Cumulative Prospect Theory in Option Valuation and Portfolio Management

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by

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ABSTRACT

This thesis deals with different models for decision-making under risk in financial applications, mainly models that incorporate irrational human behavior. First of all, traditional expected utility theory is considered. Hereafter, two models that incorporate irrational human behavior are discussed and compared: prospect theory and cumulative prospect theory. Next, these models are applied to option pricing. The influence of various levels of sentiment on the option price is investigated and the prices are compared with Black-Scholes prices. Also, a sensitivity analysis is done in order to investigate the influence of the prospect parameters on the option price. Furthermore, the different models discussed are applied to portfolio management for which the optimal wealth profiles are analyzed and compared. Moreover, a data analysis with a portfolio of stocks of different indices is done in which it is investigated whether the parameter estimates used for prospect sentiment are applicable to financial data. Finally, a hedge test under prospect sentiment is performed in order to investigate whether a delta-hedge leads to sufficient results in case of asset price paths under prospect sentiment.

Preface

I wish to acknowledge the aid and support of C.W. Oosterlee during my research. I would also like to express my thanks to him for the fruitful discussions and for commenting on earlier drafts of this report. Besides that, I would like to thank my parents for providing me support through the duration of my studies and for giving the possibility to combine my master Applied Mathematics with my piano studies at the conservatory of Rotterdam.

L.J.M. van Tol
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INTRODUCTION

In this thesis, several models for decision-making under risk are discussed and compared. The traditional expected utility theory assumes rational investors, while experiments have shown that emotions influence the decision-making process in such a way that the principles of expected utility are violated. Therefore, the main focus in this thesis lies on models that incorporate irrational behavior: prospect theory and cumulative prospect theory. These models rely on the evaluation of gains and losses instead of final positions, on different attitudes towards gains and losses, on aversion for losses and on subjective evaluation of probabilities. Then, the consequences of irrational behavior on option pricing and portfolio management are discussed and compared in several settings. Also, a data analysis is done in order to investigate whether prospect sentiment is applicable to financial data.

The outline of this thesis is as follows. In Chapter 2, different models for decision-making under risk are considered. First, the traditional expected utility theory is considered, in which investors are assumed to be rational. Then, several effects that violate the principles of expected utility theory and represent irrational human behavior are discussed. Hereafter, theories that incorporate irrational behavior are discussed: prospect theory and cumulative prospect theory. These theories account for irrational human behavior by use of a value function and a weighting function.

In Chapter 3, option pricing under different models is discussed: Black-Scholes, prospect theory and cumulative prospect theory. The relationship between these models are discussed as well as the results of computing option prices. Option prices are computed from both writer's and holder's viewpoint for different levels of sentiment. Then, it is investigated which levels of sentiment lead to a trade between holder and writer. Also, implied volatilities are considered for different levels sentiment. After that, a sensitivity analysis is done in order to measure the impact of the different prospect parameters and Black-Scholes parameters on the option price. Also, the influence of negative interest rates as well different dynamics (Geometric Brownian Motion, Heston dynamics) on the option price are considered.

In Chapter 4, portfolio management under different models is discussed. First, the traditional way of choosing an optimal portfolio which relies on the maximization of the return for a given level of risk according to Modern Portfolio Theory is considered. Then, models that incorporate individual risk profiles are discussed such expected utility theory, prospect theory without a weighting function and cumulative prospect theory. Expressions for the optimal wealth under the different models are computed and numerical examples are given in order to compare the optimal wealth profile for different levels of the state price density. The wealth profiles under the different models are compared and the differences are explained economically.

In Chapter 5, we bring together Chapter 3 and Chapter 4. It is investigated whether the estimates for the different levels of prospect sentiment as used for option valuation and portfolio management and which are derived from psychological experiments are suitable for financial applications. To this end, the prospect parameters are estimated based on historical returns of a market portfolio. Hereafter, the influence of adding sentiment to asset price paths on the hedging strategy of an option writer is considered. To this end, first a way of incorporating prospect sentiment in price path dynamics is discussed, after which the impact of adding sentiment on the hedge-effectiveness is discussed.

In Chapter 6, the conclusions are presented and recommendations for further research are given.

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Nomenclature

α	Drift of GBM price process/Heston dynamics.
χ	Volatility of volatility under Heston dynamics.
η	Parameter of the CRRA power utility function.
γ	Constant that controls for overweighting and underweighting in weighting function $w(p)$ in prospect theory and $w^+(p)$ in cumulative prospect theory.
κ	Speed at which variance v_t returns to $\overline{v}$ under Heston dynamics.
λ	Degree of loss aversion in value function v.
μ	N-dimensional vector of drift $\mu_1, \mu_2, ...\mu_N$
Ω	Set of all possible outcomes of probability space (Ω, Θ, μ).
$\omega(t)$	Vector of fractions $\omega_1(t), ..., \omega._N(t)$ invested in risky asset $S_1(t), ...S_N(t)$.
$\omega_0(t)$	Fraction of wealth invested in risk free asset at time t.
$\omega_i(t)$	Fraction of wealth invested in risky asset i at time t.
$\overline{v}$	Long term volatility under Heston dynamics.
Π_i	Portfolio value at time t_i.
$\mathbb{P}$	Objective probability measure
$\mathbb{Q}$	Risk neutral probability measure
$\Psi(\cdot)$	Cumulative standard normal distribution
$\rho(t)$	State price density or stochastic discount factor at time t.
$\rho_{x,v}$	Correlation between Brownian Motion W^x and W^v.
σ	Volatility of GBM price process.
σ_m	Volatility matrix including volatilities $\sigma_1, \sigma_2, ..., \sigma_N$ of risky assets $S_1, ..., S_N$.
Θ	Sigma-algebra on Ω with events $e \in \Theta$ in probability space (Ω, Θ, μ).
θ	Reference level to which outcomes are defined as gains or losses.
$\{p_i\}_{i=1}^n$	Chances of outcomes $x_1, ..., x_n$ of a gamble $g = (x_1, ..., x_n; p_1, ..., p_n)$.
$\{x_i\}_{i=1}^n$	Outcomes of a gamble $g = (x_1, ..., x_n; p_1, ..., p_n)$.
A	Event of terminal wealth being larger than the reference level $A = \{X(T) \geq \theta\}$.
a	Degree of risk aversion for gains in value function $v^+(x)$.
b	Degree of risk-seekingness for losses in value function $v^-(x)$.
$B(t)$	N-dimensional Brownian Motion vector with elements $B_1(t)$,...,$B_N(t)$.
c	Prospect option value of an European call at time $t = 0$.
c^h	Value of an European call option at time $t = 0$ from a holder's point of view according to cumulative prospect theory.

c^w Value of an European call option at time $t = 0$ from a writer's point of view according to cumulative prospect theory.

c_{ModTK} Cumulative prospect call option price under Moderate TK-sentiment from a writer's viewpoint.

c_{TK} Cumulative prospect call option price under TK-sentiment from a writer's viewpoint.

c_t^{BS} European call option price at time t according to Black-Scholes.

c_{zero} Cumulative prospect call option price under zero prospect sentiment from a writer's viewpoint.

CE_u Certainty equivalent for an investor with utility function u.

D_i Cash level at time t_i in a hedging strategy.

$F(\cdot)$ Cumulative distribution function

$f(\cdot)$ Probability density function

g A gamble $g \in M$ for which $g = (x_1, ..., x_n; p_1, ..., p_n)$.

G_X Quantile function of random variable X.

K Strike of an option.

K_T All contingent claims which can be replicated by a self financing portfolio with initial capital x_0.

L Lagrangian

M Choice set of random variables valued on outcomes $[x_1, ..., x_n]$.

P Probability measure $P : \Theta \to [0, 1]$ on probability space (Ω, Θ, μ).

p Prospect option value of an European put at time $t = 0$.

p^h Value of an European put option at time $t = 0$ from a holder's point of view according to cumulative prospect theory.

p^w Value of an European put option at time $t = 0$ from a writer's point of view according to cumulative prospect theory.

$r(t)$ Interest at time t.

r_f Risk free rate.

$R_u(g)$ Risk premium a gamble g for an investor with utility function u.

$r_u^{abs}(x)$ Coefficient of absolute risk-aversion (ARA).

$r_u^{rel}(x)$ Coefficient of relative risk aversion (RRA).

S Vector or risky asset prices $S_1, S_2, ..., S_N$.

s Risky asset price at time $t = 0$: $S(0) = s$

S_0 Risk free money market account

s_0 Riskless asset price at time $t = 0$: $S_0(0) = s_0$.

$S_{sent}(t)$ Asset path dynamics under prospect sentiment at time t.

S_t Stock price at time t.

$U(g)$ Expected utility defined for a gamble g: $U(g) = \sum_{i=1}^{n} p_i u(x_i)$ (discrete) and $U(g) = \int u(z) f_a(z) dz$ (continuous).

$u(x)$ Utility function representing the degree of satisfaction to an outcome x.

$v(x)$ Value function for outcome x which is given by $v^+(x)$ for gains and $v^-(x)$ for losses.

$V_{BS}(t, S_t)$ Black-Scholes price at time t.

$V_{CPT}(g)$ Discrete cumulative prospect value of a prospect $g = (x_1, ..., x_n; p_1, ..., p_n)$.

$V_{CPT}(X)$ Continuous cumulative prospect value.

V^h_{CPT} Cumulative prospect value from a holder's point of view.

V^w_{CPT} Cumulative prospect value from a writer's point of view.

V_{PT} Prospect value of a prospect $(x_1, ..., x_n; p_1, ..., p_n)$.

v_t Variance at time t under Heston dynamics.

$w(p)$ Weighting function which transforms objective probabilities p to subjective decision weights $w(p)$.

$w_+(p)$ Weighting function defined on probabilities of gains.

$w_-(p)$ Weighting function defined probabilities of losses.

$X(T)$ Value of total portfolio at time T.

$X^*(T)$ Optimal wealth at time T

x_0 Wealth at time $t = 0$.

y Lagrange multiplier

$Z(t)$ Radon-Nykodym derivative for changing from probability measure P to Q at time t

δ Constant that controls for overweighting in weighting function $w_-(p)$ in cumulative prospect theory.

Ψ_+ Derivative of weighting function w_+.

Ψ_- Derivative of weighting function w_-.

BS Black-Scholes.

CPT Cumulative Prospect Theory

EUT Expected Utility Theory

GBM Geometric Brownian Motion

k Vector of market prices of risk process or Sharp ratio $k_1, ..., k_N$ of risky assets $S_1,...,S_N$.

PT Prospect Theory

TK Tversky Kahneman sentiment

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Decision-making under risk

Within portfolio-management investors are exposed to decision-making under risk. decision-making under risk can be seen as choosing between the outcomes of gambles in which the decision-maker knows the distribution of the possible outcomes. In this context a decision-maker can have different attitudes to risk corresponding to his risk profile. For example, in case of choosing portfolios according to maximizing expected utility, the personal risk profile of an investor consists of risk-averse behavior. In this chapter several models that incorporate individual behavior in an investor's decision-making processes are considered. Firstly, expected utility theory is described which is the fundamental theory for decision-making under risk. Hereafter, irrational behavioral models are considered: prospect theory and cumulative prospect theory.

2.1. Expected utility theory

Expected utility theory (EUT) is a fundamental theory for modelling investor's preferences with respect to choices under risk. Choices under risk can be seen as choosing between the outcomes of gambles in which the decision-maker knows the distribution of the outcomes. Expected utility theory states that if an investor's preference satisfies a certain set of rational behavior hypotheses, then the investor takes decisions by maximizing expected utility. The utility represents the value assigned to an outcome; it represents the degree at which an investor is happy with the outcome and the degree of preference of that outcome to another outcome. The expected utility is computed as the sum of the utility of each outcome weighted by its probability of occurrence. In this section a description of the expected utility theory is given as described in [1].

Let (Ω, Θ, μ) be a probability space. The set Ω represents all possible outcomes, Θ is a sigma-algebra on Ω and $P : \Theta \to [0,1]$ is a probability measure. The probability of an event $e \in \Theta$ is represented by $P(e)$. Choices of an investor are identified by real random variables $g : (\Omega, \Theta) \to \mathbb{R}$ in which $\mathbb{R}$ is the space of outcomes or consequences.

Let M be a choice set of random variables valued on $[x_1, ..., x_n]$. For each $g \in M$ we have: $g \in M \iff g = (x_1, ..., x_n; p_1, ..., p_n)$ such that $p_i \geq 0$ for $i = 1, ..., n$ and $\sum_{i=1}^{n} p_i = 1$. The set M can be seen as a set of gambles or prospects g which yield outcomes $x_1, x_2, ..., x_n$ with probabilities $p_1, p_2, ..., p_n$ respectively.

The following notation is used in the remainder of this thesis:

- $x_1 \succeq x_2$: outcome x_1 is weakly preferred to x_2 or in other words x_1 is at least as preferred as x_2.
- $x_1 \sim x_2$: outcome x_1 and outcome x_2 are equally preferred.

In order to represent an investor's preference by an expected utility function, the preference relation needs to satisfy three general axioms: rationality, continuity and independence. These axioms are given below.

Axiom 1. Rationality
A preference relation is rational if it satisfies the following properties:

- Reflexivity: $\forall g \in M, g \succeq g$.
- Completeness: $\forall g_1, g_2 \in M, g_1 \succeq g_2$ or $g_2 \succeq g_1$. This property shows that either g_1 is at least as preferred as g_2 or g_2 is at least as preferred as g_2. Thus, an individual has the ability to choose between outcomes.
- Transitivity: $\forall g_1, g_2, g_3 \in M$ if $g_1 \succeq g_2$ and $g_2 \succeq g_3$ then $g_1 \succeq g_3$. This property represents the ranking of outcomes and thus ensures consistent decision-making.

Axiom 2. Continuity

A preference relation is continuous if $\forall g_1, g_2, g_3 \in M$ s.t. $g_1 \succeq g_2 \succeq g_3$, there exists a constant $\alpha \in [0,1]$ s.t. $\alpha g_1 + (1-\alpha)g_3 \sim g_2$. The continuity assumption ensures that for any gamble there exist probabilities such that the decision-maker equally prefers a combination of the most preferred and the least preferred gamble and the gamble between the most and least preferred.

Axiom 3. Independence
A preference relation is independent if $\forall g_1, g_2, g_3 \in M$ and $\forall \alpha \in [0,1]$:

$$g_1 \succeq g_2 \iff \alpha g_1 + (1-\alpha)g_3 \succeq \alpha g_2 + (1-\alpha)g_3.$$

The independence property shows that if two prospects are combined with a third, then the preference ordering of the mixture is independent of the third prospect.

The stated axioms together define a rational investor for which the main theory under EUT is defined and which is as follows:

Theorem 1. *Expected utility theory*
If a preference relation satisfies the rationality, continuity and independence conditions, then the decision-maker is rational and the preference can be represented by expected utility i.e. there exist n real numbers $u(x_i)$ s.t.

$$\forall g_1, g_2 \in M : g_1 \succeq g_2 \iff U(g_1) \geq U(g_2) \text{ where } U(g) = \sum_{i=1}^{n} p_i u(x_i)$$

In case of a continuous prospect g, the expected utility function is given by: $U(g) = \int u(z) f_g(z) dz$, in which $f_g(z)$ represents a probability density.

The expected utility theory states that if the axioms above are satisfied, then there exists a utility function such that an individual prefers gamble g_1 over gamble g_2 if and only if the expected utility of gamble g_1 is larger then the expected utility of gamble g_2. Thus, under expected utility theory decision-making is based on maximizing the expected utility. It is generally assumed that a utility function is twice continuously-differentiable, strictly increasing and strictly concave. The strictly increasing property of the utility function reflects that investors always prefer more wealth to less wealth and the strictly concave property reflects risk-aversion. The utility function can be defined either over the positive real numbers or over all real numbers.

2.1.1. ATTITUDES TOWARDS RISK

As seen, according to expected utility theory, decision-making is based not only on the probability of an outcome but also on the utility of an outcome. Individuals can have different attitudes towards risk which can be established through the utility function. The attitudes towards risk are related to two key concepts in expected utility theory: the certainty equivalent and the risk premium that are defined below.

Definition 1. *Certainty equivalent*
The certainty equivalent CE_u is the amount of money that an investor with utility function u considers as equally desirable as a risky gamble g:

$$u(CE(g)) = E[u(g)].$$

Definition 2. *Risk premium*
The risk premium R_u of a gamble g for an investor with utility function u is the maximum amount an investor is willing to pay to receive instead of the gamble, its expected value with certainty:

$$R_u(g) = E[g] - CE(g).$$

Related to the concepts of certainty equivalent and risk premium, individuals can have different attitudes towards risk. The different attitudes are risk averse, risk neutral and risk-seeking. These attitudes are related to preferences between certain and uncertain decisions and are defined as follows.

Definition 3. *Risk averse investor*
An investor is risk-averse if he always prefers a certain payment to an uncertain payment:

$$\forall g \in M, \quad E[u(g)] \leq u[E(g)].$$

A risk-averse individual attempts to minimize the degree of uncertainty: a sure and predictable payoff is preferred to an uncertain and unknown payoff with a higher expected value. An example of a risk-averse investor is one who decides to put his money into a bank account with a relatively low guaranteed interest rate rather than investing it into stocks which might result in a higher payoff but also involves the chance of losing money.

Definition 4. *Risk neutral investor*
An investor is risk neutral if he is always indifferent between a certain and an uncertain payment:

$$\forall g \in M \quad E[u(g)] = u[E(g)].$$

Definition 5. *risk-seeking investor*
An investor is risk-seeking if he always prefers an uncertain outcome over an certain payment:

$$\forall g \in M : E[u(g)] \quad \geq u[E(g)].$$

The different attitudes towards risk are related to the shape of the utility function and the concepts of certainty equivalent and risk-premium. The following propositions give the characteristics of the different risk attitudes.

Proposition 1. *Given an increasing utility function u and a random variable g, the following statements are equivalent:*

1. *An investor is risk averse;*
2. *The utility function u is concave;*
3. $CE_u(g) \leq E[g]$*;*
4. $R_u(g) \geq 0$.

Proposition 2. *Given an increasing utility function u and a random variable g, the following statements are equivalent:*

1. *An investor is risk-seeking;*
2. *The utility function u is convex;*
3. $CE_u(g) \geq E[g]$*;*
4. $R_u(g) \leq 0$.

The proofs of these theories can be found in [1].

Propositions 1 and 2 are visualized in Figure 2.1. Note that a risk neutral is between a risk-averse and risk-seeking individual: the utility function is linear, $CE_u(g) = E[g]$ and $R_u(g) = 0$.

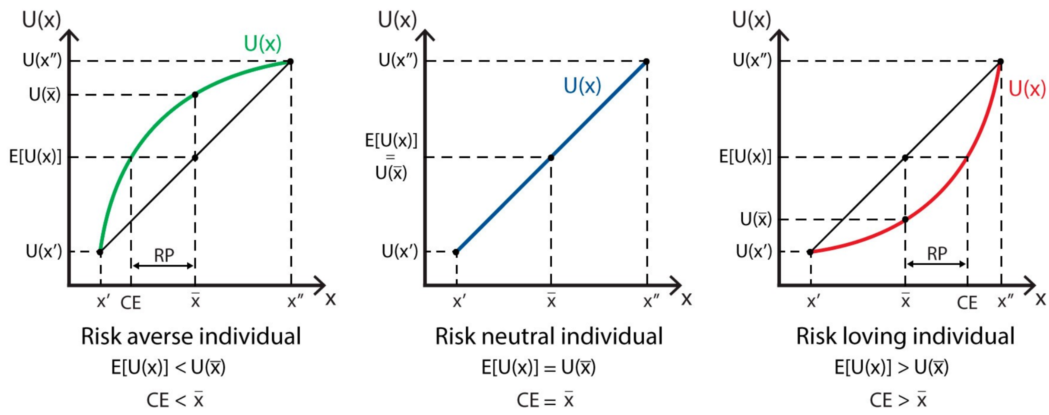


Figure 2.1: Different attitudes towards risk and corresponding properties of the risk premium (RP) and certainty equivalent (CE) (Source [2]).

As shown, investors can have different attitudes towards risk: risk averse, risk neutral or risk-seeking. The risk attitude is related to the curvature of utility function: risk averse investors have a concave utility function, risk neutral investors have a linear utility function and risk-seeking investors a convex utility function. The degree of risk-aversiness can be explained by the coefficient of absolute risk aversion (ARA) and the coefficient of relative risk aversion (RRA), which are defined below.

Definition 6. *Absolute risk-aversion*

The coefficient $r_u^{abs}(x) = -\frac{u''(x)}{u'(x)}$ *is called the coefficient of absolute risk-aversion (ARA). The absolute risk-aversion coefficient measures the risk aversion to a loss in absolute terms.*

Definition 7. *Relative risk-aversion*

The coefficient $r_u^{rel}(x) = -\frac{xu''(x)}{u'(x)}$ *is called the coefficient of relative risk-aversion (RRA). The relative risk-aversion coefficient measures the risk aversion to a loss relative to the current wealth.*

According to the character of the coefficient of absolute risk-aversion, the following classes can be distinguished:

- The utility function u has a constant absolute risk aversion coefficient (CARA): as wealth increases the investor will hold the same amount of risky assets.
- The utility function u has a decreasing absolute risk aversion coefficient (DARA): as wealth increases the investor will increase the amount of risky assets.
- The utility function u has a increasing absolute risk aversion coefficient (IARA): as wealth increases the investor will decrease the amount of risky assets.

According to the character of the coefficient of relative risk-aversion, the following classes can be distinguished:

- The utility function u has a constant relative risk aversion coefficient (CRRA): an investor will keep the same fraction of the portfolio invested in risky assets as wealth increases.
- The utility function u has a decreasing relative risk aversion coefficient (DRRA): an investor increases the fraction of the portfolio invested in risky assets as wealth increases.
- The utility function u has a increasing relative risk aversion coefficient (IRRA): an investor will decrease the fraction of the portfolio invested in risky assets as wealth increases.

2.1.2. CRITIQUES ON EXPECTED UTILITY THEORY

Expected utility theory has been the dominant approach for decision-making under risk for years. The model has been used as a descriptive model of economic behavior as well as a normative model to determine optimal decisions [3]. It relies on the assumption that investors are rational and thus their decisions are reasonable and predictable. In a perfect world, expected utility theory would be ideal, as it provides exact measurements of utility and gives perfect predictions [7]. However, results of experiments involving hypothetical choice problems have shown that risky decision-making systematically violates the principles of the expected utility theory as emotions influence the decision-making process. This leads to less predictable, inconsistent and irrational behavior [14]. In this section, first the principles of expected utility theory are summarized. Then several phenomena which are inconsistent with these principles are described as in [3] and [4].

PRINCIPLES OF EUT

The principles of EUT as discussed in Section 2.1 can be summarized as follows:

- Expected utility linear is in probabilities: $U(x_1, p_1;...;x_n, p_n) = \sum_i^n p_i u(x_i)$. The utility assigned to a prospect is the expected utility of the outcomes. This means that investors are able to evaluate probabilities objectively.
- The domain of a utility function consists of final asset positions instead of losses or gains. This means that the absolute value of an outcome is taken into account instead of the deviation of an outcome with respect to a reference point.
- Investors are always risk averse: a certain prospect with outcome x is preferred to a risky prospect with expected value $y \geq x$. This is equivalent with having a concave utility function ($u'' < 0$).

Before discussing phenomena that are inconsistent with the mentioned principles, remember M as the set of all prospect where a prospect is denoted as $(x_1, p_1;...;x_n, p_n)$. In this representation, x_i, with $i = 1,2,...n$, represent outcomes and p_i, with $i = 1,2,...,n$ represent the corresponding probabilities. For simplicity it is assumed that $x_1 \leq x_2 \leq ... \leq x_k \leq 0 \leq x_{k+1} \leq ... \leq x_n$. The outcome x_i is defined relative to reference point which is often taken as zero.

An overview of phenomena of irrational behavior that violate expected utility theory is provided below. These phenomena are the result of responses to hypothetical choice problems conducted by Kahneman and Tversky [6].

- **Non-linear decision weights**
  In expected utility theory it is assumed that expected utility is linear in probabilities. However, the Allais (1953) paradox shows that preferences can be better explained by non-linear weighted utilities. An example is the following preference: $(6000, 0.001; 0, 0.999) \succeq (3000, 0.002; 0, 0.998)$. A prospect with a probability of 0.001 of an outcome of 6000 and a probability of 0.999 of an outcome of 0 is preferred over a prospect with a probability of 0.002 of an outcome of 3000 and a probability of 0.998 of an outcome of 0. The preference shows that while both prospects have the same mean, the small probability of a large gain is exaggerated. This represents a risk-seeking attitude and explains why people participate in lotteries. The same phenomenon, namely the overweighting of small probability of large amounts, is observed in the domain of losses, which represents a risk-seeking behavior. In situations with high probabilities, an opposite effect is observed as in that case the higher probable gain is preferred: $(3000, 0.9; 0, 0.1) \succeq (6000, 0.45; 0, 0.1)$. This represents a risk averse attitude. The described examples show that subjective probabilities are evaluated differently from objective probabilities and thus non-linear decision weights are needed. The following two phenomena are in line with this effect.
- **Certainty effect / possibility effect**
  The certainty effect is another effect which violates the principles of EUT. This effect concerns the overweighting of sure outcomes relative to less sure outcomes. To illustrate this effect, the following example is considered. The results of experiments show that a sure gain of 900 is preferred over a risky prospect $(1000, 0.9; 10, 0.1)$, while there is a significant chance of a larger gain in the risky prospect and the expected values are equal. This example shows that certainty enlarges the desire of gains and as a consequence investors are risk averse in the domain of gains.

Another example of the certainty effect is shown by the Allais paradox as well: a change of probability from 0.10 to a 0.11 of winning has a smaller impact than a move from probability 0.99 to 1. Conversely, a move in probability from 0 to a 0.05 of a gain is preferred over a move from 0.05 to 0.1, the possibility effect. In other words, high probabilities are underweighted and low probabilities are overweighted. As in expected utility theory people should weight prospects by respective probabilities the described attitudes to risk can not be captured by the expected utility theory.

- **Reflection effect**
  The reflection effect is an effect related to the change in choice pattern in case all positive outcomes of a prospect are replaced by negative outcomes. In that case, the choice pattern takes the reversed form. The reflection effect implies that investors prefer risk-averse behavior in the domain of losses of low probability and risk-seeking behavior in the domain of losses of high probability. An important aspect of this effect is a reference point to which gains and losses are defined and which is described in the following point.

- **Dependence on a reference point**
  Under expected utility theory it is assumed that outcomes are considered as end positions. However, experiments have shown that people tend to think in terms of gains and losses relative to a reference point rather than outcomes in absolute terms. An example of an experiment which shows the dependence on a reference point is following:

  - **Experiment 1**

    You have a starting position of 1000.

    Choose between the following gambles: $g_{11} = (1000, 0.5; 0, 0.5)$ and $g_{12} = (500, 1)$.

  - **Experiment 2**

    You have a starting point of 2000.

    Choose between $g_{21} = (-1000, 0.5; 0, 0.5)$ and $g_{22} = (-500, 1)$.

  It turns out that in the first experiment $(500, 1) \succeq (1000, 0.5; 0, 0.5)$ while in the second experiment $(-1000, 0.5; 0, 0.5) \succeq (-500, 1)$. However, if we take the reference points into account, the gambles can be rewritten as follows: $g_{11} = (1000, 0.5; 2000, 0.5)$, $g_{12} = (1500, 1)$, $g_{21} = (1000, 0.5; 2000, 0.5)$ and $g_{22} = (1500, 1)$. This shows that gambles g_{11} and g_{21} are exactly the same and g_{21} and g_{22} as well. However, the different reference points lead to opposite preferences.

- **Loss aversion**
  Loss aversion refers to the effect that the impact of losses is greater than the impact of comparable gains: the experienced displeasure of a certain loss is greater than the pleasure of a gain of the same amount. In terms of prospects: $(y, 0.5; -y, 0.5) \succeq (x, 0.5; -x, 0.5)$ with $x > y \geq 0$ gains with respect to a reference point. The following two effects are related to the description and interpretation of decision problems.

- **Framing effect**
  In rational decision theory it is assumed that different formulations of the same choice problem should give the same preferences. However, experiments have shown that the framing of a decision problem for example in terms of gains or losses yields different preference orders. This shows that the way in which a choice problem is presented influences the choice made.

- **Heuristics**
  In order to simplify complex choices investors use heuristics such as disregarding common components and other editing operations which will be described in the following section. As problems can be represented in various ways and different representations influence the choice of heuristics, this can lead to inconsistent preferences.

2.2. PROSPECT THEORY

As seen, investor's behavior has shown several effects which are inconsistent with the principles of expected utility theory and which show that expected utility is not able to capture investor's actual behavior in risky situations. Therefore, Kahneman and Tversky [3] introduced prospect theory as an alternative model for describing decision-making under risk. Prospect theory (PT) takes into account the observed violations and incorporates irrational behavior. This theory describes in which way people choose between options and how they estimate (often in a biased way) the perceived likelihood of these options [5]. While expected utility theory is a normative model which describes how decision-makers should behave ideally, prospect theory is a descriptive model: it models how people actually choose between options and how they estimate the perceived likelihood of these options [5]. In this chapter, prospect theory is described as in [3] and [4].

Two important assumptions in prospect theory based on the described inconsistencies with expected utility are the following: individuals are risk-averse in the domain of gains and risk-seeking in the domain of losses and individuals tend to overweight low probabilities and underweight high probabilities. These two phenomena lead to the so-called fourfold pattern of risk attitudes [4]. When individuals are exposed to risky decision they have:

- A risk-averse attitude in case of **losses** of **low** probability. In case of losses of low probability people behave risk-averse; they have a fear to lose money and therefore are willing to pay for the certainty of losing nothing. This may explain why people buy insurances: a premium is paid in exchange for losing nothing for certain.
- A risk-averse attitude for **gains** of moderate and **high** probability. A sure gain is preferred over a prospect with an higher expected value. This attitude represents risk aversion; there is a fear of the disappointment of ending up with nothing.
- A risk-seeking attitude in the domain of **gains** of **low** probability. In cases of gains of low probability people are willing to pay a premium for the small possibility of a large gain. There is hope for a large gain which may explain why people participate in lotteries. This attitude represents risk-seeking behavior.
- A risk-seeking attitude in the domain of **losses** of **high** probability. In case of losses of high probability people are willing to take risk in order to avoid a sure loss.

This fourfold pattern is summarized in Table 2.1.

Table 2.1: Fourfold pattern of risk-attitudes.

	Low probability	**High probability**
Gains	Risk-seeking (lottery)	Risk-averse (gambling for gains)
Losses	Risk-averse (insurance)	Risk-seeking (gambling for losses)

The fourfold pattern of risk attitude is incorporated in prospect theory by the use of a value function and a weighting function.

- **Value function**
  The utility function from expected utility is replaced by a value function which describes the risk-attitude and the degree of loss-aversion. The value function is defined on changes (gains and losses) relative to a reference point contrary to expected utility theory in which the domain consists of final asset positions. This implies that an individual can make different choices in different situations dependent on the determined reference point. The value function is convex for gains and concave for

losses which implies an S-shape. Also, the value function is steeper for gains and less steep for losses which refers to loss-aversion.

- **Weighting function**
  While in expected utility theory the attitude towards risk is solely determined by a utility function, in prospect theory the investor's behavior is jointly determined by a value function and a weighting function. It is assumed that probabilities influence the evaluation of outcomes and therefore the decision-making. The weighting function transforms objective probabilities into subjective decision weights that reflect the attitude towards probabilities. While the utility function in expected utility theory is a linear function of outcome probabilities, the weighting function in prospect theory is non-linear.

The exact properties and formulations of both the value and weighting function will be elaborated later on. Firstly, the different stages of the decision-making process are described which is in addition to the use of the value- and weighting function another key element in prospect theory. The decision-making process can be described in two phases: an editing phase and an evaluation phase. The editing phase consists of an analysis and reformulation of prospect by using heuristics. The editing phase is meant to simplify the choice. In the evaluation phase the outcomes are evaluated and the prospect of highest value is chosen. These phases are described below.

Editing phase
In the editing phase decision-makers identify the outcomes and corresponding probabilities and restructure the problem by applying operations which transform the probabilities and outcomes of a prospect. The main mental operations in the editing phase are coding, combination, and detection of dominance.

- *Coding*

  Coding involves the determination of a subjective reference point to which outcomes can be framed as gains or losses. The identification of a reference point is also affected by the framing of the problem.

- *Combination*

  Combination is an operation in which prospects are simplified by combining probabilities corresponding to identical outcomes. For example, investors will reduce a prospect $(100, 0.25; 100, 0.25)$ to $(100, 0.5)$.

- *Segregation*

  Segregation is an operation that separates the riskless component and the risky component of a prospect. For example, a prospect (250, .80; 150, .20) is decomposed into a sure gain of 150 and the risky prospect (100, .80).

- *Cancellation*

  Operation in which the common components of prospects are ignored. For example, the choice between (150, .20; 50, .50; -100, .30) and (150, .20; 100, .50; -150, .30) can be reduced to a choice between (50,.50; -100,.30) and (100, .50; -150, .30).

- *Simplification*

  Simplification is an operation that rounds the outcomes and probabilities of the prospects.

- *Detection of dominance*

  Detection of dominance is an operation in which dominant prospects are detected and eliminated without valuation as they are worse in all perspectives. In other words, prospects that are worse in all aspect compared to the alternatives are neglected directly.

As decision problems can be represented in various ways, the same problem can be edited in different ways. However, in the remainder of this thesis the focus lies on the evaluation of prospects as it is assumed that the evaluation phase is carrying the main responsibility for the described irrational human behavior.

Evaluation phase

In the evaluation phase a decision-maker evaluates the edited prospects and selects the prospect of highest overall value. The overall value $V_{PT}(g)$ of a prospect g depends on two functions: the outcomes (gains or losses) are interpreted by a subjective value function v and the probabilities are interpreted by a weighting function w. Prospects can be categorized in three classes: strictly positive, strictly negative and regular prospect prospects that are defined for prospect of the form $(x_1, p_1; x_2, p_2)$ with at most two non-zero outcomes.

Definition 8. *Strictly positive prospect*
A prospect is strictly positive, if $x_1, x_2 > 0$ *and* $p_1 + p_2 = 1$.

Definition 9. *Strictly negative prospect*
A prospect is strictly negative, if $x_1, x_2 < 0$ *and* $p_1 + p_2 = 1$.

Definition 10. *Regular prospect*
A prospect is regular if it is neither strictly positive nor strictly negative: either $p_1 + p_2 < 1$ *or* $x_1 \geq 0 \geq x_2$ *or* $x_1 \leq 0 \leq x_2$.

If a prospect $(x_1, p_1; x_2, p_2)$ is regular then the value of the prospect equals:

$$V_{PT}(x_1, p_1; x_2, p_2) = w(p_1)v(x_1) + w(p_2)v(x_2), \tag{2.1}$$

with $v(0) = w(0) = 0$ and $w(1) = 1$. The function v represents the value function and the function w represents the weighting function which will be described in Section 2.2.1 and Section 2.2.2. Equation (2.1) can be extended to prospects with n outcomes $(x_1, p_1; x_2, p_2, ...x_n, p_n)$ and which generalizes expected utility theory:

Definition 11. *Discrete prospect value*
The discrete prospect value of a prospect $(x_1, p_1, x_2, p_2, ..., x_n, p_n)$ *is defined as:*

$$V_{PT}(x_1, p_1; x_2, p_2, ..., x_n, p_n) = \sum_{i=1}^{n} w(p_i)v(x_i). \tag{2.2}$$

Note that V_{PT} is defined on prospects while v is defined on outcomes.

2.2.1. VALUE FUNCTION

As seen, a key feature in prospect theory is the use of a value function which replaces the utility function of EUT and describes the subjective evaluation of gains and losses. The results of experiments conducted by Kahneman and Tversky [3] showed that the value function v needs to satisfy three properties:

1. Utility is defined by changes in outcomes (gains/losses) relative to a reference point instead of absolute wealth positions as in expected utility theory. This reference point corresponds to the origin of the value function. The identification of a reference point during the coding phase and thus the definition of gains and losses influences the value for the reason that gains and losses are valuated differently.

2. As individuals have different behavior for gains and losses, the value function is concave for gains and convex for losses and thus S-shaped. This implies a risk-averse attitude in the domain of gains ($v'' < 0$) and a risk-seeking attitude in the domain of losses ($v'' > 0$). The S-shape represents diminishing sensitivity: changes around the reference point have more impact than comparable changes further away from the reference point.

3. The value function is steeper for losses than for gains which reflects that the aversion of losses is greater then the pleasure of comparable gains: the marginal utility of winning 1 euro is lower than the marginal disutility of losing 1 euro. As a consequence, investors take more risk to avoid a certain loss than to obtain a comparable gain.

   In Figure 2.2 a hypothetical value function is given.

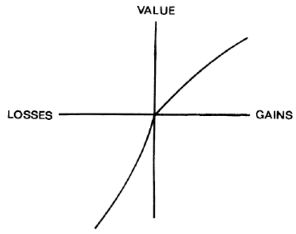


Figure 2.2: Hypothetical value function (Source [6]).

A value function $v(x)$, which has the properties described and which will be used in this thesis is defined as follows:

Definition 12. *Prospect theory value function*
The prospect value function is defined as follows:

$$v(x) = \begin{cases} v^+(x) = (x-\theta)^a & x \geq \theta \\ v^-(x) = -\lambda(\theta - x)^b & x < \theta \end{cases} \tag{2.3}$$

The PT-value function satisfies the following properties:

- Parameter $\lambda > 1$ represents the loss-aversion parameter. As $\lambda > 1$ the value function is steeper for losses than for gains. A larger value λ implies a larger degree of loss aversion.
- Parameters $0 < a \leq 1$ and $0 < b \leq 1$ are the risk attitude parameters. A larger value of a implies a lower risk averse attitude in the domain of gains and a larger value of b implies a lower risk-seeking attitude in the domain of losses. This can be derived by considering the second derivative of the value function, for example, for gains:
$$v''_+(x) = a(a-1)\left(\frac{1}{x-\theta}\right)^{2-a} < 0$$
The higher the risk aversion parameter a, the less negative $v''_+(x)$ and thus the less risk averse. This holds similarly for the risk-seeking parameter b. Note that in case $a = b = \lambda = 1$ risk and loss neutrality are implied.
- Parameter θ represents the reference point, which usually taken as zero.
- The function is twice differentiable and strictly increasing, representing the phenomenon that more wealth leads to a higher value.

In Figure 2.3 a value function is given for different values a, b and λ.

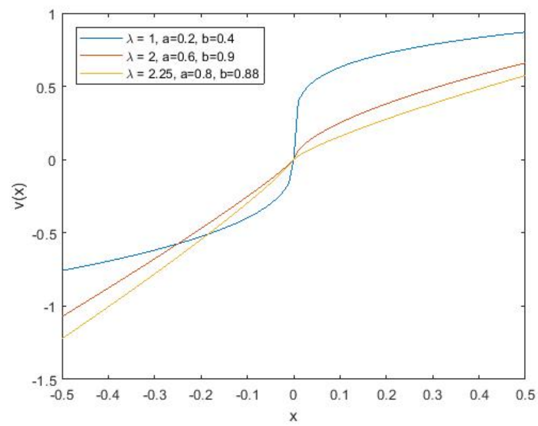


Figure 2.3: Value function (2.3) for different values of a, b and λ.

2.2.2. WEIGHTING FUNCTION

Under prospect theory a weighting function is used to transform probabilities p: $w(p) : [0,1] \to [0,1]$. Objective probabilities p_i are separately transformed to decision weights $w(p_i)$, that can be seen as individuals' subjective attitude towards probabilities. In other words, the decision weights represent the perception of actual probabilities.

A weighting function under PT is based on the following principles:

- Diminishing sensitivity: very low probabilities are overweighted and large probabilities are underweighted as described in the certainty/possibility effect. This results in a weighting function which is concave near zero (risk-seeking) and convex near one (risk-aversion). Therefore, the weighting function is an inverse S-shaped function.
- Risk-seeking behavior: people have a risk-seeking attitude in situations where the probability of winning is low, e.g. lotteries.

The weighting function $w(p)$ based on these principles is a monotonic and non-linear transformation of the probability measure. In Figure 2.4 a hypothetical probability weighting function is displayed which satisfies the described properties above. The weighting function shows the overweighting of low probabilities, the underweight of large probabilities and probabilities zero and one are perceived objectively: $w(0) = 0$ and $w(1) = 1$. In addition, it shows a concave shape near zero, a convex shape near one and the indifference for probability shift in the middle region which means that the subjective probability coincides with the objective probability.

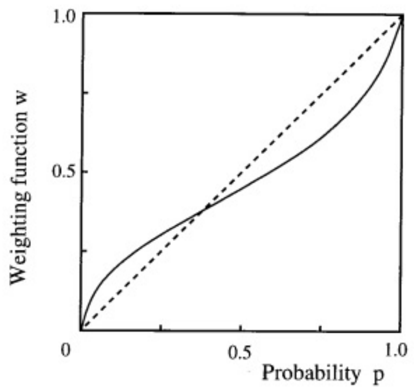


Figure 2.4: Hypothetical weighting function (Source: [6]).

A weighting function which satisfies the properties described and which will be used is the following:

Definition 13. *Prospect theory weighting function*
The prospect theory weighting function $w : [0,1] \to [0,1]$ *with* $w(0) = 0$ *and* $w(1) = 1$ *is defined as follows:*

$$w(p) = \frac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^{\frac{1}{\gamma}}}. \tag{2.4}$$

The weighting function satisfies the following properties:

- $0.28 < \gamma < 1$ is a constant that controls the over- and underweighting of small and large probabilities. The value $\gamma > 0.28$ ensures an increasing function.
- The weighting function is an increasing function in p and is concave.
- The lower the value of γ, the higher the degree of over- and underweighting.

Figure 2.5 shows the probability weighting function for different values of parameter γ.

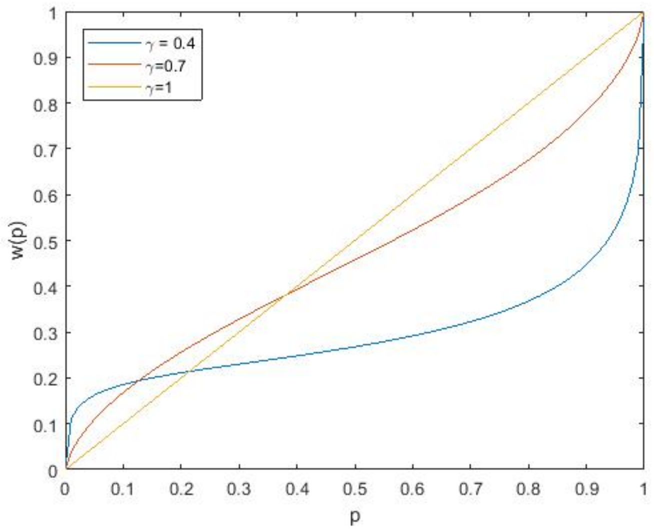


Figure 2.5: Probability weighting function (2.4) for parameter values $\gamma = 0.4$, $\gamma = 0.7$ and $\gamma = 1$.

2.3. CUMULATIVE PROSPECT THEORY

In 1992, Kahnemman and Tversky introduced an updated version of prospect theory called cumulative prospect theory (CPT) which will be described in this section according to [6]. Cumulative prospect theory overcomes the violation of stochastic dominance in prospect theory. Stochastic dominance requires that if better outcomes become more probable and worse outcomes become less probable this results in an improved prospect. Also, cumulative prospect theory is applicable to prospects with any number of outcomes and to uncertain as well as to risky prospects.

The main difference between prospect theory and cumulative prospect theory is the determination of decision weights. In cumulative prospect theory this is done by transforming the entire cumulative distribution function (CDF) instead of transforming all probabilities separately. The idea is to apply the non-linear cumulative weighting function proposed by Quiggin (1982) to gains and losses separately which allows for different attitudes towards gains and losses. This way, the weighting function can explain non-linear preferences. In cumulative prospect theory a weighting function $w_+(p)$ is defined for probabilities of positive outcomes and a weighting function $w_-(p)$ is defined for losses.

Consider a risky prospect $(x_1,...,x_n,p_1,...,p_n)$ for which $x_1 \leq x_2 \leq ... \leq x_k \leq 0 \leq x_{k+1} \leq ... \leq x_n$. The discrete CPT-value of the risky prospect with $2 \leq i \leq k$ is in this case given by:

Definition 14. *Discrete CPT prospect value*
The discrete CPT prospect value is given by:

$$V_{CPT}(x_1,..,x_n,p_1,...,p_n) = \sum_{i=1}^{k} v^-(x_i)\left(w_-(\sum_{j=1}^{i} p_j) - w_-(\sum_{j=1}^{i-1} p_j)\right) + \sum_{i=k+1}^{n} v^+(x_i)\left(w_+(\sum_{j=1}^{n} p_j) - w_+(\sum_{j=i+1}^{n} p_j)\right) \tag{2.5}$$

The quantities $w_+(p)$ and $w_-(p)$ represent strictly increasing functions from $[0,1]$ to $[0,1]$ with $w_+(0) = w_-(0) = 0$ and $w_+(1) = w_-(1) = 1$. The functions $v^-(x_i)$ and $v^+(x_i)$ represent the value function for losses and gains respectively. The exact forms of these function will be discussed below in 2.3.1.
As can be seen, the decision weights are transformed cumulative probabilities of gains and losses that represent the risk attitude towards probabilities. Note the following definitions of the boundary cases:

- In case of $i = 1$ the weighting term $w_-(\sum_{j=1}^{i} p_j) - w_-(\sum_{j=1}^{i-1} p_j)$ is defined as $w_-(p_1)$,
- In case of $i = n$ the weighting term $w_+(\sum_{j=1}^{n} p_j) - w_+(\sum_{j=i+1}^{n} p_j)$ is defined as $w_+(p_n)$.

The decision weight $w_+(\sum_{j=1}^{n} p_j) - w_+(\sum_{j=i+1}^{n} p_j)$ denotes the difference between the transformation of a probability of choosing a outcome at least as good as x_i and the transformation of a probability of choosing an outcome better than outcome x_i. Equivalently, $w_-(\sum_{j=1}^{i} p_j) - w_-(\sum_{j=1}^{i-1} p_j)$ denotes the difference between the transformation of a probability of choosing an outcome at least as bad as x_i and the transformation of a probability of choosing a worse outcome than x_i. Thus, the decision weights describe the marginal contribution of an event in terms of the functions w_+ and w_-. In other words, in the evaluation of prospects the ranking of outcomes compared to other outcomes is of importance.

2.3.1. VALUE FUNCTION

Before turning to the weighting function, first the value function under CPT is considered. In cumulative prospect theory the value function suggested has the same form as in prospect theory:

Definition 15. *Cumulative prospect theory - value function*

$$v(x) = \begin{cases} v^+(x) = (x-\theta)^a, & x \geq \theta, \\ v^-(x) = -\lambda(\theta - x)^b, & x < \theta. \end{cases} \tag{2.6}$$

In this function, $\lambda \geq 1$ is the loss-aversion parameter and $0 < a \leq 1$ and $0 < b \leq 1$ are the risk attitude parameters. A larger value λ implies a larger degree of loss aversion, a larger value of a implies a lower risk averse attitude in the domain of gains and a larger value of b implies a lower risk-seeking attitude in the domain of losses. Parameter $\theta > 0$ represents the reference level, which is usually taken as zero.

2.3.2. WEIGHTING FUNCTION

As a weighting function which reflects the mentioned properties , Kahneman and Tversky (1992) suggest the following weighting functions:

Definition 16. *Cumulative prospect theory - weighting functions The weighting function for gains and losses under CPT are as follows:*

$$w_+(p) = \frac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^{\frac{1}{\gamma}}}, \tag{2.7}$$

$$w_-(p) = \frac{p^\delta}{(p^\delta + (1-p)^\delta)^{\frac{1}{\delta}}}. \tag{2.8}$$

Note that the weighting function is equivalent to the weighting function of prospect theory, but now separately defined for gains and losses. Thus, in these formulas $0.28 < \gamma, \delta < 1$ are constants which represent the overweighting of small probabilities and underweighting of large probabilities. The lower the parameters the higher the curvature.

3
Option pricing

3.1. Option pricing under Black-Scholes

Before turning to option pricing under irrational human behavior, the classical Black-Scholes (1973) framework for option pricing is discussed [8]. Under Black-Scholes it is assumed that the price stochastic process equals:

$$dS_t = \alpha S_t dt + \sigma S_t dW_t^{\mathbb{P}} \tag{3.1}$$

The price dynamics of the bank account equals:

$$dB_t = rB_t dt. \tag{3.2}$$

Furthermore, the following assumptions are made:

- Interest rate r and volatility σ are known,
- There are no transaction costs,
- Assets can be bought or sold in any amount and in continuous time,
- There is no dividend paying,
- Short selling is allowed,
- There are no arbitrage possibilities.

The Black-Scholes price at time t is represented by $V_{BS}(t, S_t)$. Then, the following portfolio is considered with Δ_t the amount of continuously traded stocks:

$$\Pi(t, S_t) = V_{BS}(t, S_t) - \Delta_t S_t. \tag{3.3}$$

By use of Itô's lemma, the dynamics of portfolio $\Pi(t, S_t)$ can be rewritten as:

$$\begin{aligned} d\Pi &= dV_{BS} - \Delta_t dS_t \\ &= \frac{\partial V_{BS}}{\partial S_t} dS_t + \frac{\partial V_{BS}}{\partial t} dt + \frac{1}{2}\frac{\partial^2 V_{BS}}{dS_t^2} dS_t^2 - \Delta_t dS_t \\ &= \left(\alpha S_t \frac{\partial V_{BS}}{\partial S_t} + \frac{\partial V_{BS}}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V_{BS}}{\partial S_t^2}\right) dt + \sigma S_t \frac{\partial V_{BS}}{\partial S_T} dW_t - \Delta_t dS_t \\ &= \left(\alpha S_t \frac{\partial V_{BS}}{\partial S_t} + \frac{\partial V_{BS}}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V_{BS}}{\partial S_t^2}\right) dt + \sigma S_t \frac{\partial V_{BS}}{\partial S_T} dW_t - \Delta_t(\alpha S_t dt + \sigma S_t dW_t). \end{aligned} \tag{3.4}$$

Then, the Black-Scholes equation is derived by the following assumptions:

$$\Delta_t = \frac{\partial V_{BS}}{\partial S_t}, \tag{3.5}$$

$$d\Pi = r\Pi dt. \tag{3.6}$$

Then, Equation (3.4) reduces to:

$$r\Pi dt = \left(\alpha S_t \frac{\partial V_{BS}}{\partial S_t} + \frac{\partial V_{BS}}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V_{BS}}{\partial S_t^2}\right) dt - \frac{\partial V_{BS}}{\partial S_t}\alpha S_t dt, \tag{3.7}$$

which can be written as:

$$\frac{\partial V_{BS}}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V_{BS}}{\partial S_t^2} - r\Pi = \frac{\partial V_{BS}}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V_{BS}}{\partial S_t^2} - r\left(V_{BS} - \frac{\partial V_{BS}}{\partial S_t} S_t\right). \tag{3.8}$$

Finally, the Black-Scholes equation equals:

$$\frac{\partial V_{BS}}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V_{BS}}{\partial S_t^2} + \frac{\partial V_{BS}}{\partial S_t} S_t - rV_{BS} = 0. \tag{3.9}$$

A solution of the Black-Scholes equation for a European call is summarized in the following theorem [8]:

Theorem 2. *The price of a European call $c_t^{BS}(t)$ at time t and with strike K and maturity T equals:*

$$c_t^{BS}(t) = S_t N(d_1) - K\exp(-r(T-t))N(d_2), \tag{3.10}$$

with

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}(T-t)\right)}{\sigma\sqrt{T-t}}, \tag{3.11}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}, \tag{3.12}$$

with N the cumulative standard normal distribution function, i.e., $N(0,1)$.

Note that the Black-Scholes model could also be derived consistent with Section 2.1 by assuming an agent maximizing an exponential utility function of wealth, which has the property of constant absolute risk aversion preferences. However, the derivation previously discussed is more common.

3.2. OPTION PRICING UNDER PROSPECT THEORY

It is a well known fact that market option prices systematically deviate from Black-Scholes option prices. Within the Black-Scholes framework it is assumed that investors are rational and take decisions consistent with maximizing expected utility. However, as described in Section 2.1.2, observations have shown that investors actually behave irrational, which might be a possible cause for the deviations of option prices from Black-Scholes prices. In this section option pricing under prospect theory is described, as in [10]. It is assumed that the marginal investor values options under prospect theory. Within this framework the investor's attitude towards risk, loss aversion and subjective probabilities are incorporated.

3.2.1. THEORY

Consider a prospect with two possible outcomes: x_1 which is negative with probability p_1 and x_2 which is non-negative with probability p_2. Then, the prospect value of $g = (x_1, p_1; x_2, p_2)$ equals:

$$V_{PT}(g) = w(p_1)v^-(x_1) + w(p_2)v^+(x_2), \tag{3.13}$$

in which v is given by Equation (2.3) and w is given by Equation (2.4).

Before turning to the derivation of option pricing formulas under prospect theory, first the framework in which option pricing under prospect theory takes place is considered. It should be noted that this framework is not in line with the traditional framework considered in Section 3.1. In the prospect option pricing framework it is assumed that the price of an option is the result of the trading actions by participants in the market, rather than the cost of hedging as under Black-Scholes. The prospect option price is determined by marginal investors who are valuating according to prospect theory. Within this framework naked options are considered, which means that the writer's position is not covered by owning the underlying asset: an unhedged position. However, it is assumed that the writer can invest the received premium at the risk free rate. Then, the option price is determined such that the prospect value of the premium invested at the risk free rate at time T equals the subjective value of a potential loss at time T.

Consider a marginal investor who is writing a European call option on a non-dividend paying stock. At maturity ($t = T$) two states are possible. The first state is that the price of the underlying asset is higher than the exercise price ($S_T > K$) and the option will be exercised.

The probability of being exercised (p_1) is given by:

$$p_1 = P(S_T > K) = \int_K^{\infty} f(S_T) dS_T, \tag{3.14}$$

where K is the strike price, S_T the asset price at time T and $f(S_T)$ the probability density associated with the underlying asset at time T.

The expectation of the option conditional on exercising x_1 follows from the law of total expectation:

$$\begin{aligned} x_1 &= -E[max(S_T - K, 0)|S_T > K] \\ &= -\frac{E[max(S_T - K, 0)]}{P(S_T > K)} \\ &= -\frac{\int_K^{\infty}(S_T - K) f(S_T) dS_T}{\int_K^{\infty} f(S_T) dS_T}. \end{aligned} \tag{3.15}$$

The second possible outcome is that the price of the underlying asset is lower than the exercise price ($S_T \leq K$) and no exercise will take place. The probability of no exercise equals $p_2 = 1 - p_1$ and has payoff $x_2 = 0$. When an option writer evaluates a call option, x_1 is seen as a potential loss and Equation (3.13) can be simplified by using $v^+(0) = 0$:

$$\begin{aligned} V_{PT} &= w(p_1)v^-(x_1) + w(p_2)v^+(x_2) \\ &= w(p_1)v^-(x_1) + w(p_2)v^+(0) \\ &= w(p_1)v^-(x_1). \end{aligned} \tag{3.16}$$

In order to compensate for the potential loss at expiration, the writer receives an option premium c at time $t = 0$. Assuming that the writer can invest this amount at the risk-free rate its value at T equals $ce^{r_f T}$. In equilibrium the prospect value of the invested amount c at T should be equal to the prospect value of x_1 at T:

$$v^+(ce^{r_f T}) + w(p_1)v^-(x_1) = 0. \tag{3.17}$$

The substitution of the value function (2.3) into Equation (3.17) gives:

$$v^+(ce^{r_f T}) + w(p_1)v^-(x) = (ce^{r_f T})^a - w(p_1)\lambda(-x_1)^b = 0. \tag{3.18}$$

Then, the option value c at $t = 0$ equals:

$$c = e^{-r_f T}\left(w(p_1)\lambda(-x_1)^b\right)^{1/a}. \tag{3.19}$$

If the option value as in Equation (3.19) is considered all parameters are known, except for the quantities p_1 and x_1. The quantities p_1 and x_1 can be determined in case the underlying price process is known. For now it is assumed that the underlying asset price follows a Geometric Brownian Motion with drift α and volatility σ and thus the density of the underlying asset price equals:

$$f(S_T) = \exp(-(\log(S_T/S_0) - (\alpha - \sigma^2/2)T)^2/2\sigma^2 T)/S_T\sigma\sqrt{2\pi T}. \tag{3.20}$$

Under GMB, the quantity p_1 can be written as:

$$\begin{aligned} p_1 &= \int_K^{\infty} f(S_T) dS_T \\ &= \int_K^{\infty} \exp(-(\log(S_T/S_0) - (\alpha - \sigma^2/2)T)^2/2\sigma^2 T)/S_T\sigma\sqrt{2\pi T} dS_T \\ &= \frac{1}{\sqrt{2\pi}} \int_{log(K/S_0)-(\alpha-\sigma^2/2)T)/\sigma\sqrt{T}}^{\infty} e^{-\frac{1}{2}x^2} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{log(S_0/K)+(\alpha-\sigma^2/2)T)/\sigma\sqrt{T}} e^{-\frac{1}{2}x^2} dx \\ &= \Phi(\delta_{-1}). \end{aligned} \tag{3.21}$$

where Φ represents the cumulative standard normal distribution with argument $\delta_{-1} = (\log(S_0/K) + (\alpha - \sigma^2/2)T)/\sigma\sqrt{T}$. The transformation used is $x = log((S_T/S_0) - (\alpha - \sigma^2)T)/\sigma\sqrt{T}$, for which $dx = \frac{1}{\sigma\sqrt{T}S_T}dS_T$ and thus $dS_T = \sigma\sqrt{T}S_T dx$.

The equation for x_1 can be rewritten as follows:

$$\begin{aligned}
x_1 &= \frac{-\int_K^\infty (S_T - K)f(S_T)dS_T}{\int_K^\infty f(S_T)dS_T} \\
&= -\frac{\int_K^\infty (S_T - K)f(S_T)dS_T}{\Phi(\delta_{-1})} \\
&= \frac{K}{\Phi(\delta_{-1})}\int_K^\infty f(S_T)dS_T - \frac{1}{\Phi(\delta_{-1})}\int_K^\infty S_T f(S_T)dS_T \\
&= K - \frac{1}{\Phi(\delta_{-1})}\int_K^\infty \exp(-(\log(S_T/S_0) - (\alpha - \sigma^2/2)T)^2/2\sigma^2 T)/\sigma\sqrt{2\pi T}dS_T \\
&= K - \frac{1}{\Phi(\delta_{-1})}\int_{\log(K/S_0)}^\infty \exp\left(-\frac{1}{2}\left(\frac{y - (\alpha - \sigma^2/2)T}{\sigma\sqrt{T}}\right)^2\right)\frac{S_0 e^y}{\sigma\sqrt{2\pi T}}dy \\
&= K - \frac{1}{\Phi(\delta_{-1})}\frac{S_0}{\sigma\sqrt{2\pi T}}e^{(\alpha-\sigma^2/2)T+\frac{1}{2}\sigma^2 T}\int_{\log(K/S_0)}^\infty \exp\left(-\frac{1}{2}\left(\frac{y - ((\alpha - \sigma^2/2)T + \sigma^2 T)}{\sigma\sqrt{T}}\right)^2\right)dy \\
&= K - \frac{1}{\Phi(\delta_{-1})}\frac{S_0}{\sigma\sqrt{2\pi T}}e^{\alpha T}\int_{-\infty}^{\log(S_0/K)} \exp\left(-\frac{1}{2}\left(\frac{-y + ((\alpha - \sigma^2/2)T + \sigma^2 T)}{\sigma\sqrt{T}}\right)^2\right)dy \\
&= K - \frac{1}{\Phi(\delta_{-1})}S_0 e^{\alpha T}\Phi\left(\frac{\log(S_0/K) + (\alpha - \sigma^2/2)T + \sigma^2 T}{\sigma\sqrt{T}}\right) \\
&= K - \frac{1}{\Phi(\delta_{-1})}S_0 e^{\alpha T}\Phi\left(\frac{\log(S_0/K) + (\alpha + \sigma^2/2)T}{\sigma\sqrt{T}}\right) \\
&= K - S_0 e^{\alpha T}\frac{\Phi(\delta_1)}{\Phi(\delta_{-1})},
\end{aligned} \tag{3.22}$$

with

$$\delta_1 = (\log(S_0/K) + (\alpha + \sigma^2/2)T)/\sigma\sqrt{T}. \tag{3.23}$$

Note the transformation $y = log(S_T/S_0)$, $dS_T = S_0 e^y dy$ is used.

The substitution of Equations (3.21) and (3.22) in Equation (3.19) gives the final option value c:

$$\begin{aligned}
c &= e^{-r_f T}\left(w(p_1)\lambda(-x_1)^b\right)^{1/a} \\
&= e^{-r_f T}\left(w(\Phi(\delta_{-1}))\lambda(S_0 e^{\alpha T}\frac{\Phi(\delta_1)}{\Phi(\delta_{-1})} - K)^b\right)^{1/a}.
\end{aligned} \tag{3.24}$$

3.2.2. Relationship with Black-Scholes

In this section the relationship of the derived Equation (3.24) with the Black-Scholes pricing formula as described in Section 3.1 is presented. The way of evaluating options under prospect theory relies on the idea of taking the expected payoff as the value of an option. This method fits in the Black-Scholes framework in case parameters are chosen in a certain way. In this section the relationship between the two methodologies is described [12].

Suppose a European call option with exercise price K and as underlying price process a Geometric Brownian Motion with drift α and volatility σ, see Equation (3.20). The time-zero value of the option according to the discounted expected payoff method equals:

$$\begin{aligned}
c &= e^{-rT}E^{\mathbb{P}}[max(S_T - K, 0)] \\
&= e^{-rT}\int_K^\infty (S_T - K)f(S_T)dS_T.
\end{aligned} \tag{3.25}$$

The option value according to Equation (3.25) is of importance for investors holding or writing naked options. In contrast to Black-Scholes, this formula is derived without any idea of hedging in order to eliminate risk, without any assumption about the no-arbitrage principle and is dependent on the drift α. This means that this method can not be used to compute a unique fair option value. However, it can be verified that c in (3.25) equals the Black-Scholes option value when the drift α equals the risk-free rate r: the risk-neutrality assumption. This means that the real world $\mathbb{P}$-measure is changed to the risk-neutral measure $\mathbb{Q}$.

This described concept of valuating options by taking the expected payoff is also the underlying idea of option pricing under prospect theory. The premium a writer is willing to receive for the naked call option to compensate for a potential loss equals:

$$\begin{aligned} c &= e^{-rT}E[max(S_T-K,0)] \\ &= e^{-rT}E[max(S_T-K,0)|S_T>K]P(S_T>K) \\ &= e^{-rT}\frac{\int_K^\infty (S_T-K)f(S_T)dS_T}{P(S_T>K)}P(S_T>K). \end{aligned} \tag{3.26}$$

Equation (3.26) can be rewritten as:

$$ce^{rT}-\frac{\int_K^\infty (S_T-K)f(S_T)dS_T}{\int_K^\infty f(S_T)dS_T}\int_K^\infty f(S_T)dS_T = ce^{rT}+x_1p_1=0 \tag{3.27}$$

with

$$x_1=-\frac{\int_K^\infty (S_T-K)f(S_T)dS_T}{\int_K^\infty f(S_T)dS_T},$$

as in Equation (3.15) and

$$p_1=\int_K^\infty f(S_T)dS_T,$$

as in Equation (3.14).

So far, the option value is equivalent to the option value in (3.25) but presented in a slightly different form: the expectation is decomposed into a potential loss for the writer and the probability of this loss, in order to incorporate prospect theory in the valuation. Under prospect theory it is assumed that Equation (3.27) still holds but a subjective evaluation is included, which incorporates individual investor's behavior. This means that in equilibrium the prospect value of c equals the prospect value of x_1:

$$v^+(ce^{rT})+v^-(x_1)w(p_1)=(ce^{r_fT})^a-w(p_1)\lambda(-x_1)^b=0. \tag{3.28}$$

This assumption is reasonable: the value a writer assigns to the future value of the premium should be equal to the value the writer assigns to the expected loss, in order to compensate for this potential loss. The relation of option valuation under prospect theory with the Black-Scholes methodology relies on the risk-neutrality assumption, which is not made, and subjective valuation by applying the functions v^-, v^+ and w. While in the Black-Scholes framework option values should be the same regardless the risk-attitude of an individual, the option value now depends on the risk-attitude, the degree of loss-aversion and the subjective probability weighting.

Note that Equation (3.28) reduces to equation (3.27) in case parameters are chosen $a=b=\lambda=\gamma=1$ (no prospect sentiment) and thus satisfies the Black-Scholes price when $\alpha=r_f$.

This can also be verified by substitution of these parameters $a=b=\lambda=\gamma=1$ and $r_f=\alpha$ in Equation (3.24):

$$\begin{aligned} c &= e^{-r_fT}\left(\frac{\Phi(\delta_{-1})}{\Phi(\delta_{-1})+(1-\Phi(\delta_{-1}))}\right)(S_0e^{\alpha T}\Phi(\delta_1)/\Phi(\delta_{-1})-K) \\ &= S_0\Phi(\delta_1)-Ke^{-r_fT}\Phi(\delta_{-1}), \end{aligned} \tag{3.29}$$

which equals the Black-Scholes formula.

3.2.3. NUMERICAL EXAMPLES

In this section the results of computing option prices as in Equation (3.24) are discussed. The prices are computed under different prospect parameters: loss aversion λ, curvature parameters a and b and weighting parameter γ. Three levels of prospect sentiment, as estimated by Tversky and Kahneman are considered [10], that are based on the results of hypothetical choice problems under a group of individuals:

- Zero prospect sentiment: $a = b = \lambda = \gamma = 1$. This case refers to option values based on Black-Scholes: no loss aversion, no over- and underestimation, a linear value function and $\alpha = r_f$.
- Tversky Kahneman (TK)-sentiment: $a = b = 0.88$, $\lambda = 2.25$ and $\gamma = 0.61$ based on experiments by Kahneman and Tversky [4]. TK-sentiment refers to a risk-averse attitude for gains, a risk-taking attitude for losses, the overestimation of small probabilities and underestimation of large probabilities.
- Moderate TK-sentiment: $a = b = 0.988$, $\lambda = 1.125$ and $\gamma = 0.961$, reflecting 10% of the prospect sentiment: a less risk-averse attitude for gains, a less risk-taking attitude for losses and less overestimation and underestimation of low and high probabilities, respectively.

Suppose the following Black-Scholes parameters: $S_0 = 100$, $\alpha = 0.1$, $r_f = 0.1$, $\sigma = 0.2$ and $T = 1$. In Figure 3.1 call prices with respect to strikes K are given. Moderate TK-sentiment option values are a a bit higher than the Black-Scholes values. Under TK-sentiment option values are significant increased option prices compared to Black-Scholes option values. As $\alpha = r_f$ the zero prospect sentiment prices coincide with Black-Scoles prices.

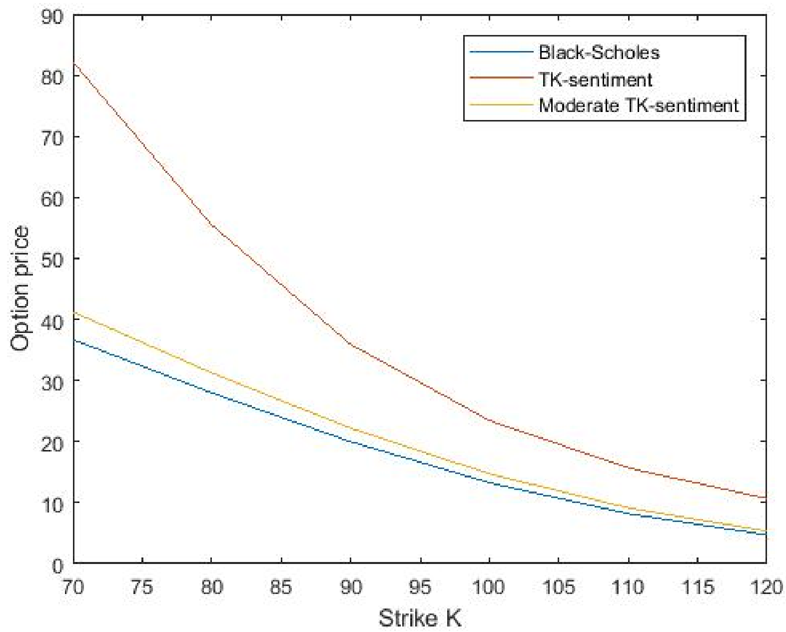


Figure 3.1: Call option prices for different strikes K. The Black-Scholes parameters used are $S_0 = 100$, $\alpha = 0.1$, $r_f = 0.1$, $\sigma = 0.2$ and $T = 1$.

3.3. OPTION PRICING UNDER CONTINUOUS CUMULATIVE PROSPECT THEORY

As already stated in Section 2.3, prospect theory does not satisfy all desired properties, as it violates stochastic dominance. Therefore, in this section option pricing under behavioral aspects is considered according to continuous cumulative prospect theory from both a writer's perspective and a holder's perspective. First, the option pricing theory under CPT is discussed in Section 2.3.1, according to [9] and [11]. Hereafter, several numerical examples are discussed (Section 2.3.2), a sensitivity analysis of the option prices with respect to several parameters is done (Section 2.3.3) and experiments with negative interest rates (Section 2.3.4) and different price dynamics are discussed (Section 2.3.5).

3.3.1. THEORY

The approach of option pricing under CPT is discussed from writer's and holder's perspective for both calls and puts. First, a continuous form of the discrete cumulative prospect value is derived. Hereafter, the option price is derived similarly to Section 3.2.1.

VALUATION FROM WRITER'S PERSPECTIVE

As already seen in Section 2.3, the discrete cumulative value of a prospect under CPT is given by:

$$V_{CPT}(x_1,..,x_n,p_1,...,p_n) = \sum_{i=1}^{k} v^-(x_i)\left(w_-(\sum_{j=1}^{i} p_j) - w_-(\sum_{j=1}^{i-1} p_j)\right) + \sum_{i=k+1}^{n} v^+(x_i)\left(w_+(\sum_{j=1}^{n} p_j) - w_+(\sum_{j=i+1}^{n} p_j)\right). \tag{3.30}$$

In the continuous case Equation (3.30) can be rewritten as:

$$\begin{aligned} V_{CPT}(X) &= \int_{-\infty}^{\theta} v^-(x)dw_-(F(x)) + \int_{\theta}^{\infty} v^+(x)dw_+(1-F(x)) \\ &= \int_{-\infty}^{\theta} v^-(x)w_-'(F(x))f(x)dx + \int_{\theta}^{\infty} v^+(x)w_+'(1-F(x))f(x)dx \\ &= \int_{-\infty}^{\theta} v^-(x)\Psi_-[F(x)]f(x)dx + \int_{\theta}^{\infty} v^+(x)\Psi_+[1-F(x)]f(x)dx, \end{aligned} \tag{3.31}$$

where θ represents the reference point, $\Psi_-(p) = \frac{dw_-(p)}{dp}$, $\Psi_+(p) = \frac{dw_+(p)}{dp}$, $F(x)$ represents the cumulative distribution fucntion (CDF) and $f(x)$ is the probability density function (PDF) of the outcomes with respect to a reference point.

Let S_t be the price at time t, $t \in [0,T]$ of the underlying asset of a European call option with maturity T. Again, it is assumed that this price process is a geometric Brownian motion. At time $t = 0$, the option writer is writing a naked European call option with premium c and strike K. Thus, the writer receives c^w at $t = 0$ and can invest this at the risk-free rate, having $c^w \exp(rT)$ at time T. At expiration the writer has to pay $max(S_T - K, 0)$. It is assumed the reference point equals zero.

- At $t = 0$ the writer receives c^w for sure which is a gain; the subjective value assigned to it equals $v^+(c^w exp(rT))$.
- At $t = T$ the option expires in-the-money or out-of-the-money. If the option expires in the money ($S_T > K$), the writer looses $K - S_T$, which happens with probability $P(S_T > K)$. Therefore, the subjective value assigned to the option equals $\int_K^{\infty} \Psi_-(1-F(S_T))f(S_T)v^-(K-S_T)dS_T$.

The cumulative prospect value follows from the following equilibrium:

$$\begin{aligned} V_{CPT}^w &= v^+(c^w e^{rT}) + \int_K^{\infty} \Psi_-(1-F(S_T))f(S_T)v^-(K-S_T)dS_T \\ &= (c^w e^{rT})^a - \lambda \int_K^{\infty} \Psi_-(1-F(S_T)f(S_T)(S_T-K)^b dS_T = 0. \end{aligned} \tag{3.32}$$

In equilibrium, Equation (3.32) gives the option price c^w:

$$c^w = e^{-rT}\left(\lambda \int_K^{\infty} \Psi_-(1-F(S_T))f(S_T)(S_T-K)^b dS_T\right)^{1/a}. \tag{3.33}$$

In this equation S_T is the underlying asset price at T, $f(S_T)$ and $F(S_T)$ are the PDF and CDF of the underlying asset price at maturity, $v^+(x)$ and $v^-(x)$ represent value functions as defined in Equation (2.6) and $w_\pm(p)$ is the weighting function as in Equation (2.8).

The derivative of $w_-(p)$ is as follows:

$$\Psi_-(p) = \frac{dw_-(p)}{dp} = \delta p^{\delta-1}[p^\delta + (1-p)^\delta]^{-1/\delta} - p^\delta[p^{\delta-1} - (1-p)^{\delta-1}][p^\delta + (1-p)^\delta]^{-(\delta+1)/\delta}. \tag{3.34}$$

The functions $F(S_T)$ and $f(S_T)$ are the CDF and PDF of the underlying asset at maturity, i.e.:

$$f(S_T) = \frac{1}{S_T\sigma\sqrt{2\pi T}} exp\left(\frac{-[log(S_T/S_0) - (\alpha - \sigma^2/2)T]^2}{2\sigma^2 T}\right), \tag{3.35}$$

$$F(S_T) = \Phi\left(\frac{log(S_T/S_0) - (\alpha - \sigma^2/2)T}{\sigma\sqrt{T}}\right), \tag{3.36}$$

with α the drift, σ the volatility, S_0 the current asset price and $\Phi(\cdot)$ the cumulative standard normal distribution.

The relation with Black-Scholes is similar to Section 3.2.2: in case $a = b = \lambda = 1$ and $\alpha = r_f$ (risk-neutral measure) Equation (3.33) reduces to:

$$c^w = e^{-rT}\left(\int_K^\infty f(S_T)(S_T - K)dS_T\right), \tag{3.37}$$

which corresponds to the Black-Scholes value

$$c_{BS} = e^{-rT}\int_0^\infty f(S_T)(S_T - K)^+ dS_T.$$

The value of a European put under cumulative prospect theory can be obtained in a similar way. In this case we call p^w the option premium at $t = 0$. Then, the prospect value from a writer's point of view follows from the following equilibrium:

$$V^w_{CPT} = v^+(p^w e^{rT}) + \int_0^K \Psi_-(F(S_T))f(S_T)v^-(S_T - K)S_T = 0, \tag{3.38}$$

from which follows:

$$V^w_{CPT} = (p^w e^{rT})^a - \lambda\int_0^K \Psi_-(F(S_T))f(S_T)(K - S_T)^b dS_T, \tag{3.39}$$

and thus

$$p^w = e^{-rT}\left(\lambda\int_0^K \Psi_-(F(S_T))f(S_T)(K - S_T)^b dS_T\right)^{1/a}. \tag{3.40}$$

VALUATION FROM HOLDER'S PERSPECTIVE

The prospect value from a holder's perspective for a call option is obtained analogously to the valuation from writer's perspective and is as follows:

$$V^h_{CPT} = v^-(-ce^{rT}) + \int_K^\infty \Psi_+(1 - F(S_T))f(S_T)v^+(S_T - K)dS_T. \tag{3.41}$$

The price c^h is obtained from $V^h_{CPT} = 0$:

$$c^h = e^{-rT}\left(\frac{1}{\lambda}\int_K^\infty \Psi_+(1 - F(S_T))f(S_T)(S_T - K)^a dS_T\right)^{1/b}. \tag{3.42}$$

Similarly, the prospect value for a put option equals:

$$V^h_{CPT} = v^-(-p^h e^{rT}) + \int_0^K \Psi_+(F(S_T)) f(S_T) v^+(K - S_T) dS_T. \tag{3.43}$$

The price p^h is obtained from $V^h_{CPT} = 0$:

$$p^h = e^{-rT}\left(\frac{1}{\lambda}\int_0^K \Psi_+(F(S_T)) f(S_T)(K - S_T)^a dS_T\right)^{1/b}. \tag{3.44}$$

3.3.2. NUMERICAL EXAMPLES

In this section the results of computing CPT option prices numerically for calls and puts for both writers and holders according to Equations (3.37), (3.40), (3.42) and (3.44) are discussed. The integrals are approximated numerically by the trapezoidal rule. The parameters used are: $S_0 = 100$, $\alpha = 0.02$, $r_f = 0.02$, $\sigma = 0.2$, $T = 1$ and the number of intervals equals 10^5 as well as the upper bound of the integral in the case of a call.

In order to verify that the correct option values are obtained by using the trapezoidal rule, first CPT-values from a writer's viewpoint under zero prospect sentiment are compared to exact Black-Scholes values. In Table 3.1 the option prices and the size of the absolute error are given. As can be seen, the option value under zero prospect sentiment is a close approximation of the exact Black-Scholes value. Note that the accuracy of the trapezoidal rule in case of puts is significantly higher due to the boundedness of the integral.

Table 3.1: Exact Black-Scholes option prices and cumulative prospect values under zero prospect sentiment. The parameters used are: $S_0 = 100$, $\alpha = 0.02$, $r_f = 0.02$, $\sigma = 0.2$ and $T = 1$.

	Call			**Put**		
Strike	Black-Scholes	Zero prospect sentiment	Absolute value error	Black-Scholes	Zero prospect sentiment	Absolute value error
70	31.58	31.58	4.74e-04	0.19	0.19	2.33e-10
80	22.54	22.54	1.09e-03	0.96	0.96	6.99e-10
90	14.81	14.80	1.57e-03	3.02	3.02	1.28e-09
100	8.92	8.91	1.63e-03	6.94	6.94	1.63e-09
110	4.94	4.94	1.32e-03	12.77	12.77	1.60e-09
120	2.55	2.55	8.94e-04	20.17	20.17	1.29e-09

WRITER'S VIEWPOINT

In Table 3.2 and corresponding Figure 3.2 and Figure 3.3 the results of computing call and put option prices from a writer's position are reported for different strikes and prospect parameters. The parameters used are: $S_0 = 100$, $\alpha = 0.02$, $r_f = 0.02$, $\sigma = 0.2$, $T = 1$ and the number of intervals equals 10^5 as well as the upper bound of the integral in the case of a call. As under prospect theory, Moderate TK-sentiment gives again a bit higher option prices compared to Black-Scholes and TK-sentiment gives significantly higher option values. This means that under TK-sentiment and Moderate TK-sentiment a writer asks a higher premium. Note that the put-call parity does not hold anymore under CPT.

Table 3.2: Option prices in the Black-Scholes model and cumulative prospect values under Tversky-Kahneman sentiment and moderate Tversky-Kahneman sentiment from a writer's viewpoint. The parameters used are: $S_0 = 100$, $\alpha = 0.02$, $r_f = 0.02$, $\sigma = 0.2$, and $T = 1$.

	Call			**Put**		
Strike	Zero prospect sentiment	TK	Moderate TK	Zero prospect sentiment	TK	Moderate TK
70	31.58	75.05	35.61	0.19	1.17	0.23
80	22.54	54.66	25.47	0.96	4.07	1.14
90	14.80	38.33	16.82	3.02	9.77	3.51
100	8.91	25.90	10.23	6.94	18.65	7.91
110	4.94	16.80	5.74	12.77	30.78	14.43
120	2.55	10.40	3.00	20.17	46.02	22.70

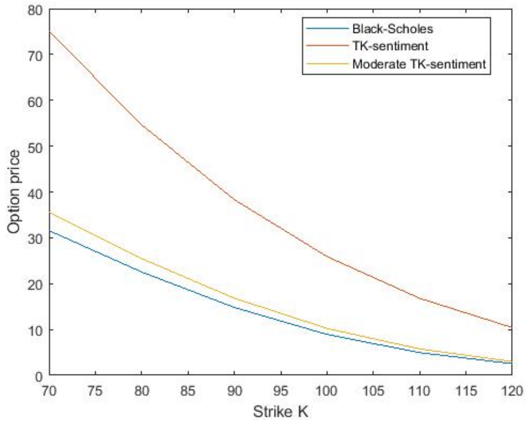


Figure 3.2: Call option prices from writer's viewpoint for different strikes. The parameters used are: $S_0 = 100$, $\mu = 0.02$, $r_f = 0.02$, $\sigma = 0.2$ and $T = 1$.

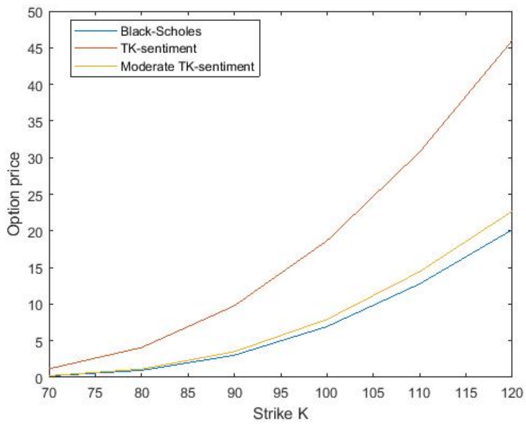


Figure 3.3: Put option prices from writer's viewpoint for different strikes. The parameters used are: $S_0 = 100$, $\alpha = 0.02$, $r_f = 0.02$, $\sigma = 0.2$ and $T = 1$.

HOLDER'S VIEWPOINT

In Table 3.3 and corresponding Figure 3.4 and Figure 3.5 the results of computing option prices from a holder's position are reported for different strikes and prospect parameters. Moderate TK-sentiment gives a bit lower option prices compared to Black-Scholes and TK-sentiment gives significantly lower option values.

Table 3.3: Black-Scholes option prices and cumulative prospect values under Tversky-Kahneman sentiment and Moderate Tversky-Kahneman sentiment from a holder's viewpoint. The parameters used are: $S_0 = 100$, $\mu = 0.02$, $r_f = 0.02$, $\sigma = 0.2$ and $T = 1$.

	Call			**Put**		
Strike	Zero prospect sentiment	TK	Moderate TK	Zero prospect sentiment	TK	Moderate TK
70	31.58	11.43	28.08	0.19	0.26	0.19
80	22.54	8.45	20.11	0.96	0.79	0.91
90	14.80	6.12	13.31	3.02	1.69	2.80
100	8.91	4.34	8.12	6.94	3.01	6.27
110	4.94	2.99	4.59	12.77	4.75	11.40
120	2.55	1.99	2.41	20.17	6.91	17.90

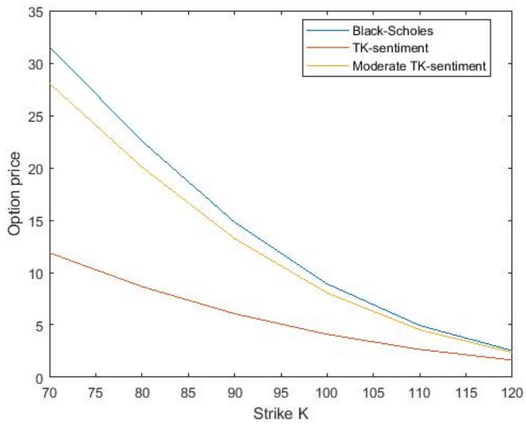


Figure 3.4: Call option prices from holder's viewpoint for different strikes and different levels of prospect sentiment. The parameters used are: $S_0 = 100$, $\alpha = 0.02$, $r_f = 0.02$, $\sigma = 0.2$ and $T = 1$.

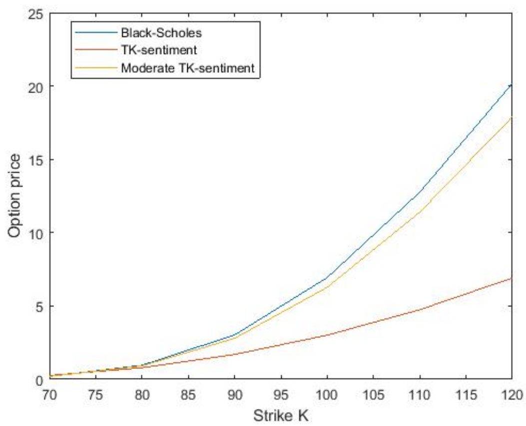


Figure 3.5: Put option prices from holder's viewpoint for different strikes and different levels of prospect sentiment. The parameters used are: $S_0 = 100$, $\alpha = 0.02$, $r_f = 0.02$, $\sigma = 0.2$ and $T = 1$.

Comparison prices from writer's and holder's viewpoint

As we now have computed prices for both calls and puts and both writers and holders the pricing problem is now considered from both the writer's and holder's perspective. In Figures 3.6 and 3.7 call and put prices under different levels are given from both writer's and holder's point of view. The parameters used are again:

$S_0 = 100$, $\alpha = 0.02$, $r_f = 0.02$, $\sigma = 0.2$, $T = 1$ and the number of intervals equals 10^5 as well as the upper bound of the integral in the case of a call. The figure shows that if an equal level of sentiment is considered for both writer and holder, there is no agreement about the option price and thus no trade. Therefore, in the following section cases in which a trade is possible are considered. Also, we see that prices from a holder's viewpoint are lower than prices from writer's viewpoint. The difference between the holder's and writer's price depends on the level of sentiment: the higher the level of sentiment the more the difference between holder's and writer's price. The Black-Scholes price lies between holder's and writer's price for all levels of sentiment.

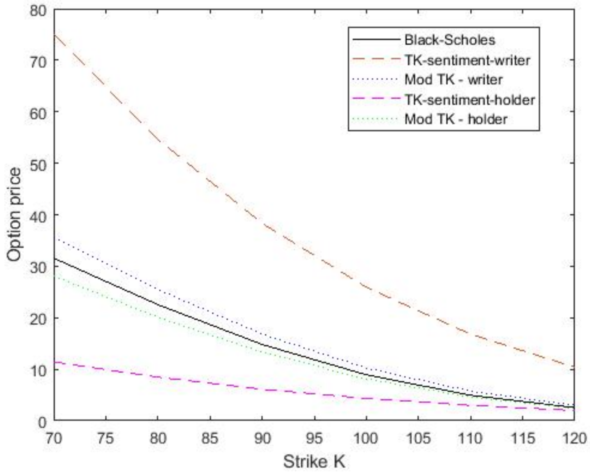


Figure 3.6: Call option prices under cumulative prospect theory from both writer's and holder's point of view and Black-Scholes prices. The Black-Scholes price always lies between the holder's and the writer's price. The more the prospect sentiment, the larger the difference between holder's and writer's price. The parameters used are: $S_0 = 100$, $\alpha = 0.02$, $r_f = 0.02$, $\sigma = 0.2$ and $T = 1$.

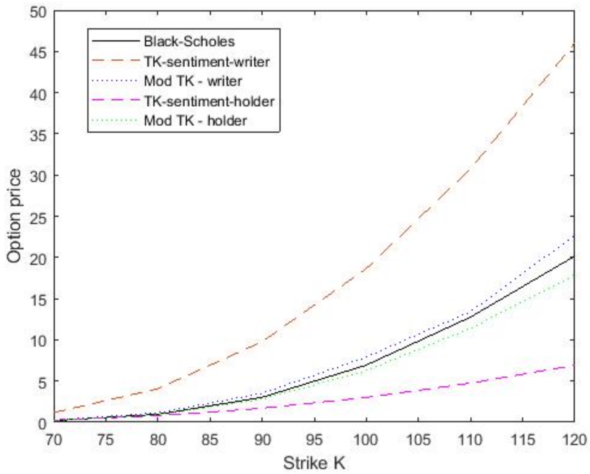


Figure 3.7: Put option prices under cumulative prospect theory from both writer's and holder's point of view. The Black-Scholes price always lies between the holder's and the writer's price. The more the prospect sentiment, the larger the difference between holder's and writer's price. The parameters used are: $S_0 = 100$, $\alpha = 0.02$, $r_f = 0.02$, $\sigma = 0.2$ and $T = 1$.

3.3.3. TRADING

As seen, for equal levels of sentiment the writer's and holder's price are different. The question then arises in which way it is possible to achieve a trade. In other words, under what conditions does $c^h = c^w$ and $p^h = p^w$ hold? We take a look at the equilibrium for the call option:

$$\begin{aligned} c^w &= e^{-rT}\left(\lambda \int_K^\infty \Psi_-(1-F(S_T))f(S_T)(S_T-K)^b dS_T\right)^{1/a} \\ &= e^{-rT}\left(\frac{1}{\lambda}\int_K^\infty \Psi_+(1-F(S_T))f(S_T)(S_T-K)^a dS_T\right)^{1/b} = c^h \end{aligned} \tag{3.45}$$

One solution to (3.45) is $a = b$, $\gamma = \delta$ and $\lambda = 1$. In order words, if the writer and holder have the same attitude towards gains, losses and probabilities and if the writer and holder are both not loss averse, the writer's price equals the holder's price. Another solution to (3.45) is $a = b$, $\gamma = \delta$ and $\lambda = -1$. In order words, if the writer and holder have the same attitude towards gains, losses and probabilities and are both gain averse, the writer's price equals the holder's price. However, this solution is not feasible as gain aversion is very unlikely and the condition $\lambda < 0$ is not satisfied. The trivial solution for $c^h = c^w$ is $a = b = \gamma = \delta = 1$ and represents the BS-framework. We take a look at the equilibrium for the first-mentioned solution: $a = b$, $\gamma = \delta$ and $\lambda = 1$.

In Figure 3.8 call option prices for which $c^h = c^w$ are given. Note that TK-sentiment now refers to $a = b = 0.88$, $\lambda = 1$ and $\gamma = \delta = 0.69$. Moderate TK-sentiment refers to $a = b = 0.988$, $\lambda = 1$ and $\gamma_- = \gamma_+ = 0.969$. As can be seen, moderate TK sentiment gives slightly higher option prices for all strikes K. Under TK-sentiment the prices are lower than the BS-prices for low strikes (in-the money options) and higher than the BS-prices for high strikes (out-of-the-money options). This can be explained by the overestimation of small probabilities, see the results of the sensitivity analysis in Section 3.3.4 for weighting parameter γ.

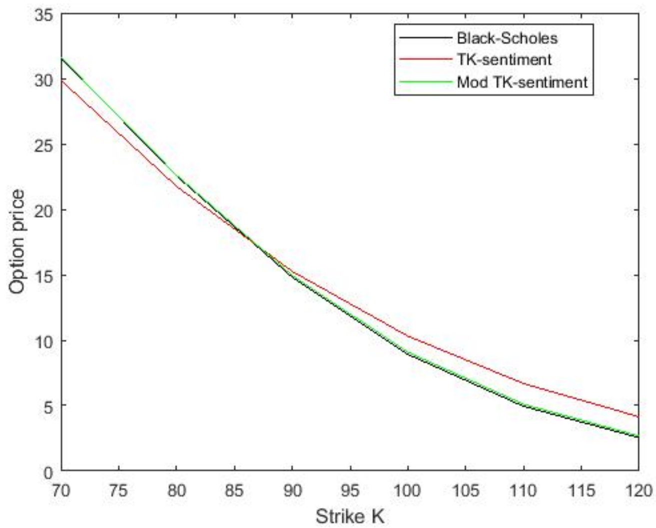


Figure 3.8: Call option prices for which $c^h = c^w$ under TK-sentiment and moderate TK-sentiment. The parameters used are: $S_0 = 100$, $\alpha = 0.02$, $r_f = 0.02$, $\sigma = 0.2$ and $T = 1$.

Note that in case of a BS-writer and a CPT-holder, an agreement about the price is only made in case of $\lambda = \gamma = a = b = 1$.

IMPLIED VOLATILITY

We now take a look at the volatility implied by Black-Scholes option prices. In Figure 3.9 the Black-Scholes implied volatility curves of call options from a writer's viewpoint are shown. The parameters used are $S_0 = 100$, $\alpha = 0.1$, $r_f = 0.05$ and $\sigma = 0.2$. As can be seen, a higher level of prospect sentiment results in a higher implied volatility values. This responds to our expectations; the implied volatility derived from the Black-Scholes equation is increasing in the option price and a higher level of prospect sentiment leads to a higher option value compared to Black-Scholes prices, see Table 3.2. The lower the strike the higher the differences between implied volatilities for the different levels of sentiment. Under TK-sentiment the implied volatility curve is a skew which is a well known effect in the market.

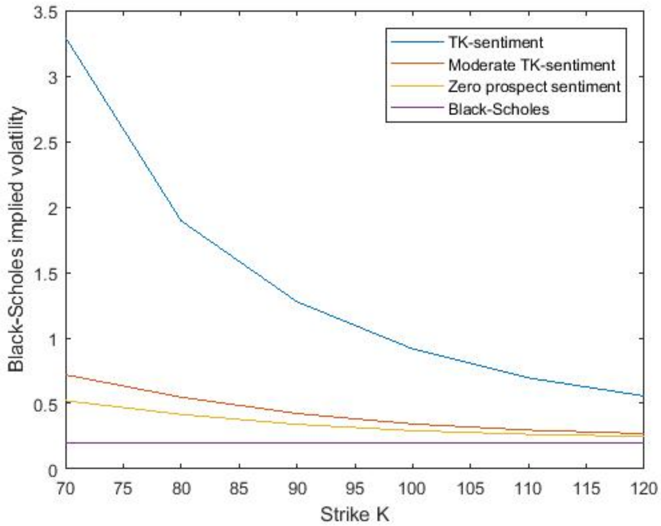


Figure 3.9: Black-Scholes implied volatility curves. The parameters used are $S_0 = 100$, $\alpha = 0.1$, $r_f = 0.05$ and $\sigma = 0.2$.

3.3.4. SENSITIVITY ANALYSIS

In this section the results of a sensitivity analysis of the cumulative prospect option price with respect to various parameters are discussed. To this end, the derivatives of the option price with respect to the prospect parameters and the Black-Scholes parameters are considered for different levels of prospect sentiment and different strikes. First, the sensitivity towards the prospect parameters is discussed: a risk averse attitude in the domain of gains (parameter a), a risk-seeking attitude towards losses (parameter b), a loss averse attitude (parameter λ) and the degree of over- and underweighting of probabilities (parameters γ and δ) are considered. Hereafter, the sensitivity towards the parameters of the Black-Scholes model are considered by means of the Greeks. The differences between the sensitivities for different levels of prospect sentiment and different strikes are discussed and explained.

SENSITIVITIES PROSPECT PARAMETERS

First, we take a look at the sensitivity of the option price towards the prospect parameters. The sensitivity of a call option price from a writer's point of view towards prospect parameters a, b, λ and γ is considered for different levels of prospect sentiment and different strikes K. Remember that the equation for a call option price from a writer's viewpoint is given by:

$$c^w = e^{-rT}\left(\lambda \int_K^\infty \Psi_-(1-F(S_T))f(S_T)(S_T-K)^b dS_T\right)^{\frac{1}{a}}. \tag{3.46}$$

The sensitivities towards the prospect parameters are derived below.

$$\frac{\partial c^w}{\partial a} = e^{-rT}\left(-\frac{(\lambda \int_K^\infty \Psi_-(1-F(S_T))f(S_T)(S_T-K)^b dS_T)^{(1/a)} log\left(\lambda \int_K^\infty \Psi_-(1-F(S_T))f(S_T)(S_T-K)^b dS_T\right)}{a^2}\right), \tag{3.47}$$

$$\frac{\partial c^w}{\partial b} = e^{-rT}\frac{1}{a}\left(\lambda \int_K^\infty \Psi_-(1-F(S_T))f(S_T)(S_T-K)^b dS_T\right)^{(1/a)-1}\left(\lambda \int_K^\infty \Psi_-(1-F(S_T))f(S_T)(S_T-K)^b log(S_T-K) dS_T\right), \tag{3.48}$$

$$\frac{\partial c^w}{\partial \lambda} = e^{-rT}\left(\frac{1}{a}\right)\lambda^{\frac{1}{a}-1}\left(\int_K^\infty \Psi_-(1-F(S_T))f(S_T)(S_T-K)^b dS_T\right)^{\frac{1}{a}}, \tag{3.49}$$

$$\frac{\partial c^w}{\partial \gamma} = e^{-rT}\left(\frac{1}{a}\right)\left(\lambda \int_K^\infty \Psi_-(1-F(S_T))f(S_T)(S_T-K)^b dS_T\right)^{(1/a)-1}\left(\lambda \int_K^\infty \frac{d\Psi_-(1-F(S_T))}{d\gamma} f(S_T)(S_T-K)^b dS_T\right). \tag{3.50}$$

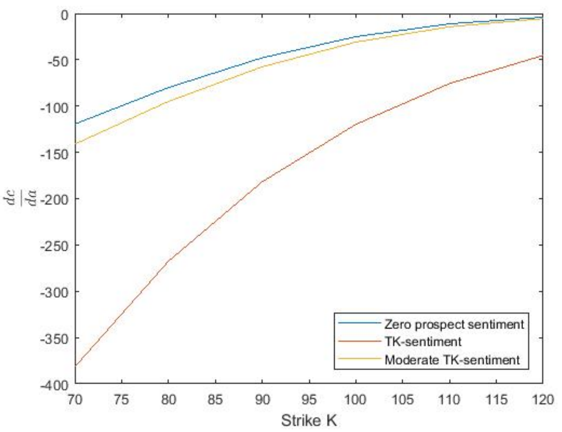
The sensitivities for the option price from a holder's viewpoint

$$c^h = e^{-rT}\left(\frac{1}{\lambda}\int_K^\infty \Psi_+(1-F(S_T))f(S_T)(S_T-K)^a dS_T\right)^{1/b}, \tag{3.51}$$

can be derived similar to the sensitivities from a writer's viewpoint.

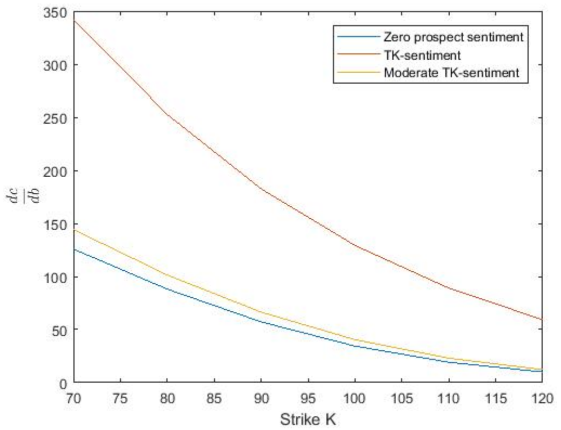
RESULTS

In this section, first the sensitivities of the call option prices from a writer's point of view towards prospect parameters a, b, λ and γ are given for different levels of prospect sentiment and different strikes K in Tables 3.10, 3.11, 3.12 and 3.13. The integrals are approximated numerically by the trapezoidal rule. Hereafter, the results are discussed.



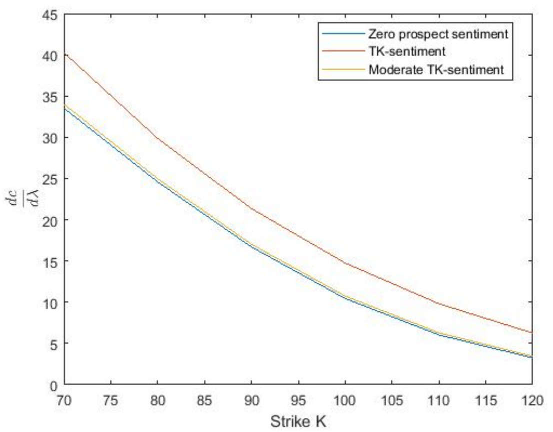
Strike	Zero	TK	Mod TK
70	-119.49	-401.19	-140.98
80	-79.97	-277.49	-94.81
90	-47.84	-182.28	-57.33
100	-25.04	-113.63	-30.60
110	-11.16	-66.62	-14.11
120	-3.99	-36.16	-5.36

Figure 3.10: Sensitivity with respect to risk averse attitude for gains a for zero prospect sentiment, TK-sentiment and Moderate TK sentiment and for different strikes from a writer's viewpoint. The Black-Scholes parameters used are $S_0 = 100$, $\alpha = 0.05$, $r_f = 0.05$, $\sigma = 0.2$ and $T = 1$.



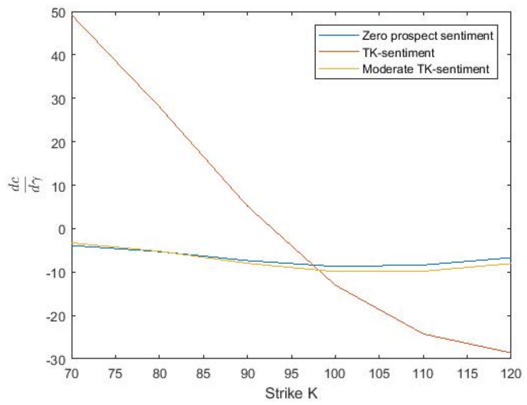
Strike	Zero	TK	Mod TK
70	125.84	350.72	143.98
80	88.08	253.73	101.08
90	57.23	177.27	66.07
100	34.42	119.63	40.13
110	19.23	77.70	22.71
120	10.05	48.38	12.05

Figure 3.11: Sensitivity with respect to risk-seeking attitude b for losses for zero prospect sentiment, TK-sentiment and Moderate TK sentiment and for different strikes from a writer's viewpoint. The Black-Scholes parameters used are $S_0 = 100$, $\alpha = 0.05$, $r_f = 0.05$, $\sigma = 0.2$ and $T = 1$.



Strike	Zero	TK	Mod TK
70	33.54	40.26	34.03
80	24.59	29.86	24.98
90	16.70	21.35	17.04
100	10.45	14.75	10.75
110	6.04	9.81	6.29
120	3.25	6.26	3.43

Figure 3.12: Sensitivity with respect to loss aversion parameter λ for zero prospect sentiment, TK-sentiment and Moderate TK sentiment and for different strikes from a writer's viewpoint. The Black-Scholes parameters used are $S_0 = 100$, $\alpha = 0.05$, $r_f = 0.05$, $\sigma = 0.2$ and $T = 1$.



Strike	Zero	TK	Mod TK
70	-3.89	30.53	-3.50
80	-5.31	13.39	-5.35
90	-7.37	-2.51	-8.03
100	-8.72	-16.76	-9.88
110	-8.39	-24.39	-9.72
120	-6.68	-25.64	-7.91

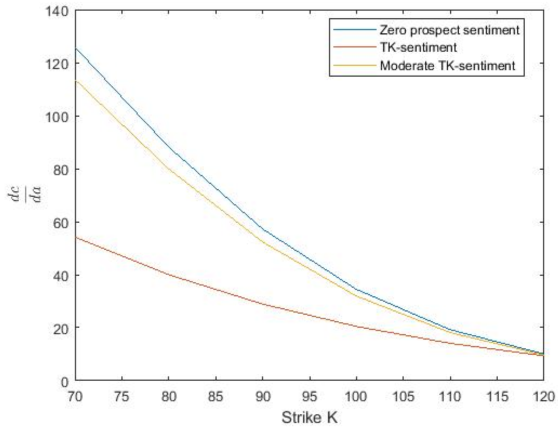
Figure 3.13: Sensitivity with respect to weighting parameter γ for zero prospect sentiment, TK-sentiment and Moderate TK sentiment and for different strikes from a writer's viewpoint. The Black-Scholes parameters used are $S_0 = 100$, $\alpha = 0.05$, $r_f = 0.05$, $\sigma = 0.2$ and $T = 1$.

The results for the sensitivities for the call option price from a writer's point are as follows:

- The higher the level of prospect sentiment the higher the absolute size of the sensitivities.
- The absolute sensitivities are highest for risk-seeking parameter b under TK-sentiment. Thus, the risk-seeking attitude in the domain of losses is most influential on the option price. Under the other levels of sentiment, the risk averse parameter a is most influential.
- The sensitivities towards risk aversion behavior in the domain of gains parameter a are negative: the smaller a (a more risk averse attitude for gains) the higher the call price. This is reasonable as the subjective value assigned to the premium c is lower and the subjective value assigned to possible loss remains the same (see Equation (3.32) or (3.46)). Therefore, the premium has to be higher in order to keep the equilibrium.
- The sensitivities towards risk-seeking behavior in the domain of losses parameter b are positive: the smaller b (a more risk-seeking attitude for losses) the lower the call option price. This is reasonable as the subjective value assigned to possibles losses is lower and thus the option premium is lower.
- The sensitivities towards loss aversion paramater λ are positive: the larger λ, which means more loss averse behavior, the higher the option price. This is reasonable as the displeasure related to the writer's loss $K - S_T$ is higher for higher levels of loss aversion and therefore the option price is higher.
- The absolute sensitivity towards weighting parameter γ is significantly lower than the absolute sensitivities towards a, b and λ which means that the impact of the parameter γ is less than the other prospect parameters. In case of a smaller value of γ the degree of overestimation of low probabilities and underestimation of high probabilities is higher. For TK-sentiment a higher level of over - and underestimation leads to lower call option prices in the case of low strikes (in-the-money options) and higher call option prices in the case of high strikes (out-of-the-money options). This can be explained by the following. An in-the-money option is a bad position for a writer without hedge, as he will loose money. As under prospect theory the probability of the transformation of an out-of-the-money option into an in-the money option is overestimated, the writer wants to have more compensation in case of a higher level of overestimation. This means that the call option price for the out-of-the-money is higher for a lower value of γ. Similarly, the probability of the transformation of an in-the-money option into an out-of-the-money option (good position) is overestimated under CPT which results in a lower in-the-money option price. The above reasoning explains why a lower value of γ, that is to say a higher level of overestimation of the probability of turning an out-of the-money option into an in-the-money option and vice versa, results in a lower option price for in-the-money options and a higher option price for out-of-the-money options. Note that the level of over- and underestimation is too low in the other levels of sentiment to see the same effect.

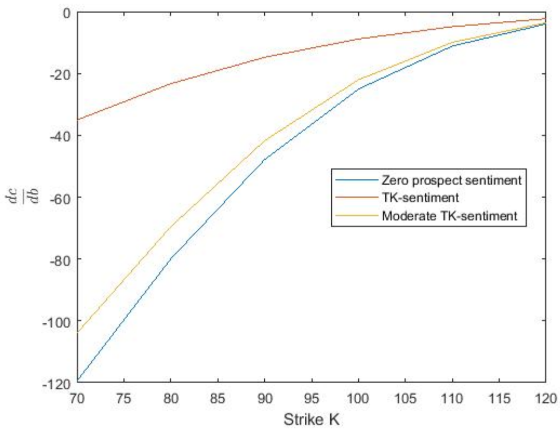
Next, the sensitivities from a holder's point of view are considered. In Tables 3.14, 3.15, 3.16 and 3.17 the sensitivities are given. The results of the sensitivities from a holder's viewpoint are in line with the sensitivities from a writer's viewpoint and are as follows:

- The higher the level of prospect sentiment, the higher the absolute size of the sensitivities. The absolute sensitivities are highest for risk-averse parameter a. In other words, the risk-averse attitude in the domain of gains is most influence on the option price. The sensitivities towards risk averse behavior in the domain of gains parameter a are positive. The sensitivities towards risk-seeking behavior in the domain of losses parameter b are negative as well as the sensitivities towards loss aversion parameter λ. The sensitivity towards weighting parameter γ are equal in sign to the sensitivities from a writer's viewpoint, which is reasonable from Equation (3.46) and (3.51). All sensitivities from a writer's viewpoint are larger than the sensitivities from a holder's viewpoint in absolute value. In other words, the writer's is more sensitive to changes in value of the prospect parameters.



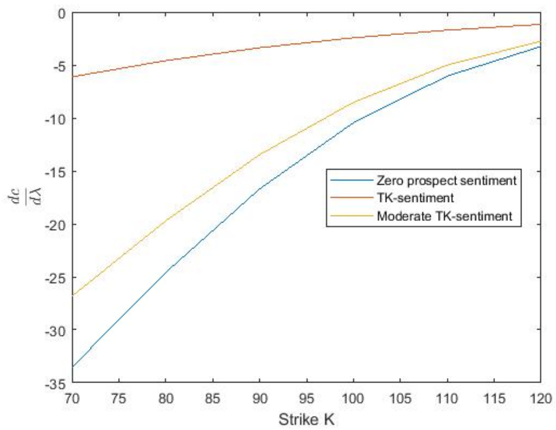
Strike	Zero	TK	Mod TK
70	125.84	54.20	113.62
80	88.08	39.99	79.87
90	57.23	28.91	52.33
100	34.42	20.46	31.90
110	19.23	14.09	18.14
120	10.05	9.39	9.68

Figure 3.14: Sensitivity with respect to risk averse attitude for gains a for zero prospect sentiment, TK-sentiment and Moderate TK sentiment and for different strikes from a holder's viewpoint. The Black-Scholes parameters used are $S_0 = 100$, $\alpha = 0.05$, $r_f = 0.05$, $\sigma = 0.2$ and $T = 1$.



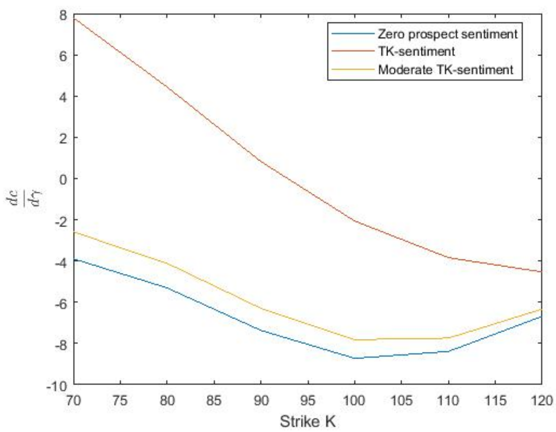
Strike	Zero	TK	Mod TK
70	-119.49	-35.06	-103.98
80	-79.97	-23.32	-69.56
90	-47.84	-14.80	-41.76
100	-25.04	-8.87	-22.05
110	-11.16	-4.89	-9.96
120	-3.99	-2.32	-3.61

Figure 3.15: Sensitivity with respect to risk-seeking attitude b for losses for zero prospect sentiment, TK-sentiment and Moderate TK sentiment and for different strikes from a holder's viewpoint. The Black-Scholes parameters used are $S_0 = 100$, $\alpha = 0.05$, $r_f = 0.05$, $\sigma = 0.2$ and $T = 1$.



Strike	Zero	TK	Mod TK
70	-33.54	-6.12	-26.83
80	-24.59	-4.59	-19.71
90	-16.70	-3.37	-13.47
100	-10.45	-2.43	-8.53
110	-6.04	-1.71	-5.01
120	-3.25	-1.17	-2.75

Figure 3.16: Sensitivity with respect to loss aversion parameter λ for zero prospect sentiment, TK-sentiment and Moderate TK sentiment and for different strikes from a holder's viewpoint. The Black-Scholes parameters used are $S_0 = 100$, $\alpha = 0.05$, $r_f = 0.05$, $\sigma = 0.2$ and $T = 1$.



Strike	Zero	TK	Mod TK
70	-3.89	7.80	-2.59
80	-5.31	4.44	-4.11
90	-7.37	0.82	-6.30
100	-8.72	-2.06	-7.81
110	-8.39	3.84	-7.74
120	-6.68	-4.53	-6.33

Figure 3.17: Sensitivity with respect to weighting parameter γ for zero prospect sentiment, TK-sentiment and Moderate TK sentiment and for different strikes from a holder's viewpoint. The Black-Scholes parameters used are $S_0 = 100$, $\alpha = 0.05$, $r_f = 0.05$, $\sigma = 0.2$ and $T = 1$.

SENSITIVITY TOWARDS BLACK-SCHOLES PARAMETERS

Now, the sensitivities of call option prices with respect to the Black-Scholes parameters S, σ, r_f and T are considered from a writer's viewpoint. First, the sensitivities are derived and hereafter the sensitivities are considered for different levels of prospect sentiment and different strikes and are explained economically.

$$\frac{\partial c^w}{\partial r} = -Te^{-rT}\left(\lambda\int_K^\infty \Psi_-(1-F(S_T))f(S_T)(S_T-K)^b dS_T\right)^{\frac{1}{a}}. \quad (3.52)$$

$$\frac{\partial c^w}{\partial T} = -re^{-rT}\left(\lambda\int_K^\infty \Psi_-(1-F(S_T))f(S_T)(S_T-K)^b dS_T\right)^{\frac{1}{a}} + e^{-rT}\frac{1}{a}\left(\lambda\int_K^\infty \Psi_-(1-F(S_T))f(S_T)(S_T-K)^b dS_T\right)^{\frac{1}{a}-1}\times$$
$$\lambda\left(\int_K^\infty \frac{d\Psi_-(1-F(S_T))}{dT}f(S_T)(S_T-K)^b dS_T + \int_K^\infty \Psi_-(1-F(S_T))\frac{df(S_T)}{dT}(S_T-K)^b dS_T\right). (3.53)$$

$$\frac{\partial c^w}{\partial \sigma} = e^{-rT}\frac{1}{a}\left(\lambda\int_K^\infty \Psi_-(1-F(S_T))f(S_T)(S_T-K)^b dS_T\right)^{\frac{1}{a}-1}\times$$
$$\lambda\left(\int_K^\infty \frac{d\Psi_-(1-F(S_T))}{d\sigma}f(S_T)(S_T-K)^b dS_T + \int_K^\infty \Psi_-(1-F(S_T))\frac{df(S_T)}{d\sigma}(S_T-K)^b dS_T\right). (3.54)$$

$$\frac{\partial c^w}{\partial S} = e^{-rT}\frac{1}{a}\left(\lambda\int_K^\infty \Psi_-(1-F(S_T))f(S_T)(S_T-K)^b dS_T\right)^{(1/a)-1}\times\lambda\left(\int_K^\infty \frac{d\Psi_-(1-F(S_T))}{dS}f(S_T)(S_T-K)^b dS_T\right.$$
$$\left.+\int_K^\infty \Psi_-(1-F(S_T))\frac{df(S_T)}{dS}(S_T-K)^b dS_T\right). (3.55)$$

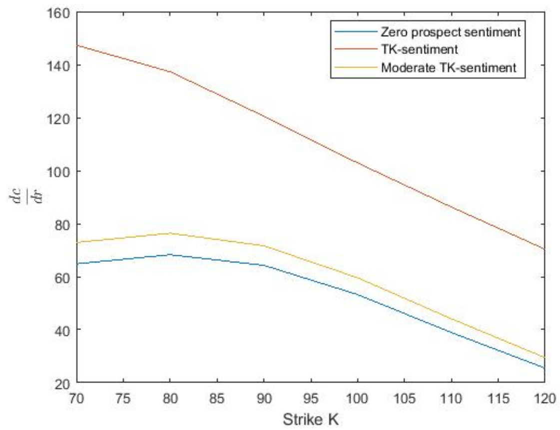
In above equations the derivatives of $f(S_T)$ and $F(S_T)$ are as follows:

$$\frac{df(S_T)}{dS} = \frac{\sqrt{2}exp\left(-(log(S_T/S_0)-T(-\sigma^2/2+\alpha))^2/(2T\sigma^2)\right)log(S_T/S_0)-T(-\sigma^2/2+\alpha))}{(S_TS_0T^{3/2}\sigma^3\sqrt{\pi}}, \quad (3.56)$$

$$\frac{dF(S_T)}{dS} = -\frac{\sqrt{2}exp(-(log(S_T/S_0)-T(-\sigma^2/2+\alpha))^2/(2T\sigma^2))}{2S_0\sqrt{T}\sigma\sqrt{\pi}}. \quad (3.57)$$

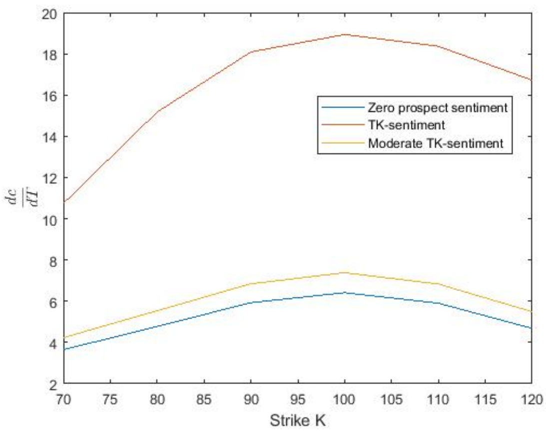
The sensitivities towards the Black-Scholes from a holder's point view can be derived in a similar way.

In Tables 3.18, 3.19, 3.20 and 3.21 the sensitivities of call option prices towards the Black-Scholes parameters are given from a writer's point of view. In Tables 3.22, 3.23, 3.24 and 3.25 the results for a holder's position are given. Hereafter, the results are discussed.



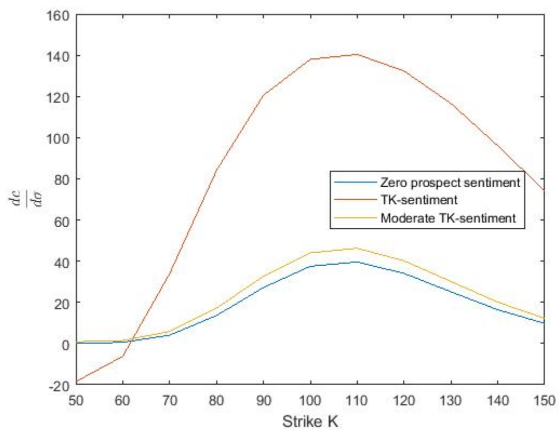
Strike	Zero	TK	Mod TK
70	64.8188	155.2994	72.9567
80	68.2814	150.4690	76.5376
90	64.2769	134.1736	71.8333
100	53.2352	113.2705	59.6364
110	38.9238	91.4325	44.0044
120	25.4689	70.2530	29.2059

Figure 3.18: Sensitivity with respect to interest rate r_f for zero prospect sentiment, TK-sentiment and Moderate TK sentiment and for different strikes from a writer's viewpoint. The Black-Scholes parameters used are $S_0 = 100$, $\alpha = 0.05$, $r_f = 0.05$, $\sigma = 0.2$ and $T = 1$.



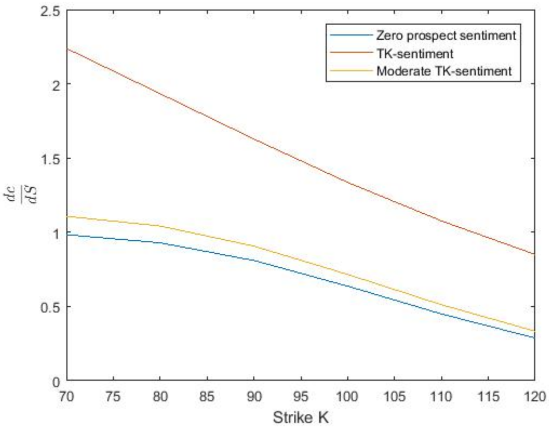
Strike	Zero	TK	Mod TK
70	3.65	10.05	4.19
80	4.78	14.17	5.49
90	5.93	17.11	6.80
100	6.41	18.07	7.34
110	5.90	17.29	6.78
120	4.68	15.24	5.43

Figure 3.19: Sensitivity with respect to time horizon T for zero prospect sentiment, TK-sentiment and Moderate TK sentiment and for different strikes from a writer's viewpoint. The Black-Scholes parameters used are $S_0 = 100$, $\alpha = 0.05$, $r_f = 0.05$, $\sigma = 0.2$ and $T = 1$.



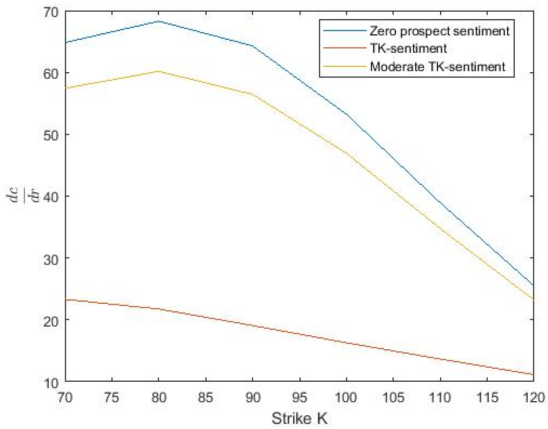
Strike	Zero	TK	Mod TK
50	0.03	-9.74	0.60
60	0.59	-4.78	1.21
70	4.09	22.85	5.44
80	13.62	66.49	16.66
90	27.17	104.01	32.05
100	37.53	124.10	43.54
110	39.58	127.19	44.78
120	34.08	117.25	39.67
130	25.12	98.75 1	29.60
140	16.42	76.48	19.63
150	9.77	54.79	11.87

Figure 3.20: Sensitivity with respect to volatility parameter σ for zero prospect sentiment, TK-sentiment and Moderate TK sentiment and for different strikes from a writer's viewpoint. The Black-Scholes parameters used are $S_0 = 100$, $\alpha = 0.05$, $r_f = 0.05$, $\sigma = 0.2$ and $T = 1$.



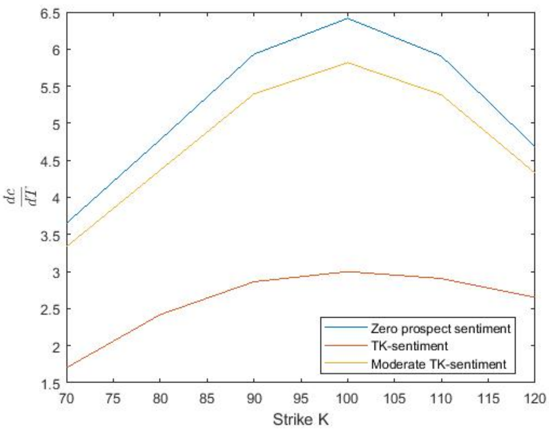
Strike	Zero	TK	Mod TK
70	0.98	2.35	1.11
80	0.93	2.10	1.04
90	0.81	1.76	0.91
100	0.64	1.42	0.72
110	0.45	1.11	0.51
120	0.29	0.83	0.33

Figure 3.21: Sensitivity with respect to parameter S for zero prospect sentiment, TK-sentiment and Moderate TK sentiment and for different strikes from a writer's viewpoint. The Black-Scholes parameters used are $S_0 = 100$, $\alpha = 0.05$, $r_f = 0.05$, $\sigma = 0.2$ and $T = 1$.



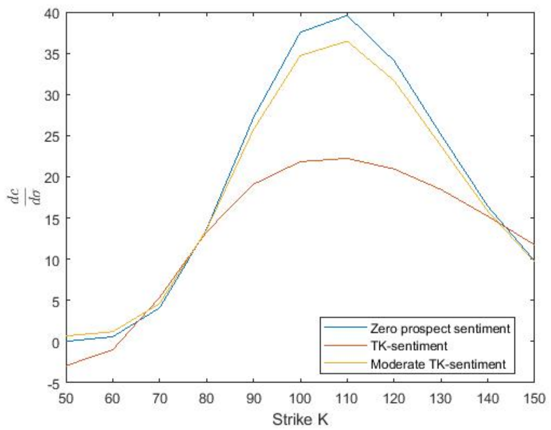
Strike	Zero	TK	Mod TK
70	64.82	23.32	57.43
80	68.28	21.75	60.17
90	64.28	19.08	71.83
100	53.24	113.27	59.64
110	38.92	91.43	44.00
120	25.47	70.25	29.21

Figure 3.22: Sensitivity with respect to interest rate r_f for zero prospect sentiment, TK-sentiment and Moderate TK sentiment and for different strikes from a holder's point of view. The Black-Scholes parameters used are $S_0 = 100$, $\alpha = 0.05$, $r_f = 0.05$, $\sigma = 0.2$ and $T = 1$.



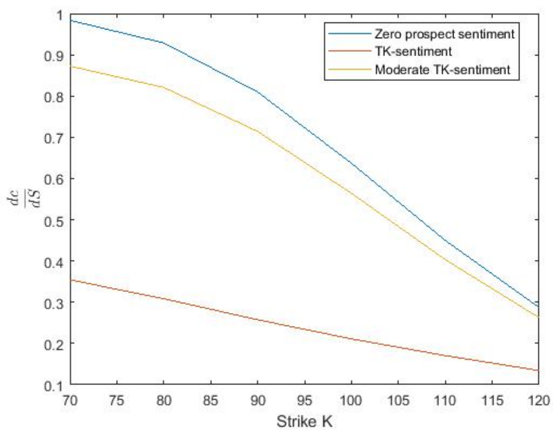
Strike	Zero	TK	Mod TK
70	3.65	1.70	3.33
80	4.78	2.42	4.37
90	5.93	2.86	5.39
100	6.41	2.00	5.82
110	5.90	2.91	5.38
120	4.68	2.65	4.33

Figure 3.23: Sensitivity with respect to time horizon T for zero prospect sentiment, TK-sentiment and Moderate TK sentiment and for different strikes from a holder's point of view. The Black-Scholes parameters used are $S_0 = 100$, $\alpha = 0.05$, $r_f = 0.05$, $\sigma = 0.2$ and $T = 1$.



Strike	Zero	TK	Mod TK
50	0.03	-2.93	0.70
60	0.59	-0.99	1.20
70	4.09	5.36	4.63
80	13.62	13.31	13.58
90	27.17	19.09	25.71
100	37.53	21.84	34.71
110	39.58	22.23	36.46
120	34.08	20.95	31.67
130	25.12	18.46 1	23.74
140	16.42	15.21	15.83
150	9.77	11.73	9.63

Figure 3.24: Sensitivity with respect to volatility parameter σ for zero prospect sentiment, TK-sentiment and Moderate TK sentiment and for different strikes from a holder's point of view. The Black-Scholes parameters used are $S_0 = 100$, $\alpha = 0.05$, $r_f = 0.05$, $\sigma = 0.2$ and $T = 1$.



Strike	Zero	TK	Mod TK
70	0.98	0.35	0.87
80	0.93	0.31	0.82
90	0.81	0.26	0.71
100	0.64	0.21	0.56
110	0.45	0.17	0.40
120	0.29	0.13	0.26

Figure 3.25: Sensitivity with respect to parameter S for zero prospect sentiment, TK-sentiment and Moderate TK sentiment and for different strikes from a holder's point of view. The Black-Scholes parameters used are $S_0 = 100$, $\alpha = 0.05$, $r_f = 0.05$, $\sigma = 0.2$ and $T = 1$.

If the sensitivities from a writer's point of view are considered, the following results can be derived:

- The higher the level of prospect sentiment, the higher the absolute value of all sensitivities.
- The sensitivities with respect to r and T are in line with the Black-Scholes framework. A higher value of the interest rate r corresponds to a lower strike K and thus an increase of the option value from a writer's viewpoint. The sensitivity with respect to r is further considered in Section 3.3.5. A longer time horizon T means more possibilities for the stock to rise and fall for which the writer wants a compensation. This results in a higher option price.
- As expected the sensitivity towards volatility σ is positive under zero prospect sentiment: higher volatility leads to a wider range of underlying asset prices which give a higher option price as out-of-the money prices have no effect while into-the-money prices lead to a higher option value. Also, under all levels of sentiment the volatility is increasing in the strike K. Now, the sensitivities towards σ for the different levels of sentiments are considered. If we look at TK-sentiment we see that for in-the-money options the sensitivity towards σ becomes negative, which can be explained by the negativity of $\frac{\partial \Psi_-(1-F(S_T))}{\partial \sigma}$ for all strikes. A possible economical explanation for this result is the following. An in-the-money position is a bad position for a writer. If the volatility increases, this leads to a wider range of asset prices. Therefore, the probability of becoming more in-the-money increases and the probability of becoming more out-of-the-money increases as well, the first probability being larger than the latter. Under increasing volatility and low strikes the probability of becoming more in-the-money is classified as a high probability and the probability of becoming out-of-the-money is classified as a low probability under increasing volatility. The relatively high chance of being deeper in-the-money (bad) is underestimated under prospect sentiment. On the other hand, the small chance of becoming out-of-the-money (good) is overestimated under prospect sentiment. Therefore, for low strikes and under TK-sentiment, a writer wants less compensation which results in a lower option price.
- The sensitivity towards price S under TK sentiment and under Moderate TK sentiment takes values above one.

If the sensitivities from a holder's point of view are considered, the following result can be derived:

- The sensitivities towards the Black-Scholes parameters from a holder's point of view are have the same sign as the sensitivities from a writer's point of view, but are all smaller. Thus, the option price from a holder's point of view is less sensitivite to changes in the Black-Scholes parameters.

In conclusion, we now have a clear view on the degree at which irrational behavior and the underlying Black-Scholes dynamics have impact on the option price for both writers and holder. Also, all sensitivities are explained from an economical/behavioral point of view.

3.3.5. NEGATIVE INTEREST RATES

In all previous experiments a risk free rate of 2% or 5% is used. However in current economic circumstances a negative interest rate is possible as well. Therefore, the influence of a negative interest rate on the option prices for different levels of prospect sentiment is considered in this section. In Figure 3.26 call option prices for different strikes, different levels of sentiment are compared for positive and negative interest rates from a writer's viewpoint. Note that the derived results are in line with the sensitivities from 3.18. It can be concluded that:

- A negative interest rate increases the call option price for all levels of sentiment.
- The increase in option price for a negative interest rate is larger for lower strikes.
- The increase of the option price under moderate and TK-sentiment relative to zero prospect sentiments is equal for positive interest rate and negative interest rates or in other words $\frac{c_{TK}-c_{zero}}{c_{zero}} = a$ for all values of r_f and $\frac{c_{ModTK}-c_{zero}}{c_{zero}} = b$ for all r_f with a and b constants. This is also reasonable from Equation (3.33).

In conclusion, the option prices under CPT for negative interest are in line with our expectations; it is possible to compute option prices with a negative interest rate, but the influence of sentiment on option prices with a negative interest rate is equal to the influence of sentiment on option prices with a positive interest rate.

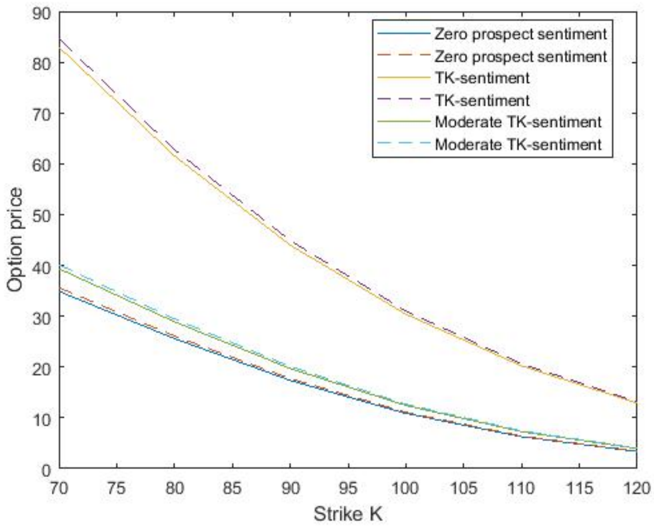


Figure 3.26: Call option prices for zero prospect sentiment, TK-sentiment and Moderate TK sentiment and for different strikes. The Black-Scholes parameters used are $S_0 = 100$, $\alpha = 0.1$, $\sigma = 0.2$ and $T = 1$. The dashed line represents a level of $r_f = -0.01$ and the solid line represents $r_f = 0.01$.

3.3.6. HESTON DYNAMICS

As theoretical option prices derived by the widely known Black-Scholes formula are not in line with market prices, in previous sections option prices under cumulative prospect theory are computed. Another possible explanation of the deviation of theoretical option prices from market prices, is the assumed underlying GBM in which the volatility is deterministic and constant until expiration. Therefore, in this section cumulative prospect option pricing is combined with Heston dynamics as underlying price process. Then, the impact of adding sentiment under Heston dynamics is compared with the impact of adding sentiment under GBM.

The Heston stochastic volatility model is given by the following dynamics:

$$\begin{aligned} dS_t &= \alpha S_t dt + \sqrt{v_t} S_t dW_t^x, \\ dv_t &= \kappa(\overline{v} - v_t)dt + \chi\sqrt{v_t} dW_t^v. \end{aligned} \tag{3.58}$$

The correlation between the Brownian motions W^x and W^v is described by $dW_t^v dW_t^x = \rho_{x,v} dt$, α is the rate of return, $\overline{v}$ is the long term average price variance, κ the speed at which v returns to $\overline{v}$ and γ the volatility of the volatility. The process v_t is strictly positive if the Feller condition holds: $2\kappa\overline{v} > \chi^2$. The correlation parameter ρ is usually taken negative.

In contrast to GBM, under Heston dynamics there are no closed forms of the density and CDF. Therefore, it is not possible to approximate the option prices by use of the trapezoidal rule as in the previous sections. In this section Monte Carlo simulation is used to obtain cumulative prospect option prices. In order to verify that the results obtained under Heston dynamics with Monte Carlo simulation are in line with the results obtained under GBM with the trapezoidal rule, firstly prices from a writer's viewpoint are computed under a reduced form of the Heston dynamics which in line with dynamics of GBM: $\chi = 0$, $k = 1$, $\overline{v} = \sigma_{GBM}^2$ and $v_0 = \sigma_{GBM}^2$. Also, the remaining parameters are chosen equally: $S_0 = 100$, $\alpha = 0.02$, $r_f = 0.02$, $\sigma = 0.2$ and $T = 1$. The number of timesteps equals $N = 10^3$ and the number of paths used equals $M = 10^4$. Table 3.5 shows that the prices obtained with Monte Carlo simulation are in line with results obtained with the trapezoidal rule.

Table 3.4: Cumulative prospect option under zero prospect sentiment, Tversky-Kahneman sentiment and Moderate Tversky-Kahneman sentiment from a writer's viewpoint and cumulative prospect option values under GBM and under Heston dynamics with $\chi = 0$, $k = 1$, $\overline{v} = \sigma^2_{GBM} = 0.04$, $v_0 = \sigma^2_{GBM} = 0.04$, $S_0 = 100$, $\alpha = 0.02$, $r_f = 0.02$, $\sigma = 0.2$ and $T = 1$.

	GBM			**Heston - 95% confidence interval**		
Strike	Zero prospect sentiment	TK	Moderate TK	Zero prospect sentiment	TK	Moderate TK
70	31.58	75.06	35.61	31.55 ± 0.34	74.74 ± 0.51	35.57 ± 0.34
80	22.54	54.66	25.47	22.51 ± 0.32	54.37 ± 0.48	25.42 ± 0.33
90	14.80	38.33	16.82	14.73 ± 0.29	37.97 ± 0.44	16.73± 0.29
100	8.91	25.90	10.23	8.82 ± 0.24	25.51 ± 0.39	10.11 ± 0.24
110	4.94	16.80	5.74	4.88 ± 0.18	16.47 ± 0.34	5.66 ± 0.19
120	2.55	10.40	3.01	2.54 ± 0.1343	10.21 ± 0.30	2.99 ± 0.14

In order to compare option prices under Heston dynamics with option prices under GBM for different levels of sentiment, the long term volatility of the Heston dynamics is chosen equal to the volatility of the GBM process. In the following table and graphs the results are presented. The parameters used are: $\kappa = 0.8$, $\chi = 0.1$, $\overline{v} = 0.04$, $v_0 = 0.04$, $\rho_{x,v} = -0.8$, $S_0 = 100$, $\alpha = 0.02$, $r_f = 0.02$, $\sigma = 0.2$, $T = 1$ and the number of paths equals $M = 10^4$.

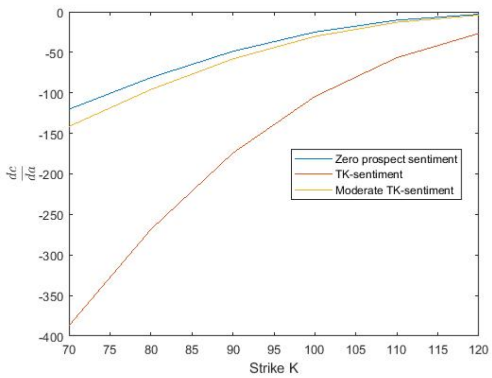
Table 3.5: Cumulative prospect option under zero prospect sentiment, Tversky-Kahneman sentiment and Moderate Tversky-Kahneman sentiment from a writer's viewpoint and cumulative prospect option values under GBM and under Heston dynamics with $\kappa = 0.2$, $\chi = 0.1$, $\overline{v} = 0.04$, $v_0 = 0.04$, $\rho_{x,v} = -0.8$, $S_0 = 100$, $\alpha = 0.02$, $r_f = 0.02$, $\sigma = 0.2$, $T = 1$ and the number of paths equals $M = 10^4$

	GBM			**Heston (95% confidence interval)**		
Strike	Zero prospect sentiment	TK	Moderate TK	Zero prospect sentiment	TK	Moderate TK
70	31.58	75.06	35.61	31.77 ± 0.32	72.97 ± 0.30	35.74 ± 0.27
80	22.54	54.66	25.47	22.83 ± 0.21	53.19 ± 0.16	25.72 ± 0.10
90	14.80	38.33	16.82	14.99 ± 0.44	36.78 ± 0.41	16.96 ± 0.37
100	8.91	25.90	10.23	8.81 ± 0.32	23.94 ± 0.27	10.06 ± 0.31
110	4.94	16.80	5.75	4.57 ± 0.32	14.47 ± 0.30	5.28 ± 0.27
120	2.55	10.40	3.01	2.08 ± 0.22	7.99 ± 0.16	2.44 ± 0.11

The following results can be derived:

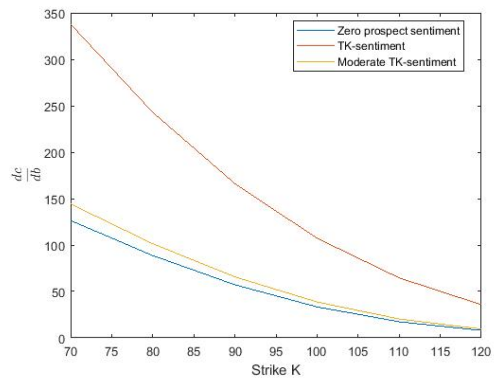
- The higher the level of sentiment the higher the option price for both GBM and Heston dynamics.
- The option price under Heston dynamics is higher than under GBM for zero prospect sentiment but lower for the other levels of sentiment.
- If the impact of adding sentiment on the option price is considered, it can be concluded that the impact of adding sentiment to the Black-Scholes framework is different from the impact of adding sentiment to the Heston framework. The relative increase in option price from zero sentiment to moderate TK sentiment and from zero sentiment to TK sentiment is smaller under Heston dynamics than under GBM. This means that sentiment has less impact on the option price under Heston dynamics than under GBM. A possible explanation for this result is the following. Under Heston dynamics a stochastic volatility is assumed instead of a constant volatility as under GBM. This means that a certain form of sentiment is already incorporated in the pricing process and thus in the option price. Therefore, the impact adding prospect sentiment is smaller under Heston dynamics than under GBM.

In the following figures and tables the results of computing the sensitivities with respect to the most important prospect parameters are presented. The parameters used are $S_0 = 100$, $\alpha = 0.05$, $r_f = 0.05$, $\sigma = 0.2$, $\chi = 0.1$, $k = 0.2$, $\rho = -0.8$ $\overline{v} = \sigma^2_{GBM} = 0.04$, $v_0 = \sigma^2_{GBM} = 0.04$ and $T = 1$.



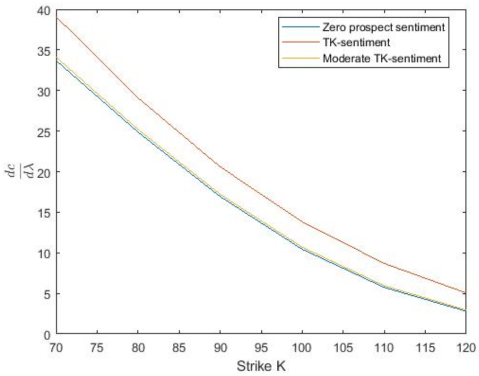
Strike	Zero	TK	Mod TK
70	-120.24	-387.20	-141.46
80	-81.135	-268.47	-95.85
90	-48.73	-174.19	-58.10
100	-24.00	-104.480	-30.34
110	-10.30	-56.57	-12.93
120	-4.41	-34.60	-5.58

Figure 3.27: Sensitivity with respect to risk averse attitude for gains a for zero prospect sentiment, TK-sentiment and Moderate TK sentiment and for different strikes under Heston dynamics. The parameters used are $S_0 = 100$, $\alpha = 0.05$, $r_f = 0.05$, $\sigma = 0.2$, $\chi = 0.1$, $k = 0.2$, $\rho = -0.8$, $\overline{v} = \sigma^2_{GBM} = 0.04$, $v_0 = \sigma^2_{GBM} = 0.04$ and $T = 1$.



Strike	Zero	TK	Mod TK
70	126.38	337.78	144.19
80	88.64	242.86	101.36
90	57.20	166.12	65.72
100	33.37	107.28	38.66
110	17.37	64.61	20.37
120	8.02	35.71	9.54

Figure 3.28: Sensitivity with respect to risk-seeking attitude b for losses for zero prospect sentiment, TK-sentiment and Moderate TK sentiment and for different strikes under Heston dynamics. The parameters used are $S_0 = 100$, $\alpha = 0.05$, $r_f = 0.05$, $\sigma = 0.2$, $\chi = 0.1$, $k = 0.2$, $\rho = -0.8$, $\overline{v} = \sigma^2_{GBM} = 0.04$, $v_0 = \sigma^2_{GBM} = 0.04$ and $T = 1$.



Strike	Zero	TK	Mod TK
70	33.70	39.11	34.12
80	24.86	29.08	25.19
90	16.93	20.60	17.21
100	10.44	13.82	10.68
110	5.73	8.69	5.94
120	2.79	5.04	2.93

Figure 3.29: Sensitivity with respect to loss aversion parameter λ for zero prospect sentiment, TK-sentiment and Moderate TK sentiment and for different strikes under Heston dyanmics. The parameters used are $S_0 = 100$, $\alpha = 0.05$, $r_f = 0.05$, $\sigma = 0.2$, $\chi = 0.1$, $k = 0.2$, $\rho = -0.8$, $\overline{v} = \sigma^2_{GBM} = 0.04$, $v_0 = \sigma^2_{GBM} = 0.04$ and $T = 1$.

The following results can be derived:

- The signs of the sensitivities as well as the parameter of most impact on the option price are in line with the results obtained under GBM.
- If compared with the results under GBM, the absolute sensitivities are smaller for the two lowest levels of sentiment and higher for TK-sentiment. In other words, for the highest level of sentiment, the influence of the prospect parameters on the option price is higher under Heston dynamics than under GBM. This result is in line with the earlier obtained result about the impact of adding sentiment on the option price.
- The relative increase in sensitivity from zero sentiment to moderate TK-sentiment and TK-sentiment is lower under Heston dynamics than under GBM. This result is again in line with the earlier obtained result about the impact of adding sentiment on the option price.

4
PORTFOLIO MANAGEMENT

Within the financial world portfolio or wealth management is an important activitiy. Portfolio managers are faced with the problem of investing a certain amount of money in different products over time such that the maximum possible degree of satisfaction is achieved. In other words, the problem of portfolio management is to invest a starting wealth in a mix of different assets in order to maximize the subjective value assigned to it. In this chapter the mathematical formulation of optimal portfolio choice, under different models is considered. Firstly, the setting under which investments take place is described in Section 4.1. Hereafter, in Section 4.2, we turn to the traditional way of choosing an optimal portfolio which relies on the maximization of the return for a given level of risk according to Modern Portfolio Theory or Mean-Variance analysis. Within this theory individual preferences and risk profiles are absent. After this, we turn to models which incorporate individual risk profiles. Firstly, in Section 4.3 the optimal portfolio choice under expected utility theory (EUT) is described. In Section 4.4 the optimal portfolio choice under prospect theory without a weighting function is presented, including numerical examples. This case may also be referred to as a portfolio choice under loss aversion. In Section 4.5 a description of the analytical optimal portfolio choice under cumulative prospect theory is given and applied to our defined value and weighting function. Finally, the optimal wealth for the different models is compared in Section 4.6.

4.1. MODEL SETTINGS

In this section the settings under which investments take place are described as in [20]. A complete market is assumed, which means that every security can be exchanged and every risk can be hedged. By the second fundamental theorem of asset pricing, this implies the existence of a unique probability measure $\mathbb{Q}$. By the first fundamental theorem of asset pricing, it follows that no arbitrage is allowed and thus all assets have a unique price at all times [18]. This implies the existence of a unique pricing kernel. When changing from the objective probability measure $\mathbb{P}$ to risk neutral measure $\mathbb{Q}$, the drift is replaced by the risk free rate.

The complete market hypothesis makes several assumptions that are described below:

- A complete probability space $(\Omega, \mathscr{F}, F, \mathbb{P})$ describes the market in which $\mathbb{P}$ is the objective probability measure. The $\mathbb{P}$-Brownian motion is N-dimensional and F is the filtration generated by the Brownian motion defined on the probability space.
- Investors make decisions in a finite time horizon $[0, T]$, with $T < \infty$. Decisions can be made at any time.
- The market consists of the risk free bank account with price S_0 and N risky non-dividend paying assets with price S_i for $i = 1, 2, \ldots N$. Investors can trade these assets continuously without transaction costs.

  The price process of the risk free money market account is represented by $S_0(t)$, for which the dynamics are given by:
$$dS_0(t) = r(t)S_0(t)dt, \tag{4.1}$$
  with $S_0(0) = s_0$ and $r(\cdot)$ the interest rate.

  The other assets $S_i(t)$, with $i = 1, 2, \ldots N$ follow Itô processes with drift $\mu_i(t)$ and volatility $\sigma_i(t)$:
$$dS_i(t) = \mu_i(t)S_i(t)dt + \sigma_i(t)S_i(t)dB(t), \text{ for } i = 1, 2, \ldots N, \tag{4.2}$$
  with $S(0) = s$, $\mu_1, \ldots, \mu_N$ and $\sigma_1, \ldots \sigma_N$ adapted processes.

  In vector notation the risky asset vector processes $S(t)$, $\mu(t)$ and $B(t)$ are given by:
$$S(t) = \begin{bmatrix} S_1(t) & S_2(t) & \cdots & S_N(t) \end{bmatrix}^T,$$

$$\mu(t) = \begin{bmatrix}\mu_1(t) & \mu_2(t) & \cdots & \mu_N(t)\end{bmatrix}^T,$$

$$B(t) = \begin{bmatrix}B_1(t) & B_2(t) & \cdots & B_N(t)\end{bmatrix}^T.$$

The volatility matrix is denoted as $\sigma_M(t)$. The dynamics of the risky asset can now be represented in vector notation as well:

$$dS(t) = S(t)\mu(t)dt + S(t)\sigma_M(t)dB(t). \tag{4.3}$$

- The fraction invested in the risky asset, $i = 1, 2, \ldots N$, at time t is represented by $\omega_i(t)$ and the fraction invested the risk free bank account is represented by $\omega_0(t)$. The fraction invested in the risky asset can be written in vector notation as:

$$\omega(t) = \begin{bmatrix}\omega_1(t) & \omega_2(t) & \cdots & \omega_N(t)\end{bmatrix}^T.$$

Then, for any self-financing portfolio the dynamics of wealth $X(t)$ at time t can be written in terms of $dB(t)$ and $S(t)$. The dynamics of wealth $X(t)$ can be described by the following stochastic process in vector notation:

$$\begin{aligned} dX(t) &= \omega_0(t)r(t)X(t)dt + \omega(t)\left(\mu(t)'X(t)dt + \sigma_M(t)'X(t)dB(t)\right) \\ &= (1 - \sum_{i>0}\omega_i(t))r(t)X(t)dt + \omega(t)\left(\mu(t)'X(t)dt + \sigma_M(t)'X(t)dB(t)\right) \\ &= r(t)X(t)dt - \sum_i \omega_i(t)r(t)X(t)dt + \omega(t)\mu(t)'X(t)dt + \omega(t)\sigma_M(t)'X(t)dB(t) \\ &= r(t)X(t)dt + (\mu(t) - \mathbb{1}r(t))'\omega(t)X(t)dt + \sigma_M(t)'\omega(t)X(t)dB(t), \end{aligned} \tag{4.4}$$

with $X(t) \geq 0$ and $\mathbb{1}$ represents a vector of $(N \times 1)$ ones. The initial wealth equals

$$X(0) = x_0 = \omega_0(0)S_0 + \omega(0)s$$

- The Radon-Nikodym derivative for changing the probability measures is defined by

$$Z(t) = \frac{d\mathbb{Q}}{d\mathbb{P}} = exp\left(-\frac{1}{2}\int_0^t ||k(s)||^2 ds - \int_0^t k(s)'dB(s)\right), \tag{4.5}$$

in which $k(s)$ denotes the market price of the risk process or the Sharpe ratio. This quantity is defined as:

$$k(t) = \sigma_M^{-1}(t)(\mu(t) - \mathbb{1}r(t)).$$

The Sharp ratio represents the return of a risky asset over a risk free asset per unit of volatility. Thus, it is a measure of the degree of compensation of the risk taken in terms of the return of the asset. Note that the volatility matrix $\sigma_M(\cdot)$ is invertible as it is assumed that the market model is complete.

4.2. MODERN PORTFOLIO THEORY

Modern Portfolio Theory (MPT) was introduced in 1952 by Markovitch [22] as a model for constructing portfolios of assets in which risk averse investors maximize the expected return for a given level of risk. This means that under MPT, an investor attempts to construct a portfolio such that risks are diversified, while the expected return is not reduced. This results in an efficient portfolio; a portfolio for which the expected return is maximized for a certain level of risk. The collection of efficient portfolios is called the efficient frontier. An investor chooses between the efficient portfolios of the efficient frontier based on his individual risk profile. The investor's individual risk preference can be represented by a function f dependent on the mean μ and standard deviation σ. It is assumed that $\frac{\partial f}{\partial \mu} > 0$ and $\frac{\partial f}{\partial \sigma} < 0$ (risk aversion). An example of pairs (μ, σ) is given in Figure 4.1. The black line represents the efficient frontier; all assets on the efficient frontier are preferred over the assets southeast from the efficient frontier as μ is lower and σ is higher for these assets. As an example, (μ, σ) pair H is preferred over all assets in the striped region. In MPT the following model assumptions are made:

- The return of a portfolio is given by: $E[r_p] = \sum_i w_i E[r_i]$. In this equation r_p represents the portfolio return, r_i the return of asset i and w_i the weight assigned to asset i.
- The variance of the portfolio return is given by: $\sigma_p^2 = \sum_i \sum_j w_i w_j \sigma_{ij}$.

  In Section 4.6 the mean-variance trade-off under MPT will be compared to a similar trade-off under PT and the differences will be discussed.

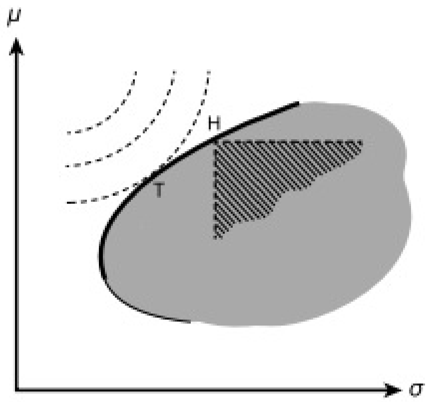


Figure 4.1: Pairs (μ, σ) with the efficient frontier under MPT.

4.3. PORTFOLIO CHOICE UNDER EXPECTED UTILITY THEORY

In this section the formulation of the portfolio choice under the traditional expected utility theory is described. An investor with initial wealth $x_0 \geq 0$ is considered at time $t = 0$. The investor creates a portfolio with an amount of $\omega_0(t)$ invested in the risk free bank account and an amount of $\omega(t)$ in the risky asset at time t. The dynamics of these assets are given by Equation (4.1) and Equation (4.2) and the wealth dynamics at time t is given by Equation (4.4). The investor aims is to maximize his wealth during the period $[0, T]$ according to his subjective value function.

It is assumed that the investor has a power utility function:

$$u(x) = \frac{1}{\eta} x^{\eta}$$

This utility function is a so-called constant relative risk aversion function (CRRA):

$$r_u^{rel}(x) = -x \frac{x^{\eta-1}}{(\eta-1)x^{\eta-2}} = (1-\eta)$$

Under the conditions mentioned in Section 4.1, the following maximization of the expect value of the value function of terminal wealth $X(T)$ is considered:

$$\begin{aligned} \max_{\omega(t)} \quad & E^{\mathbb{P}}[u(X(T))] \\ \text{subject to} \quad & dX(t) = r(t)X(t)dt + (\mu(t) - \mathbb{1}r(t))'\omega(t)X(t)dt + \sigma(t)'\omega(t)X(t)dB(t) \\ & X(t) \geq 0 \ \ \forall t \in [0,T], \\ & X(0) = x_0. \end{aligned} \tag{4.6}$$

In order to obtain the optimal wealth profile for Problem 4.6 the martingale method is used, which will be explained in the following section.

4.3.1. METHODOLOGY

In this section the martingale method to solve the optimal investment problem is explained. The method consists of first finding the wealth under the optimal strategy after which the trading strategy that corresponds to this optimal wealth can be found. The method is a probabilistic method which eliminates the dependence on the specific price dynamics and which focuses on the terminal wealth $X(T)$ and portfolio replication.

Consider all contingent claims K_T which can be replicated by a self financing portfolio with initial capital x_0. Contingent claims are securities which pay one in a particular state of the world and zero otherwise. Obviously X_T is part of this space on which we will focus. Then, Problem 4.6 can be reformulated as a problem of finding the optimal portfolio of contingent claims with the desired payoff $X(T)$ in each state. Thus, the following static problem is equivalent to Problem 4.6:

$$\begin{aligned} \max_{X(T)\geq 0} \quad & E^{\mathbb{P}}[u(X(T))], \\ \text{subject to} \quad & X(T) \in K_T. \end{aligned} \tag{4.7}$$

Note that in this formulation there is no relation with the optimal portfolio strategy but instead the focus lies on the terminal wealth $X(T)$. In order to separate the problem of determining the optimal terminal wealth from the problem of determining the optimal portfolio, the following proposition is used:

Proposition 3. *Under the assumptions described in Section 4.1, the following equivalence holds for all random variables $X(T) \in K_T$:*

$$X(T) \in K_T \iff E^{\mathbb{Q}}[e^{-\int_0^T r(s)ds} X(T)] = x_0.$$

The intuition behind this proposition is as follows. By the completeness assumption, there exists no arbitrage and all contingent claims can be replicated by the available assets. The absence of arbitrage implies a unique price for all assets at any time t. This implies the existence of a unique risk neutral probability measure $\mathbb{Q}$ under which all discounted price processes are martingales.

In the setting described, the terminal wealth $X(T)$ thus has a unique price under $\mathbb{Q}$. As the initial wealth equals x_0 and $X(T)$ is constructed from x_0, this gives $E^{\mathbb{Q}}[e^{-\int_0^T r(s)ds} X(T)] = x_0$. This constraint is called the budget constraint.

Now Problem 4.7 can be reformulated as the following static problem:

$$\begin{aligned} \max_{X(T)\geq 0} \quad & E^{\mathbb{P}}[v(X_T)], \\ \text{subject to} \quad & E^{\mathbb{Q}}[e^{-\int_0^T r(s)ds} X(T)] = x_0. \end{aligned} \tag{4.8}$$

The budget constraint can be transformed to probability measure $\mathbb{P}$ by using the transformation measure $Z(t) = \frac{d\mathbb{Q}}{d\mathbb{P}}$, i.e.,

$$\begin{aligned} E^{\mathbb{Q}}[e^{-\int_0^T r(s)ds}X(T)] &= E^{\mathbb{P}}[e^{-\int_0^T r(s)ds}\frac{d\mathbb{Q}}{d\mathbb{P}}X(T)] \\ &= E^{\mathbb{P}}[e^{-\int_0^T r(s)ds}\exp\left(-\frac{1}{2}\int_0^T ||k(s)||^2 ds - \int_0^T k(s)' dB(s)\right)X(T)] \\ &= E^{\mathbb{P}}[\rho(T)X(T)], \end{aligned} \tag{4.9}$$

in which $\rho(t)$ represents the state price density and is defined as

$$\rho(t) := \exp\left(-\int_0^t r(s)ds\right)Z(t) = \exp\left(-\frac{1}{2}\int_0^t ||k(s)||^2 ds - \int_0^t k(s)' dB(s) - \int_0^t r(s)ds\right). \tag{4.10}$$

The state price density defines the price of one unit of wealth. Note that the price density is also dependent on the state of the world ω besides its dependence on t, so $\rho = \rho(t,\omega)$. However, the dependence on the state of the world is omitted in the notation in the remainder of this thesis. Also note, that in this case ω is different from the weighting vector ω earlier defined.

The final form of the optimization problem is now as follows:

$$\begin{aligned} \max_{X(T)\geq 0} \quad & E^{\mathbb{P}}[u(X(T))] = E^{\mathbb{P}}[\frac{1}{\eta}(X(T))^{\eta}], \\ \text{subject to} \quad & E^{\mathbb{P}}[\rho(T)X(T)] = x_0. \end{aligned} \tag{4.11}$$

In order to solve this problem the Lagrangian must be maximized over $X(T)$. The Lagrangian is given by:

$$\begin{aligned} L &= E^P[u(X(T))] - y(E^P[\rho(T)X(T)] - x_0) \\ &= \int_{\Omega} u(X(T,\omega)) - y[\rho(T)X(T,\omega) - x_0]dP(\omega). \end{aligned} \tag{4.12}$$

The Lagrangian L must be maximized over $X(T)$, which implies that L can be maximized for every ω:

$$\begin{aligned} u'(X^*(T)) = y\rho(T) &\iff \\ X^*(T)^{\eta-1} = y\rho(T) &\iff \\ X^*(T) = (y\rho(T))^{\frac{1}{(\eta-1)}}&. \end{aligned}$$

The value for the Lagrange multiplier y follows from the condition $E[\rho(T)X(T)] = x_0$. Substituting the expression for the optimal wealth $X^*(T)$ gives:

$$\begin{aligned} E[\rho(T)X^*(T)] &= E[\rho(T)(y\rho(T))^{\frac{1}{\eta-1}}] \\ &= y^{\frac{1}{(\eta-1)}}E[\rho(T)^{\left(\frac{1}{\eta-1}+1\right)}] \\ &= y^{\frac{1}{(\eta-1)}}E[\rho(T)^{\left(\frac{\eta}{\eta-1}\right)}] \\ &= y^{\frac{1}{(\eta-1)}}E[e^{-rT\left(\frac{\eta}{\eta-1}\right)-\frac{k^2T}{2}\left(\frac{\eta}{\eta-1}\right)-kB(T)\left(\frac{\eta}{\eta-1}\right)}] \\ &= y^{\frac{1}{(\eta-1)}}e^{-rT\left(\frac{\eta}{\eta-1}\right)-\frac{k^2T}{2}\left(\frac{\eta}{\eta-1}\right)}E[e^{-kB(T)\left(\frac{\eta}{\eta-1}\right)}] \\ &= y^{\frac{1}{(\eta-1)}}e^{-rT\left(\frac{\eta}{\eta-1}\right)-\frac{k^2T}{2}\left(\frac{\eta}{\eta-1}\right)}e^{k^2\left(\frac{\eta^2}{(\eta-1)^2}\right)\frac{T}{2}} \\ &= x_0. \end{aligned} \tag{4.13}$$

An equation for the Lagrange multiplier y is then given by:

$$y = x_0\left(e^{rT\left(\frac{\eta}{\eta-1}\right)+\frac{k^2T}{2}\left(\frac{\eta}{\eta-1}\right)-k^2\left(\frac{\eta^2}{(\eta-1)^2}\right)\frac{T}{2}}\right)^{(\eta-1)}. \tag{4.14}$$

As we now have derived an expression for the optimal wealth $X^*(T)$, the optimal wealth can be compared to the optimal wealth under several models, which will be done in the following sections.

4.4. PORTFOLIO CHOICE UNDER PROSPECT THEORY - NO WEIGHTING FUNCTION

In this section the mathematical formulation of the portfolio choices for loss averse investors is considered [17]. Individuals who maximize the expected value of the prospect theory value function, but without a subjective weighting function are considered. Thus, the aspects of loss aversion and different risk attitudes for gains and losses as described in Section 2.2 are included, but the subjective probability weighting is not yet included. Instead of maximizing the utility function as in Section 4.3, the value function as defined in Section 2.2.1 is maximized. The martingale method is used in order to obtain the optimal wealth profile. The use of this method results in the maximization of a concave function for gains and the maximization of a convex function for losses. The optimal terminal wealth in case of the maximization of a concave function should satisfy the Lagrange condition and the optimal wealth in case of a convex function should be located at the boundaries.

Under the conditions mentioned in Section 4.1, the following maximization of the expect value of the value function of terminal wealth $X(T)$ is considered:

$$\begin{aligned} \max_{\omega(t)} \quad & E^{\mathbb{P}}[v(X(T))], \\ \text{subject to} \quad & dX(t) = r(t)X(t)dt + (\mu(t) - \mathbb{1}r(t))'\omega(t)X(t)dt + \sigma_M t)'(t)X(t)dB(t), \\ & X(t) \geq 0 \ \ \forall t \in [0,T], \\ & X(0) = x_0. \end{aligned} \tag{4.15}$$

By use of the martingale method, Problem 4.15 can be rewritten as:

$$\begin{aligned} \max_{X(T)\geq 0} \quad & E^{\mathbb{P}}[v(X(T))] = \int_{-\infty}^{\infty} v(x)dF(x) = \int_{-\infty}^{\theta} v^-(x)dF(x) + \int_{\theta}^{\infty} v^+ dF(x), \\ \text{subject to} \quad & E^{\mathbb{P}}[\rho(T)X(T)] = x_0. \end{aligned} \tag{4.16}$$

Recall that $v(x)$ is defined in Section 2.2.1 as:

$$v(x) = \begin{cases} v^+(x) = (x-\theta)^a & x \geq \theta, \\ v^-(x) = -\lambda(\theta - x)^b & x < \theta \end{cases} \tag{4.17}$$

with λ the loss-aversion parameter, a, b the risk attitude parameters and θ the reference level.

4.4.1. RESULTS

In this section, Problem 4.16 is solved in order to obtain the optimal wealth for a loss averse investor who maximizes the prospect theory value function as in Equation valuefuh2. The optimal wealth for a loss averse investor is given by the following proposition.

Proposition 4. *The optimal wealth $X^*(T)$ at time T for a loss averse investor with a value function as in Equation valuefuh2 and risk aversion parameters $0 < a < 1$ and $0 < b < 1$ equals:*

$$X^*(T) = \begin{cases} \theta + \left(\frac{y\rho(T)}{a}\right)^{1/(a-1)} & \text{if } \rho < \overline{\rho}, \\ 0 & \text{if } \rho \geq \overline{\rho}, \end{cases} \tag{4.18}$$

where $\overline{\rho}$ solves $f(\overline{\rho}) = 0$ with $f(x) = \left(\frac{1-a}{a}\right)\left(\frac{1}{yx}\right)^{a/(1-a)} a^{1/(1-a)} - \theta yx + \lambda\theta^b$ and $y \geq 0$ satisfies $E[\rho(T)X^*(T)] = x_0$.

Proposition 4 provides insight in the structure of the optimal terminal wealth without specifying the proportion of the portfolio invested in risky assets and the proportion invested in the risk free bank account. As the problem is now formulated in terms of the prospect value function $v(x)$ and the state price density $\rho(T)$, the optimal wealth is a function of those two variables. The proposition shows that the optimal wealth under the prospect value function is discontinuous. As the payoff is positive (above θ) in "good states" $\rho < \overline{\rho}$ and zero in "bad states" $\rho \geq \overline{\rho}$, it is optimal for loss averse-investors to maximize the probability of obtaining a wealth level above θ.

Proof. Let $X_+^*(T)$ be the optimal wealth for the concave value function $v_+(x)$ and $X_-^*(T)$ the optimal wealth for the convex value function $v_-(x)$.

Case $X(T) \geq \theta$
First we look at the case $X(T) \geq \theta$, in which the value function $v_+(x)$ is concave and $X_+^*(T)$ represents the optimal terminal wealth. We take a look at the Lagrangian L:

$$\begin{aligned} L &= E^{\mathbb{P}}[v_+(X(T)] - y(E^{\mathbb{P}}[\rho(T)X(T)] - x_0) \\ &= \int_\Omega v_+(X(T,\omega)) - y[\rho(T)X(T,\omega) - x_0]d\mathbb{P}(\omega). \end{aligned} \tag{4.19}$$

The Lagrangian must be maximized over $X(T)$. This implies that L can be maximized for every ω. Setting the first derivative of L equal to zero gives:

$$v'_+(X^*(T)) = y\rho(T), \tag{4.20}$$

which gives

$$X^*(T) = (v'_+)^{-1}(y\rho(T)). \tag{4.21}$$

Note that similar to $\rho(T)$, the optimal wealth $X^*(T)$ is dependent on the state of the world ω, which is omitted in the notation.

Now, the above described technique will be applied to the PT-value function as in Equation (4.17) for which $v_+(x) = (x-\theta)^a$.

The derivative of $v_+(x)$ equals:

$$v'_+(x) = a(x-\theta)^{a-1}.$$

The inverse form of $v'_+(x)$ is determined so,

$$\left(\frac{v'_+}{a}\right) = (x-\theta)^{a-1} \iff \left(\frac{v'_+}{a}\right)^{1/(a-1)} = (x-\theta) \iff x = \theta + \left(\frac{v'_+}{a}\right)^{1/(a-1)}.$$

In conclusion, the optimal terminal wealth is given by:

$$X_+^*(T) = (v'_+)^{-1}(y\rho) = \theta + \left(\frac{y\rho(T)}{a}\right)^{1/(a-1)}. \tag{4.22}$$

Case $X \leq \theta$
If $X(T) \leq \theta$, the value function $v_-(X)$ is convex and thus the optimal wealth $X_-^*(T)$ is located at $X_-^*(T) = 0$ or at $X_-^*(T) = \theta$.

Now, the local maxima X_-^* and X_+^* are compared to find a global maximum. Consider the following equation:

$$f(\rho(T)) = v_+(X_+^*(T)) - y\rho(T)X_+^*(T) - (v_-(X_-^*(T)) - y\rho(T)X_-^*(T)). \tag{4.23}$$

If $f(\rho(T)) \geq 0$, the optimal solution is given by $X_+^*(T)$. Else, the optimal solution is given by $X_-^*(T)$.

When $X_+^*(T)$ is compared to $X_-^*(T) = \theta$, the following equation holds for $f(\rho)$:

$$\begin{aligned} f(\rho(T)) &= (X_+^*(T) - \theta)^a - y\rho(T)X_+^* + y\rho(T)\theta \\ &= a\left(\frac{1-a}{a}\right)\left(\frac{a}{y\rho}\right)^{a/(1-a)} \end{aligned} \tag{4.24}$$

from which is clear that $f(\rho(T)) > 0$ for all $\rho(T)$ as $y > 0$ and $0 < a < 1$. Therefore, $X_-^* = \theta$ can not represent the optimal wealth.

When $X_+^*(T)$ is compared to $X_-^*(T) = 0$, then the following equation holds for $f(\rho(T))$:

$$f(\rho(T)) = a\left(\frac{1-a}{a}\right)\left(\frac{a}{y\rho(T)}\right)^{a/(1-a)} - y\rho(T)\theta + \lambda\theta^b. \tag{4.25}$$

It is clear that $f(\rho(T)) > 0$ holds for $\rho(T) \leq \frac{\lambda}{y}\theta^{b-1}$ since $\lambda > 0$, $y > 0$ and $0 < a < 1$. As $\lim_{x\to\infty} f(x) = -\infty$ and f is strictly decreasing, $f(\rho)$ has one zero in $[\frac{\lambda}{y}\theta^{b-1}, +\infty)$ which is denoted by $\overline{\rho}$ with $f(\overline{\rho}) = 0$. As f is strictly decreasing, $f(\rho) > 0$ gives $\rho(T) < \overline{\rho}$ and $f(\rho(T)) \leq 0$ for $\rho \geq \overline{\rho}$. Thus, $X_+^*(T)$ is optimal for $\rho(T) < \overline{\rho}$ and $X_-^*(T)$ is optimal for $\rho(T) \geq \overline{\rho}$.

□

As we now have an expression for the optimal wealth, the value of Lagrange multiplier y can be determined in order to give numerical examples in the following section. In order to determine an expression for y, we first take a look at the distribution of the pricing kernel $\rho(T)$.

Distribution pricing kernel

First, we take a look at the distribution of the pricing kernel $\rho(T)$. Recall that, assuming a constant interest rate r, $\rho(T)$ is defined as:

$$\rho(T) = e^{-\frac{1}{2}k^2T - kB(T) - rT,}$$

with $B(T)$ a Brownian motion with a normal distribution $N(0, T)$.

The expectation of $\rho(T)$ equals:

$$E[\rho(T)] = e^{-\frac{1}{2}k^2T - rT}E[e^{-kB(T)}] = e^{-\frac{1}{2}k^2T - rT}e^{\frac{1}{2}k^2T} = e^{-rT}. \tag{4.26}$$

The variance of $\rho(T)$ equals:

$$Var[\rho(T)] = e^{-k^2T - 2rT}Var(e^{-kB(T)}) = e^{-k^2T-2rT}((e^{k^2T} - 1)e^{k^2T}) = e^{k^2T - 2rT} - e^{-2rT}. \tag{4.27}$$

Thus, the distribution of $\rho(T)$ is log-normal:

$$\rho(T) \sim LogN(e^{-rT}, e^{k^2T-2rT} - e^{-2rT}). \tag{4.28}$$

By taking the natural log of $\rho(T)$, the distribution of $\rho(T)$ can be written as a normal distribution:

$$log(\rho(T)) = -\frac{1}{2}k^2T - kB(T) - rT,$$

$$E[log(\rho(T))] = -\frac{1}{2}k^2T - rT, \tag{4.29}$$

$$Var[log(\rho(T))] = k^2T. \tag{4.30}$$

Then, the distribution of $log(\rho(T))$ equals:

$$log(\rho(T)) \sim N(-\frac{1}{2}k^2T - rT, k^2T). \tag{4.31}$$

Lagrange multiplier

As we now have an expression for the distribution of pricing kernel $\rho(T)$, we can take a look at the Lagrange multiplier y. In order to obtain an expression for y the condition $E[\rho(T)X^*(T)] = x_0$ is examined. Working out this condition gives:

$$E[\rho(T)X^*(T)] = x_0 \iff$$

$$E[\rho(T)(\theta + \left(\frac{y\rho(T)}{a}\right)^{1/(a-1)})1_{\rho(T)\leq\overline{\rho}}] = x_0 \iff$$

$$\theta e^{-rT}N\left(\frac{log(\overline{\rho}) + (r - 0.5k^2)T}{k\sqrt{T}}\right) + \left(\frac{y\rho(T)}{a}\right)^{1/(a-1)} e^{\frac{a}{(1-a)}(r+0.5k^2)T + 0.5(\frac{a^2}{(1-a)^2}k^2T}N\left(\frac{\log(\overline{\rho}) + (r-0.5k^2)T}{kT} + \frac{k\sqrt{T}}{(1-a)}\right). \tag{4.32}$$

Thus in order to obtain a solution for the optimal wealth the following system of equations has to be solved for Lagrange multiplier y and $\overline{\rho}(T)$.

$$\begin{cases} \theta e^{-rT} N\left(\frac{log(\overline{\rho})+(r-0.5k^2)T}{k\sqrt{T}}\right) + \left(\frac{y\rho(T)}{a}\right)^{1/(a-1)} e^{\frac{a}{(1-a)}(r+0.5k^2)T+0.5\frac{a^2}{(1-a)^2}k^2T} N\left(\frac{\log(\overline{\rho})+(r-0.5k^2)T}{kT} + \frac{k\sqrt{T}}{(1-a)}\right) = x_0, \\ f(\overline{\rho}) = \left(\frac{1-a}{a}\right)\left(\frac{1}{y\overline{\rho}}\right)^{a/(1-a)} a^{1/(1-a)} - \theta y\overline{\rho} + \lambda\theta^b = 0. \end{cases} \tag{4.33}$$

4.4.2. NUMERICAL EXAMPLES

In this section several numerical examples of the optimal wealth profile for investors under loss aversion are considered. The optimal wealth profile will be considered for different prospect parameters in order to measure the impact of different levels of prospect sentiment on the optimal wealth profile. Before the results are considered, first an explanation is given of the way in which the results are achieved and how the results should be interpreted.

As seen, Equation (4.18) for the optimal wealth under loss aversion is in terms of state price density $\rho(T)$. Therefore, in the following examples the optimal wealth will be plotted against the state price density $\rho(T)$. Thus, the x-axis represents the price of one unit of wealth from low to high: low prices of one unit of wealth are related to good states of the world and high prices are related to bad states of the world. The y-axis represents the optimal wealth $X^*(T)$. In all examples, a constant interest rate r and a constant Sharpe ratio k over time are considered. The expression of $\rho(T)$ from Equation (4.10) reduces in this case to:

$$\rho(T) = e^{-rT-\frac{k^2T}{2}-kB(T)}.$$

The x-axis values of all figures are obtained by simulating $N = 10^5$ paths of the process $\rho(t)$ on $[0, T]$ using $n = 10^3$ steps. The possible outcomes of these simulations at final time T are displayed on the x-axis. Note that for all examples the same set of simulated paths $\rho(T)$ is used. The histogram of these simulated values is given in Figure 4.2 and shows a lognormal distribution with mean and variance as defined in Equation 4.26 and 4.27. In order to obtain the optimal wealth profile, the Lagrange multiplier y and the value of $\overline{\rho}$ are obtained by solving the equations $f(\overline{\rho}) = 0$ and $E[\rho(T)X^*(T)] = x_0$ numerically.

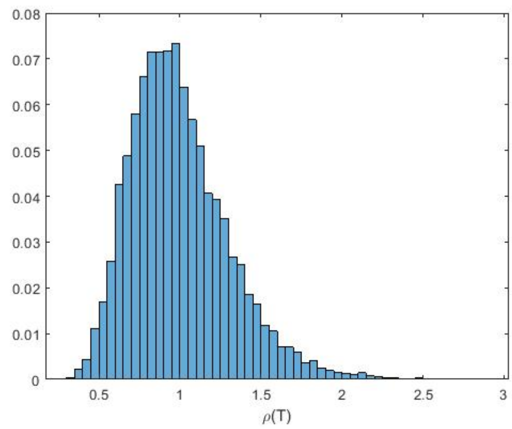


Figure 4.2: Normalized histogram of pricing kernel $\rho(T)$ with parameters $r = 0.5\%$, $k = 0.3$ and $T = 1$.

In Figure 4.3 the optimal wealth $X^*(T)$ of an investor with the prospect value function is plotted against the state price density $\rho(T)$. The parameters used in Figure 4.3 are $r = 0.5\%$, $k = 0.3$, $T = 1$, $\theta = x_0 = 1$, $a = b = 0.88$ and $\lambda = 2.25$, which refers to prospect sentiment. As expected, the wealth profile is decreasing in the state price density $\rho(T)$; if the price of one unit of wealth is higher under identical conditions for the other parameters, the obtained optimal wealth is lower. However, the wealth profile is not continuous: for bad states of the world with $\rho(T) > \overline{\rho}$ the wealth profile goes to zero discontinuously. This discontinuity is caused by the S-shape of the value function with respect to a reference point: for wealth levels above the reference point θ (gains) a concave function is optimized and for wealth levels below the reference point θ (losses) a convex function is optimized. For values of $\rho(T) < \overline{\rho}$, corresponding to a low price of one unit of wealth, the optimum of the concave function for gains dominates. This optimum is above the reference level θ. For values $\rho(T) > \overline{\rho}$, corresponding to a high price of one unit of wealth, the optimum of the convex function for losses dominates. This optimum equals zero. As the payoff is above θ in relatively good states of the world and equal to zero in relatively bad states of the world, an investor should maximize the probability of reaching the reference level θ. Note that if the value function would be globally concave, the wealth profile would continuously tend to the reference level θ:

$$\lim_{\rho(T)\to\infty} X_+^*(T) = \lim_{\rho(T)\to\infty}\left(\theta + \left(\frac{y\rho(T)}{a}\right)^{\frac{1}{(a-1)}}\right) = \lim_{\rho(T)\to\infty}\left(\theta + \left(\frac{a}{y\rho(T)}\right)^{\frac{-1}{(a-1)}}\right) = \theta \tag{4.34}$$

The last step in Equation (4.34) follows from the fact that $0 < a < 1$ and $y \geq 0$ constant.

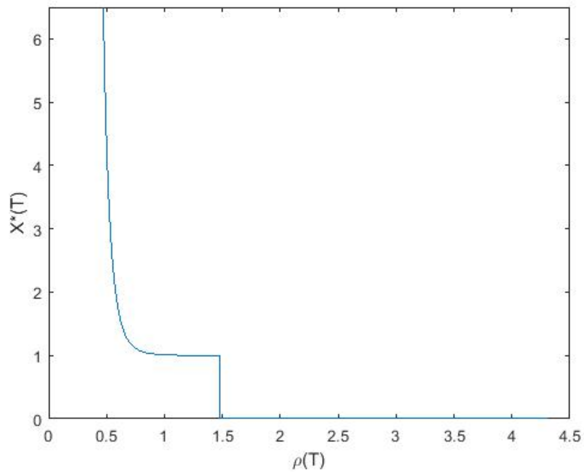


Figure 4.3: Optimal wealth $X^*(T)$ at time T against the state price density $\rho(T)$ for an investor with a prospect value function. The parameters used are: $r = 0.5\%$, $k = 0.3$, $T = 1$, $\theta = x_0 = 1$, $\lambda = 2.25$ and $a = b = 0.88$, which refers to prospect sentiment.

In Figure 4.4 the optimal wealth profile for different levels of the loss aversion λ is given under identical conditions for the other parameters. A lower level of loss aversion corresponds to a lower level at which losses are more painful than gains of comparable size are pleasant. As can be seen, a higher level of loss aversion results in a higher value of $\overline{\rho}$: there is a larger range of values $\rho(T)$ for which $X^*(T) \geq \theta$. This is in line with our expectations. If the level of loss aversion increases, the degree at which losses are painful than gains are pleasurable increases. Therefore, the optimal wealth is the optimal wealth of the gain part for a broader range of $\rho(T)$ values. In other words, the optimal wealth $X_-(T) = 0$ for losses is obtained for a worse state of the world or a higher value of the state price density $\rho(T)$. As can be seen, a higher level of loss aversion also results in a lower optimal wealth profile for values of $\rho(T) < \overline{\rho}(T)$. As the investor is more averse for losses, a larger part of the wealth is used for buying insurances to protect against potential losses. This results in a lower optimal wealth profile in good states.

In Figure 4.5 the optimal wealth profile for different levels of starting wealth x_0 is given. The higher the starting wealth the higher the optimal wealth for good states $\rho(T) < \overline{\rho}(T)$, for example for $x_0 = 1$ for $\rho(T) < 1.4$. This means that if the starting wealth increases, the investor decreases the amount of wealth invested in insurances for protecting against potential losses: the investor is less careful as his wealth is larger. On the other hand, the higher x_0 the wider the range of $\rho(T)$-values for which the investor buys insurances.

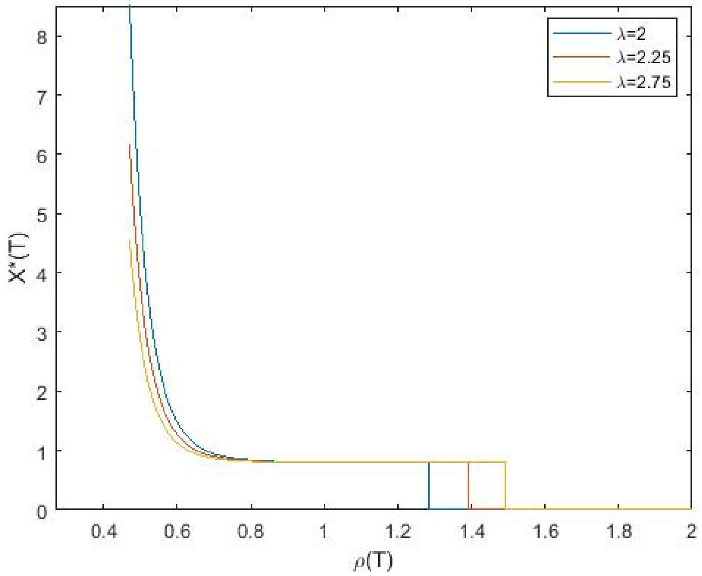


Figure 4.4: Optimal wealth at time $X^*(T)$ at time T for an investor with a prospect value function for different levels of loss aversion λ. The parameters used are: $r = 0.5\%$, $k = 0.3$, $T = 1$, $\theta = x_0 = 1$ and $a = b = 0.88$.

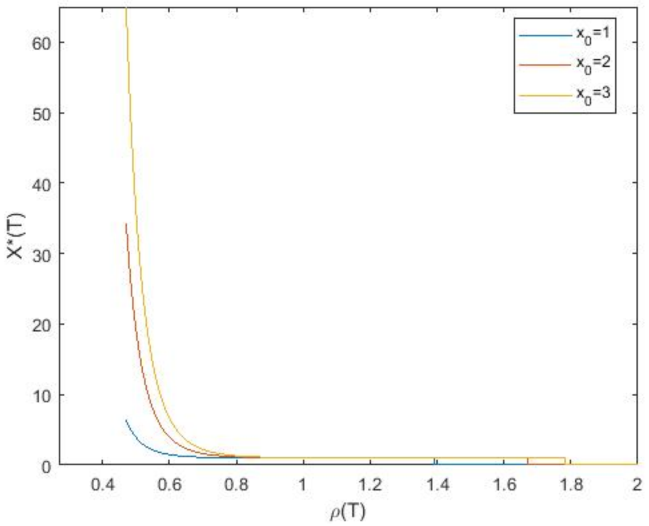


Figure 4.5: Optimal wealth $X^*(T)$ at time T for an investor with a prospect value function for different levels of starting wealth x_0. The parameters used are: $r = 0.5\%$, $k = 0.3$, $T = 1$, $\theta = x_0 = 1$, $a = b = 0.88$ and $\lambda = 2.25$.

4.5. PORTFOLIO CHOICE UNDER CUMULATIVE PROSPECT THEORY

In the previous section a solution for the optimal wealth under the prospect value function without a weighting function is obtained and numerical examples related to this wealth were considered. The phenomena of loss aversion and individual risk attitudes are incorporated, but the subjective weighting function is absent which is however an essential part of cumulative prospect theory; cumulative prospect theory describes human behavior by using an S-shaped value function, a reference point and probability weighting functions. In the context of investment problems, the use of a non-linear probability weighting function and a partly concave and partly convex value function leads to highly involved problems, that are not solvable via conventional approaches for expected utility maximization such as dynamic programming and the martingale approach. Therefore, in this section a new approach is considered.

In this section first the objective function of cumulative prospect theory without a weighting function is compared to the objective function of CPT. Then, in Section 4.5.1, the portfolio optimization problem under CPT is described. In Section 4.5.2 the difficulties related to this problem that make it impossible to use conventional approaches are discussed. In Section 4.5.4, the mathematical approach to deal with the CPT-portfolio problem is discussed according to [16] and [14]. It should be noted that the main purpose of these sections is to give a general idea of the solution procedure and the final solution of the CPT-optimization problem. Due to the difficulty of the problem, the proofs of the theories to obtain the final result are extensive. Therefore, these proofs are not included in this thesis. However, all the proofs can be found in [16] and [14]. The final goal is to apply the described solution procedure of Section 4.5.4 to our defined value and weighting functions in order to derive an expression for the optimal wealth. This is described in Section 4.5.5. Then, these results are compared with the results of Section 4.4, in Section 4.6, in order to measure the degree at which the weighting function influences the final optimal wealth profile.

Comparison

First, we take a look at the objective functions that are maximized under prospect theory without a weighting function (Section 4.4) and under cumulative prospect theory. Recall that under prospect theory without a weighting function the goal is to maximize the expected value of the value function of wealth $X(T)$ at time T. The quantity to maximize equals:

$$V_{PT}(X(T)) = E[v(X(T)] = \int_{-\infty}^{\infty} v(x)dF_X(x) = \int_{-\infty}^{\theta} v^-(x)dF_X(x) + \int_{\theta}^{\infty} v^+(x)dF_X(x). \tag{4.35}$$

As can be seen, v is a non-linear distortion on x when evaluating the mean of $X(T)$.

Now the following CPT-criterion is considered, for example, for gains:

$$\begin{aligned} V_{CPT}(X_+(T)) &= \int_{-\infty}^{+\infty} w(P(v(X^+) > v(x))dv(x) \\ &= \int_{-\infty}^{+\infty} w(P(X^+ > x)dv(x) \\ &= \int_{-\infty}^{\infty} w(1-F(x))dv(x) \\ &= \int_{-\infty}^{\infty} v(x)d[-w(1-F(x)] \\ &= \int_{-\infty}^{\infty} v(x)w'(1-F(x))dF(x). \end{aligned} \tag{4.36}$$

In this derivation, the fourth equality follows from integration by parts. In this equality the CDF is distorted by the weighting function in contrast to Equation (4.35). In the last equality, $w'(1-F_X(x))$ represents a weight assigned to $F(x)$. Thus, in the CPT-criterion the risk attitude is incorporated by the non-linear weight assigned to the decumulative distribution rather than only the value function applied to the payoff x as in Equation (4.35).

The prospect value of the terminal wealth $X(T)$ can now be written as

$$V_{CPT}(X(T)-\theta) = V_{CPT}^+(X^+(T)-\theta) - V_{CPT}^-(X^-(T)-\theta),$$

where

$$\begin{aligned} V_{CPT}(X(T)) &= V_{CPT}^+(X^+(T)) - V_{CPT}^-(X^-(T)) \\ &= \int_0^{\infty} w_+(P(v_+(X^+(T)) \geq x))dx - \int_0^{\infty} w_-(P(v_-(X^-(T)) \geq x))dx \\ &= \int_0^{\infty} w'_+(1-F(x))v^+(x)f(x)dx - \int_0^{\infty} w_-(P(v_-(X^-(T)) \geq x))dx \\ &= \int_0^{\infty} w'_+(1-F(x))v^+(x)f(x)dx + \int_{-\infty}^{0} w'_-(1-P(v_-(X^-) \geq y))dy \\ &= \int_0^{\infty} w'_+(1-F(x))v^+(x)f(x)dx + \int_{-\infty}^{0} w'_-(F(y))v^-(y)f(y)dy. \end{aligned} \tag{4.37}$$

Note that the result of Equation (4.36) is used and that definition 4.37 is equivalent to the CPT-value, as defined in Equation (3.31).

In the coming sections general definitions of the value and weighting functions are used. The CPT-agent considered has a reference point zero at time T, to which the terminal wealth $X(T)$ is evaluated as a gain or a loss. The overall value function is S-shaped and is given by:

$$v(x) = v_+(x)\mathbb{1}_{x\geq 0}(x) - v_-(x)\mathbb{1}_{x<0} \tag{4.38}$$

In this equation the functions $v_\pm(x)$ are strictly increasing and concave functions on $\mathbf{R}^+$ satisfying $v_\pm(0) = 0$ and representing risk-aversion on gains and risk-seeking on losses. Note that the value function in cumulative prospect theory as formulated in Definition 2.6 satisfies the general definition 4.38 with $v_+(x)$ from Definition 4.38 equal to $v(x)$ from Definition 2.6 and similarly $v_-(x)$ equal to $v(-x)$. The functions $w_\pm(x)$ are non-linear, nondecreasing and differentiable probability distortions on gains and losses from $[0,1]$ to $[0,1]$ satisfying $w_\pm(0) = 0$, $w_\pm(1) = 1$ and $w_\pm(p) > p$ for p close to 0, and $w_\pm < p$ for p close to 1. Note that the weighting function as formulated in Definition 2.7 satisfies above definition.

4.5.1. MODEL

The setting in which investments take place under CPT is equal to the setting as described in Section 4.1. The CPT portfolio choice problem in terminal wealth $X(T)$ can be formulated as:

$$\begin{array}{llll} \max\limits_{X(T)\geq 0} & V_{CPT}(X(T)) & & , \\ \text{subject to} & E^{\mathbb{P}}[\rho(T)X(T)] & = & x_0. \end{array} \tag{4.39}$$

Once this problem is solved with optimal value $X^*(T)$, the optimal portfolio is the portfolio which replicates $X^*(T)$. Note that in the case of $w_+(x) = x$, the equation for $V^+_{CPT}(X)$ reduces to $V^+_{CPT}(X^+(T)) = E[v_+(X(T))]$ and identically if $w_-(x) = x$ then the equation of $V^-_{CPT}(X(T))$ reduces to $V^-_{CPT}(X^-(T)) = E[v_-(X(T))]$. Note also that it is assumed that the investor's behavior only affects his value function and not the market: the condition $E[\rho(T)X(T)] = x_0$ is according to market pricing and is not influenced by an individual value function. The optimization problem 4.39 has different features compared to portfolio optimization under expected utility theory as in Section 4.3 and the optimization problem under loss aversion as in Section 4.4. In the following section it will be clear that Problem 4.39 is a difficult problem to solve. The difficulties arise from the overall S-shaped value function instead of a global convex/concave value function and the non-linear probability weighting function.

4.5.2. DIFFICULTIES

In this section the difficulties related to optimization problem 4.39 are described: time-inconsistency and absence of global concavity/convexity. Due to these difficulties, it is not possible to apply conventional techniques to solve the optimization problem.

TIME INCONSISTENCY

The use of the probability weighting function causes the phenomonon of time-inconsistency. Time-consistency holds if and only if the trade-off between CPT-values at time τ and τ' is evaluated equally at time 0 and at time t for all τ, τ' and t:

$$\frac{V^{\tau,0}_{CPT}}{V^{\tau',0}_{CPT}} = \frac{V^{\tau,t}_{CPT}}{V^{\tau',t}_{CPT}}. \tag{4.40}$$

An example of time-inconsistency under discrete prospect theory, see Definition 11, is given below. Note that under CPT the line of thought is similar.

Consider an agent at the casino. The following time positions are considered:

- τ: time after winning five bets.
- τ': time after winning three bets.
- t: time after winning two bets.

The agent gains two euros for every winning bet. When the agents enters the casino, he knows that the probability of winning five bets in a row, and hence of accumulating a total of ten euros, is very low:

$$P(\text{winning five bets in a row}) = \left(\frac{1}{2}\right)^5 = \frac{1}{32}$$

The probability of winning three bets in a row and thus of gaining six euros equals:

$$\mathbb{P}(\text{winning three bets in a row}) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}.$$

However, if the agent actually wins the first two bets, the probability of winning the fifth bet and hence of winning ten euros equals:

$$\mathbb{P}(\text{winning five bets in a row}|\text{first two bets won}) = \frac{1}{8}.$$

Similarly, the probability of winning three bets given the agent already won the first two bets equals:

$$\mathbb{P}(\text{winning three bets in a row}|\text{first two bets won}) = \frac{1}{2}.$$

The trade-off of Equation (4.40) can now be written as:

$$\frac{V_{pros}^{\tau,0}}{V_{pros}^{\tau',0}} = \frac{w(\frac{1}{32})v^+(10)}{w(\frac{1}{8})v^+(6)} \neq \frac{V_{pros}^{\tau,t}}{V_{pros}^{\tau',t}} = \frac{w(\frac{1}{8})v^+(10)}{w(\frac{1}{2})v^+(6)}. \tag{4.41}$$

The inequality is caused by the non-linear weighting function $w(x)$, as defined in Equation (2.4). As time passes, the probabilities of final outcomes change, which in turn means that the degree to which the agent under- or overweights these outcomes also changes. For example, under probability weighting, a moderate probability outcome is underweighted while a low probability is overweighted. The fact that some final outcomes are initially overweighted but subsequently underweighted, or vice-versa, means that the agent's preferences over gambling strategies change over time [23]. Due to the absence of time-consistency, it is not possible to solve Problem 4.39 via dynamic programming. First, the absence of these properties is discussed, after which the solution procedure is discussed in Section 3.5.4.

ABSENCE OF GLOBAL CONCAVITY/CONVEXITY

If we look at the objective function 4.36 of Problem 4.39, the global concavity of the left integral (gains) and the global convexity of the right integral (losses) is destroyed by the derivatives of the non-linear weighting functions $w_-(x)$ and $w_+(x)$ which will be elaborated below. Similar to the function to maximize in the proof of Proposition 4 in Section 2.1, the aim now is to maximize in a pointwise fashion,

$$w'_+(1-F(x))v_+(x) - y[\rho(T)x - x_0]. \tag{4.42}$$

However, if we look at the second derivative of $w'_+(1-F(x))v^+(x)$, this equals:

$$w'''_+(1-F(x))f(x)^2 v_+(x) + w''_+(1-F(x))f'(x)v'_+(x) + w''_+(1-F(x))f(x)v'_+(x) + w''_+(1-F(x))f(x)v'_+(x) + w'_+(1-F(x))v'_+(x) \tag{4.43}$$

The second derivative is clearly not concave in x. Therefore, the method applied as in Section 4.4 for investing under loss aversion is not applicable. The problem of non-concavity will be solved in the next sections using a transformation. This transformation turns the problem in a concave maximization problem for which the Lagrangian multiplier method can be applied.

In case of losses the aim is to maximize in a pointwise fashion:

$$w'_{-}(F(x))v^{-}(x) - y[\rho(T)x - x_0]. \tag{4.44}$$

If we look at the second derivative it is equal to:

$$w'''_{-}(F(y))f(y)^2 v^{-}(y) + w''_{-}(F(y))f(y)'v^{-}(y) + w''_{-}(F(y))f(y)v^{-\prime}(y) + w''_{-}(F(y))f(y)v'_{-}(y) + w'_{-}(F(y))v''_{-}(y). \tag{4.45}$$

The second derivative is clearly not convex in x.

4.5.3. ILL-POSEDNESS

Prior to discussing the solution procedure of Problem 4.39, first the ill-posedness of the problem is considered. A maximization problem is called ill-posed if its supremum is not finite. This means that an investor can reach an infinite value without taking into consideration the payoffs. Under the CPT-model the well-posedness is not guaranteed. The following theory holds for ill-posedness:

Theorem 3. *Problem 4.39 is ill-posed under one of the following conditions:*

1. *There exists a non-negative $\mathscr{F}_T$-measurable random variable X such that $E[\rho(T)X(T)] < \infty$ and $V^{+}_{CPT}(X(T)) = +\infty$.*
2. *$v_{+}(\infty) = +\infty$, $\overline{\rho}(T) = +\infty$ and $w_{-}(x) = x$.*

The first part of Theorem 3 states that the problem is ill-posed if one can find a claim which is non-negative and which has a finite price and an infinite cumulative prospect value. Thus, the investor can buy the claim at initial time and reach a infinite value at final time. The second part states that if the value function can reach arbitrarily large values, a probability weighting function on losses is necessary in order to have a well-posed problem. In order to exclude the ill-posedness of Problem 4.39, it is assumed that $V^{+}_{CPT}(X) < \infty$ for any nonnegative $\mathscr{F}_T$-measurable random variable X satisfying $E[\rho(T)X(T)] < +\infty$. Note that the ill-posedness as stated in part in Theorem 3 is automatically excluded by the non-linearity of the cumulative prospect weighting function.

4.5.4. SOLUTIONS

In order to solve the continuous-time CPT problem, methods are needed to handle the S-shaped value-function and the probability distortion. To handle the S-shaped value function, the problem is decomposed into a gain part problem and a loss part problem. The gain part problem is a maximisation problem involving a concave value function and a probability distortion. The probability distortion is overcome by a quantile formulation, which changes the random variable $X(T)$ to its quantile. The loss part problem is to minimise a concave function which leads to corner point solutions. Hereafter, the solutions of the gain part problem and the loss part problem are combined in an optimal way to solve the original problem.

STEP 1

In Step 1, the problem is decomposed into two subproblems. The splitting of the problem is based on the following. If $X(T)$ is feasible for Problem 4.39, then it can be split in $X^{+}(T)$, which defines an event $A = \{X(T) \geq 0\}$ and initial price $x_{+} = E[\rho(T)X^{+}(T)]$, and $X^{-}(T)$ which corresponds to A^c and $E[\rho(T)X^{-}(T)] = x_{+} - x_0$. After solving the gain- and loss part problems, their solutions are combined by optimising the parameters in the first step in order to find the optimal A and x_{+}.

- Gain part problem with parameters (A, x_{+})

$$\begin{aligned} \max_{X(T)} \quad & V^{+}_{CPT}(X(T)) = \int_0^{\infty} w_{+}(P(v^{+}(X(T)) > y))dy, \\ \text{subject to} \quad & E[\rho(T)X(T)] = x_{+},\ X(T) \geq 0 \text{ a.s}, X(T) = 0 \text{ a.s. on } A^c, \end{aligned} \tag{4.46}$$

  with $x_{+}x_0^{+} \geq 0$ and $A \in \mathscr{F}_T$ given. Note that $V^{+}_{CPT}(X(T))$ is finite for any feasible $X(T)$ as assumed. Let's denote $v_{+}(A, x_{+})$ the optimal value of Problem 4.46. The following three cases are possible:

  - If $P(A) > 0$, then the feasible region of Equation (4.46) is non-empty: let

$$X(T) = \frac{x_{+}1_A}{\rho(T)P(A)}$$

then

$$E[\rho(T)X(T)] = \frac{x_+}{P(A)}E[1_A] = x_+$$

. In this case $v_+(A, x_+)$ is defined as the supremum of Problem 4.46.

- If $P(A) = 0$ and $x_+ = 0$, then the only feasible solution is $X(T) = 0$ and thus $v_+(A, x_+) = 0$.
- If $P(A) = 0$ and $x_+ > 0$ then the problem has no feasible solution and we define $v_+(A, x_+) = -\infty$.

- Loss part problem with parameters (A, x_+)

$$\begin{aligned} \min_{X(T)} \quad & V^-_{CPT}(X(T)) = \int_0^\infty w_-(P(v_-(X(T) > y))dy, \\ \text{subject to} \quad & E[\rho(T)X(T)] = x_+ - x_0, X(T) \geq 0 \text{ a.s. }, X(T) = 0 \text{ a.s. on } A, \end{aligned} \tag{4.47}$$

with $x_+ \geq x_0^+$ and $A \in \mathscr{F}_T$ given.

Let's denote $v_-(A, x_+)$ the optimal value of problem (4.47). Again we have three cases:

- If $P(A) < 1$, then $v_-(A, x_+)$ is defined as the infimum of Problem 4.47.
- If $P(A) = 1$ and $x_+ = x_0$, then the only feasible solution is $X(T) = 0$ and then $v_-(A, x_+) = 0$.
- If $P(A) = 1$ and $x_+ \neq x_0$, then there is no feasible solution and we define $v_-(A, x_+) = \infty$.

STEP 2

In Step 2 the solutions from the gain part problem and the loss part problems are combined in an optimal way to solve the original problem. The following problem is solved:

$$\begin{aligned} \underset{(A,x_+)}{\text{maximize}} \quad & v_+(A, x_+) - v_-(A, x_+), \\ \text{subject to} \quad & \begin{cases} A \in \mathscr{F}_T,\ x_+ \geq x_0^+, \\ x_+ = 0 \text{ when } P(A) = 0,\ x_+ = x_0 \text{ when } P(A) = 1. \end{cases} \end{aligned} \tag{4.48}$$

Problem 4.48 consists of finding the optimal event A which means the optimal split between good states (gains) and bad states (losses) and the corresponding price of the gains x_+.

JUSTIFICATION

The justification of splitting the original problem in two subproblems is based on the following two propositions.

Proposition 5. *Problem 4.39 is ill-posed, iff Problem 4.48 is ill-posed.*

Proposition 6. *Given $X^*(T)$ and define $A^* = \{\omega : X^*(T) \geq 0\}$ and $x_+^*(T) = E[\rho(T)(X^*)^+(T)]$. Then $X^*(T)$ is optimal for 4.39 iff (A^*, x_+^*) are optimal for Problem 4.48 and $X_+^*(T)$ and $X_-^*(T)$ are optimal respectively for problems 4.46 and 4.47 with parameters (A^*, x_+^*).*

The implication of the propositions 5 and 6 is that the original Problem 4.39 is equivalent to the combination of problems 4.46, 4.47 and 4.48. As a consequence, a solution of 4.39 can be obtained via the solutions of problems 4.46, 4.47 and 4.48.

SIMPLIFICATION

If we look at the optimization problem of Step 2, the decision variables are a real number x_+ and a random event A. In order to solve this problem the following theorem from [14] is used:

Theorem 4. *For any feasible pair (A, x_+) of Problem 4.48, there exists a real number $d \in [\underline{\rho}, \overline{\rho}]$ such that $\overline{A} = \{\omega : \rho \leq d\}$ satisfies:*

$$v_+(\overline{A}, x_+) - v_-(\overline{A}, x_+) \geq v_+(A, x_+) - v_-(A, x_+).$$

According to Theorem 4, only events of the form $A = \{\rho(T) \leq d\}$ have to be considered with d real. Thus, the event of having gains is characterised by the pricing kernel and a threshold. In view of Theorem 4, Problem 4.48 can be replaced by a simpler problem, where $v_+(d, x_+)$ denotes $v_+(\{\omega : \rho(T) \leq d\}, x_+)$ and $v_-(d, x_+)$ denotes $v_-(\{\omega : \rho(T) \leq d\}, x_+)$:

$$\begin{aligned} &\text{maximize} \quad v_+(d, x_+) - v_-(d, x_+), \\ &\text{subject to} \quad \begin{cases} \underline{\rho} \leq d \leq \overline{\rho}, x_+ \geq x_0^+, \\ x_+ = 0 \text{ when } d = \underline{\rho}, x_+ = x_0 \text{ when } d = \overline{\rho}, \end{cases} \end{aligned} \tag{4.49}$$

with $\overline{\rho}$ and $\underline{\rho}$ the essential upper- and lower bounds.

The following theory characterizes the solution of Problem 4.49:

Theorem 5. *Given $X^*(T)$ and let $d^* = F^{-1}(P\{X^* \geq 0\})$, $x_+^* = E[\rho(X^*)^+]$, with F representing the cdf of ρ. The wealth X^* is optimal for the original Problem 4.39 iff (d^*, x_+^*) is optimal for the simplified Problem 4.49 and $(X^*)^+ \mathbb{1}_{\rho \leq d^*}$ and $(X^*)^- \mathbb{1}_{\rho > d^*}$ are optimal for the subproblems 4.46 and 4.47 with parameters $(\{\omega : \rho \leq d^*\}, x_+^*)$. In this case $\{\omega : X^* \geq 0\}$ and $\{\omega : \rho \leq d^*\}$ are identical.*

Now the gain part problem is solved in order to obtain $v_+(d, x_+)$ and the loss part problem is solved in order to obtain $v_-(d, x_+)$

GAIN PART PROBLEM

In this section, the gain part problem is solved with $A = \{\omega : \rho \leq d\}$, $\underline{\rho} \leq d \leq \overline{\rho}$ and $x_+ \geq x_0^+ > 0$. To solve the gain part problem, first a more general gain part problem is considered according to []

$$\begin{aligned} \max_{X(T)} \quad & V_{CPT}^+(X(T)) = \int_0^\infty w(P(v_+(X) > y))dy, \\ \text{subject to} \quad & E[\rho(T)X(T)] = x_+, \quad X(T) \geq 0. \end{aligned} \tag{4.50}$$

Note that it is assumed that $x_+ \geq x_0^+ > 0$ as the case $x_+ = 0$ gives the trivial solution $X^*(T) = 0$.

CHANGE OF VARIABLES: QUANTILE FORMULATION

As already seen, v_+ is a concave function but the objective function $V_{CPT}^+(X(T))$ is not concave in y due to the weighting function $w_+(x)$ resulting in a non-convex optimization problem with a constraint. To overcome this problem, the quantile formulation is used according to [15]. The decision variable $X(T)$ is changed to its quantile function G. This transformation results in concavity in terms of G.

The idea of the quantile formulation is the following: as $X \sim G_X(Z)$, with $Z \sim U(0,1)$, where G_X is the quantile function of X, X can be replaced by $G_X(Z)$ without changing the value of the objective function. In order to solve Problem 4.50 the following lemma and assumption can be used:

Lemma 6. *If an optimal solution is admitted to Problem 4.50, then it is of the form: $X^*(T) = G^{-1}(1 - F_\rho)(\rho)$ with $G(\cdot)$ the distribution function of $X^*(T)$.*

Assumption 1. *The CDF of pricing kernel ρ is continuous.*

With the use of Lemma 6 and Assumption 1 a change of variables method will be used to eliminate the negative effect of the weighting function for solving the gain part problem. Denote by $Z = 1 - F_\rho(\rho)$, then $Z \sim U(0,1)$ and $\rho = F_\rho^{-1}(1 - Z)$, due to the fact that ρ has a continuous CDF. As a result of Lemma 6 the decision variable of Problem 4.50 is of the form $G^{-1}(Z)$, with $G(\cdot)$ the distribution function of $X^*(T)$. Therefore, we will write Problem 4.50 in terms of $G(\cdot)$.

The objective function of Problem 4.50 can be rewritten as:

$$\begin{aligned}V^{+}_{CPT}(X(T)) &= \int_0^\infty w(P(v_+(X)>y))dy \\ &= \int_0^\infty v_+(x)d[-w(1-F(x)] \\ &= \int_0^\infty v_+(x)w'(1-F(x))dF(x) \\ &= \int_0^1 v_+(G_x(z))w'(1-z))dz \\ &= E[v_+(G_X(Z_\rho)w'(1-Z_\rho)].\end{aligned} \tag{4.51}$$

Thus, Problem 4.50 can be rewritten by using a quantile formulation as:

$$\begin{aligned}\max_{G_{X(T)}} \quad & V^{+}_{CPT}(X(T)) = \int_0^1 v_+(G_X(z))w'(1-z)dz = E[v_+(G_X(Z_\rho))w'(1-Z_\rho)], \\ \text{subject to} \quad & \int_0^1 G_X(z)F_\rho^{-1}(1-z)dz = E[F_\rho^{-1}(1-Z_\rho)G_X(Z_\rho)] = x_+, G\in \mathbf{G}.\end{aligned} \tag{4.52}$$

in which $\mathbf{G}$ is the set of quantile functions and F_ρ^{-1} is the quantile function of pricing kernel ρ. Because ρ has a continuous CDF, the function F_ρ^{-1} is strictly increasing. Note that while the original problem was not concave in X, the objective function of Problem 4.52 is concave in G_X. The second derivative reads:

$$v''_+(G_X(Z_\rho))w'(1-Z_\rho). \tag{4.53}$$

As $v''_+<0$ and $w'>0$ the second derivative is negative and thus the objective function is concave in G_X. The original problem is now transformed into a problem which is solvable via conventional techniques. Before solving the transformed problem, first the equivalence of the transformed and the original problem is considered. The following lemma from [14] shows the equivalence of the original Problem 4.50 and the transformed Problem 4.52.

Proposition 7. *If G^* is optimal for Problem* 4.52*, then $X^*(T)=(G^*)^{-1}(Z)$ optimal for Problem 4.50 and also conversely if X^* optimal for Problem 4.50, then its distribution function G^* is optimal for Problem 4.52 and $X^*(T)=(G^*)^{-1}(Z)$.*

In order to solve problem 4.52 the Lagrange multiplier method now can be used as the objective function is concave with respect to the quantile function and the constraint is linear. The following problem is considered:

$$\max_{G\in\mathbf{G}} L = \int_0^1 v_+(G_X(z))w'(1-z) - yG_X(z)F_\rho^{-1}(1-z)dz, \tag{4.54}$$

for some $y>0$. In order to solve this problem, the Lagrangian can be maximized in a pointwise fashion for each $z\in(0,1)$. Setting the derivative equal to zero gives:

$$v'_+(G_X(z))w'(1-z) - yF_\rho^{-1}(1-z) = 0,$$

and thus

$$G_X^*(z) = (v'_+)^{-1}\left(\frac{yF_\rho^{-1}(1-z)}{w'(1-z)}\right). \tag{4.55}$$

for $z\in(0,1)$ and $y>0$ satisfying

$$E[G^*(1-F_\rho^{-1}(\rho(T))\rho(T)] = x_+$$

It is assumed that $\frac{F_\rho^{-1}(1-z)}{w'(1-z)}$ is non-decreasing and thus the solution 4.55 is a quantile function. Once $G^*(\cdot)$ is obtained, the optimal terminal wealth $X^*(T)$ can be obtained by $X^*(T)=G^*(1-F_\rho(\rho))$, according to Lemma 6.

Now we turn back from the general gain part related to Problem 4.50 to Problem 4.46. In order to obtain an optimal solution for the positive part Problem 4.46, the general results described above for Problem 4.50 are applied to Problem 4.46 with $A=\{\rho(T)\le d\}$ and $x_+\ge x_0^+>0$. It is assumed that $P(A)>0$ or $d>\underline{\rho}$. Then, the following theory holds for the optimal solution of Problem 4.46.

Theorem 7. *Given* $A = \{\omega : \rho(T) \le d\}$ *with* $\underline{\rho} \le d \le \overline{\rho}$ *and* $x_+ \ge x_0^+ > 0$.

Then, the optimal solution for Problem 4.46 is given by:

$$X_+^*(T) = \left((v_+')^{-1}\left(\frac{y\rho}{w_+'(F_\rho(\rho))}\right)\right)\mathbb{1}_{\rho(T)\le d},$$

with optimal value

$$v_+(c, x_+) = E[v_+\left((v_+')^{-1}\left(\frac{y\rho(T)}{w_+'(F_\rho(\rho(T))}\right)\right)w_+'(F_\rho(\rho(T))\mathbb{1}_{\rho\le d}],$$

with $y > 0$ *the unique real numbers satisfying* $E[\rho(T)X^*(T)] = x_+$.

LOSS PART PROBLEM

Now we turn to solving the loss part problem. In order to solve the loss part problem, again first a more general loss part problem is considered:

$$\begin{array}{rll} \min\limits_{X(T)} & V_{CPT}^-(X(T)) & = \int_0^\infty w(P(v_-(X) > y))dy, \\ \text{subject to} & E[\rho(T)X(T)] & = x_+ - x_0, X(T) \ge 0. \end{array} \tag{4.56}$$

A quantile transformation transforms Problem 4.56 into:

$$\begin{array}{rll} \min\limits_{G} & V_{CPT}^-(X(T)) & = \int_0^1 v_-(G(z))w'(1-z)dz \\ \text{subject to} & \int_0^1 F_\rho^{-1}(z)G(z)dz & = x_+ - x_0, X(T) \ge 0, G \in \mathbf{G} \end{array} \tag{4.57}$$

The following proposition [14] ensures the equivalence between general loss part Problem 4.56 and Problem 4.57:

Proposition 8. *If* G^* *is optimal for Problem* 4.57*, then* $X^* = (G^*)(Z)$ *is optimal for Problem 4.56 and also conversely if* X^* *optimal for Problem 4.56, then its distribution function* G^* *is optimal for Problem 4.57 and* $X^* = (G^*)(Z)$.

In contrast to the gain part problem, if we use a quantile transformation, a concave function is to be minimised due to the S-shaped value function. This requires an approach different from the Lagrange multiplier-method. The problem is a combinatorial optimisation problem, which gives a corner point solution and which is characterized by the following propositon:

Proposition 9. *The optimal solution to Problem 4.57, if it exists, must be of the form*

$$G^*(z) = \frac{x_+ - x_0}{E[\rho\mathbb{1}_{\{F_\rho(\rho)>b\}}]}\mathbb{1}_{(b,1)}$$

with $z \in [0,1)$ *and* $b \in [0,1)$. *In this case, the optimal solution to Problem 4.56 equals* $X_-^*(T) = G^*(F_\rho(\rho))$.

Using the result of proposition 9, problem 4.57 can be rewritten, for some $b \in (0,1)$, as:

$$\begin{array}{rl} \min\limits_{b} & f(b) = \int_0^1 v_-(G(z))w'(1-z)dz, \\ \text{subject to} & G = \frac{x_+ - x_0}{E[\rho\mathbb{1}_{F_\rho>b}]}\mathbb{1}_{(b,1]} \text{ with } 0 \le b < 1. \end{array} \tag{4.58}$$

Theorem 8. *Problem 4.58 admits an optimal solution iff* $\min_{0\le d<\overline{\rho}} v_-\left(\frac{x_+ - x_0}{E[\rho\mathbb{1}_{\{\rho>d\}}}\right)w(P(\rho > d))$ *admits an optimal solution* d^**, in which case the optimal solution to 4.56 equals*

$$X_-^*(T) = \frac{x_+ - x_0}{E[\rho(T)\mathbb{1}_{\rho>d^*}]}\mathbb{1}_{\rho>d^*}.$$

COMBINING GAIN AND LOSS PARTS

The combination of the gain and loss parts leads to solving the following problem, which is related to Problem 4.48:

$$\begin{aligned} \max_{(d,x_+)} \quad & E\ [\mathrm{v}_+\left((v_+')^{-1}\left(\frac{y(d,x_+)\rho}{w_+'(F_\rho(\rho)}\right)\right) w_+'(F_\rho(\rho))\mathbf{1}_{\rho\le d}] - v_-\left(\frac{x_+-x_0}{E[\rho(T)\mathbf{1}_{\rho>d}]}\right) w_-(1-F_\rho(d)), \\ \text{subject to} \quad & \underline{\rho} \le d \le \overline{\rho},\ x_+ \ge x_0^+,\ x_+ = 0 \text{ in case } d = \underline{\rho} \text{ and } x_+ = x_0 \text{ in case } c = \overline{\rho}. \end{aligned} \tag{4.59}$$

in which y satisfies

$$E[(v_+')^{-1}\left(\frac{y(d,x_+)\rho}{w_+'(F_\rho(\rho)}\right)\rho 1_{\rho\le d}] = x_+.$$

The final solution for Problem 4.39 is given by the following theorem:

Theorem 9. *If (d^*, x_+^*) is optimal for Problem 4.59 then $\{X^* \ge 0\}$ and $\{\rho(T) \le d^*\}$ are identical up to a zero probability event and the final optimal solution is given by:*

$$X^* = \left[(v_+')^{-1}\left(\frac{y\rho(T)}{w'(F_{\rho(T)}(\rho))}\right)\right]\mathbb{1}_{\rho\le d^*} - \left[\frac{x_+^* - x_0}{E[\rho(T)\mathbb{1}_{\rho>d^*}]}\right]\mathbf{1}_{\rho>d^*}$$

ECONOMIC INTERPRETATION

Before applying the described method for solving problems under CPT to our specific value and weighting function, the economic interpretation of the final solution as stated in Theorem 9 is considered according to [14]. The final solution given in Theorem 9 is completely determined by the pricing kernel ρ and threshold d^*. This threshold divides the world in two states: $\rho \le d^*$ represents a good state and $\rho > d^*$ a bad state. In a good state the agent obtains a wealth above the reference wealth and in a bad state he looses a fixed constant value $\left[\frac{x_+^*-x_0}{E[\rho(T)\mathbf{1}_{\rho>d^*}]}\right]\mathbf{1}_{\rho>d^*}$. The equation for the optimal wealth shows that under the optimal strategy, the agent should buy a lottery ticket in good states ($\rho \le c^*$) with payoff $(v_+')^{-1}\left(\frac{y\rho}{w'(F_\rho)}\right)\mathbb{1}_{\rho\le c^*}$ and at cost $x_+^* \ge x_0$. As the price of gains at time T x_+^* is above the initial wealth x_0, a claim has to be sold in bad states ($\rho > c^*$), with payoff $\frac{x_+-x_0}{E[\rho\mathbf{1}_{\rho>c^*}]}\mathbb{1}_{\rho>c^*}$ in order to overcome the shortage $x_+^* - x_0$. This means that the investor borrows to invest in stocks. Thus, the optimal strategy is a gambling strategy in which the agent gambles on good states of the market and accepts a fixed loss in bad states.

4.5.5. APPLICATION

In this section the general technique of deriving the optimal wealth under cumulative prospect theory is applied to the case of our defined value and weighting functions and pricing kernel. It is assumed that the value function is of the form $v_+(x) = x^a$ and $v_-(x) = \lambda x^b$, for $x \ge 0$. This is equivalent to the value function earlier defined in Definition 2.6 with reference point taken as zero. The weighting function is as defined in Definition 2.7 and the pricing kernel as in Definition 4.10. As already shown, this value function and weighting function clearly satisfy the conditions of the general defined value function in Section 2.2. Therefore, the solution method of Section 4.5.4 is applicable.

GAIN PART

First, we take a look at the gain part problem for a given (d, x_+) with $0 < d \le \infty$ and $x_+ \ge x_0^+$, as $d = 0$ would give the trivial solution $x_+ = 0$. By the use of Theorem 7, the optimal solution can be written as:

$$X_+^*(d,x_+) = (v_+')^{-1}\left(\frac{y(d,x_+)\rho(T)}{w'(F_\rho(\rho)}\right)\mathbb{1}_{\rho(T)\le d} = \left(\frac{y(d,x_+)\rho(T)}{w'(F_\rho(\rho(T))a}\right)^{1/(a-1)}\mathbb{1}_{\rho(T)\le d},$$

as the inverse of the derivative of the value function equals:

$$(v_+')^{-1}(y) = \left(\frac{y}{a}\right)^{1/(a-1)}.$$

In order to determine the value of the Lagrange multiplier $y(d,x_+)$, we solve the condition

$$E[\rho(T)X_+^*(d,x_+)] = x_+ \iff$$

$$E[\rho(T)\left(\frac{y(d,x_+)\rho(T)}{w'(F_\rho(\rho(T))a}\right)^{1/(a-1)} 1_{\rho(T)\leq d}] = x_+ \iff$$
$$\left(\frac{y(d,x_+)}{a}\right)^{1/(a-1)} E[\left(\frac{\rho(T)}{w'(F_\rho(\rho(T))}\right)^{1/(a-1)} \rho(T)\mathbb{1}_{\rho(T)\leq d}] = x_+ \iff$$
$$y(d,x_+) = a\left(\frac{x_+}{E[\left(\frac{\rho(T)}{w'(F_\rho(\rho(T))}\right)^{1/(a-1)} \rho(T)\mathbb{1}_{\rho(T)\leq d}]}\right)^{(a-1)}.$$

Rewriting gives the expression for X_+^*:

$$\begin{aligned} X_+^* &= \left(\frac{x_+}{E\left[\left(\frac{\rho(T)}{w'(F_\rho(\rho(T))}\right)^{1/(a-1)} \rho(T)\mathbb{1}_{\rho\leq d}\right]}\right)\left(\frac{\rho(T)}{w'(F_\rho(\rho(T)))}\right)^{1/(a-1)} \mathbb{1}_{\rho(T)\leq d} \\ &= \left(\frac{x_+}{\phi(d)}\right)\left(\frac{\rho(T)}{w'(F_\rho(\rho(T)))}\right)^{1/(a-1)} \mathbb{1}_{\rho(T)\leq d}, \end{aligned} \tag{4.60}$$

with $\phi(d) = E\left[\left(\frac{\rho(T)}{w'(F_\rho(\rho(T))}\right)^{1/(a-1)} \rho(T)\mathbb{1}_{\rho(T)\leq d}\right]$.

The corresponding optimal value is given by:

$$\begin{aligned} v(d,x_+) &= E[v_+(X_+^*(d,x_+))w'(F_\rho(\rho(T)))1_{\rho(T)\leq d}] \\ &= E\left[\left(\frac{x_+}{\phi(d)}\right)^a \left(\frac{\rho(T)}{w'(F_\rho(\rho(T)))}\right)^{a/(a-1)} w'(F_\rho(\rho(T)))1_{\rho(T)\leq d}\right] \\ &= \left(\frac{x_+}{\phi(d)}\right)^a E\left[\left(\frac{\rho(T)}{w'(F_\rho(\rho(T)))}\right)^{a/(a-1)} w'(F_\rho(\rho(T)))1_{\rho(T)\leq d}\right] \\ &= \left(\frac{x_+}{\phi(d)}\right)^a E\left[\left(\frac{\rho(T)}{w'(F_\rho(\rho(T)))}\right)^{a/(a-1)} \left(\frac{\rho(T)}{w'(F_\rho(\rho(T)))\rho}\right)^{-1} 1_{\rho(T)\leq d}\right] \\ &= \left(\frac{x_+}{\phi(d)}\right)^a E\left[\left(\frac{\rho(T)}{w'(F_\rho(\rho(T)))}\right)^{a/(a-1)-1} 1_{\rho(T)\leq d}\right] \\ &= \left(\frac{x_+}{\phi(d)}\right)^a \phi(d) \\ &= \phi(d)^{1-a} x_+^a. \end{aligned} \tag{4.61}$$

Now, Problem 4.59 can be rewritten as:

$$\begin{aligned} \max_{(d,x_+)} \quad & v(d,x_+) = \phi(d)^{1-a}x_+^a - \frac{\lambda(x_+ - x_0)^b w_-(1-F_\rho(d))}{(E[\rho(T)1_{\rho(T)>d})^b}, \\ \text{subject to} \quad & 0 \leq d \leq \infty,\ x_+ \geq x_0^+,\ x_+ = 0 \text{ in case } d = 0, \text{ and } x_+ = x_0 \text{ in case } d = \infty. \end{aligned} \tag{4.62}$$

We solve Problem 4.62 in order to get the optimal wealth. The following theorem [14] characterizes the optimal wealth.

Theorem 10. *Assume* $x_0 \geq 0$, $a = b$ *and let* $v(d,x_+) = \phi(d)^{1-a}[x_+^a - k(d)(x_+ - x_0)^a]$ *with*

$$k(d) = \frac{\lambda w_-(1-F(d))}{\phi(d)^{1-a}(E[\rho(T)1_{\rho>d}])^a}.$$

Assume that $\inf_{d>0} k(d) \geq 1$. *Then, the optimal portfolio for the problem is the portfolio which replicates the following wealth:*

$$X^*(T) = \frac{x_0}{\phi(\infty)}\left(\frac{w'_+(F(\rho(T))}{\rho(T)}\right)^{1/(1-a)}, \tag{4.63}$$

with

$$\phi(d) = E\left[\left(\frac{\rho(T)}{w'(F_\rho(\rho(T))}\right)^{1/(a-1)} \rho(T)1_{\rho\leq d}\right]. \tag{4.64}$$

Proof. Before considering $\sup_{d>0,x_+>x_0}$, first the following problem is considered:

$$max_{x_+>x_0}(x^a - k(x-x_0)^a),$$

with $k \geq 1$ fixed. The derivative of the function to maximize equals $f'(x) = a(x^{a-1} - k(x-x_0)^{a-1})$. If $k \geq 1$, then $f'(x) \leq 0$ for all $x \geq x_0$ and thus $x^* = x_0$ is optimal with optimal value x_0^a. This result is used in rewriting $\sup_{d>0,x_+>x_0}$:

$$\begin{aligned} \sup_{d>0,x_+>x_0} v(d,x_+) &= \sup_{d>0}\left(\phi(d)^{1-a}\sup_{x_+\geq x_0}(x_+^a - k(d)(x_+-x_0)^b\right) \\ &= \sup_{d>0}\left(\phi(d)^{1-a}x_0^a\right) \\ &= \phi(\infty)^{1-a}x_0^a = v(\infty,x_0) \geq 0. \end{aligned} \tag{4.65}$$

From Equation (4.65) it follows that $(d^*, x_+^*) = (\infty, x_0)$ is optimal for Problem 4.62. Substituting the optimal values for (d^*, x_+^*) in Equation (4.60) gives the optimal wealth:

$$\left(\frac{x_+}{\phi(\infty)}\right)\left(\frac{\rho(T)}{w'(F_\rho(\rho(T)))}\right)^{1/(a-1)} \mathbb{1}_{\rho(T)\leq d}. \tag{4.66}$$

□

4.5.6. NUMERICAL EXAMPLE

As we have an expression for the optimal wealth under CPT, in this section a numerical example is discussed to measure the impact of the prospect parameters. It is important to note that under CPT the optimal wealth profile is independent of loss aversion parameter λ and risk-seekingness parameter b for losses. Therefore, only the impact of weighting parameter γ is considered. Remember that the lower γ the more overweighting of low probabilities and the more underweighting of high probabilities. The following conclusions can be made:

- The higher the value of γ, the higher the optimal wealth X^*_{CPT} for all values of $\rho(T)$. A possible explanation is that an attitude with less over- and underweighting is more rational, resulting in a higher optimal wealth.
- The better the state of the world $\rho(T)$ the higher the sensitivity towards γ. This means that for lower prices of the one unit of wealth, the optimal wealth is more sensitive to changes in the investor's attitude towards the over- and underweighting of probabilities.
- The higher the level of sentiment, or in other words the lower γ, the lower the sensitivity towards γ.

4.6. COMPARISON PORTFOLIO MODELS

So far, we have discussed three different models for portfolio management; portfolio management under EUT, under loss aversion and under CPT. In this section, a comparison between these models is made to investigate the influence of the prospect value function and the prospect weighting function on the optimal wealth profile. Firstly, an overview of the optimal wealth levels under the different models is given below.

- **Optimal wealth under EUT - rational investor**

$$X^*_{EUT}(\rho(T)) = \left(y\rho(T)\right)^{\frac{1}{\eta-1}},$$

with

$$y = x_0\left(e^{rT\left(\frac{\eta}{\eta-1}\right)+\frac{k^2T}{2}\left(\frac{\eta}{\eta-1}\right)-k^2\left(\frac{\eta^2}{(\eta-1)^2}\right)\frac{T}{2}}\right)^{(\eta-1)}.$$

- **Optimal wealth under loss aversion - only value function**

$$X^*_{LA}(T) = \begin{cases} X^*_+(T) = \theta + \left(\frac{y\rho(T)}{a}\right)^{1/(a-1)} & \text{if } \rho < \overline{\rho}, \\ X^*_-(T) = 0 & \text{if } \rho \geq \overline{\rho}, \end{cases}$$

with $\overline{\rho}$ so that $f(\overline{\rho}) = 0$, with

$$f(x) = \left(\frac{1-a}{a}\right)\left(\frac{1}{yx}\right)^{a/(1-a)} a^{1/(1-a)} - \theta yx + \lambda\theta^b$$

and $y \geq 0$ satisfies $E[\rho(T)X^*(T)] = x_0$.

The reference level (gains/losses) is represented by θ, a represents the risk aversion parameter in the domain of gains (the lower a the more risk averse for gains), b represents the risk-seeking parameter in the domain of losses (the lower b the more risk-seeking for gains) and λ represents the loss aversion parameter (the higher λ the higher the displeasure of a loss relative to the pleasure of a gain).

- **Optimal wealth under CPT - value and weighting function**

$$X^*_{CPT}(T) = \frac{x_0}{\phi(\infty)}\left(\frac{w'_+(F(\rho(T))}{\rho(T)}\right)^{1/(1-a)}, \tag{4.67}$$

with

$$\phi(d) = E\left[\left(\frac{\rho(T)}{w'(F_\rho(\rho(T))}\right)^{1/(a-1)} 1_{\rho\leq d}\right]. \tag{4.68}$$

The function w_+ represents the weighting function. The lower γ the higher the degree of overweighting of low probabilities and underweighting of high probabilities.

4.6.1. MODERN PORTFOLIO THEORY VS PROSPECT THEORY

The first comparison made, is the comparison between modern portfolio theory and PT. In order to compare these models, a similar concept to that of the mean-variance diagram is defined. The difference between the models is made clear by means of a fictive data set.

As seen in Section 4.2, if an MPT-investor wants to increase the portfolio's expected return, he must invest in more risky or more volatile assets. As explained, the trade-off between mean and variance can be represented in a mean-variance diagram. In case of CPT the expected return above the investor's reference point is considered instead of the expected return of an investment. This means that the average gain with respect to the investor's reference point is considered, weighted by the investor's subjective probability function. Thus, the average gain is defined as:

$$\mu_{CPT} = \sum_{s,\Delta s>0} w^+(p_s)v^+(r_s-\theta), \tag{4.69}$$

in which s represents the state. The average gain can be seen as the value assigned to gains.

Under PT instead of the deviation of the expected return as in MPT, now the expected portfolio return under the investor's reference point is considered or in other words the average loss [21]:

$$\sigma_{CPT} = -\frac{1}{\lambda} \sum_{s,\Delta s<0} w^{-}(p_s) v^{-}(\theta - r_s).$$

It is important to note that the investor's risk attitude and attitude towards probabilities is now incorporated via the value- and weighting function. The average loss can be seen as the absolute value of the normalized losses. The expression for the average loss is normalized. In Figure 4.6 the PT return-risk profile is plotted.

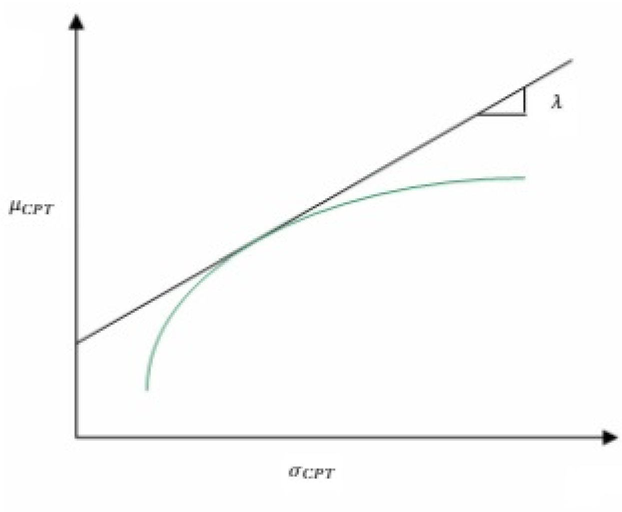


Figure 4.6: Return-risk profile under PT (Source: [21]).

In order to illustrate the difference between the trade-off between reward and risk between MPT and PT, a fictive portfolio of risky and risk free assets is considered. Consider a time period of ten years, two risky assets (for example a share index and a commodity) and one risk free asset (government bond). In Figure 4.7 a mean-variance diagram of the assets returns is given, based on fictive data. As can be seen, Asset 2 and Asset 3 are equal in mean. However, Asset 3 has a significantly lower standard deviation and is therefore preferred over Asset 2.

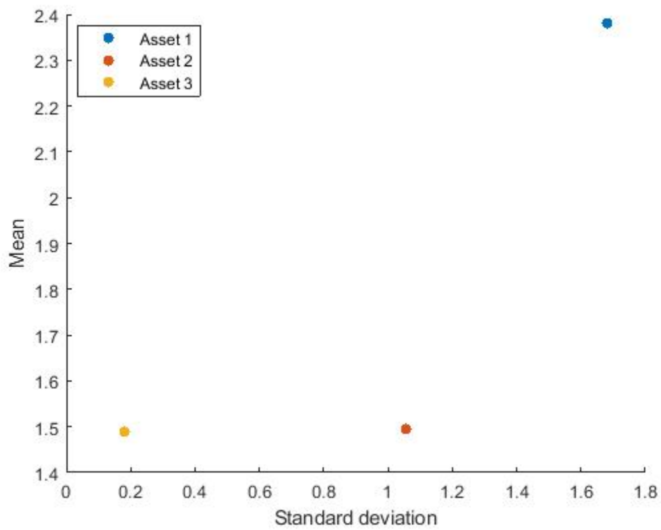


Figure 4.7: Mean-variance diagram.

In Figure 4.8 an average-gain and average-loss diagram is given. This means that the returns are considered under CPT under TK-sentiment. As can be seen, Figure 4.8 is significantly different from Figure 4.7. While in the previous case Asset 2 was preferred over Asset 3, this preference is not immediately clear as Asset 3 now has a lower average gain than Asset 2.

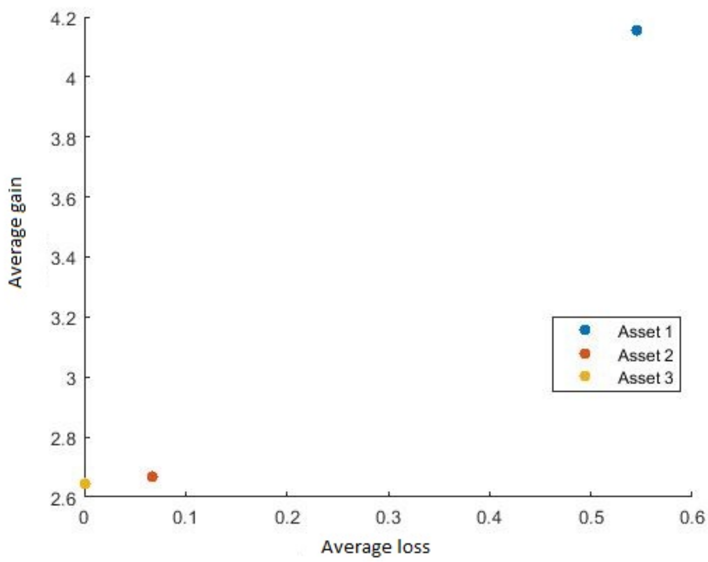


Figure 4.8: Average gain-average loss diagram under TK-sentiment.

4.6.2. Expected utility theory vs loss aversion

In this section we will discuss the results of comparing the optimal wealth profile under EUT with the optimal wealth profile under loss aversion. In Figure 4.9 the optimal wealth profile under EUT and the optimal wealth level under loss aversion are given. The level of risk aversion for gains under loss aversion equals the levels of risk aversion under EUT. As can be seen, the optimal wealth profile under EUT decreases gradually to zero for bad states of the world, while the wealth profile under loss aversion decreases discontinuously to zero because of the risk-seeking behavior for losses. Before the loss aversion profile arrives at zero it approaches the reference level θ. For low values of $\rho(T)$, or in other words, in good states of the world, the wealth profile under loss aversion is significantly lower than the wealth profile under EUT. This can be explained by the phenomenon of loss aversion; as the investor is averse for losses he invests a part of his wealth in insurances in order to protect for bad states of the world. This results in a lower optimal wealth for good states of the world.

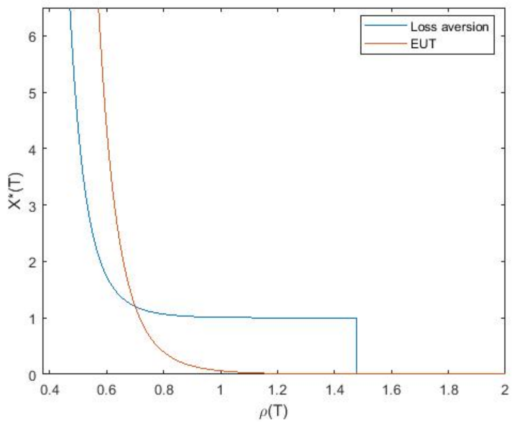


Figure 4.9: Optimal wealth profile for a loss averse investor and a EUT investor with a power utility function. The parameters used are $a = b = 0.5$, $\theta = 0.8$, $\lambda = 2.25$, $k = 0.003$, $T = 1$, $x_0 = 1$ and $\eta = 0.88$.

4.6.3. Loss aversion vs cumulative prospect theory

In this section the focus is on comparing the optimal wealth for investors under loss aversion with the optimal wealth for investors under cumulative prospect theory in order to investigate the influence of incorporating the weighting function. If a weighting function is included in the investor's behavior, the following results can be derived:

- If TK-sentiment and Moderate TK sentiment are considered, the optimal wealth under loss aversion is significantly higher than the optimal wealth profile under CPT for all values of pricing kernel $\rho(T)$ and for all values of weighting parameter γ. The better the state of the world, the larger the difference. This means that under good states of the world the weighting function has a significant negative impact on the optimal wealth profile, while under worse states the negative impact becomes smaller.
- The higher the value of γ, or in other words the smaller the over- and under weighting of probabilities, the smaller the difference between the optimal wealth under CPT and under loss aversion. This is reasonable as in case of a higher γ, the CPT framework resembles more closely the loss aversion framework.

5
DATA ANALYSIS

In the previous chapters different levels of prospect sentiment are considered: TK-sentiment, Moderate TK-sentiment and zero prospect sentiment. These levels of sentiment are used for option valuation and optimal portfolio management. It is important to note that the estimates of prospect parameters are based on psychological experiments and therefore the question is whether these estimates are suitable for financial applications. To this end, in Section 5.1 the prospect parameters are estimated based on the historical returns of a market portfolio. In Section 5.2 a hedge test is performed under prospect sentiment dynamics.

5.1. EMPIRICAL ESTIMATES PROSPECT PARAMETERS

In this section a method of estimating the prospect parameters empirically is considered based on [17]. First a method for estimating the loss aversion parameter λ from portfolio management under loss aversion is considered. Hereafter the results of applying the method to market data is considered.

5.1.1. METHODOLOGY

Again a time horizon of $[0, T]$ is considered in which a representative agent invests in N risky assets with returns $r_1, r_2, ..., r_N$ and a risk free asset with risk free asset r_0. The agents assigns weights $\omega_1, ..., \omega_N$ with the goal to maximize the expected value of the value function of his wealth at time T. Let ω be the vector of weights and let $v(x)$ be the value function as defined in Definition 2.3.

The maximization problem is then as follows:

$$\begin{aligned} \max_{\omega} \quad & E^{\mathbb{P}}\left[v(X(T)) = \begin{cases} -\lambda(\theta - X(T))^b, \text{ if } X(T) \le \theta, \\ (X(T)-\theta)^a, \text{ if } X(T) > \theta \end{cases} \right] \\ \text{subject to} \quad & X(T) \ge 0; X(T) = (r_0 + (r - \mathbb{1}r_0)'\omega)x_0 \end{aligned} \tag{5.1}$$

where $\overline{r}$ is a vector of $(N \times 1)$ returns r_k, $\overline{w}$ is a vector of weights w_k and $\mathbb{1}$ is a vector $(N \times 1)$ ones.

Then, a first-order optimality condition is considered for the excess returns of $E^M = (r - \mathbb{1}r_0)'\omega$ which represents the return of a market portfolio relative to the risk free rate. The market portfolio is considered as it is representative for an agent in the market and thus provides the optimal solution to Problem 5.1. The final wealth $X(T)$ can be written as $X(T) = (r_0 + E^M)x_0$ and the optimality conditions are as follows:

$$\int_{E^M \le 0} \lambda b(\theta - (r_0 + E^M)x_0)^{(b-1)} E^M dG(r) + \int_{E^M > 0} a((r_0 + e^M)x_0 - \theta)^{(a-1)} E^M dG(r) = 0, \tag{5.2}$$

$$\lambda \int_{E^M \le 0} b(\theta - (r_0 + E^M)x_0)^{(b-1)} E^M dG(r) = -\int_{E^M > 0} a((r_0 + E^M)x_0 - \theta)^{(a-1)} E^M dG(r).$$

Now it is possible to write the loss aversion parameter λ in terms of the distribution of the returns and risk aversion and risk-seekingness parameters a and b, i.e.,

$$\lambda = -\frac{\int_{E^M > 0} a((r_0 + E^M)x_0 - \theta)^{(a-1)} E^M dG(r)}{\int_{E^M \le 0} b(\theta - (r_0 + E^M)x_0)^{(b-1)} E^M dG(r)}. \tag{5.3}$$

With the use of the observed excess returns a sample estimator for λ in Equation (5.3) can be constructed by summing over the positive and negative excess returns. Let $\{E_t^M\}_{t=1}^n$ be the negative excess returns and $\{E_t^M\}_{t=n+1}^N$ the positive ones. A sample estimator $\hat{\lambda}$ of λ is then given by:

$$\hat{\lambda} = -\frac{\sum_{t=1}^n a((r_0 + E_t^M)x_0 - \theta)^{(a-1)} E_t^M}{\sum_{t=n+1}^N b(\theta - (r_{0t} + E_t^M)x_0)^{(b-1)} E_t^M}. \tag{5.4}$$

5.1.2. RESULTS - LOSS AVERSION PARAMETER

In this section the method described in Section 4.1.1 is applied to a set of historical returns of a market portfolio. In Figure 5.1 the results of applying the described methodology on a portfolio of stocks of different indices (NYSE, AMEX, NASDAQ) for a period from 1927 till 2017 with yearly excess returns. As an estimate of the risk free rate every year the US Treasury bill rate over one month is used. From Figure 5.1 we can conclude the following:

- The estimates of λ are quite close to the estimates derived from psychological experiments for the different levels of sentiment. If TK-sentiment is considered ($a = b = 0.88$) an estimate of $\hat{\lambda} = 2.5379$ is found.
- The lower the level of risk aversion for gains the higher the estimated value of λ. This is a reasonable result as loss- and risk aversion are similar concepts.
- The lower the level of risk-seeking for losses, the lower the estimated value of λ. This is a reasonable result as these are opposite concepts.
- A higher level of θ results in higher estimates of λ. A higher level of θ corresponds to a wider range of returns considered as losses which results in a higher level of loss aversion.

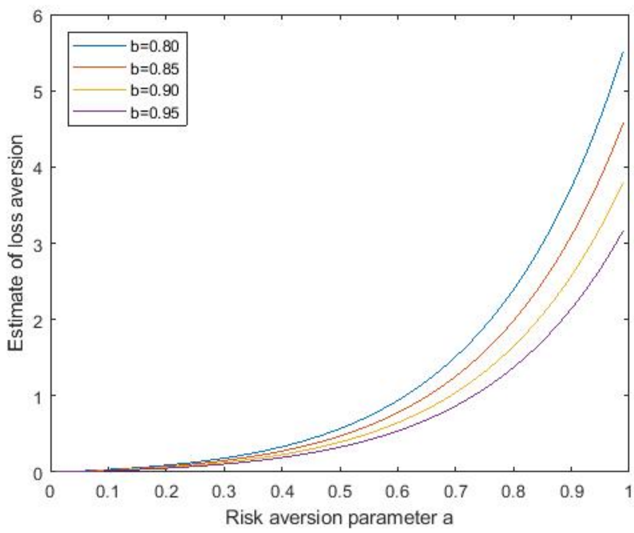


Figure 5.1: Estimates of loss aversion parameter λ from prospect value function for different levels of risk aversion a and risk-seekingness b based on historical returns from 1927 till 2017.

5.2. HEDGING STRATEGIES

In this section, firstly the idea of hedging used to derive the Black-Scholes PDE is considered [13]. Hereafter, the described hedging strategy is performed and the results are discussed. Finally, the hedge strategy for the case of asset paths in which sentiment is included is described and performed and the results are discussed.

5.2.1. THEORY

First, in this section the derivation of the hedging strategy under Black-Scholes is considered [13]. As seen in Section 3.1, the Black-Scholes option value is found by setting up a replicating portfolio consisting of assets and cash that replicates the risk of the option at all time. The portfolio value can be described by the following equation:

$$\Pi(S,t) = A(S,t)S + D(S,t), \tag{5.5}$$

in which A represents the number of assets and D the cash deposit. The number of assets is kept constant over a timestep dt. Then, the changes in the portfolio value are caused by fluctuations in the asset price and interest on the cash deposit. The portfolio dynamics can be written as:

$$d\Pi_i = A_i dS_i + rD_i dt \tag{5.6}$$

Then, the value of portfolio Π_{i+} at $t_i + dt$ is given by:

$$\Pi_{i+1} = A_i S_{i+1} + (1 + r dt) D_i \tag{5.7}$$

The number of assets is changed to A_{i+1} and the cash holding to D_{i+1}. As the system is closed, which means that no money is added or removed from the system, the new portfolio, $A_{i+1}S_{i+1} + D_{i+1}$ has to be equal to Equation (5.7), i.e.:

$$A_i S_{i+1} + (1 + r dt) D_i = A_{i+1} S_{i+1} + D_{i+1}, \tag{5.8}$$

which implies that

$$D_{i+1} = (1 + r dt) D_i + (A_i - A_{i+1}) S_{i+1}. \tag{5.9}$$

The hedging strategy under Black-Scholes can be summarized as:

- Set $A_0 = \frac{\partial V_0}{\partial S}$, $D_0 = 1$, $\Pi_0 = A_0 S_0 + D_0$.
- For each time $t = (i+1)dt$:
  - Observe S_{i+1},
  - Compute new portfolio value Π_{i+1} as in Equation (5.7),
  - Compute $A_{i+1} = \frac{\partial V_{i+1}}{\partial S}$,
  - Compute the new asset holding D_{i+1} as in Equation (5.9). Then, the new portfolio equals $A_{i+1}S_{i+1} + D_{i+1}$.

The hedging strategy is discrete as rebalancing is done at times idt. The smaller dt, the smaller the error in the risk elimination.

5.2.2. HEDGING TEST UNDER BLACK-SCHOLES

Now an example of the implementation of the described hedging test is given for a European call option. The parameters used are $S_0 = 1$, $\sigma = 0.4$, $K = 1.5$, $\alpha = 0.05$, $r = 0.03$, $T = 5$, $dt = 10^{-2}$ and the number of timestep equals $N = 500$. Figure 5.2 shows a discrete GBM asset path (t_i, S_i), the corresponding delta $(t_i, \frac{\partial C_i^{BS}}{\partial S})$ which represents the amount of assets held in the portfolio, the cash level (t_i, D_i) and the portfolio value (t_i, Π_i). As can be seen, the asset path ends in-the-money at expiry, which is in line with the delta-level of one at expiry. In Figure 5.3 another example is given, but now the strike equals $K = 2.5$. In this example the asset path expires out-of-the-money and the delta-level is zero at expiry. As can be seen, in the Black-Scholes framework the writer is able to deliver the asset in all cases.

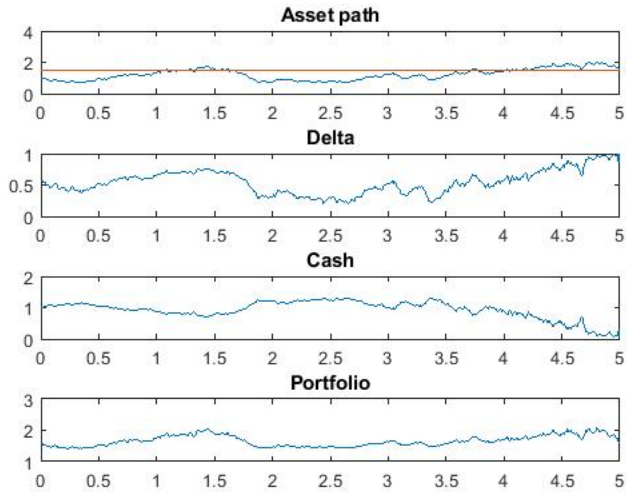


Figure 5.2: Hedging simulation. First graph from above: asset path, second graph: delta values, third graph: cash holding and fourth graph: portfolio value. The parameters used equal $S_0 = 1$, $\sigma = 0.4$, $K = 1.5$, $\alpha = 0.05$, $r = 0.03$, $T = 5$, $dt = 10^{-2}$ and $N = 500$.

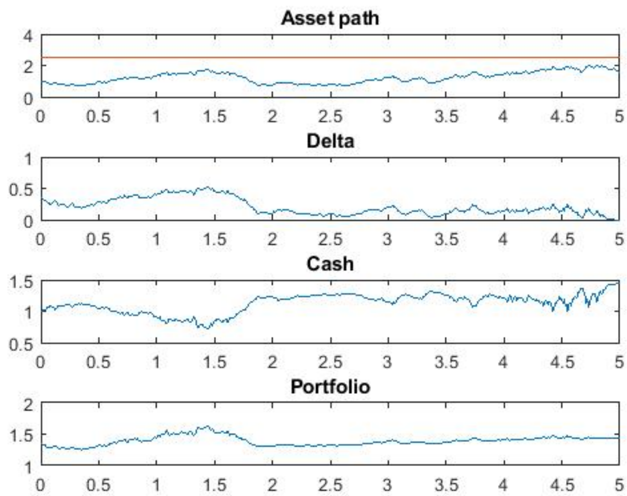


Figure 5.3: Hedging simulation. First graph from above: asset path, second graph: delta values, third graph: cash holding and fourth graph: portfolio value. The parameters used equal $S_0 = 1$, $\sigma = 0.4$, $K = 2.5$, $\alpha = 0.05$, $r = 0.03$, $T = 5$, $dt = 10^{-2}$ and $N = 500$.

5.2.3. HEDGING TEST UNDER PROSPECT SENTIMENT

In the examples discussed, GBM-dynamics are assumed for the asset price paths. An interesting case is to investigate the influence of adding sentiment to the price paths on the hedging strategy. In other words, it is assumed that people behave according to prospect theory, but the hedger is not aware of this and sets up his hedge according to the Black-Scholes framework. Then, it is considered whether the delta-hedge is sufficient or not.

Before turning to the results, first a way of including prospect theory into the asset price dynamics is discussed. To this end, GBM-dynamics are still considered, but now subjectively evaluated by prospect investors. In order to simplify the problem, a prospect investor under loss aversion is considered. This means that outcomes are evaluated by a value function which incorporates loss aversion, risk aversion for gains and risk-seekingness for losses as in Section 4.4. The prospect investor monitors fluctuations in the price path $S(t)$ every timestep t and gives a value to these fluctuations. As a reference point the asset price of the previous timestep is taken. Then, the asset path under sentiment $S_{sent}(t)$ at time t is given by:

$$S_{sent}(t) = \begin{cases} S(t-1) + (S(t) - S(t-1))^a & \text{if } S(t) \geq S(t-1), \\ S(t-1) - \lambda\,(S(t-1) - S(t))^b & \text{if } S(t) < S(t-1). \end{cases} \tag{5.10}$$

Now, comparisons between paths following GBM-dynamics and paths under prospect sentiment are considered, both using the same random numbers. The parameters used are as follows: $S_0 = 1$, $\sigma = 0.2$, $\alpha = 0.05$, $T = 5$, $dt = 10^{-2}$ and $N = 200$. In order to illustrate the impact of sentiment on the asset price path, three levels of sentiment are considered which are derived from TK-sentiment ($a = b = 0.88$, $\lambda = 2.25$):

- Sentiment 1: $a = 1$, $b = 1$ and $\lambda = 2.25$.
- Sentiment 2: $a = 1$, $b = 0.88$ and $\lambda = 1$,
- Sentiment 3: $a = 0.88$, $b = 1$ and $\lambda = 1$,

In Figures 5.4, 5.5 and 5.6 the results are given. As can be seen, under Sentiment level 1, negative changes in the value of the asset price path are enlarged. Under Sentiment level 2 and Sentiment level 3 the influence of the risk-seeking attitude for losses and risk averse attitude for gains is clearly visible.

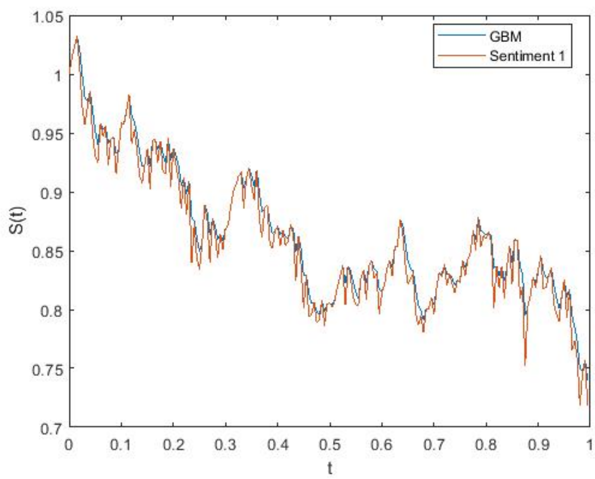


Figure 5.4: Sentiment 1 v.s. GBM. The parameters used are as follows: $S_0 = 1$, $\sigma = 0.2$, $r = 0.03$, $\alpha = 0.05$, $T = 5$, $dt = 10^{-2}$ and $N = 200$.

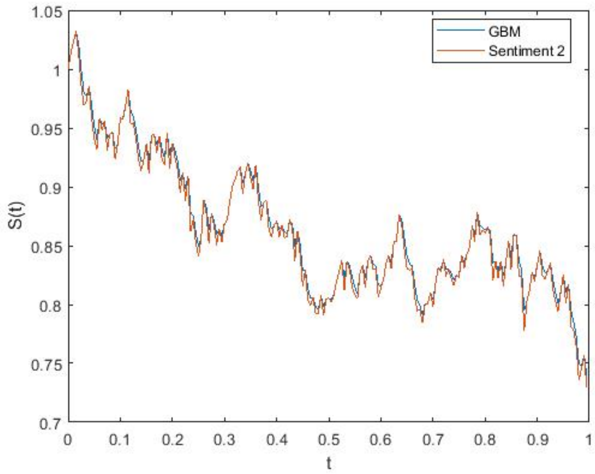


Figure 5.5: Sentiment 2 v.s. GBM. The parameters used are as follows: $S_0 = 1$, $\sigma = 0.2$, $r = 0.03$, $\alpha = 0.05$, $T = 5$, $dt = 10^{-2}$ and $N = 200$.

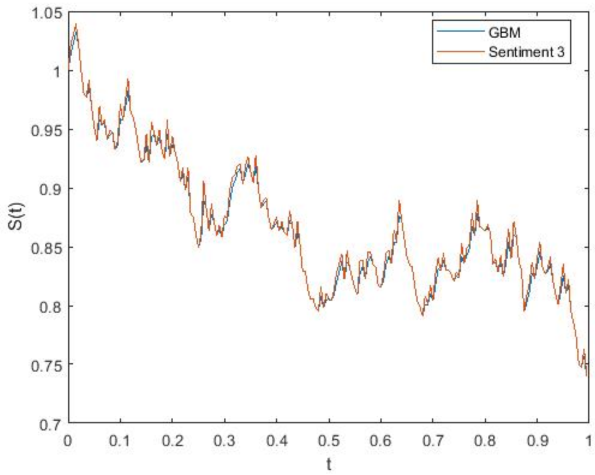


Figure 5.6: Sentiment 3 v.s. GBM. The parameters used are as follows: $S_0 = 1$, $\sigma = 0.2$, $r = 0.3$, $\alpha = 0.05$, $T = 5$, $dt = 10^{-2}$ and $N = 200$.

As we now have considered the dynamics of asset paths of prospect investors, the results of performing the hedge test as in Section 4.2.2 are now considered. The level of prospect sentiment used is TK-sentiment. The results of the quality of the hedge portfolio are compared by means of the variance of the hedge portfolio for different values of the rebalancing frequency. If we look at the portfolio value $|C(S(T),T) - \Pi(S(T),T) - (C(S_0,0) - \Pi(S_0,0))e^{rT}|$ at time T for 10^5 paths and for both GBM and sentiment paths, the results obtained are presented in Table **??**. As can be seen, while the variance of the portfolio under GBM is closer to zero for higher values of the rebalance frequency, the variance for the portfolio under TK-sentiment increases. Therefore, it can be concluded that the Black-Scholes delta-hedge is not sufficient if market prices are approximated by asset paths under sentiment.

Table 5.1: The impact of the re-balancing frequency on the variance of the hedge portfolio at time T.

Rebalance frequency	Variance portfolio under GBM	Variance portfolio under TK-sentiment
5	0.0159	0.0269
50	0.0063	0.0417
500	0.0021	0.0533

6
CONCLUSIONS

In this thesis, several models for decision-making under risk applied to option pricing and portfolio management are investigated. The traditional model for decision-making under risk assumes rational behavior, while investors typically do not behave rationally. Therefore, two models that incorporate irrational behavior are considered and compared. These models are discussed in Chapter 2 and are applied to option pricing and portfolio management.

In Chapter 3, option prices under prospect theory for different levels of prospect sentiment are discussed. Prospect sentiment includes a risk-averse attitude for gains, a risk-taking attitude for losses and overestimation of small probabilities and underestimation of large probabilities. As a result, call option prices are higher than Black-Scholes prices and are progressively higher for a higher level of sentiment.

As cumulative prospect theory is an improved version of prospect theory, the focus is turned to CPT. After having described a way of deriving option values under CPT, numerical examples of option prices are given from both writer's and holder's point of view. Again, the higher the level of sentiment, the higher the option price from a writer's viewpoint. If equal levels of sentiment are considered, the holder's price is lower than the writer's price and the Black-Scholes price is in between. This means that if a writer and a holder have equal levels of sentiment, no trade is possible. The higher the level of sentiment, the wider the difference between holder's and writer's price and thus the less likely a trade. Therefore, cases for which the holder's price equals the writer's price are considered. As a conclusion, in this case the call option price under sentiment for in-the-money options is lower than the Black-Scholes price and is higher for out-of-the-money options which is in line with the overestimation and underestimation of probabilities under sentiment. It should be noted that if the writer behaves according to Black-Scholes and the holder according to CPT, an agreement about the price is only reached in case of no sentiment. Also, the implied volatility is considered for option prices under CPT. As a result, under the highest level of sentiment, a clear volatility skew is visible which is a well known effect in the market.

Hereafter, a sensitivity analysis is done in order to measure the impact of the different aspect of prospect sentiment on the option price. First, expressions for the sensitivity with respect to the prospect parameters and Black-Scholes parameters are derived. Then, the sensitivities are computed under different levels of sentiment. For the sensitivities with respect to the prospect parameters from a writer's viewpoint, it appears that in all cases:

- A higher level of sentiment leads to a higher absolute size of the sensitivities,
- Under TK-sentiment, the risk-seeking parameter for losses is of highest influence on the option,
- The weighting parameter is of lowest influence on the option price.

All signs of the sensitivities towards prospect parameters and Black-Scholes parameters are explained economically. Besides the sensitivity analysis, the impact of negative interest rates on the option price is considered for different levels of sentiment. It can be concluded that the influence of sentiment on the option price is equal under negative and positive interest rates.

Finally, the impact of sentiment on the call option price under different dynamics are compared. Under sentiment, the Heston prices are lower than GBM prices and the impact of adding sentiment under Heston dynamics is lower than under GBM. This difference can be explained by the fact that Heston dynamics already incorporate a certain form of sentiment because of the stochastic volatility. Therefore, the impact of adding extra sentiment is smaller under Heston dynamics than under GBM. This is confirmed by results that show that for GBM the sensitivities towards prospect parameters under sentiment are higher than under Heston dynamics.

A recommendation for further research is to investigate which model under which dynamics is most in line with actual market option prices.

In Chapter 4, portfolio management under different models is discussed. Firstly, the description of portfolio management under different models is given: Modern Portfolio Theory, expected utility theory, prospect theory without weighting function and under cumulative prospect theory. For prospect theory without weighting and for cumulative prospect theory, numerical examples for the optimal wealth profile against the state price density are provided. As as result, the optimal wealth profile under loss aversion is discontinuous, which is in line with the assumed different attitudes towards gains and losses. If different levels of loss aversion are considered, it can be concluded that a higher level of loss aversion results in a larger range of values of the state price density for which the optimal wealth profile is above reference level θ and in a lower optimal wealth profile for values of the state price density below turning point $\overline{\rho}$. This in line with our expectations; if investors are more averse for losses, a larger part of their wealth is used for buying insurances to protect against potential losses. The opposite result is obtained for a higher starting wealth: the higher the starting wealth the lower the optimal wealth for values of the state price density below turning point $\overline{\rho}$; this could be explained by a more careful behavior of the investor in case of a larger start wealth.

If the optimal wealth profile under CPT is considered, the only parameter of impact is the weighting parameter. It can be concluded that for all values of the state price density, a larger value of the weighting parameter results in a higher optimal wealth. A possible explanation for this result is a more rational behavior in case of a larger value of the weighting parameter. It can also be concluded that in better states of the world, the optimal wealth is more sensitive to changes in the weighting parameter.

Finally, the wealth profiles of all models considered are compared. If MPT is compared with PT, it is concluded that the two different models lead to significant different choices of assets based on the mean-variance diagram or average gain-average loss diagram under prospect sentiment. The results could be extended to CPT, if a method similar to the mean-variance diagram or average gain-average loss diagram is derived. Then, EUT is compared with CPT without weighting function or in other words under loss aversion. A first difference is that the optimal wealth profile under EUT is continuous, while under loss aversion discontinuous due to the different risk attitudes for gains and losses under loss aversion. Also, for low values of the state price density the wealth profile under loss aversion is significantly lower than under EUT, which could be explained by the loss averse behavior that leads to extra investment in insurances. If the loss averse wealth profile is compared to the wealth profile under CPT, the wealth profile under CPT is significantly lower than under loss aversion for all values of the weighting parameter and for all values of the state price density. In case of a lower value of the weighting parameter, the differences becomes smaller, as the CPT framework resembles more closely the loss aversion framework. It is not possible to conclude which model leads in general to the highest optimal wealth profile, as it depends on the state price density.

A topic for further research would be to investigate the consequences of combining several models and to translate the different optimal wealth profiles to contents of the corresponding portfolios.

In Chapter 5, a data analysis is done in order to investigate whether the estimates of the parameter for the different levels of prospect sentiment used in the previous chapters are applicable to financial data. To this end, a portfolio of stocks of different indices is considered. The loss aversion parameter is estimated for different levels of risk aversion for gains and for different levels of risk-seeking for losses. The estimate of the loss aversion parameter for TK-sentiment turns out to be quite close the value as used under TK-sentiment. Also, the lower the level of risk aversion for gains, the higher the estimate of the loss aversion parameter, which is reasonable as these concepts are similar and thus can replace each other. Equivalently, the lower the level of risk-seeking for losses, the lower the estimated value of the loss aversion parameter.

Finally, the hedge framework under Black-Scholes is described and translated to the PT-framework. Therefore, first a way of incorporating prospect sentiment into asset path dynamics is developed. Also, numerical examples are given for different levels of sentiment, which show results in line with the earlier obtained sensitivities towards prospect parameters. Hereafter, the described delta hedge test under Black-Scholes is applied to asset paths under sentiment. If the quality of the hedge portfolio under GBM and TK-sentiment is considered by means of the variance of the portfolio, it can be concluded that the portfolio's variance under GBM decreases to zero if the rebalancing frequency increases, while the variance under TK-sentiment does not decrease. Therefore, if a writer uses delta-hedging and asset price paths are described by prospect dynamics instead of GBM, delta hedging is not sufficient. A topic for further research would be to investigate what hedging strategy could be used instead of delta hedging if sentiment is present in the asset price paths.

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