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On Green's functions, propagator matrices, focusing functions and their mutual relations

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Summary

Green's functions and propagator matrices are both solutions of the wave equation, but whereas Green's functions obey a causality condition in time ($G = 0$ for $t < 0$), propagator matrices obey a boundary condition in space. Marchenko-type focusing functions focus a wave field in space at zero time. We discuss the mutual relations between Green's functions, propagator matrices and focusing functions, avoiding up-down decomposition and accounting for propagating and evanescent waves. We conclude with discussing a Marchenko-type Green's function representation, which forms a basis for extending the Marchenko method to improve the imaging of steeply dipping flanks and to account for refracted waves.

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Introduction

An acoustic Green's function is the causal response to an impulsive point source. It is a solution of the wave equation, supplemented with a causality condition in time. A propagator matrix is a matrix which 'propagates' a wave field from one depth to another. It is a solution of a matrix-vector wave equation, supplemented with a boundary condition in space. A Marchenko-type focusing function is a wave field that focuses at a designated point in space at zero time. In this paper we briefly review these concepts, discuss their mutual relations and indicate applications of these relations. In particular we discuss a Marchenko-type Green's function representation which avoids up-down decomposition and accounts for propagating and evanescent waves. This representation is the basis for extending the Marchenko method to improve the imaging of steeply dipping flanks and to account for refracted waves.

Matrix-vector wave equation

Acoustic Green's functions and Marchenko-type focusing functions are most conveniently defined as solutions of a scalar wave equation in the space-time $(\mathbf{x},t)$ domain. Propagator matrices, on the other hand, are usually defined as solutions of a matrix-vector wave equation in the space-frequency $(\mathbf{x},\omega)$ domain. To facilitate the discussion of the mutual relations between Green's functions, propagator matrices and focusing functions, our starting point is the acoustic matrix-vector wave equation in the space-frequency domain. This equation reads (Corones, 1975; Ursin, 1983; Fishman and McCoy, 1984)

$$\partial_3\mathbf{q}-\mathbf{A}\,\mathbf{q}=\mathbf{d}, \tag{1}$$

where ∂_3 stands for differentiation in the vertical direction and where $\mathbf{q}(\mathbf{x},\omega)$, $\mathbf{d}(\mathbf{x},\omega)$ and $\mathbf{A}(\mathbf{x},\omega)$ are the wave-field vector, source vector and operator matrix, respectively, which are defined as follows

$$\mathbf{q}=\begin{pmatrix}p\\ v_3\end{pmatrix},\quad \mathbf{d}=\begin{pmatrix}f_3\\ \frac{1}{i\omega}\partial_\alpha(\frac{1}{\rho}f_\alpha)+q\end{pmatrix},\quad \mathbf{A}=\begin{pmatrix}0 & i\omega\rho\\ i\omega\kappa-\frac{1}{i\omega}\partial_\alpha\frac{1}{\rho}\partial_\alpha & 0\end{pmatrix}. \tag{2}$$

Here ∂_α (for $\alpha=1,2$) stands for differentiation in the horizontal directions, $p(\mathbf{x},\omega)$ and $v_3(\mathbf{x},\omega)$ are the acoustic pressure and vertical component of the particle velocity, $q(\mathbf{x},\omega)$ and $f_k(\mathbf{x},\omega)$ (for $k=1,2,3$) are the volume injection-rate density and external force density, $\rho(\mathbf{x})$ and $\kappa(\mathbf{x})$ are the mass density and compressibility of the 3D inhomogeneous medium, and i is the imaginary unit. We assume that the medium is lossless, hence ρ and κ are real-valued and frequency-independent. For notational convenience, we drop the argument ω in the remainder of this paper.

Green's matrix

We define the Green's matrix $\mathbf{G}(\mathbf{x},\mathbf{x}_S)$ for a 3D inhomogeneous medium as the solution of matrix-vector wave equation (1), with the source vector replaced by a unit point source matrix, hence

$$\partial_3\mathbf{G}-\mathbf{A}\mathbf{G}=\mathbf{I}\delta(\mathbf{x}-\mathbf{x}_S), \tag{3}$$

where $\mathbf{x}_S=(x_{1,S},x_{2,S},x_{3,S})$ denotes the position of the source and $\mathbf{I}$ is the identity matrix. To get a unique solution, the time-domain version of the Green's matrix is enforced to be zero at negative time, i.e.,

$$\mathbf{G}(\mathbf{x},\mathbf{x}_S,t<0)=\mathbf{O}, \tag{4}$$

where $\mathbf{O}$ is a zero matrix. We call this the causality condition. Since equations (1) and (3) are linear in terms of $\mathbf{q}$ and $\mathbf{G}$, respectively, we can use Huygens' superposition principle to express $\mathbf{q}(\mathbf{x})$ in terms of $\mathbf{G}$ and $\mathbf{d}$, according to (Rayleigh, 1878; Bleistein, 1984)

$$\mathbf{q}(\mathbf{x})=\int_{\mathbb{D}_S}\mathbf{G}(\mathbf{x},\mathbf{x}_S)\mathbf{d}(\mathbf{x}_S)\mathrm{d}^3\mathbf{x}_S, \tag{5}$$

where $\mathbb{D}_S$ denotes the domain in which the source function $\mathbf{d}(\mathbf{x})$ is non-zero. More general representations using matrix $\mathbf{G}$ are reviewed by Wapenaar (2022). The Green's matrix is partitioned as follows

$$\mathbf{G}(\mathbf{x},\mathbf{x}_S)=\begin{pmatrix}G^{p,f} & G^{p,q}\\ G^{v,f} & G^{v,q}\end{pmatrix}(\mathbf{x},\mathbf{x}_S). \tag{6}$$

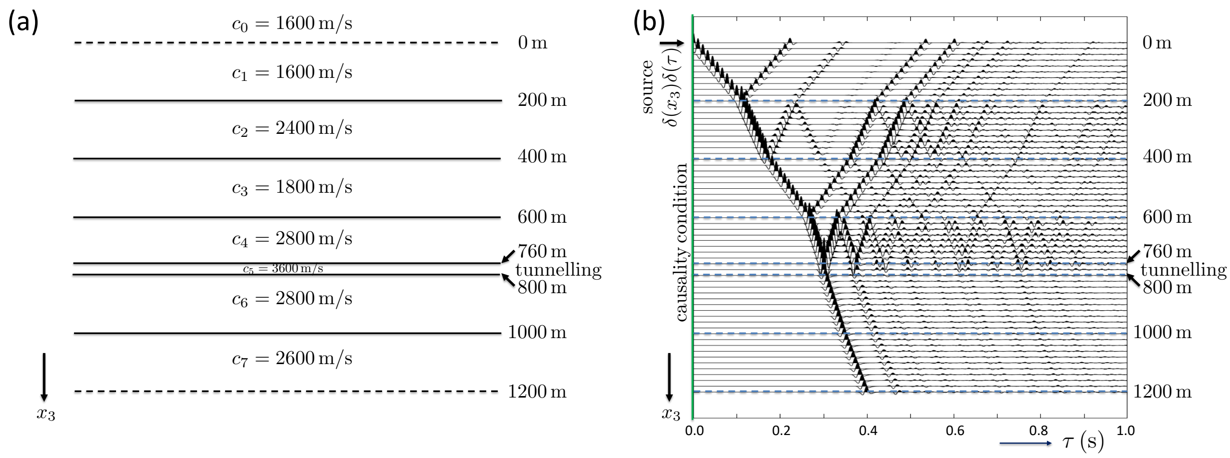

Figure 1 *(a) Horizontally layered medium. (b) Green's function* $G^{p,f}(s_1,x_3,x_{3,S},\tau)$.

From equations (2), (5) and (6) it follows that the first superscripts (p and v) refer to the wave-field quantities in $\mathbf{q}(\mathbf{x})$ and that the second superscripts (f and q) refer to the source quantities in $\mathbf{d}(\mathbf{x}_S)$.

We present an example of the component $G^{p,f}(\mathbf{x},\mathbf{x}_S)$ of the Green's matrix for the horizontally layered medium of Figure 1(a). We apply plane-wave decomposition into the horizontal slowness-intercept time (s_1,τ) domain, hence, we consider the transformed Green's function $G^{p,f}(s_1,x_3,x_{3,S},\tau)$. The causality condition of equation (4) transforms to $G^{p,f}(s_1,x_3,x_{3,S},\tau<0)=0$. We choose the source at the upper boundary, hence $x_{3,S}=0$. Moreover, we consider a single slowness $s_1 = 1/3500$ s/m. This implies that the wave field in the thin layer with velocity 3600 m/s becomes evanescent. The Green's function, convolved with a Ricker wavelet with a central frequency of 50 Hz, is shown in Figure 1(b). The vertical green line at $\tau=0$ indicates the aforementioned causality condition (i.e., the field left of this line is zero). This figure shows the evolution of the Green's wave field through space and time, including all primary and multiple reflections, and tunneling through the thin high-velocity layer.

Propagator matrix

We define the propagator matrix $\mathbf{W}(\mathbf{x},\mathbf{x}_A)$ for a 3D inhomogeneous medium as the solution of matrix-vector wave equation (1), with the source vector set to zero, hence

$$\partial_3\mathbf{W}-\mathbf{A}\mathbf{W}=\mathbf{O}, \tag{7}$$

where $\mathbf{x}_A=(x_{1,A},x_{2,A},x_{3,A})$ denotes a position in space. To get a unique solution, we imply a boundary condition at the horizontal boundary $\partial\mathbb{D}_A$ (defined as $x_3=x_{3,A}$), i.e.,

$$\mathbf{W}(\mathbf{x},\mathbf{x}_A)|_{x_3=x_{3,A}}=\mathbf{I}\delta(\mathbf{x}_\mathrm{H}-\mathbf{x}_{\mathrm{H},A}), \tag{8}$$

where $\mathbf{x}_\mathrm{H}=(x_1,x_2)$ and $\mathbf{x}_{\mathrm{H},A}=(x_{1,A},x_{2,A})$ are the horizontal coordinates of $\mathbf{x}$ and $\mathbf{x}_A$, respectively. Using Huygens' superposition principle again, we can express $\mathbf{q}(\mathbf{x})$ in terms of $\mathbf{W}$ and $\mathbf{q}$ at $\partial\mathbb{D}_A$, according to (Kennett, 1972; Woodhouse, 1974)

$$\mathbf{q}(\mathbf{x})=\int_{\partial\mathbb{D}_A}\mathbf{W}(\mathbf{x},\mathbf{x}_A)\mathbf{q}(\mathbf{x}_A)\mathrm{d}^2\mathbf{x}_A, \tag{9}$$

assuming there are no sources for $\mathbf{q}$ in the region between $x_{3,A}$ and x_3. This representation accounts for all propagation angles and for evanescent waves. Similarly, the Green's matrix can be expressed as

$$\mathbf{G}(\mathbf{x},\mathbf{x}_S)=\int_{\partial\mathbb{D}_A}\mathbf{W}(\mathbf{x},\mathbf{x}_A)\mathbf{G}(\mathbf{x}_A,\mathbf{x}_S)\mathrm{d}^2\mathbf{x}_A, \tag{10}$$

assuming $\mathbf{x}$ and $\mathbf{x}_S$ lie at opposite sides of $\partial\mathbb{D}_A$. More general representations using the propagator matrix $\mathbf{W}$ are given by Wapenaar (2022). The propagator matrix is partitioned as follows

$$\mathbf{W}(\mathbf{x},\mathbf{x}_A)=\begin{pmatrix}W^{p,p} & W^{p,v}\\ W^{v,p} & W^{v,v}\end{pmatrix}(\mathbf{x},\mathbf{x}_A). \tag{11}$$

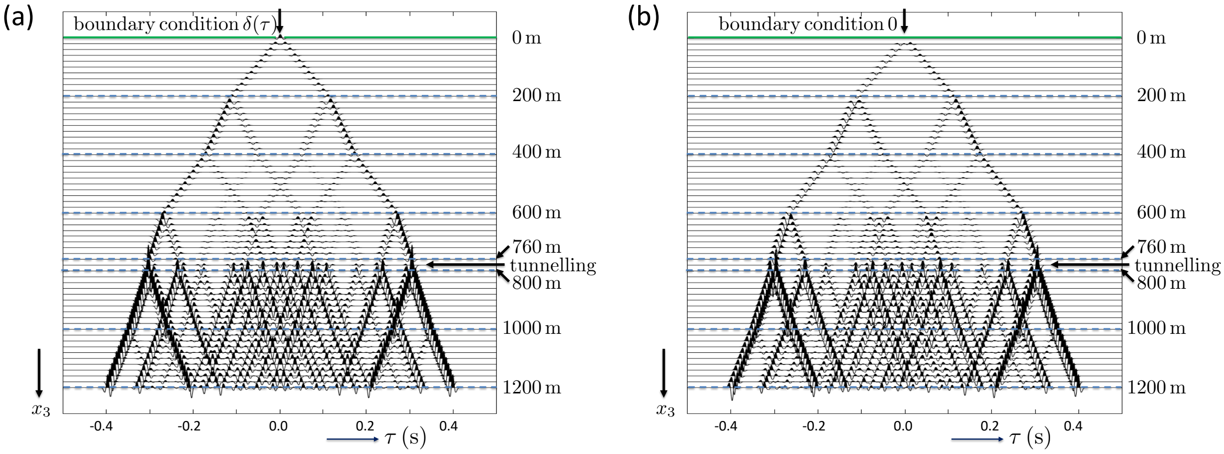

Figure 2 *(a) Propagator matrix component* $W^{p,p}(s_1,x_3,x_{3,A},\tau)$*. (b) Component* $W^{p,v}(s_1,x_3,x_{3,A},\tau)$*.*

From equations (2), (9) and (11) it follows that the first superscripts (p and v) refer to the wave-field quantities in $\mathbf{q}(\mathbf{x})$ and that the second superscripts (p and v) refer to the wave-field quantities in $\mathbf{q}(\mathbf{x}_A)$. We present an example of the components $W^{p,p}(\mathbf{x},\mathbf{x}_A)$ and $W^{p,v}(\mathbf{x},\mathbf{x}_A)$ of the propagator matrix for the horizontally layered medium of Figure 1(a). In the (s_1,τ) domain, the boundary condition of equation (8) transforms for these components to $W^{p,p}(s_1,x_{3,A},x_{3,A},\tau)=\delta(\tau)$ and $W^{p,v}(s_1,x_{3,A},x_{3,A},\tau)=0$. We choose $x_{3,A}=0$ and $s_1=1/3500$ s/m. The two components, convolved with the same Ricker wavelet, are shown in Figures 2(a) and 2(b). The horizontal Green lines at $x_3=0$ indicate the aforementioned boundary conditions. These figures show the evolution of the propagator matrix through space and time, including all primary and multiple reflections, and tunneling through the thin high-velocity layer.

Focusing functions

Consider a 3D inhomogeneous lower half-space, below a homogeneous upper half-space, separated by $\partial\mathbb{D}_A$. For this situation we previously defined a flux-normalised focusing function $f_2(\mathbf{x},\mathbf{x}_A,t)$, with $\mathbf{x}_A$ at $\partial\mathbb{D}_A$, which focuses as $f_2(\mathbf{x},\mathbf{x}_A,t)|_{x_3=x_{3,A}}=\delta(\mathbf{x}_\mathrm{H}-\mathbf{x}_{\mathrm{H},A})\delta(t)$ and subsequently propagates upward in the homogeneous upper half-space (Wapenaar et al., 2013). Here we define a pressure-normalized focusing function $F^p(\mathbf{x},\mathbf{x}_A,t)$ with the same conditions. For the horizontally layered medium of Figure 1(a), its transformed version $F^p(s_1,x_3,x_{3,A},\tau)$ is shown in Figure 3(a). Note that it resembles a part of the propagator matrix component $W^{p,p}(s_1,x_3,x_{3,A},\tau)$ of Figure 2(a). As a matter of fact, if we add its time-reversed version $F^p(s_1,x_3,x_{3,A},-\tau)$ (shown in Figure 3(b)) we get $W^{p,p}(s_1,x_3,x_{3,A},\tau)=\frac{1}{2}\{F^p(s_1,x_3,x_{3,A},\tau)+F^p(s_1,x_3,x_{3,A},-\tau)\}$, or, after a transform from τ to ω, $W^{p,p}(s_1,x_3,x_{3,A},\omega)=\Re\{F^p(s_1,x_3,x_{3,A},\omega)\}$, where $\Re$ denotes the real part. For a 3D inhomogeneous lower half-space, ignoring evanescent waves at and above $\partial\mathbb{D}_A$, this can be generalised in the space-frequency domain as

$$\mathbf{W}(\mathbf{x},\mathbf{x}_A)=\begin{pmatrix}W^{p,p} & W^{p,v}\\ W^{v,p} & W^{v,v}\end{pmatrix}(\mathbf{x},\mathbf{x}_A)=\begin{pmatrix}\Re\{F^p(\mathbf{x},\mathbf{x}_A)\} & -i\omega\rho_0\mathscr{H}_1^{-1}(\mathbf{x}_A)\Im\{F^p(\mathbf{x},\mathbf{x}_A)\}\\ i\Im\{F^v(\mathbf{x},\mathbf{x}_A)\} & -\omega\rho_0\mathscr{H}_1^{-1}(\mathbf{x}_A)\Re\{F^v(\mathbf{x},\mathbf{x}_A)\}\end{pmatrix}, \tag{12}$$

where $\Im$ denotes the imaginary part, ρ_0 is the mass density of the homogeneous upper half-space, $F^v(\mathbf{x},\mathbf{x}_A)$ is the particle velocity counterpart of the pressure-normalised focusing function $F^p(\mathbf{x},\mathbf{x}_A)$ and $\mathscr{H}_1$ is the square-root of the Helmholtz operator $\omega^2/c_0^2+\partial_\alpha\partial_\alpha$, with c_0 the propagation velocity of the upper half-space (for the derivation of equation (12) and subsequent equations, see Wapenaar (2022), but note that $F^v(\mathbf{x},\mathbf{x}_A)$ is defined differently there). Conversely, the focusing functions F^p and F^v can be expressed in terms of the components of the propagator matrix as follows

$$F^n(\mathbf{x},\mathbf{x}_A)=W^{n,p}(\mathbf{x},\mathbf{x}_A)-\frac{1}{\omega\rho_0}\mathscr{H}_1(\mathbf{x}_A)W^{n,v}(\mathbf{x},\mathbf{x}_A), \tag{13}$$

where superscript n stands for p or v. Finally, focusing and Green's functions are mutually related via

$$2G^{p,f}(\mathbf{x},\mathbf{x}_S) \;=\; \int_{\partial\mathbb{D}_A}F^p(\mathbf{x},\mathbf{x}_A)R(\mathbf{x}_A,\mathbf{x}_S)\mathrm{d}^2\mathbf{x}_A+F^{p*}(\mathbf{x},\mathbf{x}_S), \tag{14}$$

where $R(\mathbf{x}_A,\mathbf{x}_S)$ is the reflection response at $\partial\mathbb{D}_A$ of the 3D inhomogeneous lower half-space.

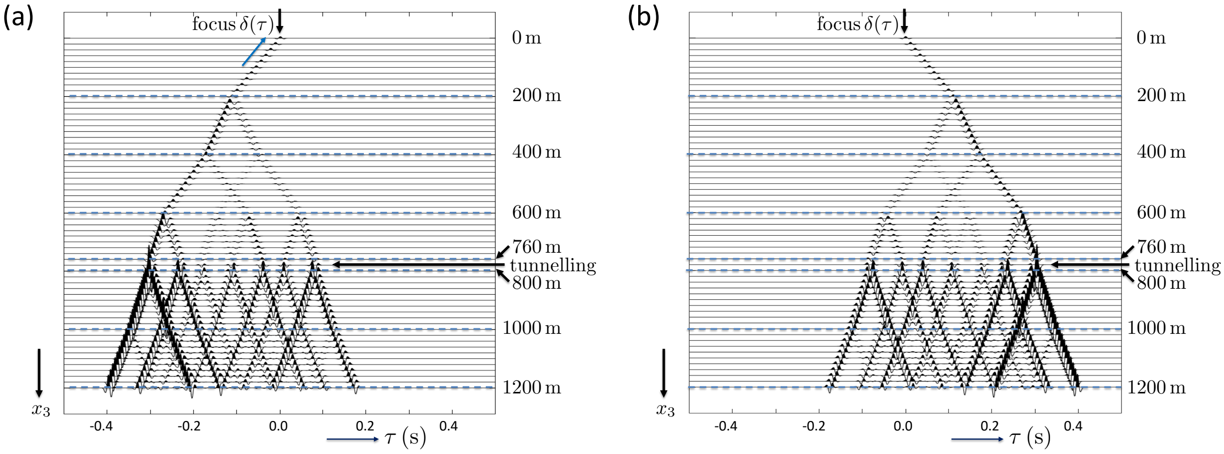

Figure 3 *(a) Focusing function* $F^p(s_1, x_3, x_{3,A}, \tau)$. *(b) Its time-reversed version* $F^p(s_1, x_3, x_{3,A}, -\tau)$.

Discussion and conclusions

In our previous work on the Marchenko method we derived expressions like equation (14) under the assumption that the focusing functions and Green's functions at **x** inside the medium can be decomposed into downgoing and upgoing components, and that evanescent waves can be ignored. Recently, Diekmann and Vasconcelos (2021) and Wapenaar et al. (2021) showed that up-down decomposition can be avoided and that evanescent waves can be taken into account. Here, we explicitly expressed the focusing functions in terms of components of the propagator matrix, which account for all propagation angles and evanescent waves (we only ignored evanescent waves at and above the boundary $\partial\mathbb{D}_A$). Hence, the representation of equation (14) forms a basis for extending the Marchenko method to improve the imaging of steeply dipping flanks and to account for refracted waves. Our current work deals with developing such methods and with pushing the limits of the Marchenko-type representations even further, by also accounting for evanescent waves in the upper half-space and for acoustic and elastic media with losses (Dukalski et al., 2022; Wapenaar et al., 2022).

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References

Bleistein, N. [1984] *Mathematical methods for wave phenomena*. Academic Press, Inc., Orlando.

Corones, J.P. [1975] Bremmer series that correct parabolic approximations. *J. Math. Anal. Appl.*, **50**, 361–372.

Diekmann, L. and Vasconcelos, I. [2021] Focusing and Green's function retrieval in three-dimensional inverse scattering revisited: A single-sided Marchenko integral for the full wave field. *Physical Review Research*, **3**, 013206.

Dukalski, M., Reinicke, C. and Wapenaar, K. [2022] Towards understanding the impact of the evanescent elastodynamic mode coupling in Marchenko equation based de-multiple methods. In: *SEG, Expanded Abstracts*. 2827–2831.

Fishman, L. and McCoy, J.J. [1984] Derivation and application of extended parabolic wave theories. I. The factorized Helmholtz equation. *J. Math. Phys.*, **25**(2), 285–296.

Kennett, B.L.N. [1972] Seismic waves in laterally inhomogeneous media. *Geophysical Journal of the Royal Astronomical Society*, **27**, 301–325.

Rayleigh, J.W.S. [1878] *The theory of sound. Volume II*. Dover Publications, Inc. (Reprint 1945).

Ursin, B. [1983] Review of elastic and electromagnetic wave propagation in horizontally layered media. *Geophysics*, **48**, 1063–1081.

Wapenaar, K. [2022] Wave-field representations with Green's functions, propagator matrices, and Marchenko-type focusing functions. *Journal of the Acoustical Society of America*, **151**(1), 587–608.

Wapenaar, K., Broggini, F., Slob, E. and Snieder, R. [2013] Three-dimensional single-sided Marchenko inverse scattering, data-driven focusing, Green's function retrieval, and their mutual relations. *Physical Review Letters*, **110**, 084301.

Wapenaar, K., de Ridder, S., Dukalski, M. and Reinicke, C. [2022] The propagator and transfer matrix for a 3D inhomogeneous dissipative acoustic medium, expressed in Marchenko focusing functions. In: *SEG, Expanded Abstracts*. 3141–3145.

Wapenaar, K., Snieder, R., de Ridder, S. and Slob, E. [2021] Green's function representations for Marchenko imaging without up/down decomposition. *Geophysical Journal International*, **227**, 184–203.

Woodhouse, J.H. [1974] Surface waves in a laterally varying layered structure. *Geophysical Journal of the Royal Astronomical Society*, **37**, 461–490.