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Absolute Stabilization of Lur'e Systems Under Event-Triggered Feedback ⋆

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Abstract: In this paper, we deal with event-triggered feedback control for Lur'e systems that consist of negative feedback interconnection of nominal linear dynamics and an unknown static nonlinearity. The unknown nonlinearity is conventionally assumed to lie in a given sector while the sector bounds are known. In the presence of event-triggered feedback mechanisms, the control input is only computed and updated when a specific event occurs. In this sense, control system resources (e.g. computation and communication capabilities) can be saved. A sufficient condition for the existence of an event-triggering condition and the corresponding even-triggered controller design are obtained by means of linear matrix inequality techniques. In addition, the avoidance of Zeno behavior is guaranteed. Furthermore, a result on the event-triggered emulation of a continuous-time feedback controller for Lur'e systems is presented. Finally, numerical simulations are given to illustrate the theoretical results along with some concluding remarks.

Keywords: Lur'e systems, absolute stabilization, event-triggered feedback, Zeno behavior, linear matrix inequalities.

1. INTRODUCTION

Feedback controllers are usually implemented using digital units, for example, digital signal processors {Kuo and Morgan (1995)}. In this case, measurements are discrete in nature and control signals are not smooth due to e.g. zero-order hold mechanisms. In order to analyze and synthesize digital control systems, sampled-data control has been popular for a long time, which uses periodic sampled information to compute and update control inputs {Chen and Francis (1995)}. As such, sampled-data control is indeed a *time-triggered* approach and the constant sampling period is an important design parameter. However, stability analysis and performance requirements typically result in conservative choices of the sampling period (i.e. small) and, therewith waste control system resources.

In contrast to constant sampling periods, adaptively updating them seems more promising {Dorf et al. (1962)}. We call such aperiodic sampled-data control approach, *event-triggered* control {Tabuada (2007)}. In this case, control inputs are only computed and updated whenever a certain error becomes large with respect to the state norm. These events determine the sequence of sampling times, usually resulting in aperiodic sampling. This time sequence is not *a priori* known, which is essentially different from time-triggered control. This also raises a critical problem: the time interval between any two events must be lower bounded by a strictly positive constant. Otherwise, the sensor may have to execute an infinite numbers of updates over a finite time interval, that is, Zeno behavior occurs {Goebel et al. (2009)}. Therefore, besides guaranteeing stability, it is also required to prevent Zeno behavior for event-triggered control systems.

We care to stress that in event-triggered control, the sensor has to monitor the plant in real-time while time-triggered control does not. Certainly, this requirement is stringent in some sense. In order to remove this demand, periodic event-triggered control was proposed, where the event-triggering condition is only checked at each sampling instant periodically {Heemels et al. (2013)}. Obviously, Zeno behavior will not occur in that case. Another strategy is to employ a variation of event-triggered control, named *self-triggered* control, in which the next control updating instant is pre-computed using previously received data and knowledge of the plant dynamics {Mazo Jr. et al. (2010)}. We just enumerate the above two methods here.

Due to the importance of handling energy, computation and communication constraints especially in the context of networked or wireless control systems, event-triggered control has attracted a lot of attention, see {Heemels et al. (2012)} and the references therein. Following the

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ISS arguments in {Tabuada (2007)}, subsequent works mainly focused on stability analysis of event-triggered control systems, see for example {Marchand et al. (2013); Girard (2015)}. Other approaches based on small gain and passivity for the stability analysis of event-triggered control systems were also exploited {Yu and Antsaklis (2013); Liu and Jiang (2015)}.

However, fewer results exist addressing event-triggered *controller design*, in particular for uncertain nonlinear systems. In {Behera and Bandyopadhyay (2016)}, event-triggered sliding mode control of linear systems against external disturbances was discussed, where the feasibility of the adopted controller depends on the sliding manifold and the control input matrix. A periodic event-triggered observer was designed for Lipschitz systems in {Etienne and Gennaro (2016)}. Although some continuous-time feedback controllers can be redesigned as event-triggered ones, more technical details will be involved and sometimes make the design procedure difficult, see e.g. {Sahoo et al. (2016)}. Hence we have to take care of event-triggered controller design case by case.

In this paper, we study the design problem of event-triggered feedback controllers for Lur'e systems, being a class of uncertain nonlinear systems. Lur'e systems can represent many physical plants, e.g. Chua's circuits, flexible joint robotic arms, drilling systems and variable-gain control systems {Liao and Yu (2008); van de Wouw et al. (2008); de Bruin et al. (2009)}. The research on absolute stability and absolute stabilization of Lur'e systems has greatly promoted the development of control theory, in particular nonlinear robust control. In {Chen and Hao (2012)}, event-triggered control for Lur'e systems is also addressed. We relax the assumption on the Lur'e-type nonlinearities in the present paper with respect to {Chen and Hao (2012)}. Besides, Zeno behavior was not excluded in {Chen and Hao (2012)} while this paper gives a strictly positive lower bound of inter-event times and its explicit computation. Moreover, we also provide an event-triggered emulation result for Lur'e systems. Also related is the work in {Seifullaev and Fradkov (2016)}, in which a switching approach has been studied to design event-triggered controllers for Lur'e systems employing a time-delayed system approach, which results in a larger number of LMI conditions.

The main contribution of this paper is an LMI-based approach for the co-design of a state feedback control law and an event-triggering condition that guarantee absolute stability of the event-triggered Lur'e control system. Moreover, it is guaranteed that using our proposed design Zeno behavior cannot occur. Finally, a result on the event-triggered emulation of a continuous-time feedback controller for Lur'e systems is presented.

The remainder of this paper is organized as follows. Some preliminaries and the problem formulation are given in Section 2. The main results are presented in Section 3. Section 4 validates the obtained theoretical results using numerical simulations. Concluding remarks close the paper in Section 5.

2. PRELIMINARIES AND PROBLEM FORMULATION

We first give some preliminaries needed in this paper.

2.1 Preliminaries

Let $\mathbb{R}$ and $\mathbb{N}$ be the fields of real numbers and nonnegative integers, respectively. Here $\mathbb{R}_0^+$ denotes the field of nonnegative real numbers. $\mathbb{R}^{m\times n}$ denotes the space of $m \times n$ real matrices. Matrices, if not explicitly stated, are assumed to have compatible dimensions. The superscripts $(\cdot)^T$ and $(\cdot)^{-1}$ denote the transpose and respectively the inverse of a real matrix. We denote by $\mathbf{0}$ and I the zero and the identity matrices of compatible dimensions.

First, we review the so-called S-procedure in a basic form.

Lemma 1. {Scherer and Weiland (2000)} Let M_1 and M_2 be real symmetric matrices. Then, $x^T M_1 x < 0$ for all real vectors $x \neq \mathbf{0}$ satisfying $x^T M_2 x \leq 0$ if there exists a positive real number τ such that $M_1 - \tau M_2 < \mathbf{0}$.

Besides the S-procedure, the Schur complement lemma is also used throughout this paper.

Lemma 2. {Scherer and Weiland (2000)} Let M be a real symmetric matrix partitioned into blocks $M = \begin{bmatrix} M_1 & M_2 \\ M_2^T & M_3 \end{bmatrix}$. Assume that M_3 is positive (negative) definite, namely $M_3 > \mathbf{0}$ ($M_3 < \mathbf{0}$). Then the following statements are equivalent:

- M is positive (negative) definite;
- The Schur complement of M_3, defined as the matrix $M_1 - M_2 M_3^{-1} M_2^T$, is positive (negative) definite.

A similar argument to M_1 and its Schur complement also holds.

2.2 Problem formulation

In this paper, we will study event-triggered control of Lur'e systems. The Lur'e system consists of a nominal linear dynamics with an *unknown* static nonlinearity around it in a negative feedback loop, as described by

$$\begin{cases} \dot{x} = Ax + Bu + Ez \\ y = Cx \\ z = -\phi(y) \end{cases}, \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^s$ are the state, the control input and the output, respectively. A, B, C and E are given constant matrices of compatible dimensions. The equation $z = -\phi(y)$ represents a memoryless, nonlinear negative feedback loop, see Fig. 1. The function $\phi(\cdot) : \mathbb{R}^s \to \mathbb{R}^s$ is an unknown static nonlinearity. It is however assumed to be sector bounded within a known sector defined as follows.

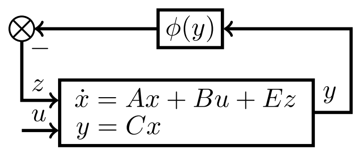

Fig. 1. Lur'e systems

Definition 3. Let $S_1,\ S_2 \in \mathbb{R}^{s\times s}$ be real symmetric matrices such that S_1 is positive semi-definite and $S_2 - S_1$ is positive definite, i.e. $\mathbf{0} \le S_1 < S_2$. Then $\phi(\cdot)$ is called sector bounded within the sector $[S_1, S_2]$ if it satisfies

$$(\phi(y) - S_1 y)^T(\phi(y) - S_2 y) \le 0 \tag{2}$$

for all $y \in \mathbb{R}^s$.

The problems of absolute stability and absolute stabilization are recalled below.

Definition 4. The Lur'e system (1) is said to be absolutely stabilized by a feedback controller if the corresponding closed-loop control system is absolutely stable, that is, it is globally asymptotically stable subject to all sector bounded $\phi(\cdot)$'s within the given sector $[S_1, S_2]$.

Here, the control input u is governed by a state feedback law and only computed and updated when needed, and is held constant by a zero-order hold until the next event occurs. Then, the considered event-triggered controller is given by

$$u(t) = Fx(t_k), \quad t \in [t_k, t_{k+1}), \tag{3}$$

where the feedback gain matrix $F \in \mathbb{R}^{m\times n}$ is to be determined. The time sequence t_k, $k \in \mathbb{N}$, is determined by the following event-triggering condition:

$$t_{k+1} = \inf\{t \ge t_k | e^T(t)e(t) = \alpha x^T(t)x(t)\}, \tag{4}$$

where $e(t) \triangleq x(t_k) - x(t)$ and the positive real constant α is the second control parameter to be designed. Obviously, $e(t_k) = \mathbf{0}$, $\forall\ k \in \mathbb{N}$, and thus $e(t)$ is right-continuous. Without loss of generality, let $t_0 \equiv 0$. Under the above event-triggering mechanism, we will always have that

$$e^T(t)e(t) \le \alpha x^T(t)x(t),\ \forall\ t \in \mathbb{R}_0^+. \tag{5}$$

By interconnecting (1) and (3), the event-triggered closed-loop Lur'e control system is derived as

$$\dot{x}(t) = (A+BF)x(t) + BFe(t) - E\phi(Cx(t)),\ t \in [t_k, t_{k+1}). \tag{6}$$

The problem we are interested in is to co-design the feedback gain matrix F and the event-triggering parameter $\alpha > 0$ such that the closed-loop control system (6) is absolutely stable for all sector bounded $\phi(\cdot)$'s within the given sector $[S_1, S_2]$.

3. MAIN RESULTS

In this section, we will present the co-design of the event-triggered controller (3) as well as the event-triggering condition (4), namely F and $\alpha > 0$. Moreover, the feasibility of the proposed sufficient absolute stabilization condition is discussed further by revealing its relations with the one for the continuous-time feedback case.

First, the constructive design of F and $\alpha > 0$ is given in the following theorem.

Theorem 5. For a given positive real constant a, if there exists a symmetric positive definite matrix $X \in \mathbb{R}^{n\times n}$, a matrix $Y \in \mathbb{R}^{m\times n}$ and positive real constants c, β such that the linear matrix inequality

$$\begin{bmatrix} \bar{A}X + BY + X\bar{A}^T + Y^TB^T + \frac{c}{2}EE^T & XC^T & X & BY \\ CX & -2c(S_2 - S_1)^{-2} & \mathbf{0} & \mathbf{0} \\ X & \mathbf{0} & -\frac{\beta}{a}I & \mathbf{0} \\ Y^TB^T & \mathbf{0} & \mathbf{0} & a(I - 2X) \end{bmatrix} < \mathbf{0} \tag{7}$$

holds, where $\bar{A} \triangleq A - \frac{1}{2}E(S_1 + S_2)C$, then the system (6) with $F = YX^{-1}$ is absolutely stable for all sector bounded $\phi(\cdot)$'s within the sector $[S_1, S_2]$ under the event-triggering condition (4) with a suitable event-triggering parameter $\alpha > 0$ given by $\alpha = 1/\beta$. Furthermore, a strictly positive lower bound of the inter-event times is given by $\frac{\sqrt{\alpha}}{l_x(1+\sqrt{\alpha})}$ when $l_x = l_e$, or by $\frac{1}{l_x - l_e} \ln \frac{l_x + l_x\sqrt{\alpha}}{l_x + l_e\sqrt{\alpha}}$ when $l_x \ne l_e$, where $l_x \triangleq \|A+BF\| + \frac{1}{2}\|E\|[\|(S_1+S_2)C\| + \|(S_2-S_1)C\|]$ and $l_e \triangleq \|BF\|$. Thus, Zeno behavior is avoided.

Proof. (*Part I: Stability Analysis*) In order to find out F and $\alpha > 0$ such that the system (6) is absolutely stable, we consider the Lyapunov function candidate $V(x) = x^TPx$ with $P = X^{-1}$, where $X > \mathbf{0}$ together with Y, $c > 0$ and $\beta > 0$ satisfying (7). Obviously, $V(x)$ is positive definite and radially unbounded. The time derivative of $V(x)$ along the trajectories of the system (6) is given by

$$\begin{aligned} \dot{V}(x) =& 2x^TP[(A + BF)x + BFe - E\phi(Cx)] \\ =& x^T\left[P(A + BF) + (A + BF)^TP\right]x \\ &+ 2x^TPBFe - 2x^TPE\phi(Cx) \\ \le& x^T\left[P(A + BF) + (A + BF)^TP\right]x \\ &+ \frac{1}{a}x^TPBFF^TB^TPx + ae^Te - 2x^TPE\phi(Cx) \\ \le& x^T\left[P(A + BF) + (A + BF)^TP\right]x \qquad \text{by (5)} \\ &+ \frac{1}{a}x^TPBFF^TB^TPx + a\alpha x^Tx - 2x^TPE\phi(Cx) \\ =& \begin{bmatrix} x \\ \phi(Cx) \end{bmatrix}^T \begin{bmatrix} P(A+BF) + (A+BF)^TP + \frac{1}{a}PBFF^TB^TP + a\alpha I & -PE \\ -E^TP & \mathbf{0} \end{bmatrix} \begin{bmatrix} x \\ \phi(Cx) \end{bmatrix}, \end{aligned} \tag{8}$$

where we use the fact that $2x^TPBFe \le \frac{1}{a}x^TPBFF^TB^TPx + ae^Te$ for any positive real constant a to derive the first inequality above. However, a has to be fixed due to the product of a and X in the block (4,4) of (7). We note that the property of sector boundedness (2) is equivalent to

$$\begin{bmatrix} x \\ \phi(Cx) \end{bmatrix}^T \begin{bmatrix} \frac{1}{2}C^T(S_1S_2 + S_2S_1)C & -\frac{1}{2}C^T(S_1 + S_2) \\ -\frac{1}{2}(S_1 + S_2)C & I \end{bmatrix} \begin{bmatrix} x \\ \phi(Cx) \end{bmatrix} \le 0.$$

Then, using the S-procedure, see Lemma 1, $\dot{V}(x)$ in (8) is negative definite for all sector bounded $\phi(\cdot)$'s within the sector $[S_1, S_2]$ if there exists a $b > 0$ such that

$$\begin{bmatrix} P(A+BF) + (A+BF)^TP + \frac{1}{a}PBFF^TB^TP + a\alpha I - bC^T(S_1S_2 + S_2S_1)C & -PE + bC^T(S_1 + S_2) \\ -E^TP + b(S_1 + S_2)C & -2bI \end{bmatrix} < \mathbf{0} \tag{9}$$

or, by the Schur complement lemma, see Lemma 2, equivalently

$$P\left(\bar{A}+BF\right)+\left(\bar{A}+BF\right)^{T}P+\frac{1}{a}PBFF^{T}B^{T}P + a\alpha I+\frac{1}{2b}PEE^{T}P+\frac{b}{2}C^{T}(S_2-S_1)^2C<\mathbf{0} \qquad (10)$$

holds.

On the other hand, from (7) and $\beta = 1/\alpha$, we have that

$$\begin{bmatrix} \bar{A}X+BY+X\bar{A}^T+Y^TB^T+\frac{c}{2}EE^T & XC^T & X & BY \\ CX & -2c(S_2-S_1)^{-2} & \mathbf{0} & \mathbf{0} \\ X & \mathbf{0} & -\frac{1}{a\alpha}I & \mathbf{0} \\ Y^TB^T & \mathbf{0} & \mathbf{0} & -aX^2 \end{bmatrix}<\mathbf{0}$$

due to the fact that $-X^2 \le I - 2X$ since $(I-X)^2 = I - 2X + X^2 \ge \mathbf{0}$. The above matrix inequality is equivalent to

$$\begin{bmatrix} \bar{A}X+BY+X\bar{A}^T+Y^TB^T+\frac{c}{2}EE^T+\frac{1}{a}BYX^{-2}Y^TB^T & XC^T & X \\ CX & -2c(S_2-S_1)^{-2} & \mathbf{0} \\ X & \mathbf{0} & -\frac{1}{a\alpha}I \end{bmatrix}<\mathbf{0},$$

$$\Leftrightarrow\begin{bmatrix} \bar{A}X+BY+X\bar{A}^T+Y^TB^T+\frac{c}{2}EE^T+\frac{1}{a}BYX^{-2}Y^TB^T+a\alpha X^2 & XC^T \\ CX & -2c(S_2-S_1)^{-2} \end{bmatrix}<\mathbf{0},$$

$$\Leftrightarrow\bar{A}X+BY+X\bar{A}^T+Y^TB^T+a\alpha X^2+\frac{c}{2}EE^T + \frac{1}{2c}XC^T(S_2-S_1)^2CX+\frac{1}{a}BYX^{-2}Y^TB^T<\mathbf{0},$$

$$\Leftrightarrow\begin{bmatrix} \bar{A}X+BY+X\bar{A}^T+Y^TB^T+a\alpha X^2+\frac{1}{2c}XC^T(S_2-S_1)^2CX+\frac{c}{2}EE^T & BY \\ Y^TB^T & -aX^2 \end{bmatrix}<\mathbf{0},$$

$$\Leftrightarrow\begin{bmatrix} \bar{A}X+BY+X\bar{A}^T+Y^TB^T+a\alpha X^2+\frac{1}{2c}XC^T(S_2-S_1)^2CX & BY & E \\ Y^TB^T & -aX^2 & \mathbf{0} \\ E^T & \mathbf{0} & -\frac{2}{c}I \end{bmatrix}<\mathbf{0}.$$

Here we applied repeatedly the Schur complement lemma. By premultiplying and postmultiplying the last matrix inequality with $\mathrm{diag}(P,P,I)$, along with $F = YP$, we get

$$\begin{bmatrix} P\left(\bar{A}+BF\right)+\left(\bar{A}+BF\right)^TP+a\alpha I+\frac{1}{2c}C^T(S_2-S_1)^2C & PBF & PE \\ F^TB^TP & -aI & \mathbf{0} \\ E^TP & \mathbf{0} & -\frac{2}{c}I \end{bmatrix}<\mathbf{0},$$

$$\Leftrightarrow\begin{bmatrix} P\left(\bar{A}+BF\right)+\left(\bar{A}+BF\right)^TP+a\alpha I+\frac{c}{2}PEE^TP+\frac{1}{2c}C^T(S_2-S_1)^2C & PBF \\ F^TB^TP & -aI \end{bmatrix}<\mathbf{0},$$

$$\Leftrightarrow P\left(\bar{A}+BF\right)+\left(\bar{A}+BF\right)^{T}P+\frac{1}{a}PBFF^{T}B^{T}P + a\alpha I+\frac{c}{2}PEE^{T}P+\frac{1}{2c}C^{T}(S_2-S_1)^2C<\mathbf{0}.$$

Therefore, (9) as well as (10) hold by taking $b = 1/c$, which implies that $\dot{V}(x)$ is negative definite. Thus the system (6) is absolutely stable.

(*Part II: Zeno Freeness*) The event-triggering condition (4) determines the inter-event time intervals $T_{k+1} \triangleq t_{k+1} - t_k$, $k \in \mathbb{N}$. In order to prevent an infinite number of samplings over a finite time interval, i.e. the so-called Zeno behavior, T_{k+1} is required to be lower bounded by a strictly positive constant for all $k \in \mathbb{N}$. Since T_{k+1} is the time it takes $\|e(t)\|$ to grow from 0 to $\sqrt{\alpha}\|x(t)\|$, $t \in [t_k, t_{k+1})$, we can follow the analysis carried out in {Tabuada (2007)}.

Before moving on, we reformulate the Lur'e-type nonlinearity $\phi(\cdot)$ using a Lipschitz-type property instead. Note that

$$\begin{aligned}&\|\phi(Cx)\|^2-\|(S_1+S_2)Cx\|\|\phi(Cx)\|+x^TC^TS_1S_2Cx\\ \le&\phi(Cx)^T\phi(Cx)-\phi(Cx)^T(S_1+S_2)Cx+x^TC^TS_1S_2Cx\\ =&(\phi(y)-S_1y)^T(\phi(y)-S_2y)\le 0. \qquad \text{by (2)}\end{aligned}$$

Then, by solving the above quadratic inequality in one variable as $\|\phi(Cx)\|$, we obtain that

$$\begin{aligned}\|\phi(Cx)\|\le&\frac{1}{2}\left[\|(S_1+S_2)Cx\|+\sqrt{\|(S_1+S_2)Cx\|^2-4x^TC^TS_1S_2Cx}\right]\\ =&\frac{1}{2}\left[\|(S_1+S_2)Cx\|+\sqrt{x^TC^T(S_1+S_2)^2Cx-2x^TC^TS_1S_2Cx-2x^TC^TS_2S_1Cx}\right]\\ =&\frac{1}{2}\left[\|(S_1+S_2)Cx\|+\sqrt{x^TC^T(S_2-S_1)^2Cx}\right]\\ =&\frac{1}{2}\left[\|(S_1+S_2)Cx\|+\|(S_2-S_1)Cx\|\right]\\ \le&\frac{1}{2}\left[\|(S_1+S_2)C\|+\|(S_2-S_1)C\|\right]\|x\|.\end{aligned}$$

Consequently, using the dynamics (6), we have that

$$\begin{aligned}\|\dot{x}(t)\|\le&\|A+BF\|\|x(t)\|+\|BF\|\|e(t)\|+\|E\|\|\phi(Cx(t))\|\\ \le&\|A+BF\|\|x(t)\|+\|BF\|\|e(t)\|+\frac{1}{2}\|E\|\left[\|(S_1+S_2)C\|+\|(S_2-S_1)C\|\right]\|x(t)\|\\ =&l_x\|x(t)\|+l_e\|e(t)\|.\end{aligned}$$

It is easily checked that $l_x > 0$ and $l_e > 0$.

Next, we lower bound T_{k+1} by studying the dynamics of $\|e\|/\|x\|$:

$$\frac{d}{dt}\frac{\|e\|}{\|x\|}\le\left(1+\frac{\|e\|}{\|x\|}\right)\frac{\|\dot{x}\|}{\|x\|} \text{ from (11) in \{Tabuada (2007)\}}$$

$$=l_x + (l_x + l_e)\frac{\|e\|}{\|x\|} + l_e\left(\frac{\|e\|}{\|x\|}\right)^2.$$

Therefore, T_{k+1}, $\forall\ k \in \mathbb{N}$, is lower bounded by the time interval T satisfying $f(T,0) = \sqrt{\alpha}$, where $f(t, f_0)$ is the solution to $\dot{f} = l_x + (l_x + l_e)f + l_e f^2$ with $f(0, f_0) = f_0$. Case 1: $l_x = l_e$, $T = \frac{\sqrt{\alpha}}{l_x(1+\sqrt{\alpha})} > 0$; Case 2: $l_x \neq l_e$, $T = \frac{1}{l_x - l_e}\ln\frac{l_x + l_x\sqrt{\alpha}}{l_x + l_e\sqrt{\alpha}} > 0$. This completes the proof. ■

Remark 6. The proof of Theorem 5 also shows that the event-triggering condition (4) with $0 < \alpha < 1/\beta$ still works. As such, Theorem 5 also expresses an event-triggered emulation result guaranteeing absolute stability under a sufficiently stringent event-triggering condition (see below for more details). However, a smaller α will reduce the lower bound T for T_{k+1} since T is a strictly increasing function of α, see Part II in the above proof. In this case, more actions of control computation and command updating might be involved. The maximization of α depends on the selection of a. Additionally the choice of a might influence the feasibility of (7). In order to find an optimal/feasible value of a one could simply employ a line-search algorithm.

Below we will discuss the feasibility of the LMI (7) in Theorem 5 by exploring relations between Theorem 5 and its counterpart regarding the continuous-time feedback case. In the case of continuous-time static state feedback, the controller for the Lur'e system (1) is given by

$$u(t) = Fx(t), \quad t \in \mathbb{R}_0^+, \tag{11}$$

where $F \in \mathbb{R}^{m\times n}$ is the feedback gain matrix, and the corresponding closed-loop control system is derived as

$$\dot{x} = (A + BF)x - E\phi(Cx). \tag{12}$$

Lemma 7. {Zhang et al. (2014)} If there exists a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$, a matrix F and a positive real constant b such that

$$P\left(\bar{A} + BF\right) + \left(\bar{A} + BF\right)^T P + \frac{1}{2b}PEE^T P + \frac{b}{2}C^T(S_2 - S_1)^2 C < \mathbf{0} \tag{13}$$

holds, then the system (12) is absolutely stable for all sector bounded $\phi(\cdot)$'s within the sector $[S_1, S_2]$.

Remark 8. Note that (13) is not an LMI. A necessary and sufficient condition for the feasibility of (13) in the form of an LMI can be found in {Zhang et al. (2014)}, see (15) therein.

Now, we can formulate the following result on the event-triggered emulation of a continuous-time controller for Lur'e systems.

Proposition 9. There exists a solution pair (P, F) to (10) if and only if there exists a solution pair (P, F) to (13).

Proof. For the 'only if' part: when (10) holds, obviously (13) holds as well due to the fact that $\frac{1}{a}PBFF^T B^T P + a\alpha I$ is always positive semi-definite. For the 'if' part: when (13) holds, (10) can also hold if $a > 0$ takes a sufficiently large value and $\alpha > 0$ takes a sufficiently small one. More precisely, let $\Sigma \triangleq P\left(\bar{A} + BF\right) + \left(\bar{A} + BF\right)^T P + \frac{1}{2b}PEE^T P + \frac{b}{2}C^T(S_2 - S_1)^2 C < \mathbf{0}$. By the Schur complement lemma, (10) becomes an LMI:

$$\begin{bmatrix} \rho PBFB^T F^T P + \Sigma & \varrho I \\ \varrho I & -\rho I \end{bmatrix} < \mathbf{0}, \tag{14}$$

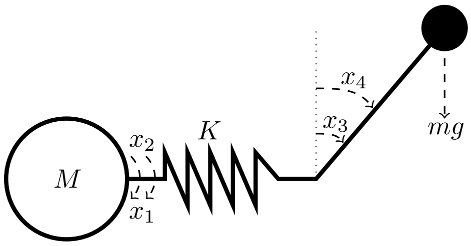

Fig. 2. Flexible joint robotic arms

where $\rho \triangleq \frac{1}{a}$ and $\varrho \triangleq \sqrt{\alpha}$. In this way, suitable a and α can be computed. The proof is complete. ■

Remark 10. It follows from Proposition 9 that, besides solving the LMI (7), there is an alternative (emulation-based) way to design an event-triggered controller of the form (3) and (4) that absolutely stabilizes the Lur'e system (1) by first solving the matrix inequality (13) using e.g. the techniques in {Zhang et al. (2014)} to compute feasible matrices $P > \mathbf{0}$, F and a constant $b > 0$, and then solving the LMI (14) to obtain a suitable α as $\alpha = \varrho^2$. The related technical details are omitted here for the sake of brevity.

Remark 11. Whereas Lemma 7 only provides a sufficient absolute stability condition for the Lur'e system (1) with the continuous-time feedback controller (11), it is commonly used in the presence of the Lur'e problem. In addition, there are no necessary and sufficient conditions on the absolute stability of the closed-loop Lur'e control system (12). In some sense, by Proposition 9, the existence of an event-triggered feedback controller (3) is equivalent to that of a continuous-time feedback controller (11) for the Lur'e system (1).

4. A SIMULATION EXAMPLE

In this section, we take a flexible joint robotic arm model as an application example of Lur'e systems and show the effectiveness of Theorem 5 obtained above.

A type of flexible joint robotic arms, see Fig. 2, can be modeled by the Lur'e system (1) {Zhang et al. (2016)}, where $x = [x_1, x_2, x_3, x_4]^T$,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -1.25 & 48.6 & 0 \\ 0 & 0 & 0 & 1 \\ 19.5 & 0 & -16.17 & 0 \end{bmatrix},$$

$B = [0\ 21.6\ 0\ 0]^T$, $C = [0\ 0\ 1\ 0]$ and $E = [0\ 0\ 0\ 3.33]^T$. Its Lur'e-type nonlinearity is described by $\phi(x_3) = x_3 + \sin x_3$, which satisfies the sector boundedness condition $\phi(x_3)(\phi(x_3) - 2x_3) \leq 0$ with $S_1 = 0$ and $S_2 = 2$.

Let $a = 4$. By solving (7) in Theorem 5 using the Matlab LMI Control Toolbox, we can compute F and α to be $F = [-0.3706\ -0.2943\ -0.5274\ -0.2650]$ and $\alpha = 0.0211$, respectively. Then the lower bound of inter-event time intervals is easily computed to be $T = 0.0017$s. Ignoring the physical quantities and units, with the initial state $x(0) = [1\ 2\ 3\ 4]^T$, the state trajectories of the system (6) are shown in Fig. 3 together with the corresponding control input. Clearly, the designed even-triggered controller (3)-(4) works well. The transient performance is also satisfactory. Meanwhile, Fig. 4 shows the inter-event times. It turns out that the actions of the control updating are indeed performed aperiodically and Zeno behavior does

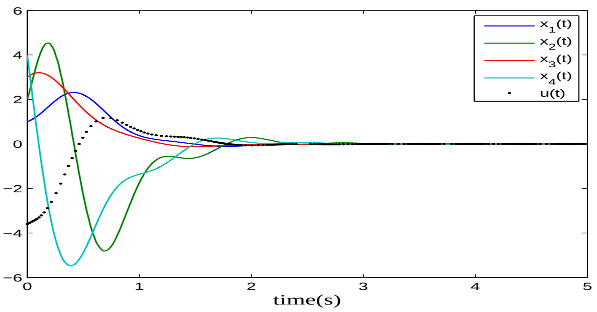

Fig. 3. The state trajectories and the corresponding control input

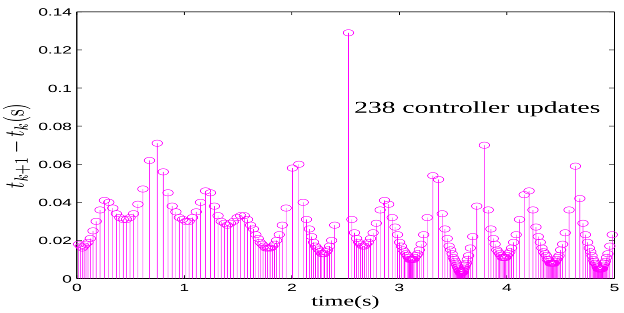

Fig. 4. The inter-event times

not occur. Moreover, it follows from Theorem 5 that these time intervals are not smaller than $0.0017s$.

5. CONCLUSIONS

A co-design approach for a state feedback control law and a triggering condition has been proposed for the event-triggered control of Lur'e systems. The obtained design condition has been formulated as an LMI. A strictly positive lower bound of the inter-event times has been provided along with its computation. In addition, an event-triggered emulation result for Lur'e systems has been presented.

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