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Successive approximations and interval halving for fractional BVPs with integral boundary conditions

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ABSTRACT

We study a system of non-linear fractional differential equations, subject to integral boundary conditions. We use a parametrization technique and a dichotomy-type approach to reduce the original problem to two "model-type" fractional boundary value problems with linear two-point boundary conditions. A numerical-analytic technique is applied to analytically construct approximate solutions to the "model-type" problems. The behaviour of these approximate solutions is governed by a set of parameters, whose values are obtained by numerically solving a system of algebraic equations. The obtained results are confirmed by an example of the fractional order problem that in the case of the second order differential equation models the Antarctic Circumpolar Current.

1. Introduction

The study of fractional boundary value problems (FBVPs) attracts a lot of attention in recent years due to their wide range of applications in mathematics and the natural sciences. In particular, these problems are able to capture the non-local nature of the physical processes, describe memory effects and abnormal diffusion, and give an additional degree of freedom to the model, which is expressed in terms of the order of the fractional derivative. This makes the FBVPs an active area of research for scientists who aim to create realistic models of complex real-world phenomena. For the recent results in the field of applied fractional dynamical systems we refer the reader to [1–8].

Besides a traditional description of the physically relevant constraints, written as periodic, anti-periodic, Dirichet and Neumann boundary conditions, a particular attention of modellers and pure mathematicians is paid to the integral restrictions. As it was shown in [9–11], phenomena, such as heat conduction, fluid flow and viscoelasticity, can be reduced to the study of such non-local problems. Here by the integral boundary conditions one understands restrictions on a physical process (e.g., a speed element of the fluid flow) over the whole interval of consideration, instead of looking only at the localized values. Most results in this direction disclose the qualitative analysis of the integral FBVPs and are based on the fractional Green's function and/or topological degree theory (see results in [12–19]). However, in the physical setting one is especially interested in the visualization of solutions that gives a better understanding of their behaviour.

Since most real-world phenomena are described by non-linear FDEs, the exact solutions to which are unavailable, this prompts the development of approximation techniques. An extensive literature analysis shows, that there are indeed well-established methods, applicable to the fractional setting. Among them are the series expansion and Grünwald–Lentikov methods, direct and indirect techniques (based on the Adams-type approximations and the quadrature-based methods),

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etc.; see [8,20]. However, all of them require a pre-knowledge of the initial values of solution to the studied FBVP, which is not available in our setting.

This triggers the further exploration of the numerical-analytic technique, that was originally suggested for the study of periodic BVPs for ordinary differential equations [21], and since then has been successfully applied to the nonlinear fractional case [22–27]. Its advantage, in comparison to the aforementioned solvers, is in its ability to incorporate complex boundary constraints via an appropriate parametrization. As a result, we derive the closed-form approximate solutions, governed by a set of parameters, that are computed numerically.

In the present work, we extend this technique to the novel investigation of the existence and construction of solutions to a system of FDEs of the Caputo type, subject to integral boundary conditions. The original FBVP is reduced to two "model-type" ones with two-point linear boundary constraints by adapting a parametrization technique used for the reduction of non-linearities in boundary conditions [28–30], and a dichotomy-type approach, based on the methodology described in [31–34]. A sequence of approximate solutions to each of the "model-type" FBVPs, which depends on vector-parameters, is constructed in analytic form. We prove the uniform convergence of the sequence to a limit function and show its connection to the original FBVP. The values of the unknown parameters are obtained by numerically solving a system of the so-called approximate determining equations at each iteration of the sequence. The obtained results are applied to an example of the gyre equation for the Antarctic Circumpolar Current, considered in the fractional setting, and subject to the integral boundary conditions (see [35–39]).

The novel technique presented in this paper has never been applied to the study of FBVPs with integral boundary conditions. It allows us to improve the convergence of the numerical-analytic technique and to sharpen the error estimates obtained in [23–27]. Additionally, the dichotomy-type approach enables application of the aforementioned method to a broader class of FDEs, in particular to those, where the right hand-side does not satisfy the Lipschitz condition on the original domain. As we will show later, this condition is essential in the application of the studied method. Together with other approximation techniques, used for solving systems of FDEs under different boundary restrictions, the approach presented here complements the fundamental study of non-linear FBVPs.

The present paper consists of 6 sections. In Section 2 we give the most general form of the problem under consideration and describe the boundary conditions parametrization and interval halving techniques. Section 3 consists of our main result on the constructive approximations and their convergence, and in Section 4 we show the relation between the original BVP and the reduced model-type problems. In Section 5 the method is applied to the Antarctic Circumpolar Current equation in the fractional setting. Section 6 presents a summary of our conclusions.

2. Problem setting and the decomposition technique

Throughout this paper we will use the following definitions of the Caputo fractional and integral operators.

Definition 1. Let $n-1 < p < n$ for some $n \in \mathbb{Z}_+$ and $f(t) : (0,\infty) \to \mathbb{R}$. Then the Caputo fractional derivative of $f(t)$ of order p is given by

$$ {}_a^C D_t^p f(t) := \frac{1}{\Gamma(n-p)} \int_a^t (t-s)^{n-p-1} f^{(n)}(s) ds. \tag{1} $$

When $p = n$, (2) reduces to the ordinary derivative of order n (see [20], Def. 2.138).

The definition of the Riemann–Liouville fractional integral is given by:

Definition 2. Let $n-1 < p < n$ for some $n \in \mathbb{Z}_+$. Then the Riemann–Liouville fractional integral of order p is given by (see [20], Def. 2.88)

$$ {}_a I_t^p f(t) := \frac{1}{\Gamma(p)} \int_a^t (t-s)^{p-1} f(s) ds. \tag{2} $$

2.1. Problem setting

We consider a system of Caputo FDEs

$$ \begin{array}{ll} {}_a^C D_t^p u_1(t) = f_1(t, u(t)), \quad u_1,\ f_1 \in \mathbb{R}^n, & \\ {}_a^C D_t^q u_2(t) = f_2(t, u(t)), \quad u_2,\ f_2 \in \mathbb{R}^m, & \end{array} \quad t \in [a,b], \tag{3} $$

for some $p, q \in (0, 1]$, subject to the integral boundary conditions

$$ Au(a) + \int_a^b P(s)u(s)ds + Cu(b) = d, \tag{4} $$

where ${}_a^C D_t^p$ denotes the Caputo fractional derivative with the lower limit at a, and $u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \in \mathbb{R}^N$, where $N = n+m$. Further we make the following assumptions:

- the unknown functions $u_1 : [a, b] \to D_1 \subset \mathbb{R}^n$ and $u_2 : [a, b] \to D_2 \subset \mathbb{R}^m$ in (3) are continuous, where D_1 and D_2 are closed and bounded subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively;
- functions $f_1 : G \to \mathbb{R}^n$ and $f_2 : G \to \mathbb{R}^m$ in the right hand-sides of (3) are, generally speaking, non-linear, with $G := [a, b] \times D_1 \times D_2$;
- matrices $A, C \in L(\mathbb{R}^N)$ in (4) are such that A is arbitrary and C is a singular matrix of the form

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & \mathbf{0}_{N-p} \end{pmatrix},$$

where C_{11} is a non-singular $p \times p$ matrix, C_{12} is a $p \times (N-p)$ matrix, C_{21} is a $(N-p) \times p$ matrix and $\mathbf{0}_{N-p}$ denotes the $(N-p) \times (N-p)$ matrix of zeros. Note, that any matrix, containing the appropriate number of zeros, can be reduced to the given block form using row operations;

and lastly,

- $d \in \mathbb{R}^N$ is a given vector and $P(\cdot)$ is a given continuous $N \times N$-dimensional matrix function.

We aim to find a continuous solution $u(t)$ of the FDS (3) that satisfies integral boundary conditions (4) in the domain $D = D_1 \times D_2$. One of the most efficient ways to deal with this task is to write (3), (4) in an equivalent integral form. In order to incorporate the integral boundary conditions (4) we first need to simplify the original FBVP to one with linear boundary constraints. This is done by an appropriate parametrization technique which is presented below.

2.2. Parametrization of the integral boundary conditions

To replace (4) by linear two-point boundary conditions, we apply a "freezing" technique, similar to [28–30]. For this we introduce the following vector-parameters

$$\begin{aligned} z &= col(z_1, z_2, \ldots, z_N), \\ \lambda &= col(\lambda_1, \lambda_2, \ldots, \lambda_N), \\ \eta &= col(\underbrace{0, 0, \ldots, 0}_{p}, \eta_{p+1}, \eta_{p+2}, \ldots, \eta_N) \end{aligned}$$

by putting

$$\begin{aligned} z &:= u(a), \\ \lambda &:= \int_a^b P(s)u(s)ds, \\ \eta_i &:= u_i(b), \quad i = p+1, p+2, \ldots, N, \end{aligned} \tag{5}$$

where, as mentioned before, $u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$. Under the parametrization (5) the integral boundary conditions (4) are re-written as

$$Au(a) + C_1 u(b) = d(\eta, \lambda), \tag{6}$$

where

$$d(\eta, \lambda) = d - \lambda + \eta,$$

$$C_1 = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & \mathbf{1}_{N-p} \end{pmatrix}, \quad \det C_1 \neq 0$$

and $\mathbf{1}_{N-p}$ denotes the $(N-p) \times (N-p)$ unit matrix.

After applying the above-described parametrization, we are going to study the family of parametrized FBVPs with linear two-point boundary conditions (3), (6), instead of the original FBVP with integral boundary conditions (3), (4). To return back to the original BVP, the values of the parameters are chosen appropriately.

Remark 1. Note, that parametrization (5) allows us not only to reduce the integral boundary conditions (4) to the two-point linear ones, but also to eliminate the singularity of the matrix C. As it will be seen in Section 3, this step is essential for constructing our iterative sequence.

2.3. Interval halving

As it will be seen in Theorem 1, one of the crucial conditions for functions f_1 and f_2 in (3) to satisfy is the Lipschitz condition. If it fails to hold in the domain of consideration, then one cannot guarantee the uniform convergence of the successive approximations technique we are talking about in this paper.

But we can overcome this difficulty by splitting the original interval $[a, b]$ (and thus the given problem (3), (6)) in such a way, that the Lipschitz condition holds on these subintervals and the convergence is guaranteed. On the other hand, even if the Lipschitz condition was not violated for the original FDS (3), the problem splitting is still beneficial since it improves the speed of convergence of the method.

Let us split the parametrized BVP (3), (6) onto two "model-type" problems, similarly to [31–34], which read

$$\begin{aligned} &{}^{C}_{a}D^{p}_{t}x_1(t) = f_1(t, x(t)), \quad f_1 \in \mathbb{R}^n, \\ &{}^{C}_{a}D^{q}_{t}x_2(t) = f_2(t, x(t)), \quad f_2 \in \mathbb{R}^m, \end{aligned} \quad t \in [a, c], \quad p, q \in [0, 1]$$
$$x_1(a) = z_1, \quad x_1(c) = \alpha_1,$$
$$x_2(a) = z_2, \quad x_2(c) = \alpha_2, \tag{7}$$

$$\begin{aligned} &{}^{C}_{c}D^{p}_{t}y_1(t) = g_1(t, x(t), y(t)), \quad g_1 \in \mathbb{R}^n, \\ &{}^{C}_{c}D^{q}_{t}y_2(t) = g_2(t, x(t), y(t)), \quad g_2 \in \mathbb{R}^m, \end{aligned} \quad t \in [c, b], \quad p, q \in [0, 1]$$
$$y_1(c) = \alpha_1, \quad y_1(b) = C_1^{-1}[d(\eta, \lambda) - Az]_1,$$
$$y_2(c) = \alpha_2, \quad y_2(b) = C_1^{-1}[d(\eta, \lambda) - Az]_2, \tag{8}$$

where $x(\cdot) := \begin{bmatrix} x_1(\cdot) \\ x_2(\cdot) \end{bmatrix}$, $y(\cdot) := \begin{bmatrix} y_1(\cdot) \\ y_2(\cdot) \end{bmatrix}$, $c := \frac{b-a}{2}$ denotes the mid-point of the interval $[a, b]$, and

$$g_1(t, x(t), y(t)) := f_1(t, y(t)) - \frac{1}{\Gamma(n-p)} \int_a^c (t-s)^{[p]-p-1} f_1^{([p])}(s, x(s))ds, \tag{9a}$$

$$g_2(t, x(t), y(t)) := f_2(t, y(t)) - \frac{1}{\Gamma(n-q)} \int_a^c (t-s)^{[q]-q-1} f_2^{([q])}(s, x(s))ds. \tag{9b}$$

Functions $x_1(t) : [a, c] \to D_1^x \subset \mathbb{R}^n$, $x_2(t) : [a, c] \to D_2^x \subset \mathbb{R}^m$, $y_1(t) : [c, b] \to D_1^y \subset \mathbb{R}^n$, $y_2(t) : [c, b] \to D_2^y \subset \mathbb{R}^m$ are continuous on their respective domains. Moreover, the domains D_i^x, D_i^y are such that $D_i^x \cup D_i^y = D_i$, $D_i^x \cap D_i^y = \emptyset$ $(i \in \{1, 2\})$.

The parameter λ in the boundary conditions of (8) is written in terms of new functions as

$$\lambda = \int_a^c P(s)x(s)ds + \int_c^b P(s)y(s)ds.$$

Remark 2. Note, that in the original system (3) we considered the Caputo derivatives ${}^{C}_{a}D^{p}_{t}$ with the lower limit at a, thus at the left end of the interval $[a, b]$, where the independent variable t was defined. After the interval splitting the Caputo derivatives in (8) are already taken with the lower limit at the middle point c. Due to the non-local nature of the Caputo fractional derivative, the right-hand side functions in the system (8), defined on the second half of the interval, need to be appropriately adjusted using the definition of the Caputo derivative [20], as it was done in (9a), (9b).

Remark 3. Another important remark is that in the boundary conditions of (7), (8) we have introduced an additional parameter $\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$, which denotes the solution value at the mid-point of the interval $[a, b]$. In order for the solution to be continuous on the entire interval $[a, b]$ we require that

$$x(c) = y(c) = \alpha.$$

3. Successive approximations on the half intervals and their convergence

In this section we present the approximating sequences and prove their uniform convergence to the exact solutions of the corresponding Cauchy problems. The equivalence of the Cauchy problems and the original BVPs is shown in Section 4.

3.1. Construction of the successive approximations

Let us consider each of the FBVPs (7) and (8) separately.

Assume that the BVP (7) satisfies the following conditions:

1(a) Functions f_1 and f_2 are bounded, i.e. they satisfy the inequalities:

$$|f_1(t, x(t))| \le M_1^x, \qquad |f_2(t, x(t))| \le M_2^x, \tag{10}$$

for all $t \in [a, c]$, $x_i \in D_i^x$ $(i \in \{1, 2\})$ and some non-negative constant vectors $M_1^x \in \mathbb{R}^n$, $M_2^x \in \mathbb{R}^m$.

2(a) Functions f_1 and f_2 satisfy the Lipschitz conditions

$$\begin{aligned} &|f_1(t,x_1^1,x_2^1)-f_2(t,x_1^2,x_2^2)| \le K_{11}|x_1^1-x_1^2| + K_{12}|x_2^1-x_2^2|,\\ &|f_2(t,x_1^1,x_2^1)-f_2(t,x_1^2,x_2^2)| \le K_{21}|x_1^1-x_1^2| + K_{22}|x_2^1-x_2^2|, \end{aligned} \tag{11}$$

for all $t \in [a,c]$, $x_i^1, x_i^2 \in D_i^x$ $(i \in \{1,2\})$ and some non-negative constant matrices $K_{lj},\ l,j \in \{1,2\}$.

3(a) The sets

$$\begin{aligned} &D_{\beta_1^x} := \{z_1 \in D_1^x : B(z_1 + 2^p(t-a)^p(c-a)^{-p}(\alpha_1 - z_1), \beta_1^x) \subset D_1^x \ \ \forall (t,\alpha_1) \in \Omega_1^x\}\\ &D_{\beta_2^x} := \{z_2 \in D_2^x : B(z_2 + 2^q(t-a)^q(c-a)^{-q}(\alpha_2 - z_2), \beta_2^x) \subset D_2^x \ \ \forall (t,\alpha_2) \in \Omega_2^x\} \end{aligned} \tag{12}$$

are non-empty, where

$$\beta_1^x = \frac{(c-a)^p M_1^x}{2^{2p-1}\Gamma(p+1)}, \quad \beta_2^x = \frac{(c-a)^q M_2^x}{2^{2q-1}\Gamma(q+1)}, \tag{13}$$

$$\Omega_1^x := [a,c] \times D_{\beta_1^y}, \quad \Omega_2^x := [a,c] \times D_{\beta_2^y}, \tag{14}$$

and the sets $D_{\beta_1^y}$ and $D_{\beta_2^y}$ are defined in (21). This means that there exist non-empty sets of initial conditions, for which the solutions remain within their corresponding domains.

4(a) The spectral radius of the matrix

$${}^xQ := K\ {}^x\Gamma_{pq} \tag{15}$$

satisfies the inequality

$$r({}^xQ) < 1, \tag{16}$$

where

$${}^x\Gamma_{pq} := \max\left\{\frac{(c-a)^p}{2^{2p-1}\Gamma(p+1)}, \frac{(c-a)^q}{2^{2q-1}\Gamma(q+1)}\right\} \tag{17}$$

and

$$K = \begin{pmatrix} K_{11} & K_{12}\\ K_{21} & K_{22} \end{pmatrix}. \tag{18}$$

Similar conditions are assumed to hold in the case of the BVP (8):

1(b) Functions g_1 and g_2 are bounded, i.e. they satisfy the inequalities:

$$|g_1(t,x(t),y(t))| \le M_1^y, \quad |g_2(t,x(t),y(t))| \le M_2^y, \tag{19}$$

for all $t \in [c,b]$, $y_i \in D_i^y$ $(i \in \{1,2\})$ and some non-negative constant vectors $M_1^y \in \mathbb{R}^n$, $M_2^y \in \mathbb{R}^m$.

2(b) Functions g_1 and g_2 satisfy the Lipschitz conditions

$$\begin{aligned} &|g_1(t,y_1^1,y_2^1) - g_1(t,y_1^2,y_2^2)| \le J_{11}|y_1^1 - y_1^2| + J_{12}|y_2^1 - y_2^2|,\\ &|g_2(t,y_1^1,y_2^1) - g_2(t,y_1^2,y_2^2)| \le J_{21}|y_1^1 - y_1^2| + J_{22}|y_2^1 - y_2^2|, \end{aligned} \tag{20}$$

for all $t \in [c,b]$, $y_i^1, y_i^2 \in D_i^y$ $(i \in \{1,2\})$, and some non-negative constant matrices $J_{lj},\ l,j \in \{1,2\}$.

3(b) The sets

$$\begin{aligned} D_{\beta_1^y} := \Bigg\{ &\alpha_1 \in D_1^y : B\left(\alpha_1 + \left(\frac{t-c}{b-c}\right)^p \{[C_1^{-1}(d(\lambda,\eta) - Az)]_1 - \alpha_1\}, \beta_1^y\right) \subset D_1^y\\ &\forall (t,z_1,\lambda,\eta) \in \Omega_1^y \Bigg\},\\ D_{\beta_2^y} := \Bigg\{ &\alpha_2 \in D_2^y : B\left(\alpha_2 + \left(\frac{t-c}{b-c}\right)^q \{[C_1^{-1}(d(\lambda,\eta) - Az)]_2 - \alpha_2\}, \beta_2^y\right) \subset D_2^y\\ &\forall (t,z_2,\lambda,\eta) \in \Omega_2^y \Bigg\} \end{aligned} \tag{21}$$

are non-empty, where

$$\beta_1^y = \frac{(b-c)^p M_1^y}{2^{2p-1}\Gamma(p+1)}, \quad \beta_2^y = \frac{(b-c)^q M_2^y}{2^{2q-1}\Gamma(q+1)}, \tag{22}$$

$$\Omega_1^y := [c,b] \times D_{\beta_1^x} \times \mathcal{P} \times D_1 \times D_2, \quad \Omega_2^y := [c,b] \times D_{\beta_2^x} \times \mathcal{P} \times D_1 \times D_2, \tag{23}$$

$$\mathcal{P} := \left\{ \int_a^c P(s)x(s)ds + \int_c^b P(s)y(s)ds,\ \ x \in C([a,b], D^x), y \in C([a,b], D^y) \right\}, \tag{24}$$

with $D^x := D_1^x \times D_2^x$, $D^y := D_1^y \times D_2^y$, and the sets $D_{\beta_1^x}$ and $D_{\beta_2^x}$ being defined in (12).

4(b) The spectral radius of the matrix

$$^yQ := J\ ^y\Gamma_{pq} \tag{25}$$

satisfies the inequality

$$r(^yQ) < 1, \tag{26}$$

where

$$^y\Gamma_{pq} := \max\left\{ \frac{(b-c)^p}{2^{2p-1}\Gamma(p+1)}, \frac{(b-c)^q}{2^{2q-1}\Gamma(q+1)} \right\}, \tag{27}$$

$$J = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}. \tag{28}$$

Let us now connect with the FBVPs (7), (8) sequences of functions $\{x_m\}$, $\{y_m\}$, given by the iterative formulas:

$$\begin{gathered} x_{1,0}(t,z,\alpha) = z_1 + \left(\frac{t-a}{c-a}\right)^p (\alpha_1 - z_1), \\ x_{1,m}(t,z,\alpha) = x_{1,0}(t,z,\alpha) + \frac{1}{\Gamma(p)} \Big[\int_a^t (t-s)^{p-1} f_1(s, x_{m-1}(s,z,\alpha))ds \\ - \left(\frac{t-a}{c-a}\right)^p \int_a^c (c-s)^{p-1} f_1(s, x_{m-1}(s,z,\alpha))ds \Big], \end{gathered} \tag{29}$$

$$\begin{gathered} x_{2,0}(t,z,\alpha) = z_2 + \left(\frac{t-a}{c-a}\right)^q (\alpha_2 - z_2), \\ x_{2,m}(t,z,\alpha) = x_{2,0}(t,z,\alpha) + \frac{1}{\Gamma(q)} \Big[\int_a^t (t-s)^{q-1} f_2(s, x_{m-1}(s,z,\alpha))ds \\ - \left(\frac{t-a}{c-a}\right)^q \int_a^c (c-s)^{q-1} f_2(s, x_{m-1}(s,z,\alpha))ds \Big], \end{gathered} \tag{30}$$

for $m \in \mathbb{Z}^+$ and $t \in [a,c]$, and

$$\begin{gathered} y_{1,0}(t,z,\alpha,\lambda,\eta) = \alpha_1 + \left(\frac{t-c}{b-c}\right)^p \{[C_1^{-1}(d(\lambda,\eta) - Az)]_1 - \alpha_1\}, \\ y_{1,m}(t,z,\alpha,\lambda,\eta) = y_{1,0}(t,z,\alpha,\lambda,\eta) \\ + \frac{1}{\Gamma(p)} \Big[\int_c^t (t-s)^{p-1} g_1(s, x_{m-1}(s,z,\alpha,\lambda,\eta), y_{m-1}(s,z,\alpha,\lambda,\eta))ds \\ - \left(\frac{t-c}{b-c}\right)^p \int_c^b (b-s)^{p-1} g_1(s, x_{m-1}(s,z,\alpha,\lambda,\eta), y_{m-1}(s,z,\alpha,\lambda,\eta))ds \Big], \end{gathered} \tag{31}$$

$$\begin{gathered} y_{2,0}(t,z,\alpha,\lambda,\eta) = \alpha_2 + \left(\frac{t-c}{b-c}\right)^q \{[C_1^{-1}(d(\lambda,\eta) - Az)]_2 - \alpha_2\}, \\ y_{2,m}(t,z,\alpha,\lambda,\eta) = y_{2,0}(t,z,\alpha,\lambda,\eta) \\ + \frac{1}{\Gamma(q)} \Big[\int_c^t (t-s)^{q-1} g_2(s, x_{m-1}(s,z,\alpha,\lambda,\eta), y_{m-1}(s,z,\alpha,\lambda,\eta))ds \\ - \left(\frac{t-c}{b-c}\right)^q \int_c^b (b-s)^{q-1} g_2(s, x_{m-1}(s,z,\alpha,\lambda,\eta), y_{m-1}(s,z,\alpha,\lambda,\eta))ds \Big] \end{gathered} \tag{32}$$

for $m \in \mathbb{Z}^+$ and $t \in [c,b]$.

Note, that every function in the sequences (29)–(30) and (31)–(32) *is constructed to satisfy the parametrized boundary conditions of the corresponding problems* (7), (8).

Remark 4. It follows from the definitions (12) of $D_{\beta_1^x}$ and $D_{\beta_2^x}$ that the values of $x_{1,0}(t,z,\alpha)$, $x_{2,0}(t,z,\alpha)$ in (29), (30) do not escape D_1^x and D_2^x respectively, for any $z_1 \in D_{\beta_1^x}$, $z_2 \in D_{\beta_2^x}$, $\alpha_1 \in D_{\beta_1^y}$, $\alpha_2 \in D_{\beta_2^y}$. Similar conclusion holds for the values of $y_{1,0}(t,z,\alpha,\lambda,\eta)$, $y_{2,0}(t,z,\alpha,\lambda,\eta)$, defined in (31), (32), with respect to the sets $D_{\beta_1^y}$ and $D_{\beta_2^y}$ of the form (21), for any $\alpha_1 \in D_{\beta_1^y}$, $\alpha_2 \in D_{\beta_2^y}$, $z_1 \in D_{\beta_1^x}$, $z_2 \in D_{\beta_2^x}$, $\lambda \in \mathcal{P}$, $\eta \in D_1 \times D_2$.

Next we will prove, that under conditions **1(a)–4(a)** and **1(b)–4(b)** the sequences of functions (29)–(30) and (31)–(32) converge uniformly to the corresponding limit functions. But first we need the following lemmas.

3.2. Auxiliary statements

Lemma 1 ([29]). *Let $f(t)$ be a continuous function on $t \in [a, c]$. Then, for all $t \in [a, c]$, the following estimate is true*

$$\frac{1}{\Gamma(r)}\left| \int_a^t (t-s)^{r-1} f(s)ds - \left(\frac{t-a}{\mathcal{I}_x}\right)^r \int_a^{a+\mathcal{I}_x} (a+\mathcal{I}_x - s)^{r-1} f(s)ds \right| \\ \leq {}^x\alpha_1^r(t, a, \mathcal{I}) \max_{t\in[a,a+\mathcal{I}_x]} |f(s)|, \tag{33}$$

where $r \in \{p, q\}$, $\mathcal{I}_x := \frac{c-a}{2}$,

$${}^x\alpha_1^r(t, a, \mathcal{I}_x) := \frac{2(t-a)^r}{\Gamma(r+1)}\left(1 - \frac{t-a}{\mathcal{I}_x}\right)^r. \tag{34}$$

Lemma 2 ([29]). *Let $\{{}^x\alpha_m^r(\cdot, a, \mathcal{I}_x)\}_{m\in\mathbb{N}}$ be a sequence of continuous functions on the interval $[a, b]$ given by*

$${}^x\alpha_m^r(t, a, \mathcal{I}_x) := \frac{1}{\Gamma(r)}\left| \int_a^t \left((t-s)^{r-1} - \left(\frac{t-a}{\mathcal{I}_x}\right)^r (a+\mathcal{I}_x - s)^{r-1}\right) {}^x\alpha_{m-1}^r(s, a, \mathcal{I}_x)ds \right. \\ \left. - \left(\frac{t-a}{\mathcal{I}_x}\right)^r \int_t^{a+\mathcal{I}_x} (a+\mathcal{I}_x - s)^{r-1} {}^x\alpha_{m-1}^r(s, a, \mathcal{I}_x)ds \right|, m \in \mathbb{N}, \tag{35}$$

where $r \in \{p, q\}$, ${}^x\alpha_0^r(\cdot, a, \mathcal{I}_a) := 0$ and ${}^x\alpha_1^r(\cdot, a, \mathcal{I}_x)$ is defined in (34). Then the following estimate holds

$${}^x\alpha_m^r(t, a, \mathcal{I}_x) \leq \frac{\mathcal{I}_x^{(m-1)r}}{2^{(m-1)(2r-1)}[\Gamma(r+1)]^{m-1}} {}^x\alpha_1^r(t, a, \mathcal{I}_x) \\ \leq \frac{\mathcal{I}_x^{mr}}{2^{m(2r-1)}[\Gamma(r+1)]^m} \tag{36}$$

for all $m \in \mathbb{Z}$.

Analogous statements hold for $t \in [c, b]$, where $r \in \{p, q\}$, $\mathcal{I}_y := \frac{b-c}{2}$, and

$${}^y\alpha_1^r(t, c, \mathcal{I}_y) := 0, \\ {}^y\alpha_1^r(t, c, \mathcal{I}_y) := \frac{2(t-c)^r}{\Gamma(r+1)}\left(1 - \frac{t-c}{\mathcal{I}_y}\right)^r, \\ {}^y\alpha_m^r(t, c, \mathcal{I}_y) := \frac{1}{\Gamma(r)}\left| \int_c^t \left((t-s)^{r-1} - \left(\frac{t-c}{\mathcal{I}_y}\right)^r (c+\mathcal{I}_y - s)^{r-1}\right) {}^y\alpha_{m-1}^r(s, a, \mathcal{I}_y)ds \right. \\ \left. - \left(\frac{t-c}{\mathcal{I}_y}\right)^r \int_t^{c+\mathcal{I}_y} (c+\mathcal{I}_y - s)^{r-1} {}^y\alpha_{m-1}^r(s, c, \mathcal{I}_y)ds \right|, m \in \mathbb{N}. \tag{37}$$

For proofs of Lemmas 1 and 2 we refer to [29].

3.3. Convergence results

Let us consider the FBVP (7).

Theorem 1. *Assume that the BVP (7) satisfies conditions 1(a)–4(a). Then for all fixed $z_1 \in D_{\beta_1^x}$, $z_2 \in D_{\beta_2^x}$, $\alpha_1 \in D_{\beta_1^y}$, $\alpha_2 \in D_{\beta_2^y}$ it holds:*

1. *Functions of the sequences (29), (30) are continuous and satisfy the parametrized boundary conditions*

$$x_{1,m}(a, z, \alpha) = z_1, \quad x_{1,m}(c, z, \alpha) = \alpha_1, \tag{38}$$

$$x_{2,m}(a, z, \alpha) = z_2, \quad x_{2,m}(c, z, \alpha) = \alpha_2. \tag{39}$$

2. *The sequences of functions (29), (30) for $t \in [a, c]$ converge uniformly as $m \to \infty$ to the limit functions*

$$x_{1,\infty}(t, z, \alpha) = \lim_{m\to\infty} x_{1,m}(t, z, \alpha), \tag{40}$$

$$x_{2,\infty}(t, z, \alpha) = \lim_{m\to\infty} x_{2,m}(t, z, \alpha). \tag{41}$$

3. *The limit functions (40), (41) satisfy the parametrized boundary conditions*

$$x_{1,\infty}(a, z, \alpha) = z_1, \quad x_{1,\infty}(c, z, \alpha) = \alpha_1, \tag{42}$$

$$x_{2,\infty}(a,z,\alpha)=z_2, \quad x_{2,\infty}(c,z,\alpha)=\alpha_2. \tag{43}$$

4. The limit functions (40), (41) are the unique continuous solutions to the integral equations

$$x_1(t)=z_1+\left(\frac{t-a}{c-a}\right)^p(\alpha_1-z_1)+\frac{1}{\Gamma(p)}\Bigg[\int_a^t(t-s)^{p-1}f_1(s,x(s))ds \\ -\left(\frac{t-a}{c-a}\right)^p\int_a^c(c-s)^{p-1}f_1(s,x(s))ds\Bigg], \tag{44}$$

$$x_2(t)=z_2+\left(\frac{t-a}{c-a}\right)^q(\alpha_2-z_2)+\frac{1}{\Gamma(q)}\Bigg[\int_a^t(t-s)^{q-1}f_2(s,x(s))ds \\ -\left(\frac{t-a}{c-a}\right)^q\int_a^c(c-s)^{q-1}f_2(s,x(s))ds\Bigg], \tag{45}$$

or equivalently, they are the unique continuous solutions to the Cauchy problems

$${}_a^C D_t^p x_1(t)=f_1(t,x(t))+\Delta^{px}(z,\alpha), \quad x_1(a)=z_1, \tag{46}$$

$${}_a^C D_t^q x_2(t)=f_2(t,x(t))+\Delta^{qx}(z,\alpha), \quad x_2(a)=z_2, \tag{47}$$

where

$$\Delta^{px}(z,\alpha)=\frac{\Gamma(p+1)}{(c-a)^p}(\alpha_1-z_1)-\frac{p}{(c-a)^p}\int_a^c(c-s)^{p-1}f_1(s,x(s))ds, \tag{48}$$

$$\Delta^{qx}(z,\alpha)=\frac{\Gamma(q+1)}{(c-a)^q}(\alpha_2-z_2)-\frac{q}{(c-a)^q}\int_a^c(c-s)^{q-1}f_2(s,x(s))ds. \tag{49}$$

5. The following error estimate holds

$$\begin{pmatrix}|x_{1,\infty}(t,z,\alpha)-x_{1,m}(t,z,\alpha)|\\|x_{2,\infty}(t,z,\alpha)-x_{2,m}(t,z,\alpha)|\end{pmatrix}\le {}^x\Gamma_{pq}\,{}^xQ^m(I-{}^xQ)^{-1}\begin{pmatrix}M_1^x\\M_2^x\end{pmatrix}, \tag{50}$$

where $t\in[a,c]$, ${}^xQ$ is defined by (15), and I is a unit N-dimensional matrix.

Proof. The first statement follows directly from computations, since the sequences of functions (29), (30) are constructed in such a way that they satisfy the parametrized boundary conditions (38), (39).

Next, we show that $x_{1,m}(t,z,\alpha)\in D_1^x$, $x_{2,m}(t,z,\alpha)\in D_2^x$ for arbitrary $(t,z_1,\alpha_1)\in[a,c]\times D_{\beta_1^x}\times D_{\beta_1^y}$, $(t,z_2,\alpha_2)\in[a,c]\times D_{\beta_2^x}\times D_{\beta_2^y}$. By applying Lemma 1 to

$$|x_{1,m}(t,z,\alpha)-x_{1,0}(t,z,\alpha)|= \\ =\frac{1}{\Gamma(p)}\Bigg|\int_a^t\left[(t-s)^{p-1}-\left(\frac{t-a}{c-a}\right)^p(c-s)^{p-1}\right]f_1(s,x_{m-1}(s,z,\alpha))ds \\ -\left(\frac{t-a}{c-a}\right)^p\int_t^c(c-s)^{p-1}f_1(s,x_{m-1}(s,z,\alpha))ds\Bigg|$$

and

$$|x_{2,m}(t,z,\alpha)-x_{2,0}(t,z,\alpha)|= \\ =\frac{1}{\Gamma(q)}\Bigg|\int_a^t\left[(t-s)^{q-1}-\left(\frac{t-a}{c-a}\right)^q(c-s)^{q-1}\right]f_2(s,x_{m-1}(s,z,\alpha))ds \\ -\left(\frac{t-a}{c-a}\right)^q\int_t^c(c-s)^{q-1}f_2(s,x_{m-1}(s,z,\alpha))ds\Bigg|,$$

respectively, it follows that

$$|x_{1,m}(t,z,\alpha)-x_{1,0}(t,z,\alpha)|\le {}^x\alpha_1^p(t)\max_{a\le t\le c}|f_1(t,x_{m-1}(t,z,\alpha))| \\ ={}^x\alpha_1^p(t)M_1^x$$

$$|x_{2,m}(t,z,\alpha)-x_{2,0}(t,z,\alpha)|\le {}^x\alpha_1^q(t)\max_{a\le t\le c}|f_2(t,x_{m-1}(t,z,\alpha))| \\ ={}^x\alpha_1^q(t)M_2^x.$$

Applying Lemma 2 with $m = 1$ to the last two inequalities and using the definitions of β_1^x and β_2^x in (13) yields

$$|x_{1,m}(t,z,\alpha) - x_{1,0}(t,z,\alpha)| \le \beta_1^x$$
$$|x_{2,m}(t,z,\alpha) - x_{2,0}(t,z,\alpha)| \le \beta_2^x.$$

Thus, we have shown that $x_{1,m}(t,z,\alpha) \in D_1^x$, $x_{2,m}(t,z,\alpha) \in D_2^x$ for arbitrary $(t,z_1,\alpha_1) \in [a,c] \times D_{\beta_1^x} \times D_{\beta_1^y}$, $(t,z_2,\alpha_2) \in [a,c] \times D_{\beta_2^x} \times D_{\beta_2^y}$.

Next, we set

$$\mu_1^x(t) := \alpha_1^p(t), \quad \nu_1^x(t) := \alpha_1^q(t) \tag{51}$$

$$\begin{aligned}\mu_m^x(t) := \frac{1}{\Gamma(p)} \max \Bigg\{ &\int_a^t \Big[(t-s)^{p-1} - \Big(\frac{t-a}{c-a}\Big)^p (c-s)^{p-1}\Big]\mu_{m-1}^x(s)ds \\ &+ \Big(\frac{t-a}{c-a}\Big)^p \int_t^c (c-s)^{p-1}\mu_{m-1}^x(s)ds, \\ &\int_a^t \Big[(t-s)^{p-1} - \Big(\frac{t-a}{c-a}\Big)^p (c-s)^{p-1}\Big]\nu_{m-1}^x(s)ds \\ &+ \Big(\frac{t-a}{c-a}\Big)^p \int_t^c (c-s)^{p-1}\nu_{m-1}^x(s)ds \Bigg\},\end{aligned} \tag{52}$$

$$\begin{aligned}\nu_m^x(t) := \frac{1}{\Gamma(q)} \max \Bigg\{ &\int_a^t \Big[(t-s)^{q-1} - \Big(\frac{t-a}{c-a}\Big)^q (c-s)^{q-1}\Big]\nu_{m-1}^x(s)ds \\ &+ \Big(\frac{t-a}{c-a}\Big)^q \int_t^c (c-s)^{q-1}\nu_{m-1}^x(s)ds, \\ &\int_a^t \Big[(t-s)^{q-1} - \Big(\frac{t-a}{c-a}\Big)^q (c-s)^{q-1}\Big]\mu_{m-1}^x(s)ds \\ &+ \Big(\frac{t-a}{c-a}\Big)^q \int_t^c (c-s)^{q-1}\mu_{m-1}^x(s)ds \Bigg\},\end{aligned} \tag{53}$$

$\forall m \in \mathbb{Z}^+$, and use induction to show that

$$\begin{aligned}&|x_{1,m}(t,z,\alpha) - x_{1,m-1}(t,z,\alpha)| \le (M_1^x)^m \mu_m^x(t), \\ &|x_{2,m}(t,z,\alpha) - x_{2,m-1}(t,z,\alpha)| \le (M_2^x)^m \nu_m^x(t).\end{aligned} \tag{54}$$

When $m = 1$ it is clear from the previous calculations that (54) holds. Now assume (54) holds for some arbitrary $m > 1$ and consider

$$\begin{aligned}&|x_{1,m+1}(t,z,\alpha) - x_{1,m}(t,z,\alpha)| \\ \le &\frac{1}{\Gamma(p)} \int_a^t \Big[(t-s)^{p-1} - \Big(\frac{t-a}{c-a}\Big)^p (c-s)^{p-1}\Big] |f_1(t, x_m(t,z,\alpha)) - f_1(t, x_{m-1}(t,z,\alpha))| ds \\ &- \Big(\frac{t-a}{c-a}\Big)^p \int_t^c (c-s)^{p-1} |f_1(t, x_m(t,z,\alpha)) - f_1(t, x_{m-1}(,z,\alpha))| ds \\ &\le \frac{K_{11}(M_1^x)^m}{\Gamma(p)} \Bigg\{ \int_a^t \Big[(t-s)^{p-1} - \Big(\frac{t-a}{c-a}\Big)^p (c-s)^{p-1}\Big]\mu_m^x(s)ds \\ &\qquad + \Big(\frac{t-a}{c-a}\Big)^p \int_t^c (c-s)^{p-1}\mu_m^x(s)ds \Bigg\} \\ &+ \frac{K_{12}(M_2^x)^m}{\Gamma(p)} \Bigg\{ \int_a^t \Big[(t-s)^{p-1} - \Big(\frac{t-a}{c-a}\Big)^p (c-s)^{p-1}\Big]\nu_m^x(s)ds \\ &\qquad + \Big(\frac{t-a}{c-a}\Big)^p \int_t^c (c-s)^{p-1}\nu_m^x(s)ds \Bigg\},\end{aligned}$$

where we used the first Lipschitz condition in (11) and the induction hypothesis. Now applying definition (52) of $\mu_m^x(t)$ yields

$$\begin{aligned}|x_{1,m+1}(t,z,\alpha) - x_{1,m}(t,z,\alpha)| &\le [K_{11}(M_1^x)^m + K_{12}(M_2^x)^m]\mu_{m+1}^x(t) \\ &= (M^x)^{m+1}\mu_{m+1}^x(t).\end{aligned}$$

Applying the same reasoning to

$$|x_{2,m+1}(t,z,\alpha) - x_{2,m}(t,z,\alpha)|$$

yields the estimate in (54). Applying Lemma 2 to (54) and using definitions (15), (17) gives

$$\begin{pmatrix} |x_{1,m+1}(t,z,\alpha)-x_{1,m}(t,z,\alpha)| \\ |x_{2,m+1}(t,z,\alpha)-x_{2,m}(t,z,\alpha)| \end{pmatrix} \le {}^{x}\Gamma_{pq}^{m+1} \begin{pmatrix} (M_1^x)^{m+1} \\ (M_2^x)^{m+1} \end{pmatrix} = {}^{x}\Gamma_{pq}^{m+1} K^m \begin{pmatrix} M_1^x \\ M_2^x \end{pmatrix} = {}^{x}\Gamma_{pq}\, {}^{x}Q^m \begin{pmatrix} M_1^x \\ M_2^x \end{pmatrix},$$

which implies that

$$\begin{aligned} \begin{pmatrix} |x_{1,m+j}(t,z,\alpha)-x_{1,m}(t,z,\alpha)| \\ |x_{2,m+j}(t,z,\alpha)-x_{2,m}(t,z,\alpha)| \end{pmatrix} &= \begin{pmatrix} \sum_{i=1}^{j} |x_{1,m+i}(t,z,\alpha)-x_{1,m+i-1}(t,z,\alpha)| \\ \sum_{i=1}^{j} |x_{2,m+i}(t,z,\alpha)-x_{2,m+i-1}(t,z,\alpha)| \end{pmatrix} \\ &\le {}^{x}\Gamma_{pq}\, {}^{x}Q^m \sum_{i=0}^{j-1} {}^{x}Q^i \begin{pmatrix} M_1^x \\ M_2^x \end{pmatrix}. \end{aligned} \tag{55}$$

From (16) it follows that

$$\sum_{i=0}^{j-1} {}^{x}Q^i \le (I - {}^{x}Q)^{-1}, \quad \lim_{m\to\infty} {}^{x}Q^m = O,$$

where O denotes the matrix of zeros. Passing in (55) to the limit as $j \to \infty$ we obtain the error estimate (50). Thus, the sequences of functions in (29), (30) converge uniformly to the limit functions (40), (41) in their domains $[a,c]\times D_{\beta_1^x}$ and $[a,c]\times D_{\beta_2^x}$.

The functions $x_{1,\infty}(t,z,\alpha)$ and $x_{2,\infty}(t,z,\alpha)$ are the limits to sequences of functions, all of which satisfy the boundary conditions (38), (39), therefore, the limit functions also satisfy the same boundary conditions.

To prove *St. 4* of the Theorem we suppose $(x_1^1(t), x_1^2(t))$ and $(x_2^1(t), x_2^2(t))$ are two pairs of functions, both of which are solutions to the integral equations (44) and (45). Let

$$m_1 := \max_{a\le t\le c} |x_1^1(t) - x_1^2(t)|, \quad m_2 := \max_{a\le t\le c} |x_2^1(t) - x_2^2(t)|,$$

and consider

$$\begin{aligned} &|x_1^1(t) - x_1^2(t)| \le \\ &\le \frac{K_{11}}{\Gamma(p)}\left\{\int_a^t \left[(t-s)^{p-1} - \left(\frac{t-a}{c-a}\right)^p (c-s)^{p-1}\right]ds + \left(\frac{t-a}{c-a}\right)^p \int_t^c (c-s)^{p-1}ds\right\}m_1 \\ &+\frac{K_{12}}{\Gamma(p)}\left\{\int_a^t \left[(t-s)^{p-1} - \left(\frac{t-a}{c-a}\right)^p (c-s)^{p-1}\right]ds + \left(\frac{t-a}{c-a}\right)^p \int_t^c (c-s)^{p-1}ds\right\}m_2 \\ &\le \frac{K_{11}}{\Gamma(p)} {}^{x}\Gamma_{pq} m_1 + \frac{K_{12}}{\Gamma(p)} {}^{x}\Gamma_{pq} m_2, \end{aligned}$$

$$\begin{aligned} &|x_2^1(t) - x_2^2(t)| \le \\ &\le \frac{K_{21}}{\Gamma(q)}\left\{\int_a^t \left[(t-s)^{q-1} - \left(\frac{t-a}{c-a}\right)^q (c-s)^{q-1}\right]ds + \left(\frac{t-a}{c-a}\right)^q \int_t^c (c-s)^{q-1}ds\right\}m_1 \\ &+\frac{K_{22}}{\Gamma(q)}\left\{\int_a^t \left[(t-s)^{q-1} - \left(\frac{t-a}{c-a}\right)^q (c-s)^{q-1}\right]ds + \left(\frac{t-a}{c-a}\right)^q \int_t^c (c-s)^{q-1}ds\right\}m_2 \\ &\le \frac{K_{21}}{\Gamma(p)} {}^{x}\Gamma_{pq} m_1 + \frac{K_{22}}{\Gamma(p)} {}^{x}\Gamma_{pq} m_2. \end{aligned}$$

Thus,

$$\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \le {}^{x}Q \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}$$

holds for all $t \in [a,c]$, and since $r({}^{x}Q) < 1$, $m_1 = m_2 = 0$, which implies that $x_1^1(t) = x_1^2(t)$ and $x_2^1(t) = x_2^2(t)$. Hence, $x_{1,\infty}(t,z,\alpha,\lambda)$ and $x_{2,\infty}(t,z,\alpha,\lambda)$ are the unique solutions to integral equations (44) and (45). Moreover, the Cauchy problems (46), (47) are equivalent to the integral equations

$$\begin{aligned}
x_1(t) =&z_1 + \frac{1}{\Gamma(p)}\int_a^t (t-s)^{p-1}[f_1(s,x(s)) + \Delta^{p_x}]ds \\
=&z_1 + \Big(\frac{t-a}{c-a}\Big)^p(\alpha_1 - z_1) + \frac{1}{\Gamma(p)}\Big[\int_a^t (t-s)^{p-1}f_1(s,x(s))ds \\
&-\Big(\frac{t-a}{c-a}\Big)^p\int_a^c (c-s)^{p-1}f_1(s,x(s))ds\Big] \\
x_2(t) =&z_2 + \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1}[f_2(s,x(s))ds + \Delta^{q_x}] \\
=&z_2 + \Big(\frac{t-a}{c-a}\Big)^q(\alpha_2 - z_2) + \frac{1}{\Gamma(q)}\Big[\int_a^t (t-s)^{q-1}f_2(s,x(s))ds \\
&-\Big(\frac{t-a}{c-a}\Big)^q\int_a^c (c-s)^{q-1}f_2(s,x(s))ds\Big],
\end{aligned} \tag{56}$$

where Δ^{p_x} and Δ^{q_x} are given in (48) and (49). From comparing the integral equations in (56) to (44) and (45) and knowing that $x_{1,\infty}(t,z,\alpha)$ and $x_{2,\infty}(t,z,\alpha)$ are the unique continuous solutions to (44) and (45), it follows that they are also the unique continuous solutions to the Cauchy problems (46) and (47). This completes the proof. □

Similar result holds for the second BVP (8). The outline of the proof is the same as in Theorem 1, so we will leave for the reader.

Theorem 2. *Assume that conditions 1(b)–4(b) for BVP (8) are true. Then for all fixed $z_1 \in D_{\beta_1^x}$, $z_2 \in D_{\beta_2^x}$, $\alpha_1 \in D_{\beta_1^y}$, $\alpha_2 \in D_{\beta_2^y}$, $\lambda \in \mathcal{P}$, $\eta \in D_1 \times D_2$ it holds:*

1. *Functions of the sequences (31), (32) are continuous and satisfy the parametrized boundary conditions*

$$y_{1,m}(c,z,\alpha,\lambda,\eta) = \alpha_1, \quad y_{1,m}(b,z,\alpha,\lambda,\eta) = [C_1^{-1}(d(\lambda,\eta) - Az)]_1$$

$$y_{2,m}(c,z,\alpha,\lambda,\eta) = \alpha_2, \quad y_{2,m}(b,z,\alpha,\lambda,\eta) = [C_1^{-1}(d(\lambda,\eta) - Az)]_2$$

2. *The sequences of functions (31), (32) for $t \in [c,b]$ converge uniformly as $m \to \infty$ to the limit functions*

$$y_{1,\infty}(t,z,\alpha,\lambda,\eta) = \lim_{m\to\infty} y_{1,m}(t,z,\alpha,\lambda,\eta) \tag{57}$$

$$y_{2,\infty}(t,z,\alpha,\lambda,\eta) = \lim_{m\to\infty} y_{2,m}(t,z,\alpha,\lambda,\eta). \tag{58}$$

3. *The limit functions (57), (58) satisfy the parametrized boundary conditions*

$$y_{1,\infty}(c,z,\alpha,\lambda,\eta) = \alpha_1, \quad y_{1,\infty}(b,z,\alpha,\lambda,\eta) = [C_1^{-1}(d(\lambda,\eta) - Az)]_1$$

$$y_{2,\infty}(c,z,\alpha,\lambda,\eta) = \alpha_2, \quad y_{2,\infty}(b,z,\alpha,\lambda,\eta) = [C_1^{-1}(d(\lambda,\eta) - Az)]_2.$$

4. *The limit functions $y_{1,\infty}(t,z,\alpha,\lambda,\eta)$, $y_{2,\infty}(t,z,\alpha,\lambda,\eta)$ are the unique continuous solutions to the integral equations*

$$\begin{aligned}
y_1(t) =&\alpha_1 + \Big(\frac{t-c}{b-c}\Big)^p\{[C_1^{-1}(d(\lambda,\eta) - Az)]_1 - \alpha_1\} \\
&+ \frac{1}{\Gamma(p)}\Big[\int_c^t (t-s)^{p-1}g_1(s,x(s),y(s))ds \\
&-\Big(\frac{t-c}{b-c}\Big)^p\int_c^b (b-s)^{p-1}g_1(s,x(s),y(s))ds\Big]
\end{aligned} \tag{59}$$

$$\begin{aligned}
y_2(t) =&\alpha_2 + \Big(\frac{t-c}{b-c}\Big)^q\{[C_1^{-1}(d(\lambda,\eta) - Az)]_2 - \alpha_2\} \\
&+ \frac{1}{\Gamma(q)}\Big[\int_c^t (t-s)^{q-1}g_2(s,x(s),y(s))ds \\
&-\Big(\frac{t-c}{b-c}\Big)^q\int_c^b (b-s)^{q-1}g_2(s,x(s),y(s))ds\Big],
\end{aligned} \tag{60}$$

or equivalently, they are the unique continuous solutions to the Cauchy problems

$${}_a^C D_t^p y_1(t) = g_1(t,x(t),y(t)) + \Delta^{p_y}(z,\alpha,\lambda,\eta), \quad y_1(c) = \alpha_1 \tag{61}$$

$${}_a^C D_t^q y_2(t) = g_2(t,x(t),y(t)) + \Delta^{q_y}(z,\alpha,\lambda,\eta), \quad y_2(c) = \alpha_2, \tag{62}$$

where

$$\begin{aligned}\Delta^{p_y}(z,\alpha,\lambda,\eta) = & \frac{\Gamma(p+1)}{(b-c)^p}\{[C_1^{-1}(d(\lambda,\eta)-Az)]_1-\alpha_1\} \\ & -\frac{p}{(b-c)^p}\int_c^b (b-s)^{p-1}g_1(s,x(s),y(s))ds\end{aligned} \tag{63}$$

and

$$\begin{aligned}\Delta^{q_y}(z,\alpha,\lambda,\eta) = & \frac{\Gamma(q+1)}{(b-c)^q}\{[C_1^{-1}(d(\lambda,\eta)-Az)]_2-\alpha_2\} \\ & -\frac{q}{(b-c)^q}\int_c^b (b-s)^{q-1}g_2(s,x(s),y(s))ds.\end{aligned} \tag{64}$$

5. The following error estimate holds

$$\begin{pmatrix}|y_{1,\infty}(t,z,\alpha,\lambda,\eta)-y_{1,m}(t,z,\alpha,\lambda,\eta)| \\ |y_{2,\infty}(t,z,\alpha,\lambda,\eta)-y_{2,m}(t,z,\alpha,\lambda,\eta)|\end{pmatrix} \le {}^y\Gamma_{pq}{}^yQ^m(I-{}^yQ)^{-1}\begin{pmatrix}M_1^y \\ M_2^y\end{pmatrix}. \tag{65}$$

Remark 5. Theorems 1 and 2 guarantee that under the assumed conditions 1(a)–4(a) and 1(b)–4(b) the functions

$$\begin{aligned}x_{1,\infty}(t,z,\alpha) &: \Omega_1^x \times D_{\beta_1^x} \to D_1^x, \\ x_{2,\infty}(t,z,\alpha) &: \Omega_2^x \times D_{\beta_2^x} \to D_2^x, \\ y_{1,\infty}(t,z,\alpha,\lambda,\eta) &: \Omega_1^y \times D_{\beta_1^y} \to D_1^y, \\ y_{2,\infty}(t,z,\alpha,\lambda,\eta) &: \Omega_2^y \times D_{\beta_2^y} \to D_2^y\end{aligned}$$

are well-defined for all sets of artificially introduced parameters $(z,\alpha) \in D_{\beta^x} \times D_{\beta^y}$ and $(\lambda,\eta) \in \mathcal{P} \times D_1 \times D_2$. By putting

$$u_{1,\infty}(t,z,\alpha,\lambda,\eta) = \begin{cases} x_{1,\infty}(t,z,\alpha), & t \in [a,c] \\ y_{1,\infty}(t,z,\alpha,\lambda,\eta), & t \in [c,b]\end{cases} \tag{66}$$

and

$$u_{2,\infty}(t,z,\alpha,\lambda,\eta) = \begin{cases} x_{2,\infty}(t,z,\alpha), & t \in [a,c] \\ y_{2,\infty}(t,z,\alpha,\lambda,\eta), & t \in [c,b]\end{cases} \tag{67}$$

we obtain the well-defined continuous functions $u_{1,\infty}(t,z,\alpha,\lambda,\eta)$ and $u_{2,\infty}(t,z,\alpha,\lambda,\eta)$, which at $t=c$ coincide:

$$\begin{aligned}u_{1,\infty}(c,z,\alpha,\lambda,\eta) &= x_{1,\infty}(c,z,\alpha) = y_{1,\infty}(c,z,\alpha,\lambda,\eta) = \alpha_1 \\ u_{2,\infty}(c,z,\alpha,\lambda,\eta) &= x_{2,\infty}(c,z,\alpha) = y_{2,\infty}(c,z,\alpha,\lambda,\eta) = \alpha_2.\end{aligned}$$

4. Relation between the parametrized and original BVPs

In this section, we show the connection between the solutions of the Cauchy problems (46)–(47), (61)–(62) and the solutions to the "model"-type BVPs (7), (8).

4.1. Initial value problem and its BVP equivalence

Consider the fractional initial value problems (FIVP) with constant perturbation terms χ^{x_p}, χ^{y_p}, χ^{x_q}, and χ^{y_q}:

$$\begin{aligned}{}_0^CD_t^p x_1(t) &= f_1(t,x(t)) + \chi^{x_p}, \qquad t \in [a,c], \\ x_1(a) &= z_1,\end{aligned} \tag{68}$$

$$\begin{aligned}{}_0^CD_t^q x_2(t) &= f_2(t,x(t)) + \chi^{x_q}, \qquad t \in [a,c], \\ x_2(a) &= z_2,\end{aligned} \tag{69}$$

and

$$\begin{aligned}{}_0^CD_t^p y_1(t) &= g_1(t,x(t),y(t)) + \chi^{y_p}, \qquad t \in [c,b], \\ y_1(c) &= \alpha_1,\end{aligned} \tag{70}$$

$$\begin{aligned}{}_0^CD_t^q y_2(t) &= g_2(t,x(t),y(t)) + \chi^{y_q}, \qquad t \in [c,b], \\ y_2(c) &= \alpha_2,\end{aligned} \tag{71}$$

where $\chi^{x_p} = (\chi_1^{x_p}, \chi_2^{x_p}, \dots, \chi_n^{x_p})^T$, $\chi^{y_p} = (\chi_1^{y_p}, \chi_2^{y_p}, \dots, \chi_n^{y_p})^T \in \mathbb{R}^n$ and $\chi^{x_q} = (\chi_1^{x_q}, \chi_2^{x_q}, \dots, \chi_m^{x_q})^T$, $\chi^{y_q} = (\chi_1^{y_q}, \chi_2^{y_q}, \dots, \chi_m^{y_q})^T \in \mathbb{R}^m$ are referred to as control parameters.

Theorem 3. *Suppose $z \in D_{\beta^x}$, $\alpha \in D_{\beta^y}$, $\lambda \in \mathcal{P}$, $\eta \in D_1 \times D_2$ and assume the conditions of Theorem 1 hold.*
Then the solutions $x_1(\cdot, z, \alpha)$ and $x_2(\cdot, z, \alpha)$ of the FIVPs (68)–(69), satisfy conditions

$$x_1(c, z, \alpha) = \alpha_1, \quad x_2(c, z, \alpha) = \alpha_2, \tag{72}$$

i.e. they are solutions to the decomposed FBVPs (7) with parametrized boundary conditions on the subinterval $[a, c]$, if and only if the control parameters χ^{x_p} and χ^{x_q} in (68), (69) are given by

$$\chi^{x_p} := \frac{\Gamma(p+1)}{(c-a)^p}(\alpha_1 - z_1) - \frac{p}{(c-a)^p}\int_a^c (c-s)^{p-1} f_1(s, x_\infty(s))ds, \tag{73}$$

$$\chi^{x_q} := \frac{\Gamma(q+1)}{(c-a)^q}(\alpha_2 - z_2) - \frac{q}{(c-a)^q}\int_a^c (c-s)^{q-1} f_2(s, x_\infty(s))ds, \tag{74}$$

where $x_\infty(\cdot) = \begin{bmatrix} x_{1,\infty}(\cdot, z, \alpha) \\ x_{2,\infty}(\cdot, z, \alpha) \end{bmatrix}$ are the limit functions in (40) and *(41).*

Proof. Sufficiency: Suppose the control parameters in (68), (69) are given by (73) and (74) respectively. Then, according to Theorem 1, the limit functions (40), (41) of the sequences in (29), (30) are the unique solutions to BVP (7). That is, they satisfy the initial conditions in (68) and (69), which means that they are solutions to the Cauchy problems (68), (69) with χ^{x_p} and χ^{x_q}, defined as in (73) and (74). Thus, $x_1(\cdot, z, \alpha) = x_{1,\infty}(\cdot, z, \alpha)$ and $x_2(\cdot, z, \alpha) = x_{2,\infty}(\cdot, z, \alpha)$.

Necessity: Suppose that there exist control parameters $\bar{\chi}^{x_p}$ and $\bar{\chi}^{x_q}$, such that the functions $\bar{x}_1(t, z, \alpha)$ and $\bar{x}_2(t, z, \alpha)$ are solutions to the FIVPs (68), (69), which also satisfy conditions (72). Then $\bar{x}_1(t, z, \alpha)$ and $\bar{x}_2(t, z, \alpha)$ are continuous solutions to the integral equations

$$\begin{aligned}\bar{x}_1(t) &= z_1 + \frac{1}{\Gamma(p)}\int_a^t (t-s)^{p-1} f_1(s, \bar{x}(s))ds + \frac{(t-a)^p}{\Gamma(p+1)}\bar{\chi}^{x_p}\\ \bar{x}_2(t) &= z_2 + \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1} f_2(s, \bar{x}(s))ds + \frac{(t-a)^q}{\Gamma(q+1)}\bar{\chi}^{x_q}\end{aligned} \tag{75}$$

Using conditions (72) in (75) and re-arranging the terms yields

$$\bar{\chi}^{x_p} = \frac{\Gamma(p+1)}{(c-a)^p}(\alpha_1 - z_1) - \frac{p}{(c-a)^p}\int_a^c (c-s)^{p-1} f_1(s, \bar{x}(s))ds,$$

$$\bar{\chi}^{x_q} = \frac{\Gamma(q+1)}{(c-a)^q}(\alpha_2 - z_2) - \frac{q}{(c-a)^q}\int_a^c (c-s)^{p-1} f_2(s, \bar{x}(s))ds.$$

This implies that

$$\begin{aligned}\bar{x}_1(t) =& z_1 + \left(\frac{t-a}{c-a}\right)^p(\alpha_1 - z_1) + \frac{1}{\Gamma(p)}\Bigg[\int_a^t (t-s)^{p-1} f_1(s, \bar{x}(s))ds\\ &-\left(\frac{t-a}{c-a}\right)^p \int_a^c (c-s)^{p-1} f_1(s, \bar{x}(s))ds\Bigg],\end{aligned} \tag{76}$$

$$\begin{aligned}\bar{x}_2(t) =& z_2 + \left(\frac{t-a}{c-a}\right)^a(\alpha_2 - z_2) + \frac{1}{\Gamma(q)}\Bigg[\int_a^t (t-s)^{q-1} f_2(s, \bar{x}(s))ds\\ &-\left(\frac{t-a}{c-a}\right)^q \int_a^c (c-s)^{p-1} f_2(s, \bar{x}(s))ds\Bigg].\end{aligned} \tag{77}$$

Since $z_1 \in D_{\beta_1^x}$, $z_2 \in D_{\beta_2^x}$, according to the integral equations above and the definitions of the sets $D_{\beta_1^x}$ and $D_{\beta_2^x}$, it can be shown that $\bar{x}_1(t, z, \alpha) \in D_1^x$ and $\bar{x}_2(t, z, \alpha) \in D_2^x$. Eqs. (76) and (77) are equivalent to (44) and (45) respectively, hence, by part 4 of Theorem 1 it follows that $x_1(\cdot, z, \alpha) = x_{1,\infty}(\cdot, z, \alpha)$, $x_2(\cdot, z, \alpha) = x_{2,\infty}(\cdot, z, \alpha)$ and $\bar{\chi}^{x_p} = \chi^{x_p}$, $\bar{\chi}^{x_q} = \chi^{x_q}$, where χ^{x_p} and χ^{x_q} are given by (73) and (74), respectively. This completes the proof of the theorem. □

Similar result holds for the FIVP (70), (71).

Theorem 4. *Suppose $z \in D_{\beta^x}$, $\alpha \in D_{\beta^y}$, $\lambda \in \mathcal{P}$, $\eta \in D_1 \times D_2$ and assume the conditions of Theorem 2 hold.*
Then the solutions $y_1(\cdot, z, \alpha, \lambda, \eta)$, $y_2(\cdot, z, \alpha, \lambda, \eta)$ of the FIVPs (70), (71) satisfy conditions

$$y_1(b, z, \alpha, \lambda, \eta) = [C_1^{-1}(d(\lambda, \eta) - Az)]_1, \quad y_2(b, z, \alpha, \lambda, \eta) = [C_1^{-1}(d(\lambda, \eta) - Az)]_2,$$

i.e. they are solutions to the decomposed FBVPs (8) with parametrized boundary conditions on the subinterval $[c, b]$ if and if only if the control parameters χ^{y_p}, χ^{y_q} in (70), (71) are given by

$$\begin{aligned}\chi^{y_p} :=& \frac{\Gamma(p+1)}{(b-c)^p}\{[C_1^{-1}(d(\lambda, \eta) - Az)]_1 - \alpha_1\}\\ &- \frac{p}{(b-c)^p}\int_c^b (b-s)^{p-1} g_1(s, x_{\infty,}(s), y_\infty(s))ds,\end{aligned} \tag{78}$$

$$\chi^{y_q} := \frac{\Gamma(q+1)}{(b-c)^q}\{[C_1^{-1}(d(\lambda,\eta)-Az)]_2-\alpha_2\} \\ -\frac{q}{(b-c)^q}\int_c^b (b-s)^{q-1} g_2(s,x_{\infty,}(s),y_\infty(s))ds, \tag{79}$$

where $y_\infty(\cdot,z,\alpha,\lambda,\eta)=\begin{bmatrix} y_{1,\infty}(\cdot,z,\alpha,\lambda,\eta) \\ y_{2,\infty}(\cdot,z,\alpha,\lambda,\eta)\end{bmatrix}$ *are the limit functions in* (57) and (58)*, respectively.*

Proof. The proof of Theorem 4 follows the lines of the proof of Theorem 3. □

4.2. Main result

The following theorem demonstrates the connection between the limit functions (66) and (67) and the solutions to the original BVP (3), (4).

Theorem 5. *Suppose the conditions of Theorems 1 and 2 hold. Then the functions* $u_{1,\infty}(\cdot,z,\alpha,\lambda,\eta)$ *and* $u_{2,\infty}(\cdot,z,\alpha,\lambda,\eta)$*, defined in* (66) and (67) *are continuous solutions to the original FBVP* (3), (4)*, if and only if the following system of algebraic or transcendental equations is satisfied*

$$\begin{aligned} \Delta^{p_x}(z,\alpha)&=0,\\ \Delta^{q_x}(z,\alpha)&=0,\\ \Delta^{p_y}(z,\alpha,\lambda,\eta)&=0,\\ \Delta^{q_y}(z,\alpha,\lambda,\eta)&=0,\\ V(z,\alpha,\lambda,\eta)-\lambda&=0,\\ y_i(b,z,\alpha,\lambda,\eta)-\eta_i&=0,\ \ i=p+1,\ldots,N, \end{aligned} \tag{80}$$

where $\Delta^{p_x}(z,\alpha)$*,* $\Delta^{q_x}(z,\alpha)$*,* $\Delta^{p_y}(z,\alpha,\lambda,\eta)$*, and* $\Delta^{q_y}(z,\alpha,\lambda,\eta)$ *are given by* (48), (49), (63), *and* (64)*, and* $V(z,\alpha,\lambda,\eta)$ *is defined as*

$$V(z,\alpha,\lambda,\eta) := \int_a^c P(s)x_\infty(s,z,\alpha)ds + \int_c^b P(s)y_\infty(s,z,\alpha,\lambda,\eta). \tag{81}$$

Proof. Since the conditions of Theorems 1 and 2 hold, we can apply Theorems 3 and 4. The perturbed IVPs (46)–(47) and (61)–(62) coincide with BVPs (7), (8) if and only if

$$\begin{aligned} \Delta^{p_x}(z,\alpha)&=0,\\ \Delta^{q_x}(z,\alpha)&=0,\\ \Delta^{p_y}(z,\alpha,\lambda,\eta)&=0,\\ \Delta^{q_y}(z,\alpha,\lambda,\eta)&=0. \end{aligned}$$

Moreover, from the definition of λ in (5), it follows that in order for $x_1(\cdot,z,\alpha)$, $x_2(\cdot,z,\alpha)$, $y_1(\cdot,z,\alpha,\lambda,\eta)$, and $y_2(\cdot,z,\alpha,\lambda,\eta)$ to coincide with the solutions of (7) and (8), it must hold that

$$\begin{aligned} &\int_a^c [P_{11}(s)x_{1,\infty}(s)+P_{12}(s)x_{2,\infty}(s)]ds \\ &\qquad + \int_c^b [P_{11}(s)y_{1,\infty}(s)+P_{12}(s)y_{2,\infty}(s)]ds-\lambda_1=0,\\ &\int_a^c [P_{21}(s)x_{1,\infty}(s)+P_{22}(s)x_{2,\infty}(s)]ds \\ &\qquad + \int_c^b [P_{21}(s)y_{1,\infty}(s)+P_{22}(s)y_{2,\infty}(s)]ds-\lambda_2=0,\\ &\qquad y_{i,\infty}(b,z,\alpha,\lambda,\eta)-\eta_i=0, \end{aligned}$$

for $i=p+1,\ldots,N$, where the notation $y_{i,\infty}(b,z,\alpha,\lambda,\eta)$ refers to the ith component of the vector $y_\infty(b,z,\alpha,\lambda,\eta)$. Thus, $x_{1,\infty}(\cdot,z,\alpha)$, $x_{2,\infty}(\cdot,z,\alpha)$, $y_{1,\infty}(\cdot,z,\alpha,\lambda,\eta)$, and $y_{2,\infty}(\cdot,z,\alpha,\lambda,\eta)$ are the solutions of (7) and (8) if an only if the equations in (80) are satisfied. This completes the proof of the theorem. □

Remark 6. Theorem 5 gives necessary and sufficient conditions on the solvability of the system of FBVPs (7), (8) and the construction of their solutions, however, a difficulty of its application arises from the fact that the explicit forms of the exact functions Δ^{p_x}, Δ^{q_x}, Δ^{p_y}, Δ^{q_y}, V, and $y(b,z,\alpha,\lambda,\eta)$ are unknown. In order to overcome this complication, in practice we solve an approximate system of determining equations

$$\begin{aligned} \Delta_m^{p_x}(z,\alpha) &= 0,\\ \Delta_m^{q_x}(z,\alpha) &= 0,\\ \Delta_m^{p_y}(z,\alpha,\lambda,\eta) &= 0,\\ \Delta_m^{q_y}(z,\alpha,\lambda,\eta) &= 0\\ V_m(z,\alpha,\lambda,\eta) &= 0,\\ y_{m,i}(b,z,\alpha,\lambda,\eta) &= 0,\ \ i=p+1,\dots,N, \end{aligned} \tag{82}$$

that only depends on the mth terms in the functional sequences (29)–(32), and can therefore be constructed explicitly. In particular, the equations in (82) at the mth iteration are given by:

$$\begin{aligned} \Delta_m^{p_x}(z,\alpha) &:= \frac{\Gamma(p+1)}{(c-a)^p}(\alpha_1 - z_1) - \frac{p}{(c-a)^p}\int_a^c (c-s)^{p-1} f_1(s,x_m(s))ds,\\ \Delta_m^{q_x}(z,\alpha) &:= \frac{\Gamma(q+1)}{(c-a)^q}(\alpha_2 - z_2) - \frac{q}{(c-a)^q}\int_a^c (c-s)^{q-1} f_2(s,x_m(s))ds,\\ \Delta_m^{p_y}(z,\alpha,\lambda,\eta) &:= \frac{\Gamma(p+1)}{(b-c)^p}\{[C_1^{-1}(d(\lambda,\eta)-Az)]_1 - \alpha_1\}\\ &\quad - \frac{p}{(b-c)^p}\int_c^b (b-s)^{p-1} g_1(s,x_m(s),y_m(s))ds,\\ \Delta_m^{q_y}(z,\alpha,\lambda,\eta) &:= \frac{\Gamma(q+1)}{(b-c)^q}\{[C_1^{-1}(d(\lambda,\eta)-Az)]_2 - \alpha_2\}\\ &\quad - \frac{q}{(b-c)^q}\int_c^b (b-s)^{q-1} g_2(s,x_m(s),y_m(s))ds,\\ V_m(z,\alpha,\lambda,\eta) &:= \int_a^c P(s)x_m(s,z,\alpha)ds + \int_c^b P(s)y_m(s,z,\alpha,\lambda,\eta). \end{aligned} \tag{83}$$

5. Example

Motivated by [36,40] we consider a BVP for the non-linear fractional differential equation

$${}_0^C D_t^{\frac{3}{2}} u(t) = \frac{-2e^t}{(1+e^t)^2}\left[\frac{u(t)}{16}\right]^2 - \frac{2\omega e^t(1-e^t)}{(1+e^t)^3} \ (:= f(t,u(t))), \quad t\in[0,1], \tag{84}$$

subjected to the integral boundary conditions

$$\begin{aligned} u(0)+\dot u(0) &= -\int_0^1 u(s)d\zeta(s),\\ u(1)+\dot u(1) &= \int_0^1 u(s)d\eta(s), \end{aligned} \tag{85}$$

where $\zeta(t)$ and $\eta(t)$ are nondecreasing, right-continuous on $t\in[0,1)$ and left continuous at $t=1$, and $\int_0^1 u(s)d\zeta(s)$, $\int_0^1 u(s)d\eta(s)$ denote the Riemann–Stieltjes integrals of u with respect to $\zeta(t)$ and $\eta(t)$, [40]. For simplicity, we take $\zeta(t)=\eta(t)=t$, hence $d\zeta(t)=d\eta(t)=1$ and BCs (85) become

$$\begin{aligned} u(0)+\dot u(0) &= -\int_0^1 u(s)ds,\\ u(1)+\dot u(1) &= \int_0^1 u(s)ds. \end{aligned} \tag{86}$$

In (84) ω is a scalar which in the context of the flow of the Antarctic Circumpolar Current corresponds to the dimensionless Coriolis parameter being equal to 4649.56.

Eq. (84) can be written as a system of a first order ODE and a FDE of order $q=1/2$ by letting

$$u_1(t) := u(t), \qquad u_2(t) := \dot u(t) = \dot u_1(t). \tag{87}$$

Substituting (87) into (84) results in the following system

$$\begin{cases} \dot u_1(t) = u_2(t) \ \ (:= f_1(t,u(t))),\\ {}_0^C D_t^{\frac{1}{2}} u_2(t) = \frac{-2e^t}{(1+e^t)^2}\left[\frac{u_1(t)}{16}\right]^2 - \frac{2\omega e^t(1-e^t)}{(1+e^t)^3} \ \ (:= f_2(t,u(t))), \end{cases} \tag{88}$$

and the boundary conditions in (86) are transformed into

$$u_1(0)+u_2(0)=-\int_0^1 u_1(s)ds, \tag{89}$$

$$u_1(1)+u_2(1)=\int_0^1 u_1(s)ds. \tag{90}$$

We apply the parametrization technique, described in Section 2.2, by introducing

$$z_1 := u_1(0),\;\; z_2 := u_2(0),\;\; \lambda_1 := -\int_0^1 u_1(s)ds,$$
$$\lambda_2 := \int_0^1 u_1(s)ds,\;\; \eta := \begin{pmatrix} u_2(1) \\ 0 \end{pmatrix}.$$

With the given parametrization, boundary conditions (86) are re-written as

$$Az + C_1 u(1) = d(\eta,\lambda), \tag{91}$$

where

$$A := \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},\;\; C_1 := \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},\;\; d(\eta,\lambda) := \eta+\lambda,\;\; u(1) := \begin{pmatrix} u_1(1) \\ u_2(1) \end{pmatrix}.$$

This allows us to express the values of $u_1(1)$ and $u_2(1)$ as

$$u_1(1) = [C_1^{-1}(d(\eta,\lambda)-Az)]_1 = -\eta_1-\lambda_1+z_1+z_2+\lambda_2,$$
$$u_2(1) = [C_1^{-1}(d(\eta,\lambda)-Az)]_2 = \eta_1+\lambda_1-z_1-z_2.$$

The BVP is considered on the domain

$$D_1 := \{u_1 : -855.04 \le u_1 \le 183.69\}, \qquad t \in [0,1]$$
$$D_2 := \{u_2 : -701.03 \le u_2 \le 1248.85\}, \qquad t \in [0,1],$$

on which the right-hand side function $f(t,u_1(t),u_2(t)) = \begin{pmatrix} f_1(t,u_1(t),u_2(t)) \\ f_2(t,u_1(t),u_2(t)) \end{pmatrix}$ satisfies the Lipschitz condition with a constant matrix $\tilde{K} = \begin{pmatrix} 0 & 1 \\ 1.46 & 0 \end{pmatrix}$. The matrix Q

has spectral radius $r(Q) \approx 1.36 > 1$. That is, condition 4(a) is not satisfied, and hence the numerical-analytic technique cannot be used for constructing approximate solutions system (88) on the whole interval $t \in [0,1]$. Therefore, it is necessary to apply the interval halving technique, described in Section 2.3.

Let $c = 1/2$ and

$$u_1(t) = \begin{cases} x_1(t), & t \in [0,1/2] \\ y_1(t), & t \in [1/2,1] \end{cases} \qquad u_2(t) = \begin{cases} x_2(t), & t \in [0,1/2] \\ y_2(t), & t \in [1/2,1]. \end{cases}$$

Introducing an additional parameter

$$\alpha := \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} x_1(1/2) \\ x_2(1/2) \end{pmatrix} = \begin{pmatrix} y_1(1/2) \\ y_2(1/2) \end{pmatrix}$$

allows us to decompose BVP (88), (91) into the following two BVPs

$$\begin{cases} \dot{x}_1(t) = x_2(t) := f_1(t,x(t)), \\ {}^{C}_{0}D_t^{\frac{1}{2}} x_2(t) = \frac{-2e^t}{(1+e^t)^2}\left[\frac{x_1(t)}{16}\right]^2 - \frac{2\omega e^t(1-e^t)}{(1+e^t)^3} := f_2(t,x(t)), \\ x_1(0) = z_1, \;\; x_1(1/2) = \alpha_1, \\ x_2(0) = z_2, \;\; x_2(1/2) = \alpha_2; \end{cases} \tag{92}$$

$$\begin{cases} \dot{y}_1(t) = y_2(t) := g_1(t,x(t),y(t)), \\ {}^{C}_{1/2}D_t^{\frac{1}{2}} y_2(t) = \frac{-2e^t}{(1+e^t)^2}\left[\frac{y_1(t)}{16}\right]^2 - \frac{2\omega e^t(1-e^t)}{(1+e^t)^3} - \frac{1}{\Gamma(1/2)}\int_0^{1/2}(s-t)^{-1/2}\dot{x}_2(t)dt, \\ \qquad := g_2(t,x(t),y(t)), \\ y_1(1/2) = \alpha_1, \;\; y_1(1) = [C_1^{-1}(d(\eta,\lambda)) - Az]_1, \\ y_2(1/2) = \alpha_2, \;\; y_2(1) = [C_1^{-1}(d(\eta,\lambda)) - Az]_2. \end{cases} \tag{93}$$

The adjustment in the right-hand side function $g(t,x(t),y(t))$ in the BVP (93) follows from the considerations presented in Remark 2.

Let BVPs (92), (93) be defined on the domains

$$D^x := \{(x_1, x_2) : -494 \le x_1 \le -365.74,\ \ -144.05 \le x_2 \le 318.15\},\ t \in [0, 1/2],$$
$$D^y := \{(y_1, y_2) : -713.02 \le y_1 \le 87.35,\ \ -528.35 \le y_2 \le 1208.61\},\quad t \in [1/2, 1],$$

respectively.

The right-hand side functions $f(t, x_1(t), x_2(t))$ and $g(t, y_1(t), y_2(t))$ satisfy conditions 1(a), 2(a) and 1(b), 2(b), respectively, with

$$M^x = \begin{pmatrix} M_1^x \\ M_2^x \end{pmatrix} = \begin{pmatrix} 114.72 \\ 289.64 \end{pmatrix},\quad K = \begin{pmatrix} 0 & 1 \\ 1.46 & 0 \end{pmatrix},$$
$$M^y = \begin{pmatrix} M_1^y \\ M_2^y \end{pmatrix} = \begin{pmatrix} 586.75 \\ 779.43 \end{pmatrix},\quad J = \begin{pmatrix} 0 & 1 \\ 1.35 & 0 \end{pmatrix}.$$

The constants in (13), (22) and the spectral radii (16), (26) are calculated to be

$$\beta_1^x = 28.68,\quad \beta_2^x = 231.1,\qquad r(^xQ) = 0.96,$$
$$\beta_1^y = 293.38,\quad \beta_2^y = 621.87,\quad r(^yQ) = 0.93.$$

Since $r(^xQ) < 1$ and $r(^yQ) < 1$, the functions $f(t, x_1(t), x_2(t))$, $g(t, y_1(t), y_2(t))$ are bounded and satisfy Lipschitz conditions with constant matrices K and J respectively, conditions 1(a)–4(a) and 1(b)–4(b) are satisfied, therefore we can apply the numerical-analytic technique for constructing sequences of approximations of the solutions to BVPs (92) and (93).

Solving the system of approximate equations (83) at iterations $m = 0,\ 1,\ 2$ and applying Maple yields the following values of the artificially introduced parameters (see Table 1):

Plots of the first three iterates ($m = 0,\ 1,\ 2$) of the first and second components are shown in Fig. 1. To verify how well the approximations satisfy systems (92), (93), we computed the first derivatives of $x_{1,m}(t)$, $y_{1,m}(t)$ and the Caputo derivatives of $x_{2,m}(t)$, $y_{2,m}(t)$ for $m = 0,\ 1,\ 2$ and graphically compared them to the right-hand sides of the equations in (92), (93). These plots are shown in Figs. 2–4. With each successive iteration, the computed approximations satisfy the equations more accurately. The larger error in the approximations at the initial iterations results in discontinuities in the right-hand sides of the equations. At $m = 2$ the equations are already well satisfied. Continuing calculations, one can obtain approximations with higher precision.

Table 1
Numerically calculated parameter values for $m = 0, 1, 2$.

Parameter	$m = 0$	$m = 1$	$m = 2$
$z_{1,m}$	−465.3220981	−378.2519730	−369.7740162
$z_{2,m}$	87.05267109	37.01220483	31.28948606
$\alpha_{1,m}$	−419.6409477	−372.9985154	−365.7399766
$\alpha_{2,m}$	93.51711567	115.1684774	114.7173335
$\lambda_{1,m}$	−378.2694271	−341.2397681	−338.4845302
$\lambda_{2,m}$	378.2694271	341.2397681	338.4845302
$\eta_{1,m}$	586.7431416	547.8912348	544.5100375

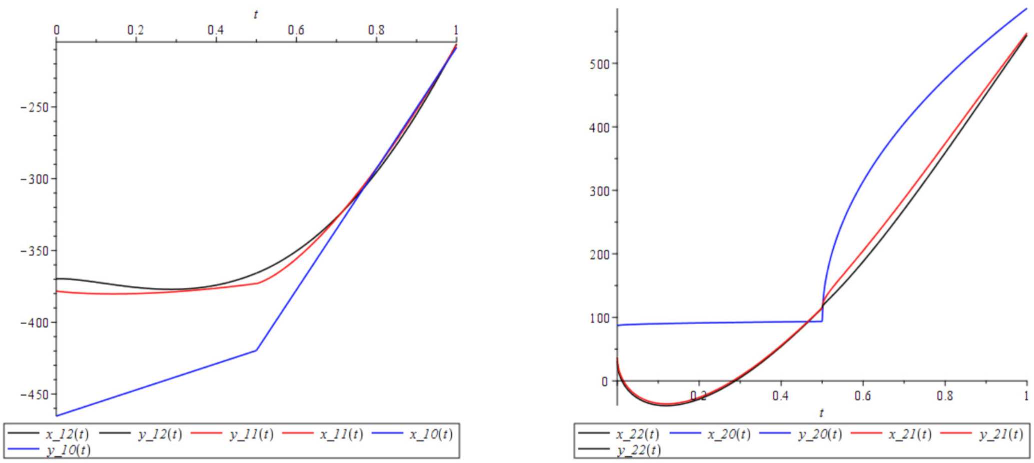

Fig. 1. Plots of components $x_{1,m}(t)$, $y_{1,m}(t)$ (left) and $x_{2,m}(t)$, $y_{2,m}(t)$ (right) for $m = 0,\ 1,\ 2$ over $t \in [0, 1]$.

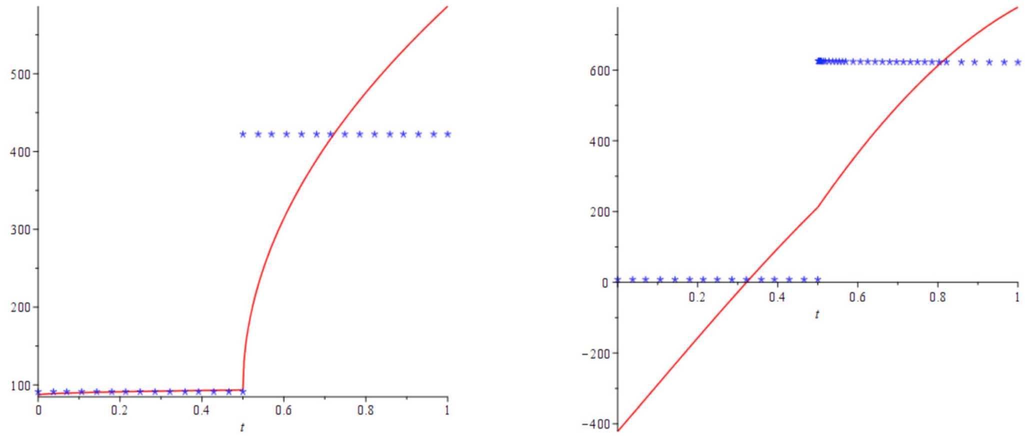

Fig. 2. Left- (blue dotted lines) and right- (red solid lines) hand sides of system (88) for $m = 0$. The left panel shows plots for the first equation and the right panel shows plots for the second one.

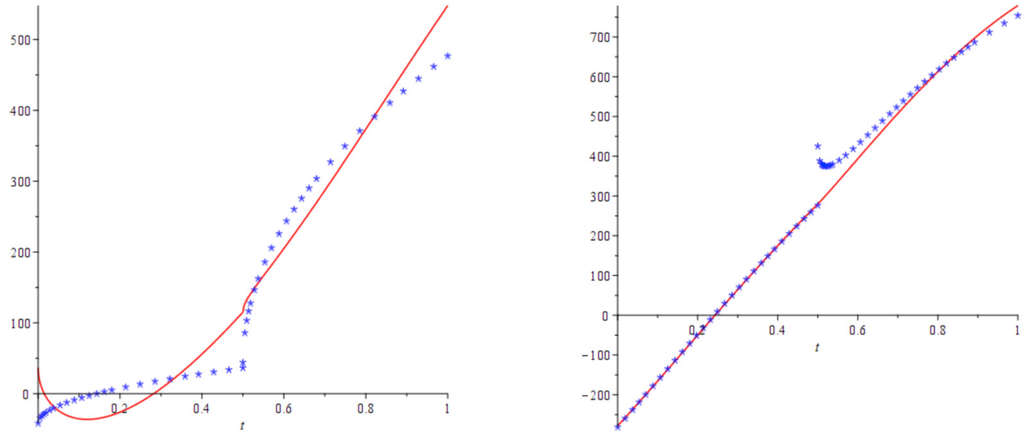

Fig. 3. Left- (blue dotted lines) and right- (red solid lines) hand sides of system (88) for $m = 1$. The left panel shows plots for the first equation and the right panel shows plots for the second one.

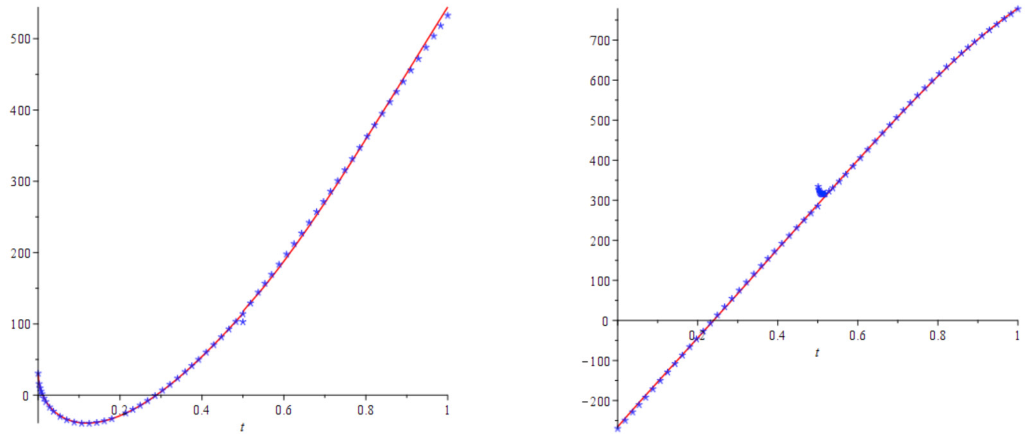

Fig. 4. Left- (blue dotted lines) and right- (red solid lines) hand sides of system (88) for $m = 2$. The left panel shows plots for the first equation and the right panel shows plots for the second one.

6. Conclusion

In this paper we presented a novel approach for the construction of approximate solutions to systems of non-linear FDEs of a mixed real order, subject to integral boundary constrains. The novelty consists in extending the applicability of the studied method to systems of FDEs with the special type integral boundary conditions. The boundary restrictions are re-written as two-point linear boundary conditions using a parametrization technique. A dichotomy-type approach is applied to transform the original BVP into two BVPs, each defined on an interval with half the length of the original problem. This modification reduces the error estimate of the method, or can be applied to problems for which the approximation technique does not converge on the entire interval. Sequences of approximations are constructed in an analytic form using the numerical-analytic technique and the values of a set of parameters which govern the behaviour of the solutions are computed by numerically solving a system of algebraic equations. The technique is applied to a particular case of the BVP (3), (4), obtained from the equation modelling the motion of a gyre in the Southern hemisphere. The validity of the obtained results is demonstrated by comparing the left and right hand-sides of the original system on each iteration step.

Data availability

No data was used for the research described in the article.

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