Technische Universiteit Delft
Faculteit Elektrotechniek, Wiskunde en Informatica
Delft Institute of Applied Mathematics

De elementaire deeltjes en het wortel systeem E_8.

(Engelse titel: The elementary particles and the root system E_8.)

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list of variables

Variables	Description
Φ	Root system
Δ	Base
$\vec{\alpha}, \vec{\beta}$	Roots
$\vec{\gamma}$	Simple Roots
$\vec{\lambda}$	Weights
$\mathscr{W}$	Weyl Group
σ, τ	Operator / Weyl Group element

Contents

Abstract

E_8 is a famous root system in mathematics. Lisi claims in his paper "An Exceptionally Simple Theory of Everything"[1] that there is a connection between the standard model of the elementary particles and this root system. He claims that standard model has the same structure as the root system E_8. If this is true, there must be a mapping from the standard model to E_8. This map must identify each elementary particle with an unique root of E_8. Also the four charges (weak isospin, weak hypercharge, g_3 and g_8) of the standard model must have corresponding vectors in $\mathbb{R}^8$. These four charge vectors must have the property that taking the inner product of the charge vector and the root identified with an elementary particle, it would result in the charge of that particle. This is needed to guarantee that we have conservation of charge. In this paper we have shown that it is not possible to find a map with four charge vectors. This was done by studying root systems and how different root systems are present in E_8 as sub root systems. We also proved that the intersection of E_8 with the orthogonal complement of a charge, must be a root system. This limits the possibilities for the charge vectors. For these possible charges it has been checked whether there are enough roots for a given charge to identify the elementary particles. Since this was not the case, we concluded that no mapping with four charge vectors exists.

Chapter 1

Introduction

In 1869 Dmitri Mendeleev published his periodic table of the elements. Mendeleev constructed the periodic table of the elements by listing all the elements in columns ordered by atomic weight. Starting a new row or column when the characteristics of the elements began to repeat. The ordering of atomic weight led Mendeleev to be able to accurately predict missing elements which later have been discovered. In 1913 Henry Moseley used X-ray spectroscopy to show that the rows in Mendeleev's table are actually ordered based on atomic number which is closely related to the atomic mass Mendeleev used to order his periodic table of the elements. Mendeleev his idea was not based on experimental results or theoretical knowledge, but was driven by trying to see the structure behind the atoms. This pragmatic way of ordering the elements eventually led to a better understanding about atoms and how they are ordered.

Lisi had a similar idea for the ordering of all the elementary particles. In his paper "An Exceptionally Simple Theory of Everything" he connects all the elementary particles with the root system E_8 [1]. Similar to Mendeleev, he tries to find the structure which connects all elementary particles together. When he published his ideas it was quickly shown that this was not possible by multiple papers [2].

All root systems are very important to mathematics. Every simple Lie group is categorized by a root system. The root system E_8 is the largest and most symmetric irreducible exceptional root system. It has 240 roots in $\mathbb{R}^8$ and these roots give the solutions to the kissing problem (the maximum number of spheres kissing one sphere), as well as the the sphere packing problem (most dense packing of spheres). In physics, the elementary particles from the standard model are represented by the Lie groups $U(1), SU(2), SU(3)$. The root systems associated with these Lie groups are sub root systems of E_8.

We will take a look to the idea of trying to match all the elementary particles to the roots of E_8 together with four charge vectors. These charge vectors have the property that an inner product of the charge vector with a root identified with a particle, results in the charge. We will try to show that this is not possible by just looking at the charge values of the elementary particles and theorems about root systems. Until now it has been shown that this is not possible via physics arguments[2]. In this paper we will show it is not possible by keeping the particle physics on a basic level and try to reason from the mathematical properties that this is not possible. For this we will first discuss which elementary particles there are and what charges they have. In the third chapter we will study root systems in general followed in chapter 4 where the link between root systems and the elementary particles is made. In chapter 5 my own theorems are stated with their proofs where we will explore how to determine sub root systems in a larger root system. In chapter 6 the results of exploring the sub root systems of E_8 is given. In chapter 7 these results together with the theorems in chapter 5 come together to form an argument why it is not possible to place all the elementary particles in E_8 followed by a discussion in chapter 8.

Chapter 2

The Standard Model

Elementary particles are the building blocks of nature. They can not be split up in more fundamental components. All matter and radiation we observe, is made of a collection of elementary particles. The elementary particles and their interactions is a very large part of Physics. There are 4 known fundamental interactions, these are Gravity, Electromagnetic force, the Weak force and the Strong force. Each fundamental interaction acts on a certain set of particles. These interactions are mediated by the elementary particles themselves. Each fundamental force has its own mediator particles which are used to transfer the force. How these particles interact will be described in Chapter 3 where the link between the particles, Lie algebra's and root systems is studied.

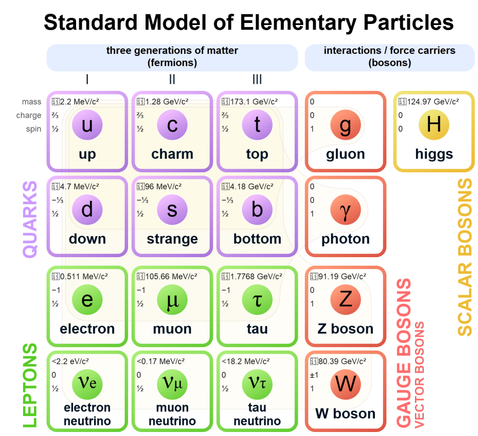


Figure 2.1: The different particles catagorized. The catagorization will be discussed in the following pages. Note that the gravition is not in this figure as it is still theoretical.

2.1 Particles

Each elementary particle has an anti-particle. An anti-particle is similar to the particle but has opposite charges. Also the elementary particles can be split up in groups. These groups will be discussed with the help of to Figure 2.1. However the gravitons are not shown since they are still theoretical. All the particles will be discussed based on their properties. The first distinction we will make is between bosons and fermions. Bosons have integer spin (0,1,2) and are the force carriers while fermions are spin half ($\frac{1}{2}$) and make up matter.

2.1.1 Bosons

The bosons are the force carriers. For example photons carry the electromagnetic force. The other forces are gravity, the strong force, the weak force. The electromagnetic force and weak force have been united and are in fact two different aspects of the same force: the electroweak force. For this reason these two forces will be discussed as one. The bosons will be introduced in the following, according to the type of force.

Electroweak force

There are four particles that mediate the electroweak force. Namely the photon and the 3 vector bosons Z^0, W^+, W^-. These all have spin 1. The superscript denotes the electric charge of a particle. So the W^+ is a positively electrically charged particle with an electric charge of 1e[1]. Out of these 4 particles the photon and the Z^0 vector boson are their own anti-particles. The W^+ and W^- are anti-particles of each other. So the anti-particle of a W^+ is a W^- particle and the anti-particle of a W^- particle is a W^+ particle.

An annihilation of a W^+ and W^- vector boson which result in a photon can be seen in Figure 2.2. This Figure shows a Feynman diagram of this annihilation. The direction of time is here from left to right, first the two vector bosons come together and interact to form a photon in the end. The arrows indicate the "flow" of charge of each particle. The direction of time can be chosen arbitrary in this graph (see Figure 2.3). One can also start with a photon and a W^+ vector boson. In this case the W^- particle would "flow" out of the interaction. Now the arrow of the W^- particle would need to "flip" since the charge "flows" out of the interaction. This "flip" results that instead of the W^- particle is produced the anti-particle is actually produced. The anti-particle of a W^- particle is a W^+ particle.

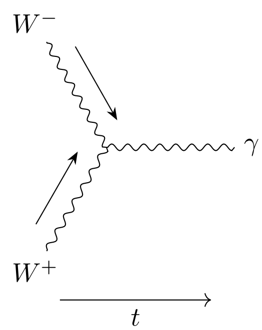


Figure 2.2: A W^- and a W^+ intermediate vector boson interact to form a photon.

Weak Isospin & Weak Hyper charge

The weak isospin charge denoted by T_3, It's value quantifies the coupling strength of the weak interaction. For example in Figure 2.2 a W^- vector boson interacts with a W^+ vector boson to produce an photon. Here the W^- has weak isospin -1 and the W^+ has weak isospin 1. Isospin is conserved as the photon has 0 weak isospin. Furthermore, the weak isospin is also conserved under electromagnetic force and the strong force. It is thus one of the conserved charges.

The weak hypercharge is another coupling strength of the electric interaction. The weak hypercharge is denoted by Y_w and it's total value is also conserved. In Figure 2.2 the W^- has weak hypercharge 0 and the W^+ has weak hypercharge 0. The resulting photon has weak hypercharge 0. The weak isospin

[1] The electron has charge $-e$ and by definition e is the elementary electric charge.

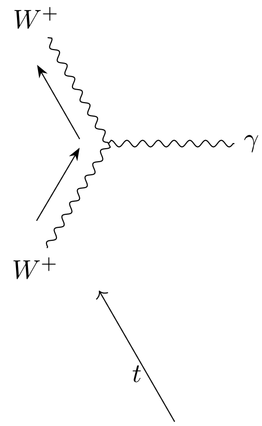


Figure 2.3: A W^+ absorbs a photon.

and weak hypercharge arise from representation theory[2]. T_3 and Y_w relate to the electric charge of a particle by the formula:

$$Q = T_3 + \frac{1}{2}Y_w$$

See Figure 2.2. The electric charge must also be conserved as both T_3 and Y_w where conserved. This can also be calculated as the W^- has $-1 + \frac{1}{2}0 = -1$ and the W^+ has $1 + \frac{1}{2}0 = 1$. The resulting photon has electric charge $0 + \frac{1}{2}0 = 0$ and total electric charge is thus conserved in the interaction showing in Figure 2.2. For an overview of the calculations see Table 2.3.

Strong force

There are 3 different colors in the strong force. Namely red, green and blue. Note that these colors have nothing to do with the actual color of a particle. The reason these charges have color names is that the charges are very similar to a color wheel, e.g. the charges red added to green and blue results in a particle which is neutral (or for the comparison to a color wheel simply white). Besides these 3 colors there are 3 anti colors (for the anti-particles). Namely anti-red, anti-green and anti-blue. Similar to the color wheel red and anti-red combine to form a neutral color. The colors can be arranged in a 2D plane as shown in Figure 2.4. In this figure two vectors g_3 and g_8 are also shown. These two vectors can be used to represent the color charges in the 2D plane and thus fully describe the color charges.

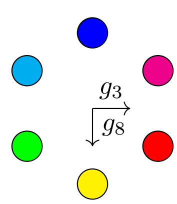


Figure 2.4: The 6 different color charges. The three charges red, green and blue and there anti charges anti-red (colored turquoise), anti-green (colored magenta) and anti-blue (colored yellow).

[2]Representation theory will be discussed in Chapter 3.

gluons

The strong force is mediated by 8 different gluons. Gluons are spin 1 particles and the 8 gluons always have a color and anti-color, they carry the color between quarks[3] (Quarks have a single color charge as opposed to gluons). However the color is conserved so when a red quarks emits a gluon and turns green, the gluon must carry the color red and anti-green so that the total charge is conserved. For this reason gluons always carries a color and anti-color. This would mean that there are a total of 3×3 gluons. However there does not exist a gluon in the singlet state so that in total there are just 8 gluons. The singlet state is given by:

$$\frac{1}{\sqrt{3}}\left(g\bar{g} + r\bar{r} + b\bar{b}\right)$$

In Figure 2.5 a red anti-blue gluon interacts with a green anti-red gluon. Since charge is conserved this interaction must result in a green anti-blue gluon.

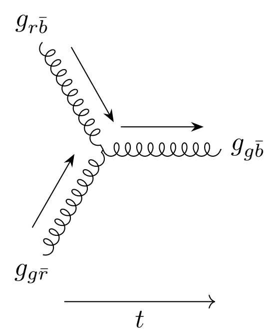


Figure 2.5: A red anti-blue gluon and a green anti-red gluon interact to form a green anti-blue gluon.

Gravity

There is only one particle which mediates the gravity force: the graviton. This particle is however hypothetical. The graviton is theorized to be it's own anti-particle and to have spin 2. An interaction of a graviton and a vector boson can be seen in Figure 2.6. Here the vector boson absorbs the graviton. The vector boson can be replaced with any other elementary particle (including the graviton itself) because any particle with energy interacts with gravity.

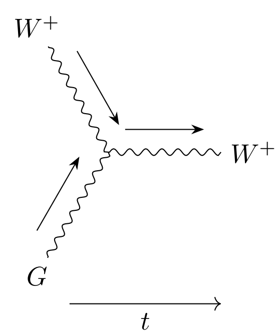


Figure 2.6: A vector boson interacts with a graviton.

[3]Quarks will be discussed in the next section about fermions.

Higgs

In order to get the standard model to work (and to be able to renormalize the states), the Higgs field was introduced with 4 degrees of freedom. These degrees of freedom are all Higgs particles so we count 4 different Higgs particles. However when the universe cooled down, the Higgs particle was forced to choose a lower point in the higgs potential which broke the symmetry of it's original position. This is called symmetry breaking. Due to symmetry breaking the Higgs field obtains a now zero value. This means that particles that interact with the Higgs field obtain mass. Due to this symmetry breaking 3 of the 4 degrees of freedom merge with the 3 vector bosons W^+, W^-, Z^0. This is why in our present day we only observe one Higgs particle with spin 0. In figure 2.7 a Higgs can be seen interacting with an up quark. An up quark is a fermion and will be discussed in the following chapter. The up quark in this figure can be replaced by any other particle with mass.

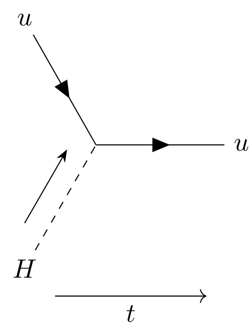


Figure 2.7: An up quark interacts with a Higgs.

2.1.2 Fermions

As discussed in the beginning of the chapter fermions are spin half particles and make up matter. There are two types of fermions: quarks and leptons.

Quarks

Quarks are the building blocks of protons and neutron (and many other types of Baryons). Quarks are also the only elementary particles which participate in all the Fundamental interactions. Also, they are the only elementary particles with an electric charge which is not an integer multiple of the elementary electric charge[4]. In total there are six different quarks which can be seen in Table 2.1. In this table the quarks are ordered in three different columns and two different rows. The rows are based on the electric charge while the columns are based on "generation" number.

Table 2.1: The six different types of quarks. The rows are ordered based on the electric charge while the columns are ordered based on the "generation."

generation	I	II	III
electric charge: $\frac{2}{3}e$	up	charm	top
electric charge: $-\frac{1}{3}e$	down	strange	bottom

Several quarks combine together to form a Baryon. For example, a proton is made up of two up quarks and one down quarks while the neutron is made up of one up quarks and two down quarks (protons and neutrons are thus examples of Baryons). Many other combinations of quarks will create other

[4]the elementary electric charge (the charge of an electron) was discovered before the quarks. Instead of redefining the elementary charge, physicists kept the convention and the elementary charge is thus not the smallest unit of charge found in the universe.

particles. Quarks can change type by interacting with vector bosons. One of these interactions can be seen in Figure 2.8. Here a down quark interacts with a W^+ vector boson to form an up quark. In this figure the down quark can also be replaced by a strange or bottom quark while the up quark can be replaced by a charm and top quark.

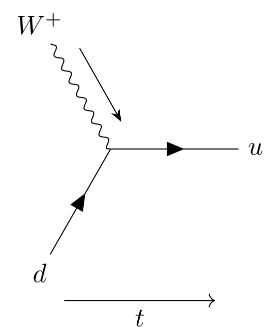


Figure 2.8: A down quark and a W^+ vector boson interact to form a up quark.

As discussed before quarks also have a single color charge which means that in total there are $6 \times 6 = 36$ quarks (6 types of quarks with all 6 possible color charges). These quarks are denoted by the first letter of the type (from Table 2.1) and a sub script of the letter of the color. So a red up quark is u_r. Note that for a anti-quark the anti is depicted as a bar above the letter of the color. So an anti-red up quark is $u_{\bar{r}}$.

Quarks and gluons can interact as in Figure 2.9 where a red up quark interacts with a green up quark by emiting a red and anti-green gluon. By emiting this gluon the red up quark turns into a green up quark. The gluon eventualy interacts with the green up quark which turns into a red up quark.

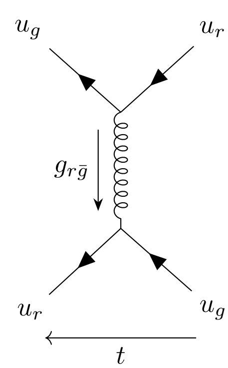


Figure 2.9: A Feynman diagram of a red up quark interacting with a green up quark. By exchanging a gluon the color charge of the quarks swap. Note that we have two interaction here. Each vertex in the diagram depicts an interaction. These combined interactions describe the process of how quarks interact via the strong interaction.

Leptons

Leptons are elementary particles with halfspin that do not interact with the strong force. The best example of a lepton is the electron. The electron has spin $\frac{1}{2}$ and does not interact with the strong force. In total there are six different leptons which can be divided in three electrically charged leptons (e.g. the electron) and 3 leptons without an electric charge. The leptons with electric charge are also called electron-like leptons and the leptons without charge are called neutrinos. The six different leptons can be seen in Table 2.2. The rows are again ordered on electric charge and the columns on generations.

Table 2.2: The six different leptons divided by charge and "generation' number'.

generation	I	II	III
electric charge: $-1e$	electron	muon	tau
electric charge: 0	electron neutrino	muon neutrino	tau neutrino

The total amount of Leptons is thus 12 as each Lepton has a unique anti-particle. Leptons can interact with bosons as seen in Figure 2.10. Here a W^- vector boson decays into an electron neutrino and an electron.

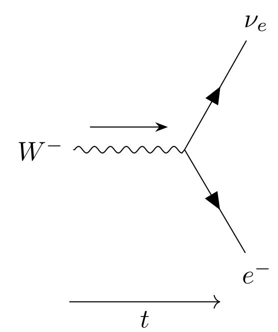


Figure 2.10: A W^- vector boson decays in to electron neutrino and a electron.

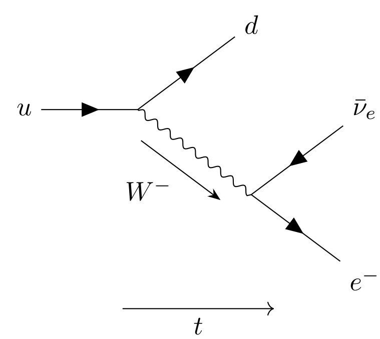


Figure 2.11: Beta decay is the process of a up quark changing in to a down quark while emitting a electron and neutrino. This is the combination of the two interactions depicted in Figure 2.8 and Figure 2.10.

generations

The quarks, electron-like leptons and the neutrinos can each be subdivided in to three generations. These generations can be seen in Figure 2.1 and the Tables 2.1 and 2.2. This division is based on an increase in mass of particles in higher generations for the same electric charge (so a quark with 2/3 electric charge from generation 2 has more mass than a quark from generation 1 with 2/3 electric charge). Since the particles have more mass (in a higher generation for particles with the same electric charge) they will decay in to lower generations (with less mass). The ordering of the quarks and leptons based on generations might seem strange. There is in fact no compelling reason to put the quarks and leptons in the same generation [3]. This ordering might however show to be useful when leptons and quarks are better understood.

2.1.3 Helicity

The Helicity of a particle is the projection of the spin onto the direction of motion for a given particle. This doubles the particles in to right handed, if the spin and motion are in the same direction, or left handed, if the spin and motion are opposite. This effectively doubles the number of particles as each particle can be left handed or right handed. Neutrino's are the exception as for the neutrino's only left handed neutrino's and right handed anti-neutrino's have been observed. The Higgs particles do not have Helicity since they have spin 0 and have no projection.

If we count all the helicity states and anti-particles as different single particle states, there are 30 bosons and 90 fermions. All the elementary particles with there respective charges are given in Table 2.3. Note that all the charge values have been normalized so that all the charges are integer values. This scaling is no problem as will become evident later on.

By now counting all the particle states we find that there are a total of 120 elementary particles in the standard model. All these particles can be seen in Table 2.3.

Table 2.3: All 120 elementary particles with the given charges. Note that the values in the table are scaled to be integer.

Type	Number	What	H	$2T_3$	$3Y_w$	$2g_3/\sqrt{3}$	$2g_8$	Note
Higgs bosons	1	Higgs		1	3	0	0	
	2	Higgs		1	-3	0	0	
	3	Higgs		-1	-3	0	0	
	4	Higgs		-1	3	0	0	
vector bosons	5	W+	L	2	0	0	0	
	6	W+	R	2	0	0	0	
	7	W-	L	-2	0	0	0	
	8	W-	R	-2	0	0	0	
	9	Z	L	0	0	0	0	Neutral
	10	Z	R	0	0	0	0	Neutral
gluons	11	gredantiblue	L	0	0	1	3	
	12	gredantiblue	R	0	0	1	3	
	13	gredantigreen	L	0	0	2	0	
	14	gredantigreen	R	0	0	2	0	
	15	gblueantigreen	L	0	0	1	-3	
	16	gblueantigreen	R	0	0	1	-3	
	17	gblueantired	L	0	0	-1	-3	
	18	gblueantired	R	0	0	-1	-3	
	19	ggreenantired	L	0	0	-2	0	
	20	ggreenantired	R	0	0	-2	0	
	21	ggreenantiblue	L	0	0	-1	3	
	22	ggreenantiblue	R	0	0	-1	3	
	23	g3	L	0	0	0	0	Neutral
	24	g3	R	0	0	0	0	Neutral
	25	g8	L	0	0	0	0	Neutral
	26	g8	R	0	0	0	0	Neutral
photons	27	gam	L	0	0	0	0	Neutral
	28	gam	R	0	0	0	0	Neutral
gravitons	29	G	L	0	0	0	0	Theoretical/Neutral
	30	G	R	0	0	0	0	Theoretical/Neutral
neutrinos	31	nue+	L	1	-3	0	0	
	32	nue-	R	-1	3	0	0	
	33	numu+	L	1	-3	0	0	
	34	numu-	R	-1	3	0	0	
	35	nutau+	L	1	-3	0	0	
	36	nutau-	R	-1	3	0	0	
leptons	37	e+	L	1	3	0	0	
	38	e+	R	0	6	0	0	
	39	e-	L	-1	-3	0	0	
	40	e-	R	0	-6	0	0	
	41	mu+	L	1	3	0	0	
	42	mu+	R	0	6	0	0	
	43	mu-	L	-1	-3	0	0	
	44	mu-	R	0	-6	0	0	
	45	tau+	L	1	3	0	0	
	46	tau+	R	0	6	0	0	
	47	tau-	L	-1	-3	0	0	
	48	tau-	R	0	-6	0	0	
quarks	49	upred	L	1	1	1	1	
	50	upred	R	0	4	1	1	
	51	upgreen	L	1	1	-1	1	
	52	upgreen	R	0	4	-1	1	
	53	upblue	L	1	1	0	-2	
	54	upblue	R	0	4	0	-2	
	55	upantired	L	-1	-1	-1	-1	
	56	upantired	R	0	-4	-1	-1	
	57	upantigreen	L	-1	-1	1	-1	
	58	upantigreen	R	0	-4	1	-1	
	59	upantiblue	L	-1	-1	0	2	
	60	upantiblue	R	0	-4	0	2	
quarks	61	downred	L	-1	1	1	1	
	62	downred	R	0	-2	1	1	
	63	downgreen	L	-1	1	-1	1	
	64	downgreen	R	0	-2	-1	1	
	65	downblue	L	-1	1	0	-2	
	66	downblue	R	0	-2	0	-2	
	67	downantired	L	1	-1	-1	-1	
	68	downantired	R	0	2	-1	-1	
	69	downantigreen	L	1	-1	1	-1	
	70	downantigreen	R	0	2	1	-1	
	71	downantiblue	L	1	-1	0	2	
	72	downantiblue	R	0	2	0	2	
	73	charmred	L	1	1	1	1	
	74	charmred	R	0	4	1	1	
	75	charmgreen	L	1	1	-1	1	
	76	charmgreen	R	0	4	-1	1	
	77	charmblue	L	1	1	0	-2	
	78	charmblue	R	0	4	0	-2	
	79	charmantired	L	-1	-1	-1	-1	
	80	charmantired	R	0	-4	-1	-1	
	81	charmantigreen	L	-1	-1	1	-1	
	82	charmantigreen	R	0	-4	1	-1	
	83	charmantiblue	L	-1	-1	0	2	
	84	charmantiblue	R	0	-4	0	2	
	85	strangered	L	-1	1	1	1	
	86	strangered	R	0	-2	1	1	
	87	strangegreen	L	-1	1	-1	1	
	88	strangegreen	R	0	-2	-1	1	
	89	strangeblue	L	-1	1	0	-2	
	90	strangeblue	R	0	-2	0	-2	
	91	strangeantired	L	1	-1	-1	-1	
	92	strangeantired	R	0	2	-1	-1	
	93	strangeantigreen	L	1	-1	1	-1	
	94	strangeantigreen	R	0	2	1	-1	
	95	strangeantiblue	L	1	-1	0	2	
	96	strangeantiblue	R	0	2	0	2	
	97	topred	L	1	1	1	1	
	98	topred	R	0	4	1	1	
	99	topgreen	L	1	1	-1	1	
	100	topgreen	R	0	4	-1	1	
	101	topblue	L	1	1	0	-2	
	102	topblue	R	0	4	0	-2	
	103	topantired	L	-1	-1	-1	-1	
	104	topantired	R	0	-4	-1	-1	
	105	topantigreen	L	-1	-1	1	-1	
	106	topantigreen	R	0	-4	1	-1	
	107	topantiblue	L	-1	-1	0	2	
	108	topantiblue	R	0	-4	0	2	
	109	bottomred	L	-1	1	1	1	
	110	bottomred	R	0	-2	1	1	
	111	bottomgreen	L	-1	1	-1	1	
	112	bottomgreen	R	0	-2	-1	1	
	113	bottomblue	L	-1	1	0	-2	
	114	bottomblue	R	0	-2	0	-2	
	115	bottomantired	L	1	-1	-1	-1	
	116	bottomantired	R	0	2	-1	-1	
	117	bottomantigreen	L	1	-1	1	-1	
	118	bottomantigreen	R	0	2	1	-1	
	119	bottomantiblue	L	1	-1	0	2	
	120	bottomantiblue	R	0	2	0	2	

One way to analyse this table is in Figure 2.12. In this Figure one can see the particles grouped on the condition that they are neutral to some charges. For example there are 36 particles neutral to the charge g_3 and g_8.

2.2 Charge Spectrum

A charge spectrum is defined as:

Definition 2.1. *A* ***spectrum*** *(or a* ***charge spectrum****) of a charge Q is defined as the set of all possible outcomes of that charge with their multiplicity the total number of particles that have that charge.*

What a charge spectrum is easiest demonstrated by an example. The charge spectrum of the charge g_3 is given in figure 2.13 for all 120 elementary particles from Table 2.3. Further note that this will be the convention for a spectrum. A vertical number line with the charge value on the left and the amount of particles as a value on the right. Note that the actual charge value here is the normalized values from Table 2.3.

In a similar fashion we can also create a 2D charge spectrum. In a two dimensional charge spectrum we have a two different charges on the two different axis. In the case of the 2D charge spectrum the actual charge values will be dropped since they clutter the figure of the charge spectrum while not being necessary. To explain why the actual charge value is unnecessary note that we can regard every particle in Table 2.3 as a vector in a $4D$ space with each dimension being a charge. This 4D space can be transformed by a change of basis resulting in 120 different vectors/particles with different charge values. These 120 transformed particle charges are off course still the same particles but the position and structure seemed to have changed. This is however not the case as the original structure can be obtained by inverting the change of basis. So as long as the actual change of basis is known, one can always go back to the original structure. Note that scaling of an axis is again a change of basis and thus is the absolute value of the charge not important as long as the change of basis is known. The 2D charge spectrum's will thus be shown without the actual charges. All different 2D charge spectra can be seen in Figure 2.14 where a change of basis has already been applied. Note that since the change of basis is known (e.g. in Figure 2.14(a) we have the charges T_3 and $-T_3 + Y_w$ as the x and y axis) one can back calculate the to the actual charges (e.g. T_3 is the x axis and Y_w is the y axis plus the x axis).

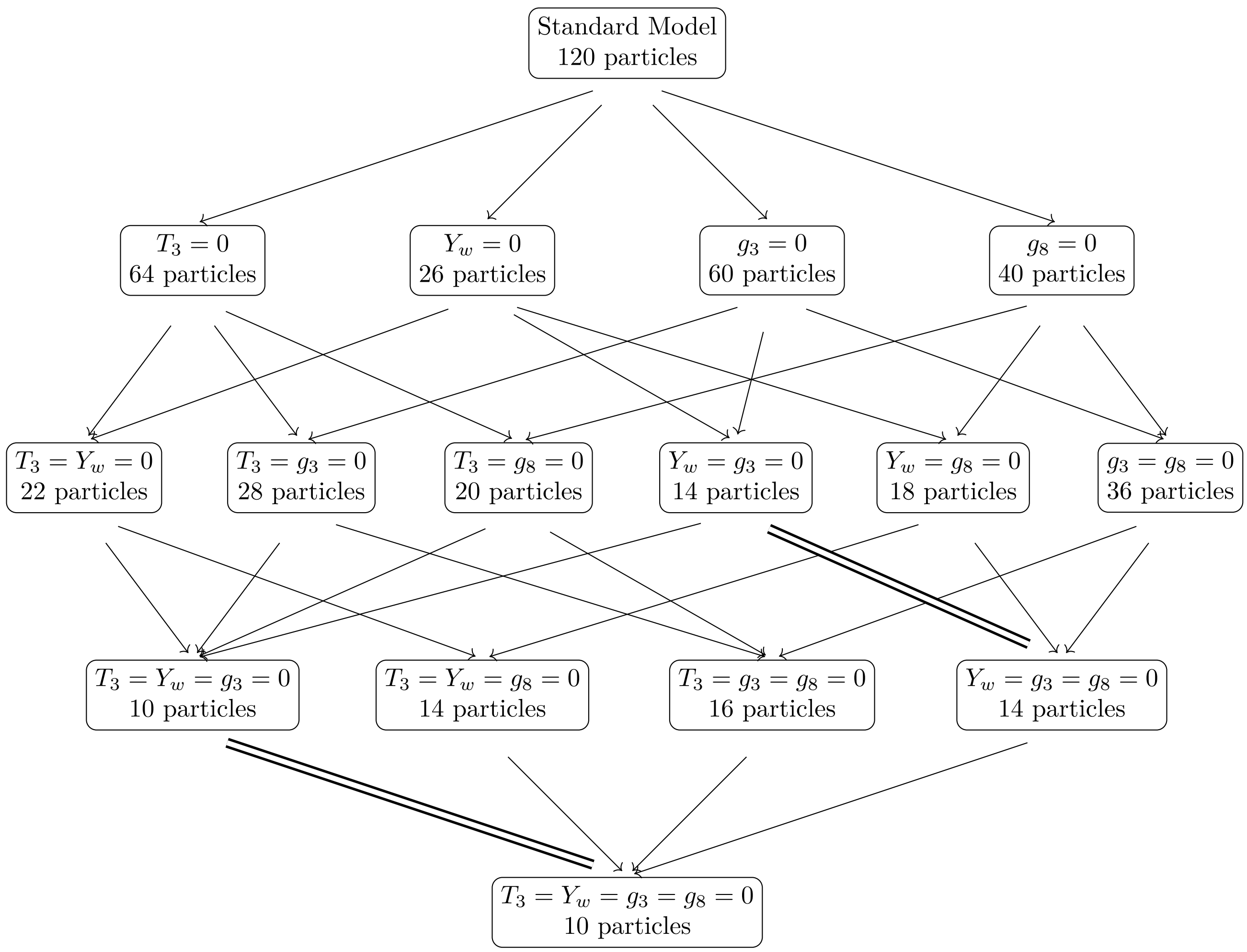


Figure 2.12: A graph of how many elementary particles are neutral to a given charge. Every layer deeper an extra charge has been set to zero. This reduces the amount of particles with the condition of those charges being zero. Note that two lines are double lines instead of arrows. Here we have that the condition $Y_w = g_3 = 0$ also means that $g_8 = 0$. There are no particles with $Y_w = g_3 = 0$ and a non zero g_8 charge.

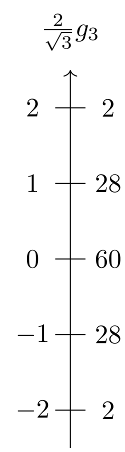
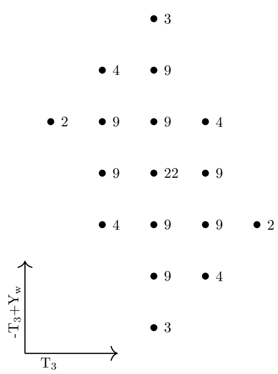
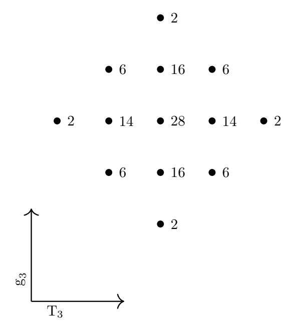


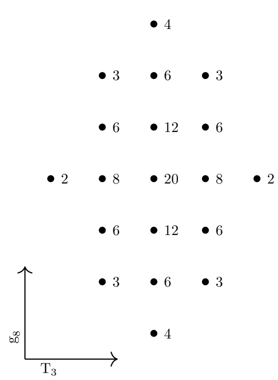
Figure 2.13: The spectrum of the charge g_3. Charge value range from -1 to 2 on the left. Multiplicities on the right.



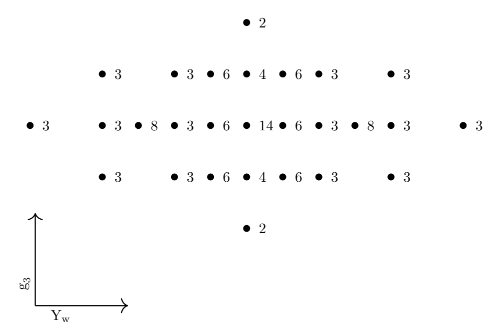
(a) T_3 against $-T_3 + Y_w$



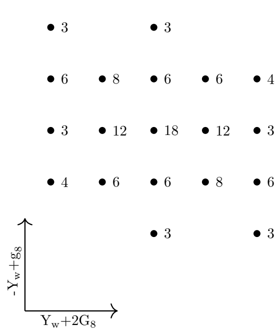
(b) T_3 against g_3



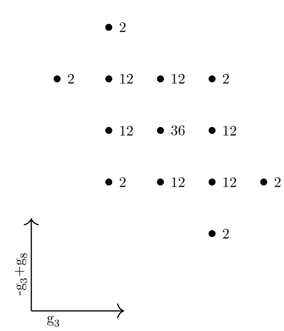
(c) T_3 against g_8



(d) Y_w against g_3



(e) $Y_w - 2g_8$ against $-Y_w + g_8$



(f) g_3 against $g_3 + g_8$

Figure 2.14: All the different charge spectrum's for all the combinations of the charges T_3, Y_w, g_3, g_8. The numbers indicate the amount of particles that have the same charge outcome.

Chapter 3

Root systems

In this section we will introduce root systems and some basic properties of root systems. Root systems are a finite set of vectors in an Euclidean space $\mathbb{R}^n$ (or more general in an Euclidean space V) with inner product $(\cdot,\cdot)$. For this reason we will assume to always work in a Euclidean space $\mathbb{R}^n$ with the standard inner product.

3.1 Root Systems

Before we can give the definition of a root systems we first have to introduce reflections in hyper-planes. A hyperplane is defined as the subspace of the euclidean space $\mathbb{R}^n$ consisting of the elements which are orthogonal to a non-zero vector $\vec{\alpha} \in \mathbb{R}^n$. Since this $\vec{\alpha}$ characterizes the hyper-plane we will denote the hyper-plane who's elements are orthogonal to $\vec{\alpha}$ as $H_{\vec{\alpha}}$. This hyper-plane is thus given by the set $H_{\vec{\alpha}} = \{\vec{\beta} \in \mathbb{R}^n | (\vec{\alpha}, \vec{\beta}) = 0\}$.

We are now at the stage where we can introduce the reflection operation. The reflection operation, just as a hyperplane, is linked to a non-zero vector $\vec{\alpha} \in \mathbb{R}^n$. The reflection operation $\sigma_{\vec{\alpha}}$ is the operation which leaves $H_{\vec{\alpha}}$ point-wise fixed and sends any vector orthogonal to the hyper-plane into its negative. This operation is defined as (3.1).

$$\sigma_{\vec{\alpha}}(\vec{\beta}) = \vec{\beta} - \frac{2(\vec{\beta}, \vec{\alpha})}{(\vec{\alpha}, \vec{\alpha})}\vec{\alpha} \tag{3.1}$$

Since the fraction $\frac{2(\vec{\beta},\vec{\alpha})}{(\vec{\alpha},\vec{\alpha})}$ occurs frequently we will abbreviate this to $\left\langle \vec{\beta}, \vec{\alpha} \right\rangle$. An simple example of the reflection operation can be seen in Figure 3.1.

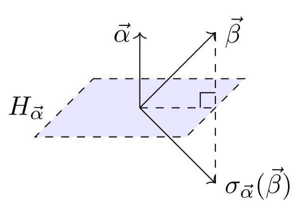


Figure 3.1: A vector $\vec{\alpha}$ and the hyperplane $H_{\vec{\alpha}}$ of the vector $\vec{\alpha}$. The reflection operation $\sigma_{\vec{\alpha}}$ acts here on a vector $\vec{\beta}$ and reflects the vector around the hyperplane to the vector $\sigma_{\vec{\alpha}}(\vec{\beta})$.

With the reflection operation defined we are ready to give the definition of a root system.

Definition 3.1. *A set Φ of the Euclidean space $\mathbb{R}^n$ (or Euclidean space V) is called a* **root system** *in $\mathbb{R}^n$ (or V) if the following axioms are satisfied:*

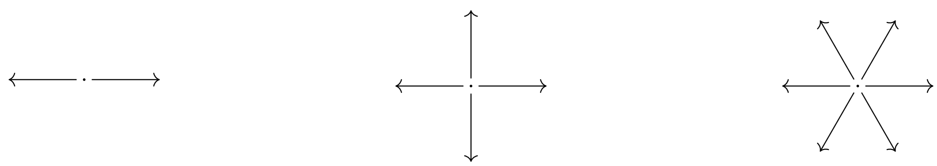
(R1) *Φ is finite, spans $\mathbb{R}^n$, and does not contain $\vec{0}$.*

(R2) *If $\vec{\alpha} \in \Phi$, the only multiples of $\vec{\alpha}$ in Φ are $\pm\vec{\alpha}$.*

(R3) *If $\vec{\alpha} \in \Phi$, the reflection $\sigma_{\vec{\alpha}}$ leaves Φ invariant.*

(R4) *If $\vec{\alpha}, \vec{\beta} \in \Phi$, then $\left\langle \vec{\alpha}, \vec{\beta} \right\rangle \in \mathbb{Z}$.*

The elements of a given root system Φ in $\mathbb{R}^n$ are called the **roots**. From R2 we have that only the multiples of a root $\vec{\alpha}$ are it's positive ($\vec{\alpha}$) and it's negative ($-\vec{\alpha}$). This is one reason we are interested in root systems for the elementary particles since each particle only has itself and its anti particle as the multiples of the properties of the particle. Three different root systems can be seen in Figure 3.2. Here A_1 is the smallest root system as it only consists of 2 elements. We will comeback to the naming of these root systems.



(a) A_1 (b) A_1A_1 (c) A_2

Figure 3.2: Three different root systems. Namely A_1, A_1A_1 and A_2. Here A_1 is the smallest irreducible root system and A_1A_1 is the smallest reducible root system. We will come back on what it means to be a reducible or irreducible root system.

Because a root system is a geometric object, we can introduce the **Weyl group** of a root system. The Weyl group which we will denote as $\mathscr{W}$ is the group generated by all the reflection $\sigma_{\vec{\alpha}}$ for $\vec{\alpha} \in \Phi$. Note that all elements of the Weyl group are made up of reflections around roots of the root systems, because R3 all leave the root system invariant so thus must all the elements of the Weyl group also leave the root system invariant. Not all the elements in the Weyl group are reflections. For example some of the operations in the Weyl group are rotations. An example is given in Figure 3.3 where two reflections after each other results in a rotation

The following Theorem from Humphreys [4] is a theorem I use for further proofs. This theorem relates how different operations in $GL(\mathbb{R}^n)$ (General Linear Group) and the reflection operations act together. The $GL(\mathbb{R}^n)$ is the group of all invert-able $n \times n$ matrices. For root systems we are not interested in the group $GL(\mathbb{R}^n)$ but in the Weyl group. This theorem is however usefull to us as the Weyl group is a subset of the General Linear Group and can thus also be applied to $\tau \in \mathscr{W}$ (again since we have already stated that all elements of $\mathscr{W}$ leave Φ invariant).

Theorem 3.2 (Humphreys [4] Theorem 9.2)**.** *Let Φ be a root system in $\mathbb{R}^n$, with Weyl group $\mathscr{W}$. If $\tau \in GL(\mathbb{R}^n)$ leaves Φ invariant , then $\tau\sigma_{\vec{\alpha}}\tau^{-1} = \sigma_{\tau(\vec{\alpha})}$ for all $\vec{\alpha} \in \Phi$, and $\left\langle \vec{\beta}, \vec{\alpha} \right\rangle = \left\langle \tau(\vec{\beta}), \tau(\vec{\alpha}) \right\rangle$ for all $\vec{\alpha}, \vec{\beta} \in \Phi$.*

3.2 Base

For a given root system we can introduce the concept of a base. This base is very important as we will later on see the base can fully specify the roots in a root system. The base of a root system is defined as follows:

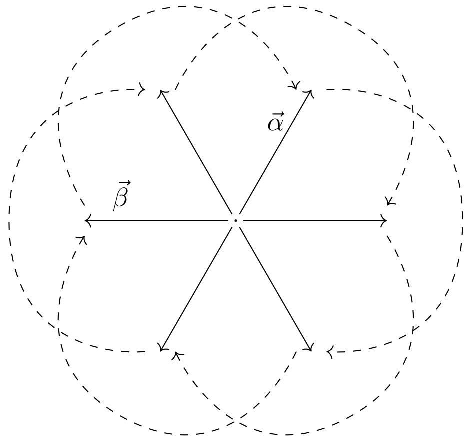


Figure 3.3: By first applying $\sigma_{\vec{\alpha}}$ and then $\sigma_{\vec{\beta}}$ in system A_2, a rotation operation is created which rotates all roots clockwise over 120 degrees. So $R_{120} = \sigma_\beta \cdot \sigma_\alpha$.

Definition 3.3. *A subset Δ of a root system Φ in $\mathbb{R}^n$ (or V) is called a* **base** *if:*

(B1) *Δ is a basis of $\mathbb{R}^n$ (or V) (here with basis we mean that Δ forms a basis of $\mathbb{R}^n$ in linear algebra terms).*

(B2) *Each root $\vec{\beta} \in \Phi$ can be written as:*

$$\vec{\beta} = \sum_{\vec{\gamma} \in \Delta} k_{\vec{\gamma}} \vec{\gamma} \tag{3.2}$$

with integer coefficients $k_{\vec{\gamma}}$ all non-negative or all non-positive.

We will call the roots of Δ **simple** (or **simple roots**). Note that since Δ is a basis for $\mathbb{R}^n$ the expression for $\vec{\beta}$ in (B2) is unique (here with basis we mean that Δ forms a basis of $\mathbb{R}^n$ in linear algebra terms). An example of a base can be seen in Figure 3.4. Here the roots $\vec{\gamma}_1$ and $\vec{\gamma}_2$ are a base of the root system A_2. Note that $\Delta = \{\vec{\gamma}_1, \vec{\gamma}_2\}$ satisfies the definition of a base.

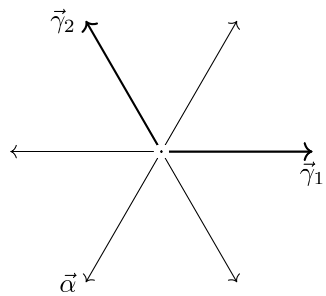


Figure 3.4: The simple roots $\vec{\gamma}_1$ and $\vec{\gamma}_2$ of the root system A_2. Note that the two roots span the whole space $\mathbb{R}^2$ and each roots is either a positive linear combination or a negative linear combination of the two simple roots. E.g. $\vec{\alpha} = -\vec{\gamma}_1 - \vec{\gamma}_2$.

Note however that the base of a root system is not unique. There are multiple different bases for a given root system. In our previous example in Figure 3.4 we could also take $\vec{\gamma}_1$ and $\vec{\alpha}$ as a base. The definition of a base does not imply that for a given root system a base exists. This is however the case and to show this we will present a way to find a base of a given root system. We start by taking a $\vec{\gamma} \in \mathbb{R}^n$. This $\vec{\gamma}$ can either be **regular** or **singular**. $\vec{\gamma}$ is singular if it is an element of the subset $\cup_{\vec{\alpha}\in\Phi} H_{\vec{\alpha}}$ (the union of all the hyper-planes of each root in Φ). Otherwise $\vec{\gamma}$ is regular. Let $\vec{\gamma} \in \mathbb{R}^n$ be regular and define the set $\Phi^+(\vec{\gamma}) = \{\vec{\alpha} \in \Phi | (\vec{\gamma}, \vec{\alpha}) > 0\}$. Next we will call $\vec{\alpha} \in \Phi^+(\vec{\gamma})$ **decomposable** if $\vec{\alpha} = \vec{\beta}_1 + \vec{\beta}_2$ for some $\vec{\beta}_i \in \Phi^+(\vec{\gamma})$, and **indecomposable** otherwise. In this case Theorem 3.4 from Humphreys [4] is enough to show that a base always exists.

Theorem 3.4 (Humphreys [4] Theorem 10.1)**.** *Let $\vec{\gamma} \in \mathbb{R}^n$ be regular. Then the set $\Delta(\vec{\gamma})$ of all decomposable roots in $\Phi^+(\vec{\gamma})$ is a base of Φ, and every base is obtainable in this manner.*

We have shown that each root system has a base. This means that $\Phi = \Phi^+ \cup \Phi^-$. If this were not the case, there is a root which disobeys (B2). But then no base exists which is a contradiction to Theorem 3.4. We can also define the root system by the base. The following Theorem shows this.

Theorem 3.5 (Humphreys [4] Theorem 10.3)**.** *For a root system Φ in a Euclidean space $\mathbb{R}^n$. Let $\mathscr{W}$ be the Weyl group and let Δ be a base of Φ. Then:*

(1) *If $\vec{\gamma} \in \mathbb{R}^n$, $\vec{\gamma}$ regular, there exists $\sigma \in \mathscr{W}$ such that $(\sigma(\vec{\gamma}), \vec{\alpha}) > 0$ for all $\vec{\alpha} \in \Delta$ (so $\mathscr{W}$ acts transitively on Weyl chambers).*

(2) *If Δ' is another base of Φ, then $\sigma(\Delta') = \Delta$ for some $\sigma \in \mathscr{W}$ (so $\mathscr{W}$ acts transitively on bases).*

(3) *If $\vec{\alpha}$ is any root from Φ with base Δ, then there exist a $\sigma \in \mathscr{W}$ such that $\sigma(\vec{\alpha}) \in \Delta$.*

(4) *$\mathscr{W}$ is generated by the $\sigma_{\vec{\alpha}}$ ($\vec{\alpha} \in \Delta$).*

(5) *If $\sigma(\Delta) = \Delta$, $\sigma \in \mathscr{W}$, then $\sigma = Id$ (identity element) (so $\mathscr{W}$ acts freely on bases).*

Theorem 3.5(3) shows that every root can be reached by applying the Weyl group to the all the simple roots (the roots of the base). Next Theorem 3.5(4) shows us how to produce the Weyl group from the Base. This means that the base fully describes the root system. We can recap this in the following corollary:

Corollary 3.6. *Given a base Δ. This base fully describes the root system Φ.*

Proof. We start by noting that according to Theorem 3.5(4) the Weyl group can be constructed from the base. Next note that according to Theorem 3.5(3) all roots can be created by applying one of the Weyl group elements on a particular base. By applying all the elements of the Weyl group to the base we find the root system. Note that the base is a subset of the root system and the Weyl group leaves the root system invariant thus this procedure can never generate a vector which is not a root. $\square$

3.3 Weyl chambers

In the previous section we have seen that the subset $\cup_{\vec{\alpha}\in\Phi} H_{\vec{\alpha}}$ determines if a vector is regular or singular. We will now extend this concept. Each hyper-plane $H_{\vec{\alpha}}$ partitions $\mathbb{R}^n$ in two sets, in a similar fashion $\cup_{\vec{\alpha}\in\Phi} H_{\vec{\alpha}}$ also partitions $\mathbb{R}^n$ in multiple sets. The connected components of these partitions are called the **Weyl chambers**. This means that every regular $\vec{\gamma}$ belongs to one Weyl chamber. We denote this Weyl chamber by $\mathfrak{C}(\vec{\gamma})$. For two regular vectors $\vec{\gamma}, \vec{\gamma}'$ to be in the same Weyl chamber we must have that the two regular vectors lie on the same side of each hyper plane $H_{\vec{\alpha}}$. This means that $\Phi^+(\vec{\gamma}) = \Phi^+(\vec{\gamma}')$ and we must have that $\Delta(\vec{\gamma}) = \Delta(\vec{\gamma}')$. This shows that Weyl chambers are in 1-1 correspondence with bases. This means that for a given root system Φ with base Δ there is a regular $\vec{\gamma}$ such that $\Delta = \Delta(\vec{\gamma})$ and we call the Weyl chamber to this $\vec{\gamma}$ the **Fundamental Weyl chamber** relative to the base Δ. An example of Weyl chambers in the root system A_2 can be seen in Figure 3.5.

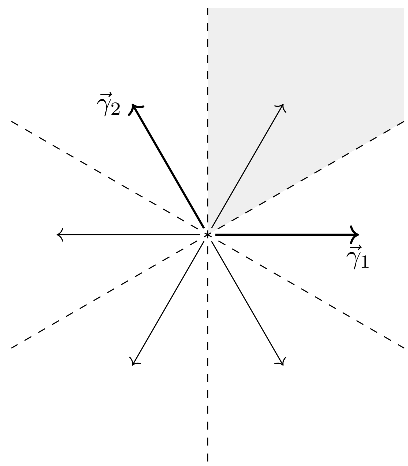
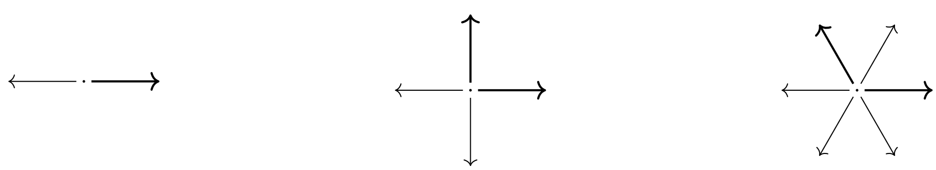


Figure 3.5: The six different Weyl chambers of A_2 and the Fundamental Weyl chamber colored grey with respect to the base $\{\vec{\gamma}_1, \vec{\gamma}_2\}$.

3.4 Cartan matrix

A base is defined as a specific set of roots that satisfy Definition 3.3. Because a set does not have an order (we cannot speak of the first, second or last simple root of a base). An order is however needed for the following section. For now will not pay attention to the actual order. So label a simple root of the base as the "first" simple root as the first simple root, a "second" simple root the second simple root until the last simple root has been labelled the according "last" simple root. Thus for a given root system Φ in $\mathbb{R}^n$ with base Δ we have an ordered base of the simple roots Δ' consisting of n simple roots $\{\vec{\gamma}_1, \vec{\gamma}_2, ..., \vec{\gamma}_n\}$. For this ordered base the Cartan matrix C is defined as an $n \times n$ matrix where each entry is defined as $C_{i,j} = \langle \vec{\gamma}_i, \vec{\gamma}_j \rangle$. This matrix depends on the root system and on the order of a base. We will denote the Cartan matrix as $C_{\Phi,\Delta'}$. Later on we will create a standard base for root systems and the base will be dropped so that the notation becomes C_Φ. In Figure 3.6 a base is given for the root systems A_1, A_1A_1, A_2 and the Cartan matrices are given.



(a) A_1 (b) A_1A_1 (c) A_2

Figure 3.6: Three different root systems A_1, A_1A_1, A_2 depicted as vectors with a base denoted by the thicker drawn vectors.

$$C_{A_1,\Delta'} = \begin{pmatrix} 2 \end{pmatrix} \qquad C_{A1A1,\Delta''} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \qquad C_{A_2,\Delta'''} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \tag{3.3}$$

By definition of a base, the n simple roots span $\mathbb{R}^n$. This means that that the Cartan matrix must be non singular. The next proposition shows that the Cartan matrix determines the root system up to isomorphisms.

Proposition 3.1 (Humphreys [4] Proposition 11.1)**.** *For a root system Φ with base Δ and another root system Φ' and base Δ' both with rank n. Suppose $\langle \vec{\alpha}_i, \vec{\alpha}_j \rangle = \langle \vec{\alpha}_i‘, \vec{\alpha}_j‘ \rangle$ The bijection ϕ which maps $\vec{\gamma}_i \to \vec{\gamma}_i'$ for all $i = 1, 2, ..., n$ extends to a bijection which maps Φ to Φ'. Therefore the Cartan matrix determines Φ up to isomorphisms.*

3.5 Dynkin diagram

The Dynkin diagram is a graph that represents the Cartan Matrix. Consider a root system Φ in $\mathbb{R}_n$ with base Δ and Cartan matrix $C_{\Phi,\Delta}$. It is a graph of n vertices where the vertices i and j have $|\langle\vec{\gamma}_i,\vec{\gamma}_j\rangle|$ edges with $i \neq j$. The Dynkin diagrams of the root systems in Figure 3.6 are given in Figure 3.7. A Dynkin diagram coveys the same information as the Cartan matrix and also encodes the properties of the base.

(a) A_1 (b) A_1A_1 (c) A_2

Figure 3.7: The Dynkin diagrams of three different root systems A_1, A_1A_1, A_2.

3.6 Type of root systems

We start this section with the distinction between reducible and irreducible root systems. The following definition tells us when a root system is reducible.

Definition 3.7. *A root system is **reducible**, if it can be partitioned into the union of two sub sets, such that the roots of one set are orthogonal to all the roots of the other set.*

A root system is**irreducible** when it is not reducible. In this research we are only interested in so called simply-laced root systems. A simply-laced root system has additional properties which are defined by:

Definition 3.8. *Given an irreducible root system Φ. The root system is **simply-laced** if the length*[1] *of each root is $\sqrt{2}$.*

This additional restriction implies that for any two roots $\vec{\alpha}$ and $\vec{\beta}$, the value $<\vec{\alpha},\vec{\beta}>$ is limited to $-2,-1,0,1,2$. The main root system discussed in this paper E_8 is a simply-laced root system. This means that all the root systems that can be found as a sub root system E_8 must also be simply-laced. For this reason we will only concern us with these simply-laced root systems. Since we will only discuss the simply-laced root systems (and the union there of) I will drop the simply-laced label as it is an global assumption in this paper.

We can "combine" two root systems ($\tilde{\Phi}$ and $\hat{\Phi}$) if the bases of the two root systems span two orthogonal spaces. In this case we can create[2] the new root system $\Phi = \tilde{\Phi} \cup \hat{\Phi}$. We will check the axioms of a root system to see that Φ is indeed a root system.

1. Both $\tilde{\Phi}$ and $\hat{\Phi}$ are finite and not empty so Φ must be finite as well. Besides the zero vector can not be in Φ since it is not in $\tilde{\Phi}$ or $\hat{\Phi}$.
2. Since the bases span orthogonal spaces, a multiple of $\alpha \in \tilde{\Phi}$ cannot be an element of $\hat{\Phi}$ and vice versa. So we conclude that the only multiples of $\alpha \in \Phi$ must be the multiples of α in $\tilde{\Phi}$ or $\hat{\Phi}$ which are root systems. So the only multiples are $\pm\alpha$.
3. Since the two root systems $\tilde{\Phi}$ and $\hat{\Phi}$ span orthogonal spaces we must have for $\alpha \in \tilde{\Phi}$ and $\beta \in \hat{\Phi}$ that these two vectors are orthogonal to each other. So for a given $\alpha \in \tilde{\Phi}$ we have that σ_α leaves $\tilde{\Phi}$ invariant. Besides $\hat{\Phi} \subset H_\alpha$ so σ_α leaves $\hat{\Phi}$ invariant. A similar argument exists why for a given

[1] as calculated by the square root of inner-product $(\cdot,\cdot)$. Note that this is a different operation than the $<\cdot,\cdot>$.

[2] The combination of the two root systems is a direct sum. For the direct sum, we have that the dimension of the Euclidean space for the combined root system is the sum of each dimension of the separate root systems. Note that the two root systems are not in the same Euclidean space and we can freely merge them.

$\beta \in \hat{\Phi}$ σ_β leaves both $\tilde{\Phi}$ and $\hat{\Phi}$ invariant. This means for a given $\alpha \in \Phi$ we have that σ_α leaves Φ invariant.

4. Given two roots α, β in Φ. These vectors are either both an element of $\tilde{\Phi}$ or $\hat{\Phi}$ in which case we are done, or both in different root systems. In the last case the $<\alpha, \beta>= 0$ (see point 3).

We can thus "combine" root systems to form a new root system. When a root system is made up of multiple root systems we call that root systems a **reducible root system**. If a root system is not made up of multiple root system we call this root system an **irreducible root system**.

3.6.1 Irreducible root systems

We will now discuss irreducible root systems. It happens to be that there are only 3 types of (simply-laced) root systems[3]. These three types are characterized by A_n, D_n and E_n. The subscript n denotes the rank, or the number of simple roots for the given root system and the letter denotes the shape of the Dynkin diagram. The Dynkin diagrams of all different types can be seen in Figure 3.8.

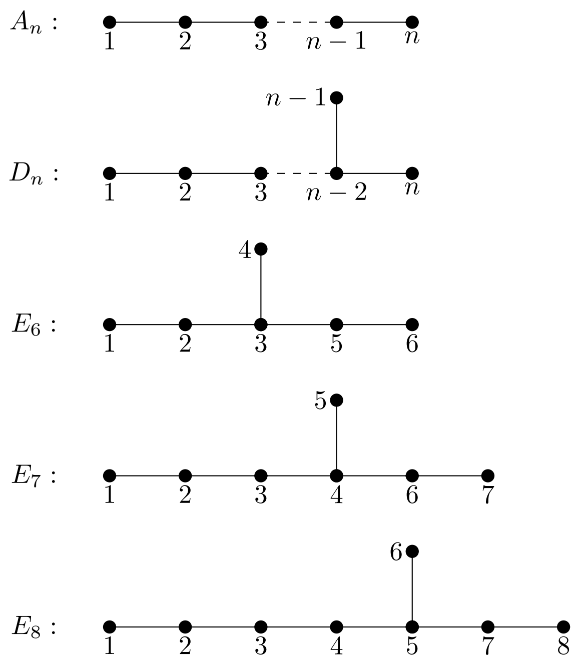


Figure 3.8: The different types of irreducible root systems A_n, D_n and E_n. Since there are only three root systems of the type E_n all three are given.

A_n :

In Figure 3.8 the Dynkin diagram can be seen of the root systems of type A_n. It's Dynkin diagram is a chain of n nodes. For the root systems of type A_n we must also have that $n \geq 1$. Two examples of root systems of type A_n can be seen in Figure 3.2.

D_n :

In Figure 3.8 the Dynkin diagram can be seen of the root systems of type D_n. The Dynkin diagram is a chain of $n-1$ nodes with an appendix of a single node 1 node from the end. For the root systems of

[3]There are in fact 7 total different types of root systems. The restriction of a simply-laced root system reduces the possible irreducible root systems to 3.

type D_n we must also have that $n \geq 4$. If we would allow $n = 3$ we have that D_3 would be the same root system as A_3.

E_n :

In Figure 3.8 the Dynkin diagram can be seen of the root systems of type E_n. The Dynkin diagram is a chain of $n-1$ nodes. For the root systems of type E_n we must also have that $8 \geq n \geq 6$. This means that there are only three types of E_n, all three are given in Figure 3.8.

3.6.2 Reducible root systems

We have already discussed how to "combine" different root systems to form a reducible root system. We have seen that $\tilde{\Phi}$ added to $\hat{\Phi}$ is a new root system Φ when the bases span orthogonal spaces. This new root system cannot be classified as a irreducible root system. The new root system will have a new type. So if $\tilde{\Phi}$ is of the type A_3 and $\hat{\Phi}$ is of the type D_4 we will call the type of Φ to be D_4A_3. Note that the order of D_4 and A_3 has no significance. Therefore we will introduce our own order. We will always denote a reducible root system by the letters from E to D to A, then by number (the number of simple roots)[4]. We have already seen an example of a reducible root system in Figure 3.2 where A_1A_1 is given.

3.6.3 E_8

Since the focus is E_8 we will also give a construction on the root systems of E_8. We will give the root system in the so called even coordinate system. In this coordinate system E_8 is given by either integer or half integer vectors of length $\sqrt{2}$. There are 112 roots in the form of:

$$\begin{pmatrix} \pm 1 & \pm 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T$$

and 128 roots in the following form with an even number of minus signs:

$$\begin{pmatrix} \pm\frac{1}{2} & \pm\frac{1}{2} & \pm\frac{1}{2} & \pm\frac{1}{2} & \pm\frac{1}{2} & \pm\frac{1}{2} & \pm\frac{1}{2} & \pm\frac{1}{2} \end{pmatrix}^T$$

This means that there are a total of 240 roots in E_8. There are different methods to build the E_8 root system but we will always use the even coordinate system as it has some nice properties that all coordinates are either integer or half-integer.

3.7 Weights

For a root system Φ in $\mathbb{R}^n$ with a base Δ the set of all vectors $\vec{\lambda} \in \mathbb{R}^n$ for which $(\vec{\lambda}, \vec{\alpha}) \in \mathbb{Z}$ with $\vec{\alpha} \in \Phi$ is called the set of **weights**. Denote this set by Λ. Note that this set contains an infinite number of elements under the condition that this set is non empty. Simply choose a $\vec{\lambda} \in \Lambda$ then for all $n \in \mathbb{Z}$ we have that $n\vec{\lambda} \in \Lambda$.

Define **fundamental dominant weights relative to** Δ as all the vectors $\vec{\lambda}_i \in \mathbb{R}^n$ so that $(\vec{\lambda}_i, \vec{\gamma}_j) = \delta_{ij}$. Since the base spans $\mathbb{R}^n$ we can find n fundamental dominant weights that satisfy the condition.

$$\begin{bmatrix} | & \cdots & | \\ \vec{\lambda}_1 & \cdots & \vec{\lambda}_n \\ | & \cdots & | \end{bmatrix}^T \begin{bmatrix} | & \cdots & | \\ \vec{\gamma}_1 & \cdots & \vec{\gamma}_n \\ | & \cdots & | \end{bmatrix} = I_n \tag{3.4}$$

Here I_n is the $n \times n$ identity matrix. This also states how to calculate the fundamental dominant weights. An example for A_2 is

[4]E.g. $E_8E_6D_7D_5A_7A_2$

$$\begin{bmatrix} | & | \\ \vec{\gamma}_1 & \vec{\gamma}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} \sqrt{2} & -\frac{1}{2}\sqrt{2} \\ 0 & \frac{1}{2}\sqrt{6} \end{bmatrix} \implies \begin{bmatrix} | & | \\ \vec{\lambda}_1 & \vec{\lambda}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\sqrt{2} & 0 \\ \frac{1}{6}\sqrt{6} & \sqrt{\frac{2}{3}} \end{bmatrix} \tag{3.5}$$

From this we find that the first fundamental dominant weight is $\vec{\lambda}_1 = \begin{pmatrix} \frac{1}{2}\sqrt{2} \\ \frac{1}{6}\sqrt{6} \end{pmatrix}$ and the second fundamental dominant weight is $\vec{\lambda}_2 = \begin{pmatrix} 0 \\ \sqrt{\frac{2}{3}} \end{pmatrix}$. The fundamental dominant weights can be seen in Figure 3.9 together with the root system A_2.

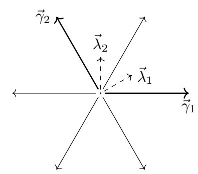


Figure 3.9: The fundamental dominant weights $\vec{\lambda}_1$ and $\vec{\lambda}_2$ of the root system A_2 with the base $\{\vec{\gamma}_1, \vec{\gamma}_2\}$. Note how $\vec{\lambda}_1$ is orthogonal to $\vec{\gamma}_2$ and $\vec{\lambda}_2$ is orthogonal to $\vec{\gamma}_1$ as the definition of the fundamental dominant weights states.

We are only concerned with the fundamental dominant weights so we will refer to the fundamental dominant weights as just the weights.

3.7.1 Root system spectrum

In Chapter 1 the concept of a charge spectrum of the particles has been introduced. A similar concept will now be introduced for a weight for a root system.

Definition 3.9. *given a root system Φ and a vector $\vec{v}$, a* ***root system spectrum*** *is the collection of all possible outcomes when taking the inner product between $\vec{v}$ and $\vec{\alpha}$ for all $\vec{\alpha} \in \Phi$ and their multiplicities.*

Note the similarity between the definition of a root systems spectrum and a charge spectrum. The simplest example is to take the weights of a root system as the vector. In the case of A_2 and the weights give root system spectrum's as given in Figure 3.10.

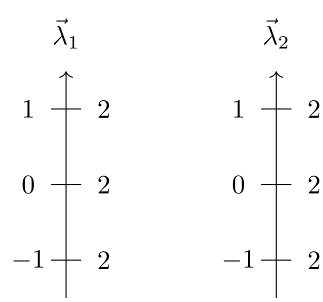


Figure 3.10: The root system spectrum of the weights $\vec{\lambda}_1$ and $\vec{\lambda}_2$ of A_2.

Again we introduce a 2D spectrum similar to the 2D charge spectrum. This can be seen in Figure 3.11.

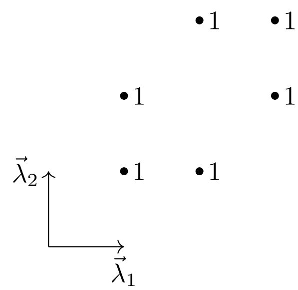


Figure 3.11: The 2D root system spectrum of the root system A_2 for the two weights.

3.8 Sub root systems

A sub root system Φ' is a subset of a root system Φ which on itself is also a root system. The simplest examples have already been discussed when dealing with reducible root systems. It is easy to check that A_1 is a sub root system of the root system A_1A_1. Not all sub root systems are this easy to identify. For example in the root system A_2 there are 3 different sub root systems (all of the type A_1). This can be seen in Figure 3.12. It is easy to see that in a root system with n roots there are $n/2$ different sub root systems of type A_1. In the next chapter methods will be discussed on how to identify find the sub root systems of a root system.

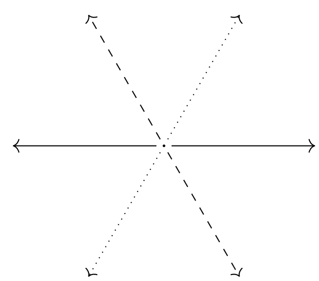


Figure 3.12: For the given root system A_2 the set of dashed vectors is a sub root system of type A_1. The set of dotted vectors or the set of normal vectors are also sub root systems of type A_1. In total there are 3 different sub root systems of type A_1 in the (larger) root system A_2.

Chapter 4

Root systems and particles

In this chapter the link between the elementary particles and root systems will be discussed. This is done in two steps. First the link between particles and Lie groups will be discussed, after which the link from a Lie group to Lie algebra and finally a root system will be discussed. Since this is a broad topic we will first focus on the weak interaction and how quarks and the vector bosons interact. This will turn out to be connected to the smallest root system A_1. Next we will show how the color charges are connected to the root system A_2 and to conclude the research question will be introduced.

4.1 Representation Theory

4.1.1 Symmetry group $SU(2)$

To introduce the symmetry group $SU(2)$ the definition of a group will be discussed first. After which the symmetry group $SU(2)$ will be introduced and the definition will be used to check that the symmetry group $SU(2)$ is indeed a group.

Definition 4.1. *Given a set G with associative operation $f: G \times G \to G$ the set G is a group with operation if:*

1. $\exists e \implies f(e,g) = f(g,e) = g \quad \forall g \in G$
2. $\forall g \exists g^{-1}$ *such that* $f(g,g^{-1}) = f(g^{-1},g) = e$

The symmetry group $SU(2)$ is the set of all unitary 2×2 matrices which have determinant 1 with the standard matrix multiplication as operation. All these matrices are of the form:

$$g = \begin{pmatrix} \alpha & -\beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \text{ with } \alpha\bar{\alpha} + \beta\bar{\beta} = 1 \tag{4.1}$$

To check if this is indeed a group, note that matrix multiplication is an associative operation. Note that the identity matrix is an element of $SU(2)$ and is thus the e which has to exists according to 1. For the second constraint we simply take the hermitian conjugate of the element g from which constraint 2 can be found:

$$\bar{g} = \begin{pmatrix} \bar{\alpha} & \beta \\ -\bar{\beta} & \alpha \end{pmatrix} \implies g\bar{g} = \bar{g}g = I_2 = e \tag{4.2}$$

Thus the group $SU(2)$ is indeed a group according to the definition.

4.1.2 Representation of $SU(2)$

In the previous section the symmetry group $SU(2)$ was introduced as a group. In the next section the group will act on a vector space as a matrix multiplication to see how it behaves. It will become clear in the following sections why this will be useful. This action on a vector space is called a representation of the group and the definition is:

Definition 4.2. *A **representation** of a group G on a vector space V is an association between an element $g \in G$ with a linear map $\rho(g)$ working on the vector space V which respects the structure:*

1. $\rho(g_1 g_2) = \rho(g_1)\rho(g_2)$
2. $\rho(g^{-1}) = \rho(g)^{-1}$
3. $\rho(e) = 1_V$ *where e is the identity element of G and 1_V is the identity of V.*

The simplest example of a representation on $SU(2)$ is given by taking $\rho(g) = g$ and as an action on $\mathbb{C}^2$. This representation is called the defining representation and can be seen in (4.3). Note that this satisfies all the constraints and is thus a valid representation.

$$g = \begin{pmatrix} \alpha & -\beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in SU(2) \quad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{C}^2 \implies \rho(g)\vec{v} = g\vec{v} = \begin{pmatrix} \alpha & -\beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \alpha v_1 - \beta v_2 \\ \bar{\beta} v_1 + \bar{\alpha} v_2 \end{pmatrix} \quad (4.3)$$

However the following representation on the tensor product[1] $\mathbb{C}^2 \otimes \bar{\mathbb{C}}^2$ is one of the representations which will become rather useful. Here $\mathbb{C}^2 \otimes \bar{\mathbb{C}}^2$ is a vector space with each element $u \in \mathbb{C}^2 \otimes \bar{\mathbb{C}}^2$ denoted as $u = v \otimes \bar{w}$ with $v, \bar{w} \in \mathbb{C}^2$ and where $\bar{w}$ the hermitian conjugate of the vector w. Some simple examples are:

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \otimes \begin{pmatrix} 2 \\ 3 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1+5i \\ 2-3i \end{pmatrix} \otimes \begin{pmatrix} 2i \\ 3+5i \end{pmatrix}$$

A basis can be created for $\mathbb{C}^2 \otimes \bar{\mathbb{C}}^2$ by taking a basis for $\mathbb{C}^2$ (e.g. the two elements $b_1 = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$ and $\bar{b}_2 = \begin{pmatrix} 0 & 1 \end{pmatrix}^T$) and determining all possible combinations (e.g.: $b_1 \otimes \bar{b}_1, b_2 \otimes \bar{b}_1, b_1 \otimes \bar{b}_2, b_2 \otimes \bar{b}_2$). The representation of $SU(2)$ on $\mathbb{C}^2 \otimes \bar{\mathbb{C}}^2$ is now given by (4.4).

$$\rho(U)(v \otimes \bar{w}) = Uv \otimes \bar{U}\bar{w} \tag{4.4}$$

We check the requirements of the definition in order to conclude that (4.4) is a representation:

$$1: \qquad \rho(U_1U_2)(v \otimes \overline{w}) = U_1U_2v \otimes \bar{U}_1\bar{U}_2\bar{w} = \rho(U_1)(U_2v \otimes \bar{U}_2\bar{w}) = \rho(U_1)\rho(U_2)(v \otimes \overline{w})$$

$$3: \qquad \rho(e)(v \otimes \bar{w}) = \rho(I_2)(v \otimes \bar{w}) = v \otimes \bar{w}$$

$$2: \rho(U)\rho(U^{-1}) = \rho(UU^{-1}) = \qquad 1_V = \rho(e) = \rho(U)\rho(U)^{-1} \implies \rho(U^{-1}) = \rho(U)^{-1}$$

This means that (4.4) is a representation of $SU(2)$ on $\mathbb{C}^2 \otimes \bar{\mathbb{C}}^2$. The second representation of $SU(2)$ is given by (4.5) on the 2×2 complex matrices.

$$\rho(U)(M) = UMU^{-1} \tag{4.5}$$

Here M is an element of the 2×2 complex matrices. For this we will associate the vector space $M_2(\mathbb{C})$ as the 2×2 matrices with basis:

$$e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

To see that (4.5) is an actual representation the definition is checked:

$$1: \qquad \rho(U_1U_2)(M) = (U_1U_2)M(U_1U_2)^{-1} = U_1(U_2MU_2^{-1})U_1^{-1} = \rho(U_1)\rho(U_2)(M)$$

[1] The tensor product $\mathbb{C}^2 \otimes \bar{\mathbb{C}}^2$ can simply be understood as a space which stasisfies $w \otimes (\overline{av}) = a(w \otimes \bar{v}) = (aw) \otimes \bar{v}$ where $v, w \in \mathbb{C}^2$ and $a \in \mathbb{C}$ just a constant and $w_1 \otimes \bar{v}_1 + w_2 \otimes \bar{v}_2 + w_1 \otimes \bar{v}_2 + w_2 \otimes \bar{v}_1 = (w_1 + w_2) \otimes (\bar{v}_1 + \bar{v}_2)$.

$$2: \qquad \rho(U^{-1})\rho(U)(M) = U^{-1}UMU^{-1}U = M \implies \rho(U^{-1}) = \rho(U)^{-1}$$
$$3: \qquad \rho(I_2)(M) = I_2 M I_2^{-1} = M \implies \rho(I_2) = 1_V$$

In the literature this representation is known as the adjoint representation. The adjoint representation will show its usefulness in a later section. Note that $\mathbb{C}^2 \otimes \bar{\mathbb{C}}^2$ is isomorphic to $M_2(\mathbb{C})$. For this note that every matrix $U \in M_2(\mathbb{C})$ can be written as the sum of terms of the form $v\bar{w}^T$ for $v, w \in \mathbb{C}^2$. From this we can make the identification from $\mathbb{C}^2 \otimes \bar{\mathbb{C}}^2$ to $M_2(\mathbb{C})$ as $v \otimes \bar{w} \leftrightarrow v\bar{w}^T$. Next we have the following theorem in Suijlekom [5]:

Theorem 4.3 (Theorem 3.4 in Suijlekom [5])**.** *The representation of $SU(2)$ on $\mathbb{C}^2 \otimes \bar{\mathbb{C}}^2$ is isomorphic with the representation of $SU(2)$ on $M_2(\mathbb{C})$.*

Definition 4.4. *Given two representations V_1, V_2 of a group G. The linear operation $\Phi : V_1 \to V_2$ is an **intertwiner** if:*

$$\Phi(ga) = g\Phi(a) \tag{4.6}$$

For $g \in G$ and $a \in V_1$.

A good example is given in 4.3 where the function Φ is the unique map that satisfies:

$$\Phi(v \otimes \bar{w}) = v\bar{w}^T \tag{4.7}$$

Which conforms to (4.6) by (4.8):

$$\Phi(Uv \otimes \bar{U}\bar{w}) = Uv\left(\bar{U}\bar{v}\right)^T = Uv\bar{w}^T U^{-1} \tag{4.8}$$

These representations can be made clearer by introduction diagrams. In these diagram a group is depicted with all the different representations as directed lines where a intertwiner is depicted as a vertex between multiple representations. The diagram of $SU(2)$ can be seen in Figure 4.1. Note that in Figure 4.1 the representation $\mathbb{C}^2 \otimes \bar{\mathbb{C}}^2$ is split up into two different representations. Namely $\mathbb{C}^2$ and $\bar{\mathbb{C}}^2$. The tensor product "adds" two representations together to form a new representation. However in the diagram the different representations are drawn separately.

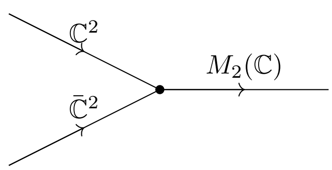


Figure 4.1: The diagram of $SU(2)$ of the intertwiner given by (4.8).

4.2 Particle interactions

In the previous section representations of a group have been introduced. In this section these representations will be used to show how different particles interact by associating particles with elements of the different representations. In quantum theory the intertwiners describe interactions of different particles and thus link the elements of different representations together. For example in the previous section we have seen two representations of the group $SU(2)$. Namely $\mathbb{C}^2 \otimes \bar{\mathbb{C}}^2$ and $M_2(\mathbb{C})$. As noted in the previous section the representation $\mathbb{C}^2 \otimes \bar{\mathbb{C}}^2$ is actually two separate representations. The representation $\mathbb{C}^2$ can be seen as the quarks. For example the element $(1 \quad 0)^T$ can be thought of as a left handed up quark. The representation $\bar{\mathbb{C}}^2$ is similar to $\mathbb{C}^2$ but the bar indicates that the representation is used for the anti quarks. For example $(0 \quad 1)^T$ can be associated with the left handed anti-down quark. For the representation $M_2(\mathbb{C})$ a smaller representation is looked at. Namely the subset of

$M_2(\mathbb{C})$ which all have trace zero. This set is called $su(2)$. In $su(2)$ the W and Z vector bosons are represented as matrices. The W^+ vector boson is represented by the matrix:

$$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{C}^2 \qquad \bar{d} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \bar{\mathbb{C}}^2 \qquad W^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in su(2)$$

By now combining these particles with (4.7) we find:

$$\Phi(u \otimes \bar{d}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}^T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = W^+$$

This shows how the intertwiner from $\mathbb{C}^2 \otimes \bar{\mathbb{C}}^2$ to $M_2(\mathbb{C})$ describes the interaction. Every interaction can be described in a similar fashion with an intertwiner. This result can also be seen in Figure 4.2 as a diagram.

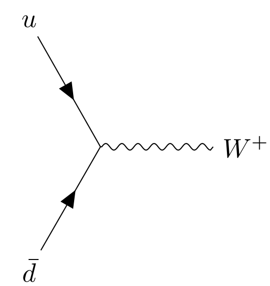


Figure 4.2: The interaction of a left handed up quark and left handed anti down quark which results in the W^+ vector boson.

4.3 Root systems

In the previous section we have seen how a group ($SU(2)$) can be used to represent particles and also be used to describe interactions. For this the group $SU(2)$ was limited to the Lie algebra $su(2)$ of all 2×2 complex matrices with trace 0. In the following section the link between a Lie algebra and a root system will be made. For this the following operation will plays a large roll in Lie algebras and will thus briefly be discussed.

$$[a, b] = ab - ba \tag{4.9}$$

This operation is called the **commutator**. A examples is:

$$\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} \right] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ -2 & 2 \end{pmatrix}$$

For any two elements a, b of a set for which $[a, b] = 0$ we say that the elements **commute**. For a given Lie algebra, the smallest subalgebra of which all elements commute which each other will play an important roll in deriving the root system from a Lie algebra.

Definition 4.5. *For a given Lie algebra G the **Cartan subalgebra** H is the subalgebra of the Lie algebra such that it is the smallest subalgebra for which all elements commute with each other.*

In the case of $su(2)$ the Cartan subalgebra only has one element. To see why note that the commutator is bilinear[2]. Besides note that any element of $su(2)$ is a linear combination of the three matrices:

[2]This simply means that the operator is linear in both the first and second argument.

$$a_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad a_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad a_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

For these matrices we have $[a_1, a_2] = a_3$ and any other permutation can be calculated as a cyclic permutation of $1 \to 2 \to 3 \to 1$ combined with the property[3] that $[a_i, a_j] = -[a_j, a_i]$. This means that for any $[x, y] = 0$ we have that $[x_1a_1 + x_2a_2 + x_3a_3, y_1a_1 + y_2a_2 + y_3a_3] = x_1y_2a_3 + x_2y_3a_1 + x_3y_1a_2 - x_1y_3a_2 - x_2y_1a_3 - x_3y_2a_1 = (x_2y_3 - x_3y_2)a_1 + (x_3y_1 - x_1y_3)a_2 + (x_1y_2 - x_2y_1)a_3 = 0$. We find that:

$$\frac{x_2}{x_3} = \frac{y_2}{y_3} \qquad \frac{x_3}{x_1} = \frac{y_3}{y_1} \qquad \frac{x_1}{x_2} = \frac{y_1}{y_2}$$

This implies that $y = cx$ where c is a complex constant. Thus in the case of $su(2)$ the Cartan subalgebra is span of a single element. For simplicity take a_3 as this element so that the Cartan subalgebra H is:

$$H = \{ca_3 | c \in \mathbb{C}\}$$

Note that the Cartan subalgebra is always the span of n elements of the Lie algebra. So that $H = span\{h_1, \dots, h_n\}$. In the case of $su(2)$ this would be $span\{a_3\}$. Next we will introduce a linear function from the elements of $span\{a_3\}$ to the complex numbers. This function is a linear function where we sum over all the matrix elements. So the function λ can be thought of as a vector $\vec{\lambda}$ of 4 elements acting on a vector of the 4 elements of the matrix. An example can be seen below:

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \vec{\lambda} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} \qquad \vec{x} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \to \lambda(x) = \vec{\lambda}^T \vec{x} = a\lambda_1 + b\lambda_2 + c\lambda_3 + d\lambda_4 \in \mathbb{C}$$

Now note that the matrices of the $span\{a_3\}$ just have diagonal elements and trace zero. So all elements are of the form:

$$\begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix} \qquad a \in \mathbb{C}$$

We can thus reduce the λ function to work on these elements and get a vector of 2 elements where the first element works on the $(1,1)$ matrix element and the second element works on the $(2,2)$ element. For this we can rewrite the matrix h into a 2 dimensional vector $\vec{h}$. Thus we get:

$$h = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \to \vec{\lambda} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \qquad \vec{h} = \begin{pmatrix} a \\ -a \end{pmatrix} \to \lambda(x) = \vec{\lambda}^T \vec{h} = a\lambda_1 - a\lambda_2 \in \mathbb{C}$$

So for any vector $\vec{\lambda} \in \mathbb{R}^n$ a function has been defined on the elements of $span\{a_3\}$. Next define the set g_λ for all $\vec{\lambda} \in \mathbb{R}^n$ so that:

$$g_\lambda = \{x \in G | [h, x] = \lambda(h)x \ \forall h \in H\}$$

Where G is the Lie algebra (e.g. $su(2)$) and H is the Cartan subalgebra of G (e.g. $span\{a_3\}$). In this case $\vec{\lambda}$ is a **root** if the set g_λ is non empty and $\lambda \neq \vec{0}$.

Now note that h is a diagonal matrix with diagonal elements a and $-a$ so after calculations we get:

$$[h, a_1] = ha_1 - a_1h = a \times a_1 + a \times a_1 = 2a \times a_1 = \lambda(h)a_1$$

$$[h, a_2] = ha_2 - a_2h = -a \times a_2 - a \times a_2 = -2a \times a_2 = \lambda(h)a_2$$

[3] This follows directly from (4.9).

We are thus looking for two vectors $\vec{\lambda}$ so that $\lambda(h) = \lambda_1 a - \lambda_2 a = \pm 2a$. This means that two solutions are:

$$\vec{\lambda}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \vec{\lambda}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

We can plot the two roots in Figure 4.3 from where it is clear that this is the root system A_1 which was already discussed in chapter 2.

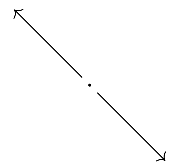


Figure 4.3: The roots of the Lie algebra $SU(2)$. Note how this is the root system A_1.

To conclude the Lie algebra $su(2)$ is connected to the root system A_1. The same operation can be done to $su(3)$ (the 3×3 complex matrices with trace 0) from which the root system A_2 emerges. To see how this root system connects to the Lie algebra and the particles note that the root 2 connects to the Lie algebra element a_1, as a_1 is an element of g_2. Furthermore in the previous section the link between the element a_1 and the vector boson W^+ was made.

4.4 Representation of $SU(3)$

The same operation as described above which results in the root system A_1 on the group $SU(2)$ results in the root system A_2 for the group $SU(3)$. $SU(3)$ is the group of all unitary 3×3 matrices with determinant 1.

A representation of $SU(3)$ can be created on the tensor product of $\mathbb{C}^3 \otimes \bar{\mathbb{C}}^3$. This is similar to the representation of $SU(2)$ on $\mathbb{C}^2 \otimes \bar{\mathbb{C}}^2$.

$$\rho(U)(v \otimes \bar{w}) = Uv \otimes \bar{U}\bar{w} \tag{4.10}$$

The second representation of $SU(3)$ can be created on the 3×3 complex matrices $M_3(\mathbb{C})$ by:

$$\rho(U)(M) = UMU^{-1} \tag{4.11}$$

Similar to Theorem 28 4.3 the representation on $\mathbb{C}^3 \otimes \bar{\mathbb{C}}^3$ and $M_3(\mathbb{C})$ are isomorphic. Thus meaning that:

$$\Phi(v \otimes w) = v\bar{w}^T \tag{4.12}$$

With these representations we can again associate particles to elements of the different representations. The representation $\mathbb{C}^3$ can be seen as the color of quarks. For example $(100)^T \in \mathbb{C}^3$ is a red quark and $(010)^T \in \bar{\mathbb{C}}^3$ a anti green quark. In a similar fashion the subset of $M_3(\mathbb{C})$ $su(3)$ (containing all the elements of $M_3(\mathbb{C})$ with trace zero) can be seen as the gluons. For example the gluon with color red and anti-green can be represented as an element of $su(3)$.

$$r = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{C}^3 \qquad \bar{g} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \in \bar{\mathbb{C}}^3 \qquad g_{r\bar{g}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in su(3)$$

Combining this with (4.12) we obtain:

$$\Phi(r \otimes \bar{g}) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}^T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

To find the root system A_2 we simply have to find the Cartan subalgebra. For this note that there Note that the all the elements of $su(3)$ are a linear combination of the following 8 elements:

$$a_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad a_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad a_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad a_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$a_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad a_6 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad a_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad a_8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

From these 8 we find that the Cartan subalgebra is the set of all linear combinations from a_1 and a_5. The elements of the Cartan subalgebra are of the form:

$$h = \begin{pmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{pmatrix} \qquad b_1 + b_2 + b_3 = 0$$

So the λ function is a vector of 3 elements since it only acts on the diagonal elements. The other 6 a_n matrices can be characterized as $E_{i,j}$ where i and j determines on which position the 1 is placed with the condition that $i \neq j$. The commutator of an element from $su(3)$ with an element from the Cartan subalgebra gives:

$$[h, E_{i,j}] = hE_{i,j} - E_{i,j}h = (b_i - b_j)E_{i,j}$$

From this we find that $\lambda(h) = b_i\lambda_i - b_j\lambda_j = b_i - b_j$ so that the roots $\vec{\lambda} = e_i - e_j$ where e_i is the unit vector for one of the three different axises. This gives us the following 6 roots:

$$\vec{\lambda}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \vec{\lambda}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \quad \vec{\lambda}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \vec{\lambda}_4 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \vec{\lambda}_5 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad \vec{\lambda}_6 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

These 6 roots are actually the 6 roots of the root system A_2.[4] Note that these roots are actually 6 different gluons. These are plot in Figure 4.4. The other 2 gluons (since there are 8 in total) are associated to the elements of the Cartan subalgebra and are "neutral" in this plot since for the elements in the Cartan subalgebra the $\vec{\lambda} = 0$.

4.5 E_8 and the elementary particles

In the previous section the link between the elementary particles and root systems was made. However the root system A_1 to represent the vector bosons does not contain the quarks or any other elementary particle. For this a larger root system is needed. The root system E_8 was chosen as the larger root system as it has 240 roots (double the amount of elementary particles) and has lots of sub root systems. For example A_1 is a sub root system of E_8. However the elementary particles should not be placed randomly in E_8. The map must conform to the rule that when the two roots of two particles are added, they add up to the root of the particle which is the result of the interaction of the two beginning particles. For example a u quark and $\bar{d}$ quark still add up to become a W^+ vector boson. This can be seen in Figure 4.5.

[4] A_2 was introduced as a root system in the 2D plane while there roots are actually part of a 3D space. This is however not an issue as these roots lie in the same 2D plane. This plane has been used to plot the root system in Figure 4.4.

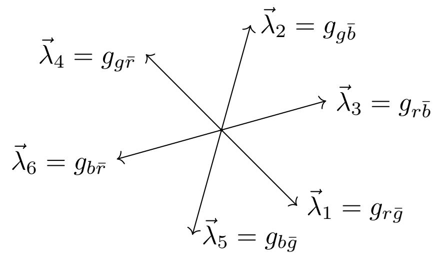


Figure 4.4: The roots of the Lie algebra $su(3)$. Note how this is the root system A_2. Furthermore the roots have also been linked to the actual particle they represent.

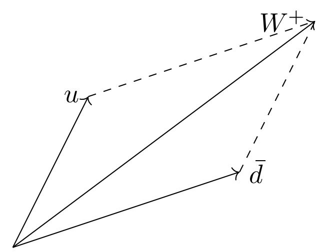


Figure 4.5: The roots of u and $\bar{d}$ add together to the root of W^+ just as the interaction in Figure 4.2 states.

We can introduce 4 charge functions that give the charge of a given particle. For example the weak isospin function acting on a left handed W^+ vector boson would return 1 T_3 while the same function would return $\frac{1}{2}$ T_3 for a left handed down quark and $\frac{1}{2}$ T_3 for a anti left handed up quark. Since all the elementary charges are conserved during an interaction, we must have that the charge functions are linear. Since functions act on E_8 which is a subspace of $\mathbb{R}^8$ and the functions are linear it must be the case that the functions are inner products for different vectors $\vec{w}_i$[5] so that the functions become of the form $f_i(\vec{v}) = \vec{w}_i^T\vec{v}$. This means that the functions can be described by a vector $\vec{w}$ and thus are the 4 charge functions equivalent to 4 charge vectors. If the original assertion that the particles can be mapped to E_8 is valid and different particles add up to the correct root given by the interaction, these 4 charge vectors must exist. This alters the problem of finding a map of all elementary particles to E_8 while all interactions are conserved to finding the map and finding the charge vectors. From this the research question becomes:

Research question 1. *Do there exist 4 (charge) vectors in $\mathbb{R}^8$ and a map of the elementary particles to the roots of E_8 so that the inner product of the root of a particle with the charge vector is the particle charge?*

[5]The subscript i implies that the vector $\vec{w}$ is different for each charge function.

Chapter 5

Identifying root systems

This chapter is an extension to chapter 3, I will go into depth in Theorems proven by the author, which are important to the research. For this we will see how to determine sub root systems inside a larger root system and we will introduce an algorithm to find (all) sub root systems. Finally we will see which sub root systems are present in the larger root systems.

5.1 Sub root systems

We will start this section with the definition of a sub root system.

Definition 5.1. *A root system Φ' is **sub root system** of a root system Φ if Φ' is root system and is a subset of Φ.*

The simplest examples of a sub root system is A_1 which is a sub root system of any root system. To see why, note that A_1 is the set of roots $\vec{\alpha}$ and $-\vec{\alpha}$. Since for any root $\vec{\beta}$ of any root system the only multiples are $\vec{\beta}$ and $-\vec{\beta}$ we can conclude that $A_1 = \{\vec{\beta}, -\vec{\beta}\}$ is a sub root system of every root system. To see which sub root systems there are of a root system the following theorem states that it is enough to find the base (of the sub root system) in the larger root system instead of all the elements of the sub root system.

Theorem 5.2. *Let Δ' be a base of Φ', if $\Delta' \subset \Phi$ then $\Phi' \subset \Phi$.*

Proof. Let Δ be a base for Φ and let $\mathscr{W}$ be the Weyl group for Φ and let $\mathscr{W}'$ be the Weyl group for Φ'.

First we show that $\mathscr{W}' \subset \mathscr{W}$. We have $\Delta' \subset \Phi$ and from Theorem 3.5(3) we have that for all $\vec{\alpha}' \in \Delta' \subset \Phi$ that $\exists \tau \in \mathscr{W}$ and $\exists \vec{\alpha} \in \Delta$ such that $\tau(\vec{\alpha}') = \vec{\alpha} \Leftrightarrow \vec{\alpha}' = \tau(\vec{\alpha})$. From Theorem 3.5(4) we have that $\mathscr{W}'$ is generated from $\sigma_{\vec{\alpha}'} = \sigma_{\tau(\vec{\alpha})}$. Since $\tau \in \mathscr{W}$ it leaves Φ invariant thus we can apply Theorem 3.2 which implies that $\sigma_{\tau(\vec{\alpha})} = \tau\sigma_{\vec{\alpha}}\tau^{-1}$. Note that $\tau \in \mathscr{W}$ and $\sigma_{\vec{\alpha}} \in \mathscr{W}$ thus we conclude that $\mathscr{W}' \subset \mathscr{W}$.

Next we show that every element of Φ' is also an element of Φ. Let $\vec{\beta}'$ be an element of Φ'. Then we can write $\vec{\beta}'$ as $\vec{\beta}' = \sigma(\vec{\alpha}')$ with $\sigma \in \mathscr{W}' \subset \mathscr{W}$ and $\vec{\alpha}' \in \Delta'$ by using Theorem 3.5(3). From our assumption that $\Delta' \subset \Phi$ we conclude that $\vec{\beta}' = \sigma(\vec{\alpha}') \in \Phi$. □

The next theorem shows why we are interested in sub root systems for the research question. The theorem states that given a root system Φ and the span of a set of (charge) vectors $\vec{v}_1, \ldots, \vec{v}_i$, the intersection of the two must be a root system Φ' and thus a sub root system of Φ.

Theorem 5.3. *Let Φ be a root system in $\mathbb{R}^n$. Given $\vec{v}_1, ..., \vec{v}_i \in \mathbb{R}^n$ such that $V = span\{\vec{v}_1, ..., \vec{v}_i\}$, then $\Phi \cap V$ is either a root system in $span\{\Phi \cap V\}$ or the empty set.*

Proof. Denote $\tilde{\Phi} = \Phi \cap V$. If $\tilde{\Phi}$ is empty we are done. Other wise we need to check Definition 3.1, if $\tilde{\Phi}$ is a root system:

(R1) We assumed $\tilde{\Phi}$ to be not empty and since Φ is a root system (and thus finite) $\tilde{\Phi}$ must also be finite. From this we can conclude that $\tilde{\Phi}$ satisfies $R1$ for a given set $\tilde{E} = span\{\tilde{\Phi}\}$

(R2) Let $\vec{\alpha} \in \tilde{\Phi}$. Then $\vec{\alpha} \in V$ and so $-\vec{\alpha} \in V$ since V is the span of multiple vectors. This means that $-\vec{\alpha} \in \tilde{\Phi}$ since $-\vec{\alpha} \in \Phi$. Also note that $\pm\vec{\alpha}$ are the only multiples of $\vec{\alpha}$ in $\tilde{\Phi}$ since they are the only multiples of $\vec{\alpha}$ in Φ.

(R3) Since V is the span of multiple vectors a reflection of a vector $\vec{\beta}$ in V acting on an element of V must stay within V $(\sigma_{\vec{\beta}}(\vec{\alpha}) = \vec{\alpha} - < \vec{\alpha}, \vec{\beta} > \vec{\beta} \in V$ since $\vec{\alpha}, \vec{\beta} \in V)$. This means that given $\vec{\alpha}, \vec{\beta} \in \Phi \cap V$ we have that $\sigma_{\vec{\beta}}(\vec{\alpha}) \in V$ and $\sigma_{\vec{\beta}}(\vec{\alpha}) \in \Phi$ so we have $\sigma_{\vec{\beta}}(\vec{\alpha}) \in \tilde{\Phi}$.

(R4) Since $R4$ is satisfied for Φ it must also be satisfied for $\tilde{\Phi}$.

We conclude that $\tilde{\Phi}$ is a root system or the empty set. □

This theorem can be extended to a corollary which shows that the elements of the root system which are orthogonal to a linear space must also be part of a root system.

Corollary 5.4. *Let Φ be a root system in $\mathbb{R}^n$. Given $\vec{v}_1, ..., \vec{v}_i \in \mathbb{R}^n$. All the elements of Φ that are orthogonal to the vectors $\vec{v}_1, \ldots, \vec{v}_i$ form a root system.*

Proof. Let $n = rank(\Phi)$ and $M = span\{\Phi\}$. The vectors $\vec{v}_1, ..., \vec{v}_i$ can be extended to a set of n vectors $\vec{v}_1, ..., \vec{v}_i, \vec{w}_{i+1}, ..., \vec{w}_n$ so that $\vec{v}_l \perp \vec{w}_k$ and that the set of vectors span the space of M. We can now apply Theorem 5.3 to Φ and $W = span\{\vec{w}_{i+1}, ..., \vec{w}_n\}$ to find that $\tilde{\Phi} = \Phi \bigcap W$ is a root system. All elements of Φ which are orthogonal to the vectors $\vec{v}_1, ..., \vec{v}_i$ must thus be elements of W. This implies that all the elements orthogonal to $\vec{v}_1, ..., \vec{v}_i$ of Φ are part of the root system $\tilde{\Phi}$. □

This corollary will proof to be useful for this research. Given a 2D root system spectrum of E8 for two vectors. For example the weights $\vec{\gamma}_1$ and $\vec{\gamma}_2$, which can be seen in Figure 5.1. All the elements which are orthogonal to these two vectors must be a root system according to Corollary 5.4. In Figure 5.1 this must mean that the 72 elements of E_8 which are orthogonal with respect to $\vec{\gamma}_1$ and $\vec{\gamma}_2$ are a root system[1].

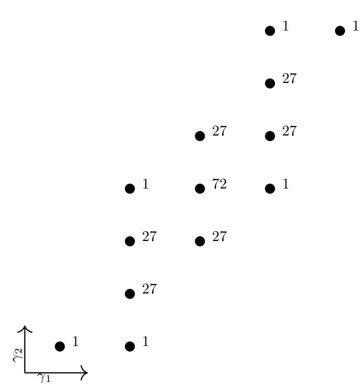


Figure 5.1: Root system Spectrum of $\vec{\gamma}_1$ and $\vec{\gamma}_2$.

[1]Since the rank of the space of this sub root system can maximally be 6, one can already deduce that this root system must be E_6 as it is the only root system of rank 6 or less with 72 elements.

5.2 Identifying sub root systems via Determinants

From Theorem 5.2 we have seen that finding a base of a sub root system in a larger root system is sufficient to conclude that the sub root system is present in the larger root system. In the following section we will look into how to identify a base of a sub root system as efficient as possible. Our first idea was to look at the determinant of the Cartan matrix to see if these are unique. The following theorems will give us the determinant of the root systems A_n and D_n for $n \geq 1$.

Lemma 5.5. *Let C_{A_n} be the Cartan matrix of root system A_n, then $det\,(C_{A_n}) = n + 1$.*

Proof. The Cartan matrix of root system A_n is given by the matrix in (5.1). Note that C_{A_n} is a $n \times n$ matrix.

$$C_{A_n} = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix} \tag{5.1}$$

By now taking the determinant of C_{A_n} and applying the Laplace expansion we can formulate an expression for $det\,(C_{A_n}) = |C_{A_n}|$:

$$\begin{aligned} |C_{A_n}| &= 2 \begin{vmatrix} 2 & -1 & & & \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{vmatrix} + \begin{vmatrix} -1 & & & & \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{vmatrix} \\ &= 2|C_{A_{n-1}}| - \begin{vmatrix} 2 & \ddots & & \\ \ddots & \ddots & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{vmatrix} = 2|C_{A_{n-1}}| - |C_{A_{n-2}}| \end{aligned} \tag{5.2}$$

In (5.2) we have found a recursive formula for calculating the determinant. We will now proof that $|C_{A_n}| = n + 1$ using induction. For this we first show that $|C_{A_1}| = 2$ and $|C_{A_2}| = 3$:

$$|C_{A_1}| = \begin{vmatrix} 2 \end{vmatrix} = 2 \qquad |C_{A_2}| = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$$

Now assume that $|C_{A_n}| = n + 1$ holds for $k \leq n$

$$|C_{A_{n+1}}| = 2|C_{A_n}| - |C_{A_{n-1}}| = 2\,(n + 1) - (n - 1 + 1) = n + 1$$

Since $|C_{A_n}| = n + 1$ hold for $k = n - 1$ and $k = n$. This implies that it also hold for $n + 1$. Thus from induction we conclude that $|C_{A_n}| = n + 1$ holds for all $n \geq 1$. $\square$

Lemma 5.6. *Let C_{D_n} be the Cartan matrix of root system D_n ($n \geq 4$), then $det\,(D_n) = 4$.*

Proof. The Cartan matrix of root system D_n is given by the matrix in (5.3). Note that C_{D_n} is a $n \times n$ matrix.

$$C_{D_n}: \begin{pmatrix} 2 & -1 & & & & & \\ -1 & 2 & \ddots & & & & \\ & \ddots & \ddots & -1 & & & \\ & & -1 & 2 & -1 & & \\ & & & -1 & 2 & -1 & -1 \\ & & & & -1 & 2 & 0 \\ & & & & -1 & 0 & 2 \end{pmatrix} \tag{5.3}$$

By taking the determinant of C_{D_n} and applying the Laplace expansion on the last column we find the following expression for the determinant:

$$|C_{D_n}| = \begin{vmatrix} 2 & -1 & & & & & \\ -1 & 2 & \ddots & & & & \\ & \ddots & \ddots & -1 & & & \\ & & -1 & 2 & -1 & & \\ & & & -1 & 2 & -1 & -1 \\ & & & & -1 & 2 & 0 \\ & & & & -1 & 0 & 2 \end{vmatrix} = 2|C_{A_{n-1}}| - \begin{vmatrix} 2 & -1 & & & & \\ -1 & 2 & \ddots & & & \\ & \ddots & \ddots & -1 & & \\ & & -1 & 2 & -1 & \\ & & & & -1 & 2 \\ & & & & -1 & 0 \end{vmatrix}$$

$$= 2|C_{A_{n-1}}| + 2 \begin{vmatrix} 2 & -1 & & & \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & -1 & \\ & & -1 & 2 & -1 \\ & & & & -1 \end{vmatrix} = 2|C_{A_{n-1}}| - 2 \begin{vmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{vmatrix}$$

$$= 2|C_{A_{n-1}}| - 2|C_{A_{n-3}}|$$

Where C_{A_n} is the Cartan matrices of root system A_n. From Lemma 5.5 we find:

$$|C_{D_n}| = 2\,(n) - 2\,(n-2) = 4$$

□

The determinants of E_6, E_7 and E_8 have been numerically computed via Matlab and gave the following results:

$$|C_{E_6}| = 3 \qquad |C_{E_7}| = 2 \qquad |C_{E_8}| = 1$$

From these lemmas we can conclude that only considering the determinant is not sufficient to uniquely identify an irreducible root system. According to Lemma 5.6 the determinant of the Cartan matrix of D_n is always 4. This problem can be resolved by also checking the dimension of the Cartan matrix. These conditions can be seen in Table 5.1.

Table 5.1: The determinants of the Cartan matrix of all the simply-laced root systems.

	A_n	D_n	E_6	E_7	E_8
$\|C\|$	$n+1$	4	3	2	1
$dim\,(C)$	n	n	6	7	8

This table indicates that irreducible simply-laced root systems have a unique determinant. As a theorem:

Theorem 5.7. *All the irreducible simply-laced root systems are uniquely determined by the determinant of the Cartan matrix and its dimension.*

Proof. This proof will be based on table 5.1. Note that D_n $\forall n \geq 4$,E_6,E_7 and E_8 are all unique compared to each other. It suffices to check that all A_n $\forall n \geq 1$ are unique compared to the other root systems.

For a given root system A_n with $n \geq 4$ the determinant is sufficient for uniqueness. This means we will need to check the $n = 1, 2, 3$ cases:

n=1: For $n = 1$ we have that $|C_1| = 2$ and this is the same as the determinant of the Cartan matrix of E_7. But the dimensions are different.

n=2: For $n = 2$ we have that $|C_2| = 3$ and this is the same as the determinant of the Cartan matrix of E_6. But the dimensions are different.

n=3: For $n = 3$ we have that $|C_3| = 4$ and this is the same as the determinant of the Cartan matrix of D_n $n \geq 0$. But the dimensions are different.

So we conclude that all the irreducible simply-laced root systems can be uniquely identified by the determinant of the Cartan matrix and its dimension. □

Theorem 5.7 gives us the necessary tools to identify irreducible simply-laced root systems. To check if we can identify reducible simply-laced root systems we need to be able to calculate the determinant of the Cartan matrix of a reducible simply-laced root system. The following lemma and corollary give us the tools to do so:

Lemma 5.8. *Given a reducible root system A and two root systems A_1 and A_2 such that $A = A_1A_2$, the determinant of the Cartan matrix is the product of the determinant of the two sub root systems:*

$$det\,(C_A) = det\,(C_{A_1})\,det\,(C_{A_2})$$

Proof. The Cartan Matrix C_A of A can be written in block form[2] as:

$$C_A = \begin{pmatrix} C_{A_1} & 0 \\ 0 & C_{A_2} \end{pmatrix} = \begin{pmatrix} C_{A_1} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & C_{A_2} \end{pmatrix}$$

Next we use the rule that the determinant of a product of matrices is the product of the determinant of each matrix, giving us:

$$det\,(C_A) = det \begin{pmatrix} C_{A_1} & 0 \\ 0 & I \end{pmatrix} det \begin{pmatrix} I & 0 \\ 0 & C_{A_2} \end{pmatrix} = det\,(C_{A_1})\,det\,(C_{A_2})$$

□

Corollary 5.9. *Given a reducible root system A and irreducible root systems $A_1, A_2, \ldots, A_n$ such that $A = A_1A_2 \ldots A_n$ the determinant of the Cartan matrix of A is the same as the product of the determinants of the irreducible root systems:*

$$det\,(C_A) = det\,(C_{A_1})\,det\,(C_{A_2}) \ldots det\,(C_{A_n})$$

Proof. We will proof the assertion by repeatedly applying Lemma 5.8 until all the smaller root systems are irreducible. □

[2]Note that permuting the columns and rows does not change the Determinant.

From the corollary we find that the determinant of the Cartan matrix of a reducible root system is the product of the determinants of Cartan matrices of the irreducible sub root systems. This is however not enough to identify reducible simply-laced root systems as for example the reducible root systems A_3A_3 and $D_4A_1A_1$ the determinant of the Cartan matrices are both 16 and the dimension of both root systems is 6. The benefit this method would have had is that calculating a determinant and noting its dimension is relatively easy. Since this method does not identify the reducible simply-laced sub root systems we did not further investigate this method.

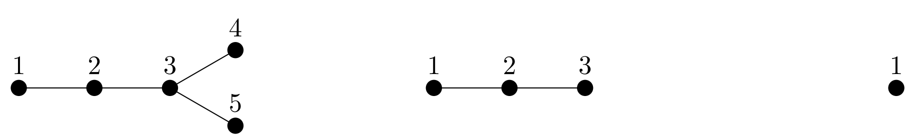
5.3 Identifying root systems via the Dynkin diagram

As seen in the previous section we cannot identify a reducible simply-laced root system by calculation of the determinant and the dimension. In this section we will look at the Cartan Matrix and how it is correlated to the Dynkin diagram.

Given a Cartan Matrix C (e.g. Figure 5.3), all the diagonals elements of the Cartan Matrix of a simply-laced root system are 2. The other elements are either 0 or -1. An element is 0 if the two simple roots are orthogonal and -1 if the angle between the two roots is $\frac{2\pi}{3}$.

Given a Dynkin diagram the nodes correspond to the simple roots. Two nodes are connected if the angle between two nodes (the two simple roots) is $\frac{2\pi}{3}$ and no connection if the two nodes (the two simple roots) are orthogonal.

The Cartan matrix C and Dynkin diagram are related by the matrix $I - 2C$ which is the adjacency matrix of the Dynkin diagram. Here I is the Identity matrix of the same dimension as the Cartan Matrix. The Dynkin diagrams for the root systems D_5, A_3 and A_1 can be seen in Figure 5.2, the Cartan Matrices for these root systems can be seen in Figure 5.3 and the Adjacency matrix can be seen in Figure 5.4. The idea is now to determine the adjacency matrix of the Cartan Matrix. After that, we determine the connected parts of the Dynkin diagram via the adjacency matrix. Lastly we determine the irreducible root system from each connected part.



(a) D_5 (b) A_3 (c) A_1

Figure 5.2: The Dynkin diagram for three different root systems.

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{pmatrix}$$

(a) D_5

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

(b) A_3

$$\begin{pmatrix} 2 \end{pmatrix}$$

(c) A_1

Figure 5.3: The Cartan Matrix matrix of three different root systems given in Figure 5.2.

The nodes in the Dynkin diagram of a root system can have at most three connections (as seen in Figure 5.2a). Because of this we start with the nodes in the adjacency matrix that have three connections, if these exists (Note that when a node is found with 4 or more connections it cannot be a

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

(a) D_5

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

(b) A_3

$$\begin{pmatrix} 0 \end{pmatrix}$$

(c) A_1

Figure 5.4: The adjacency matrix of three different root systems given in Figures 5.2 and 5.3.

Cartan matrix and we are done). This node has three connections with three new nodes. This can be seen in Figure 5.2a as node 3. Call these three connections (the connections from node 3 to node 2,4 and 5) the three different arms of the starting node. On each of these arms we find a new node which has either two connections or one connection. For example in 5.2a node 2 is connected to node 3 and 1 while node 4 is only connected to node 3. When a node only has one connection the whole arm has been explored and we stop. If a node has two connections we continue the process by looking at the unexplored node. For example in 5.2a node 2 is connected to node 3 (already explored) and node 1 (not explored yet). We continue the process on node 1. Node 1 has just one connection and thus we are done. By now applying this to all of the arms and keeping number of the amount of new nodes we find in each arms (so the lengths of the arms in 5.2a are $1, 1$ and 2), we can identify this connected part of the adjacency matrix with a root system. Namely a root system of type D_n or E_n. This identification is really straightforward. We simply order the lengths of the three arms. The smallest length must be one in either of the two cases. The second smallest length can either be 2 (in the case of a E_n root system) or 1 (in the case of an D_n root system). The dimension identification is also straightforward as it must be the sum of all the lengths plus one. This works since the lengths tell us how many nodes each arm has and we add one for the starting node.

The next type of root systems, A_n $n \geq 2$, will also be identified in a similar fashion. To do this we will first remove all the nodes found to be part of a root system of type D_n or E_n. This can easily be done by keeping track of the nodes that are part of those root systems. The Dynkin diagram of the root systems of type A_n $n \geq 2$ is a single strand of nodes with two ends. The nodes inside the strand all have two connections while the ends have only one connection (as can be seen in 5.2b). This means we can identify these root systems starting at an end (e.g. node 1 in 5.2b) and walking down the whole strand till we find an other end. By keeping track of all the nodes we find in the string we can identify the base of the root system. The number of nodes found (including the starting node) must be the dimension of the root system and we have thus found a root system of type A_n $n \geq 2$.

All the other types of graphs we find, e.g. a cycle or nodes with more than 3 arm, cannot be a adjacency matrix of a Cartan matrix. These adjacency matrices can thus not be connected to a root system. This adjacency matrix can thus be neglected.

This leaves us only to identify the root systems of the type A_1. The identification of the root system A_1 is really trivial. The simple root of the root system A_1 is orthogonal to all other simple roots. This means that in the adjacency matrix all nodes with no connections must be a root system of the type A_1.

We have now only discussed irreducible root systems. But the algorithm will also work in reducible root systems. For this we simply loop over all the connected parts of the graph by keeping track of which nodes we have already visited.

5.4 Orbits of a Sub Root System

We have found different sub root systems in larger root systems. In this section we will consider if two sub root systems of the same type in a larger root system are related by a map of the Weyl group. This relation is an **orbit** and will be illustrated with the help of Figure 5.5.

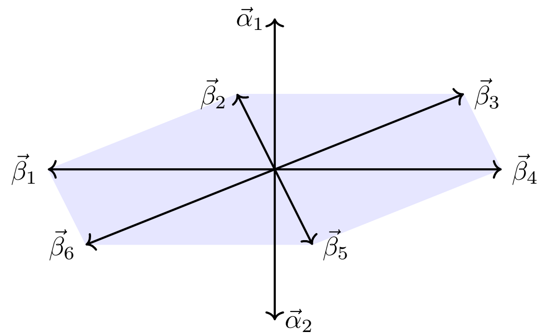


Figure 5.5: The root system A_2A_1 with Weyl group $\mathscr{W}$, the roots $\vec{\alpha}_1, \vec{\alpha}_2$ form A1 while the roots $\vec{\beta}_1, \vec{\beta}_2, \ldots, \vec{\beta}_6$ form A2 (inside the blue hexagon).A_1 occurs four times as a sub root system of A_2A_1. Every root can be used as a base for the sub root system A_1 however there exists no element in the Weyl group $\mathscr{W}$ that maps the $\vec{\alpha}$ roots to the $\vec{\beta}$ roots. There are two orbits of A_1 in A_2A_1, namely the trivial orbit of A_1 in A_1 and the orbit of A_1 in A_2 (these are the three different ways to find A_1 in A_2 which is the blue shaded region).

Definition 5.10. *An* ***Orbit*** *of a sub root system $\Phi' \subset \Phi$ is the set of sub root systems generated by applying all the elements of the Weyl group of Φ to Φ'.*

In Figure 5.5 one can see the root system A2A1. Off course all 8 roots can be a base for the sub root system A1 in A2A1. If we take the sub root system Φ' as $\{\vec{\beta}_1, \vec{\beta}_4\}$, the orbit of this sub root system Φ' would be the set $\{\{\vec{\beta}_1, \vec{\beta}_4\}, \{\vec{\beta}_2, \vec{\beta}_5\}, \{\vec{\beta}_3, \vec{\beta}_6\}\}$. Note that $\{\vec{\alpha}_1, \vec{\alpha}_2\}$ is not an element of the orbit of $\{\vec{\beta}_1, \vec{\beta}_4\}$. There exists no element in the Weyl Group of the root system A2A1 that maps $\{\vec{\alpha}_1, \vec{\alpha}_2\}$ to $\{\vec{\beta}_1, \vec{\beta}_4\}$. There are maps that map $\{\vec{\alpha}_1, \vec{\alpha}_2\}$ to $\{\vec{\beta}_1, \vec{\beta}_4\}$. However these elements are not part of the Weyl group since they do not leave A_2A_1 invariant. In this example we have two different orbits of the sub root system A1. We have already seen that a root system is fully determined by the base. Instead of looking at the orbit of a sub root system we can limit ourself to looking at the the orbit of the base of the sub root system.

Definition 5.11. *A* ***Base Orbit*** *of a sub root system $\Phi' \subset \Phi$ is the set of bases which is generated by applying all the elements of the Weyl group of Φ to the base for the sub root system Φ'.*

The base orbit is closely related to the orbit of a sub root system. The difference being that the orbit of a sub root system is a set of root systems, while the base orbit is a set of bases. The base orbit will be explained with the help of Figure 5.5. Here all roots are a base for the sub root system A_1. However there only exist elements in the Weyl group that map $\vec{\beta}$ roots to other $\vec{\beta}$ roots and elements that map $\vec{\alpha}$ roots to other $\vec{\alpha}$ roots. There are two base orbits of A_1 in A_2A_1. Namely the base orbit $\{\{\vec{\alpha}_1\}, \{\vec{\alpha}_2\}\}$ and the base orbit $\{\{\vec{\beta}_1\}, \{\vec{\beta}_2\}, \{\vec{\beta}_3\}, \{\vec{\beta}_4\}, \{\vec{\beta}_5\}, \{\vec{\beta}_6\}\}$. Note that the base orbits have more elements than the orbits of root systems. In the case of A_1 this is due to the fact that both $\vec{\alpha}_1$ and $\vec{\alpha}_2$ are a base for A_1. This doubles the amount of elements in each orbit.

Till now we have neglected the ordering of the base of a root system. When talking about a set the ordering is not important. For example the base $\{\vec{\alpha}_1, \vec{\beta}_1, \vec{\beta}_3\}$ for A_2A_1 is the same base as $\{\vec{\beta}_3, \vec{\alpha}_1, \vec{\beta}_1\}$

for A_2A_1. However when we try to represent the base on the computer there must be a first simple root, a second and a last. We have denoted the base before as $\Delta = \{\vec{\alpha}_1, \vec{\alpha}_2, \ldots, \vec{\alpha}_n\}$. When we order the base, this means that $\vec{\alpha}_1$ is the first simple root and $\vec{\alpha}_n$ is the last simple root. Fixing the order of the base is not unique however. In the case of D_4, the base consists of 4 roots. These 4 roots can be ordered in 6 different ways so that they produce the same Dynkin diagram. This can be seen in Figure 5.6. We will refer to these different orderings as the **symmetries of the Cartan Matrix**. When two simple roots of the Cartan symmetry are switched (for example simple roots 1 and 3 in Figure 5.6(a)) the Dynkin diagram and Cartan matrix stays the same.

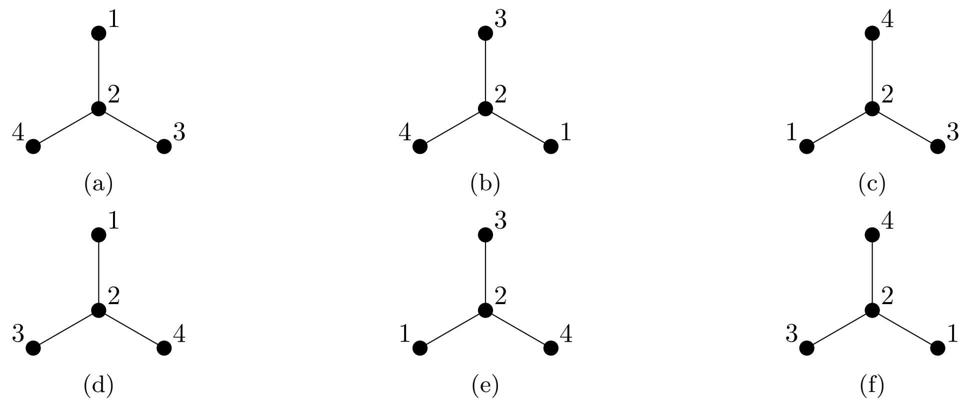


Figure 5.6: The 6 different orderings of the base of D_4. We have chosen the middle root to be the second since this will give us the standard Cartan Matrix of D_4.

In the case of a fixed base our definition of a base orbit is not inclusive enough. Since we are now concerned with the order of the base we will need to include all the different symmetries of the Cartan Matrix. This gives us the following definition:

Definition 5.12. *An **Ordered Base Orbit** of a sub root system $\Phi' \subset \Phi$with a fixed base is the set of bases generated by applying all the elements of the Weyl group of Φ to all the different ordered bases of the sub root system Φ'.*

The question we want to answer in this section is how can we determine the amount of orbits a sub root system has in the larger root system. For this we will introduce the concept of a representative of the sub root system.

Definition 5.13. *Given a root system Φ and sub root system $\tilde{\Phi}$ with an ordered base $\tilde{\Delta} = \{\vec{\alpha}_1, \vec{\alpha}_2, \ldots, \vec{\alpha}_n\}$, the **representative** of the sub root system is a vector given by the formula:*

$$\vec{w} = \sum_{i=1}^{n} 5^i \vec{\alpha}_i$$

As we can see the ordering of the base is important here. Different orderings will result in different representatives of the root system. The representative is rather useful. We have seen that the elements of the Weyl group permute the Weyl chambers, so when the representative is an element of a Weyl chamber, the Weyl group moves the representative to different Weyl chambers. There must also be an element that maps the representative to the fundamental Weyl chamber (here we mean the fundamental representative with respect to the larger root system Φ). If the representative does not lie in a Weyl chamber a map still exists in the Weyl group so that the representative is mapped to the closed fundamental Weyl chamber. (The fundamental Weyl chamber is the set of vectors which all have a strict positive inner product with all the simple roots, the closed fundamental Weyl chamber is the set of vectors which all have positive or zero inner product with the simple roots).

The following lemma and theorem shows that for all root systems which are in the same orbit, the fundamental representatives must be the same. For this we will first introduce Lemma 5.14 which will be used in the proof of Theorem 5.15.

Lemma 5.14. *Given the root system E_8 with roots given as in Section 3.6.3. Any element σ of the Weyl group of E_8 applied to a any root $\vec{\beta}$ implies that the components of $2\sigma(\vec{\beta})$ are not a multiples of 5.*

Proof. Note that all the element of the roots can only take the values $0, \pm\frac{1}{2}, \pm 1$. This means that the operation $\sigma(\vec{\beta})$ goes to a root with those values. By now multiplying by 2 we see that the elements of $2\sigma(\vec{\beta})$ can only be $0, \pm 1, \pm 2$ which are not multiples of 5. Also note that the zero vector cannot be a root. □

Theorem 5.15. *Let Φ be the root system of E_8 with roots given as in Section 3.6.3 with 2 sub root systems Φ' and Φ'' of the same type. Then:*
The two sub root systems are in the same ordered base orbit
$\Longleftrightarrow$
there exists orderings on the bases of Φ' and Φ'' so that the fundamental representatives of the two sub root systems are the same ($\vec{w}'_f = \vec{w}''_f$).

Proof. We need to proof the implication in both directions:

$\Rightarrow$:
Two sub root systems are in the same ordered orbit if there exists a $\sigma \in \mathscr{W}$ and an order on $\vec{\alpha}'_i$ and $\vec{\alpha}''_i$ such that $\sigma(\vec{\alpha}'_i) = \vec{\alpha}''_i$ for all elements $\vec{\alpha}'_i \in \Delta'$ and $\vec{\alpha}''_i \in \Delta''$. In this case we must also have that $\sigma(\vec{w}') = \vec{w}''$. Note there exists a σ_1 such that $\sigma_1(\vec{w}') = \vec{w}'_f$. In this case we have that $\vec{w}'_f = \sigma_1(\sigma^{-1}(\vec{w}''))$. This means that $\sigma_1(\sigma^{-1}(\vec{w}''))$ lies in the fundamental Weyl Chamber and thus must be the Fundamental representative of $\vec{w}''$ and thus we have that $\vec{w}''_f = \sigma_1(\sigma^{-1}(\vec{w}'')) = \vec{w}'_f$. Thus we have found the fundamental representatives of the two sub root systems.

$\Leftarrow$:
Since the the Fundamental representatives are the same we can define the maps σ' and σ'' such that $\sigma'(\vec{w}') = \vec{w}'_f$ and $\sigma''(\vec{w}'') = \vec{w}''_f$. Now we define $\sigma = \sigma''^{-1}\sigma'$ so that $\sigma(\vec{w}') = \vec{w}''$. To proof the assertion we need to show that $\sigma(\vec{\alpha}'_i) = \vec{\alpha}''_i$ for all i. For this note that as by the definition of a representative we have:

$$\vec{w}' = \vec{\alpha}'_1 + 5\vec{\alpha}'_2 + \cdots + 5^{n-1}\vec{\alpha}'_n$$
$$\vec{w}'' = \vec{\alpha}''_1 + 5\vec{\alpha}''_2 + \cdots + 5^{n-1}\vec{\alpha}''_n$$

Now we have that $2\sigma(\vec{w}') = 2\vec{w}''$ which we can also rewrite as:

$$2\sigma(\vec{w}') = 2\sigma(\vec{\alpha}'_1 + 5\vec{\alpha}'_2 + \cdots + 5^{n-1}\vec{\alpha}_n)' = 2\sigma(\vec{\alpha}'_1) + 5 \cdot 2\sigma(\vec{\alpha}'_2) + \cdots + 5^{n-1} \cdot 2\sigma(\vec{\alpha}'_n) =$$
$$2\vec{w}'' = 2\vec{\alpha}''_1 + 5 \cdot 2\vec{\alpha}''_2 + \cdots + 5^{n-1} \cdot 2\vec{\alpha}''_n$$

Which we can rewrite into:

$$2\sigma(\vec{\alpha}'_1) + 5 \cdot 2\sigma(\vec{\alpha}'_2) + \cdots + 5^{n-1} \cdot 2\sigma(\vec{\alpha}'_n) = 2\vec{\alpha}''_1 + 5 \cdot 2\vec{\alpha}''_2 + \cdots + 5^{n-1} \cdot 2\vec{\alpha}''_n \tag{5.4}$$

Now we take mod 5 of the equation from which we find:

$$2\sigma(\vec{\alpha}'_1) = 2\vec{\alpha}''_1 \qquad \text{mod } 5$$

Note however that $2\vec{\beta}$ (with $\vec{\beta}$ any root in E_8) can only take the values $0, \pm 1, \pm 2$. So this means that $2\sigma(\vec{\alpha}'_1) = 2\vec{\alpha}''_1$. without the modulus and we find that $\sigma(\vec{\alpha}'_1) = \vec{\alpha}''_1$. We can now subtract $2\sigma(\vec{\alpha}'_1) = 2\vec{\alpha}''_1$ from (5.4) and divide by 5 to find:

$$2\sigma(\vec{\alpha}'_2) + \cdots + 5^{n-2} \cdot 2\sigma(\vec{\alpha}'_n) = 2\vec{\alpha}''_2 + \cdots + 5^{n-2} \cdot 2\vec{\alpha}''_n \tag{5.5}$$

By again taking mod 5 we find:

$$2\sigma(\vec{\alpha}_2') = 2\vec{\alpha}_2'' \qquad \text{mod } 5$$

And by the same argument as before we see that $\sigma(\vec{\alpha}_2') = \vec{\alpha}_2''$. We can repeat this process to show that $\sigma(\vec{\alpha}_i)' = \vec{\alpha}_i''$ for any $i \in \{1, 2, \dots, n\}$. Note that this also forces the ordering on the bases.

□

We have thus found a way to check if two root systems are in the same base orbit. Namely checking if the two fundamental representatives are the same.

5.4.1 Finding Fundamental Representatives

Next we will determine how to find the Fundamental Representative $\vec{w}_f$ of a sub root system Φ' from a representative $\vec{w}$ in the root system Φ with basis $\Delta = \{\vec{\gamma}_1, \dots, \vec{\gamma}_n\}$. This is done with the help of Algorithm 1 where the vector $\vec{w}$ is the Fundamental representative when the algorithm finishes.

Algorithm 1 An algorithm to find the Fundamental Representative $\vec{w}_f$ of a sub root system Φ' with representative $\vec{w}$ in a root system Φ with base $\Delta = \{\vec{\gamma}_1, \dots, \vec{\gamma}_n\}$.

$\vec{w} \leftarrow$ representative of *sub root system*
$\vec{\gamma}_i \leftarrow$ simple roots of *root system*
while any $\vec{\gamma}_i^T \vec{w} < 0$ **do**
  $j \leftarrow$ smallest i for which $\vec{\gamma}_i^T \vec{w} < 0$
  $\vec{w} \leftarrow \sigma_{\vec{\gamma}_j}(w)$
end while

The Fundamental representative is by definition the vector $\vec{w}$ for which the while condition is *false* and for which the algorithm finishes. To show that Algorithm 1 returns the Fundamental representative it needs to finish (in finite steps). We will show this with the help of a cost function. This cost function will be a inner product with the vector $\vec{\gamma}_C$. This vector is a linear combination of the simple roots of the larger root system Φ.

$$\vec{\gamma}_C = \sum_{i=1}^{n} c_i \vec{\gamma}_i = [\vec{\gamma}_1 \vec{\gamma}_2 \dots \vec{\gamma}_n]\vec{c} \tag{5.6}$$

This vector $\vec{c}$ is given by the following formula:

$$\vec{c} = \left([\vec{\gamma}_1 \vec{\gamma}_2 \dots \vec{\gamma}_n]^T [\vec{\gamma}_1 \vec{\gamma}_2 \dots \vec{\gamma}_n]\right)^{-1} \vec{1} \tag{5.7}$$

Where $\vec{1}$ is a vector of n elements which are all 1. Note that this condition implies that:

$$(\vec{\gamma}_i, \vec{\gamma}_C) = (\vec{\gamma}_C, \vec{\gamma}_i) = \vec{\gamma}_i^T \vec{\gamma}_C = \vec{\gamma}_i^T [\vec{\gamma}_1 \vec{\gamma}_2 \dots \vec{\gamma}_n]\vec{c} = 1 \tag{5.8}$$

Now consider a Representative $\vec{w}'$ of a sub root system Φ' with base Δ'. The cost of this fundamental representative can be calculated of this vector $\vec{w}'$ by taking the inner product of $\vec{w}'$ with $\vec{\gamma}_C$. Note that in the while loop we must have that $\vec{\gamma}_i^T \vec{w} < 0$ other wise the algorithm does not update the vector $\vec{w}$. In this case the algorithm updates $\vec{w}'$ by mirroring around the hyper-plane of $\vec{\gamma}_i$ (or by applying $\sigma_{\vec{\gamma}_i}$). Next we will calculate what happens to the cost function when we update the vector $\vec{w}$:

$$(\vec{\gamma}_C, \sigma_{\vec{\gamma}_i}(\vec{w})) - (\vec{\gamma}_C, \vec{w}') = (\vec{\gamma}_C, \sigma_{\vec{\gamma}_i}(\vec{w}') - \vec{w}') \tag{5.9}$$

Note however that $\sigma_{\vec{\gamma}_i}(\vec{w}') = \vec{w} - \vec{\gamma}_i^T \vec{w}' \vec{\gamma}_i$ (see equation (3.1)). This reduces equation (5.9) to:

$$(\vec{\gamma}_C, \sigma_{\vec{\gamma}_i}(\vec{w}')) - (\vec{\gamma}_C, \vec{w}') = (\vec{\gamma}_C, -\vec{\gamma}_i^T \vec{w}' \vec{\gamma}_i) = -\vec{\gamma}_i^T \vec{w}' (\vec{\gamma}_C, \vec{\gamma}_i) \tag{5.10}$$

Next we use the property of $\vec{\gamma}_C$ that $(\vec{\gamma}_i, \vec{\gamma}_C) = 1$. This means that the every time the vector $\vec{w}$ gets updated, it increases in cost of the cost function. Namely by the amount of $-\vec{\gamma}_i^T \vec{w}' > 0$. This means that the algorithm strictly increases the cost every iteration.

We will now introduce a pseudo representative. A pseudo representatives is similar to a representative but instead of a sum over the m simple roots of the base, it is a sum over of m roots of the sub root system. A pseudo representative is thus a more general case of a representative. When we apply a $\sigma \in \mathscr{W}$ to a pseudo representative, we map the m roots from which we made the pseudo representative to m other (or possibly the same) roots. These roots also produce a pseudo representative. This means that any $\sigma \in \mathscr{W}$ maps a pseudo representative to a different pseudo representative. For a given root system Φ with N roots a total of $\frac{N!}{(N-m)!}$ possible pseudo representatives are possible of m roots.

Coming back to the assertion that Algorithm 1 finishes in a finite amount of iterations. The algorithm starts with the original representative $\vec{w}$. Suppose the Algorithm never stops. This means there exists a sequence of pseudo representatives $(\vec{w})_n$ with $\vec{w}_0$ the starting representative. Since there are a finite amount of pseudo representatives the sequence must pass a pseudo representative at least twice. Denote this pseudo representative in the sequence with index k and l with $k \neq l$ (so $\vec{w}_k = \vec{w}_l$). Since the pseudo representative is the same at index k and l, the cost of the cost function at index k and l must be the same. However the cost function is a strict increasing function so when $k > l$ we must have that the cost of $\vec{\gamma}_C^T \vec{w}_k > \vec{\gamma}_C^T \vec{w}_l$. This contradiction forces us to conclude that it is not possible for the algorithm to produce an infinite sequence. Thus the algorithm must stop in a finite number of steps.

Chapter 6

Sub Root Systems

The search for all the irreducible root systems in E_8 has resulted in multiple data sets of sub root systems and the number of orbits of each sub root system. In the case of E_8 the root system proved to be to large to search through entirely[1]. For this reason only the sub root systems of dimension 1 to 6 have been searched in the case of E_8. Besides the results of E_8 the sub root systems of the smaller irreducible root systems of dimension 7 and less have been searched. The found sub root systems of E_8 can be found in the following table while the other results can be found in the Appendix.

Table 6.1: E8

Name	Counts	Orbit
A_1	120	1
A_2	1120	1
A_1^2	3780	1
A_3	7560	1
A_2A_1	40320	1
A_1^3	37800	1
D_4	3150	1
A_4	24192	1
A_3A_1	151200	1
A_2^2	67200	1
$A_2A_1^2$	302400	1
A_1^4	122850	2
D_5	7560	1

Name	Counts	Orbit
D_4A_1	37800	1
A_5	40320	1
A_4A_1	241920	1
A_3A_2	302400	1
$A_3A_1^2$	529200	2
$A_2^2A_1$	403200	1
$A_2A_1^3$	604800	1
A_1^5	113400	1
E_6	1120	1
D_6	3780	1
D_5A_1	45360	1
D_4A_2	50400	1
$D_4A_1^2$	56700	1

Name	Counts	Orbit
A_6	34560	1
A_5A_1	161280	1
A_4A_2	241920	1
$A_4A_1^2$	362880	1
A_3^2	189000	2
$A_3A_2A_1$	604800	1
$A_3A_1^3$	453600	3
A_2^3	44800	1
$A_2^2A_1^2$	604800	1
$A_2A_1^4$	151200	1
A_1^6	56700	1
$Dim > 6$	?	?
?	?	?

[1]this was due to both time and memory constraints.

Chapter 7

The elementary Particles and E8

In this chapter we will show that the elementary particles **cannot** be identified by roots of the root system E_8 with 4 charge vectors so that the inner product of a charge vector with the identified root of a particle produces that particles charge. If it were the case that these 4 charge vectors exist it means that given a root systems spectrum of these charge vectors, the actual charge spectrum fits inside this root system spectrum as for a given charge it must be that there are at least as many or more roots with this charge than actual elementary particles with this charge. The proof uses the charge spectrum of the charges g_3 and g_8. The 2D charge spectrum of g_3 and g_8 can be seen in Figure 7.2. Note that this is not the pure g_3 and g_8 spectrum but a linear combination of the the charges. This is simply done to make the charge spectrum easier to work with. One can easily transform back to the original g_3 and g_8 charge vectors from the g_3 and $-g_3 + g_8$ vectors. The charge vectors g_3 and g_8 span a 2D space which we will denote by W. According to Corollary 5.4, all the vectors orthogonal to the sub space W of E_8 must be a root system (and thus a sub root system of E_8). We will call this root system Φ:

$$\Phi = E_8 \bigcap W^{\perp} = E_8 \bigcap (span\{g_3, g_8\})^{\perp} \tag{7.1}$$

Here we take Φ to be the intersection of E_8 with the complement of the space W. The space W is the span of g_3 and g_8 thus the resulting root system Φ must be orthogonal to g_3 and g_8. This sub root system has the following properties. Since Φ is orthogonal to a 2D space in E_8 (8 dimensional), the dimension of Φ is at most 6. Next there are 36 elementary particles with $g_3 = g_8 = 0$, therefore the sub root system must have at least 36 elements. See Figure 2.12. These 36 elementary particles must thus be mapped to 36 different roots in this sub root system. There are exactly 5 possible root systems which satisfy these conditions. Namely: D_5, D_5A_1, A_6, D_6 and E_6. These sub root systems of E_8 all have the property that they only have one orbit (as can be seen from Table 6.1). This means that there is exactly one way to produce the Dynkin diagram of these root systems from the Dynkin diagram of E_8. The Dynkin diagrams of each sub root system can be seen in Figure 7.3.

One can also look at different charge combinations. In this case we will get the Figure 7.1. In this Figure, different combinations of charges have been set to zero. The particles which are neutral to these charges must again be a root system. The possible irreducible root systems have been denoted in the Figure for the types A_n, D_n, E_n. Only the smallest root system is denoted but larger root systems are also possible. Note that for n charges equal to zero, the root system can be at most $8 - n$ dimensional. So $T_3 = g_3 = g_8 = 0$ can be at most 5 dimensional. All possible irreducible root systems are then D_5, D_4, A_5, A_4.

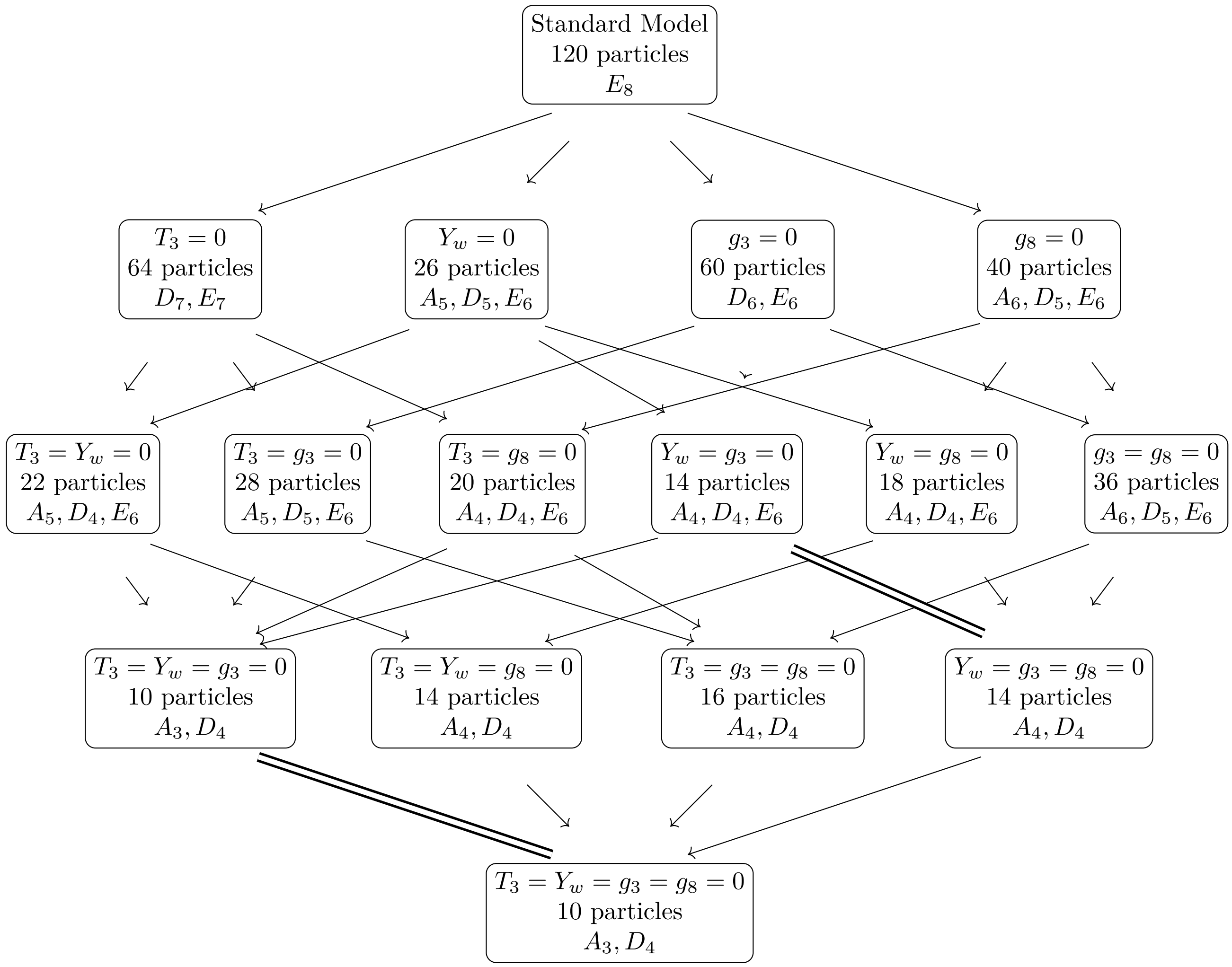


Figure 7.1: A graph of how many elementary particles are neutral to a given charge. Every layer deeper an extra charge has been set to zero. This reduces the amount of particles with the condition of those charges being zero. Note that two lines are double lines instead of arrows. Here we have that the condition $Y_w = g_3 = 0$ also means that $g_8 = 0$. There are no particles with $Y_w = g_3 = 0$ and a non zero g_8 charge. The smallest possible root system has been denoted for the following type of root systems: A_n, D_n, E_n.

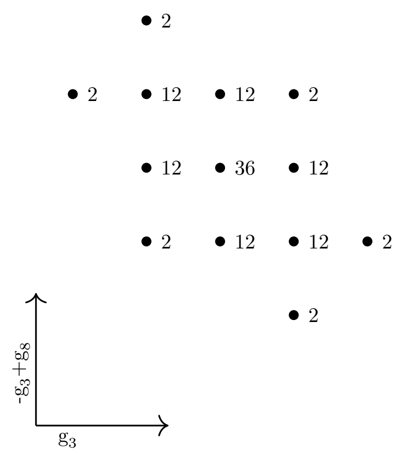


Figure 7.2: The 2D charge spectrum of g_3 and g_8.

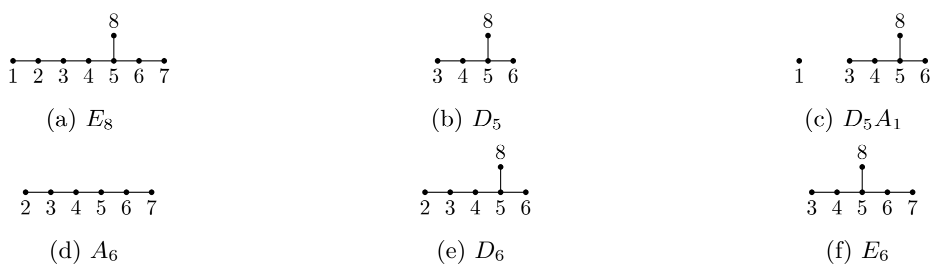
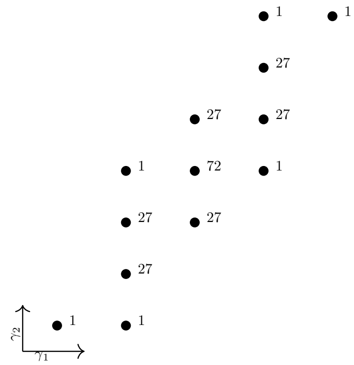


Figure 7.3: The simple roots of E_8 and how to construct $D_5, D_5A_1, A_6, D_6, E_6$ from these simple roots.

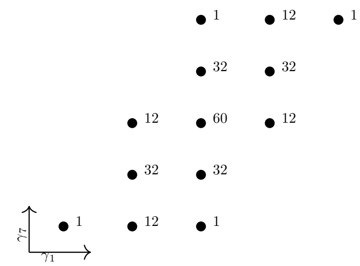
The Dynkin diagram of the sub root systems D_5A_1, A_6, D_6 and E_6 can be created from the Dynkin diagram of E_8. This can be seen in Figure 7.3. In the case of E_6 the simple roots $\vec{\gamma}_1$ and $\vec{\gamma}_2$ can be disregarded from the Dynkin diagram of E_8, to form the Dynkin diagram of E_6. Recall that for the fundamental weight $\vec{\lambda}_j$ we have $<\vec{\gamma}_i, \vec{\lambda}_j>= \delta_{ij}$. This implies that the fundamental weights $\vec{\lambda}_1$ and $\vec{\lambda}_2$ are orthogonal to all the (simple) roots of E_6. Furthermore the span of $\vec{\lambda}_1$ and $\vec{\lambda}_2$ is orthogonal to E_6. This is also the set of all vectors in $\mathbb{R}^8$ that are orthogonal to E_8. This means that:

$$E_6 = E_8 \bigcap (span\{\vec{\lambda}_1, \vec{\lambda}_2\})^{\perp} \tag{7.2}$$

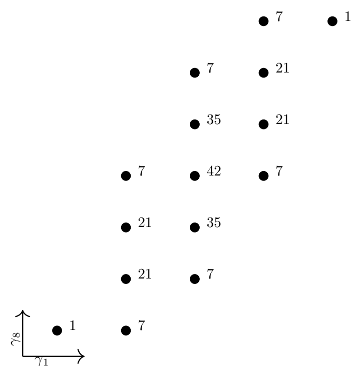
This way of producing E_6 as a sub root system in E_8 is however not unique. There are infinitely many vectors (not necessarily fundamental weights) $\vec{\lambda}_1$ and $\vec{\lambda}_2$ in $\mathbb{R}^8$ so that we have (**??**). However we have shown in Table 6.1 that E_6 only has one orbit in E_8. This means that for any two sub root systems E_6‘ and E_6“, there exists an element σ in the Weyl group of E_8 so that E_6“ $= \sigma(E_6$‘$)$. Besides it must also be the case that this σ maps the span of all orthogonal complement of E_6‘, to the span of the orthogonal complement of E_6“. In other words $span\{\vec{\lambda}_1$“$, \vec{\lambda}_2$“$\} = \sigma(span\{\vec{\lambda}_1$‘$, \vec{\lambda}_2$‘$\})$. This means that we only have to check the span of $\vec{\lambda}_1$ and $\vec{\lambda}_2$ (now referring to the fundamental weights of E_8). Every other span can be mapped to this span by an element of the Weyl group. When we compare (7.1) to (7.2) we must have that the charge vectors of g_3, g_8 are elements of $span\{\vec{\lambda}_1, \vec{\lambda}_2\}$ (or can be mapped to vectors in this span). This span must now be the space W. We can do the same for the sub root



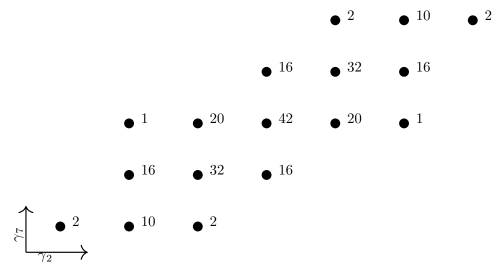
(a) Root system Spectrum of γ_1 and γ_2. Corresponds to E_6.



(b) Root system Spectrum of γ_1 and γ_7. Corresponds to D_6.



(c) Root system Spectrum of γ_1 and γ_8. Corresponds to A_6.



(d) Root system Spectrum of γ_2 and γ_7. Corresponds to D_5A_1.

Figure 7.4: The root system spectrum corresponding to the complement of various sub root systems of E_8.

systems D_5A_1, A_6, D_6. We will find that the charge vectors of g_3 and g_8 are elements of the span of fundamental weights. These fundamental weights can be determined from Figure 7.3. For E_6 we have to take the weights $\vec{\lambda}_1$ and $\vec{\lambda}_2$, for D_6 we have to take $\vec{\lambda}_1$ and $\vec{\lambda}_7$, for A_6 we will take $\vec{\lambda}_8$ and $\vec{\lambda}_1$ and for D_5A_1 we will take $\vec{\lambda}_7$ and $\vec{\lambda}_2$. Since the charge vectors g_3 and g_8 are elements of the span, they must in fact be linear combinations of these fundamental weights. To check whether there is a possible charge vector, the root system spectra of these fundamental weights will be investigated. Here we will try to find a linear combination of the fundamental weights, so that the charge spectrum of Figure 7.2 fits. In Figure 7.4 it is obvious that there is no linear combination possible of the chosen weights so that the g_3 and g_8 spectrum fits.

We thus conclude that if the elementary particles fit inside E_8, the orthogonal complement of the span of g_3 and g_8 (the particles with $g_3 = g_8 = 0$) must fit inside D_5. Our previous argument is less obvious since there are now three weights to choose from. Namely $\vec{\gamma}_1, \vec{\gamma}_2$ and $\vec{\gamma}_7$. From this we get a 3D root system spectrum. This 3D root system spectrum can be seen in Figure 7.5. This means that the 2D charge spectrum of g_3 and g_8 is a projection of the 3D root system spectrum. This complicates things as there are a lot of different projections.

This problem can be split up in to two smaller problems. In the first sub problem the assumption is

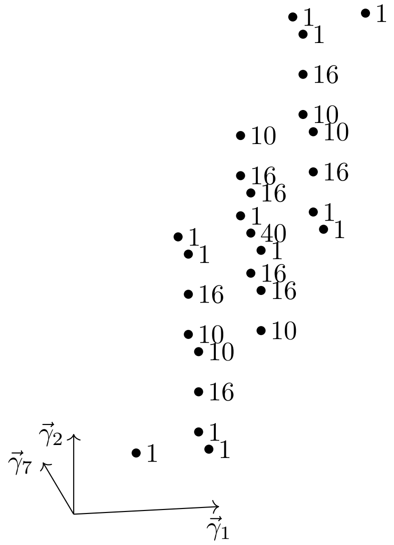


Figure 7.5: The 3D root system spectrum of the complement of D_5 of the weights of $\vec{\gamma}_1, \vec{\gamma}_2, \vec{\gamma}_7$.

made that in the projection there are no two different 3D spectrum points mapped to the same 2D spectrum point. An example can be seen in Figure 7.5[1]. In this first problem we are only concerned with the already existing quantities which cannot change. In the second problem we will focus on the case when two 3D spectrum points are mapped to the same 2D spectrum point by the projection. In this case we need to add the quantities, but since there are only a finite amount of 3D spectrum points, a finite amount of these type of projections exist.

We will first discuss the second problem. In this case we have that two points ($\vec{x}$ and $\vec{y}$) in 3D space are projected on the same point in 2D. In this case the vector $\vec{x} - \vec{y}$ is the vector perpendicular to the 2D surface on which the points are projected. This means that the projection surface is the span of the vectors perpendicular to $\vec{x} - \vec{y}$. These can be calculated by the creation of the matrix $N = (\vec{x} - \vec{y})^T$ and by then determining the null space of N (here we regard N as a 1×3 matrix). The result of this calculation results in two vectors which are perpendicular to $\vec{x} - \vec{y}$.

$$N = (\vec{x} - \vec{y})^T \qquad M = null(N) = [\vec{v}_1 \vec{v}_2] \tag{7.3}$$

There are $N = 27$ points in the 3D space there are a total of $N(N-1) = 702$ possible vectors of this type, so a maximum of 702 possible projections for which two points are mapped to the same point. Numerically these projections have been determined to check whether they can contain the spectrum of g_3 and g_8. The possible projections to be checked can be reduced by noticing in Figure 7.2 that there must be at least 7 points with 12 or more counts and 13 points with 2 or more counts. This reduces the to be checked projections to 19. These were checked and none of them are possible candidates.

This leaves us with the first smaller problem. In this option the 8 points of 16 counts must make up the 6 points of 12 counts in the g_3 and g_8 spectrum. We will create a map that maps three of these points to the upper right three points of 12 in the spectrum of g_3 and g_8 (Figure 7.2). For this map

[1]A projection is necessary in order to print a 3D cloud of points on a 2D sheet of paper.

we have that:

$$M\vec{v}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad M\vec{v}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} \qquad M\vec{v}_3 = \begin{pmatrix}1\\1\\0\end{pmatrix}$$

We can combine this to:

$$M\begin{pmatrix}| & | & |\\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3\\ | & | & |\end{pmatrix} = \begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ 0 & 0 & 0\end{pmatrix} \quad \text{or} \quad MV = B \tag{7.4}$$

This means that the matrix we are interested in is given by $M = BV^{-1}$, if the inverse exists. If it does not exists, it means that the vectors $\vec{v}_i$ are linear dependent. Of the 8 3D points with a count of 16 there are actually 4 distinct points as the other 4 are the centre symmetric counter parts of the 4 distinct points. We simply need to check that a combination of 3 of these 4 distinct points cannot be linear dependent. This is not the case. This means that when a combination of the 8 points (so the 4 distinct points and there 4 centre symmetric points) is linear dependent we must have a distinct point and its symmetric counter part. This would mean that the map we are looking for does not exist since this map would map the two centre symmetric points to two non centre symmetric points, and would thus not be a linear map. We can simply ignore the cases when the inverse of V does not exist.

In the case that the matrix is invertible we have that $M = BV^{-1}$ maps the three points in the 3D spectrum to the correct points in the g_3 and g_8 spectrum. We now need to apply this map to all the points of the 3D spectrum to see if the g_3 and g_8 spectrum fits. In total there are $8 \cdot 7 \cdot 6 = 336$ different vector sets. We have checked all 336 different maps and none were found to give a spectrum so that the charge spectrum would fit in the root system spectrum. We thus conclude that the elementary particles do not fit in the root system E_8.

To conclude, there do not exists 4 weight vectors so that their spectrum for 120 roots of E_8 is equal to the charge spectrum of the elementary particles.

Chapter 8

Conclusion

To conclude it has been shown that there exists **no** map from the elementary particles to the root system E_8 such that there exist 4 charge vectors for the charges: weak isospin, weak hypercharge, and the two color charges g_3 and g_8. These charge vectors would have had the property that for a root identified with an elementary particle via the map, the inner product of this root and the charge vector would result in the charge of this elementary particle.

The proof relies mostly on Corollary 5.4. This corollary states that the intersection of E_8 and the space orthogonal to a linear space W must again be a root system Φ. This corollary was applied to the linear space W of the still to be determined charge vectors of g_3 and g_8. Since there are 36 particles which are neutral in both g_3 and g_8, the root system of Φ has to contain at least 36 roots if the map and the four charge vectors exists.

Since E_8 is 8 dimensional, the root system Φ can at most be 6 dimensional (since it is orthogonal to two (charge) vectors). Together with the condition that the root system Φ must have at least 36 roots, the possible root systems which are candidate root systems for Φ are the following 5: $E_6, D_6, A_6, D_5A_1, D_5$.

These 5 candidate root systems for Φ in turn limited the possible candidate vectors for the charge vectors g_3 and g_8. These limitations have been found by checking all the possible sub root systems of dimension 5 and 6 with a computer script. From this it was possible to determine all the vectors which could be candidates for g_3 and g_8. To find these candidate vectors, a method to check if two sub root systems are related by an element of the Weyl group (thus in the same orbit) has been created and was run on all the candidate root system found in E_8. From this it was determined that for each of the candidate root systems for Φ, there was only one distinct way to create Φ as the orthogonal complement of two or three vectors. In this case it must be that the charge vectors g_3 and g_8 are a linear combination of these 2 or 3 vectors. All the linear combinations of these vectors are thus candidate vectors for g_3 and g_8.

All the candidate vectors for g_3 and g_8 have been checked from which it was concluded that none of the candidate vectors are possible. We checked the candidate vectors if there are enough roots for a given charge outcome of g_3 and g_8 to facilitate the number of elementary particles with these charges. For all of the candidate vectors it was found that there was always a charge outcome of g_3 and g_8 where there were to few roots with the same charge outcome. Or in other words there were always more elementary particles than candidate roots for a given charge combination of g_3 and g_8. From this it was concluded that all the candidate vectors cannot be the charge vectors of g_3 and g_8. Since these were all the candidates it was concluded that it is not possible to map the elementary particles to E_8 together with four charge vectors for the charges weak isospin, weak hypercharge, g_3 and g_8.

This result implies that the root system E_8 is not the structure that connects all the elementary particles together. However we have made the assertion that all the elementary particles have a nice structure, like e.g. E_8. It might however be the case that we have to distinguish between fermions and bosons to be able to find a map and charge vectors in either E_8 or smaller root systems not in E_8.

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Chapter 9

Appendix

A_1

Name	Counts	Orbits
A_1	1	1

A_2

Name	Counts	Orbits
A_1	3	1
A_2	1	1

A_3

Name	Counts	Orbits
A_1	6	1
A_2	4	1
A_1^2	3	1
A_3	1	1

A_4

Name	Counts	Orbits
A_1	10	1
A_2	10	1
A_1^2	15	1
A_3	5	1
A_2A_1	10	1
A_4	1	1

A_5

Name	Counts	Orbits
A_1	15	1
A_2	20	1
A_1^2	45	1
A_3	15	1
A_2A_1	60	1
A_1^3	15	1
A_4	6	1
A_3A_1	15	1
A_2^2	10	1
A_5	1	1

A_6

Name	Counts	Orbits
A_1	21	1
A_2	35	1
A_1^2	105	1
A_3	35	1
A_2A_1	210	1
A_1^3	105	1
A_4	21	1
A_3A_1	105	1
A_2^2	70	1
$A_2A_1^2$	105	1
A_5	7	1
A_4A_1	21	1
A_3A_2	35	1
A_6	1	1

A_7

Name	Counts	Orbits
A_1	28	1
A_2	56	1
A_1^2	210	1
A_3	70	1
A_2A_1	560	1
A_1^3	420	1
A_4	56	1

Name	Counts	Orbit
A_3A_1	420	1
A_2^2	280	1
$A_2A_1^2$	840	1
A_1^4	105	1
A_5	28	1
A_4A_1	168	1
A_3A_2	280	1

Name	Counts	Orbit
$A_3A_1^2$	210	1
$A_2^2A_1$	280	1
A_6	8	1
A_5A_1	28	1
A_4A_2	56	1
A_3^2	35	1
A_7	1	1

A_8

Name	Counts	Orbits
A_1	36	1
A_2	84	1
A_1^2	378	1
A_3	126	1
A_2A_1	1260	1
A_1^3	1260	1
A_4	126	1
A_3A_1	1260	1
A_2^2	840	1
$A_2A_1^2$	3780	1

Name	Counts	Orbit
A_1^4	945	1
A_5	84	1
A_4A_1	756	1
A_3A_2	1260	1
$A_3A_1^2$	1890	1
$A_2^2A_1$	2520	1
$A_2A_1^3$	1260	1
A_6	36	1
A_5A_1	252	1
A_4A_2	504	1

Name	Counts	Orbit
$A_4A_1^2$	378	1
A_3^2	315	1
$A_3A_2A_1$	1260	1
A_2^3	280	1
A_7	9	1
A_6A_1	36	1
A_5A_2	84	1
A_4A_3	126	1
A_8	1	1

D_4

Name	Counts	Orbits
A_1	12	1
A_2	16	1
A_1^2	18	1
A_3	12	3
A_1^3	12	1
D_4	1	1
A_1^4	3	1

D_5

Name	Counts	Orbits
A_1	20	1
A_2	40	1
A_1^2	70	2
A_3	50	2
A_2A_1	80	1
A_1^3	60	1
D_4	5	1
A_4	16	1
A_3A_1	20	1
$A_2A_1^2$	40	1
A_1^4	15	1
D_5	1	1
$A_3A_1^2$	10	1

D_6

Name	Counts	Orbits
A_1	30	1
A_2	80	1
A_1^2	195	2
A_3	140	2
A_2A_1	480	1
A_1^3	300	3
D_4	15	1
A_4	96	1
A_3A_1	360	3
A_2^2	160	1
$A_2A_1^2$	240	1

Name	Counts	Orbit
A_1^4	225	2
D_5	6	1
D_4A_1	30	1
A_5	32	2
A_3A_2	80	1
$A_3A_1^2$	180	3
A_1^5	90	1
D_6	1	1
$D_4A_1^2$	15	1
A_3^2	10	1
A_1^6	15	1

D_7

Name	Counts	Orbits
A_1	42	1
A_2	140	1
A_1^2	441	2
A_3	315	2
A_2A_1	1680	1
A_1^3	1260	2
D_4	35	1
A_4	336	1
A_3A_1	2100	2
A_2^2	1120	1
$A_2A_1^2$	2520	2
A_1^4	1365	2

Name	Counts	Orbit
D_5	21	1
D_4A_1	210	1
A_5	224	1
A_4A_1	672	1
A_3A_2	1680	2
$A_3A_1^2$	1470	3
$A_2A_1^3$	1680	3
A_1^5	630	1
D_6	7	1
$D_5A_1^2$	21	1
D_4A_2	140	1
$D_4A_1^2$	105	1

Name	Counts	Orbit
A_6	64	1
A_5A_1	42	1
$A_4A_1^2$	336	1
A_3^2	350	2
$A_3A_1^3$	420	3
$A_2A_1^4$	420	2
A_1^6	105	1
D_7	1	1
D_4A_3	35	1
$A_3A_1^4$	105	2

D_8

Name	Counts	Orbits
A_1	56	1
A_2	224	1
A_1^2	868	2
A_3	616	2
A_2A_1	4480	1
A_1^3	4200	2
D_4	70	1
A_4	896	1
A_3A_1	7840	2
A_2^2	4480	1
$A_2A_1^2$	15680	2
A_1^4	6930	4
D_5	56	1
D_4A_1	840	1
A_5	896	1
A_4A_1	5376	1
A_3A_2	11200	2
$A_3A_1^2$	14000	5
$A_2^2A_1$	8960	1

Name	Counts	Orbit
$A_2A_1^3$	13440	3
A_1^5	5880	2
D_6	28	1
D_5A_1	336	1
D_4A_2	1120	1
$D_4A_1^2$	2688	2
A_6	512	1
A_5A_1	1792	2
A_4A_2	3584	1
$A_4A_1^2$	2688	1
A_3^2	4760	4
$A_3A_2A_1$	4480	1
$A_3A_1^3$	10080	9
$A_2^2A_1^2$	4480	1
$A_2A_1^4$	3360	2
A_1^6	2940	2
D_7	8	1
D_6A_1	56	1
D_5A_2	224	1

Name	Counts	Orbit
$D_5A_1^2$	168	1
D_4A_3	840	2
$D_4A_1^3$	840	3
A_7	128	2
$A_5A_1^2$	896	2
A_4A_3	896	1
$A_3^2A_1$	560	1
$A_3A_2A_1^2$	2240	1
$A_3A_1^4$	2520	6
A_1^7	840	4
D_8	1	1
$D_6A_1^2$	28	1
D_5A_3	56	1
D_4^2	35	1
$D_4A_1^4$	210	1
$A_3^2A_1^2$	280	1
A_1^8	105	1

E_6

Name	Counts	Orbit
A_1	36	1
A_2	120	1
A_1^2	270	1
A_3	270	1
A_2A_1	720	1
A_1^3	540	1
D_4	45	1

Name	Counts	Orbit
A_4	216	1
A_3A_1	540	1
A_2^2	120	1
$A_2A_1^2$	1080	1
A_1^4	135	1
D_5	27	1
A_5	36	1

Name	Counts	Orbit
A_4A_1	216	1
$A_3A_1^2$	270	1
$A_2^2A_1$	360	1
E_6	1	1
A_5A_1	36	1
A_2^3	40	1

E_7

Name	Counts	Orbit
A_1	63	1
A_2	336	1
A_1^2	945	1
A_3	1260	1
A_2A_1	4095	2
A_1^3	4095	1
D_4	315	1
A_4	2016	1
A_3A_1	8820	2
A_2^2	3360	1
$A_2A_1^2$	15120	1
A_1^4	4725	2
D_5	378	1

Name	Counts	Orbit
D_4A_1	945	1
A_5	1344	2
A_4A_1	6048	1
A_3A_2	5040	1
$A_3A_1^2$	11340	3
$A_2^2A_1$	10080	1
$A_2A_1^3$	5040	1
A_1^5	2835	1
E_6	28	1
D_6	63	1
D_5A_1	378	1
$D_4A_1^2$	945	1
A_6	288	1
A_5A_1	2016	2

Name	Counts	Orbit
A_4A_2	2016	1
A_3^2	630	1
$A_3A_2A_1$	5040	1
$A_3A_1^3$	3780	3
A_2^3	1120	1
A_1^6	945	1
E_7	1	1
D_6A_1	63	1
$D_4A_1^3$	315	1
A_7	36	1
A_5A_2	336	1
$A_3^2A_1$	630	1
A_1^7	135	1

E_8

Name	Counts	Orbit
A_1	120	1
A_2	1120	1
A_1^2	3780	1
A_3	7560	1
A_2A_1	40320	1
A_1^3	37800	1
D_4	3150	1
A_4	24192	1
A_3A_1	151200	1
A_2^2	67200	1
$A_2A_1^2$	302400	1
A_1^4	122850	2
D_5	7560	1

Name	Counts	Orbit
D_4A_1	37800	1
A_5	40320	1
A_4A_1	241920	1
A_3A_2	302400	1
$A_3A_1^2$	529200	2
$A_2^2A_1$	403200	1
$A_2A_1^3$	604800	1
A_1^5	113400	1
E_6	1120	1
D_6	3780	1
D_5A_1	45360	1
D_4A_2	50400	1
$D_4A_1^2$	56700	1

Name	Counts	Orbit
A_6	34560	1
A_5A_1	161280	1
A_4A_2	241920	1
$A_4A_1^2$	362880	1
A_3^2	189000	2
$A_3A_2A_1$	604800	1
$A_3A_1^3$	453600	3
A_2^3	44800	1
$A_2^2A_1^2$	604800	1
$A_2A_1^4$	151200	1
A_1^6	56700	1
$Dim > 6$	?	?
?	?	?