar number of iterations as well as computational times.

7.6 Concluding Remarks

This work formulated an optimization problem for a day-ahead prediction plan of smart thermal grids with uncertain local demands. Smart thermal grids refer to energy networks whose main goal is to provide and distribute thermal energy among their local agents. This formulation leads to a multistage chance-constrained mixed-integer linear program. We proposed a unified framework, namely a robust randomized MPC approach to solve such a problem, while providing a-priori guarantees for the chance constraint fulfillment. Additionally, we also propose to apply a distributed optimization method based on ADMM to solve the robust randomized MPC program. Our current work focuses on incorporating aquifer thermal energy storage (ATES) systems in the developed framework.

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